GROUPOIDS WITH ROOT SYSTEMS IN REAL VECTOR SPACES

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Abstract

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The study of root systems attached to groupoids that resemble Coxeter groups has seen many recent developments. We take the notion of signed groupoid set, introduced by Dyer to abstract primitive features of such groupoids with root systems, as our general framework.

The main purpose of this thesis is to discuss realizations of root systems of signed groupoid sets in real vector spaces. Inversion sets and the weak and dominance preorders play a major role, and we discuss realizations in relation to these notions. Compressed signed groupoid sets, for which the dominance preorder is a partial order, are particularly important and we give a construction attaching a compressed realized signed groupoid set to an uncompressed one under certain hypotheses.

In some cases, we establish a correspondence between the inversion sets and hyperplanes that separate the roots of a realization of a compressed signed groupoid set. We then focus on the groupoids defined by Brink and Howlett in their study of normalizers of parabolic subgroups of Coxeter groups. The strongest results we obtain hold when the Coxeter group is finite, in which case we give an isomorphism between a realization of the universal covering of the corresponding signed groupoid set and a realized signed groupoid set arising from a simplicial hyperplane arrangement.
CHAPTER 5: HYPERPLANES AND THE UNIVERSAL COVER  . . . . . . . . 60
  5.1 Signed groupoid sets of hyperplane arrangements . . . . . . . 60
  5.2 Universal Coverings . . . . . . . . . . . . . . . . . . . . . . . . 63
  5.3 Realizations of $\mathcal{H}$ signed groupoid sets . . . . . . . . 67

CHAPTER 6: COXETER GROUPOIDS AND SIGNED GROUPOID SETS . 69
  6.1 Free signed groupoid sets . . . . . . . . . . . . . . . . . . . . 70
  6.2 Coxeter groups . . . . . . . . . . . . . . . . . . . . . . . . . . . 73
  6.3 Signed groupoid sets arising from Coxeter groups . . . . . . . 76
  6.4 Properties of $(\hat{W}(\Pi), \hat{\Phi})$ and its connected components . . . . . . 78
  6.5 Signed groupoid sets associated to finite Coxeter groups . . . . 81
  6.6 Hyperplanes and the universal covering . . . . . . . . . . . . . 84
  6.7 A realized signed groupoid set from $D_4$ . . . . . . . . . . . . 87

BIBLIOGRAPHY . . . . . . . . . . . . . . . . . . . . . . . . . . . 98
FIGURES

6.1 The Coxeter graph of $(W, S)$ .................................................. 79
6.2 The simple morphisms of $(\hat{W}(\{\alpha\}), \Phi_{\{\alpha\}})$ ......................... 80
6.3 Coxeter graph for $D_4$ ................................................................. 88
6.4 The simple morphisms of $(\hat{W}(\Delta), \Phi_\Delta)$ .............................. 90
TABLES

6.1 THE SIMPLE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta)^{\text{re}}$ .......................... 92
6.2 THE POSITIVE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta)^{\text{re}}$ .......................... 93
6.3 THE SIMPLE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta/\Pi)$ ................................. 94
6.4 THE POSITIVE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta/\Pi)$ ................................. 95
6.5 THE SIMPLE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta^*/\Pi)$ ............................... 96
6.6 THE POSITIVE ROOTS OF $\hat{W}(\Delta, \Phi_\Delta^*/\Pi)$ ............................... 96
INTRODUCTION

In this thesis we study Coxeter group-like groupoids with root systems. Recently work on groupoids with root systems that resemble Coxeter groups has been done by a number of people in different contexts. One line of development comes from the work by Deodhar in [8], Howlett in [13], Brink in [3], and Brink-Howlett in [4] and [5]. Another course of investigation involves Heckenberger and Yamane in [12] and Cuntz and Heckenberger in [7]. A framework which includes this work, while abstracting and extending many of these developments is sketched, partly without proof, by Dyer in [10], [11], and [9].

The notions of a root systems in a real vector space for a groupoid is taken as part of the definition in [12] and [7], though not considered in [8], [13], [3], [4] and [5]. In [10], [11], [9], several abstract notions of root systems are considered, but their realizability in real vector spaces is not guaranteed. Thus it remains an interesting question to ask to what extent are the properties of these realization preserved by the constructions of Dyer in [10], [11], [9]. When studying root systems of a groupoid, one major difference from studying root systems in a Coxeter group is the fact that for every object of the groupoid there is a distinct component on the root system, where each component lies in a different vector space. A notion of signed groupoid sets was introduced by Dyer in order to abstract the most primitive common features of groupoids with root systems.

In this thesis we consider various types of realizations of signed groupoid sets in real vector spaces. We describe these realized root systems for groupoids in the context of the construction of Brink and Howlett found in [4] and [5] and produce an
example that demonstrates that in such a root system, the set of roots do not necessarily lie in the span of the simple roots. Our main result concerns the construction of Brink and Howlett in the case that the Coxeter group is finite. In this situation the set of roots lie in the span of the simple roots. We then show that the universal cover of any component of this realized signed groupoid set is isomorphic, up to rescaling of roots, to the signed groupoid set attached to a simplicial hyperplane arrangement consisting of hyperplanes orthogonal to the roots.

Chapters 2-4 describe basic properties of signed groupoid sets, including their weak order, dominance order and the notion of compression. Our development mostly follows that of [10], [11] and [9] with some differences, particularly in our notion of simply generated and principal signed groupoid sets.

Chapter 1 introduces signed sets and signed groupoid sets and lays the foundation for the results in this thesis. From the definition of a signed groupoid set we attach a root system to a groupoid. We give special attention to properties inversion sets in signed groupoid sets, as well as introducing the cocyle map and the weak right preorder on morphisms in a signed groupoid set. We conclude Chapter 1 by briefly discussing faithful signed groupoid sets.

In Chapter 2 we discuss the dominance preorder on a root system of a signed groupoid set. We begin by introducing real and imaginary roots and classify the real roots in terms of inversion sets. Then we introduce the dominance preorder and relate it to properties of inversion sets. Following this, we discuss equivalences classes induced by the dominance preorder on the root system. This allows us to construct new signed groupoid sets from the equivalence classes. We conclude Chapter 2 by introducing compressed signed groupoid sets where each equivalence class is a singleton. Additionally we provide a method of real compression that allows us to take an signed groupoid set and produce a similar real compressed signed groupoid set. Of special importance in this section is a discussion regarding the properties that this
real compression preserves.

Chapter 3 discusses signed groupoid sets that have a generating set, known as a simple set, which satisfy certain conditions. This are called simply generated signed groupoid sets. Since we have a generating set for our groupoid this allows to examine properties of a simply generated signed groupoid by studying its generating set. From a simple set we use their inversion sets to produce a set of simple roots for a root system. We spend a significant portion of this chapter discussing the properties of the dominance order and inversion sets in terms of the simple morphisms. Each simple root is a simple equivalence class. Thus when studying compressed signed groupoid sets, we have a set of simple roots that consists simply of roots in the root system. We conclude Chapter 3 by introducing principal signed groupoid sets, which allow us to study inversion sets of morphisms in terms of inversion sets of simple roots.

Chapters 4-6 contain the main new contributions of this thesis to the study of realizations in real vector spaces of root systems of signed groupoid sets. In Chapter 4 we define a notion of a realization of the (abstract) root systems of a signed groupoid set in real vector spaces. Each object of the groupoid has its own root system realized in a its own vector space. We first discuss properties of real inner product spaces necessary to construct this realization. We then introduce realized signed groupoid sets. For the rest of the chapter we discuss various properties of realized signed groupoid sets. We specifically discuss the compressed realized signed groupoid sets and produce a method of compressing a realized signed groupoid set. Compressed realized signed groupoid sets allow us to consider the simple roots as a set of vectors, and thus there are various geometric properties involving this. A special type of this is a conical realized signed groupoid set, which occurs when the positive roots of a root system lie in the positive span of the simple roots. We conclude the chapter by studying the relationship between hyperplanes in a realized signed groupoid set and its inversion sets. Of note are Theorem 4.6.1 and Theorem 4.6.2 that give a
correspondence between hyperplanes separating subsets of the positive roots and the
inversion sets of the signed groupoid set. In particular, the result in Theorem 4.6.2
is critical when discussing constructions involving Coxeter groups in Chapter 6.

In Chapter 5 we discuss properties signed groupoid sets and hyperplane arrange-
ments. Here we produce a realized signed groupoid set that comes from a hyperplane
arrangement. This construction of the underlying signed groupoid set may be found
in [10]. In this the objects of the groupoid are chambers of hyperplane arrange-
ment, while morphisms are maps between chambers. The root system of the signed
groupoid set comes from vectors normal to the hyperplanes. Also in this chapter we
discuss the universal covering of a groupoid and produce a universal covering for a
signed groupoid set and its realization.

Chapter 6 introduces groupoids that come from Coxeter systems, as found in [4]
and [5] and studies properties of their root systems. This chapter contains many
of the main results of this thesis. We begin by introducing these constructions and
reviewing properties of Coxeter groups and their roots systems. The Brink Howlett
construction attaches a groupoid to a Coxeter group. The objects of this groupoid are
subsets of the simple roots in which a morphism $\Delta \to \Delta'$ is a Coxeter group element
$w$ where $w(\Delta) = \Delta'$ for objects $\Delta, \Delta'$. We consider the connected components of this
groupoid, paying special attention to the case where these groupoids come from a
finite Coxeter group. We show in Theorem 6.5.1 that when we study the compression
of these realized signed groupoid sets in the quotient vector space, we must have a
conical realization. In Theorem 6.5.2 we give a classification of hyperplanes in this
realization in terms of inversion sets. Then in Theorem 6.6.1 we show that the
realization from universal covering is isomorphic to the realized signed groupoid set
that arises from the chambers of a hyperplane arrangement. We conclude Chapter
6 by discussing an example of a realized signed groupoid set that comes from the
Coxeter group $D_4$ which embodies these properties.
A notion of simply generated signed groupoid sets is fundamental to our results. We take simple generation as a hypothesis in our formulations here. It is a much deeper matter to show that natural examples are simply generated. In this thesis we mainly consider examples of constructions from Brink and Howlett. However, in [10], [11] and [9], Dyer sketches, without proof, several results showing that such groupoid sets are very ubiquitous and are preserved by many natural categorical constructions. Thus there exist many interesting examples this framework may be applied to which are not discussed in this thesis.
CHAPTER 1

SIGNED GROUPOID SETS

This first chapter introduces the most general objects and constructions discussed in this thesis. We use the language of category theory to introduce these. We first discuss the category of signed groupoid sets and discuss many of their properties in sections 1.1 - 1.3. In Section 1.4 we construct a root system for a signed groupoid set and introduce the inversion sets in the root system. Section 1.5 introduces the cocycle map on a root system which allows us to study properties of inversion sets further. In section 1.6 we introduce the weak order on the set of morphisms of a groupoid in a signed groupoid set. The weak order further reveals properties of the morphisms in a signed groupoid set as well as provides a useful tool when studying the behavior of different morphisms within the root system in a signed groupoid set. Section 1.7 classifies the special case of faithful signed groupoid sets which allows us to refine our study of inversion sets of a root system. These constructions are introduced by Dyer and studied in further generality in [10], [11] and [9].

1.1 Signed Sets

In order to construct a signed groupoid set we first introduce introduce the sign group and the free left action of a group on a set. The sign group is an interoperation of the cyclic group of order 2. The free left action of the sign group on a set is critical when discussing signed sets which provide the framework for signed groupoid sets. The action of the sign group on a set is studied in numerous ways throughout this thesis with many properties relying on specifications of the this action.
Definition 1.1.1. (a) The sign group \{±\} is the group of order 2. The elements of \{±\} are \{+, −\} where + is the identity.

(b) An action of the group \(H\) on a set \(Θ\) is said to be free if whenever \(h \in H, x \in Θ\) such that \(h.x = x\), then \(h\) is the identity element of the group \(H\) (i.e. all the isotropy groups are trivial).

The free action of the sign group allows us to define signed sets. We first introduce indefinitely signed sets which allows us to discuss definitely signed sets. Definitely signed sets distinguish elements as either positive or negative. Definitely signed sets are required when discussing signed groupoid sets. Much of the work in this thesis regarding signed groupoid sets discusses the positive and negative elements of definitely signed sets.

Definition 1.1.2. (a) An indefinitely signed set is a set \(Θ\) together with a free left action of the sign group.

(b) A definitely signed set is a pair \((Θ, Θ^+)\) where \(Θ\) is an indefinitely signed set and \(Θ^+ \subseteq Θ\) is a set of \{±\}-orbit representatives of \(Θ\).

For a definitely signed set \((Θ, Θ^+)\) we will refer to \(Θ^+\) as the positive representatives of \(Θ\). Additionally we write \(Θ^- := −Θ^+ = \{-s : s ∈ Θ^+\}\), which we refer to as the negative representatives of \(Θ\).

An important fact that is that \(Θ = Θ^+ ∪ Θ^-\). Thus it is the case that an element of a definitely signed set must either be positive or negative, but not both. This fact enables us to discuss the sign of an element of a definitely signed set, and allows us to study properties determined by its sign. Throughout this thesis we may refer to a definitely signed set \((Θ, Θ^+)\) simply as the signed set \(Θ\). When this occurs we implicitly assume that a set of positive representatives \(Θ^+\) is defined, although it may not be explicitly stated.

Now that we have defined signed sets we will discuss functions that map signed sets to signed sets. When considering these functions it is useful to pay special attention to how these functions behave with respect to the sign of an element. A
special case of this is a functions that sends positive elements to positive elements and negative elements to negative elements. This is called a positivity preserving function and is defined below.

**Definition 1.1.3.** A positivity preserving function between signed sets is a function $f: \Theta \to \Theta'$ where $\Theta, \Theta'$ are signed sets such that for every $\zeta \in \Theta$, we have $f(\Theta_+) \subseteq \Theta'_+$ and $f(\epsilon \zeta) := \epsilon f(\zeta)$ for any $\epsilon \in \{\pm\}$.

Since the sign of an element in signed groupoid set plays a vital role in the study of signed groupoid sets, it is useful to know when a function that does not change the sign of an element. The positivity preserving functions play a special role in signed groupoid sets.

1.2 Some categories involving signed sets

Since the language of category theory is useful when discussing the behavior of algebraic objects, we will use category theory throughout this thesis to discuss our constructions and how they relate to each other. Many properties of the objects studied in this thesis are made evident thorough the language of category theory. In order to discuss category theory we use the following conventions for a category $C$:

- $\text{Ob}(C)$ denotes the objects of $C$
- $\text{Mor}(C)$ denotes the morphisms of $C$.
- If $f \in \text{Mor}(C)$, $\text{dom}(f)$ denotes the domain of $f$
- $\text{cod}(f)$ denotes the codomain of $f$.
- $aC := \{g \in \text{Mor}(C) : \text{cod}(g) = a\}$
- $C_b := \{g \in \text{Mor}(C) : \text{dom}(g) = b\}$.
- $aC_b := \{g \in \text{Mor}(C) : \text{dom}(g) = b, \text{cod}(g) = a\}$.
Recall that a category where every morphism is invertible is called a groupoid. Since this thesis mainly studies properties of groupoids, we now introduce the representation of a groupoid in a category. This allows us to develop constructions involving groupoids as well as discuss properties of a groupoid using properties of its representation in a category.

**Definition 1.2.1.** \( \Gamma : G \to C \) is a representation of groupoid \( G \) in a category \( C \) if \( \Gamma \) is a functor.

Since groupoids play such a crucial role in this thesis it is necessary to discuss subgroupoids and the functors we use for them. The following is a natural category theory construction that we specifically use when discussing groupoids.

**Definition 1.2.2.** Let \( C, G \) be categories such that \( G \) is a groupoid, and let \( H \) be a subgroupoid of \( G \), and let \( F : G \to C \) be a functor.

(a) \( i_H : H \to G \) is the inclusion functor.

(b) \( F_H : H \to C \), the restriction of \( F \) to \( H \), is the functor \( F_H := F \circ i_H \).

Now we introduce some categories involving signed sets. Many of the constructions in this thesis involve these categories and discuss representations of groupoids in the these categories.

**Definition 1.2.3.**

(a) \( \text{Set} \) is the usual category whose objects are sets and whose morphisms are functions between sets.

(b) The category of indefinitely signed sets \( \text{Set}_{\{\pm\}} \) is the category such that \( \text{Ob}(\text{Set}_{\{\pm\}}) \) consists of the indefinitely signed sets and \( \text{Mor}(\text{Set}_{\{\pm\}}) \) consists of the \( \{\pm\} \)-equivariant functions where composition of morphisms in \( \text{Set}_{\{\pm\}} \) is composition of functions.

(c) \( \text{Set}_\pm \), the category of definitely signed sets, is the category where \( \text{Ob}(\text{Set}_\pm) \) consists of the definitely signed sets and \( \text{Mor}(\text{Set}_\pm) \) consists of \( \{\pm\} \)-equivariant functions where composition of morphisms in \( \text{Set}_\pm \) is composition of functions.

(d) \( \text{Set}_{+, -} \) is the subcategory of \( \text{Set}_\pm \) where \( \text{Ob}(\text{Set}_{+, -}) = \text{Ob}(\text{Set}_\pm) \) and \( \text{Mor}(\text{Set}_{+, -}) \) are the sets of positivity preserving functions in \( \text{Mor}(\text{Set}_\pm) \).
In particular \( \text{Set}_\pm \), the category of definitely signed sets plays a major role in the study of signed groupoid sets. Throughout this thesis we will further discuss the categories \( \text{Set}_\pm \) and \( \text{Set}_{+-} \), and study properties of representations of groupoids in these categories.

1.3 Signed groupoid sets

We are now ready to define signed groupoid sets, one of the main objects studied in this thesis. Signed groupoid sets give a representation of a groupoid in the category of definitely signed sets. Using this representation we can study properties of a groupoid based on its action on signed sets. These properties of signed sets attached to a groupoid in a signed groupoid set allow us to introduce root systems to signed groupoid sets. Thus many of these properties of signed groupoid sets are discussed using the language of root systems.

**Definition 1.3.1.** The pair \( (G, \Phi) \) is a signed groupoid set if \( G \) is a groupoid and \( \Phi : G \to \text{Set}_\pm \) is a functor.

**Remark 1.3.1.** Whenever we discuss a signed groupoid set \( (G, \Phi) \), unless otherwise noted we assume that for every distinct \( a, b \in \text{Ob}(G) \) that \( \Phi(a) \cap \Phi(b) = \emptyset \).

Now that we have given a formal definition of a signed groupoid set, we may discuss the category of signed groupoid sets. Morphisms and isomorphisms of signed groupoid sets play a crucial role in studying the properties of signed groupoid sets. Throughout this thesis we take special note of properties that are preserved by morphisms or isomorphisms.

**Definition 1.3.2.** \( \text{Gpd} - \text{Set}_\pm \) is the category where \( \text{Ob}(\text{Gpd} - \text{Set}_\pm) \) consists of all signed groupoid sets and \( \text{Mor}(\text{Gpd} - \text{Set}_\pm) \) consists of the maps \( f : (G, \Phi) \to (H, \Psi) \) for \( (G, \Phi), (H, \Psi) \in \text{Ob}(\text{Gpd} - \text{Set}_\pm) \), where \( f := (\alpha, \nu) \) such that
i. \( \alpha : G \to H \) is a functor

ii. \( \nu : \Psi \alpha \to \Phi \) is a natural transformation of functors \( G \to \text{Set}_\pm \) where for each \( a \in \text{Ob}(G) \) the component \( \nu_a : \Psi(\alpha(a)) \to \Phi(a) \) is a positivity preserving map.

The composite of two morphisms \( f : (G, \Phi) \to (H, \Psi) \) and \( g : (H, \Psi) \to (K, \Lambda) \) in \( \text{Gpd} - \text{Set}_\pm \) where \( f := (\alpha, \nu) \) and \( g := (\beta, \mu) \) is defined as \( gf := (\beta, \mu)(\alpha, \nu) : (G, \Phi) \to (K, \Lambda) \) such that \( gf := (\beta \alpha, \nu(\mu \alpha)) \) with for each \( a \in \text{Ob}(G) \) \( (\nu(\mu \alpha))_a = \nu_a \mu(\alpha)_a \).

One checks that an isomorphism in \( \text{Gpd} - \text{Set}_\pm \) is a morphism \( (\mathcal{F}, \nu) : (G, \Phi) \to (H, \Psi) \) in \( \text{Gpd} - \text{Set}_\pm \) such that \( \mathcal{F} : G \to H \) is an isomorphism and for every \( a \in \text{Ob}(G) \), the component \( \nu_a \in \text{Mor}(\text{Set}_\pm) \) is a positivity preserving isomorphism in \( \text{Set}_\pm \), (i.e. \( \nu \) is a natural isomorphism).

It is useful to consider different representations of the same groupoid in \( \text{Set}_\pm \). This lead us to introduce the following category consisting of the signed groupoid sets of the same groupoid.

**Definition 1.3.3.** For a groupoid \( G \), \( G - \text{Set}_\pm \) is the subcategory of \( \text{Gpd} - \text{Set}_\pm \) consisting of all objects \( (G, \Phi) \) for a fixed \( G \), and all morphisms of the form \( (\text{Id}_G, \nu) \in \text{Mor}(\text{Gpd} - \text{Set}_\pm) \).

**Remark 1.3.2.** Observe that for an isomorphism of \( G - \text{Set}_\pm \) of the form \( f := (\text{Id}_G, \nu) : (G, \Phi) \to (G, \Psi) \) we have that \( f^{-1} := (\text{Id}_G, \nu^{-1}) \) where \( (\nu^{-1})_a = (\nu_a)^{-1} \) for every \( a \in \text{Ob}(G) \).

Throughout this thesis we discuss many of the ways signed groupoid sets may be studied. There are many different properties we use to classify signed groupoid sets. We discuss faithful signed groupoid sets in Section 1.7 while Chapter 2 discusses real and compressed signed groupoid sets and Chapter 3 discusses different aspects of simply generated signed groupoid sets. Although these are classifications are not mutually exclusive and some are more general than others, these classifications provide useful tools for applications of signed groupoid sets.
1.4 Root systems and inversion sets

The first construction relating to signed groupoid sets that we will introduce is that of a root system attached to a signed groupoid set. For a signed groupoid set \((G, \Phi)\), the functor \(\Phi\) enables us to define a root system for every object of \(G\). Of special interest are the positive and negative roots which correspond to the positive and negative subsets of the signed set.

**Definition 1.4.1.** Let \((G, \Phi) \in \text{Ob}(\text{Gpd} - \text{Set}_\pm)\) with \(a, b \in \text{Ob}(G), g \in \text{aG}_b\).

\[
\begin{align*}
(a) & \ \ \ a\Phi := (\Phi(a), \Phi(a)^+) \in \text{Ob}(\text{Set}_\pm) \\
(b) & \ \ \ a\Phi^+ := \Phi(a)^+ \text{ and } a\Phi^- := -(a\Phi^+) \text{ We call } a\Phi^+ \text{ the positive roots at } a, \text{ and } a\Phi^- \text{ the negative roots at } a.
\end{align*}
\]

When studying positive and negative roots of a root system for a signed groupoid set it is useful to consider how morphisms act on root systems based on the sign on an element. Inversion sets allow us to study exactly this. The inversion set of a morphism in the representation is defined as the set of positive roots in the codomain that are the image of a negative root under the morphism. We also define a generalized notion of an inversion set. Although a more general property, this notion preserves properties that the standard inversion set does not, particularly when studying composition of morphisms in a signed groupoid set.

**Definition 1.4.2.** Let \((G, \Phi) \in \text{Ob}(\text{Gpd} - \text{Set}_\pm)\) with \(a, b \in \text{Ob}(G), g \in \text{aG}_b\).

\[
\begin{align*}
(a) & \ \ \ \Phi_g := a\Phi^+ \cap \Phi(g)(b\Phi^-) \\
(b) & \ \ \ \Phi_g := \Phi_g \cup (-\Phi_g)
\end{align*}
\]

For \(x \in a\Phi, \ g \in G_a\) we will write \(\Phi(g)(x) = gx\). Similarly for \(A \subseteq a\Phi\) we may write \(\Phi(g)(A) = g(A)\). For example, we may rewrite Definition 1.4.2(c) as \(\Phi_g := a\Phi^+ \cap g(b\Phi^-)\). We call \(a\Phi\) the root system of \((G, \Phi)\) at \(a\), while \(\Phi_g\) is the inversion set for \(g\).
Remark 1.4.1. Let \((G, \Phi)\) be a signed groupoid set.

(a) If \(a, b \in \text{Ob}(G), g \in {}_aG_b\), then since \(G\) is a groupoid we see that \(g(\Phi(b)) = \Phi(a)\).

(b) \(\Phi_{\text{Id}} = \emptyset\) for every identity morphism of \(G\).

(c) If \(H\) is a subgroupoid of \(G\), then \((H, \Phi_H)\) is a signed groupoid set. Additionally \((i_H, \text{Id}) : (H, \Phi_H) \to (G, \Phi) \in \text{Mor}(\text{Gpd} - \text{Set}_\pm)\).

This next lemma demonstrates how inversion sets behave with respect to morphisms in \(\text{Gpd} - \text{Set}_\pm\).

Lemma 1.4.1. If \((\rho, \nu) : (G, \Phi) \to (H, \Psi)\) is a morphism in \(\text{Gpd} - \text{Set}_\pm\) with \(g \in \text{Mor}(G)\) then \(\nu^{-1}_a(\Phi(g)) = \Psi(\rho(g))\).

Proof. Let \(a, b \in \text{Ob}(G)\) such that \(g \in {}_aG_b\). Since \(\nu : \Psi \alpha \to \Phi\) is a natural transformation we see that the diagram below commutes.

\[
\begin{array}{ccc}
_b\Phi & \xrightarrow{\Phi(g)} & a\Phi \\
\downarrow{\nu_b} & & \downarrow{\nu_a} \\
\Psi(\rho(b)) & \xrightarrow{\Psi(\rho(g))} & \rho(a)\Psi
\end{array}
\]

Thus we see that

\[
\nu^{-1}_a(\Phi(g)) = \nu^{-1}_a(a\Phi^+ \cap \Phi(g)b\Phi^-) = \nu^{-1}_a(a\Phi^+) \cap \nu^{-1}_a(\Phi(g)(b\Phi^-)),
\]

and since \(\nu_a\) and \(\nu_b\) are positivity preserving functions we see that both \(\nu^{-1}_b(b\Phi^-) = \rho(b)\Phi^-\) and \(\nu^{-1}_a(a\Phi^+) = \rho(a)\Psi^+\). Observe that \(\Phi(g)\) and \(\Psi(\rho(g))\) are bijections, and their inverses are \(\Phi(g^{-1})\) and \(\Psi(\rho(g)^{-1})\) respectively. Therefore

\[
\nu^{-1}_a(\Phi(g)(b\Phi^-)) = (\Phi(g^{-1})\nu_a)^{-1}(b\Phi^-)
\]

\[
= (\nu_b(\Psi(\rho(g)^{-1})))^{-1}(b\Phi^-)
\]

\[
= \Psi(\rho(g))\nu^{-1}_b(b\Phi^-) = \Psi(\rho(g))(\rho(b)\Psi^-),
\]
and so we see that $\nu_{\rho(a)}^{-1}(\Phi_g) = \rho(\Psi(\rho(g))(r_{\rho(b)}\Psi^+ - \Psi(\rho(g))).$

While studying inversion sets, such as $\Phi_g$ or $\hat{\Phi}_g$, it can be helpful to consider these sets as an element of the power set of the root system. Thus it is necessary to discuss the symmetric difference and some its properties which play a key role in the study of inversion sets.

**Remark 1.4.2.** Let $X$ be a set and $\mathcal{P}(X)$ the power set of $X$. For $A, B \in \mathcal{P}(X)$ we define the symmetric difference of $A$ and $B$ to be $A + B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. For $A, B, C \in \mathcal{P}(X)$ we have the following:

(i) $A + A = \emptyset$

(ii) $A + B = B + A$

(iii) $A \cap (B + C) = (A \cap B) + (A \cap C)$

(iv) $A + (B + C) = (A + B) + C$.

In fact, $\mathcal{P}(X)$ is a boolean ring where $+$ is addition, $\cap$ is multiplication and $\emptyset$ is the identity element.

In the next few lemmas we demonstrate the usefulness of the above properties when studying inversion sets. This first lemma shows the role of the symmetric difference in inversion sets.

**Lemma 1.4.2.** If $(G, \Phi)$ is a signed groupoid set with $g, h \in \text{Mor}(G)$ such that $\text{dom}(g) = \text{cod}(h)$ then $\Phi_{gh} \subseteq \Phi_g + g(\Phi_h)$

**Proof.** Let $a, b, c \in \text{Ob}(G)$ such that $g \in aG_b, h \in gG_c$. Observe that

$$\Phi_{gh} \setminus \Phi_g = (a\Phi^+ \cap gh(c\Phi^-)) \setminus (a\Phi^+ \cap g(b\Phi^-)) = a\Phi^+ \cap gh(c\Phi^-) \cap g(b\Phi^+) = \Phi_{gh} \cap g(\Phi_h).$$
Thus $\Phi_{gh} \subseteq \Phi_g \cup g(\Phi_h)$. Additionally we see that $\emptyset = \Phi_g \cap (\Phi_{gh} \cap g(\Phi_h)) = \Phi_{gh} \cap (\Phi_g \cap g(\Phi_h))$, and so $\Phi_{gh} \subseteq \Phi_g + g(\Phi_h)$.

Although the above lemma gives some useful information regarding inversion sets, the following lemma provides stronger results about inversion sets using the symmetric difference. In order to reach this conclusions of this lemma it is necessary to consider generalized inversion sets of the form $\hat{\Phi}_g$. In particular, the result from part (c) demonstrates the usefulness of the generalized inversion set with regard to composition.

**Lemma 1.4.3.** Let $(G, \Phi)$ be a signed groupoid set with $a, b, c \in \text{Ob}(G)$, $g \in {}_a G_b$ and $h \in {}_b G_c$. Then

(a) $\hat{\Phi}_g = {}_a \Phi^+ + g(\Phi^-)$

(b) $g^{-1}(\hat{\Phi}_g) = \hat{\Phi}_{g^{-1}}$

(c) $\hat{\Phi}_{gh} = \hat{\Phi}_g + g(\Phi_h)$.

**Proof.** Since $\hat{\Phi}_g = {}_a \Phi^+ \cap g(\Phi^-)$ and $\hat{\Phi}_g = \Phi_g \cup (-\Phi_g)$ we see that

$$
\hat{\Phi}_g = ({}_a \Phi^+ \cap g(\Phi^-)) \cup (- {}_a \Phi^+ \cap g(\Phi^-))
\quad = ({}_a \Phi^+ \cap g(\Phi^-)) \cup (\Phi^- \cap g(\Phi^+))
\quad = (\Phi^+ \setminus g(\Phi^+)) \cup (g(\Phi^+ \setminus \Phi^+) = {}_a \Phi^+ + g(\Phi^+).
$$

proving (a). From (a) we see that

$$
g^{-1}(\hat{\Phi}_g) = g^{-1}({}_a \Phi^+ + g(\Phi^+)) = g^{-1}({}_a \Phi^+) + g(\Phi^+) = \hat{\Phi}_{g^{-1}}
$$
proving (b). Additionally

$$\hat{\Phi}_g + g(\hat{\Phi}_h) = (a\Phi^+ + g(b\Phi^+)) + g(h(c\Phi^+))$$

$$= a\Phi^+ + g(b\Phi^+) + g(h(c\Phi^+)) = \Phi^+ + gh(c\Phi^+) = \hat{\Phi}_{gh}$$

which proves (c).

These properties of inversion sets provide us with the ability to discuss the manner in which different properties of inversion sets are related to each other. The following lemma outlines many useful properties of inversion sets and relates them directly to the symmetric difference. In particular we see that (i) is a property regarding inversion sets while (ii) - (vi) are properties involving the generalized notion of an inversion set. It should be noted that Lemma 1.5.1 (b) provides another equivalent condition that is not listed here.

**Lemma 1.4.4.** Let \((G, \Phi)\) be a signed groupoid set with \(g, h \in \text{Mor}(G)\) such that \(\text{dom}(g) = \text{cod}(h)\). Then the following statements are equivalent:

(i) \(\Phi_g \subseteq \Phi_{gh}\)

(ii) \(\hat{\Phi}_g \subseteq \hat{\Phi}_{gh}\)

(iii) \(\hat{\Phi}_g \cap g(\hat{\Phi}_h) = \emptyset\).

(iv) \(\hat{\Phi}_{g^{-1}} \cap \hat{\Phi}_h = \emptyset\)

(v) \(\hat{\Phi}_{h^{-1}} \cap h^{-1}(\hat{\Phi}_{g^{-1}}) = \emptyset\)

(vi) \(\hat{\Phi}_{h^{-1}} \subseteq \hat{\Phi}_{h^{-1}g^{-1}}\).

Proof. Since \(\hat{\Phi}_g = \Phi_g \cup -\Phi_g\) and \(\hat{\Phi}_{gh} = \Phi_{gh} \cup -\Phi_{gh}\) and \(\Phi_g \cap -\Phi_{gh} = \emptyset\) we see that \(\Phi_g \subseteq \Phi_{gh}\) if and only if \(\hat{\Phi}_g \subseteq \hat{\Phi}_{gh}\) and thus showing equivalence of (i) and (ii). Lemma 1.4.3 (c) shows equivalence of (ii) and (iii). Similarly (v) and (vi) are
equivalent. From Lemma 1.4.3 (b) we see that (iii) and (iv) are equivalent, and the
equivalence of (iv) and (v) follows similarly.

1.5 The cocycle map

In this section we introduce the cocycle map. The cocycle map is very similar
to the generalized inversion set. The cocycle map allows us to relate the statements
in Lemma 1.4.4 more directly to inversion sets we introduce the cocycle map. An
advantage of the cocycle map over inversion sets is that the cocycle map allows us
to study roots in a root system without considering the sign. The properties of the
cocycle map allow us to observe properties of inversion sets of signed groupoid sets.

In order to construct the cocycle map we must first consider the following equivalence relation on the root system at an object.

**Definition 1.5.1.** Let \((G, \Phi)\) be a signed groupoid set with \(a \in \text{Ob}(G)\).

(a) Let \(\sim_a^\pm\) be the equivalence relation on \(a\Phi\) such that \(\alpha \sim_a^\pm \beta\) if and only if \(\alpha = \pm \beta\)
for \(\alpha, \beta \in a\Phi\).

(b) Let \(a\Phi/\{\pm\} := \bigcup_{\alpha \in a\Phi} \{\alpha, -\alpha\}\) be the set of \(\sim_a^\pm\) equivalence classes.

(c) We define the map \(\pi : a\Phi \to a\Phi/\{\pm\}\) such that \(\alpha \mapsto \{\alpha, -\alpha\}\) for \(\alpha \in a\Phi\).

The following remark provides a useful bijection that plays an integral role in
relating the cocycle map to generalized inversion sets.

**Remark 1.5.1.** Observe that \(a\pi \mid_{a\Phi^+} : a\Phi^+ \to a\Phi/\{\pm\}\), the restriction of \(a\pi\) to \(a\Phi^+\),
is a bijection.

We now introduce the cocycle map \(N\). Observe that from the definitions in (a)
and (b), we may consider the cocycle map as a representation of the groupoid.

**Definition 1.5.2.** Let \((G, \Phi)\) be a signed groupoid set with \(a, b \in \text{Ob}(G)\).
(a) \( aN : aG \to \mathcal{P}(a\Phi/\{\pm\}) \) is defined such that \( g \mapsto a\pi(\hat{\Phi}_g) \) for \( g \in aG \).

(b) For \( h \in bG_a \) and \( \Gamma \subseteq a\Phi/\{\pm\} \), we define \( h(\Gamma) := \{h(x) | x \in \Lambda\} \subseteq b\Phi/\{\pm\} \).

(c) The cocyle map \( N : \text{Mor}(G) \to \Phi/\{\pm\} \) is the map such that for \( g \in \text{Mor}(G) \), \( g \mapsto aN(g) \) where \( a = \text{cod}(g) \).

**Remark 1.5.2.** Let \((G, \Phi)\) be a signed groupoid set with \( a, b \in \text{Ob}(G) \).

(a) For \( g \in aG \), \( h \in bG_a \) we see that \( h(aN(g)) \) allows us to extend the map \( a \mapsto a\Phi/\{\pm\} \) to a representation \( G \to \text{Set} \).

(b) For a signed groupoid set \((G, \Phi)\) with \( g, h \in \text{Mor}(G) \) such that \( \text{dom}(h) = \text{cod}(g) \) we see from Lemma 1.4.3(c) that \( N(gh) = N(g) + gN(h) \). This is known as the cocyle condition.

(c) Since the restriction of \( a\pi \) to \( a\Phi^+ \) is a bijection we see that \( a\pi \) bijectively maps

\[
\{\hat{\Phi}_g | g \in aG\} \longrightarrow \{N(g) | g \in aG\}
\]

\[
\{\hat{\Phi}_g | g \in aG\} \longrightarrow \{N(g) | g \in aG\}.
\]

(d) For \( g \in \text{Mor}(G) \) the statements in Lemma 1.4.4 for the sets \( \hat{\Phi}_g \) also hold for the sets \( N(g) \).

Note that the bijections in Remark 1.5.1 and Remark 1.5.2 are two distinct conditions. Both are useful and help us to relate the cocyle map \( N \) to the study of inversion sets.

The lemma below demonstrates how studying the cocycle map can directly give us information about inversion sets. In particular part (b) relates directly to Lemma 1.4.4 from the previous section.

**Lemma 1.5.1.** If \((G, \Phi)\) is a signed groupoid set with \( a, b \in \text{Ob}(G) \), \( g \in aG_b \), \( h \in bG \) then

(a) \( |\Phi_g| = |aN(g)| \)

(b) \( \Phi_g \subseteq \Phi_{gh} \) if and only if \( N(gh) = N(g) \cup gN(h) \), and thus is equivalent to the statements in Lemma 1.4.4.
Proof. Since \( \pi|_{\Phi^+} : \Phi^+ \to \Phi/\{\pm\} \) is a bijection we see that \( \pi(\Phi_g) = \pi(\hat{\Phi}_g) = aN(g) \) which gives us that \( |\Phi_g| = |aN(g)| \) as in (a).

If \( \Phi_g \subseteq \Phi_{gh} \) then \( N(g) \subseteq N(gh) = N(g) + gN(h) \), and thus \( N(gh) = N(g) \cup gN(h) \). Now suppose \( N(gh) = N(g) \cup gN(h) \). Then \( \hat{\Phi}_{gh} = \hat{\Phi}_g \cup g(\hat{\Phi}_h) \), and so

\[
\Phi_{gh} = a \Phi^+ \cap (\hat{\Phi}_g \cup g(\hat{\Phi}_h)) = \Phi_g \cup (a \Phi^+ \cap g(\hat{\Phi}_h))
\]

Thus \( \Phi_g \subseteq \Phi_{gh} \), proving (b).

Since nearly all the properties of generalized inversion sets also hold for the cocycle map, there are many cases where either method may be used to study properties of signed groupoid sets. Typically we attempt to use the method that appears to most closely relate to the topic we are studying, but there are times when these notions may be used interchangeably.

1.6 Weak Order

In this section we introduce the notion of the weak order on morphisms in a signed groupoid set. The weak order defined here is discussed in further detail in [10]. In order to discuss this need to first consider the preorder relation on a set.

Remark 1.6.1. A preorder on a set \( A \) is a a binary relation that is both reflexive and transitive. A partial order is a preorder that is antisymmetric (as a relation) on \( A \).

Now we introduce the weak order on morphisms in a groupoid in a signed groupoid set \((G, \Phi)\). In the weak order we may only compare morphisms that have the same codomain, and thus we define the weak order on the set \( aG \) for \( a \in \text{Ob}(G) \). Although
we give special attention to the weak right preorder, there are other preorders and partial orders on \(_aG\) associated to the weak order.

**Definition 1.6.1.** The weak right preorder \(_a \leq\) on \(_aG\) for a signed groupoid set \((G, \Phi)\) with \(a \in \text{Ob}(G)\) is the preorder on \(_aG\) such that for \(g, h \in _aG\) we have \(g \_a \leq h\) if \(a N(g) \subseteq _a N(h)\).

**Remark 1.6.2.** Recall that Lemma 1.4.4 shows that \(a N(g) \subseteq _a N(h)\) if and only if \(\Phi_g \subseteq \Phi_h\), and thus this is equivalent to \(g \_a \leq h\).

The following lemma, from [10], provides some properties of the weak right preorder. In particular, observe that in certain cases we may compare the weak right preorders at different objects of the groupoid.

**Lemma 1.6.1.** Let \((G, \Phi)\) be a signed groupoid set with \(a, b, c, d \in \text{Ob}(G)\), \(v \in _aG_d\), \(x \in _aG_b\), \(y \in _bG_c\) and \(w \in _bG\). Then

(a) \(\text{Id}_a \_a \leq x\)

(b) If \(x \_a \leq xy\), then \(y^{-1}c \leq y^{-1}x^{-1}\)

(c) If \(x \_a \leq xy\) and \(x \_a \leq xw\), then \(xy \_a \leq xw\) if and only if \(y \_b \leq w\).

(d) If \(v^{-1}d \_a \leq v^{-1}x\), \(v \_a \leq xy\) and \(y^{-1}c \leq y^{-1}w\), then \(v^{-1}d \_a \leq v^{-1}xw\).

(e) If \(y \_b \leq w\) and \(w \_b \leq y\), then \(xy \_a \leq xw\).

**Proof.** (a) is true since \(\Phi_{\text{Id}} = \emptyset\), and (b) follows from Lemma 1.4.4. For (c), since \(x \_a \leq xy\) and \(x \_a \leq xw\), by Lemma 1.4.4 we see that \(N(x) \cap xN(y) = \emptyset\) and \(N(x) \cap xN(w) = \emptyset\). Thus \(N(y) \subseteq N(w)\) if and only if \(xN(y) \subseteq xN(w)\), which occurs if and only if

\[N(xy) = N(x) \cup xN(y) \subseteq N(x) \cup xN(w) = N(xw)\]

by Lemma 1.4.3

For (d), Lemma 1.4.3 shows that since \(N(v^{-1}) \subseteq N(v^{-1}x)\) we have that \(N(v^{-1}) \cap v^{-1}N(x) = \emptyset\), and so \(N(v) \cap N(x) = \emptyset\). Similarly since \(N(y) \cap N(w) = \emptyset\) we see that
$xN(y) \cap xN(w) = \emptyset$. Since $N(v) \subseteq N(xy) = N(x) + xN(y)$ and $N(x) \cap N(v) = \emptyset$, it follows that $N(v) \subseteq xN(y)$. Therefore
\[
N(v) \cap N(xw) = N(v) \cap (N(x) + xN(w))
\]
\[
= (N(v) \cap N(x)) + (N(v) \cap xN(w))
\]
\[
= (N(v) \cap xN(w)),
\]
and since
\[
N(v) \cap xN(w) \subseteq xN(y) \cap xN(w) = x(N(y) \cap N(w)) = \emptyset
\]
we see that $N(v) \cap N(xw) = \emptyset$. Thus from Lemma 1.4.3 gives us that $N(v^{-1}) \cap v^{-1}N(xw) = \emptyset$, and so from Lemma 1.4.4 we see that $N(v^{-1}) \subseteq N(v^{-1}xw)$.

For (e) we see that since $N(y) \subseteq N(w)$ and $N(w) \subseteq N(y)$ we have $N(y) = N(w)$. Thus from Lemma 1.4.3 we see that $N(xy) = N(x) + xN(y) = N(x) + xN(w) = N(xw)$.

\[\square\]

1.7 Faithful signed groupoid sets

This section we discusses the notion of a faithful signed groupoid set. Faithful signed groupoid sets have a stronger correspondence between their morphisms and their inversion sets, and thus studying the properties of their inversion sets reveals more information regarding properties of morphisms and root systems. This means that in some circumstances the properties of faithful signed groupoid sets more accessible to us.

**Definition 1.7.1.** A signed groupoid set $(G, \Phi)$ is called faithful if for every $g \in \text{Mor}(G)$, we have that $\Phi_g = \emptyset$ if and only if $g$ is an identity morphism.
In the following Lemma we demonstrate some properties of faithful signed groupoid sets. When discussing faithful signed groupoid sets we typically refer to one of the equivalent statements in this lemma. Recall the properties of inversion sets and the cocycle map the we have already discussed.

**Lemma 1.7.1.** Let \((G, \Phi)\) be a signed groupoid set. The following statements are equivalent

(i) \((G, \Phi)\) is faithful.

(ii) For all \(a \in \text{Ob}(G)\) and \(g \in aG\), one has \(g \leq a\text{Id}_a\) only if \(g = \text{Id}_a\).

(iii) For all \(a \in \text{Ob}(G)\), \(g, h \in aG\) with \(_aN(g) = _aN(h)\), one has \(g = h\).

(iv) For all \(a \in \text{Ob}(G)\), the weak right preorder \(\leq a\) is a partial order.

**Proof.** Let \(a \in \text{Ob}(G)\). First we assume \((G, \Phi)\) is faithful as in (i). If \(g \in aG\), we see that \(g \leq a\text{Id}_a\) if and only if \(_aN(g) \subseteq \emptyset\), which means that \(_aN(g) = \emptyset\). Since \((G, \Phi)\) is faithful, we see that \(g = \text{Id}_a\). So we see that (i) implies (ii).

Now assume (ii). Let \(g, h \in aG\) such that \(_aN(g) = _aN(h)\). This means that \(a \leq h\) and \(b \leq a\). Lemma 1.6.1 (e) gives us that \(h^{-1}g \leq h^{-1}h = \text{Id}_b\), where \(b := \text{dom}(h) \in \text{Ob}(G)\). Thus \(h^{-1}g = \text{Id}_b\), and so \(g = h\) as in (iii).

Now assume (iii). Clearly \(\leq a\) is reflexive and transitive. The assumption of (iii) gives us that \(\leq a\) is antisymmetric, and we see that (iii) implies (iv).

Now assume (iv). Let \(g \in aG\) such that \(\Phi_g = \emptyset\). Since \(\Phi_{\text{Id}_a} = \emptyset\) we get that \(g \leq a\text{Id}_a\) and \(\text{Id}_a \leq g\). Since we assumed that \(\leq a\) is a partial order, this gives us that \(g = \text{Id}_a\), and so we get that (iv) implies (i).

The following remark demonstrates an interesting property of faithful signed groupoid sets. Many of the results in this thesis regarding faithful signed groupoid sets come from this fact or concepts of similar nature.

**Remark 1.7.1.** Observe that if \((G, \Phi)\) is a faithful signed groupoid set with \(a, b \in \text{Ob}(G)\), then...
\[ \text{Ob}(g), \ g \in \mathcal{G}, \ h \in \mathcal{G} \text{ and } x \in \mathcal{G} \text{ then } \mathcal{N}(x) = \mathcal{N}(g) + g(h) \] if and only if \( x = gh \).
CHAPTER 2

THE DOMINANCE PREORDER

In this chapter we discuss properties of individual roots in a root system of a signed groupoid set. We begin first discuss real and imaginary roots. At every object of a signed groupoid set we are able to classify each root of its root system as either real or imaginary. This is determined by the way morphisms act on the root. From here we go on to define a real signed groupoid set consisting of only real roots and classify the real roots through inversion sets.

In section 2.2 we introduce the dominance preorder on a root system. The dominance preorder is a generalization of the dominance order on the root system of a Coxeter group discussed in [1], [5], and [14]. The dominance preorder allows us to further study the behavior of an individual root based on how morphisms act on it. Properties of the dominance preorder allow us to study other properties of root systems including the study of inversion sets. The dominance preorder naturally induces equivalence classes on a root system, which is discussed in section 2.3. Roots that lie in the same equivalence class have many of the same properties. These equivalence classes lead to the notion of compressed signed groupoid sets. In a compressed signed groupoid set each equivalence class is a singleton. The result of this is that the properties studied in this chapter have stronger correspondence with individual roots. In section 2.4 we examine situations where properties of signed groupoid sets also hold for a compressed signed groupoid set of the same form.
2.1 Real and imaginary roots

In this section we introduce the concept of real and imaginary roots. Every root may be classified as real or imaginary based on the manner in which the morphisms in a signed groupoid set act on it.

Definition 2.1.1. Let \((G, \Phi)\) be a signed groupoid set with \(a \in \text{Ob}(G)\).

(a) \(\alpha \in \Phi^a\) is an imaginary root if \(g(\alpha)\) has the same sign as \(\alpha\) (positive or negative) for every \(g \in G_a\).

(b) \(\alpha \in \Phi^a\) is a real root if it is not an imaginary root

(c) \(\Phi^\text{im}_a\) is the set of imaginary roots of \(\Phi^a\)

(d) \(\Phi^\text{re}_a\) is the set of real roots of \(\Phi^a\).

Since only the real roots in a root system may change sign when acted on by a morphism, many interesting properties of roots are contained within the set of real roots. We see in the following lemma that restriction of a root system to its real roots gives us a signed groupoid set. Thus we may study signed groupoid sets by considering signed groupoid sets that contain only real roots.

Lemma 2.1.1. Let \((G, \Phi)\) be a signed groupoid set. If \(\Phi^\text{re} : G \rightarrow \text{Set}_\pm\) is the map such that for \(a \in \text{Ob}(G), g \in G_a\)

\[
\begin{align*}
  a & \mapsto (\Phi^\text{re}_a, \Phi^\text{re}_a \cap \Phi^+_a) \\
  g & \mapsto g|_{\Phi^\text{re}_a}
\end{align*}
\]

where \(g|_{\Phi^\text{re}_a}\) is the restriction of \(g\) to \(\Phi^\text{re}_a\), then \((G, \Phi^\text{re})\) is a signed groupoid set.

Proof. Let \(a, b \in \text{Ob}(G), g \in bG_a\). Observe that for \(\alpha \in \Phi^a\) we have that \(\alpha \in \Phi^\text{re}_a\) if and only if for every \(b \in \text{Ob}(G), g \in bG_a\) we have that \(g(\alpha) \in \Phi^\text{re}_b\). Thus it is the case that \(g(\Phi^\text{re}_a) = \Phi^\text{re}_b\). Additionally we see that \(\alpha \in \Phi^\text{re}_a\) if and only if \(-\alpha \in \Phi^\text{re}_a\) and so we see that \((\Phi^\text{re}_a, \Phi^\text{re}_a \cap \Phi^+_a) \in \text{Ob}((\text{Set}_\pm), \Phi^\text{re}(g) \in \text{Mor}(\text{Set}_\pm)\). Therefore
it follows that $\Phi^{re} : G \to \textbf{Set}_{\pm}$ is a representation of $G$ in $\textbf{Set}_{\pm}$, and so $(G, \Phi^{re})$ is a signed groupoid set.

A consequence of the above lemma is that we can define real signed groupoid sets and provide a method of studying the properties of signed groupoid sets by examining only real signed groupoid sets. Thus we introduce the following definitions give us this framework.

**Definition 2.1.2.** Let $(G, \Phi)$ be a signed groupoid set, and $\Phi^{re} : G \to \textbf{Set}_{\pm}$ is the map such that for $a \in \text{Ob}(G), g \in G_a$

\[
a \mapsto (a\Phi^{re}, a\Phi^{re} \cap a\Phi^+) \\
g \mapsto g|_{a\Phi^{re}}
\]

where $g|_{a\Phi^{re}}$ is the restriction of $g$ to $a\Phi^{re}$. We define the following:

(a) $(G, \Phi)^{re} := (G, \Phi^{re})$.

(b) $(G, \Phi)$ is a real signed groupoid set if for every $a \in \text{Ob}(G), a\Phi = a\Phi^{re}$.

The next lemma provides a useful classification of the real roots of a signed groupoid set using inversion sets. The lemma shows that inversion sets contain only real roots and additionally that every real root is contained in an inversion set. This allows us to discuss the behavior of real roots by the studying properties of inversion sets. We use this technique throughout this thesis.

**Lemma 2.1.2.** If $(G, \Phi)$ be a signed groupoid set with $a \in \text{Ob}(G)$, then

\[
a\Phi^+ \cap a\Phi^{re} = \bigcup_{g \in aG} \Phi_g.
\]
Proof. Let $\alpha \in _a\Phi^+ \cap _a\Phi^{re}$. Since $\alpha \in _a\Phi^{re}$ there exists $b \in \text{Ob}(G)$, $h \in _bG_a$ such that $h(\alpha) \in _b\Phi^-$. Thus $\alpha \in _a\Phi^+ \cap h^{-1}(_b\Phi^-) = \Phi_{h^{-1}}$, and so we see that $\Phi^+ \cap _a\Phi^{re} \subseteq \bigcup_{g \in _aG_h} \Phi_g$.

Now suppose $\beta \in \Phi_g$ for some $c \in \text{Ob}(G)$, $g \in _cG_c$. Since $\beta \in _a\Phi^+$, and $g^{-1}(\beta) \in _c\Phi^-$ we see that $\beta \in _a\Phi^{re}$ and thus $\bigcup_{g \in _aG_h} \Phi_{h^{-1}} \subseteq _a\Phi^+ \cap _a\Phi^{re}$.

2.2 Dominance Preorder

We now introduce a preorder on a root system, which we call the dominance preorder. The dominance preorder classifies roots in a root system of a signed groupoid set based on how its morphisms act on it. This dominance preorder discussed here is a natural generalization of the dominance order on the root system of a Coxeter group described in [1], [5] and [14].

Definition 2.2.1. The dominance preorder $\preceq_a$ on $a\Phi$, where $(G, \Phi)$ is a signed groupoid set with $a \in \text{Ob}(G)$, is defined such that for $\alpha, \beta \in a\Phi$, $\alpha \preceq_a \beta$ if for every $b \in \text{Ob}(G)$ whenever $g \in _bG_a$ with $g(\beta) \in _b\Phi^-$ one has that $g(\alpha) \in _b\Phi^-$ as well.

Many properties of signed groupoid sets may be studied using the dominance order. The lemma below introduces some some elementary but useful properties of the dominance preorder on roots of a signed groupoid set. In particular part (a) shows that the imaginary roots exist only at either the top or the bottom of the dominance preorder, while other properties show that the sign of a root restricts the roots that it may dominate. We also see that part (e) gives a relationship between inversion sets and the dominance preorder.

Lemma 2.2.1. Let $(G, \Phi)$ be a signed groupoid set with $a, b \in \text{Ob}(G), g \in _bG_a$, and $\alpha, \beta \in a\Phi$.

(a) If $\alpha \in _a\Phi^{im} \cap _a\Phi^+$ then $\beta \preceq_a \alpha$. 

27
(b) If $\alpha \preceq \beta$ then $g(\alpha) \preceq g(\beta)$.

(c) $\alpha \preceq \beta$ if and only if $-\beta \preceq -\alpha$.

(d) If $\alpha \in a\Phi^+$ and $\beta \in a\Phi^-$ it is never the case that $\alpha \preceq \beta$.

(e) If $\alpha \in a\Phi^+, \beta \in \Phi_{g^{-1}}$ with $\alpha \preceq \beta$ then $\alpha \in \Phi_{g^{-1}}$.

Proof. If $\alpha \in a\Phi^+ \cap a\Phi^+$ the for all $g \in aG$ we see that $g(\alpha)$ is never negative, and so $\beta \preceq \alpha$ proving (a). Now assume $\alpha \preceq \beta$. Thus for all $h \in G_b$ if $hg(\beta) \in a\Phi^-$, then $hg(\alpha) \in a\Phi^-$. This shows $g(\alpha) \preceq g(\beta)$ as in (b). Since $b\Phi = b\Phi^+ \cup - (b\Phi^+)$ we see that if $g(\alpha) \in b\Phi^+$ then $g(\beta) \in b\Phi^+$. Thus if $g(-\alpha) = -g(\alpha) \in b\Phi^-$ then $g(-\beta) = -g(\beta) \in b\Phi^-$, and so $-\beta \preceq -\alpha$ proving (c).

Now if $\alpha \in a\Phi^+$ and $\beta \in a\Phi^-$ we see that $\text{Id}_G(\beta) = \beta \in a\Phi^-$ and $\text{Id}_G(\alpha) = \alpha \in a\Phi^-$ contradicting the assumption that $\alpha \preceq \beta$, and so proving (d). Now if $\beta \in \Phi_{g^{-1}} = a\Phi^+ \cap g^{-1}(b\Phi^-)$ we see that $g(\beta) \in g(a\Phi^+) \cap b\Phi^-$. Since $\alpha \preceq \beta$, $g(\alpha) \in b\Phi^-$, and so $\alpha \in g^{-1}(b\Phi^-)$. Thus if $\alpha \in a\Phi$ we get (e).

The next lemma shows that morphisms in $\textbf{Gpd} - \textbf{Set}_\pm$ preserve the dominance preorder on signed groupoid sets. This becomes a useful fact when introducing further constructions involving signed groupoid sets.

**Lemma 2.2.2.** Let $(\rho, \nu) : (G, \Phi) \to (H, \Psi)$ be a morphism in $\textbf{Gpd} - \textbf{Set}_\pm$ and $a \in \text{Ob}(G)$. Then $\nu_a : \rho(a)\Psi \to a\Phi$ preserves the dominance preorder.

**Proof.** Let us assume that $\nu_a : \rho(a)\Psi \to a\Phi$ does not preserve the dominance preorder. Then there exists $\alpha, \beta \in \rho(a)\Psi$ such that $\alpha \preceq \rho(a)\beta$ and $\nu_a(\alpha) \not\preceq \nu_a(\beta)$. Therefore there must exist $b \in \text{Ob}(G), g \in bG_a$ where $\Phi(g)(\nu_a(\beta)) \in b\Phi^-$ and $\Phi(g)(\nu_a(\alpha)) \in b\Phi^+$. Since $\nu_b$ is a natural transformation we observe the following commutative diagram
and so we observe the following equalities:

\[(\Phi(g) \circ \nu_a)(\alpha) = (\nu_b \circ \Psi\rho(g))(\alpha)\]

\[(\Phi(g) \circ \nu_a)(\beta) = (\nu_b \circ \Psi\rho(g))(\beta).\]

Since \(\nu_b : \Psi\rho(b) \to \Phi(b)\) is positivity preserving we see that \(\Psi\rho(g)(\alpha) \in \rho(b)\Psi^+\) and \(\Psi\rho(g)(\beta) \in \rho(b)\Psi^-\) which contradicts the assumption that \(\alpha \rho(b) \preceq \beta\).

2.3 Equivalence classes

The dominance preorder discussed in section 2.2 is not necessarily a partial order on the root system. Since the dominance preorder is a generalization of the dominance order on a root system for a Coxeter group, which is a partial order, we now consider the possibility of a partial ordering on a root system. We may achieve this by studying the equivalence classes induced by the dominance preorder on a root system. These equivalence classes allow us to study properties of the dominance preorder on a root system of a signed groupoid set, while at same time obtaining a structure where dominance preorder gives partial ordering on these equivalence classes. Additionally, since roots of the same equivalence class share many properties, we may study properties of equivalence classes to identify roots that have these same properties.

**Definition 2.3.1.** Let \((G, \Phi)\) be a signed groupoid set with \(a \in \text{Ob}(G)\) and \(\alpha, \beta \in \Phi_a\).

(a) \(a \sim\) is the associated equivalence relation of the preorder \(\preceq\) on \(\Phi_a\), i.e. \(\alpha \sim \beta\).
if both $\alpha \leq_a \beta$ and $\beta \leq_a \alpha$. We use $[a][\alpha]$ to denote the equivalence class of $\alpha$ under $a \sim$.

(b) $[[a]\Phi] := [a\Phi]/a \sim = \{[a][\alpha] : \alpha \in a\Phi\}$ is the set of equivalence classes on $a\Phi$.

(c) $[[a]\Phi]^+ := \{[a][\alpha] : \alpha \in a\Phi^+\}$, and $[[a]\Phi]^− := \{[a][\alpha] : \alpha \in a\Phi^-\}$

We remark which that the set of equivalence classes on a root system is a signed set. This will allow us to use the set of equivalence classes to construct new a signed groupoid set that exhibits many of the same properties.

**Remark 2.3.1.** Let $(g, \Phi)$ be a signed groupoid set with $a \in \text{Ob}(G)$.

(a) By Lemma 2.2.1(d) we see that if $\alpha \in [a] \Phi^+$ then $[a][\alpha] \subseteq [a] \Phi^+$, and if $\alpha \in [a] \Phi^−$ then $[a][\alpha] \subseteq [a] \Phi^−$. Thus it is the case that $[[a]\Phi] = [[a]\Phi]^+ \cup ([a]\Phi)^−$.

(b) Additionally we observe that $([a]\Phi], [[a]\Phi]^+)$ is a definitely signed set where the $\{\pm\}$ action is given by $- [a][\alpha] = [a][-\alpha]$.

In the next lemma we discuss some properties of equivalence classes. Many of these are evident from properties of the dominance preorder, as they exhibit similar behavior.

**Lemma 2.3.1.** Let $(G, \Phi)$ be a signed groupoid set with $a, b \in \text{Ob}(G)$ and $\alpha, \beta \in a\Phi$.

(a) The following statements are equivalent:

(i) $\alpha \sim_a \beta$

(ii) $g(\alpha) \sim_b g(\beta)$ for all $g \in bG_a$.

(iii) For every $g \in aG$, $\alpha \in a\Phi_g$ if and only if $\beta \in a\Phi_g$.

(b) Suppose $x, y \in a\Phi$ such that $x \sim_a \alpha$ and $y \sim_a \beta$. If $\alpha \leq_a \beta$, then $x \leq_a y$.

(c) If $\alpha, \beta \in a\Phi^\text{im} \cap a\Phi^+$ then $\alpha \sim_a \beta$.

**Proof.** We first prove (a). Statements (i) and (ii) are equivalent by Lemma 2.2.1(b).

Now we assume $\alpha \sim_a \beta$ as in (i). If $\alpha \in a\Phi_g$, Lemma 2.2.1(e) guarantees that $\beta \in a\Phi_g$. Similarly we see that if $\beta \in a\Phi_g$, $\alpha \in a\Phi_g$ as well, thus giving us (iii). Now
suppose that for every \( g \in aG, \alpha \in \Phi_g \) if and only if \( \beta \in \Phi_g \) as in (iii). Thus for \( g \in aG, g(\alpha) \in b\Phi^- \) if and only if \( g(\beta) \in b\Phi^- \), and so \( g(\alpha) \sim g(\beta) \) as in (ii).

In (b), since \( x \sim \alpha \) we have \( x \leq \alpha \). Since \( \alpha \leq \beta \) we have \( x \leq \beta \). Similarly since \( y \sim \beta \) we have \( \beta \leq y \) and so we get \( x \leq y \). (c) is a direct consequence Lemma 2.2.1(a).

From Lemma 2.3.1 we see that the equivalence classes on a root system completely determine how morphisms of the underlying groupoid act on the roots with respect to sign.

2.4 Real Compression

We already noted the roots of the same equivalence class share many properties. This is in large part due to the fact that the roots of a single equivalence class in a root system cannot be distinguished by sign with regard to how morphisms act on them. Thus it can be beneficial to study signed groupoid sets where each equivalence class is a singleton. We call such signed groupoid sets compressed. Since each equivalence class preserves many properties of the root in the root system, we are able to produce the compression of a signed groupoid set, which gives us a compressed signed groupoid set and preserves many of the properties of the original signed groupoid set.

**Definition 2.4.1.** Let \( R := (G, \Phi) \) be a signed groupoid set.

(a) \( R \) is compressed if for all \( a \in \text{Ob}(G), \alpha, \beta \in a\Phi \), whenever \( a[\alpha] = a[\beta] \) it is the case that \( \alpha = \beta \) (i.e. \( |a[\alpha]| = 1 \)).

(b) \( R^c \), the compression of \( R \) is the signed groupoid set \( R^c := (G, \Psi) \) such that for all \( a, b \in \text{Ob}(G), g \in aG_b, \Psi(a) := [[a\Phi]] \) and \( g(a[\alpha]) := a[g(\alpha)] \).

The following natural transformation will be considered below.
Remark 2.4.1. Observe that if \( R := (G, \Phi) \) and \( R^c := (G, \Psi) \) there is a natural transformation \( \pi : \Phi \to \Psi \) with components \( \pi_a : a\Phi \to a\Psi \) in \( \text{Set}_{+,-} \) given by \( \alpha \mapsto _a [\alpha] \).

This next lemma, found in [9] without proof, demonstrates the extent to which some properties of signed groupoid sets are represented their compressions. Due to the extent that properties of a signed groupoid set are preserved by compression. Since compression preserves many properties, it is often more efficient to study compressed signed groupoid sets.

Lemma 2.4.1. Let \( R = (G, \Phi) \) be a signed groupoid set and \( R^c := (G, \Psi) \) with \( a \in \text{Ob}(G) \). Let \( \pi : \Phi \to \Psi \) be the natural transformation in Remark 2.4.1.

(a) The map \( \pi_a : a\Phi \to a\Psi \) where for all \( g \in aG \), \( \Phi_g \mapsto \Psi_g \) induces a bijection \( \{\Phi_g | g \in aG\} \to \{\Psi_g | g \in aG\} \).

(b) The map \( \Phi_g \mapsto \Psi_g \), for \( g \in aG \) induces a preorder isomorphism from the weak preorder of \( R \) at \( a \) to the weak preorder of \( R^c \) at \( a \).

(c) The dominance preorder on \( a\Psi \) coincides with the partial order associated to the dominance preorder on \( a\Phi \). In particular, the dominance preorder on \( a\Psi \) is a partial preorder.

(d) \( a\Phi^{im} \neq \emptyset \) if and only if there exists a maximum (resp. minimum) element of the dominance order on \( a\Psi \). Furthermore \( a\Psi^{im} \cap a\Psi^+ \) (resp. \( a\Psi^{im} \cap a\Psi^- \)) is the maximum (resp. minimum) element of the dominance order on \( a\Psi \).

Proof. We see from Lemma 2.3.1 (a) that for every \( g \in aG \) that \( \pi_a(\Phi_g) = \Psi_g \) and so \( \pi \) bijectively maps \( \Phi_g \mapsto \Psi_g \) proving (a). From the proof of part (a) we see that if \( g, h \in aG \) the \( \Phi_g \subseteq \Phi_h \) if and only if \( \Psi_g \subseteq \Psi_h \) and so we get (b). (c) follows directly from Lemma 2.3.1.

To show (d), first suppose that \( a\Phi^{im} \neq \emptyset \) and let \( \alpha \in a\Phi^{im} \cap a\Phi^+ \). By Lemma 2.3.1(c) \( a\Psi^{im} \cap a\Psi^+ = \{a[\alpha]\} \), and Lemma 2.2.1 (a) tells us that \( a[\alpha] \) is maximal in the dominance order on \( a\Psi \).

Now suppose there exists \( \beta \in a\Phi \) such that \( _a[\beta] \) is maximal in the dominance order on \( a\Psi \). Thus \( _a[-\beta] \preceq _a[\beta] \), and so Lemma 2.2.1 (b) tells us that for every
$b \in \text{Ob}(G), g \in {}_a G_b$ we have that $g([_{a}[-\beta]]) \preceq g([_{a}\beta])$. Therefore we see by Lemma 2.2.1(d) that $[_{a}\beta] \in {}_a \Psi^\text{im} \cap {}_a \Psi^+$. 

\[\square\]

The next lemma shows that isomorphisms of a signed groupoid sets preserve compression.

**Lemma 2.4.2.** Suppose $(\rho, \nu) : (G, \Phi) \rightarrow (H, \Psi)$ is a isomorphism in $\text{Gpd} - \text{Set}$. If $(G, \Phi)$ is a compressed signed groupoid set then $(H, \Psi)$ is compressed as well.

**Proof.** Let $a \in \text{Ob}(G)$ and $\alpha, \beta \in {}_{\rho(a)}\Psi$ such that $\alpha \sim_{\rho(a)} \beta$. By Lemma 2.2.2 $\nu_a(\alpha) \sim \nu_a(\beta)$. Since $(G, \Phi)$ is compressed $\nu_a(\alpha) = \nu_a(\beta)$, and since $\nu$ is an isomorphism $\alpha = \beta$. 

\[\square\]

The above lemmas demonstrate that many properties of signed groupoid sets are preserved or similarly represented in its compression. Although the compression of a signed groupoid set does not provide an exact representation, compressed signed groupoid sets do provide a useful construction for studying signed groupoid sets. Throughout this thesis we give special attention to the manner in which aspects of signed groupoid sets are represented in their compression.
CHAPTER 3
SIMPLY GENERATED SIGNED GROUPOID SETS

In this chapter we discuss simply generated signed groupoid sets. Signed groupoid sets that are simply generated contain a set, known as a simple set, which generates the morphisms of the groupoid. A simple set is required to satisfy other conditions as well, which are laid out in Section 3.1. Properties of a signed groupoid set may be observed from properties its generating set in a similar manner to the way that properties of a group may be observed from its generating set.

Since simply generated signed groupoid sets have a simple set that generates the entire groupoid we may introduce length function in section 3.1. Section 3.2 discusses the dominance order and equivalence classes for simply generated signed groupoid sets. We show correspondence between equivalence classes and inversion sets of simple roots. Section 3.2 studies this correspondence applies it to the length function, inversion sets and the cocycle map. In section 3.3 discusses the properties of compressed simply generated signed groupoid sets.

In section 3.4 we introduce the notion of principal signed groupoid sets, also found in [10] and [11]. Principal signed groupoid sets have a stronger correspondence between the length function and inversion sets. Thus when a signed groupoid set is principal there are more properties we can gather from it.

3.1 Simply generated signed groupoid sets

This section introduces simply generated signed groupoid sets. In order for a signed groupoid set to be simply generated, there must exist a set of simple mor-
phisms satisfying a number of conditions. One of these conditions is that the simple morphisms generate the groupoid, while the other conditions involve the functor \( \Phi : G \to \text{Set}_+ \). We will see that the simple set of a simply generated signed groupoid set behaves similarly to set of the simple reflections in a root system for a Coxeter group. Thus allows us to discuss the ways in which properties of a simple set demonstrate further properties of the signed groupoid set. Note that a signed groupoid set does not necessarily contain a simple set. In fact finding a simple set for an arbitrary signed groupoid set, or showing that a signed groupoid set is simply generated can be quite challenging. In Chapter 6 we discuss examples of signed groupoid sets which are known to be simply generated.

**Definition 3.1.1.** \((G, \Phi)\) is a simply generated signed groupoid set if there exists a set \( S \subseteq \text{Mor}(G) \) such that

1. \( S \) doesn’t contain any identity morphisms
2. If \( s \in S \), then \( s^{-1} \in S \)
3. \( < S > = \text{Mor}(G) \), i.e. \( S \) generates all the morphisms of \( G \)
4. For \( a \in \text{Ob}(G) \), \( s \in S \cap aG \), \( g \in aG \), either \( \Phi_s \subseteq \Phi_g \) or \( \Phi_s \cap \Phi_g = \emptyset \).
5. For every \( s \in S \), we have that \( |\Phi_s| < \infty \).

For \( a \in \text{Ob}(G) \), we write \( _aS := S \cap aG \).

For a simply generated signed groupoid set \((G, \Phi)\) with simple set \( S \), we may refer to \((G, \Phi)\) as the signed groupoid set generated by \( S \). This assumes that \( S \) is a simple set for \((G, \Phi)\). Additionally \( S \) may also be referred to as the set of simple morphisms of \((G, \Phi)\).

In root systems it is natural to consider simple roots in a root system. For a root system of a signed groupoid set we are able to use the simple morphisms to give us a set of simple roots. We define the set of simple roots of a simply generated signed groupoid set \((G, \Phi)\) with simple set \( S \) as subsets of the root system that correspond
to inversion sets of simple morphisms. Note that a simple root of a root system at an object is a subset of $a\Phi$, not necessarily a single root. This contrary to the notion of simple roots in the study of root systems of Coxeter groups or Lie groups.

**Definition 3.1.2.** Let $(G, \Phi)$ be a signed groupoid set simply generated by $S$ and $a \in \text{Ob}(G)$.

(a) $S$ is the set of simple morphisms of $(G, \Phi)$. If $s \in S$, then $s$ is called a simple morphism.

(b) $a\Pi := \{ \Phi_s : s \in aS \}$ is the set of simple roots of $a\Phi$

(c) $\Pi := \bigcup_{b \in \text{Ob}(G)} b\Pi$ are the simple roots of $\Phi$.

Since a simple set provides a generating set for a simply generated signed groupoid set, it makes sense to discuss the length of a morphism. The length function of the groupoid of a signed groupoid set is defined using the standard convention.

**Definition 3.1.3.** The length function $l_S : \text{Mor}(G) \to \mathbb{N}$ for a simply generated signed groupoid set $(G, \Phi)$ with simple set $S$ is the defined such that for every identity morphism $\text{Id} \in \text{Mor}(G)$ we have the $l_S(\text{Id}) = 0$ and otherwise we have that

$$l_S(g) := \min(\{ n \in \mathbb{N} : g = s_1s_2\ldots s_n, s_i \in S \}),$$

where $g \in \text{Mor}(G)$ such that is not an identity morphism.

The length function of a signed groupoid set behaves similarly to the length function of a Coxeter group found in [1]. Many properties of the length function for Coxeter groups and their root systems may be generalized when discussing properties about simply generated signed groupoid sets and their root systems. The relationship between the length function of signed groupoid set and its root system is not equivalent to that for Coxeter groups, and so we make special note of circumstances that give stronger correspondences between these signed groupoid sets and Coxeter groups.
3.2 Equivalence classes of simply generated signed groupoid sets

Section 2.3 introduced equivalence classes on a root system induced by the dominance preorder. Since these equivalence classes play a special role when studying a simply generated signed groupoid set, we study the equivalence classes for simply generated signed groupoid sets and their additional properties in this section. Note that there is a strong correspondence between inversion sets and the equivalence classes. In particular the inversion set of a simple morphism coincides with the the roots in an equivalence class. Additionally the orbits of simple roots coincide with equivalence classes on the root system. This first lemma below demonstrates these properties.

**Lemma 3.2.1.** Let \((G, \Phi)\) be a simply generated signed groupoid set with simple set \(S\) and let \(a \in \text{Ob}(G)\), \(g, h \in G_a\) and \(r, s \in aS\).

(a) If \(s \in aS\) and \(\alpha \in \Phi_s\), then \([\alpha]_a = \Phi_s\).

(b) If \(g(\Phi_r) \cap h(\Phi_s) \neq \emptyset\) then \(g(\Phi_r) = h(\Phi_s)\).

**Proof.** Lemma 2.3.1 (a) tells us that if \(\beta \sim_a \alpha\) and \(\alpha \in \Phi_s\) then \(\beta \in \Phi_s\) as well. Thus \([\alpha] \subseteq \Phi_s\). Since for every \(g \in aG\) either \(\Phi_s \subseteq \Phi_g\) or \(\Phi_s \cap \Phi_g = \emptyset\), we see that for all \(\alpha, \alpha' \in \Phi_s\) that \(\alpha \in \Phi_g\) if and only if \(\alpha' \in \Phi_g\). Thus by Lemma 2.3.1 (a) we see that \(\alpha' \in [\alpha]\) and so \(\Phi_s \subseteq [\alpha]\), which completes the proof of (a). Thus there exist \(\alpha, \beta \in \Phi\) such that \(\Phi_r = \Phi_s = [\alpha]\) and \(\Phi_s = [\beta]\). Thus \(g(\Phi_r) = g([\alpha])\) and \(h(\Phi_s) = h([\beta])\) are equivalence classes, and so it follows that if \(g(\Phi_r) \cap h(\Phi_s) = \emptyset\) then \(g(\Phi_r) = h(\Phi_s)\) as in (b).

The next lemma demonstrates the relationship between inversion sets and the equivalence classes. Equivalence classes, given by the orbits of simple roots, completely describe the inversion sets of morphisms of a simply generated signed groupoid
set. Properties of the cocyle map, including part (a) of this lemma, allow us to see this. We conclude by giving statements describing the set of real roots of a root system using equivalence classes.

**Lemma 3.2.2.** Let \((G, \Phi)\) be a simply generated signed groupoid set with simple set \(S\) and let \(a \in \text{Ob}(G)\).

(a) If \(g \in \text{Mor}(G)\) with \(l_S(g) = n\) such that \(g = s_1s_2\ldots s_n\), then
\[
N(g) = N(s_1) + s_1N(s_2) + \cdots + s_1s_2\ldots s_{n-1}N(s_n).
\]

(b) There exist \(r_1, r_2, \ldots, r_k \in S\), \(h_1, h_2, \ldots, h_k \in \text{Mor}(G)\) such that
\[
\Phi_g = h_1(\Phi_{r_1}) \cup h_2(\Phi_{r_2}) \cup \cdots \cup h_k(\Phi_{r_k}).
\]

Additionally \(k \leq l_S(g)\).

(c) For every \(\alpha \in {}_a\Phi^\text{re}\) there exists \(s \in S\), \(h \in {}_aG\) such that \([a][\alpha] = h(\Phi_s)\).

(d) We have that
\[
{}_a\Phi^\text{re} = \bigcup_{\substack{b \in \text{Ob}(G), \\ g \in {}_aG \cap {}_bS}} g(\Phi_s).
\]

**Proof.** (a) follows from Lemma [1.4.3] Thus it is the case that
\[
\Phi_g = {}_a\Phi^+ \cap ({}_a\Phi_{s_1} + s_1(\hat{\Phi}_{s_2}) + \cdots + s_1s_2\ldots s_{n-1}(\hat{\Phi}_{s_n}))
\]
\[
= ({}_a\Phi^+ \cap \hat{\Phi}_{s_1}) + ({}_a\Phi^+ \cap s_1(\hat{\Phi}_{s_2})) + \cdots + ({}_a\Phi^+ \cap s_1\ldots s_{n-1}(\hat{\Phi}_{s_n})).
\]

Since each \({}_a\Phi^+ \cap s_1\ldots s_{i-1}(\hat{\Phi}_{s_i})\) is an equivalence class, they must be either pairwise disjoint or equal. Thus there exists \(r_1, r_2, \ldots, r_k \in S\), \(h'_1, h'_2, \ldots, h'_k \in \text{Mor}(G)\) with \(k \leq n\) such that
\[
\Phi_g = ({}_a\Phi^+ \cap h'_1(\hat{\Phi}_{r_1})) + ({}_a\Phi^+ \cap h'_2(\hat{\Phi}_{r_2})) + \cdots + ({}_a\Phi^+ \cap h'_k(\hat{\Phi}_{r_k})).
\]

Since for every \(1 \leq i \leq k\) we have that either \({}_a\Phi^+ \cap h'_i(\hat{\Phi}_{r_i}) = h'_i(\Phi_{r_i})\) or \({}_a\Phi^+ \cap h'_i(\hat{\Phi}_{r_i}) = h'_i(-\Phi_{r_i})\), and \(-\Phi_{r_i} = r_i(\Phi_{r_i})\) we get the result in (b).
For (c) we see that if \( \alpha \in _a\Phi^+ \) there must be \( h' \in _aG \) such that \( h'(\alpha) \in _a\Phi^+ \). Thus we may assume without loss of generality that \( \alpha \in _a\Phi^+ \cap _a\Phi^{re} \). Lemma 2.1.2 shows that \( \alpha \in \Phi_k \) for some \( k \in \text{Mor}(G) \), and part (b) gives the desired result in (c). (d) follows directly from (c).

3.3 Compressed signed groupoid sets

Recall from Section 2.4 that compressed signed groupoid sets are signed groupoid sets where each equivalence class has exactly one element. Since the equivalence classes of simply generated signed groupoid sets give us additional properties about the root system, we may study many properties of compressed simply generated signed groupoid sets as well. Since each simple root contains at most one element we are able to study the simple roots may more precisely. This classification allows us to relate the cardinality of an inversion set to the length function. This first lemma gives us the cardinality of the inversion set of a simple root.

**Lemma 3.3.1.** If \((G, \Phi)\) is a simply generated signed groupoid set with simple set \( S \), then \((G, \Phi^{re})\) is compressed if and only if \(|\Phi_s| \leq 1\) for every \( s \in S \).

**Proof.** Let \( a \in \text{Ob}(G) \), \( r \in _aS \). We first assume that \((G, \Phi^{re})\) is compressed and \( \Phi_s \neq \emptyset \). If \( \alpha \in \Phi_r \) we see from Lemma 3.2.1 (a) that \( _a[\alpha] = \Phi_r \). Since \((G, \Phi)\) is compressed we see that \(|\Phi_r| = 1\).

Now suppose that \(|\Phi_s| \leq 1\) for every \( s \in S \). Let \( \beta \in _a\Phi^{re} \). By Lemma 3.2.2 (c) there exists \( s \in S, h \in _aG \) such that \( _a[\beta] = h(\Phi_s) \). Thus \(|_a[\beta]| = |\Phi_s| \leq 1\), and so \((G, \Phi)\) is compressed.

**Remark 3.3.1.** Lemma 3.3.1 shows us that for a compressed simply generated signed groupoid set \((G, \Phi)\) with simple set \( S \) we may describe the simple roots at \( a \in \text{Ob}(G) \)
The previous section showed that the real roots of a root system are given by orbits of the simple roots. Since a compressed simply generated signed groupoid sets allow us to study the simple roots more explicitly, we can give a further description of the real roots in this situation. This description enables us to discuss the relationship between the length function and the cardinality of an inversion set. In particular we give an upper bound of cardinality of an inversion set in terms of the length function.

**Lemma 3.3.2.** Let \((G, \Phi)\) be a compressed signed groupoid set that is simply generated with simple set \(S\) and let \(a \in \text{Ob}(G)\). Then

\[
(a) \quad a^\Phi_{\text{re}} = \bigcup_{g \in S, \alpha \in \Pi} g(\alpha)
\]

\[
(b) \quad |\Phi_g| \leq l_S(g).
\]

**Proof.** (a) follows directly from Lemma 3.2.2(d). Lemma 3.3.1 tells us that for all \(s \in S\) we have that \(|\Phi_s| \leq 1\), and thus we see from Lemma 3.2.2 (b) that \(|\Phi_g| \leq l_S(g)|

Signed groupoid sets that are faithful and compressed have many useful properties. When considering simply generated signed groupoid sets that are faithful and compressed this remains true. The next lemma introduces some properties of these that relate to the cardinality of elements from both the inversion sets and the cocycle map. Note that these properties only follow because the simple roots of a compressed signed groupoid set have cardinality \(\geq 1\).

**Lemma 3.3.3.** Let \((G, \Phi)\) be a simply generated signed groupoid set that is faithful and compressed. Let \(S\) be a simple set for \((G, \Phi)\), and let \(a, b \in \text{Ob}(G), \ g \in {}_{a}G_{b}, \ s \in S_{a}\) and \(r \in {}_{b}S\). Then
(a) \(|N(s)| = 1\) for \(s \in S\).

(b) \(|N(sg)| = |N(g)| \pm 1\) and \(|N(gr)| = |N(g)| \pm 1\)

(c) \(|\Phi_g| - 1 \leq |\Phi_{sg}| \leq |\Phi_g| + 1\) and \(|\Phi_g| - 1 \leq |\Phi_{gr}| \leq |\Phi_g| + 1\).

Proof. Since \((G, \Phi)\) is compressed Lemma 3.3.1 shows that \(|\Phi_s| = |N(s)| \leq 1\). Since \((G, \Phi)\) is faithful it is the case that \(N(s) \neq \emptyset\), and so \(|N(s)| = 1\) proving (a). Part (b) follows from (a) and Lemma 1.4.3 (c), while (c) follows from part (b) and Lemma 1.5.1.

3.4 Principal signed groupoid sets

In this section we introduce principal signed groupoid sets. A more detailed description of them may be found in [10] and [11]. Recall that Lemma 3.3.3 demonstrated properties of the length function as they relate to the root system. When studying root systems of Coxeter groups, these properties give slightly stronger results, some of which may be found in [1] and [15]. Principal signed signed groupoid sets more closely resemble Coxeter groups, and thus have slightly stronger results than those from Lemma 3.3.3 about signed groupoid sets.

In the definition of principal signed groupoid sets given below, it should be noted that we require such a signed groupoid set to be simply generated.

Definition 3.4.1. A real signed groupoid set \((G, \Phi)\) is principal if it is simply generated with simple set \(S\) and for every \(g \in \text{Mor}(G)\) we have that \(l_S(g) = |\Phi_g|\).

An important fact regarding principal signed groupoid sets is that they are both faithful and compressed. This allows us to refine the results of simply generated signed groupoid sets when they are principal. It should be noted that in the definition of principal signed groupoid sets found in [10] requires faithfulness which is proved in the lemma below.
Lemma 3.4.1. If $(G, \Phi)$ is a principal signed groupoid set then $(G, \Phi)$ is faithful and compressed.

Proof. Let $S$ be a simple set for $(G, \Phi)$. Since $(G, \Phi)$ is principal we see that for $g \in \text{Mor}(G)$, $\Phi_g = \emptyset$ if and only if $l_S(g) = 0$ and thus $g$ is an identity morphism. Thus by Lemma 1.7.1 we see that $(G, \Phi)$ is faithful. To show that $(G, \Phi)$ is compressed let $a \in \text{Ob}(G)$, $\alpha \in _a\Phi$. From Lemma 3.2.2 (c) we see that there exists $h \in _aG$, $s \in S$ such that $g(\Phi_s) = _a[\alpha]$. Since $(G, \Phi)$ is principal, it we see that

$$|_a[\alpha]| = |g(\Phi_s)| = |\Phi_s| = l_S(s) = 1,$$

and so $(G, \Phi)$ is compressed. 

This next lemma give some equivalent conditions of principal signed groupoid sets. When discussing principal signed groupoid sets we frequently refer to these equivalent statements here.

Lemma 3.4.2. Let $(G, \Phi)$ be a simply generated signed groupoid set with simple set $S$ that is faithful and compressed. Then the following are equivalent

(i) $(G, \Phi)$ is principal

(ii) For every $g \in \text{Mor}(G)$ where $l_S(g) = n$ with $g = s_1s_2\ldots s_n$ for $s_i \in S$ we have that

$$N(g) = N(s_1) \cup s_1N(s_2) \cup \cdots \cup s_1s_2\ldots s_{n-1}N(s_n).$$

(iii) For every $g \in \text{Mor}(G)$, if $g = s_1s_2\ldots s_n$ where $s_i \in S$ such that $l_S(g) = n$, then $\Phi_{s_1} \subseteq \Phi_g$.

(iv) For every $a \in \text{Ob}(G)$, $g \in _aG$ where $\Phi_g \neq \emptyset$, there exists $s \in _aS$ with $\Phi_s \subseteq \Phi_g$.

Proof. Suppose $(G, \Phi)$ is a principal, and let $g \in \text{Mor}(G)$, $s_1s_2,\ldots s_n \in S$ such that $g = s_1s_2\ldots s_n$ and $l_S(g) = n$. Observe that $l_S(g) = |\Phi_g| = n$, and Lemma 3.2.2 (a) shows that

$$N(g) \subseteq N(s_1) \cup s_1N(s_2) \cup \cdots \cup s_1s_2\ldots s_{n-1}N(s_n).$$
Since Lemma 3.3.3 (a) tells us that $|N(s_i)| = 1$ for $1 \leq i \leq n$ we see that

$$n = |\Phi_g| \leq |N(s_1) \cup s_1N(s_2) \cup \cdots \cup s_1s_2 \cdots s_{n-1}N(s_n)| \leq n,$$

and so $N(g) = N(s_1) \cup s_1N(s_2) \cup \cdots \cup s_1s_2 \cdots s_{n-1}N(s_n)$, which gives us that (i) implies (ii).

Now assume (ii), and let $a \in \text{Ob}(G)$, $g \in _aG$, $s_1s_2 \ldots s_n \in S$ such that $g = s_1s_2 \ldots s_n$ and $l_S(g) = n$. Thus we see that $\hat{\Phi}_{s_1} \subseteq \hat{\Phi}_g$, and so

$$\Phi_{s_1} = _a\Phi^+ \cap \hat{\Phi}_{s_1} \subseteq _a\Phi^+ \cap \hat{\Phi}_g = \Phi_g,$$

giving us that (ii) implies (iii).

Now assume (iii). Let $h \in \text{Mor}(G)$ such that $l_S(h) = n$ and $h = s_1s_2 \ldots s_n$ where $s_i \in S$. From Lemma 1.4.3 we see that

$$N(h) = N(s_1) + s_1N(s_2) + \cdots + s_1s_2 \ldots s_{n-1}N(s_n)$$

We see from Lemma 3.3.2 (b) that $|N(h)| \leq n$. Suppose $|N(h)| < n$. Then there must exist $1 \leq i < j \leq n$ such that $s_is_{i+1} \ldots s_{i-1}N(s_i) = s_is_2 \ldots s_{j-1}N(s_j)$, and so

$$N(s_i) = s_is_{i+1} \ldots s_jN(s_j).$$

Let us choose such an $i, j$ such that $j$ is minimal. By the assumption in (iii) we see that

$$N(s_i) \subseteq N(s_is_{i+1} \ldots s_j)$$

$$= N(s_i) + s_iN(s_{i+1} \ldots n_{j-1}) + s_is_{i+1} \ldots s_{j-1}N(s_j)$$

$$= s_iN(s_{i+1} \ldots n_{j-1})$$

$$= s_iN(s_{i+1}) + s_is_{i+1}N(s_{i+1}) + \cdots + s_is_{i+1} \ldots s_{j-1}N(s_j),$$

and so there must exist $i < m < j$ such that $N(s_i) = s_is_{i+1} \ldots s_{m-1}N(s_m)$. This
contradicts the minimality of \( j \), and thus \( |N(h)| = n \). There we see that \((G, \Phi)\) is principal, completing the proof of (i) implies (iii).

(iii) implies (iv) follows trivially. Now assume (iv). Since \( |\Phi_g| < \infty \) we may use induction to show that \( |\Phi_g| = l_S(g) \). From Lemma 3.3.2(b) we know that \( |\Phi_g| \leq l_S(g) \).

Thus if \( |\Phi_g| = 1 \) it is the case that \( \Phi_g = \Phi_s \). Since \((G, \Phi)\) is faithful we see that \( g = s \) and so \( l_S(g) = 1 \).

Now suppose \( |\Phi_g| = n \geq 1 \). Since \( \Phi_s \subseteq \Phi_g \) we see from Lemma 1.4.4 that \( N(g) = N(s) \cup sN(s^{-1}g) \) and thus \( |N(s^{-1}(g))| = n - 1 \). If \( b := \text{dom}(s) \) we see that \( |\Phi_{s^{-1}g}| = n - 1 \) and \( \Phi_{s^{-1}g} \subseteq b\Phi^+ \). Thus by induction there exists \( s_2, s_3, \ldots, s_n \in S \) such that \( s^{-1}g = s_2s_3 \ldots s_n \) and

\[
N(s^{-1}g) = N(s_2) \cup s_2N(s_3) \cup \cdots \cup s_2s_3 \ldots s_{n-1}N(s_n).
\]

Let \( s_1 := s \). Then we have \( g = s_1s_2 \ldots s_n \) and

\[
N(g) = N(s_1) \cup s_1N(s_2) \cup \cdots \cup s_1s_2 \ldots s_{n-1}N(s_n).
\]

Thus we see that

\[
n = |\Phi_g| \leq l_S(g) \leq n,
\]

and thus \( l_S(g) = n \). Therefore \((G, \Phi)\) is principal, and so we have that (iv) implies (i).

\( \square \)

**Remark 3.4.1.** Note that from Lemma 3.4.2 we can conclude that if \((G, \Phi)\) is a principal signed groupoid set with \( g \in \text{Mor}(G) \) such that \( l_S(g) = n \) and \( g = s_1s_2 \ldots s_n \) where \( s_i \in S \), then we have that

(a) \( \Phi_{s_1} \subseteq \Phi_{s_1s_2} \subseteq \cdots \subseteq \Phi_{s_1s_2 \ldots s_n} = \Phi_g \).
(b) \( N(s_1) \subseteq N(s_1s_2) \subseteq \cdots \subseteq N(s_1s_2\ldots s_n) = N(g) \).

Observe that (a) and (b) are equivalent statements.
CHAPTER 4

REALIZED SIGNED GROUPOID SETS

A root system of a Coxeter system naturally exists in a real vector space. In fact, nearly the entire study of root systems of Coxeter systems studies exactly this. This section introduces a realization of signed groupoid sets in a real vector product space. In order to this, we must first discuss real vector spaces in section 4.1. Properties involving hyperplanes, half spaces and a total ordering on a vector space are particularly useful here. In this thesis we only use finite dimensional vector spaces, thus the properties discussed here may not hold in general. In section 4.2 we introduce realized signed groupoid sets. A realized signed groupoid set is a realization of a root system for a signed groupoid set in a real vector space.

Section 4.3 discusses realizations of compressed signed groupoid sets. In the same way we discuss individual roots of compressed signed groupoid sets is we also discuss these properties in their realizations. In Section 4.4 we introduce a way to compress a realized signed groupoid set. This construction is a realization for the compression of a signed groupoid set. It should be noted that we may not always be able to compress a realized signed groupoid set in this way; thus we introduce the property of compressibility. In the same way that properties of a signed groupoid set may be studied using its compression, we may also study properties of a realized signed groupoid set by considering its compression.

In Section 4.5 we discuss realizations of simply generated signed groupoid sets. Hyperplanes in a vector space play a key role in this situation. Here we introduce hyperplanes in a realization. Section 4.6 is devoted to the theory of hyperplanes in
simply generated realized signed groupoid sets. We study the relationship between
inversion sets and half spaces of a vector space of a realized signed groupoid set. Of
note are Theorem 4.6.1 and Theorem 4.6.2 that give a correspondence between hy-
perplanes separating subsets of the positive roots and the inversion sets of the signed
groupoid set. In particular, the result in Theorem 4.6.2 is critical when discussing
construction involving Coxeter groups in Chapter 6.

4.1 Real vector spaces

In this section we discuss properties of finite dimensional real vector spaces that
are useful in the study of realized signed groupoid sets. Properties of hyperplanes,
inner product spaces and vector space total orderings are discussed here.

Definition 4.1.1. Vect is the category of finite dimensional vector spaces. The
objects of Vect are real finite dimensional vector spaces and its morphisms are the
linear maps between vector spaces.

We will study representation of a groupoid $G$ in Vect that satisfy certain condi-
tions to enable us to discuss realized signed groupoid sets. Properties of hyperplanes
and open half spaces are very important in this thesis. Thus we define these notions
here.

Definition 4.1.2. Let $V$ be a vector space

(a) A hyperplane of a vector space is a linear subspace $H$ of $V$ with codimension 1.

(b) An open half space of $V$ is a connected component of $V \setminus H$ where $H$ is a hyper-
plane of $V$.

This next remark provides us with some useful properties of hyperplanes in a
vector space. The properties here are integral to many results in this thesis.

Remark 4.1.1. Let $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ be an inner product on $V$. 47
(a) If $H$ is a hyperplane in $V$, then there exists $x \in V$ such that $H := \{v \in V : (x, v) = 0\}$. In particular, $x$ is a vector orthogonal to $H$.

(b) A hyperplane $H$ in a vector space $V$ partitions $V \setminus H$ into two open half spaces.

(c) If $H$ is a hyperplane in $V$ and $x \in V$ such that $H := \{v \in V : (x, v) = 0\}$, then $H^+ := \{v \in V : (x, v) > 0\}$ and $H^- := \{v \in V : (x, v) < 0\}$ are two distinct halfspaces of $V$ separated by $H$.

Note that the choice of $x$ distinguishes $H^+$ and $H^-$. 

(d) When discussing half planes we often do not explicitly choose $H^+$ or $H^-$, but rather imply a choice

When considering hyperplanes in a vector space it is natural to consider a hyperplane arrangement. A hyperplane arrangement consists of chambers the are found in the intersection of open half spaces. We further study signed groupoid sets related to hyperplane arrangement in Chapter 5.

**Definition 4.1.3.** Let $V$ be a vector space.

(a) $\mathcal{H}$ is a hyperplane arrangement if it is a set of distinct hyperplanes in $V$.

(b) An open chamber $C$ of a hyperplane arrangement $\mathcal{H}$ a nonempty intersection of the form $C := \bigcap_{H_i \in \mathcal{H}} H_i^*$ where for each $H_i \in \mathcal{H}$ either $H_i^* = H_i^+$ or $H_i^* = H_i^-$.

(c) $C$ and $C'$ are adjacent chambers of a hyperplane arrangement $\mathcal{H}$ if $\overline{C} \cup \overline{C'}$ is a convex set, where $\overline{C}, \overline{C'}$ represent the closures of $C$ and $C'$ respectively.

We now introduce terminology that allows us to discuss properties involving hyperplanes.

**Definition 4.1.4.** Let $V$ be vector space with a hyper plane $H$, and let $A, B \subset V$ such that $A \cap H = \emptyset$ and $B \cap H = \emptyset$.

(a) $H^+$ and $H^-$ are the open half spaces bounded by $H$. We may use $H^*$ to denote an arbitrary open half space bounded by $H$ (either $H^+$ or $H^-$).

(b) $\overline{H^*}$ is a closed half space of $V$ bounded by $H$.

(c) $A$ and $B$ are on the same side of $H$ is they are contained in the same open half space bounded by $H$. 

48
(d) $H$ separates $A$ and $B$ if $A$ and $B$ are contained in distinct open half spaces bounded by $H$.

Vector space total orderings are an important aspect of a the realization of a signed groupoid set in a real vector space. We introduce the concept here.

**Definition 4.1.5.** $\preceq$ is a vector space total ordering for a vector space $V$ in $\mathbb{R}$ if $\preceq$ is a total ordering of $V$ such that for every $u, v \in V$ we have

i. If $0 \preceq u$ and $0 \preceq v$, then $0 \preceq u + v$

ii. If $0 \preceq u$ and $c \in \mathbb{R}_{\geq 0}$, then $0 \preceq cu$.

When discussing a vector space $V$ that has a vector space total ordering $\preceq$, we say that $V$ is vector space totally ordered with $\preceq$.

**Definition 4.1.6.** $C \subseteq V$ is a polyhedral cone of a vector space $V$ if $C := \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \alpha_i$ for some $\alpha_i \in V$.

The following theorem is a standard result of convex geometry and is necessary for our results. A proof of this fact may be found in [6] or another introductory text on convex geometry.

**Theorem 4.1.1.** If $C$ is a polyhedral cone in a finite dimensional vector space $V$ with $C \cap -C = \{0\}$, then there is a hyperplane in $V$ separating $C \setminus \{0\}$ and $-C \setminus \{0\}$.

**Remark 4.1.2.** Let $V$ be finite dimensional real inner product space with $A \subseteq V$. If $U := \text{Span}_\mathbb{R}(A)$ and $U^\perp := \{v \in V | (v, w) = 0 \text{ for all } w \in U\}$ it is the case that for every $x \in V$, $x = w + w_0$ where $w \in U, w_0 \in U^\perp$.

4.2 Realized signed groupoid sets

Root systems of Coxeter systems that exist in finite dimensional vector spaces have been studied extensively. Although the root system of a signed groupoid set
remains an abstract notion, we may still consider realizations of a root system in real vector space. Since root systems that exist in a vector space have many useful applications, the remainder of this thesis is devoted to studying these realized signed groupoid sets.

**Definition 4.2.1.** A realized signed groupoid set is an ordered triple \((G, \Phi, \mathcal{V})\) such that \((G, \Phi)\) is a signed groupoid set, \(\mathcal{V} : G \to \textbf{Vect}\) is a representation of \(G\) in \(\textbf{Vect}\), and for every \(a, b \in \text{Ob}(G)\) the following conditions are satisfied:

1. \(\Phi(a) \subseteq \mathcal{V}(a)\) such that the action by \(\{\pm\}\) on \(\Phi(a)\) is induced by scalar multiplication by \(\{\pm 1\}\) on \(\mathcal{V}(a)\) (i.e. for every \(\alpha \in \Phi(a)^+, \ -\alpha = (-1)\alpha \in \mathcal{V}(a)\)).
2. \(\mathcal{V}(a)\) has a vector space total ordering \(\leq\) such that for every \(\alpha \in \Phi(a)^+\) we have \(0 \leq \alpha\).
3. For every \(g \in aG_b\) the diagram below commutes.

\[
\begin{array}{ccc}
\Phi(a) & \to & \mathcal{V}(a) \\
\Phi(g) \downarrow & & \downarrow \mathcal{V}(g) \\
\Phi(b) & \to & \mathcal{V}(b)
\end{array}
\]

i.e. the action of \(g\) on \(\Phi\) in \(\textbf{Set}_\pm\) agrees with the action of \(g\) on \(\mathcal{V}\) in \(\textbf{Vect}\).

When discussing realized signed groupoid sets it is important to consider the signed groupoid set that is realized. This notion is formally defined below.

**Definition 4.2.2.** A realization of a signed groupoid set \((G, \Phi)\) is a realized signed groupoid set \((G, \Psi, \mathcal{V})\) together with \((\text{Id}_G, \nu) : (G, \Psi) \to (G, \Phi) \in \text{Mor}(\textbf{Gpd} - \textbf{Set}_\pm)\) such that \((\text{Id}_G, \nu)\) is an isomorphism in \(\textbf{Gpd} - \textbf{Set}_\pm\).

Note that a realized signed groupoid set \((G, \Phi, \mathcal{V})\) is a realization of \((G, \Phi)\). When discussing a realized signed groupoid set \((G, \Phi, \mathcal{V})\) we often study how the properties of \((G, \Phi)\) affect its realization \((G, \Phi, \mathcal{V})\).

We next define an isomorphism of realized signed groupoid sets. Note that there is not a natural notion of the category of realized signed groupoid, and thus the
isomorphism defined below is not an isomorphism in the category theory sense. An isomorphism of realized signed groupoid sets essentially describes two realized signed groupoid sets contain the same information.

**Definition 4.2.3.** The map \((\Theta, \nu, \mu) : (G, \Phi, V) \to (H, \Psi, U)\) is an isomorphism between realized signed groupoid sets \((G, \Phi, V), (H, \Psi, U)\) if

(i) \(\Theta : G \to H\) is a groupoid isomorphism

(ii) \(\mu : U\Theta \to V\) is a natural isomorphism of functors \(G \to \text{Vect}\)

(iii) \(\nu : \Psi\Theta \to \Phi\) is a natural isomorphism of functors

such that for every \(a, b \in \text{Ob}(G)\), \(g \in G_{ab}\) we have that

\[
\begin{array}{ccc}
\Phi(a) & \xrightarrow{\nu_a} & V(a) \\
\Psi(\Theta(a)) & \xrightarrow{\mu_a} & U(\Theta(a))
\end{array}
\]

is a commutative diagram.

### 4.3 Compressed realized signed groupoid sets

In this section we study realizations of compressed signed groupoid sets. In the same the that compressed signed groupoid sets are useful to study, their realizations reveal similar properties as well.

**Definition 4.3.1.** A realized signed groupoid set \((G, \Phi, V)\) is compressed if \((G, \Phi)\) is a compressed signed groupoid set.

This remark allows us to formally describe what it means for a realized signed groupoid set to be compressed.

**Remark 4.3.1.** Lemma 2.4.2 shows that if \((G, \Psi, V)\) is a realization of \((G, \Phi)\) and \((G, \Phi)\) is a compressed signed groupoid set, then \((G, \Psi, V)\) is a compressed realized signed groupoid set.
Realized signed groupoid sets that arise from real signed groupoid sets are of special interest to us. The following lemma allows us to provide a construction to study only the real roots of a root system.

**Lemma 4.3.1.** If \((G, \Phi, \mathcal{V})\) is a realized signed groupoid set, then \((G, \Phi^{\text{re}}, \mathcal{V})\) is also a realized signed groupoid set.

**Proof.** Since \((G, \Phi, \mathcal{V})\) is a realized signed groupoid set, \(\mathcal{V} : G \to \text{Vect}\) is a representation of \(G\) in \(\text{Vect}\).

For every \(a \in \text{Ob}(G)\), \(\Phi^a \subseteq \Phi\), and so \(\Phi^{\text{re}}(a) \subseteq \mathcal{V}(a)\). Since for every \(b \in \text{Ob}(G), g \in \Phi^a\) we have that \(g(-\alpha) = -g(\alpha)\) we see that the action by \(\{\pm\}\) is induced by scalar multiplication by \(\{\pm\}\) on \(\mathcal{V}(a)\). Since \(\Phi^{\text{re}} \subseteq \Phi\), we see that \((G, \Phi^{\text{re}}, \mathcal{V})\) is a realized signed groupoid set.

Now we may formally define a real realized signed groupoid set. This becomes a useful notion for later discussions in this thesis.

**Definition 4.3.2.** Let \((G, \Phi, \mathcal{V})\) be a realized signed groupoid set.

(a) \((G, \Phi, \mathcal{V})\) is a real realized signed groupoid set if \((G, \Phi)\) is a real signed groupoid set.

(b) \((G, \Phi, \mathcal{V})^{\text{re}} := (G, \Phi^{\text{re}}, \mathcal{V})\). We say that \((G, \Phi, \mathcal{V})^{\text{re}}\) is a real realization of \((G, \Phi)\).

**Remark 4.3.2.** Observe that if \((G, \Phi, \mathcal{V})\) is a realized signed groupoid set, then \((G, \Phi, \mathcal{V})^{\text{re}}\) is a real realized signed groupoid set. When \(\mathcal{X} := (G, \Phi, \mathcal{V})\) we may denote \((G, \Phi, \mathcal{V})^{\text{re}}\) as \(\mathcal{X}^{\text{re}}\).

4.4 Compressible realized signed groupoid sets

In section 2.4, when discussing the equivalence classes induced by dominance preorder in Definition 2.4.1, we studied properties of real compression. Real compression gave us a natural signed groupoid set. This section we studies realized signed
groupoid sets that arise from natural signed groupoid sets. In particular, we discuss a method of compressing a realized signed groupoid set. This produced a realization of the compression of the signed groupoid set, and is thus quite useful.

This method of compressing a realized signed groupoid set is always possible to apply. Thus we introduce the notion of compressible, where we may apply our method.

**Definition 4.4.1.** A realized signed groupoid set \((G, \Phi, V)\) is compressible if for all \(a \in \text{Ob}(G), \alpha \in \Phi_{\cdot a}, a[\alpha]\) is a finite set.

This next definition lays out the tools for provide a compression of a realized signed groupoid set.

**Definition 4.4.2.** Let \((G, \Phi, V)\) be a compressible realized signed groupoid set with \(a \in \text{Ob}(G), \alpha \in \Phi_{\cdot a}\).

\(\begin{align*}
(a) \quad \overline{\alpha} & := \sum_{\beta \in a[\alpha]} \beta \in V(a) \\
(b) \quad \overline{\Phi}_{\cdot a} & := \{\alpha : \alpha \in \Phi_{\cdot a}\} \subset V(a), \text{ and } \overline{\Phi}^{+}_{\cdot a} := \{\alpha : \alpha \in \Phi^{+}_{\cdot a}\} \subset V(a) \\
(c) \quad \omega_{\cdot a} : [\Phi] \rightarrow \overline{\Phi}_{\cdot a} \text{ is the map such that } a[\gamma] \mapsto \overline{\gamma} \text{ for } \gamma \in \Phi_{\cdot a}.
\end{align*}\)

This next lemma demonstrates how the definitions above produce a compressed signed groupoid set.

**Lemma 4.4.1.** Let \((G, \Phi, V)\) be a compressible realized groupoid set with \(a, b \in \text{Ob}(G), \alpha, \beta \in \Phi_{\cdot a}\).

\(\begin{align*}
(a) \quad V(g)(\overline{\alpha}) & = \omega_{\cdot b}(g[a[\alpha]]) \\
(b) \quad \overline{\alpha} = \overline{\beta} \text{ if and only if } \alpha \sim_{\cdot a} \beta.
\end{align*}\)
Proof. For (a), by definition $\alpha = \sum_{\gamma \in [\alpha]} a_\gamma$, and so we see that

$$\omega(g_a[\alpha]) = \omega(g_b[\alpha]) = g\alpha = \sum_{\delta \in [g(\alpha)]} \delta$$

$$= \sum_{\delta \in g(\alpha)} \delta = \sum_{\gamma \in [\alpha]} g(\gamma) = V(g)(\overline{\alpha}).$$

If $\alpha \sim a \beta$ it is clear that $\overline{\alpha} = \overline{\beta}$. Now suppose $\overline{\alpha} = \overline{\beta}$. Thus for every $h \in G_a$ we have that $V(h)(\overline{\alpha}) = V(h)(\overline{\beta})$. Since $\overline{\alpha} = \overline{\beta}$ it must be the case that $\alpha$ and $\beta$ have the same sign (either both positive or both negative). Since $V(h)(\overline{\alpha}) = V(h)(\overline{\beta})$ it must also be the case that $h(\alpha)$ and $h(\beta)$ have the same sign. Thus it for every $h \in G_a$ we have that $\alpha \in \Phi_h$ if and only if $\beta \in \Phi_h$. Lemma 2.3.1(a) tells us that this occurs if and only if $\alpha \sim a \beta$.

This next remark demonstrates that we have produced a compressed signed groupoid set that may be considered in a realization.

**Remark 4.4.1.** Let $(G, \Phi, V)$ be a compressible realized signed groupoid set with $a, b \in \text{Ob}(G)$.

(a) Since for all $\alpha, \beta \in \Phi_a$ we have that $\overline{\alpha} = \overline{\beta}$ if and only if $\alpha \sim a \beta$ it follows that the map $\omega : a[\Phi] \rightarrow a[\Phi]$ is a bijection.

(b) From (a) we see that $(\Phi_a, a[\Phi^+])$ is a definitely signed set where the $\{\pm\}$ action is induced by scalar multiplication of $\{\pm 1\}$.

Now we may define the compression of a realized signed groupoid set.

**Definition 4.4.3.** Let $(G, \Phi, V)$ be a compressible signed groupoid set.

(a) $\overline{\Phi} : G \longrightarrow \text{Set}_\pm$ is the functor such that for $a, b \in \text{Ob}(G), g \in aG_b$

$$a \mapsto \overline{a \Phi}$$

$$g \mapsto (V(g)|_{\overline{a \Phi}} : \overline{a \Phi} \rightarrow \overline{a \Phi})$$
where $\mathcal{V}(g)|_{\bar{b}\Phi}$ is the restriction of $\mathcal{V}(g)$ to $\bar{b}\Phi \subseteq \mathcal{V}(b)$.

(b) $(G, \Phi, \mathcal{V})^c := (G, \bar{\Phi}, \mathcal{V})$ is the compression of $(G, \Phi, \mathcal{V})$.

This lemma shows that the realized signed groupoid set we construction does in fact satisfy our requirements.

**Lemma 4.4.2.** Let $(G, \Phi, \mathcal{V})$ be a compressible realized signed groupoid set with $a, b \in \text{Ob}(G)$, $g \in _aG_b$. Then

(a) $(G, \bar{\Phi}, \mathcal{V})$ is a realized signed groupoid set

(b) $(G, \bar{\Phi}, \mathcal{V})$ is a realization of $(G, \Phi)^c$.

(c) There exists $f \in \text{Mor}(G - \text{Set}_\pm)$ such that $f : (G, \bar{\Phi}) \to (G, \Phi)$.

**Proof.** Remark 4.4.1 shows that $\mathcal{V} : G \to \text{Vect}$ is a representation of $G$ and that for $\bar{\Phi} : G \to \text{Set}_\pm$ the action of $\{\pm\}$ is induced by scalar multiplication by $\{\pm 1\}$ on $\mathcal{V}(a)$. Since for $x \in \bar{\Phi}(a) \subseteq \mathcal{V}(a)$ we have $\bar{\Phi}(g)(x) = \mathcal{V}(g)(x)$ we see that

\[
\begin{array}{ccc}
\Phi(a) & \rightarrow & \mathcal{V}(a) \\
\Phi(g) \downarrow & & \downarrow \mathcal{V}(g) \\
\Phi(b) & \rightarrow & \mathcal{V}(b)
\end{array}
\]

is a commutative diagram. From Lemma 2.4.1(c) we see that the vector space total ordering $\preceq_a$ of $\mathcal{V}(a)$ induced by $(G, \Phi, \mathcal{V})$ is a sufficient ordering to get that $(G, \bar{\Phi}, \mathcal{V})$ is a realized signed groupoid set proving (a).

To show (b), let $(G, \Psi) := (G, \Phi)^c$ and let $\omega : \Psi \to \bar{\Phi}$ be the natural transformation such that for $a \in \text{Ob}(G)$, the component $\omega_a : _a\Psi \to _a\bar{\Phi}$ is the map $\omega_a : [_[a]\Phi] \to _a\bar{\Phi}$ from Definition 4.4.2 (c). We see that $(\text{Id}_G, \omega) \in \text{Mor}(G - \text{Set}_\pm)$, and from Remark 4.4.1 it is the case that $(\text{Id}_G, \omega) : (G, \bar{\Phi}) \to (G, \Psi)$ is an isomorphism. Par (c) follows from (b) and Lemma 2.4.1(d).
4.5 Simply generated realized signed groupoid sets

This section discusses realizations of simply generated signed groupoid sets and their properties.

**Definition 4.5.1.** A realized signed groupoid set \((G, \Phi, V)\) is simply generated if it is a realization of a simply generated signed groupoid set.

**Remark 4.5.1.** Let \((G, \Phi, V)\) be a simply generated realized signed groupoid set with simple set \(S\). Since we have that for every \(s \in S\), \(|\Phi_s| < \infty\), Lemma 3.2.2(b) tells us that \((G, \Phi, V)^r\) is compressible.

When discussing root systems in a vector space, it can be beneficial to study the case where the roots are a spanning set of the vector space. When this occurs for a realized signed groupoid set, we call it reduced.

**Definition 4.5.2.** A realized signed groupoid set \((G, \Phi, V)\) is reduced if for every \(a \in \text{Ob}(G)\), \(\Phi^a\) spans \(V(a)\).

The following lemma relates realized signed groupoid sets with hyperplanes. This is the first of many properties of realized signed groupoid sets involving hyperplanes that are discussed in this thesis.

**Lemma 4.5.1.** Let \((G, \Phi, V)\) be a realized signed groupoid set that is reduced with \(a \in \text{Ob}(G)\). If \(H\) is a hyperplane in \(V(a)\) then \(H^+ \cap \Phi^a = -(H^- \cap \Phi^a)\).

**Proof.** From [4.1.1](a) we see that there exists \(x \in V(a)\) such that \(H := \{v \in V(a) : (x, v) = 0\}\) and \(H^+ = \{v \in V(a) : (x, v) > 0\}\). Since \((G, \Phi, V)\) is reduced we see that \(\Phi^a \cap H^+ = \{\alpha \in \Phi^a : (x, \alpha) > 0\}\) and thus

\[-(\Phi^a \cap H^+) = \{\alpha \in \Phi^a : (x, -\alpha) > 0\} = \{\alpha \in \Phi^a : (x, \alpha) < 0\} = \Phi^a \cap H^-.
\]
4.6 Hyperplanes

This section gives results involving hyperplanes in realizations of signed groupoid sets. We show hyperplanes split the positive roots of a root system into subsets that correspond to inversion sets. In order to relate to half spaces of hyperplanes we discuss the case when the positive roots lie in a polyhedral cone.

**Definition 4.6.1.** A realized signed groupoid set \((G, \Phi, V)\) is polyhedral if for every \(a \in G\), \(\mathbb{R}_{\geq 0}(a\Phi^+)\) is a polyhedral cone in \(V\).

**Remark 4.6.1.** Note that if \((G, \Phi, V)\) is a realized signed groupoid such that for every \(a \in G\) it is the case that \(|a\Phi| < \infty\), then \((G, \Phi, V)\) is polyhedral.

This next lemma shows that every inversion set in a realized signed groupoid sets is separated from the other positive roots by a hyperplane.

**Lemma 4.6.1.** If \((G, \Phi, V)\) is a polyhedral realized signed groupoid set with \(a \in \text{Ob}(G)\), \(g \in _a G\) then there exists a hyperplane in \(V(a)\) that separates \(\Phi_g\) and \(_a \Phi^+ \setminus \Phi_g\).

**Proof.** Let \(b := \text{dom}(g)\). Since \(\mathbb{R}_{\geq 0}(b\Phi^+) \subseteq \{v | v \preceq b \leq 0\}\) and \(\mathbb{R}_{\geq 0}(b\Phi^-) \subseteq \{v | 0 \preceq b \leq v\}\), we see that \(\mathbb{R}_{\geq 0}(b\Phi^+) \cap \mathbb{R}_{\geq 0}(b\Phi^-) = \{0\}\). Since \((G, \Phi, V)\) is polyhedral, Theorem [4.1.1](#) says that there must exist a hyperplane \(H\) in \(V(b)\) such that \(H\) separates \(_b \Phi^+\) and \(_b \Phi^-\). Since \(V(g)\) is a vector space isomorphism, we see that \(V(g)(H)\) is a hyperplane in \(V(a)\) that separates \(g(b\Phi^+)\) and \(g(b\Phi^-)\). Since \(\Phi_g \subseteq g(b\Phi^-)\) and \(_a \Phi^+ \setminus \Phi_g \subseteq g(b\Phi^+)\), we get that \(V(g)(H)\) separates \(\Phi_g\) and \(_a \Phi^+ \setminus \Phi_g\).  

It is typical to consider the case when a root system lies in the span of its simple roots. When this occurs for a realized simply generated signed groupoid set, we call it a conical realized signed groupoid set.
Definition 4.6.2. A realized compressed simply generated signed groupoid set \((G, \Phi, V)\) is conical if for every \(a \in \text{Ob}(G)\), \(a\Phi^+ \subseteq \mathbb{R}_{\geq 0}(a\Pi)\), (i.e. every positive root at \(a\) is a positive linear combination of the simple roots at \(a\)).

This next theorem shows that for a conical realized signed groupoid set, the converse of Lemma 4.6.1 holds.

Theorem 4.6.1. Let \((G, \Phi, V)\) be a conical realized signed groupoid set with \(a \in \text{Ob}(G)\). Let \(A \subseteq a\Phi\) such that there exists a hyperplane in \(V(a)\) that separates \(A\) and \(a\Phi^+ \setminus A\).

(a) If \(A \neq \emptyset\), then there must exist \(s \in aS\) such that \(\Phi_s \subseteq A\).

(b) If \(|A| \leq \infty\), then exists \(g \in aG\) such that \(\Phi_g = A\).

Proof. Let \(H\) be hyperplane in \(V(a)\) that separates \(A\) and \(a\Phi^+ \setminus A\). By Remark 4.1.1 there exists \(x \in V(a)\) such that \(A := \{\alpha \in a\Phi^+|(x, \alpha) > 0\}\). Since \((G, \Phi)\) is conical we see that for \(\alpha \in A\) there exist \(\alpha_1, \alpha_2 \ldots \alpha_k \in a\Pi\), such that \(\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k\) where \(c_i \geq 0\) for \(1 \leq i \leq k\). Since \((x, \alpha) > 0\) there must exist \(\alpha_i \in a\Pi\) such that \((x, \alpha_i) > 0\). Thus \(\alpha_i \in A\) and so there exists \(s \in aS\) such that \(\Phi_s = \{\alpha_i\} \subseteq A\) proving (a).

We will use induction on \(|A|\) to prove (b). If \(|A| = 0\) then \(\Phi_{Id_a} = \emptyset = A\). Now suppose \(|A| = n\) for \(n \geq 1\). From (a) we see that there exists \(s_1 \in S\) such that \(\Phi_{s_1} \subseteq A\). Let \(b := \text{dom}(s_1)\) and \(\{\beta\} := \Phi_{s_1}\).

Since \(\{\beta\} = a\Phi^+ \cap s_1(b\Phi^-)\) we see that \(a\Phi^+ \setminus \{\beta\} \subseteq s_1(b\Phi^+)\), and so \(s_1^{-1}(A \setminus \{\beta\}) \subseteq b\Phi^+\). Thus if \(A' := s_1^{-1}(A \setminus \{\beta\})\) we observe that \(A' \subseteq b\Phi^+, |A'| = n - 1,\) and \(s_1^{-1}(H)\) is a hyperplane in \(V(b)\) separating \(A'\) and \(b\Phi^+ \setminus A'\).

Thus by induction we see that there exist \(s_2, s_3, \ldots, s_n \in S, c := \text{dom}(s_n) \in \text{Ob}(G)\) such that

\[A = \Phi_{s_1} \cup s_1(\Phi_{s_2s_3 \ldots s_n}) = \Phi_{s_1s_2 \ldots s_n}\]
Lemma 4.6.1 and Theorem 4.6.1 lead to the following result.

**Theorem 4.6.2.** Let \((G, \Phi, V)\) be a conical realized signed groupoid set such that for every \(a \in \text{Ob}(G)\), \(|a\Pi| < \infty\). Let \(a \in \text{Ob}(G)\) and \(A \subseteq _a\Phi\). Then there exists a hyperplane \(H\) in \(V(a)\) that separates \(A\) and \(_a\Phi^+ \setminus A\) if and only if there exists \(g \in _aG\) such that \(\Phi_g = A\).

**Proof.** If \(H\) is a hyperplane in \(V(a)\) that separates \(A\) and \(_a\Phi^+ \setminus A\), then Theorem 4.6.1 shows that there exists \(g \in _aG\) such that \(\Phi_g = A\).

Now suppose that there exists \(g \in _aG\) such that \(\Phi_g = A\). Since \((G, \Phi, V)\) is conical, we see that for every \(b \in \text{Ob}(G)\) we have that \(\mathbb{R}_{\geq 0}(a\Phi^+) = \mathbb{R}_{\geq 0}(a\Pi)\) is a polyhedral cone. Thus \((G, \Phi, V)\) is polyhedral. Therefore Lemma 4.6.1 gives the desired result.

Another interesting fact of conical realized signed groupoid sets is that it is the realization of a principal signed groupoid set.

**Lemma 4.6.2.** Let \((G, \Phi, V)\) be a realized faithful signed groupoid set. If \((G, \Phi, V)\) is conical then \((G, \Phi)\) is a principal signed groupoid set.

**Proof.** Let \(a \in \text{Ob}(G)\), \(g \in _aG\). From Lemma 4.6.1 and Theorem 4.6.1 we see that if \(\Phi_g \neq \emptyset\), then there must exist \(s \in _aS\) such that \(\Phi_s \subseteq \Phi_g\). Thus from Lemma 3.4.2 shows that \((G, \Phi)\) is principal.

An interesting follow-up question to Lemma 4.6.2 is whether a realization of a principal signed groupoid set must be conical. We conclude this chapter by explicitly describing the subsets of roots that are separated by hyperplanes.

**Remark 4.6.2.** Let \((G, \Phi, V)\) be a realized signed groupoid set with \(a, b \in \text{Ob}(G)\) and \(g \in _aG_b\). Observe that if \(H\) is a hyperplane such that \(H\) separates \(\Phi_g\) and \(_a\Phi^+ \setminus \Phi_g\), then \(H\) separates \(g(\Phi^+)\) and \(g(\Phi^-)\) where \(_a\Phi = g(\Phi^+) \cup g(\Phi^-)\).
Section 4.6 discussed hyperplanes in vector spaces with in the context of the realization of a signed groupoid set. Chapter 5 discusses hyperplane arrangements related to realized signed groupoid sets.

In section 5.1 we construct a signed groupoid set associated to a hyperplane arrangement. The objects of this signed groupoid set are chambers while the morphisms are maps between chambers. Section 5.2 discusses the universal cover of a groupoid and introduces a similar concept for signed groupoid sets. In particular, the universal cover of signed groupoid set is a signed groupoid set associated to the universal cover of the groupoid. In section 5.3 we produce a realization of the signed groupoid set associated to a hyperplane arrangement. This realized signed groupoid set is used in section 6.6 when we discuss a realization of the universal cover of the signed groupoid sets given in section 5.2.

5.1 Signed groupoid sets of hyperplane arrangements

For a hyperplane arrangement \( \mathcal{H} \), we may consider a connected and simply connected groupoid where the objects are given by the chambers of \( \mathcal{H} \). Associated to this groupoid is a set of vectors normal to the hyperplanes of \( \mathcal{H} \).

**Definition 5.1.1.** Let \( \mathcal{H} \) be a hyperplane arrangement in a finite dimensional real inner product space \( V \).

(a) \( \Lambda_\mathcal{H} := \{ x_i : x_i \text{ is a unit normal vector for some } H \in \mathcal{H} \} \)
(b) A groupoid $G$ is an $\mathcal{H}$ groupoid if

i. $\text{Ob}(G) = \{C : C$ is a chamber of $\mathcal{H}\}$

ii. for $a, b \in \text{Ob}(G)$ we have $|aG_b| = 1$.

(c) $C(\Lambda) := \{x \in \Lambda_{\mathcal{H}}| (\rho, x) > 0\}$ where $C \in \text{Ob}(G), \rho \in C$ for an $\mathcal{H}$ groupoid $G$ (i.e. $C(\Lambda_{\mathcal{H}})$ is the set of unit normals of the hyperplanes in $\mathcal{H}$ that point towards $C$).

Remark 5.1.1. Let $\mathcal{H}$ be a hyperplane arrangement.

(a) Observe that for $C$, a chamber of $\mathcal{H}$, if $\rho_1, \rho_2 \in C$, then $\{x \in \Lambda_{\mathcal{H}}| (\rho_1, x) > 0\} = \{x \in \Lambda_{\mathcal{H}}| (\rho_2, x) > 0\}$ (i.e. $C(\Lambda_{\mathcal{H}})$ does not depend on the choice of $\rho$).

(b) We also observe that the set $\Lambda_{\mathcal{H}}$ is closed under scalar multiplication by $\{\pm 1\}$. This allows us to regard $(\Lambda_{\mathcal{H}}, C(\Lambda_{\mathcal{H}}))$ as a definitely signed set for every chamber $C$ of $\mathcal{H}$.

The following lemma attaches a root system to an $\mathcal{H}$ groupoid and thus gives us a signed groupoid set. This signed groupoid set is faithful, compressed and simply generated.

Lemma 5.1.1. Let $G$ be an $\mathcal{H}$ groupoid, and let $\Phi : G \to \textbf{Set}_\pm$ be the functor such that for $C, D \in \text{Ob}(G), g \in cG_D$

$$C \mapsto (\Lambda_{\mathcal{H}}, C(\Lambda_{\mathcal{H}}))$$

$$g \mapsto (\text{Id} : \Lambda_{\mathcal{H}} \to \Lambda_{\mathcal{H}}).$$

(a) Then $(G, \Phi)$ is a signed groupoid set that is both faithful and compressed.

(b) $(G, \Phi)$ is also a simply generated signed groupoid set.

Proof. It is clear that $(G, \Phi)$ is a signed groupoid set. Let $C \in \text{Ob}(G)$. Thus we observe that $C = \{v \in V|(v, \alpha) > 0$ for all $\alpha \in C(\Lambda_{\mathcal{H}})\}$. Since given $C, C' \in \text{Ob}(G)$, we have that $C(\Lambda_{\mathcal{H}}) = C'(\Lambda_{\mathcal{H}})$ if and only if $C = C'$, it follows that $(G, \Phi)$ is faithful.

To show that $(G, \Phi)$ is compressed, it is sufficient to show that for every $\alpha, \beta \in \Lambda_{\mathcal{H}}$ where $\alpha \in C(\Lambda_{\mathcal{H}})$ if and only $\beta \in C(\Lambda_{\mathcal{H}})$ for every chamber $C \in \text{Ob}(G)$, it is the case that $\alpha = \beta$. 

61
Suppose, to the contrary, that there exists $\alpha, \beta \in \Lambda_{3\xi}$ such that $\alpha \neq \beta$ and for every chamber $C \in \text{Ob}(G)$ we have that $\alpha \in C(\Lambda_{3\xi})$ if and only $\beta \in C(\Lambda_{3\xi})$. Since $\Lambda_{3\xi} = \bigcup_{C \in \text{Ob}(G)} C(\Lambda_{3\xi})$, there exists a chamber $C$ where $\alpha, \beta \in C(\Lambda_{3\xi})$. Since $\alpha \in C(\Lambda_{3\xi})$ it is the case that $-\alpha \notin C(\Lambda_{3\xi})$, and so $\alpha \neq -\beta$. Thus $\alpha \neq c\beta$ for any scalar $c$. Since $\alpha \neq \beta$, we see that $\emptyset \neq \{v \in V|(v, \alpha) > 0, (v, \beta) < 0\} \subseteq V$. Also since $\alpha, \beta \in \Lambda_{3\xi}$ we see that $\{v \in V|(v, \alpha) > 0, (v, \beta) < 0\}$ must contain a chamber $D \in \text{Ob}(G)$. Thus $\alpha \in D(\Lambda_{3\xi})$ and $\beta \notin D(\Lambda_{3\xi})$, which gives us a contradiction. Therefore $(G, \Phi)$ is compressed and so we complete the proof of (a).

To prove that that $(G, \Phi)$ is simply generated, let

$$S := \{g \in cG_{C'} : C, C' \text{ are adjacent chambers}\}.$$ 

We show that $S$ satisfies conditions (i)-(v) of Definition 3.1.1. Clearly $S$ contains no identity morphisms. It is easily shown that for any $s \in S$, $s^{-1} \in S$, and thus (i) and (ii) are satisfied. To show (iii), note that [2] demonstrates that given any two chambers $C, D$ of $\mathcal{H}$, there exists a sequence of chambers

$$C = C_0, C_1, \ldots, C_n = D$$

where $C_{i-1}$ and $C_i$ are adjacent chambers for $1 \leq i \leq n$. Therefore $<S> = \text{Mor}(G)$ and so (iii) is satisfied.

For $C, D \in \text{Ob}(G), g \in cG_D$ we see that the inversion set of $g$ is

$$\Phi_g = C(\Lambda_{3\xi}) \cap -D(\Lambda_{3\xi}) = C(\Lambda_{3\xi}) \setminus D(\Lambda_{3\xi}).$$

This means that $\Phi_g$ is the set of unit normal vectors of $\mathcal{H}$ that point towards $C$ that separate $C$ and $D$. Thus $|\Phi_g| = 1$ if and only if $C$ and $D$ are separated by a unique hyperplane in $\mathcal{H}$. A well known result from [2] shows that two chambers $C$ and $D$
are separated by a unique hyperplane in \( \mathcal{H} \) if and only if \( C \) and \( D \) are adjacent. Thus
\[ |\Phi_g| = 1 \] if and only if \( g \in S \), and so (iv) follows immediately. (v) holds since \( a\Phi \) is finite. Therefore, since conditions (i)-(v) are satisfied, we see that \( S \) is a simple set for \((G, \Phi)\).

Lemma 5.1.1 allows us to define a signed groupoid set arising from a hyperplane arrangement.

**Definition 5.1.2.** An \( \mathcal{H} \)-signed groupoid set for a hyperplane arrangement \( \mathcal{H} \) is a signed groupoid set \((G, \Phi)\) where

(i) \( G \) is an \( \mathcal{H} \)-groupoid
(ii) \( \Phi : G \to \text{Set}_\pm \) is the functor such that for \( C, D \in \text{Ob}(G) \), \( g \in cG_D \)

\[ C \mapsto (\Lambda_{\mathcal{H}}, C(\Lambda_{\mathcal{H}})) \]
\[ g \mapsto \text{Id}_{\Lambda_{\mathcal{H}}} : \Lambda_{\mathcal{H}} \to \Lambda_{\mathcal{H}}. \]

**Remark 5.1.2.** Let \( \mathcal{H} \) be a hyperplane arrangement and let \((G, \Phi)\) be an \( \mathcal{H} \) groupoid. If \( C \in \text{Ob}(G) \) we see that \( c\Phi = \Lambda_{\mathcal{H}} \) and \( c\Phi^+ := C(\Lambda_{\mathcal{H}}) \). We may refer to these using either notation.

### 5.2 Universal Coverings

In this section we discuss the universal covering of a groupoid. We then give an occurrence of the universal covering and use this to produce a universal covering for signed groupoid sets and their realizations.

**Definition 5.2.1.** A universal cover of a connected groupoid \( G \) is a groupoid \( H \) together with a groupoid homomorphism \( \Theta : H \to G \) such that

(i) For every \( a \in \text{Ob}(H) \) the map \( aH \to \Theta(a)G \) induced by \( \Theta \) is a bijection.
(ii) \( \Theta : \text{Ob}(H) \to \text{Ob}(G) \) is surjective.
(iii) $H$ is connected and simply connected (i.e. for every $a, b \in \text{Ob}(G)$, $|_{a}H_{b}| = 1$).

We now construct a groupoid and a groupoid homomorphism from a connected groupoid. We will show that is construction is a universal cover of the groupoid.

**Definition 5.2.2.** Let $G$ be a connected groupoid with $a \in \text{Ob}(G)$.

(a) $G/a$ is the groupoid such that

(i) $\text{Ob}(G/a)$ is the collection of maps $a \xleftarrow{g} b$ where $b \in \text{Ob}(G), g \in _{a}G_{b}$.

(ii) For $x := a \xleftarrow{g} b, y := a \xleftarrow{g'} b' \in \text{Ob}(G/a)$, the morphisms are given by $x(G/a)_{y} := \{h \in _{b}G_{b'}|g' = gh\}$.

(b) The covering for $G/a$ is the functor $\Theta : G/a \to G$ such that for $a \xleftarrow{g} b \in \text{Ob}(G/a), h \in \text{Mor}(G/a)$

$$(a \xleftarrow{g} b) \mapsto b \in \text{Ob}(G) \quad h \mapsto h \in \text{Mor}(G).$$

This next lemma shows that the above construction is indeed a universal cover of the connected groupoid.

**Lemma 5.2.1.** If $G$ is a connected groupoid with $a \in \text{Ob}(G)$ and $\Theta : G/a \to G$ the covering for $G/a$, then $G/a$ with $\Theta$ is a universal cover of $G$.

**Proof.** Let $x, y \in \text{Ob}(G/a)$, where $x := a \xleftarrow{g} b, y := a \xleftarrow{g'} b'$ for $a, b, b' \in \text{Ob}(G), g, g' \in \text{Mor}(G)$. Since $g^{-1}g' \in _{b}G_{b'}$, we see that $g^{-1}g' \in _{x}(G/a)_{y}$. Since the morphism $g^{-1}g'$ always exists and is unique, it is the case that $G/a$ is a connected and simply connected groupoid.

We will now show that for every $x \in \text{Ob}(G)$ where $x := a \xleftarrow{g} b$ the map $x\Theta : x(G/a) \to _{b}G$ is a bijection. Let $h \in x\Theta, y := a \xleftarrow{g'} b' = \text{dom}(h)$. Thus it is the case that $h \in _{b}G_{b'}$ where $g' = gh$. Therefore $h := g^{-1}g' \in _{b}G$, and so $x\Theta$ identifies with the identity map $_{b}G \to _{b}G$, and is trivially a bijection. Since $G$ is a connected groupoid it follows that $\Theta : \text{Ob}(G/a) \to G$ is surjective, and thus $G/a$ with $\Theta$ is a universal cover of $G$. 

64
The universal cover of a groupoid gives rise to the universal covering of a signed groupoid set and a realized signed groupoid set. The following lemma gives us these constructions.

**Lemma 5.2.2.** Let $G$ be a connected groupoid and let the groupoid $H$ together with $\Theta : H \to G$ be a universal cover for $G$.

(a) If $(G, \Phi)$ is a signed groupoid set then $(H, \Phi \Theta)$ is a signed groupoid set

(b) If $\text{Id} : \Phi \Theta \to \Phi \Theta$ is the identity map, then $(\Theta, \text{Id}) : (H, \Phi \Theta) \to (G, \Phi)$ is a morphism of $\text{Gpd} - \text{Set}_\pm$

(c) If $(G, \Phi, \mathcal{V})$ is a realized signed groupoid set, then $(H, \Phi \Theta, \mathcal{V} \Theta)$ is a realized signed groupoid set.

**Proof.** (a) Let $a, b \in \text{Ob}(H)$, $h \in aH_b$. Since $\Theta(a), \Theta(b) \in \text{Ob}(G)$, we have that $\Phi(\Theta(a)), \Phi(\Theta(b)) \in \text{Ob}(\text{Set}_\pm)$. Also since $\Phi : G \to \text{Set}_\pm$ is functor we see that $\Theta(h) \in \Theta(a)G_{\Theta(b)}$, and so $\Phi \Theta : H \to \text{Set}_\pm$ is a representation of $H$ in $\text{Set}_\pm$ giving us (a). By definition, $\Theta : H \to G$ is a functor, and thus (b) follows from the fact that the natural transformation $\text{Id} : \Phi \Theta \to \Phi \Theta$ is positivity preserving.

For (c) let $a \in \text{Ob}(H)$, $h \in aH$. Observe that $\Theta(a) \in \text{Ob}(G)$ and $\Theta(h) \in aG$. Since $(G, \Phi, \mathcal{V})$ is a realized signed groupoid set, $\mathcal{V} : G \to \text{Vect}$ is functor, and so $\mathcal{V} \Theta : H \to \text{Vect}$ is a representation of $H$ in $\text{Vect}$, with $\Phi \Theta(a) \subseteq \mathcal{V} \Theta(a)$ where the $\{\pm\}$ action is induced by scalar multiplication of $\{\pm 1\}$. Since $aH \to \Theta(a)G$ is a bijection we see that the total ordering on $\mathcal{V}(\Theta(a))$ holds for $\mathcal{V} \Theta(a)$, and thus $(H, \Phi \Theta, \mathcal{V} \Theta)$ is a realized signed groupoid set proving (c).

From the lemma above we may define the universal cover of a signed groupoid set and the universal cover of a realized signed groupoid set.
**Definition 5.2.3.** Let $G$ be a connected groupoid such that the groupoid $H$ with $\Theta : H \to G$ is a universal cover of $G$, and let $(G, \Phi, V)$ be a realized signed groupoid set.

(a) We call $(H, \Phi \Theta)$ the universal cover of $(G, \Phi)$.

(b) We call $(H, \Phi \Theta, \forall \Theta)$ the universal cover of $(G, \Phi, V)$.

The following lemma is useful when considering the universal cover of groupoid.

**Lemma 5.2.3.** Let $G$ be a connected and simply connected groupoid and $C$ a category. If $F : G \to C$ is a representation of $G$ in $C$, then $F$ is naturally isomorphic to the constant functor (i.e. every representation of $G$ in $C$ is trivial).

**Proof.** Let $a \in \text{Ob}(G)$ be fixed. Since $G$ is connected and simply connected we have that for any $b \in \text{Ob}(G)$, there exists a unique $g \in {}_aG_a$. Thus for every $b \in \text{Ob}(G)$ we may define $v_b := F(g)^{-1} = F(g^{-1})$ where $g \in {}_aG_b$ is the unique such morphism.

Now let $M := F(a)$, and let $F_M$ the constant functor with value $M$. One readily checks that for $b \in \text{Ob}(G)$, the $v_b$ constitute the components of a natural transformation $\nu : F \to F_M$. \hfill \Box

**Corollary 5.2.1.** Let $G$ be a groupoid that is connected and simply connected.

(a) If $(G, \Phi)$ is a signed groupoid set, then $(G, \Phi)$ is isomorphic to a signed groupoid set $(G, \Psi)$ where $\Psi : G \to \text{Set}_\pm$ has an underlying constant functor $G \to \text{Set}$.

(b) If $(G, \Phi, V)$ is a realized signed groupoid set, then $(G, \Phi, V)$ is isomorphic to a realized signed groupoid set $(G, \Psi, V')$ where $\Psi : G \to \text{Set}$ has an underlying constant functor $G \to \text{Set}_\pm$ and $V' : G \to V'$ is a constant function.

This tells us that for a realization of a connected and simply connected groupoid, the set of roots (and ambient vector space) attached to an object may be chosen independently of the object. Note that the sets of positive roots may not be taken independently of the object since the definition of an isomorphism of signed groupoid sets requires that a natural transformations of functors to signed sets has positivity preserving components.
Remark 5.2.1. In general, for any connected and simply connected groupoid $G$ with $a, b \in \text{Ob}(G)$, we will write $aG_b = \{a g_b\}$ (i.e. $a g_b$ is the unique morphism in $aG_b$).

5.3 Realizations of $\mathcal{H}$ signed groupoid sets

In this section we discuss a natural realization of a signed groupoid set associated to a hyperplane arrangement. This first lemma gives us the desired realized signed groupoid set.

Lemma 5.3.1. Let $\mathcal{H}$ be a hyperplane arrangement in an inner product space $V$, and let $(G, \Phi)$ be an $\mathcal{H}$ signed groupoid set. If $\mathcal{V} : G \to \textbf{Vect}$ is the map such that for $C, D \in \text{Ob}(G)$, $g \in cG_d$ we have that

\[
\mathcal{V}(C) : = V,
\]
\[
\mathcal{V}(g) : = \text{Id}_V : V \to V,
\]

then $(G, \Phi, \mathcal{V})$ is a realized signed groupoid set.

Proof. Let $a \in \text{Ob}(G)$, $g \in \text{Mor}(G)$. Remark 5.1.1 tells us that $\Phi(a) \subseteq V(a)$ where the $\{\pm\}$ action on $\Phi(a)$ is induced by scalar multiplication by $\{\pm 1\}$ on $V(a)$, and since $\mathcal{V}(g)$ is a vector space isomorphism, the action of $g$ on $\Phi$ agrees with the action of $g$ on $\mathcal{V}$.

If $a \in \text{Ob}(G)$, where $a$ is the chamber $C$, it is the case that $\Phi(a)^+ = \{\alpha \in \Phi(a)|(\rho, \alpha) > 0\}$ for any $\rho \in C$. Thus the hyperplane $\alpha^\perp$ strictly separates $\Phi(a)^+$ and $\Phi(a)^-$, and so there is a vector space total ordering $\preceq$ of $V$ where $\Phi(a)^+ \subseteq \{v \in V|0 \preceq v\}$.

One such vector space total ordering may be seen as follows. Let $\{\rho_1, \rho_2, \ldots, \rho_n\}$ be a basis of $V$ where $\rho_1 := \rho$. The lexicographic ordering on the coordinate vectors $((\rho_1, v), (\rho_2, v), \ldots, (\rho_n, v))$ for $v \in V$ gives such a total ordering. From this we see that $(G, \Phi, \mathcal{V})$ is a realized signed groupoid set.

67
Lemma 5.3.1 allows us define the realized signed groupoid set given below.

**Definition 5.3.1.** We call \((G, \Phi, \mathcal{V})\) the \(\mathcal{H}\) realized signed groupoid set for a hyperplane arrangement \(\mathcal{H}\) in an inner product space \(V\) where \(\mathcal{V} : G \to \text{Vect}\) is the map such that for \(C, D \in \text{Ob}(G), g \in cG_D\), we have that \(\mathcal{V}(C) := V\) and \(\mathcal{V}(g) := \text{Id}_V : V \to V\).
CHAPTER 6

COXETER GROUPOIDS AND SIGNED GROUPOID SETS

In this chapter we introduce singed groupoid sets arising from Coxeter systems. These constructions may be found in [4] and [5]. We discuss these constructions in the framework of the previous chapters and study their properties. Many of the main results of this thesis are found in this chapter.

In Section 6.1 we introduce a generalized construction of the groupoids and root systems studied in [4] and [5]. We study simply generated instances of these and produce new signed groupoid sets. Section 6.2 introduces the standard theory of Coxeter systems and their root systems. A more thorough description may be found in an introductory text on Coxeter groups such as [15] [1]. In Section 6.3 we introduces the one object groupoid that is a Coxeter group. The root system arising Coxeter system gives us a realized signed groupoid set. We note that this signed groupoid set is simply generated and principal and that its realization is conical.

In section 6.4 we study the Brink and Howlett construction, which is known to be a simply generated signed groupoid set. We give a realization of this signed groupoid set that is not conical. Section 6.5 considers this construction in the case of finite Coxeter groups. We show that this is a real realized signed groupoid set and that the compression of this realization is conical. Results of note in this section include Theorem 6.5.1 which shows that this compression is conical, and Theorem 6.5.2 which classifies hyperplanes in this realization in terms of inversion sets of the signed groupoid set.

Section 6.6 studies the universal covering of these realized signed groupoid sets.
We relate it to realized signed groupoid sets that come from hyperplane arrangements as discussed in section 5.3. Of note in this section is Theorem 6.6.1 which shows that the universal covering of a realized signed groupoid set is isomorphic, up to rescaling of roots, to the realized signed groupoid set associated to a simplicial hyperplane arrangement.

In section 6.7 we study an example of this construction that arises from the Coxeter group $D_4$. We produce a root system has 14 roots, 3 simple roots, and 32 morphisms at each object. We verify that this is a real, simply generated, conical realized signed groupoid set.

6.1 Free signed groupoid sets

In this section we give a generalization of the construction of Brink and Howlett in [4] and [5]. Our construction produces a new signed groupoid set from a simply generated signed groupoid set. We first introduce the notion of a free signed groupoid set.

**Definition 6.1.1.** A realized signed groupoid set $(G, \Phi, V)$ is free if it is simply generated and for every $a \in \text{Ob}(G)$, $a\Pi$ is a linearly independent set in $V(a)$.

We now introduce a new groupoid and signed groupoid set arising from a free signed groupoid set. Brink and Howlett studied this construction in terms of Coxeter groups; this is a slight generalization of their construction.

**Definition 6.1.2.** Let $(G, \Phi)$ be a simply generated signed groupoid set with simple set $S$ and simple roots $\Pi$.

(a) $G(\Pi)$ is the groupoid such that

i. $\text{Ob}(G(\Pi)) := \{(a, \Delta) | a \in \text{Ob}(G), \Delta \subseteq a\Pi\}$,

ii. $\Delta(G(\Pi))_y := \{(x, g, y) | g(\Delta_2) = \Delta_1\}$ where $x := (a, \Delta_1), y := (b, \Delta_2) \in \text{Ob}(G(\Pi)), g \in {}_aG_b$. 

70
(b) $\mathcal{F}_\Pi : G(\Pi) \rightarrow G$ is the functor such that for $(a, \Delta) \in \text{Ob}(G(\Pi))$, $g \in \text{Mor}(G(\Pi))$, $\mathcal{F}_\Pi((a, \Delta)) := a$ and $\mathcal{F}_\Pi(g) := g$.

(c) $(G(\Pi), \tilde{\Phi})$ is the signed groupoid set where $\tilde{\Phi}$ is the functor such that $\tilde{\Phi} := \Phi \circ \mathcal{F}_\Pi$

**Remark 6.1.1.** If $(G, \Phi)$ is a simply generated signed groupoid set with simple roots $\Pi$, then $(\mathcal{F}_\Pi, \text{Id}) : (G(\Pi), \tilde{\Phi}) \rightarrow (G, \Phi)$ is a morphism in the category $\text{Gpd} - \text{Set}_\pm$.

The signed groupoid set defined above gives rise to a realized signed groupoid set.

**Lemma 6.1.1.** If $(G, \Phi, \mathcal{V})$ is a simply generated realized signed groupoid set with simple roots $\Pi$, then $(G(\Pi), \tilde{\Phi}, \mathcal{V} \circ \mathcal{F}_\Pi)$ is also a realized signed groupoid set

**Proof.** Since $(G, \Phi, \mathcal{V})$ is a realized signed groupoid set and $\mathcal{F}_\Pi$ is a functor such that for every $(a, \Delta) \in \text{Ob}(G(\Pi)) \in \text{Mor}(G(\Pi))$, it is the case that $\mathcal{F}_\Pi((a, \Delta)) = a \in \text{Ob}(G)$ and $\mathcal{F}_\Pi(g) = g \in \text{Mor}(G)$. Thus $(G(\Pi), \tilde{\Phi}, \mathcal{V} \circ \mathcal{F}_\Pi)$ is also a realized signed groupoid set.

From the realized signed groupoid set given in the previous lemma, we produce another realization of that same signed groupoid set. This realization is contained within a quotient vector space. The quotient vector space allows us to observe other properties of these roots in the vector space.

**Definition 6.1.3.** Let $(G, \Phi, \mathcal{V})$ be a free realized signed groupoid set that is conical with simple roots $\Pi$. Let $(a, \Delta) \in \text{Ob}(G(\Pi))$.

(a) $\mathcal{V}(a)/(\mathbb{R}\Delta)$ is the quotient vector space. It has basis $\{\gamma + \mathbb{R}\Delta | \gamma \in \Pi \setminus \Delta\}$

(b) $\mathcal{V}/\Pi : G(\Pi) \rightarrow \text{Vect}$ is the functor such that all $x := (a, \Delta), y := (a', \Delta') \in \text{Ob}(G(\Pi)), g \in \mathcal{V}(G(\Pi))_x$

$$
\mathcal{V}(a) \mapsto \mathcal{V}(a)/(\mathbb{R}\Delta)
$$

$$(g : x \rightarrow y) \mapsto g : (\mathcal{V}(a)/(\mathbb{R}\Delta) \rightarrow \mathcal{V}(a')/(\mathbb{R}\Delta'))$$ is the induced map.

Note that since $g \in \mathcal{V}(G(\Pi))_x$ it is the case that $g(\Delta) = \Delta'$.
(c) \((\mathcal{V}/\Pi)^a : \mathcal{V}(a) \to \mathcal{V}(a)/\Pi\) is the quotient map such that for \(v \in \mathcal{V}(a)\) where \(v := \sum_{c_i \in \mathbb{R}, \gamma_i \in \Pi} c_i \gamma_i\)

\[
v \mapsto (\sum_{\gamma_i \in \Pi \setminus \Delta} c_i \gamma_i) + \mathbb{R}\Delta.
\]

(d) \(\hat{\Phi}/\Pi : G(\Pi) \to \text{Set}_\pm\) is the functor such that for \((a, \Delta) \in \text{Ob}(\Pi)\) with \(a \in \text{Ob}(G), \Delta \subseteq \Pi\)

\[
(a, \Delta) \mapsto (A^+, A)
\]

where \(A^+ := (\mathcal{V}/\Pi)^a(a\hat{\Phi}^+) \setminus \{0 + \mathbb{R}(\Delta)\}\) and \(A := (\mathcal{V}/\Pi)^a(a\hat{\Phi}) \setminus \{0 + \mathbb{R}(\Delta)\}\)

Lemma 6.1.2. Let \((G, \Phi, \mathcal{V})\) be a free realized signed groupoid set that is conical with simple roots \(\Pi\). For every \((a, \Delta) \in \text{Ob}(G(\Pi)), \Delta \subseteq \Pi\)

\[
(a, \Delta) \mapsto (\mathcal{V}(a)/\Pi)^{\Phi}(a\hat{\Phi}) \setminus \{0\}, \text{ and } (\mathcal{V}(a)/\Pi)^{\Phi} = (\mathcal{V}(a)/\Pi)^{(a\hat{\Phi})^+} \setminus \{0\}.
\]

(b) Let with \(\alpha, \beta \in a\hat{\Phi}^+\) such that \((\mathcal{V}/\Pi)^a(\alpha), (\mathcal{V}/\Pi)^a(\beta) \in x(\hat{\Phi}/\Pi)^+\) (i.e. \((\mathcal{V}/\Pi)^a(\alpha) \neq 0 \neq (\mathcal{V}/\Pi)^a(\beta)\) ). Then \(\alpha \leq \beta\) if and only if \((\mathcal{V}/\Pi)^a(\alpha) \leq (\mathcal{V}/\Pi)^a(\beta)\)

(c) \((G(\Pi), \hat{\Phi}/\Pi, \mathcal{V}/\Pi)\) is a realized signed groupoid set.

Proof. Since \((G, \Phi, \mathcal{V})\) is conical, \(a\hat{\Phi}^+ \subseteq \mathbb{R}_{\geq 0}(\Pi)\) and thus (a) holds. (b) follows from part (a). For (c), the definition gives us that \((G(\Pi), \hat{\Phi}/\Pi)\) is a signed groupoid set. Additionally, we see that for every \(x \in G(\Pi)\) it is the case that \(\hat{\Phi}/\Pi(x) \subseteq \mathcal{V}(a)/\Pi\) where the \(\{\pm\}\) action is induced by scalar multiplication by \(\{\pm 1\}\). (a) and (b) show that for every \(x \in \text{Ob}(G(\Pi))\) there exists a total ordering on \((\mathcal{V}/\Pi)(x)\) where the positive (respectively negative) roots are positive (respectively negative) in that ordering.

This allows us to define the following realized signed groupoid sets.
Definition 6.1.4. Let $R := (G, \Phi, V)$ be a free realized signed groupoid set that is conical with simple roots $\Pi$.

(a) We define $R/\Pi := (G(\Pi), \hat{\Phi}/\Pi, V/\Pi)$.

(b) If $(a, \Delta) \in \text{Ob}(G(\Pi))$ we define $R(\Delta)/\Pi$ to be realized signed groupoid set of the connected component of $G(\Pi)$ containing $(a, \Delta)$.

6.2 Coxeter groups

Here we discuss Coxeter systems. Much of the motivation for our discussion arises from results involving Coxeter systems. Coxeter systems attach a root system to a Coxeter group. These root systems contain positive and negative roots. Root systems of a Coxeter System exist in an inner product space. A Coxeter system defines a group and its generating set. This allows us to discuss discuss Coxeter groups and their root systems.

Definition 6.2.1. A pair $(W, S)$ is a Coxeter system if $S \subset W$ and $W$ is the group generated by the set $S$ subject only to relations of the form $(ss')^{m(s,s')} = 1$ for $s, s' \in S$ such that $m(s, s) = 1$ and $m(s, s') \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq s'$. If $m(s, s') = \infty$, then $s$ and $s'$ have no relation in $W$.

Remark 6.2.1. If $(W, S)$ is a Coxeter System it is common to refer to $W$ as a Coxeter group. When we refer to a Coxeter group $W$ we will assume that there is an underlying Coxeter system $(W, S)$, although we may not explicitly state this.

Observe the resemblance between the generating set $S$ of a Coxeter system $(W, S)$, and the simple set of a simply generated signed groupoid set. Late results establish this similarity further.

Definition 6.2.2. The Coxeter graph $\Gamma_{(W, S)}$ for a Coxeter system $(W, S)$ is the simple undirected graph with partial edge labeling such that for $s_i, s_j \in S$
(i) \( S \) is the vertex set of \( \Gamma_{(W,S)} \).

(ii) There is an edge joining \( s_i \) and \( s_j \) if and only if \( m(s_i, s_j) \neq 2 \).

(iii) If \( m(s_i, s_j) \geq 4 \), then the corresponding edge is given the label \( m(s_i, s_j) \). This includes the case where \( m(s_i, s_j) = \infty \).

We next introduce the standard notion of a root system for a Coxeter system. This provides us with a representation of a Coxeter group in a real vector space. Since Coxeter groups may be considered as abstract groups that admit a formal description in terms of reflections, we define the root system in terms of reflections in a real inner product space.

**Definition 6.2.3.** The reflection about a vector \( \alpha \) in a vector space \( V \) with a symmetric bilinear form \( (\cdot, \cdot) : V \times V \to \mathbb{R} \) is defined for \( (\alpha, \alpha) > 0 \) to be the linear map \( r_{\alpha} : V \to V \) such that for \( v \in V \),

\[
    r_{\alpha}(v) := v - 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \alpha.
\]

**Remark 6.2.2.** Let \( \alpha, v \in V \), \( m \in \mathbb{R} \) with \( (\alpha, \alpha) > 0 \) where \( V \) a vector space with a symmetric bilinear form. The following properties about reflections are well known.

(a) \( r_{\alpha}(\alpha) := -\alpha \).

(b) \( r_{\alpha} \) is an involution.

(c) \( r_{\alpha}(v) := v \) if and only if \( v \in \alpha^\perp \).

(d) If \( (\alpha, \beta) = -\cos \left( \frac{\pi}{m} \right) \), then the linear map \( r_{\alpha} \circ r_{\beta} : V \to V \) has order \( m \).

A root system of a Coxeter system is a definitely signed set in an inner product space. This resembles a realized signed groupoid set.

**Definition 6.2.4.** \( \Phi \) is a root system for a Coxeter system \( (W,S) \) in a real vector space \( V \) with an inner product \( (\cdot, \cdot) : V \times V \to \mathbb{R} \) if \( \Phi \subset V \) such that there exists \( \Pi \subset \Phi \) such that:
(i) $\Pi$ is a linearly independent set.

(ii) There exists a bijection $\hat{\Pi} : S \rightarrow \Pi$

(iii) For $\alpha, \beta \in \Pi$ where $s_\alpha := \hat{\Pi}^{-1}(\alpha)$ and $s_\beta := \hat{\Pi}^{-1}(\beta)$ we have that

\[
(\alpha, \beta) = -\cos\left(\frac{\pi}{m(s_\alpha, s_\beta)}\right),
\]

where we use the convention that $\frac{\Pi}{\infty} = 0$.

(iv) For every $\alpha \in \Phi$, $(\alpha, \alpha) > 0$.

(v) $\Phi := \{r_{\alpha_1}r_{\alpha_2}\ldots r_{\alpha_n}(\alpha)|\alpha_1, \alpha_2, \ldots, \alpha_n \in \Pi, \text{ and } \alpha \in \Pi\}$.

and $\Phi^+ := \Phi \cap \mathbb{R}_{\geq 0}(\Pi)$.

$\Phi^+$ is called the positive roots of $\Phi$, while $\Pi$ is called the simple roots of $\Phi$.

Associated with the root system of a Coxeter system is the action of the group on the vector space. This action is described below.

**Definition 6.2.5.** If $\Phi$ is a root system for a Coxeter system $(W, S)$ where $\Pi$ the set of simple roots for $\Phi$, then we may consider the action of $W$ on $V$ such that for $\alpha \in V$, $w \in W$ we have the following linear map

\[
w : V \rightarrow V
\]

\[
\alpha \mapsto r_{\alpha_1}r_{\alpha_2}\ldots r_{\alpha_n}(\alpha)
\]

where $w = s_1s_2\ldots s_n$ and $\alpha_i = \hat{\Pi}(s_i)$ for $1 \leq i \leq n$.

**Remark 6.2.3.** Observe that for a Coxeter system $(W, S)$ with root system $\Phi$ where $\Pi$ is the set of simple roots for $\Phi$, we have that

(a) $\Phi = \Phi^+ \cup -(\Phi^+)$.

(b) $W\Pi = \Phi$.

(c) $W\Phi = \Phi$. 

75
6.3 Signed groupoid sets arising from Coxeter groups

Since a Coxeter group may be considered as a one object groupoid, we may study a Coxeter system with an associated root system as a signed groupoid set.

Lemma 6.3.1. Let \((W, S)\) be Coxeter system and let \(\Phi\) be a root system for \((W, S)\) with simple roots \(\Pi\) in an inner product space \(V\). Then

(a) \((\Phi^+, \Phi)\) is a definitely signed set.

(b) If \(\hat{W}\) is the one object groupoid such that for \(a \in \text{Ob}(G)\), \(a \hat{W}_a = W\), and \(\hat{\Phi} : G \rightarrow \text{Set}_\pm\) is the functor such that for \(a \in \text{Ob}(\hat{W})\), \(w \in \text{Mor}(\hat{W})\)

\[
\begin{align*}
    a & \mapsto (\Phi^+, \Phi) \\
    w & \mapsto (w : \Phi \rightarrow \Phi),
\end{align*}
\]

then \((\hat{W}, \hat{\Phi})\) is a signed groupoid set.

(c) \((\hat{W}, \hat{\Phi})\) is a principal signed groupoid set.

(d) If \(\hat{V} : \hat{W} \rightarrow \text{Vect}\) is the functor such that for \(a \in \text{Ob}(\hat{W})\), \(w \in \text{Mor}(\hat{W})\)

\[
\begin{align*}
    a & \mapsto V \\
    w & \mapsto (w : V \rightarrow V),
\end{align*}
\]

then \((\hat{W}, \hat{\Phi}, \hat{V})\) is a realized signed groupoid set.

(e) \((\hat{W}, \hat{\Phi}, \hat{V})\) is a conical realized signed groupoid set.

(f) If \(|\Pi| < \infty\) then \((\hat{W}, \hat{\Phi}, \hat{V})\) is a polyhedral realized signed groupoid set.

Proof. Since \(\Phi = \Phi^+ \cup (\Phi^+)\) scalar multiplication by \(\{\pm 1\}\) gives the \(\{\pm\}\) action on \((\Phi^+, \Phi)\). Thus \((\Phi^+, \Phi)\) is a definitely signed set giving us (a). Since for every \(w \in W\) we have that \(w(\Phi) = \Phi\) we see that \(\hat{\Phi}(w)\) is an isomorphism in \(\text{Set}_\pm\), and so \((\hat{W}, \hat{\Phi})\) is a signed groupoid set giving us (b).

It is well known that for \(w \in W\), \(|\Phi_w| = l_S(w)\) proving (c). For (d), observe that \(\hat{V} : \hat{W} \rightarrow \text{Vect}\) is a representation of \(\hat{W}\) such that for \(a \in \text{Ob}(\hat{W})\),

\[
\hat{\Phi}(a) = \Phi \subseteq V = \hat{V}(a).
\]
Since \( \Phi^+ \subseteq \mathbb{R}_{\geq 0}(\Pi) \), there exists a total ordering on \( V \) that respects the sign of \( \Phi \), and so \((\hat{W}, \tilde{\Phi}, \hat{V})\) is a conical realized signed groupoid set. This proves (d) and (e). If \( |\Pi| < \infty \) then \( \mathbb{R}_{\geq 0}(\Pi) \) is a polyhedral cone, and since \( \Phi^+ \subseteq \mathbb{R}_{\geq 0}(\Pi) \) it is the case that \((\hat{W}, \tilde{\Phi}, \hat{V})\) is polyhedral, giving us (f).

Thus we may naturally consider the signed groupoid set of a Coxeter system.

**Definition 6.3.1.** If \((W, S)\) is a Coxeter system where \( \Phi \) is a root system for \((W, S)\) with simple roots \( \Pi \) we define the following:

(a) \( \hat{W} \) is the one object groupoid such that for \( a \in \text{Ob}(G), a \hat{W}_a = W \).

(b) \( \Phi : G \to \text{Set}_\pm \) is the functor such that for \( a \in \text{Ob}(\hat{W}), \ w \in \text{Mor}(\hat{W}) \)

\[
\begin{align*}
a &\mapsto (\Phi^+, \Phi) \\
w &\mapsto (w : \Phi \to \Phi).
\end{align*}
\]

(c) \((\hat{W}, \Phi)\) is the natural signed groupoid set of \( W \)

Since a Coxeter system gives us a simply generated signed groupoid and realization, we may consider the constructions from Section 6.1 in terms of Coxeter systems.

**Definition 6.3.2.** Let \((W, S)\) be a Coxeter system and let \( \Phi \) be a root system for \((W, S)\) with simple roots \( \Pi \).

(a) We call \((\hat{W}(\Pi), \tilde{\Phi})\) is the full \((W, S)\) signed groupoid set for the root system \( \Phi \).

(b) For \( a \in \text{Ob}(\hat{W}), \Delta \subseteq \Pi \), we define the groupoid \( \hat{W}(\Delta) \) to be the connected component of \( \hat{W}(\Pi) \) where \((a, \Delta) \in \text{Ob}(\hat{W}(\Delta))\).

(c) We call \((\hat{W}(\Delta), \Phi_\Delta)\), the \( \Delta \) -signed groupoid set for \((W, S)\) with root system \( \Phi \).

**Remark 6.3.1.** Observe that since \((\hat{W}, \tilde{\Phi}, \hat{V})\) is a realized signed groupoid set, Lemma 6.1.1 shows that \((\hat{W}, \tilde{\Phi}, \hat{V} \circ \mathcal{F}_\Pi)\) is also a realized signed groupoid set. Additionally if \( \hat{V} \circ \mathcal{F}_\Pi|_\Delta \) is the restriction of \( \hat{V} \circ \mathcal{F}_\Pi \) to \( \hat{W}(\Delta) \) then we see that \((\hat{W}(\Delta), \Phi_\Delta, \hat{V} \circ \mathcal{F}_\Pi|_\Delta)\) is also a realized signed groupoid set.
6.4 Properties of \((\hat{W}(\Pi), \hat{\Phi})\) and its connected components

In this section we introduce constructions of Brink and Howlett from [4] and [5] that involve signed groupoid sets associated to Coxeter groups. We present results about these constructions and specialize them for the constructions in this thesis.

In this section we let \((W, S)\) be a Coxeter system with root system \(\Phi\) that has simple roots \(\Pi\). For \(\Delta \subseteq \Pi\), we study the signed groupoid set \((\hat{W}(\Pi), \hat{\Phi})\) by focusing on the connected component \((\hat{W}(\Delta), \Phi_\Delta)\). Since \(\hat{W}\) has a single object, every object of \(\hat{W}(\Pi)\) is of the form \((a, \Delta)\) for some \(\Delta \subseteq \Pi\). Thus we use \(\Delta\) to refer to the object \((a, \Delta)\) of \(\hat{W}(\Pi)\) allowing us to associate \(\text{Ob}(\hat{W}(\Pi))\) with subsets of \(\Pi\).

**Definition 6.4.1.** Let \(\Delta \subseteq \Pi\).

(a) \(W_\Delta\) is the standard parabolic subgroup of \((W, S)\).

(b) If \(|W_\Delta| < \infty\), the we define \(w_\Delta\) to be the longest element of \(W_\Delta\) (Note that such an element exists if and only if \(|W_\Delta| < \infty\)).

(c) If \(a \in \Pi \setminus \Delta\) such that \(|W_{\Gamma_a}| < \infty\) where \(\Gamma_a\) is the connected component in the Coxeter graph of \(\Delta \cup \{a\}\) containing \(a\), then we define \(\nu(\Delta, a) := w_{\Gamma_a} w_{\Gamma_a \setminus \{a\}} \in \hat{W}\).

**Remark 6.4.1.** Let \(\Delta \subseteq \Pi\) where \(W_\Delta\) is finite and \(a \in \Pi \setminus \Delta\). If \(\nu(\Delta, a)\) exists, then let \(\Delta' := \nu(\Delta, a)(\Delta) \subseteq \Pi\). Thus we see that \((\Delta', \nu(\Delta, a), \Delta) \in \text{Mor}(\hat{W}(\Pi))\).

We now give a result of Brink and Howlett found in [5]. This allows us to discuss simply generated signed groupoid sets in this situation. Rather than presenting a full proof of this result we present the set of simple morphisms and provide a reference to a detailed version of the proof.

**Theorem 6.4.1.** If \((W, S)\) is a Coxeter system that has a root system \(\Phi\) with simple roots \(\Pi\), then \((\hat{W}(\Pi), \hat{\Phi})\) is simply generated. Thus it is the case for every \(\Delta \subseteq \Pi\) we have that \((\hat{W}(\Delta), \Phi_\Delta)\) is also simply generated.

**Proof.** We define \(\hat{S}\) to be the set of morphisms of \(\hat{W}(\Pi)\) such that

\[
\hat{S} := \{(\Gamma', \nu(\Gamma, a), \Gamma) \mid \text{there exists } \Delta \subseteq \Pi \text{ such that } (\Gamma', \nu(\Gamma, a), \Gamma) \in \text{Mor}(\hat{W}(\Delta))\}.
\]
It is the case that $\hat{S}$ is a simple set for $(\hat{W}(\Pi), \hat{\Phi})$. This follows from the work of Brink and Howlett from Theorem A and its proof which may be found in [5].

Remark 6.4.2. Let $\Lambda \subseteq \Pi$. If $\Delta \in \text{Ob}(\hat{W}(\Lambda))$ then $\Delta' \in \text{Ob}(\hat{W}(\Lambda))$ and $g \in \Delta \hat{W}(\Lambda)_{\Delta}$, we have that $g(\Phi_{\Delta})$ consists of positive roots. Thus $\Phi_{\Delta} \subseteq \Delta \hat{\Phi}^{\text{im}}$ and also $(\mathbb{R}(\Delta) \cap \Delta \Phi) \subseteq \Delta \hat{\Phi}^{\text{im}}$.

Now we see that for $\Delta \subseteq \Pi$, $(\hat{W}(\Delta), \Phi_{\Delta})$ is a simply generated signed groupoid set. Note that although $(\hat{W}, \hat{\Phi})$ is compressed it does not necessarily hold that $(\hat{W}(\Delta), \Phi_{\Delta})$ is compressed. Thus we will study $(\hat{W}(\Delta), \Phi_{\Delta})^{c}$, the compression of $(\hat{W}(\Delta), \Phi_{\Delta})$. Recall that Remark 6.3.1 tells us that $(\hat{W}(\Delta), \Phi_{\Delta}, \hat{V} \circ \mathcal{F}_{\Pi}|_{\Delta})$ is realization of $(\hat{W}(\Delta), \Phi_{\Delta})$. This allows us to consider the compressed realized signed groupoid set $(\hat{W}(\Delta), \Phi_{\Delta}^{\text{c}}, \hat{V} \circ \mathcal{F}_{\Pi}|_{\Delta})$. We use the notation $\hat{V}_{\Delta}$ to refer to the representation $\hat{V} \circ \mathcal{F}_{\Pi}|_{\Delta}$.

Recall from Lemma 4.4.2 that $(\hat{W}(\Delta), \Phi_{\Delta}^{\text{c}}, \hat{V}_{\Delta})$ is a realization of $(\hat{W}(\Delta), \Phi_{\Delta})^{c}$, and thus $(\hat{W}(\Delta), \Phi_{\Delta})^{c}$ is isomorphic to $(\hat{W}(\Delta), \Phi_{\Delta})^{c}$. Since $(\hat{W}, \hat{\Phi}, \hat{V})$ is conical, it is reasonable to hope that $(\hat{W}(\Pi), \hat{\Phi}, \hat{V} \circ \mathcal{F}_{\Pi})$ is also conical. Unfortunately this is not the case, as demonstrated by the following example.

Example 6.4.1. Suppose $(W, S)$ is the Coxeter system where $S := \{r, s, t\}$ where $m(r, s) = m(s, t) = 3$ and $m(r, t) = \infty$ whose Coxeter graph is given in Figure 6.1.

Let $\Phi$ be a root system for $(W, S)$ with simple roots $\Pi := \{\alpha_{r}, \alpha_{s}, \alpha_{t}\}$ such that $\hat{\Pi}(r) = \alpha_{r}, \hat{\Pi}(s) = \alpha_{s}$ and $\hat{\Pi}(t) = \alpha_{t}$ where $\hat{\Pi}$ is the bijection given in Definition 6.2.4.
Consider $\hat{W}(\{\alpha, r\})$, the connected component of $\hat{W}(\Pi)$. Observe that

$$\text{Ob}(\hat{W}(\{\alpha, r\})) = \{\{\alpha, r\}, \{\alpha, s\}, \{\alpha, t\}\},$$

while the set of simple morphisms $\hat{S}$ from Theorem 6.4.1 for $(\hat{W}(\{\alpha, r\}), \Phi_{\{\alpha\}})$ is

$$\hat{S} := \{\{\alpha, r\}, rs, \{\alpha, s\}\}, \{\{\alpha, r\}, sr, \{\alpha, t\}\}, \{\{\alpha, s\}, st, \{\alpha, t\}\}, \{\{\alpha, s\}, ts, \{\alpha, s\}\}.$$ 

Figure 6.2 gives a directed graph of the groupoid $\hat{W}(\{\alpha\})$ where the edges are the objects of $\hat{W}(\{\alpha\})$ and the edges are the simple morphisms of $(\hat{W}(\{\alpha\}), \Phi_{\{\alpha\}})$.

Observe that since $m(r, t) = \infty$ we have that $\{\alpha, r\} \cap \hat{S} = \emptyset$.

We now consider $(\hat{W}(\{\alpha, r\}), \Phi_{\{\alpha, r\}}, \hat{V}_{\{\alpha, r\}})$ which is a realization of $(\hat{W}(\{\alpha, r\}), \Phi_{\{\alpha, r\}})^c$.

**Lemma 6.4.1.** $(\hat{W}(\{\alpha, r\}), \Phi_{\{\alpha, r\}}, \hat{V}_{\{\alpha, r\}})$, which is simply generated and compressed, is not conical. Thus $(\hat{W}(\Pi), \Phi, \hat{V} \circ \mathcal{F}_\Pi)$ is not conical.

**Proof.** Observe in Example 6.4.1 that $|\{\alpha, r\}| = 1$ and $|\{\alpha, s\}| = 2$. If $(\hat{W}(\Pi), \Phi, \hat{V} \circ \mathcal{F}_\Pi)$ is conical, then it must be that $|\{\alpha, r\} \Phi^+| = 1$. Since

$$(\hat{V} \circ \mathcal{F}_\Pi(sr))(\{\alpha, r\} \Phi^+) \subseteq \{\alpha, r\} \Phi^+$$

and $|\{\alpha, r\} \Phi^+| \geq 2$ it cannot be the case that $\hat{V} \circ \mathcal{F}_\Pi(sr)$ is a vector space isomorphism,
which gives a contradiction.

6.5 Signed groupoid sets associated to finite Coxeter groups

Here we study the realized signed groupoid sets constructed by Brink and Howlett in the case of a finite Coxeter group. We conclude a result involving the correspondence between inversion sets and hyperplanes in this situation. Throughout this section we assume that \((W, S)\) is a Coxeter system with root system \(\Phi\) that has simple roots \(\Pi\) where \(W\) is a finite Coxeter group.

**Lemma 6.5.1.** If \(R := (\hat{W}, \bar{\Phi}, \bar{V})\) where \((W, S)\) is a Coxeter system with root system \(\Phi\) that has a linearly independent simple roots \(\Pi\) where \(W\) is a finite Coxeter group, then \(R/\Pi\) is a real-realized signed groupoid set.

**Proof.** Let \(\Lambda \subseteq \Pi\). We will show that \(R(\Lambda)/\Pi\) is a real-realized signed groupoid set.

Let \(\Delta \in \text{Ob}(\hat{W}(\Lambda))\). Since \(W\) is finite the morphism \(w_{\Pi\Delta} \in \hat{W}\) exists. Note that \(w_{\Pi}(\Delta) \subseteq -\Pi\). If \(\Delta' := w_{\Pi}(-\Delta) \in \text{Ob}(\hat{W}(\Lambda))\) we see that

\[
w_{\Pi\Delta}(\Delta) = w_{\Pi}(-\Delta) = \Delta',
\]

and so \(w_{\Pi\Delta} \in \Delta\hat{W}(\Lambda)\). Thus

\[
\bar{\Phi}_{w_{\Pi\Delta}} = \bar{\Phi}^+ \cap w_{\Delta}w_{\Pi}(\bar{\Phi}^-) = \bar{\Phi}^+ \cap w_{\Delta}(\bar{\Phi}^+) = \bar{\Phi}^+ \setminus \mathbb{R}\Delta.
\]

Note that since \(\hat{W}\) is a groupoid with one object, we need not specify an object when considering the root systems above. Since Remark 6.4.2 tells us that \(\Delta\bar{\Phi} \cap \mathbb{R}\Delta \subseteq \Delta\bar{\Phi}^\text{im},\)
it is the case that $\Delta \Phi \setminus \mathbb{R}\Delta = \Delta \Phi^{re}$. Thus

$$\Delta \Phi^{+} \setminus \mathbb{R}\Delta \subseteq \Delta \Phi^{+} \setminus \mathbb{R}\Delta \subseteq \Phi_{w\Pi \Delta},$$

and so

$$\Delta(\Phi/\Pi)^{+} = \Delta \Phi^{+} + \mathbb{R}\Delta \subseteq \Phi_{w\Pi \Delta} + \mathbb{R}\Delta = (\Phi \setminus \Pi)_{w\Pi \Delta}.$$ 

Therefore $\Delta(\Phi/\Pi)^{+}$ is a real root and so $R/\Pi$ is a real realized signed groupoid set.

Another important result is that this compression quotient realized signed groupoid set is conical. From this there are many other Coxeter group-like properties that follow.

Theorem 6.5.1. If $R := (\hat{W}, \overline{\Phi}, \hat{V})$ where $(W, S)$ is a Coxeter system with root system $\Phi$ that has simple roots $\Pi$ where $W$ is a finite Coxeter group, then $R/\Pi$ is conical.

Proof. Let $\Psi := \hat{\Phi}/\Pi$ and $V := \hat{V}/\Pi$. Thus $R/\Pi = (\hat{W}(\Pi), \Psi, V)$. Suppose that $\Delta \subseteq \Pi$ and let $\epsilon \in \Delta \Psi^{+}$. Then there exists $\gamma \in \hat{\Phi}^{+}$ such that $\epsilon = [\gamma] + \mathbb{R}\Delta$. Since $(\hat{W}, \hat{\Phi}, \hat{V})$ is conical, $\gamma \in \mathbb{R}_{\geq 0}(\Pi)$ and so

$$[\gamma] = \sum_{\alpha \in [\gamma]} \alpha = \sum_{\beta \in \Pi} c_{\beta} \beta,$$

where $c_{\beta} \in \mathbb{R}_{\geq 0}$. Lemma 6.5.1 shows that $\epsilon \in \Delta \Psi^{re}$. Thus there exists at least one $\beta \in \Pi \setminus \Delta$ where $c_{\beta} > 0$. Therefore

$$\epsilon = [\gamma] + \mathbb{R}\Delta = \sum_{\beta \in \Pi \setminus \Delta} d_{\beta}(\beta + \mathbb{R}\Delta)$$

where $d_{\beta} \geq 0$. 

82
Now observe that for every $\beta \in \Pi \setminus \Delta$, it is the case that $\nu(\Delta, \beta) = w_{\Delta, \beta} w_\Delta \in W_{\Delta, \beta} \setminus W_\Delta$. Thus $\Phi_{\nu(\Delta, \beta)} \subseteq R_{\geq 0} \beta + R\Delta$ and Theorem 6.4.1 shows that $\nu(\Delta, \beta)$ is a simple morphism of $(G(\Pi), \Psi)$ giving us that $|\Psi_{\nu(\Delta, \beta)}| = 1$.

If $x_\beta \in R_{\geq 0}(\beta)$ such that $\Psi_{\nu(\Delta, \beta)} = \{x_\beta + R\Delta\}$, we see that

$$R_{\geq 0}(\beta + R\Delta) \subseteq R_{\geq 0}(x_\beta + R\Delta),$$

and so

$$\epsilon \in R_{\geq 0}(\{x|x \in \Psi_s \text{ where } s \text{ is a simple morphism of } (G(\Pi), \Psi)\}).$$

Therefore $R/\Pi$ is a conical realized signed groupoid set.

Since the compression quotient realized signed groupoid set is both real and conical we may conclude the theorem below.

**Theorem 6.5.2.** Let $R := (\hat{W}, \Phi, \hat{V})$ where $(W, S)$ is a Coxeter system with root system $\Phi$ that has simple roots $\Pi$ where $W$ is a finite Coxeter group, and let $\Psi := \hat{\Phi}/\Pi, \mathcal{V} := \hat{V}/\Pi$ so that $R/\Pi := (\hat{W}(\Pi), \Psi, \mathcal{V})$.

Then for every $\Delta \in \text{Ob}(\hat{W}(\Pi))$, $A \subseteq \Delta \Psi^+$, it is the case that $\Psi_w = A$ for some $w \in \Delta \hat{W}(\Pi)$ if and only if there exists a hyperplane in $\mathcal{V}(\Delta)$ that separates $A$ and $\Delta \Psi^+ \setminus A$.

**Proof.** Observe that $R/\Pi$ satisfies the hypotheses for Theorem 4.6.2 and thus this gives us the result.

This theorem provides a correspondence between hyperplanes and inversion sets in this instance of realized signed groupoid sets. This provides us with some very interesting facts regarding the geometry of such root systems.
6.6 Hyperplanes and the universal covering

In this section we study the relationship between the universal covering of the realized signed groupoid sets that arise from finite Coxeter groups and the realized signed groupoid set arising from hyperplane arrangements discussed in Chapter 5. Throughout this section we assume the \((W, S)\) is a Coxeter system with root system \(\Phi\) that has simple roots \(\Pi\), and that \(W\) is a finite Coxeter group. Additionally we let \(R := (\hat{W}, \bar{\Phi}, \hat{V})\) and \(\Psi := \hat{\Phi} / \Pi\), \(V := \hat{V} / \Pi\) which gives us that \(R / \Pi := (\hat{W}(\Pi), \Psi, V)\).

Fix \(\Lambda \in \text{Ob}(\hat{W}(\Pi))\), and let \(V' := \hat{V} \circ \mathcal{F}_\Pi(\Lambda)\). Let \((\cdot, \cdot) : V' \times V' \to \mathbb{R}\) be an inner product on \(V'\). Let \(R' := (G', \Phi', V')\) be a realized signed groupoid set isomorphic to the universal covering of the component of \(R / \Pi\) containing \(\Lambda\).

**Remark 6.6.1.** Corollary \([5.2.1]\) tells us that the choice of universal covering does not matter, and thus without loss of generality we may assume \(R'\) is chosen so that \(V'\) is constant. Observe that this also means that the functor \(G' \to \text{Set}_\pm\) underlying \(\Phi'\) is constant.

**Remark 6.6.2.** Recall that since \((G', \Phi')\) is compressed we have that if \(\alpha, \beta \in \Lambda \Phi'\) with \(\alpha = c\beta\) for some \(c \in \mathbb{R}\), then it must be the case that \(c \in \{\pm 1\}\).

Observe that if we replace every \(\alpha \in \Lambda \Phi'\) with \(\hat{\alpha}\), the unit vector in the direction of \(\alpha\), we may assume that every \(\alpha \in \Lambda \Phi'\) is a unit vector. Hence we may study the root system \(\Psi'\) for \(G'\) where every root is unit vector. Let \(\hat{\Lambda} := \{\hat{\alpha} | \alpha \in \Lambda\}\), and note that every \(\alpha \in \Lambda \Psi'\) is a unit vector.

**Remark 6.6.3.** Note that although the rescaling of a root system to unit vectors changes the isomorphism type of the realized signed groupoid set \(R'\), it remains that \((G', \Psi')\) is isomorphic to \((G', \Psi')\) as a signed groupoid set.

Let \(\mathcal{H}\) be the hyperplane arrangement \(\mathcal{H} := \{\alpha^\perp | \alpha \in \Lambda \Psi'\}\) where \(\alpha^\perp := \{v \in V' | (\alpha, v) = 0\}\) is the hyperplane in \(V'\) orthogonal to \(\alpha\). This gives us the following lemma.
Lemma 6.6.1. \( \mathcal{H} = \{ \alpha^+ | \alpha \in \Delta \Psi' \} \) for any \( \Delta \in \text{Ob}(G') \) for any object \( \Delta \) of \( G' \).

Proof. Since
\[
\Delta \Psi' = \Delta \Psi' + \cup \Delta \Psi' - = \Delta \Psi' + \cup (-\Delta \Psi')
\]
we see that for every \( \Delta \in \text{Ob}(G') \)
\[
\mathcal{H} = \{ \alpha^+ | \alpha \in \Delta \Psi' \} = \{ \alpha^+ | \alpha \in \Delta \Psi' \}.
\]
\( \Box \)

Now we let \((G'', \Psi'', V'')\) be the realized signed groupoid set for \( \mathcal{H} \) as found in Definition 5.3.1

Remark 6.6.4. Note that for \( C \in \text{Ob}(G'') \) it is the case the \( \_ \Psi'' = \_ \Psi' \) and
\[
\_ \Psi''^+ = \{ \alpha \in \_ \Psi'' | (\alpha, \rho) > 0 \text{ for some } \rho \in C \}.
\]
Note that for every \( \rho \in C \), \( \{ \alpha \in \_ \Psi'' | (\alpha, \rho) > 0 \} \) is same set.

We now give some properties \((G', \Phi')\) that allow us to see the connection between the universal cover and hyperplane arrangements.

Lemma 6.6.2. For \( \Delta \in \text{Ob}(G') \), let \( C_\Delta := \{ q \in V' | (q, \alpha) > 0 \text{ for all } \alpha \in \_ \Psi' \} \).
Then the following are true.

(a) \( C_\Delta \) is a chamber of \( \mathcal{H} \).
(b) \( C_\Delta \) is a simplicial cone.
(c) The map \( \Theta : \text{Ob}(G') \to \text{Ob}(G'') \) where \( \Theta(\Delta) := C_\Delta \) is a bijection.
(d) The map \( \Theta \) from part (c) extends uniquely to a groupoid isomorphism \( \Theta : G' \to G'' \).
(e) It is the case that \( \Psi'' \Theta = \Psi' \) as functors \( G' \to \text{Set}_\pm \).
(f) \( (\Theta, \text{Id}_\Psi, \text{Id}_V) \) is an isomorphism of realized signed groupoid sets.
Proof. To prove (a), recall from Theorem 4.1.1 that there exists a hyperplane \( H \) in \( V' \) separating \( \Delta \Psi''^+ \) and \( \Delta \Psi''^- \). Let \( q \in V' \) such that \( q^\perp = H \) and for every \( \alpha \in \Delta \Psi''^+ \) we have that \( (q, \alpha) > 0 \). Thus \( q \in C_\Delta \) and so \( C_\Delta \neq \emptyset \). Observe that \( C_\Delta \) consists of all points in \( V' \) that are on the same side of \( H' \) as \( q \) for every hyperplane \( H' \in \mathcal{H} \). Thus we see that \( C_\Delta \) is a chamber of \( \mathcal{H} \) proving (a). We see from Theorem 6.5.1 that \( \Delta \Psi''^+ = \Delta \Psi' \cap \mathbb{R}_{\geq 0}(\Delta \Pi') \) where \( \Delta \Pi' \) are the simple roots of \( \Delta \Psi''^+ \). Thus

\[
C_\Delta = \{ q \in V' | (q, \alpha) > 0 \text{ for all } \alpha \in \Delta \Pi' \}.
\]

Since \( \Delta \Pi' \) is an \( \mathbb{R} \) basis of \( V' \), we see from the definition of a simplicial cone that \( C_\Delta \) is a simplicial cone proving (b). Let \( \Delta \in \text{Ob}(G') \) and let \( C := C_\Delta \), a chamber of \( \mathcal{H} \), and let \( q_0 \in C \). We see that \( (q_0, \alpha) > 0 \) for all \( \alpha \in \Delta \Psi' \). Thus

\[
c \Psi''^+ = \{ \alpha \in \Delta \Psi' | (q_0, \alpha) > 0 \} \supseteq \Delta \Psi'^+.
\]

Since for every \( \gamma \in \{ \alpha \in \Delta \Psi' | (q_0, \alpha) > 0 \} \) it is the case that \( -\gamma \notin \{ \alpha \in \Delta \Psi' | (q_0, \alpha) > 0 \} \), we see that \( c \Psi''^+ = \Delta \Psi'^+ \). Thus it follows that \( \Theta \) is injective.

Now to show that \( \Theta_{\text{Ob}} \) is surjective, let \( C \) be any chamber of \( \mathcal{H} \), and let \( q \in C \). Then \( q^\perp \cap \Lambda \Psi' = \emptyset \), and so \( q^\perp \) strictly separates \( \{ \alpha \in \Lambda \Psi' | (q, \alpha) > 0 \} \) and \( \{ \alpha \in \Lambda \Psi' | (q, \alpha) < 0 \} \).

From Theorem 6.5.2 we see that there must exist a morphism \( g \in \Lambda G'_{\Delta} \) such that

\[
\Psi'_g = \{ \alpha \in \Lambda \Psi'^+ | (q, \alpha) < 0 \},
\]

and so

\[
\{ \alpha \in \Lambda \Psi'^+ | (q, \alpha) < 0 \} = \Lambda \Psi'^+ \cap \Lambda \Psi'^-.\]

This implies that

\[
\Delta \Psi'^+ = \{ \alpha \in \Lambda \Psi'^+ | (q, \alpha) > 0 \}.
\]
Since \( C_{\Delta} = \{ v \in V' | (v, \alpha) > 0 \text{ for all } \alpha \in \Delta \Psi' + \} \), we see that \( q \in C_{\Delta} \). Since both \( C \) and \( C_{\Delta} \) are chambers of \( \mathcal{H} \) that contain \( q \), they must be the same chamber. Thus \( C = C_{\Delta} = \Theta(\Delta) \), and therefore \( \Theta \) is surjective completing the proof of (c).

For (d), since \( G' \) and \( G'' \) are connected and simply connected we see that for \( \Delta, \Delta' \in \text{Ob}(G') \), using the notation from Remark 5.2.1 that \( \Theta \) maps \( \Delta g_{\Delta} \mapsto \Theta(\Delta) g_{\Theta(\Delta')} \) is well defined.

For (e), observe that the underlying functors to \( \text{Set} \) of \( \Psi' : G' \to \text{Set} \) and \( \Psi'' \Theta : G'' \to \text{Set} \) with value \( \Lambda \Psi' \). Thus it is sufficient to check that for every \( \Delta \in \text{Ob}(G') \) it is the case that \( \Delta \Psi' + = \Delta \Psi'' + \), where \( C := C_{\Delta} \). This is shown in the proof that \( \Theta \) is injective in part (c). (f) follows directly from part (e) and definitions.

Recall that a hyperplane arrangement is said to be simplicial if every chamber is a simplicial cone. The preceding results may be less precisely summarized in the following theorem.

**Theorem 6.6.1.** Up to rescaling of positive roots, the universal cover of any component of \( R/\Pi \) is isomorphic to an \( \mathcal{H} \) realized signed groupoid set, where \( \mathcal{H} \) some real, simplicial hyperplane arrangement.

### 6.7 A realized signed groupoid set from \( D_4 \)

We conclude this thesis by demonstrating some of the results in this chapter in the case of a realized signed groupoid set that arises from the Coxeter group \( D_4 \). We study one of the connected components of the groupoid constructed by Brink and Howlweit.

**Definition 6.7.1.** The Coxeter system \( D_n \) for \( n \geq 4 \) is the Coxeter system \( (W, S) \) such that

(i) \( S := \{s_1, s_2, \ldots, s_n\} \)
(ii) \( W \) is the group \( D_n \) such that \( D_n := \langle S \mid (s_i s_j)^{m(s_i, s_j)} = 1 \rangle \), where for \( s_i, s_j \) where \( i \leq j \)

\[
m(s_i, s_j) = \begin{cases} 
1 & : \text{if } i = j \\
3 & : \text{if } j = i + 1, j \neq n - 1 \\
3 & : \text{for } i = n - 2, j = n \\
2 & : \text{otherwise}
\end{cases}
\]

Remark 6.7.1. The following is a root system for \( D_n \).

Let \( V := \mathbb{R}^n \) with standard basis \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \). Then if \( \Phi := \{\pm \epsilon_i \pm \epsilon_j \mid \text{for } 1 \leq i < j \leq n\} \)

where \( \Pi := \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{\epsilon_{n-1} + \epsilon_n\} \) it is the case that \( \Phi \) is a root system for \( D_n \) with simple roots \( \Pi \).

Observe that \( \Phi^+ := \{\epsilon_i \pm \epsilon_j \mid \text{for } 1 \leq i < j \leq n\} \).

Since we will examine the realized signed groupoid set arising from \( D_4 \), in the remainder of this chapter we will use the following definitions and notation.

Definition 6.7.2. (i) \((W, S)\) is the \( D_4 \) Coxeter system

(ii) \( S := \{r, s, t, u\} \)

(iii) \( \Phi \) is the root system for \((W, S)\) given in Remark 6.7.1

(iv) \( \Pi \) is the simple roots for \( \Phi \) where \( \Pi := \{\alpha_r, \alpha_s, \alpha_t, \alpha_u\} \) such that \( \alpha_r := \epsilon_1 - \epsilon_2, \alpha_s := \epsilon_2 - \epsilon_3, \alpha_t := \epsilon_3 - \epsilon_4, \alpha_u := \epsilon_3 + \epsilon_4. \)

Figure 6.3 gives the Coxeter graph for \( D_4 \). The signed groupoid set we will study is \((\hat{W}(\Pi), \Phi(\Pi))\) with realization \((\hat{W}(\Pi), \Phi(\Pi), \hat{V})\). Since \((\hat{W}(\Pi), \Phi(\Pi))\) and its realization consists of different components of the form \((\hat{W}(\Delta), \Phi_\Delta)\) for \( \Delta \subseteq \Pi \), we
will study the specific component $(\hat{W}(\Delta), \Phi_{\Delta})$ with realization $(\hat{W}(\Delta), \Phi_{\Delta}, \hat{V} \circ \mathcal{F}_\Pi)$, where $\Delta := \{\alpha_r\} \subseteq \Pi$.

Observe that the objects of $\hat{W}(\Delta)$ consist of all the one object subsets of $\Pi$.

**Lemma 6.7.1.** \(\text{Ob}(\hat{W}(\Delta)) := \{\{\alpha_r\}, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_u\}\}\).

**Proof.** Since \(|\Delta| = 1\) we that for every $\Delta' \in \text{Ob}(\hat{W}(\Delta))$ it is the case that \(|\Delta'| = 1\).

Observe that

\[
rs(\{\alpha_r\}) = \{\alpha_s\},
strs(\{\alpha_r\}) = \{\alpha_t\},
surs(\{\alpha_r\}) = \{\alpha_u\}.
\]

Since this includes every one element subset of $\Pi$, it must be that \(\text{Ob}(\hat{W}(\Delta)) := \{\{\alpha_r\}, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_u\}\}\).

\[\square\]

Theorem 6.4.1 tells us that $(\hat{W}(\Delta), \Phi_{\Delta})$ is simply generated and describes the simple morphisms. Let $\hat{S}$ be the set of simple morphisms of $(\hat{W}(\Delta), \Phi_{\Delta})$. The directed graph in Figure 6.4 demonstrates the structure of the groupoid $\hat{W}(\Delta)$ in terms of the simple morphisms. In this graph the set of vertices correspond to the elements of $\text{Ob}(\hat{W}(\Delta))$, while the edges correspond to elements of $\hat{S}$.

For every object of $\hat{W}(\Delta)$ there are 32 morphisms mapping into object. We list all all the morphisms of $\hat{W}(\Delta)$ by codomain. Note that in our list below, we give a single reduced expression for each morphism, It should be noted that due to the existence of braid relations in $\hat{W}(\Delta)$, the reduced expression given may not be unique. The
morphisms with codomain \( \{ \alpha_r \} \) are

\[
\{ \alpha_r \} \hat{W}(\Delta) = \{ \text{Id}_{\{ \alpha_r \}}, t, u, (sr), tu, t(sr), u(sr), (sr)(ts), (sr)(us), ut(sr), t(sr)(ts), \\
u(sr)(ts), (sr)(ts)u, t(sr)(us), u(sr)(us), (sr)(us)t, (sr)(ts)u(st), \\
(ut(sr)(ts), (sr)(ts)ur, u(sr)(ts)u, ut(sr)(us), (sr)(us)tr, t(sr)(us)t, \\
(sr)(ts)u(st)(rs), t(sr)(ts)u(st)(rs), u(sr)(ts)u(st), ut(sr)(ts)u(st), ut(sr)(ts)u(st)(rs) \},
\]

Figure 6.4. The simple morphisms of \((\hat{W}(\Delta), \Phi_\Delta)\)
the morphisms with codomain \{\alpha_s\} are

\{\alpha_s\}\mathcal{W}(\Delta) = \{\text{Id}_{\{\alpha_s\}}, (ts), (us), (rs), (ts)r, (ts)u, (us)r, (us)t, (rs)t, (rs)u, (rs)t(sr),
(rsu)= (su)r= (ts)u(st), (ts)ru, (us)rt, (rs)tu, (rs)tu(sr), (ts)ru(st),
(us)rt(su), (rs)u(sr)(ts), (rs)t(sr)(us), (ts)u(st)(rs), (rs)u(sr)(ts)r,
(rs)u(sr)(ts)u, (rs)t(sr)(us)r, (rs)t(sr)(us)t, (ts)u(st)(rs)t, (ts)u(st)(rs)u,
(rs)u(sr)(ts)r, (rs)t(sr)(us)rt, (ts)u(st)(rs)tu, (ts)u(st)(rs)tu(sr)\},

the morphisms with codomain \{\alpha_t\}

\{\alpha_t\}\mathcal{W}(\Delta) = \{\text{Id}_{\{\alpha_t\}}, r, u, (st), ru, r(st), u(st), (st)(rs), (st)(us), ur(st), r(st)(rs),
(st)u(st)(rs), (st)u(st)(us), (st)(us)r, (st)(rs)u(sr),
ur(st)(rs), (st)(rs)ut, u(st)(rs)u, ur(st)(us), (st)(us)rt, r(st)(us)r,
(st)(rs)u(sr)(ts), r(st)(rs)u(sr), u(st)(rs)u(sr), ur(st)(rs)u, ur(st)(us)r,
r(st)(rs)u(sr)(ts), u(st)(rs)u(sr)(ts), ur(st)(rs)u(sr), ur(st)(rs)u(sr)(ts)\},

and the morphisms with codomain \{\alpha_u\}:

\{\alpha_u\}\mathcal{W}(\Delta) = \{\text{Id}_{\{\alpha_u\}}, r, t, (su), rt, r(su), t(su), (su)(rs), (su)(ts), tr(su), r(su)(rs),
t(su)(rs), (su)(rs)t, r(su)(ts), t(su)(ts), (su)(ts)r, (su)(rs)t(sr),
tr(su)(rs), (su)(rs)tu, t(su)(rs)t, tr(su)(ts), (su)(ts)ru, r(su)(ts)r,
(su)(rs)t(sr)(us), r(su)(rs)t(sr), t(su)(rs)t(sr), tr(su)(rs)t, tr(su)(ts)r,
r(su)(rs)t(sr)(us), t(su)(rs)t(sr)(us), tr(su)(rs)t(sr), tr(su)(rs)t(sr)(us)\}

Observe that the number of morphisms with of a particular length varies depend-
TABLE 6.1
THE SIMPLE ROOTS OF $(\hat{W}(\Delta), \Phi_\Delta)^{re}$

<table>
<thead>
<tr>
<th>${\alpha_r} \Pi_\Delta$</th>
<th>${\alpha_s} \Pi_\Delta$</th>
<th>${\alpha_t} \Pi_\Delta$</th>
<th>${\alpha_u} \Pi_\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_r + 2\alpha_s$</td>
<td>$2\alpha_r + \alpha_s$</td>
<td>$\alpha_r$</td>
<td>$\alpha_r$</td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>$\alpha_s + 2\alpha_t$</td>
<td>$2\alpha_s + \alpha_t$</td>
<td>$2\alpha_s + \alpha_u$</td>
</tr>
<tr>
<td>$\alpha_u$</td>
<td>$\alpha_s + 2\alpha_u$</td>
<td>$\alpha_u$</td>
<td>$\alpha_t$</td>
</tr>
</tbody>
</table>

ing on the codomain. For example there exists 5 unique morphisms of length 2 with codomain $\{\alpha_r\}$, while there 6 unique morphisms of length 2 with codomain $\{\alpha_s\}$.

Observing that the simple morphisms with codomain $\{\alpha_r\}$ are $\{\alpha_r\}^\hat{S} := \{t, u, sr\}$, and so the simple roots at $\{\alpha_r\}$ are $\{\alpha_r\} \Pi := \{\Phi_{\Delta t}, \Phi_{\Delta u}, \Phi_{\Delta sr}\}$ where

$$\Phi_{\Delta t} = \{\alpha_t\}$$
$$\Phi_{\Delta u} = \{\alpha_u\}$$
$$\Phi_{\Delta sr} = \{\alpha_s, \alpha_r + \alpha_s\}$$

Since $|\Phi_{\Delta sr}| = 2$. $(\hat{W}(\Delta), \Phi_\Delta)$ is not compressed. Thus we consider the realized signed groupoid set $(\hat{W}, \Phi_\Delta, \hat{V} \circ F_{\Pi})$, which is a realization of $(\hat{W}, \Phi_\Delta)^c$. Just as we computed the simple roots at $\{\alpha_r\}$ we may do the same for every object of $\hat{W}$. Table 6.1 gives the simple roots of $(\hat{W}(\Delta), \Phi_\Delta)^{re}$.

We also may compute the complete set of the roots for $(\hat{W}(\Delta), \Phi_\Delta)^{re}$. Recall that the simple morphisms generate all the morphisms of $\hat{W}(\Delta)$, and every real root of $(\hat{W}(\Delta), \Phi_\Delta)^{re}$ must lie in an inversion set. Table 6.2 lists all the positive roots of $(\hat{W}(\Delta), \Phi_\Delta)$.
Observe that for every $a \in \text{Ob}(\hat{W}(\Delta))$, $|a\Phi^{\text{re}}_\Delta| = 7$, and so $|a\Phi^{\text{re}}_\Delta| = 14$.

We now consider these root systems in the quotient space, $(\hat{W}(\Delta), \Phi^{\Delta}/\Pi, \hat{V}/\Pi)$. Recall that the quotient vector space at each is defined as follows.

\[
\begin{align*}
(\hat{V}/\Pi)(\{\alpha_r\}) &:= V/\mathbb{R}\{\alpha_r\} \\
(\hat{V}/\Pi)(\{\alpha_s\}) &:= V/\mathbb{R}\{\alpha_s\} \\
(\hat{V}/\Pi)(\{\alpha_t\}) &:= V/\mathbb{R}\{\alpha_t\} \\
(\hat{V}/\Pi)(\{\alpha_u\}) &:= V/\mathbb{R}\{\alpha_u\}
\end{align*}
\]
TABLE 6.3

THE SIMPLE ROOTS OF \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi)\)

<table>
<thead>
<tr>
<th>{\alpha_r} \Pi_{\Delta}/{\alpha_r}</th>
<th>{\alpha_s} \Pi_{\Delta}/{\alpha_s}</th>
<th>{\alpha_t} \Pi_{\Delta}/{\alpha_t}</th>
<th>{\alpha_u} \Pi_{\Delta}/{\alpha_u}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2\beta_s</td>
<td>2\beta_r</td>
<td>\beta_r</td>
<td>\beta_r</td>
</tr>
<tr>
<td>\beta_t</td>
<td>2\beta_t</td>
<td>2\beta_s</td>
<td>2\beta_s</td>
</tr>
<tr>
<td>\beta_u</td>
<td>2\beta_u</td>
<td>\beta_u</td>
<td>\beta_t</td>
</tr>
</tbody>
</table>

For \(\omega \in \text{Ob}(\hat{W}(\Delta))\) we define the following vectors in the quotient spaces:

\[
\begin{align*}
\beta_r & := \alpha_r + R\{\omega\} \\
\beta_s & := \alpha_s + R\{\omega\} \\
\beta_t & := \alpha_t + R\{\omega\} \\
\beta_u & := \alpha_u + R\{\omega\}
\end{align*}
\]

Table 6.3 gives the simple roots of \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\), and Table 6.4 gives the positive roots of \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\).

Without loss of generality we may rescale the simple roots giving an isomorphic signed groupoid set. Thus for \(\{\alpha_r\} \Pi'_{\Delta}/\{\alpha_r\}\) we use the following rescaling:

\[
\begin{align*}
\gamma_s & := 2\beta_s \\
\gamma_t & := \beta_t \\
\gamma_u & := \beta_u.
\end{align*}
\]
Using this convention for rescaling we give the simple roots of \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\) in Table 6.5, and the positive roots of \((\hat{W}(\Delta), \Phi_{\Delta}^+ / \Pi, \hat{V}/\Pi)\) in Table 6.6.

Remark 6.7.2. These tables show us that \((\hat{W}, \Phi_{\Delta}, \hat{V} \circ F_{\Pi})\) is conical, and Lemma 4.6.2 tells us that \((\hat{W}, \Phi_{\Delta})\) is principal signed groupoid set.

Remark 6.7.3. (a) Recall that Theorem 6.5.2 tells us that the inversion sets of the morphisms of a fixed codomain in \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\) correspond to hyperplanes that split the positive roots.

(b) Note that in \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\), the inversion sets of the 32 morphisms of a fixed codomain correspond to a simplicial hyperplane arrangement in 3 dimensions that consists of 7 hyperplaneness and 32 chambers.

Observe that in \((\hat{W}(\Delta), \Phi_{\Delta}/\Pi, \hat{V}/\Pi)\) that not only do we have that the positive roots lie in the positive linear span of the simple roots, but also we see that they lie in the span of positive integer multiplies of the simple roots. Note that this not happen in \((\hat{W}(\Delta), \Phi_{\Delta}^+ / \Pi, \hat{V}/\Pi)\), and thus it is due to our rescaling of the simple roots.
TABLE 6.5
THE SIMPLE ROOTS OF $(\hat{W}(\Delta), \Phi_{\Delta}/\Pi)$

<table>
<thead>
<tr>
<th>${\alpha_r} \Pi_\Delta/{\alpha_r}$</th>
<th>${\alpha_s} \Pi_\Delta/{\alpha_s}$</th>
<th>${\alpha_t} \Pi_\Delta/{\alpha_t}$</th>
<th>${\alpha_u} \Pi_\Delta/{\alpha_u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_s$</td>
<td>$\gamma_r$</td>
<td>$\gamma_r$</td>
<td>$\gamma_r$</td>
</tr>
<tr>
<td>$\gamma_t$</td>
<td>$\gamma_t$</td>
<td>$\gamma_s$</td>
<td>$\gamma_s$</td>
</tr>
<tr>
<td>$\gamma_u$</td>
<td>$\gamma_u$</td>
<td>$\gamma_u$</td>
<td>$\gamma_t$</td>
</tr>
</tbody>
</table>

TABLE 6.6
THE POSITIVE ROOTS OF $(\hat{W}(\Delta), \Phi_{\Delta}/\Pi)$

<table>
<thead>
<tr>
<th>${\alpha_r} \Phi_{\Delta}^+/{\alpha_r}$</th>
<th>${\alpha_s} \Phi_{\Delta}^+/{\alpha_s}$</th>
<th>${\alpha_t} \Phi_{\Delta}^+/{\alpha_t}$</th>
<th>${\alpha_u} \Phi_{\Delta}^+/{\alpha_u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_s$</td>
<td>$\gamma_r$</td>
<td>$\gamma_r$</td>
<td>$\gamma_r$</td>
</tr>
<tr>
<td>$\gamma_t$</td>
<td>$\gamma_t$</td>
<td>$\gamma_s$</td>
<td>$\gamma_s$</td>
</tr>
<tr>
<td>$\gamma_u$</td>
<td>$\gamma_u$</td>
<td>$\gamma_u$</td>
<td>$\gamma_t$</td>
</tr>
<tr>
<td>$\gamma_s+\gamma_t$</td>
<td>$\gamma_r+\gamma_t$</td>
<td>$\gamma_r+\gamma_s$</td>
<td>$\gamma_r+\gamma_s$</td>
</tr>
<tr>
<td>$\gamma_s+\gamma_u$</td>
<td>$\gamma_r+\gamma_u$</td>
<td>$\gamma_s+\gamma_u$</td>
<td>$\gamma_s+\gamma_t$</td>
</tr>
<tr>
<td>$\gamma_s+\gamma_t+\gamma_u$</td>
<td>$\gamma_r+\gamma_u$</td>
<td>$\gamma_r+\gamma_s+\gamma_u$</td>
<td>$\gamma_r+\gamma_s+\gamma_t$</td>
</tr>
<tr>
<td>$2\gamma_s+\gamma_t+\gamma_u$</td>
<td>$\gamma_r+\gamma_t+\gamma_u$</td>
<td>$\gamma_r+2\gamma_s+\gamma_u$</td>
<td>$\gamma_r+2\gamma_s+\gamma_t$</td>
</tr>
</tbody>
</table>
It remains an interesting question whether one can always arrange, by rescaling, that the positive roots are all integer combinations of simple roots if one begins with a Weyl group.


