CHARACTERIZING FORKING IN VC-MINIMAL THEORIES

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

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Notre Dame, Indiana
May 2012
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Abstract

by

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We consider the class of VC-minimal theories, as introduced by Adler in [2]. After covering some basic results, including a notion of generic types, we consider two kinds of VC-minimal theories: those whose generating directed families are unpackable and almost unpackable. We introduce two new decompositions of definable sets in VC-minimal theories, the layer decomposition and the irreducible decomposition, which allow for more precision than the standard Swiss cheese decomposition with regard to parameters. Finally, after introducing a slight generalization of the classical notions of forking and dividing, we prove that in any VC-minimal or quasi-VC-minimal theory whose generating family is unpackable or almost unpackable, forking of formulae over a model $M$ is equivalent to containment in a global $M$-definable type, generalizing a result of Dolich on o-minimal theories in [3].
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During the last fifty years, a major area of research within model theory has been the identification of theories in which there is a succinct description of all definable subsets of models. Significant topics explored have included the o-minimal theories, generalizing the behavior of ordered real closed fields; the strongly minimal theories, which parallel the algebraically closed fields; and the C-minimal theories, in which definable sets mimic those of an algebraically closed valued field.

In 2008, Hans Adler introduced VC-minimality (see [2]), seeking to generalize several notions of minimality previously studied. The resulting definition is well situated in the hierarchy of model-theoretic notions of well-behavedness. The strongly minimal, o-minimal, and C-minimal theories are all subclasses of the VC-minimal theories, providing a number of widely studied examples of VC-minimal theories. As VC-minimality is sufficiently strong to imply dp-minimality, and through it NIP and NTP$_2$, the tools and results developed for those classes of theories are all available in the VC-minimal setting.

In this dissertation, we give an overview of VC-minimality, developing new tools with the aim of characterizing forking over models in VC-minimal theories. We begin in Chapter 1 with a summary of some non-logical topics useful in studying VC-minimality: Vapnik-Chervonenkis dimension and codimension, and the related subject of directed families of sets. We define two types of directed
families, those that are *unpackable* and those that are *almost unpackable*, whose properties will prove to be particularly useful in the context of VC-minimality.

Chapter 2 includes the basic results of VC-minimality. We develop a notion of generic types, and adapt the definitions of unpackable and almost unpackable to a VC-minimal setting. A key tool in working with VC-minimal theories is the ability to decompose definable sets into finitely many sets which are structurally very simple; the standard decomposition of definable sets is into sets called *Swiss cheeses*. We generalize this to two new decompositions of definable sets, the *layer decomposition* and the *irreducible decomposition*; although somewhat more abstract than the Swiss cheese decomposition, these allow us to control more tightly the parameters used to define the pieces in each decomposition. In the last section of Chapter 2 we consider the relationships between VC-minimality and several other model-theoretic notions. In Chapter 3 we present a number of examples of VC-minimal theories.

Chapter 4 is devoted to the topics of forking and dividing. We list a number of results related to these which apply in VC-minimal theories, and establish that in VC-minimal theories with unpackable generating families, forking equals dividing over any set. We then introduce the notions of \( \beta \)-dividing and \( \beta \)-forking, generalizations of dividing and forking which allow us to incorporate a parameter tuple \( \beta \) chosen definably from outside of our ambient model. By adapting results of [6], we demonstrate that \( \beta \)-forking equals \( \beta \)-dividing over models, a result which will serve to bridge a gap in the main proof of Chapter 5.

Lastly, we present, in three different settings, the proof of the main result: that for any formula \( \varphi(\overline{x},\overline{c}) \) and any model \( M \), the formula \( \varphi(\overline{x},\overline{c}) \) does not fork over \( M \) if and only if there is a global \( M \)-definable type containing \( \varphi(\overline{x},\overline{c}) \). First, in
Chapter 5 we prove it for VC-minimal theories in which the generating family \( \Psi \) is unpackable (see Theorem 5.2.1). This is in some ways the most intuitive setting for the proof. Unpackability is a notion which occurs naturally, being equivalent to uniqueness of Swiss cheese decompositions; it applies to many VC-minimal theories, particularly the strongly minimal and o-minimal theories, and assuming that \( \Psi \) is unpackable allows us to explicitly build the definable type needed in the proof.

Next, in Section 6.1 we re-present the proof under the assumption that \( \Psi \) is almost unpackable, in Theorem 6.1.1. Almost unpackability is less intuitive than unpackability, and is strictly weaker; however, it applies to a number of interesting theories. Some sections of the original proof carry over to the almost unpackable setting with very little adjustment needed. However, the proof does become more complex; most notably, the set of formulae constructed in the original proof in order to generate a definable type need not be consistent when \( \Psi \) is not unpackable, forcing us to abandon it and construct an entirely new type.

Finally, in Section 6.2 we consider quasi-VC-minimality, a generalization of VC-minimality. Although the definition of quasi-VC-minimality seems less natural than that of VC-minimality, it does apply to some noteworthy non-VC-minimal theories - in particular, Presburger Arithmetic. After establishing analogues to some of the structural results from Chapter 2 in the quasi-VC-minimal setting, we briefly outline in Theorem 6.2.13 what changes are needed for the proof of the main result in Section 6.1 to apply to quasi-VC-minimal theories.
ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Sergei Starchenko, without whom this thesis would never have come to be. It has been a privilege to work with him, and I am deeply grateful for his vast knowledge, endless patience, and unfailing support.

The Notre Dame Logic Group has been an invaluable resource for the last five years. I thank Julia Knight and Peter Cholak for their support throughout my time at Notre Dame. I am particularly grateful to Joe Flenner for his many insights and suggestions on this thesis. Thanks also to Don Brower and Vince Guingona, who always have interesting ideas, both mathematically and otherwise.

I am grateful to many in the wider mathematical community who have influenced me over the years. I particularly want to thank Bernadette Boyle and Margaret Thomas for their friendship and encouragement. I thank Uri Andrews and Jim Freitag for our many interesting mathematical discussions in the past several years. Thank you to Dan Velleman, who introduced me to logic, and to Wing Mui for encouraging me to consider a career in mathematics.

I am tremendously grateful to my family, for the love and encouragement they have always provided.

Finally, I want to thank my fiancé, Justin, for his love, his support, and his endless determination to make me laugh.
### SYMBOLS

- $\mathcal{P}(X)$: power set of $X$
- $|X|$: cardinality of $X$
- $x, y, a, b$: single variables and elements
- $\bar{x}, \bar{y}, \bar{a}, \bar{b}$: finite tuples of variables and elements
CHAPTER 1

NON-LOGICAL BACKGROUND

We begin by considering two combinatorial notions we will use later on: Vapnik-Chervonenkis dimension and codimension, which give VC-minimality its name, and directed families of sets, which give VC-minimal theories their structure.

1.1 VC-dimension and VC-codimension

In [23], Vapnik and Chervonenkis devised a measure of dimension for families of sets, which attempts to measure the complexity of the intersections among those sets. The resulting notions, VC-dimension and VC-codimension, have applications to a variety of areas of mathematics, from probability and learning theory (23) to computational geometry (19) to logic (17).

We work throughout with a collection $\mathcal{F}$ of subsets of some base set $X$.

Definition 1.1.1. Let $X$ be any set, and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of $X$.

1. For any $A \subseteq X$, we denote by $\mathcal{F} \cap A$ the family of sets $\{F \cap A : F \in \mathcal{F}\}$.
   Note that $\mathcal{F} \cap A \subseteq \mathcal{P}(A)$.

2. $\mathcal{F}$ shatters $A$ if $\mathcal{F} \cap A = \mathcal{P}(A)$, i.e. if every subset of $A$ appears as $F \cap A$ for some $F \in \mathcal{F}$. 

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3. The *shatter function* of $\mathcal{F}$ is a function $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$, given by

$$\pi_{\mathcal{F}}(n) = \max \left\{|\mathcal{F} \cap A| : A \in X, |A| = n\right\}.$$ 

Note that $1 \leq \pi_{\mathcal{F}}(n) \leq 2^n$ for any $n$, with $\pi_{\mathcal{F}}(n) = 2^n$ iff $\mathcal{F}$ shatters some set of size $n$.

4. The *VC-dimension* of $\mathcal{F}$ is

$$\text{vc}(\mathcal{F}) = \max \left\{n : \pi_{\mathcal{F}}(n) = 2^n\right\}$$

if this maximum exists, and $\text{vc}(\mathcal{F}) = \infty$ otherwise.

So, the VC-dimension of $\mathcal{F}$ is the size of the largest finite set all of whose subsets can be distinguished by $\mathcal{F}$; if $\mathcal{F}$ can distinguish all subsets for arbitrarily large finite sets, then the VC-dimension of $\mathcal{F}$ is infinite. A *VC-class* is a family $\mathcal{F}$ with finite VC-dimension.

If a family $\mathcal{F}$ has infinite VC-dimension, then $\pi_{\mathcal{F}}(n) = 2^n$ for all $n$. A strong upper bound on the growth of $\pi_{\mathcal{F}}$ when $\mathcal{F}$ is a VC-class was established independently by Sauer [20] and by Shelah [21]:

**Theorem 1.1.2.** If the VC-dimension of $\mathcal{F}$ is $d$, then $\pi_{\mathcal{F}}(n) \leq n^d$ for all sufficiently large $n$.

In addition to VC-dimension, we have the dual notion of VC-codimension, which also measures complexity of intersections in $\mathcal{F}$.

**Definition 1.1.3.** Given finitely many sets $F_1, \ldots, F_n \in \mathcal{F}$ and points $x, y \in X$, say that $x$ and $y$ are *equivalent* with respect to $F_1, \ldots, F_n$ if for all $i = 1, \ldots, n$ we have that $x \in F_i$ iff $y \in F_i$. 

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So, points $x$ and $y$ are equivalent with respect to $F_1, \ldots, F_n$ if they are in the same region of the Venn diagram on $F_1, \ldots, F_n$.

**Definition 1.1.4.**

1. For any $F_1, \ldots, F_n \in \mathcal{F}$, let $\#\mathcal{E}(F_1, \ldots, F_n)$ be the number of equivalence classes of the equivalence relation in Definition 1.1.3.

2. The dual shatter function of $\mathcal{F}$ is a function $\pi^*_\mathcal{F} : \mathbb{N} \to \mathbb{N}$, given by

   $$\pi^*_\mathcal{F}(n) = \max \{ \#\mathcal{E}(F_1, \ldots, F_n) : F_1, \ldots, F_n \in \mathcal{F} \}.$$  

   Note that $1 \leq \pi^*_\mathcal{F}(n) \leq 2^n$ for any $n$, with $\pi^*_\mathcal{F}(n) = 2^n$ iff some $n$ sets from $\mathcal{F}$ form a complete Venn diagram.

3. The VC-codimension of $\mathcal{F}$ is

   $$\text{vc}^*(\mathcal{F}) = \max \{ n : \pi^*_\mathcal{F}(n) = 2^n \}$$

   if this maximum exists, and $\text{vc}^*(\mathcal{F}) = \infty$ otherwise.

The VC-codimension of $\mathcal{F}$, then, is the largest number of sets in $\mathcal{F}$ which form a complete Venn diagram, with $\text{vc}^*(\mathcal{F}) = \infty$ if sets in $\mathcal{F}$ form arbitrarily large complete Venn diagrams. We have the following relationship between VC-dimension and VC-codimension:

**Proposition 1.1.5** ([10], Theorem 9.3.2). For any family $\mathcal{F}$,

$$\text{vc}(\mathcal{F}) \leq 2^{\text{vc}^*(\mathcal{F})} \quad \text{and} \quad \text{vc}^*(\mathcal{F}) \leq 2^{\text{vc}(\mathcal{F})}.$$  

So in particular, $\text{vc}(\mathcal{F})$ is finite iff $\text{vc}^*(\mathcal{F})$ is finite.
1.2 Directed families of sets

Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a collection of subsets of $X$.

**Definition 1.2.1.** $\mathcal{F}$ is a *directed family* if for any two sets $B_0, B_1 \in \mathcal{F}$, one of the following holds:

(i) $B_0 \subseteq B_1$; or,

(ii) $B_1 \subseteq B_0$; or,

(iii) $B_0 \cap B_1 = \emptyset$.

We call the sets in a directed family *balls*.

In other words, $\mathcal{F}$ is a directed family if and only if no two balls from $\mathcal{F}$ form a complete Venn diagram. As an immediate result, we get the following:

**Proposition 1.2.2.** If $\mathcal{F} \subseteq \mathcal{P}(X)$ is a directed family, then $\text{vc}^*(\mathcal{F}) \leq 1$.

**Remark** 1.2.3. For notational simplicity, we will assume that $X \in \mathcal{F}$ for any directed family $\mathcal{F}$ on $X$.

Given a directed family $\mathcal{F}$ on a base set $X$, we can use the sets in $\mathcal{F}$ to describe certain subsets of $X$ in a finite way. We call these subsets of $X$ *constructible*.

**Definition 1.2.4.** A set $Y \subseteq X$ is *constructible* from $\mathcal{F}$ if $Y$ is a finite boolean combination of balls in $\mathcal{F}$.

**Definition 1.2.5.** Let $\mathcal{F}$ be a directed family. A *Swiss cheese* is a set of the form $S = A \setminus (B_1 \cup \cdots \cup B_n)$, where $A \in \mathcal{F}$ is a ball and $B_1, \ldots, B_n \in \mathcal{F}$ are balls properly contained in $A$. The set $A$ is called an *outer ball* of $S$, while $B_1, \ldots, B_n$ are called *holes* of $S$. 
Remark 1.2.6. Any ball $A$ is also said to be a Swiss cheese, with no holes.

Theorem 1.2.7 ([11], Lemma 2.1). Let $F$ be a directed family. Then every constructible set is the union of finitely many disjoint Swiss cheeses.

If a constructible set $Y$ is the disjoint union of Swiss cheeses $S_1, \ldots, S_n$, we call the set $\{S_1, \ldots, S_n\}$ a Swiss cheese decomposition of $Y$. We say that the decomposition of $Y$ into $S_1, \ldots, S_n$ is non-trivial if we can write these Swiss cheeses as $S_1 = A_1 \setminus (B_1^1 \cup \cdots \cup B_{k_1}^1), \ldots, S_n = A_n \setminus (B_1^n \cup \cdots \cup B_{k_n}^n)$, such that there do not exist indices $i, j_1, \ldots, j_l$ for which the union of the outer balls $A_{j_1}, \ldots, A_{j_l}$ equals the union of some of the holes of $S_i$.

In general, Swiss cheese decompositions (and even non-trivial Swiss cheese decompositions) need not be unique. However, Flenner and Guingona’s main result in [11] establishes a canonical way of breaking down constructible sets in any directed family.

Theorem 1.2.8. Let $F$ be a directed family. For every constructible set $Y$ there is a canonical way of choosing finite sets of balls $\text{Lev}_0(Y), \text{Lev}_1(Y), \ldots, \text{Lev}_{2n+1}(Y)$ such that:

- for $i > 0$, each ball in $\text{Lev}_i(Y)$ is properly contained in a ball of $\text{Lev}_{i-1}(Y)$;
- \[ Y = \bigcup_{i=0}^{n} \left( \bigcup \text{Lev}_{2i}(Y) \setminus \bigcup \text{Lev}_{2i+1}(Y) \right). \]

There may be multiple sets of balls with these properties for a given constructible set $Y$; however, the choice of the sets $\text{Lev}_0(Y), \ldots, \text{Lev}_{2n+1}(Y)$ is canonically determined by $Y$. The sets $\bigcup \text{Lev}_{2i}(Y) \setminus \bigcup \text{Lev}_{2i+1}(Y)$ are called the layers of $Y$. For some Swiss cheese decomposition of $Y$, the set $\bigcup \text{Lev}_0(Y) \setminus \bigcup \text{Lev}_1(Y)$ is the union of those Swiss cheeses that are not contained in the holes of any other
Swiss cheeses, the set $\bigcup \text{Lev}_2(Y) \setminus \bigcup \text{Lev}_3(Y)$ is the union of those Swiss cheeses of $Y \setminus (\bigcup \text{Lev}_0(Y) \setminus \bigcup \text{Lev}_1(Y))$ not contained in the holes of any other, and so on.

1.2.1 Unpackability

We now consider additional conditions on directed families which result in bounds on the variation of Swiss cheese decompositions.

**Definition 1.2.9.** Let $\mathcal{F}$ be a directed family.

1. $\mathcal{F}$ is **packable** if some ball is the union of finitely many smaller balls, that is, there exist balls $A, B_1, \ldots, B_n$ such that $B_i \subsetneq A$ for each $i$ and $A = \bigcup_{i=1}^{n} B_i$.

2. $\mathcal{F}$ is **unpackable** if it is not packable.

The following was originally an unpublished result of Dolich, adapting a result of Holly on valued fields in [13]. A direct proof is now available in Theorem 2.2 of [11].

**Theorem 1.2.10.** The following are equivalent for any directed family $\mathcal{F}$:

1. $\mathcal{F}$ is unpackable;

2. if $A, B_1, \ldots, B_n$ are balls and $A \subseteq \bigcup_{i=1}^{n} B_i$, then $A \subseteq B_i$ for some $i$;

3. if $S$ is a Swiss cheese, then there exist a unique ball $A$ and a unique set of non-empty balls $\{B_1, \ldots, B_n\}$, each properly contained in $A$, such that $S = A \setminus (B_1 \cup \ldots \cup B_n)$;

4. every constructible set has a unique non-trivial Swiss cheese decomposition.

The following slight weakening of unpackability will also prove to be useful.
Definition 1.2.11.

1. \( \mathcal{F} \) \textit{partitions} (or \textit{breaks}) a constructible set \( Y \) into \( n \) pieces, where \( n \geq 2 \), if there are balls \( B_1, \ldots, B_n \) such that \( Y \cap B_1, \ldots, Y \cap B_n \) are pairwise disjoint non-empty sets and \( Y = \bigcup_{i=1}^{n} Y \cap B_i \).

2. A constructible set \( Y \) is \textit{unbreakable} if \( \mathcal{F} \) does not partition \( Y \) into any finite number of pieces.

3. \( \mathcal{F} \) is \textit{almost unpackable} if it partitions every constructible set into finitely many unbreakable pieces. These pieces, \( Y \cap B_1, \ldots, Y \cap B_n \), are called \textit{unbreakable components of} \( Y \), and they are uniquely determined by \( \mathcal{F} \).

Remark 1.2.12. A directed family is unpackable if and only if every ball is unbreakable.

Theorem 1.2.13. The following are equivalent for any directed family \( \mathcal{F} \):

(1): \( \mathcal{F} \) is almost unpackable;

(2): every ball is the union of finitely many unbreakable balls.

Proof. If \( \mathcal{F} \) is almost unpackable, then for any ball \( B \), the unbreakable components of \( B \) must be unbreakable balls whose union is \( B \).

Conversely, suppose every ball is the union of finitely many unbreakable balls. Let \( Y \) be any constructible set, and consider the finite set of balls \( \text{Lev}_0(Y) \), as given in Theorem 1.2.8. Letting \( B_1, \ldots, B_n \) enumerate the balls in \( \text{Lev}_0(Y) \), every point of \( Y \) is in one of the balls \( B_i \). For each \( i \), let \( C^1_i, \ldots, C^{k_i}_i \) be the unbreakable balls of \( B_i \). Then the sets \( Y \cap C^j_i \), where \( 1 \leq i \leq n \) and \( 1 \leq j \leq k_i \), are disjoint unbreakable sets whose union is \( Y \), giving us unbreakable components for \( Y \). \( \Box \)
CHAPTER 2

VC-MINIMALITY

This chapter covers a number of basic results about VC-minimality. We develop some tools for working with VC-minimal theories, including a notion of generic types and several decompositions of definable sets, and consider the relationships between VC-minimal theories and several other classes of theories.

We work throughout in a complete elementary first-order theory $T$. We will not generally distinguish between a formula $\varphi(\bar{x}, \bar{a})$ and the set $B$ which it defines, writing $B = \varphi(\bar{x}, \bar{a})$ or $B \subseteq \varphi(\bar{x}, \bar{a})$ where convenient.

2.1 Definition and basic results

**Definition 2.1.1.** A complete theory $T$ is *VC-minimal* if there exists a family $\Psi = \{\psi_i(x, \bar{y}_i) : i \in I\}$ of formulae, each in a single variable $x$ and some finite tuple of variables $\bar{y}_i$, such that for any model $M$ of $T$:

(i) the family of sets $\{\psi(x, \bar{b}) : \psi \in \Psi, \bar{b} \in M\}$ defined by instances of $\Psi$ is a directed family;

(ii) any definable subset of $M$ is a finite boolean combination of sets of the form $\psi(x, \bar{b})$, where $\psi \in \Psi$ and $\bar{b} \in M$.

This $\Psi$ is called a *generating directed family* for $T$. 

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Remark 2.1.2. This differs slightly from Adler’s original definition of “directed VC-minimal theories” in [2].

Notation 2.1.3.

• Similarly to Remark 1.2.3, for notational simplicity, we will generally assume that the formula $x = x$ is in any generating directed family $\Psi$.

• An instance of $\Psi$ is a formula $\psi(x, \bar{a})$, where $\psi(x, \bar{y}) \in \Psi$ and the parameters $\bar{a}$ are taken from some model of $T$.

So, a ball is now a set defined by an instance of $\Psi$, a Swiss cheese is a set of the form $A \setminus (B_1 \cup \cdots \cup B_n)$, where each of $A, B_1, \ldots, B_n$ is defined by an instance of $\Psi$, and the definable sets in a single $x$ variable are exactly the constructible sets with respect to the directed family of balls defined by $\Psi$.

Remark 2.1.4. If $\Psi$ is finite, then by combining the formulae and possibly introducing dummy variables, we may assume that $|\Psi| = 1$.

Definition 2.1.5. Let $A \subseteq B$ be subsets of a model $M$, and let $p \in S_\varphi(B)$. The type $p$ is definable over $A$, or $A$-definable, if for every formula $\varphi(x, \bar{y})$ there exists a formula $\delta_\varphi(\bar{y})$, with parameters in $A$, such that for all tuples $\bar{c} \in B$,

$$\varphi(x, \bar{c}) \in p \iff M \models \delta_\varphi(\bar{c}).$$

Proposition 2.1.6. Let $T$ be a VC-minimal theory, with generating directed family $\Psi$, and let $M \models T$. Suppose that $p^* \in S^\Psi_1(M)$ is a complete $\Psi$-type over $M$. Then $p^*$ generates a unique complete type $p \in S_1(M)$. Further, if $p^*$ is definable over a set $A \subseteq M$, then so is $p$.

Proof. The existence and uniqueness of $p$ follow directly from the definition of VC-minimality. Definability follows from Lemma 2.3.1 of [14]; alternately, definability can be shown by a method similar to that used in Proposition 2.2.18.
In a VC-minimal theory, we also have a notion of generic types for balls. Recall from Definition 1.2.11 that a ball is unbreakable if it cannot be written as the union of finitely many smaller balls. If we restrict ourselves to unbreakable balls, we get the type defined in the following result:

**Proposition 2.1.7.** Let $T$ be VC-minimal, with generating directed family $\Psi$, and let $M \models T$. Let $B$ be an unbreakable ball, defined by $\psi(x, \overline{b})$, where $\psi \in \Psi$ and $\overline{b} \in M$. Then there is a unique complete type $p_B \in S_1(M)$ with the property that for any ball $C$ which is definable with parameters in $M$, $C$ is in $p_B$ iff $B \subseteq C$.

($p_B$ is called the generic type of (the interior of) $B$ over $A$, and is definable over any set containing $\overline{b}$.)

**Proof.** Begin by building a $\Psi$-type $p_B^*(x)$ over $M$: for any $\psi'(x, \overline{y}) \in \Psi$ and any $\overline{c} \in M$, the formula $\psi'(x, \overline{c})$ goes into $p_B^*(x)$ if $\psi(x, \overline{b}) \rightarrow \psi'(x, \overline{c})$; otherwise, the formula $\neg \psi'(x, \overline{c})$ is in $p_B^*(x)$.

If $p_B^*$ were inconsistent, then there would be finitely many balls $A_1, \ldots, A_m, B_1, \ldots, B_n$, each definable over $M$, such that the set of formulae

$$\{ x \in A_i : i = 1, \ldots, m \} \cup \{ x \notin B_i : i = 1, \ldots, n \}$$

is inconsistent. Since $B \subseteq A_i$ for all $i$, the set $\{ x \in B \} \cup \{ x \notin B_i : i = 1, \ldots, n \}$ must also be inconsistent. Each ball $B_i$ must be either disjoint from $B$ or a proper subset of $B$; if $B_1, \ldots, B_k$ is a list of those balls $B_i$ which are proper subsets of $B$, then the set $\{ x \in B \} \cup \{ x \notin B_i : i = 1, \ldots, k \}$ is also inconsistent. But then $B$ is covered by the finitely many smaller balls $B_1, \ldots, B_k$, contradicting its unbreakability.

Thus, $p_B^*$ is a complete, consistent $\Psi$-type, and by Proposition 2.1.6 generates a unique complete type $p_B$, which is definable over $\overline{b}$. \qed
Working with unbreakable balls gives us several other results, as well.

**Proposition 2.1.8.** Let \( \varphi(x, \bar{y}) \) be any formula and \( \bar{c} \) any parameters such that \( \varphi(x, \bar{c}) \) is unbreakable. Then for any \( \bar{c}' \equiv_\emptyset \bar{c} \), the set \( \varphi(x, \bar{c}') \) is unbreakable.

**Proof.** Suppose that \( \Psi \) breaks \( \varphi(x, \bar{c}') \) into \( n \) pieces, via the balls \( \psi_1(x, b_1), \ldots, \psi_n(x, b_n) \). Then the formula \( \exists \bar{z}_1 \cdots \exists \bar{z}_n [\varphi(x, \bar{y}) \leftrightarrow \bigvee_i \varphi(x, \bar{y}) \land \psi_i(x, \bar{z}_i)] \) is in \( \text{tp}(\bar{c}'/\emptyset) = \text{tp}(\bar{c}/\emptyset) \), and so \( \varphi(x, \bar{c}) \) must also break into \( n \) pieces. \( \square \)

**Corollary 2.1.9.** Any automorphism maps unbreakable balls to unbreakable balls.

Additionally, if a set \( S \) is a Swiss cheese, then \( S \) is unbreakable if and only if we can write it as \( S = A \setminus (B_1 \cup \cdots \cup B_n) \), where \( A, B_1, \ldots, B_n \) are balls and \( A \) is unbreakable.

**Proposition 2.1.10.** Any 2-consistent family of unbreakable Swiss cheeses is consistent.

**Proof.** Let \( F = \{S_1, \ldots, S_n\} \) be a finite family of unbreakable Swiss cheeses, and suppose \( F \) is 2-consistent. Write each Swiss cheese as \( S_i = A_i \setminus (B_{i1} \cup \cdots \cup B_{kn}) \), where each ball \( A_i \) is unbreakable.

Since \( F \) is 2-consistent, the set of outer balls \( \{A^1, \ldots, A^n\} \) is 2-consistent; as they are balls, this means they must form a chain. Possibly after renumbering, we may assume \( A^1 \subseteq A^2 \subseteq \ldots \subseteq A^n \).

If the family \( F \) were inconsistent, then we would have \( A_1 \subseteq \bigcup B_{j} \); since \( A_1 \) cannot be the union of any number of the balls \( B_{j} \), \( A_1 \) would be contained in one of them, \( A_1 \subseteq B_j \). But then \( S_1 \cap S_i = \emptyset \), violating 2-consistency. \( \square \)
2.1.1 Unpackability

As in Section 1.2, we will find it useful to have a notion of unpackability for the VC-minimal setting.

**Definition 2.1.11.** Suppose $T$ is VC-minimal. A generating directed family $\Psi$ is *unpackable* if the directed family of sets generated by $\Psi$ is unpackable, i.e. no instance of $\Psi$ is the union of finitely many smaller instances of $\Psi$.

**Remark 2.1.12.** Note that unpackability is a property of the family $\Psi$, not of the theory $T$. In Chapter 3, we will see examples of theories in which the VC-minimality of $T$ is witnessed by multiple generating families, some of which are unpackable and some of which are not (for instance, any o-minimal theory, as in Theorem 3.1.2). When working in a theory where there is more than one choice for $\Psi$, whenever possible, we will choose to work with one which is unpackable.

Recall that in an unpackable directed family, every ball is unbreakable. Thus, if $\Psi$ is unpackable, we have the following results:

- For every ball $B$, there is a generic type $p_B$.
- Any 2-consistent family of Swiss cheeses is consistent.

**Definition 2.1.13.** A generating directed family $\Psi$ is *almost unpackable* if the directed family of sets generated by $\Psi$ is almost unpackable, i.e. every instance of $\Psi$ is the union of finitely many unbreakable instances of $\Psi$.

**Remark 2.1.14.** Again, this is a property of $\Psi$ and not of $T$. In Example 3.1.7 we will see an example of a theory where there are multiple options for $\Psi$, some of which are almost unpackable and some of which are not.
If \( \Psi \) is not almost unpackable, then there exist balls which can be written as the union of arbitrarily many smaller balls. In an attempt to formalize the arbitrary breaking of balls seen when \( \Psi \) is not almost unpackable, we introduce the notion of *branching*.

**Definition 2.1.15.** A formula \( \varphi(x, \overline{y}) \) **branches** if there exist parameter tuples \( \{ \overline{a}_t : t \in S \} \) indexed by a tree \( S \), such that:

1. each node in \( S \) has at most finitely many children, and no node has exactly one child;
2. \( S \) is infinite;
3. if \( t \in S \) is a node which has children \( t_1, \ldots, t_n \), then the set \( \varphi(x, \overline{a}_t) \) is the disjoint union of the non-empty sets \( \varphi(x, \overline{a}_{t_1}), \ldots, \varphi(x, \overline{a}_{t_n}) \).

**Remark 2.1.16.** If \( \Psi \) is a finite generating directed family, then we will say that \( \Psi \) branches if the single-formula representative of \( \Psi \), as in Remark 2.1.4, branches.

**Proposition 2.1.17.** If \( \Psi \) is a generating directed family, and some formula in \( \Psi \) branches, then \( \Psi \) is not almost unpackable.

*Proof.* If \( \psi \in \Psi \) branches, and \( \overline{a} \) is the parameter corresponding to the root of the witnessing tree, then \( \psi(x, \overline{a}) \) is a ball which can be written as the union of arbitrarily large numbers of balls. \( \square \)

**Corollary 2.1.18.** If \( \Psi \) is a finite generating directed family, then \( \Psi \) is almost unpackable iff \( \Psi \) does not branch.

If \( \Psi \) is infinite, then it is possible that no formula in \( \Psi \) branches, but \( \Psi \) is not almost unpackable - for instance, see Example 3.1.7.
2.2 Decompositions of definable sets

A key tool in working with VC-minimal theories is the ability to break definable sets into well-behaved pieces. We consider several such decompositions here. Throughout this section, assume that $T$ is a VC-minimal theory with generating directed family $\Psi$, and fix a large saturated model $U$ of $T$.

2.2.1 Swiss cheeses

The standard decomposition of definable sets follows directly from the definition of VC-minimality and Theorem 1.2.7.

**Theorem 2.2.1** (Swiss cheese decomposition). For any formula $\varphi(x, \bar{a})$ and any $\bar{a} \in U$, the set $\varphi(x, \bar{a})$ is the disjoint union of finitely many Swiss cheeses.

We can also restate some of the equivalent conditions from Theorem 1.2.10 in logical terms:

**Corollary 2.2.2.** The following are equivalent for any generating family $\Psi$:

(1): $\Psi$ is unpackable;

(2): for all $\psi, \psi_1, \ldots, \psi_n \in \Psi$ and all $\bar{a}, \bar{b}_1, \ldots, \bar{b}_n \in U$, if $\psi(x, \bar{a}) \rightarrow \bigvee_{i=1}^{n} \psi_i(x, \bar{b}_i)$ then for some $i$, $\psi(x, \bar{a}) \rightarrow \psi_i(x, \bar{b}_i)$;

(3): every definable set has a unique non-trivial Swiss cheese decomposition.

Thus, in the case where $\Psi$ is unpackable, it makes sense to talk about the Swiss cheeses of a definable set $\varphi(x, \bar{a})$. In this case, we may refer to the (uniquely determined) outer balls of those Swiss cheeses as the outer balls of $\varphi(x, \bar{a})$.

If $\Psi$ is almost unpackable, then Swiss cheese decompositions will not be unique; in general, any definable set $\varphi(x, \bar{a})$ will have at most finitely many non-trivial
Swiss cheese decompositions. (We will see in Example 3.1.8 that even with an almost unpackable $\Psi$, there can be a formula $\varphi(x, y)$ such that different instances of $\varphi$ can be written non-trivially as the union of arbitrarily many Swiss cheeses.) However, the unbreakable components of any definable set will be uniquely determined. In particular, if $B \subseteq U$ is a ball (or even the union of finitely many balls), then the finitely many unbreakable balls whose union is $B$ are uniquely determined, and so it makes sense to talk about the unbreakable balls of $B$.

**Proposition 2.2.3.** Suppose that $\Psi$ is almost unpackable. Then for any formula $\varphi(x, y)$ and any $\bar{a} \in U$, the set $\varphi(x, \bar{a})$ is the disjoint union of finitely many unbreakable Swiss cheeses.

**Proof.** Let $\{S_1, \ldots, S_n\}$ be any Swiss cheese decomposition of $\varphi(x, \bar{a})$. For each $i$, let $S_i \cap B_1^i, \ldots, S_i \cap B_k^i$ be the unbreakable components of $S_i$; each of them must be an unbreakable Swiss cheese. It follows that $\{S_1 \cap B_1^1, S_1 \cap B_2^1, \ldots, S_n \cap B_k^n\}$ is a set of disjoint unbreakable Swiss cheeses whose union is $\varphi(x, \bar{a})$. \square

If $\Psi$ is not almost unpackable, there will exist a formula $\varphi(x, y)$ and parameters $\bar{a}$ such that $\varphi(x, \bar{a})$ can be written non-trivially as the union of arbitrarily many Swiss cheeses. However, via Compactness, we have the following result guaranteeing a uniform bound on how complex a Swiss cheese decomposition is needed to express any instance of a formula $\varphi(x, y)$.

**Theorem 2.2.4.** For every formula $\varphi(x, y)$, there are a finite set $\Psi_0 \subseteq \Psi$ and natural numbers $n_1$ and $n_2$ such that for every parameter tuple $\bar{a}$, $\varphi(x, \bar{a})$ can be decomposed non-trivially as the union of at most $n_1$ disjoint Swiss cheeses, each of them having at most $n_2$ holes, such that all balls appearing in the decomposition are instances of formulae in $\Psi_0$. 15
Swiss cheese decompositions give us a way to break any definable set $\varphi(x, \bar{a})$ into finitely many disjoint definable sets, each of very simple configuration. However, in general we do not know what parameters are needed to define these smaller sets; all we can guarantee is that parameters defining the Swiss cheeses can be chosen from any model containing $\bar{a}$. A need for more precision with regard to parameters leads us to consider several further decompositions.

2.2.2 Layers

Subsequent decompositions of definable sets seek to generalize the notion of “Swiss cheese”. This leads us to consider uballs, finite unions of balls, in place of balls.

**Definition 2.2.5.**

1. A definable set $C$ is a *uball* if $C = B_0 \cup \cdots \cup B_n$, where $B_0, \ldots, B_n$ are balls.

2. A definable set $D$ is a *ucheese* if $D = C_1 \setminus C_2$, where $C_1$ and $C_2$ are uballs.

**Remark 2.2.6.**

- The intersection of two uballs is a uball.
- If $B$ is a uball and $S \not\subseteq B$ is a Swiss cheese, then $S \setminus B$ is a Swiss cheese.
- A ucheese is the union of finitely many Swiss cheeses whose outer balls are disjoint.
- Suppose $\Psi$ is almost unpackable. If $D$ is a non-empty ucheese, then there exist unique uballs $C_1$ and $C_2$ such that $D = C_1 \setminus C_2$ and each unbreakable ball of $C_2$ is properly contained in an unbreakable ball of $C_1$. The set $C_1$ is called the *outer uball* of $D$, and $C_2$ is the *inner uball* of $D$. 
Theorem 2.2.7 (Layer Decomposition). For every formula $\varphi(x, \bar{y})$ there exist formulae $\lambda_0^+(x, \bar{y}), \lambda_0^-(x, \bar{y}), \lambda_1^+(x, \bar{y}), \ldots, \lambda_s^-(x, \bar{y})$ over $\emptyset$ such that for any parameter tuple $\bar{a} \in U$:

- each $\lambda_i^+(x, \bar{a})$ and each $\lambda_i^-(x, \bar{a})$ defines a uball;
- $\lambda_s^-(x, \bar{a}) \subseteq \lambda_{s-1}^+(x, \bar{a}) \subseteq \cdots \subseteq \lambda_1^+(x, \bar{a}) \subseteq \lambda_0^-(x, \bar{a}) \subseteq \lambda_0^+(x, \bar{a})$;
- $\varphi(x, \bar{a})$ is the disjoint union of the sets $\lambda_i^+(x, \bar{a}) \setminus \lambda_i^-(x, \bar{a})$, for $i = 0, \ldots, s$.

Proof. By Theorem 1.2.8, for any instance $\varphi(x, \bar{a})$ the sets $\bigcup \text{Lev}_i$ are canonically determined for the constructible set $\varphi(x, \bar{a})$; these uballs are unchanged by automorphisms fixing $\bar{a}$, and thus are defined over $\bar{a}$, by some formulae $\lambda^+_{\pi,0}(x, \bar{a}), \lambda^-_{\pi,0}(x, \bar{a}), \ldots, \lambda^-_{\pi,s}(x, \bar{a})$. By Compactness, we may assume that as the parameters $\pi$ vary, only finitely many such formulae are used. By combining these in a disjunction encompassing all of the finitely many possibilities, we get a uniform set of formulae which work for all instances of $\varphi(x, \bar{y})$. \hfill \qed

The sets $L_i^\pi(x, \bar{a}) = \lambda_i^+(x, \bar{a}) \setminus \lambda_i^-(x, \bar{a})$ are called the layers of $\varphi(x, \bar{a})$, as in the terminology from Theorem 1.2.8. Each layer is a ucheese which is contained in the union of the holes of the previous layer. Note that if the formula $\varphi(x, \bar{a})$ defines a uball, then $\lambda_0^+(x, \bar{a}) = \varphi(x, \bar{a})$ and $\lambda_0^-(x, \bar{a}) = \emptyset$.

Remark 2.2.8. In the case where $\Psi$ is unpackable, we have an explicit definition of the layer decomposition. Assume $\Psi$ is unpackable, and hence that Swiss cheese decompositions are unique. Fix a formula $\varphi(x, \bar{y})$, and fix the finite set $\Psi_0 \subseteq \Psi$ as in Theorem 2.2.4. We construct formulae $\lambda_1^+(x, \bar{y})$ and $\lambda_1^-(x, \bar{y})$ defining the outer and inner uballs of each layer, respectively. We need two claims.

Claim 2.2.9. Suppose $\Psi$ is unpackable. For every formula $\tau(x, \bar{z}) \in \Psi$, there
is a formula $O_{\varphi,\tau}(\overline{y}, \overline{z})$ such that for any parameter tuples $\overline{a}$ and $\overline{b}$, the formula $O_{\varphi,\tau}(\overline{a}, \overline{b})$ holds iff $\tau(x, \overline{b})$ defines the outer ball of a Swiss cheese of $\varphi(x, \overline{a})$.

Proof. Any outer ball of a Swiss cheese of $\varphi(x, \overline{a})$ must be an instance of $\Psi_0$. So, to specify that $\tau(x, \overline{b})$ defines an outer ball, it suffices to say that for any instance $\psi(x, \overline{c})$ of $\Psi_0$:

- if $\psi(x, \overline{c})$ properly contains $\tau(x, \overline{b})$, then $\psi(x, \overline{c}) \backslash \tau(x, \overline{b})$ contains a point not in $\varphi(x, \overline{a})$; and

- if $\psi(x, \overline{c})$ is properly contained in $\tau(x, \overline{b})$, then $\tau(x, \overline{b}) \backslash \psi(x, \overline{c})$ contains a point from $\varphi(x, \overline{a})$.

Each of these conditions is expressible by a first-order formula. \(\square\)

Claim 2.2.10. Suppose $\Psi$ is unpackable. For every formula $\tau(x, \overline{z}) \in \Psi$, there is a formula $H_{\varphi,\tau}(\overline{y}, \overline{z})$ such that for any parameter tuples $\overline{a}$ and $\overline{b}$, $H_{\varphi,\tau}(\overline{a}, \overline{b})$ holds iff $\tau(x, \overline{b})$ defines a hole in a Swiss cheese of $\varphi(x, \overline{a})$.

Proof. Similar to the previous claim. \(\square\)

Continuing with the construction, the formulae $\lambda_i^+$ and $\lambda_i^-$ can be written in terms of the formulae $O$ and $H$. Let $\lambda^+_0(x, \overline{y})$ be the formula:

$$\bigvee_{\psi(x, \overline{z}) \in \Psi_0} \exists \overline{z} \left[ O_{\varphi,\psi}(\overline{y}, \overline{z}) \land \psi(x, \overline{z}) \right]$$

and let $\lambda^-_0(x, \overline{y})$ be the formula:

$$\bigvee_{\psi(x, \overline{z}) \in \Psi_0} \exists \overline{z} \left[ H_{\varphi,\psi}(\overline{y}, \overline{z}) \land \psi(x, \overline{z}) \right].$$
Then for any \( \bar{a} \), the set \( L_0(x, \bar{a}) = \lambda_0^+(x, \bar{a}) \setminus \lambda_0^-(x, \bar{a}) \) will be a single ucheese, made up of the “outer” Swiss cheeses of \( \varphi(x, \bar{a}) \) - those which are not contained in the holes of any other Swiss cheese of \( \varphi(x, \bar{a}) \).

For the next layer, use the same formulae, but relativized to \( \varphi(x, y) \land \neg L_0(x, y) \).

So, take \( \lambda_1^+(x, \bar{y}) \) to be the formula \( \bigvee_{\psi(x, z) \in \Psi_0} \exists z \left[ O_{\varphi \land \neg L_0, \psi}(\bar{y}, z) \land \psi(x, z) \right] \), and take \( \lambda_1^-(x, \bar{y}) \) to be the formula \( \bigvee_{\psi(x, z) \in \Psi_0} \exists z \left[ H_{\varphi \land \neg L_0, \psi}(\bar{y}, z) \land \psi(x, z) \right] \). Continuing in this fashion, the formulae \( \lambda_i^+(x, \bar{y}) \) and \( \lambda_i^-(x, \bar{y}) \) will have the desired properties.

### 2.2.3 Irreducible pieces

In the case where \( \Psi \) is almost unpackable, we get one more decomposition of definable sets, this time into pieces which are definable over any set we choose.

**Definition 2.2.11.** Let \( A \subseteq U \) be a small set.

1. A uball \( B \) is **definable over** \( A \) if there is a formula over \( A \) defining \( B \).

2. A uball \( B \) which is definable over \( A \) is **reducible over** \( A \) if there are non-empty disjoint uballs \( C_1 \) and \( C_2 \), each definable over \( A \), such that \( B = C_1 \cup C_2 \).

3. A uball \( B \) which is definable over \( A \) is **irreducible over** \( A \) (or \( A \)-irreducible) if it is not reducible over \( A \).

4. An **\( A \)-irreducible ucheese** is a set of the form \( C_1 \setminus C_2 \), where \( C_1 \) and \( C_2 \) are uballs definable over \( A \) and \( C_1 \) is irreducible over \( A \).

**Remark 2.2.12.** If \( B \) is an unbreakable ball which is definable over \( A \), then \( B \) is an \( A \)-irreducible uball.
Proposition 2.2.13. Suppose $\Psi$ is almost unpackable. Then any uball $B$ which is definable over $A$ is the disjoint union of finitely many uballs which are irreducible over $A$.

Proof. Follows from the previous remark by induction on the number of unbreakable balls making up $B$. □

Theorem 2.2.14 (Irreducible decomposition). Suppose $\Psi$ is almost unpackable. Let $A \subseteq U$ be any set. Let $\varphi(x, \bar{y})$ be any formula, and let $\bar{a} \in A$ be any parameters. Then $\varphi(x, \bar{a})$ is the disjoint union of finitely many $A$-irreducible ucheeses.

Proof. Fix a set $\varphi(x, \bar{a})$ which is definable over $A$, and let $\left(\lambda_0^+(x, \bar{a}) \setminus \lambda_0^-(x, \bar{a})\right), \left(\lambda_1^+(x, \bar{a}) \setminus \lambda_1^-(x, \bar{a})\right), \ldots, \left(\lambda_s^+(x, \bar{a}) \setminus \lambda_s^-(x, \bar{a})\right)$ be its layers. Since each set $\lambda_i^+(x, \bar{a})$ is an $A$-definable uball, we can write it as the disjoint union of $A$-irreducible uballs $B_{i_0}^0, B_{i_1}^1, \ldots, B_{i_s}^n$. Then $\left\{B_{i_j}^j \setminus \lambda_i^-(x, \bar{a}) : 0 \leq i \leq s, 0 \leq j \leq n_i\right\}$ is a set of disjoint $A$-irreducible ucheeses whose union is $\varphi(x, \bar{a})$. □

If the set $A$ is a model of $T$, then any $A$-irreducible uball is actually a ball, and the irreducible decomposition is just a Swiss cheese decomposition. The following results provide further parallels between balls and irreducible uballs, suggesting that irreducible ucheeses are in some sense the “correct” generalization of Swiss cheeses.

Proposition 2.2.15. Suppose $\Psi$ is almost unpackable. If $\varphi_0(x, \bar{a}_1)$ and $\varphi_1(x, \bar{a}_1)$ define $A$-irreducible uballs, then one of the following holds:

(i) $\varphi_0(x, \bar{a}_0) \subseteq \varphi_1(x, \bar{a}_1)$;

(ii) $\varphi_1(x, \bar{a}_1) \subseteq \varphi_0(x, \bar{a}_0)$;

(iii) $\varphi_0(x, \bar{a}_0) \cap \varphi_1(x, \bar{a}_1) = \emptyset$. 

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Proof. Suppose $\varphi_0(x, \overline{a}_0) \cap \varphi_1(x, \overline{a}_1) \neq \emptyset$. Then one of the unbreakable balls making up $\varphi_0(x, \overline{a}_0)$ intersects one of the unbreakable balls making up $\varphi_1(x, \overline{a}_1)$. Since they are balls, one is contained in the other; without loss of generality, some unbreakable ball of $\varphi_0(x, \overline{a}_0)$ is contained in an unbreakable ball of $\varphi_1(x, \overline{a}_1)$.

Let $B$ be the union of those unbreakable balls of $\varphi_0(x, \overline{a}_0)$ which are contained in some unbreakable ball of $\varphi_1(x, \overline{a}_1)$. Consider any $\sigma \in \text{Aut}(\mathbb{U}/A)$. Since $\sigma$ permutes the unbreakable balls of $\varphi_0(x, \overline{a}_0)$ and $\varphi_1(x, \overline{a}_1)$, and preserves subsets, it follows that $B$ is fixed by $\sigma$. So, $B$ must be definable over $A$ (see, for instance, Lemma III.2.3 of [22]), making $B$ and $\varphi_0(x, \overline{a}_0) \setminus B$ two $A$-definable disjoint uballs whose union is $\varphi_0(x, \overline{a}_0)$. To avoid violating irreducibility, $\varphi_0(x, \overline{a}_0) \setminus B$ must be empty; thus, $B = \varphi_0(x, \overline{a}_0)$, and so $\varphi_0(x, \overline{a}_0) \subseteq \varphi_1(x, \overline{a}_1)$.

Corollary 2.2.16. Suppose that $\Psi$ is almost unpackable. Then the intersection of two $A$-irreducible ucheeses is an $A$-irreducible ucheese.

Proposition 2.2.17. Suppose that $\Psi$ is almost unpackable. Then any 2-consistent family of $A$-irreducible ucheeses is consistent.

Proof. Let $\{S_1, S_2, \ldots, S_n\}$ be a 2-consistent set of $A$-irreducible ucheeses. Write $S_i = B_i \setminus C_i$, where $B_i$ and $C_i$ are $A$-definable uballs and $B_i$ is $A$-irreducible. The set $\{B_1, \ldots, B_n\}$ must be 2-consistent, so by Proposition 2.2.15 possibly after renumbering, $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$.

For each $i$, let $D_i = B_1 \cap C_i$. Because $B_1 \setminus C_i \neq \emptyset$ by 2-consistency, each $D_i$ is an $A$-definable uball and a proper subset of $B_1$. Since $B_1$ is $A$-irreducible, each unbreakable ball of $D_i$ is a proper subset of an unbreakable ball of $B_1$ (if not, “the unbreakable balls of $B_1$ which equal an unbreakable ball of $D_i$” and “the unbreakable balls of $B_1$ which do not” would be non-empty $A$-definable uballs whose union is $B_1$).
If $B_1 \setminus (D_1 \cup \cdots \cup D_n)$ were empty, then each unbreakable ball of $B_1$ would be covered by finitely many unbreakable balls from the sets $D_i$, a contradiction. So, take $b \in B_1 \setminus (D_1 \cup \cdots \cup D_n)$. Then $b \in D_i$ for all $i$, and $b \notin C_i$ for all $i$, so $b \in \bigcap S_i$.

Finally, we also have a notion of generic types for irreducible uballs.

**Proposition 2.2.18.** Suppose that $A$ and $B$ are sets, with $B \subseteq A$. Let $C$ be a uball which is definable over $B$ and which is irreducible over $A$. Then there is a unique type $p_C \in S_1(A)$, the generic type of the interior of $C$, which is definable over $B$, with the property that for any $A$-definable uball $D$, $D$ is in $p_C$ iff $C \subseteq D$.

**Proof.** Let $\tau(x, \bar{b})$ be a formula defining $C$, where $\bar{b} \in B$. Begin by building a partial type $p^*(x)$ over $A$, composed of $A$-definable uballs and their complements. If $\sigma(x, \bar{a})$ defines a uball, where $\bar{a} \in A$, then:

- if $\tau(x, \bar{b}) \rightarrow \sigma(x, \bar{a})$, then the formula $\sigma(x, \bar{a})$ goes in $p^*$;
- otherwise, the formula $\neg \sigma(x, \bar{a})$ goes in $p^*$.

If $p^*$ is not consistent, then there are finitely many $A$-definable uballs $D_1, \ldots, D_{m_1}$, $E_1, \ldots, E_{m_2}$ such that the set of formulae

$$\{ x \in D_i : i = 1, \ldots, m_1 \} \cup \{ x \notin E_i : i = 1, \ldots, m_2 \} \subseteq p^*$$

is inconsistent, and so $\bigcap_{i=1}^{m_1} D_i \subseteq \bigcup_{i=1}^{m_2} E_i$. But since $C$ is a subset of $\bigcap_{i=1}^{m_1} D_i$, it follows that $C$ is properly covered by some of the finitely many $A$-definable uballs $E_i$, contradicting the irreducibility of $C$.

Thus, $p^*$ is a $B$-definable partial type. It follows from Theorem [2.2.7] that $p^*$ generates a unique complete type $p_C$ over $A$: for any formula $\varphi(x, \bar{y})$, the layer
decomposition provides a uniform way to write each $A$-definable instance $\varphi(x, \bar{a})$ in terms of $A$-definable uballs. Define the type $p_C$ in terms of the definition scheme for $p^*$:

$$
\varphi(x, \bar{a}) \in p_C \iff \bigvee_{i=0}^{s} (\lambda_i^+(x, \bar{a}) \in p^*) \land (\lambda_i^-(x, \bar{a}) \notin p^*)
$$

Then $p_C \in S(A)$ is a $B$-definable complete type, which contains exactly those $A$-definable uballs that contain $C$ as a subset.

2.3 Relation to other model-theoretic notions

**Definition 2.3.1.** The VC-dimension of a formula $\varphi(\bar{x}, \bar{y})$, $\text{vc}(\varphi)$, is the VC-dimension of the family of all sets defined by instances $\varphi(\bar{x}, \bar{a})$ of $\varphi$; the VC-codimension $\text{vc}^*(\varphi)$ of $\varphi(\bar{x}, \bar{y})$ is defined similarly.

So, in a VC-minimal theory, every formula in a generating directed family $\Psi$ has VC-codimension 0 or 1.

We take as a definition for the independence property the following characterization, given by Laskowski in [17]:

**Definition 2.3.2.**

1. A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* if $\text{vc}(\varphi) = \infty$.

2. A complete theory $T$ is *NIP* if no formula $\varphi(x, \bar{y})$, where $x$ is a single variable, has the independence property.

The set of formulae which do not have the independence property is closed under boolean combinations (see, for instance, Remark 9 of [1]), so we immediately get the following:

**Proposition 2.3.3.** *Every VC-minimal theory is NIP.*
In fact, since dp-minimal theories are NIP, we can show something stronger.

**Definition 2.3.4.** A theory \( T \) is **dp-minimal** if there do not exist a model \( M \) of \( T \), formulae \( \varphi(x, y) \) and \( \psi(x, y) \), and mutually indiscernible sequences \( \langle a_i : i \in \omega \rangle \) and \( \langle b_i : i \in \omega \rangle \) in \( M \) such that for all \( i, j \in \omega \), the set

\[
\{ \varphi(x, a_i) \} \cup \{ \psi(x, b_j) \} \cup \{ \neg \varphi(x, a_k) : k \neq i \} \cup \{ \neg \psi(x, b_l) : l \neq j \}
\]

is consistent.

**Proposition 2.3.5** ([9], Theorem 4.2). Every VC-minimal theory is dp-minimal.

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**Figure 2.1.** Placement of VC-minimality among model-theoretic notions.

All implications are strict.

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Several other model-theoretic notions will appear as examples of VC-minimal theories in Chapter 3. As a result, we get the chart of implications given in Figure 2.1 (NTP₂ theories will appear in Chapter 4).

Finally, in VC-minimal theories we have a simplified method for determining whether or not the theory is stable.

**Definition 2.3.6.** A formula \( \varphi(x, y) \) has the strict order property if there exists a sequence \( \langle a_i : i \in I \rangle \) indexed by an infinite linear order \( I \) such that for all \( i < j \), we have that \( \varphi(x, a_i) \subsetneq \varphi(x, a_j) \).

**Definition 2.3.7.** A theory \( T \) is stable if for every formula \( \varphi(x, y) \), where \( x \) is a single variable, \( \varphi \) does not have the independence property and \( \varphi \) does not have the strict order property. (If some formula has the independence property or the strict order property, we say that \( T \) is unstable.)

**Proposition 2.3.8.** Suppose \( T \) is VC-minimal. Then \( T \) is stable iff no formula in \( \Psi \) has the strict order property.

**Proof.** By Proposition 2.3.3, \( T \) is stable iff no formula has the strict order property.

The left-to-right direction is clear.

Right-to-left: suppose \( T \) is unstable, and let \( \varphi \) be a formula which has the strict order property, witnessed by a sequence of tuples \( \langle a_i : i \in I \rangle \). Without loss of generality, assume each instance of \( \varphi \) is a single Swiss cheese. Fix the finite set \( \Psi_0 \subseteq \Psi \) as in Theorem 2.2.4, for each \( i \in I \), write the set \( \varphi(x, a_i) \) as \( B^i \setminus (C_1^i \cup \cdots \cup C_k^i) \), where \( B^i, C_1^i, \ldots, C_k^i \) are instances of \( \Psi_0 \).

If for infinitely many \( i \) we have that \( B^i \subsetneq B^{i+1} \), then since \( \Psi_0 \) is finite, there are a formula \( \psi \in \Psi_0 \) and infinitely many indices \( i_1, i_2, \ldots \) such that for each \( j \), the set \( B^{i_j} \) is an instance of \( \psi \) and \( B^{i_j} \subsetneq B^{i_{j+1}} \). Thus \( \psi \) has the strict order property.
If not, then the sequence of outer balls $B^i$ is eventually constant. It follows that (possibly after renumbering), for infinitely many $i$, $C^i_1 \supsetneq C^{i+1}_1$. It follows similarly that one formula $\psi \in \Psi_0$ defines infinitely many of these, and thus has the strict order property.

**Proposition 2.3.9.** If there exists a formula which branches, then $T$ is unstable.

*Proof.* If $\varphi$ branches, witnessed by parameters $\{a_t : t \in S\}$, let $\langle a_i : i \in \omega \rangle$ be parameters corresponding to any infinite path through the tree $S$. Then $\langle a_i \rangle$ witnesses that $\varphi$ has the strict order property. $\square$

**Corollary 2.3.10.** Suppose that $T$ is VC-minimal and stable. Then any finite generating directed family for $T$ must be almost unpackable.

*Proof.* Follows from Proposition 2.3.9 and Corollary 2.1.18. $\square$

**Remark 2.3.11.** If $T$ is unstable, then it is possible that some formula $\varphi$ may branch while $\Psi$ is almost unpackable – for instance, in any o-minimal theory (see Theorem 3.1.2), the formula $\varphi(x, y_1, y_2) = y_1 < x \leq y_2$ branches.
In this chapter, we present various examples of VC-minimal theories. We first look at several classes of theories which are VC-minimal: the strongly minimal, o-minimal, weakly o-minimal, and C-minimal theories, all of which appear in the chart of implications in Figure 2.1. We then present some individual theories which are VC-minimal and which exhibit interesting properties. Finally, for contrast, we end by noting several theories which have been shown not to be VC-minimal.

3.1 Examples

The following two theorems are widely known, having first appeared in [2].

**Theorem 3.1.1.** Every strongly minimal theory is VC-minimal.

*Proof.* Let $\Psi = \{x = y\}$. Each instance of $\Psi$ is a single point; these form a directed family, and generate all of the finite and cofinite sets. $\square$

**Theorem 3.1.2.** Every o-minimal theory is VC-minimal.

*Proof.* Let $\Psi = \{x > y, x \geq y\}$. Instances of $\Psi$ are sets of the form $(a, \infty)$ and $[a, \infty)$. Every point is constructible from $\Psi$, as $[a, \infty) \setminus (a, \infty)$, and every interval is constructible from $\Psi$, as $(a, b) = (a, \infty) \setminus [b, \infty)$, and so every definable set is constructible from $\Psi$. $\square$
In any o-minimal theory, the generating family $\Psi = \{x > y, x \geq y\}$ is unpackable. The set $\Psi' = \{x > y, x \geq y, x = y\}$ also serves as a generating directed family for any o-minimal theory, and $\Psi'$ is packable: for any $a$, the ball $[a, \infty)$ is the union of the balls $(a, \infty)$ and $\{a\}$. (However, $\Psi'$ is still almost unpackable.)

Every weakly o-minimal theory is also VC-minimal. Unlike the o-minimal theories, no single generating directed family works for every weakly o-minimal theory: the choice of $\Psi$ depends on the specific theory.

**Example 3.1.3.** Let $T = \text{Th}(\mathbb{Q}, <, P)$, where $P x$ is interpreted as $e < x < \pi$.

There are $\emptyset$-definable formulae determining whether or not $x > e$ and $x > \pi$:

\[
x > e \iff \exists y [y < x \land Py]
\]

\[
x > \pi \iff \exists y \exists z [y < z < x \land Py \land \neg Pz]
\]

Using these, the theory $T$ is VC-minimal, with generating directed family

\[
\Psi = \{x > y, x \geq y, x > e, x > \pi\}.
\]

As in the o-minimal case, for any weakly o-minimal theory, we can always choose the generating directed family $\Psi$ to be unpackable.

**Remark 3.1.4.** Adler notes in [2] that in any weakly o-minimal theory, we can take as a generating directed family the set of all formulae $\psi(x, \bar{y})$ with the property that every instance of $\psi$ is closed downward.

The following result is also widely known.

**Theorem 3.1.5.** Every $C$-minimal theory is VC-minimal.
Proof. By Example 2.5 of [18], the definable sets in any C-minimal theory are finite boolean combinations of what Adler in [2] calls an undirected VC-minimal instantiable family. By Proposition 6 of [2], any such family can be made into a directed family, and thus any C-minimal theory is VC-minimal.

Some C-minimal theories have unpackable generating directed families, while others appear not to. For example, the theory of any algebraically closed valued field, in the language \{+,’, 0, 1, v(x) ≥ v(y)\}, is C-minimal and hence VC-minimal; its standard generating family is the set of all open and closed valuation balls, i.e. instances of \{v(x - y_1) > v(y_2), v(x - y_1) ≥ v(y_2)\}, which is unpackable. The theory of the additive group of \(p\)-adic numbers, Th(\(\mathbb{Q}_p\), +, v(x) ≥ v(y)), is also C-minimal, and also uses the collection of valuation balls as a generating family, via \(\Psi = \{v(x - y_1) ≥ v(y_2)\}\). This \(\Psi\) is not even almost unpackable: every valuation ball in the \(p\)-adics is the union of \(p\) smaller balls, each of which is the union of \(p\) smaller balls, and so on. It seems unlikely that any alternate choice of \(\Psi\) could avoid this problem. However, the question of how to prove that no almost unpackable generating family for \(\mathbb{Q}_p\) exists is currently open.

3.1.1 Individual examples

Example 3.1.6. (from [3]) The theory of the additive group of integers, Th(\(\mathbb{Z}\), +), is VC-minimal.

Proof. This theory eliminates quantifiers in the language \(\mathcal{L} = \{+, P_2, P_3, P_4, \ldots\}\), where the predicate \(P_n\) denotes that \(x\) is divisible by \(n\). Consequently, all definable sets in a single \(x\) variable are eventually periodic.
For any $n \in \omega$, the relation "$x$ is equivalent to $y$ mod $n$" is definable:

$$x \equiv y \mod n \iff \exists z [x + z + z + \cdots + z = y]$$

If we let $\Psi$ be the set of formulae

$$\Psi = \{x = y\} \cup \{x \equiv y \mod n! : n \in \omega\},$$

then $\Psi$ generates a directed family, and the sets constructible from $\Psi$ are exactly those sets which differ from a periodic set at only finitely many points. \qedsymbol

$\text{Th}(\mathbb{Z}, +)$ is interesting as a natural example of a VC-minimal theory which is not strongly minimal, o-minimal, or C-minimal. The generating family $\Psi$ given above is infinite and not almost unpackable; any infinite subset of $\Psi$ gives a generating directed family with the same properties. No almost unpackable generating family for this theory is known.

**Example 3.1.7.** Consider the set $2^\omega$ of all infinite binary strings. Let $\mathcal{L}$ be a language consisting of a predicate $P_\sigma$ for each finite binary string $\sigma \in 2^{<\omega}$, where $P_\sigma x$ is interpreted as "$\sigma$ is an initial segment of $x$". Then $\text{Th}(2^\omega, \mathcal{L})$ is VC-minimal.

*Proof.* This theory eliminates quantifiers, and the predicates in $\mathcal{L}$ form a directed family, so we can use $\Psi = \{x = y\} \cup \{P_\sigma x : \sigma \in 2^{<\omega}\}$ as a generating directed family. This family is packable: every ball which is not a single point is the union of two smaller balls.

Many subsets of $\Psi$ also work as generating directed families for $T$; in particular, we can use the set $\Psi' = \{x = y\} \cup \{P_\sigma x : \sigma \in 2^{<\omega}, \sigma \text{ ends in } 0\}$ as a generating family, which is unpackable. \qedsymbol
This example demonstrates that it is possible for a VC-minimal theory to have more than one possible choice for its generating directed family, some of which are almost unpackable (as the set $\Psi'$ is above), and some of which are not (such as the set $\Psi$ given above). It is currently unknown whether any VC-minimal theory which has a finite almost unpackable generating family $\Psi$ can also have another finite generating family $\Psi'$ which fails to be almost unpackable.

**Example 3.1.8.** Let $\mathcal{L}$ be a language consisting of a single equivalence relation $E$, and let $T$ be a complete theory which states that for every natural number $n$, $E$ has exactly $n$ equivalence classes of size $n$. Then $T$ is VC-minimal.

*Proof.* Since this theory eliminates quantifiers, the set $\Psi = \{x = y,xEy\}$ is a generating directed family. Additionally, for any $n \in \omega$, let $P_n(x)$ be the formula which says “$x$ is equivalent to exactly $n$ objects”, let $Q_n(x)$ be the formula which says “$x$ is equivalent to at least $n$ objects”, and let $R_n(x)$ be the formula which says “$x$ is equivalent to at most $n$ objects”. Then if we add to $\Psi$ any subset of $\{P_n(x) : n \in \omega\} \cup \{Q_n(x) : n \in \omega\}$, we get another generating directed family for $T$; similarly, we get a new generating family if we instead add to $\Psi$ any subset of $\{P_n(x) : n \in \omega\} \cup \{R_n(x) : n \in \omega\}$. 

Each of the generating families given here for the theory $T$ is almost unpackable, but not unpackable; the unbreakable balls are the points and the infinite equivalence classes. The formula $xEy$ has both instances which are breakable balls and instances which are unbreakable balls. Additionally, since there are arbitrarily large finite equivalence classes, different instances of the formula $xEy$ are the unions of arbitrarily many unbreakable balls.
3.2 Non-examples

By Proposition 2.3.5, any theory which is not dp-minimal is not VC-minimal. However, showing that a dp-minimal theory is not VC-minimal is generally very difficult; for some time, only one example of such a theory was known, a theory in an uncountable language constructed in Proposition 3.6 of [9]. More recently, several natural examples of dp-minimal, non-VC-minimal theories have been found.

**Proposition 3.2.1.** *Presburger Arithmetic, the theory of $(\mathbb{Z}, +, <)$, is dp-minimal but not VC-minimal.*

This was shown independently in [3] and [12]. Also in [12] is the following example:

**Proposition 3.2.2.** *The theory of the $p$-adic field, $\text{Th}(\mathbb{Q}_p, +, \cdot, v(x) \geq v(y))$, is dp-minimal but not VC-minimal.*
This chapter covers material on dividing and forking which will be necessary as background for the main proofs of Chapters 5 and 6. In Section 4.1 we define dividing and forking, and list basic results describing their behavior in various classes of theories. We conclude by showing that in a VC-minimal theory with an unpackable generating directed family, forking and dividing are the same over any set. In Section 4.2 we define new generalizations of these notions, called $\beta$-dividing and $\beta$-forking, which allow us to work with dividing and forking while accommodating a tuple $\beta$ chosen from outside of our ambient model. We establish that, provided that $\beta$ is chosen carefully, $\beta$-forking and $\beta$-dividing behave very similarly to regular forking and dividing.

4.1 Definitions and basic results

Fix a complete theory $T$, and let $U$ be a large saturated model of $T$.

**Definition 4.1.1.** Fix a small set $A \subseteq U$. A formula $\varphi(\bar{x}, \bar{a})$ divides over $A$ if there is an $A$-indiscernible sequence $\langle \bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots \rangle$, where $\bar{a}_0 = \bar{a}$, such that the set $\{ \varphi(\bar{x}, \bar{a}_i) : i \in \omega \}$ is $k$-inconsistent for some $k$.

If a formula $\varphi(\bar{x}, \bar{a})$ divides over $A$, then by taking the images of the parameter tuple $\bar{a}$ under automorphisms fixing $A$, we can get infinitely many almost disjoint
copies of the set defined by \( \varphi \). The set of dividing formulae is not generally closed under disjunctions; for this reason, we often work instead with the related notion of forking.

**Definition 4.1.2.** Fix a small set \( A \subseteq U \). A formula \( \varphi(\overline{x}, \overline{a}) \) *forks over* \( A \) if there exist finitely many \( \varphi_i(\overline{x}, \overline{a}_i) \) such that each formula \( \varphi_i(\overline{x}, \overline{a}_i) \) divides over \( A \) and

\[
\varphi(\overline{x}, \overline{a}) \vdash \bigvee_{i=0}^{n} \varphi_i(\overline{x}, \overline{a}_i).
\]

There is also a notion of forking and dividing for types, in place of formulae.

**Definition 4.1.3.** Fix small sets \( A \subseteq B \). If \( p \) is a partial type over \( B \), then \( p \) *divides over* \( A \) (*forks over* \( A \)) if \( p \) implies a formula which divides (forks) over \( A \).

4.1.1 Forking in NTP\(_2\) theories

Much effort has gone into cataloguing the behavior of forking and dividing in various classes of theories. A recent, significant paper in the area is \([6]\), which deals with forking and dividing in NTP\(_2\) theories.

**Definition 4.1.4.** \( T \) has TP\(_2\), the *tree property of the second kind*, if there exist a formula \( \varphi(\overline{x}, \overline{y}) \), a natural number \( k \) and tuples of elements \( \langle \overline{a}_i^j : i, j \in \omega \rangle \) such that when we consider the array of formulae \( \langle \varphi(\overline{x}, \overline{a}_i^j) : i, j \in \omega \rangle \):

- every row is \( k \)-inconsistent: for all \( i \in \omega \), the set \( \{ \varphi(\overline{x}, \overline{a}_i^j) : j \in \omega \} \) is \( k \)-inconsistent;

- every vertical path through the array is consistent: for every \( \eta : \omega \to \omega \), the set \( \{ \varphi(\overline{x}, \overline{a}_{\eta(i)}^j) : i \in \omega \} \) is consistent.

**Definition 4.1.5.** We say that \( T \) is NTP\(_2\) if \( T \) does not have TP\(_2\).
The class of NTP\(_2\) theories includes all simple and NIP theories. In the paper [6], Chernikov and Kaplan cover a variety of results comparing dividing and forking in NTP\(_2\) theories.

**Definition 4.1.6.** A set \(A \subseteq U\) is an extension base for non-forking if types over \(A\) do not fork over \(A\).

The following is the main result of [6]. We omit any proof here; however, the main proof of Section 4.2 below will closely mirror it.

**Theorem 4.1.7.** Let \(T\) be NTP\(_2\), and suppose that \(A \subseteq U\) is an extension base for non-forking. Then forking over \(A\) equals dividing over \(A\): if \(\varphi(\overline{x}, \overline{a})\) is any formula, then \(\varphi(\overline{x}, \overline{a})\) forks over \(A\) iff it divides over \(A\).

**Corollary 4.1.8.** Let \(T\) be NTP\(_2\), and let \(M\) be any model of \(T\). Then forking over \(M\) equals dividing over \(M\).

### 4.1.2 Forking in dp-minimal theories

The following results on forking and consistency in dp-minimal theories are widely known; however, we were unable to find a reference for them. We include proofs here for completeness.

**Proposition 4.1.9.** Let \(T\) be dp-minimal. Let \(A \subseteq U\) be a small set, \(\varphi(\overline{x}, \overline{y})\) a formula over \(A\), and \(I = \langle \overline{b}_i : i \in \omega \rangle\) an \(A\)-indiscernible sequence. If the set \(\{\varphi(\overline{x}, \overline{b}_i) : i \in \omega\}\) is \((l(\overline{x}) + 1)\)-consistent, then it is consistent.

**Proof.** First, consider any tuple \(\overline{a} \in U\), and let \(m = l(\overline{a})\). By Theorem 2.7 of [16], the dp-rank of \(\text{tp}(\overline{a}/A)\) is at most \(m\). It follows from Proposition 2.3 of [16] that for any family \(I_0, \ldots, I_m\) of mutually \(A\)-indiscernible sequences, at least one of the sequences \(I_k\) is indiscernible over \(A\overline{a}\).
Let \( n = l(\bar{x}) \). Assume that for an \( A \)-indiscernible sequence \( I = \langle \bar{b}_i : i \in \omega \rangle \) the set \( \{ \varphi(\bar{x}, \bar{b}_i) : i \in \omega \} \) is \((n + 1)\)-consistent; we will show that it is consistent.

Let \( \langle \bar{c}_i : i \in Q \rangle \) be an \( A \)-indiscernible sequence with the same EM-type over \( A \) as \( I \): for any \( i_0 < i_1 < \cdots < i_k \in Q \) we have that \( \text{tp}(\tau_{i_0} \cdots \tau_{i_k}/A) = \text{tp}(\bar{b}_{i_0} \cdots \bar{b}_k/A) \).

We know that the set of formulae \( \{ \varphi(\bar{x}, \bar{c}_i) : i \in Q \} \) is \((n + 1)\)-consistent.

Pick \( \bar{a} \in U \) such that \( U \models \bigwedge_{i=0}^{n} \varphi(\bar{a}, \bar{c}_i) \).

For \( k = 0, \ldots, n \), let \( I_k \) be the indiscernible sequence \( \langle \bar{c}_i : k - \frac{1}{2} < i < k + \frac{1}{2} \rangle \).

This gives us \( n + 1 \) mutually \( A \)-indiscernible sequences \( I_0, \ldots, I_n \). At least one of them, say \( I_0 \), must be indiscernible over \( A \bar{a} \). So we have that \( U \models \varphi(\bar{a}, \bar{c}_i) \) for all \( \bar{c}_i \in I_0 \), and hence the set \( \{ \varphi(\bar{x}, \bar{c}_i) : i \in I_0 \} \) is consistent. Since \( I_0 \) has the same EM-type over \( A \) as \( I \), the set \( \{ \varphi(\bar{x}, \bar{b}_i) : i \in \omega \} \) is consistent as well. \( \square \)

**Corollary 4.1.10.** Suppose \( T \) is dp-minimal. Let \( M \subset U \) be a small model, \( \varphi(\bar{x}, \bar{y}) \) a formula, and \( \bar{b} \in U \). Then the following conditions are equivalent:

1. the formula \( \varphi(\bar{x}, \bar{b}) \) does not divide over \( M \);
2. the formula \( \varphi(\bar{x}, \bar{b}) \) does not fork over \( M \);
3. the set of formulae \( \{ \varphi(\bar{x}, \bar{b}') : \bar{b}' \equiv_M \bar{b}, \bar{b}' \in U \} \) is consistent;
4. the set of formulae \( \{ \varphi(\bar{x}, \bar{b}') : \bar{b}' \equiv_M \bar{b}, \bar{b}' \in U \} \) is \( (l(\bar{x}) + 1)\)-consistent.

**Proof.** The fact that (1), (2), and (3) are equivalent follows from Theorem 1 and Corollary 3.31 in [6].

Obviously (3) implies (4), and (4) implies (1) by the previous proposition. \( \square \)

### 4.1.3 Forking in VC-minimal theories

Finally, the following results establish that in VC-minimal theories with unpackable generating directed families, every set is an extension base for non-
forking, and thus forking equals dividing over any set.

**Proposition 4.1.11.** Let $T$ be a VC-minimal theory, with generating directed family $\Psi$, and suppose that $\Psi$ is unpackable. Let $A$ be any set, $\varphi(x, \overline{y})$ any formula, and $\overline{a} \in A$. Then $\varphi(x, \overline{a})$ does not fork over $A$.

**Proof.** Let $S_1, \ldots, S_n$ be the Swiss cheeses of $\varphi(x, \overline{a})$, and let $B_1, \ldots, B_n$ be their outer balls. For each $i = 1, \ldots, n$ let $q_i$ be the generic type $p_{B_i}$ of $B_i$ over $U$ as in Definition 2.1.7. Note that for each $i$ the formula $x \in S_i$ is in $q_i$, and so $\varphi(x, \overline{a}) \in q_i$.

Consider any $\sigma \in \text{Aut}(U/A)$. Since $\varphi(x, \overline{a})$ is over $A$, it is invariant under $\sigma$; since Swiss cheese decompositions are unique, $\sigma$ must permute $S_1, \ldots, S_n$. In particular, any such $\sigma$ permutes $B_1, \ldots, B_n$. So, the orbit of $q_1$ under $\text{Aut}(U/A)$ is a subset of $\{q_1, \ldots, q_n\}$, and is therefore finite. It follows that $q_1$ must not divide over $A$. Since $U$ is saturated, this implies that $q_1$ does not fork over $A$, and so neither does $\varphi(x, \overline{a})$.

**Corollary 4.1.12.** If $T$ is VC-minimal, with generating directed family $\Psi$, and $\Psi$ is unpackable, then every set is an extension base for non-forking, and hence forking equals dividing over any set.

**Proof.** By transitivity of forking, it suffices to check that 1-types over $A$ do not fork over $A$, which follows from Proposition 4.1.11.

### 4.2 $\beta$-dividing and $\beta$-forking

In proving the main result of Chapter 5, we will also need the following slight generalization of the standard definitions of forking and dividing.
Definition 4.2.1. Fix a $|U|^+$-saturated model $V \succ U$, and a small model $M \ll U$ of $T$. Fix also a tuple $\overline{\beta} \in V$ such that $\text{tp}(\overline{\beta}/U)$ is definable over $M$.

1. For any formula $\sigma(\overline{x}, \overline{\beta}, d)$, where $d \in U$, we say that $\sigma(\overline{x}, \overline{\beta}, d) \beta$-divides over $M$ if there is an $M$-indiscernible sequence $\langle d_i : i < \omega \rangle$ in $U$, with $d_0 = d$, such that $\{ \sigma(\overline{x}, \overline{\beta}, d_i) : i < \omega \}$ is $k$-inconsistent for some $k$.

2. For any formula $\sigma(\overline{x}, \overline{\beta}, d)$, where $d \in U$, we say that $\sigma(\overline{x}, \overline{\beta}, d) \beta$-forks over $M$ if it can be written as the disjunction of finitely many formulae $\sigma_i(\overline{x}, \overline{\beta}, \overline{c}_i)$, where $\overline{c}_i \in U$, each of which $\beta$-divides over $M$.

Notice that in the above definition we are permitted to take $M$-indiscernible sequences $\langle d_i \rangle$ from $U$ only, while $\overline{\beta}$ does not have to be in $U$. (If $\overline{\beta}$ is in $U$, then this definition coincides exactly with the original definitions of forking and dividing.) If a formula $\sigma(\overline{x}, \overline{\beta}, d) \beta$-divides over $M$, then it divides over $M\overline{\beta}$, but the converse generally does not hold.

Remark 4.2.2. If $\overline{\beta}$ is chosen as in Definition 4.2.1, then because the type $\text{tp}(\overline{\beta}/U)$ is definable over $M$, for any $d, d' \in U$ we get that $d \equiv_M d'$ iff $d \equiv_M \overline{\beta} d'$.

In order to prove our main result about $\overline{\beta}$-dividing below, we will need some background information about invariant types.

Definition 4.2.3. Let $q(\overline{y}) \in S(U)$ be an $M$-invariant type. A Morley sequence in $q$ over $M$ is a sequence $\langle \overline{\pi}_i : i \in \omega \rangle$ such that $\overline{\pi}_0 \models q \upharpoonright M$, $\overline{\pi}_1 \models q \upharpoonright M\overline{\pi}_0$, $\overline{\pi}_2 \models q \upharpoonright M\overline{\pi}_0\overline{\pi}_1$, and so on, with $\overline{\pi}_n \models q \upharpoonright M\overline{\pi}_0\ldots\overline{\pi}_{n-1}$ for any $n \in \omega$. Because $q$ is $M$-invariant, any such sequence is $M$-indiscernible.

Proposition 4.2.4. Let $q \in S(U)$ be invariant over $M$. If $I$ and $J$ are two Morley sequences in $q$ over $M$, then $I \equiv_M J$. 
Definition 4.2.5. A type \( q(\overline{y}) \in S(\overline{U}) \) is called strictly invariant over \( M \) if it is \( M \)-invariant and for any set \( B \supseteq M \) and any realization \( \overline{\pi} \) of \( q \upharpoonright B \), the type \( \text{tp}(B/M\overline{\pi}) \) does not fork over \( M \).

Claim 4.2.6. Suppose that \( T \) is an NTP\(_2 \) theory, and let \( M, \forall, \text{ and } \overline{\beta} \) be as in Definition 4.2.1. Suppose that \( \varphi(\overline{x}, \overline{\beta}, \overline{c}) \) \( \overline{\beta} \)-divides over \( M \), and let \( q(\overline{y}) \in S(\overline{U}) \) be a type extending \( \text{tp}(\overline{c}/M) \) which is strictly invariant over \( M \). If \( \langle \overline{b}_i : i \in \omega \rangle \subseteq \overline{U} \) is a Morley sequence in \( q(\overline{y}) \) over \( M \), then the set of formulae \( \{ \varphi(\overline{x}, \overline{\beta}, \overline{b}_i) : i \in \omega \} \) is inconsistent.

Proof. This proof is modeled after Adler’s exposition of Lemma 3.14 in \([6]\).

Let \( I_0 = \langle \overline{\tau}_i^0 : i \in \omega \rangle \) be any sequence witnessing the \( \overline{\beta} \)-dividing of \( \varphi(\overline{x}, \overline{\beta}, \overline{c}) \): \( I_0 \subseteq \overline{U} \) is an \( M \)-indiscernible sequence, \( \overline{\tau}_0^0 = \overline{\tau} \), and the set \( \{ \varphi(\overline{x}, \overline{\beta}, \overline{\tau}_i) : i \in \omega \} \) is \( k \)-inconsistent for some \( k \). We will build a sequence of sequences \( \langle I_i : i \in \omega \rangle \) beginning with \( I_0 \), where each \( I_i = \langle \overline{\tau}_j^i : j \in \omega \rangle \), such that each sequence \( I_i \) is indiscernible over \( MI_0I_1 \cdots I_{i-1} \) and \( I_i \equiv_M I_0 \) for all \( i \).

Let \( \overline{\tau}_0 = \overline{\tau} \), and let \( \overline{\tau}_1 \models q \upharpoonright MI_0 \). Because \( \overline{\tau}_1 \equiv_M \overline{\tau}_0 \), there is an \( M \)-indiscernible sequence \( J \) beginning with \( \overline{\tau}_1 \) witnessing \( \overline{\beta} \)-dividing: just take any \( \sigma \in \text{Aut}(\overline{U}/M) \) mapping \( \overline{\tau}_0 \) to \( \overline{\tau}_1 \), and let \( J = \sigma(I_0) \).

By strict invariance, we know that \( \text{tp}(I_0/M\overline{\tau}_1) \) does not fork over \( M \). So, by Lemma 3.1 of \([5]\), there is a sequence \( I_1 \equiv_{MI_0} J \) such that \( I_1 \) is indiscernible over \( MI_0 \). Because \( J \) is \( MI_0 \)-indiscernible and \( \overline{\tau}_1 \models q \upharpoonright MI_0 \), for all \( i \in \omega \) we get that \( \overline{\tau}_i \models q \upharpoonright MI_0 \).

Continue in this fashion: suppose we have \( n \) sequences \( I_0, \ldots, I_{n-1} \), each \( M \)-equivalent to \( I_0 \), such that \( I_i \) is indiscernible over \( MI_0 \cdots I_{i-1} \) for all \( i < n \). Let \( \overline{\tau}_n \models q \upharpoonright MI_0I_1 \cdots I_{n-1} \), and choose an \( M \)-indiscernible sequence \( J \equiv_M I \) beginning with \( \overline{\tau}_n \). Using Lemma 3.1 from \([5]\), we get an \( MI_0 \cdots I_{n-1} \)-indiscernible
sequence \( I_n \) such that \( I_n \equiv_{MI_n} J \), and each tuple in the sequence \( I_n \) realizes the type \( q \restriction MI_0 \cdots I_{n-1} \).

Consider the array \( \langle c^i_j : i, j \in \omega \rangle \). Every vertical path through this array is a Morley sequence in \( q \) over \( M \): for any \( \eta : \omega \to \omega \), we know that \( c^i_{\eta(i)} \) realizes the type \( q \restriction MI_0 \cdots I_{n-1} \), so it certainly realizes the subtype \( q \restriction MI_{\eta(0)} \ldots c^{i-1}_{\eta(i-1)} \).

In the array of formulae \( \langle \varphi(x, \beta, c^i_j) : i, j \in \omega \rangle \), each row \( \{ \varphi(x, \beta, c^i_j) : j \in \omega \} \) is \( k \)-inconsistent, since \( I_i \equiv_M I_0 \) and \( I_0 \) witnesses \( \beta \)-dividing. If every vertical path through the array were consistent, it would violate NTP\(_2\). So, for some \( \eta : \omega \to \omega \), the set \( \{ \varphi(x, \beta, c^i_{\eta(i)}) : i \in \omega \} \) is inconsistent: there exist indices \( i_1, i_2, \ldots, i_m \) such that \( \forall \models \neg \exists x \left[ \bigwedge_{i=1}^m \varphi(x, \beta, c^i_{\eta(i,j)}) \right] \).

Therefore, we get that the formula \( \neg \exists x \left[ \bigwedge_{j=1}^m \varphi(x, \beta, c^i_{\eta(j)}) \right] \) is in the type \( \text{tp}(\beta/\mathbb{U}) \); since this type is definable over \( M \), there is a formula \( d(z_1, \ldots, z_m) \) with parameters in \( M \) defining it, and \( \mathbb{U} \models d\left( c^i_{\eta(i_1)}, \ldots, c^i_{\eta(i_m)} \right) \). It follows that \( d(z_1, \ldots, z_m) \) is in the type of the Morley sequence \( \langle c^i_j : i \in \omega \rangle \) over \( M \).

\( b_i : i \in \omega \) is also a Morley sequence in \( q \) over \( M \), so by Proposition 4.2.4 the formula \( d(z_1, \ldots, z_m) \) is in the type of \( \langle b_i \rangle \) over \( M \), too. Therefore the set \( \{ \varphi(x, \beta, b_i) : i \in \omega \} \) is inconsistent, and so \( \langle b_i \rangle \) also witnesses \( \beta \)-dividing.

**Proposition 4.2.7.** Let \( T \) be an NTP\(_2\) theory, and let \( M, \mathbb{V}, \) and \( \bar{\beta} \) be as in Definition 4.2.1. Let \( \bar{a}_1, \bar{a}_2 \in \mathbb{U} \), and suppose that \( \sigma_1(\bar{x}, \bar{\beta}, \bar{a}_1) \) and \( \sigma_2(\bar{x}, \bar{\beta}, \bar{a}_2) \) each \( \bar{\beta} \)-divide over \( M \). Then \( \sigma_1(\bar{x}, \bar{\beta}, \bar{a}_1) \lor \sigma_2(\bar{x}, \bar{\beta}, \bar{a}_2) \) also \( \bar{\beta} \)-divides over \( M \).

**Proof.** Taking \( \bar{a} = \bar{a}_1 \bar{a}_2 \) if necessary, we may assume that \( \bar{a}_1 = \bar{a}_2 = \bar{a} \). Since \( M \) is a model, by Corollary 3.29 of [6], there is a global type \( q(\bar{y}) \in S(\mathbb{U}) \) extending \( \text{tp}(\bar{a}/M) \) which is strictly invariant over \( M \). Let \( I = \langle b_i : i \in \omega \rangle \subseteq \mathbb{U} \) be a Morley sequence in \( q(\bar{y}) \) over \( M \).
Assume $\sigma_1(\bar{x}, \bar{\beta}, \bar{a}) \lor \sigma_2(\bar{x}, \bar{\beta}, \bar{a})$ does not $\bar{\beta}$-divide over $M$. Then the set of formulae \{\$\sigma_1(\bar{x}, \bar{\beta}, \bar{b}_i) \lor \sigma_2(\bar{x}, \bar{\beta}, \bar{b}_i): i \in \omega\}$ is consistent, and we can find $\bar{a} \in \mathcal{V}$ realizing this set.

Then one of the sets $\{i \in \omega: \mathcal{V} \models \sigma_1(\bar{a}, \bar{\beta}, \bar{b}_i)\}$ or $\{i \in \omega: \mathcal{V} \models \sigma_2(\bar{a}, \bar{\beta}, \bar{b}_i)\}$ must be infinite, and hence by Claim 4.2.6, one of $\sigma_1$ and $\sigma_2$ does not $\bar{\beta}$-divide over $M$.

\begin{flushright}
$\square$
\end{flushright}

**Corollary 4.2.8.** Suppose that $T$ is an NTP$_2$ theory, let $M$, $\mathcal{V}$, and $\bar{\beta}$ be as in Definition 4.2.1, and let $\bar{a} \in \mathcal{U}$. Then a formula $\sigma(\bar{x}, \bar{\beta}, \bar{a})$ $\bar{\beta}$-forks over $M$ if and only if it $\bar{\beta}$-divides over $M$. 

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In this chapter, we give an alternate characterization of forking over models for formulae in VC-minimal theories with unpackable generating directed families. Throughout, let $T$ be a complete first-order theory, and let $U$ be a large saturated model of $T$.

5.1 History

The origins of this characterization of forking are in stable theories, where forking is generally well-behaved. There are many equivalent definitions of stability, including the following:

**Theorem 5.1.1.** The following are equivalent for any complete theory $T$:

1. $T$ is stable;
2. every complete type over a model is definable;
3. a global type $p \in S(U)$ is definable over a small model $M \preceq U$ iff it does not fork over $M$.

Resulting from this, we get an alternate characterization of forking over models in stable theories.
Theorem 5.1.2. Suppose that $T$ is stable. Then a formula $\varphi(\overline{x}, \overline{c})$ does not fork over a model $M$ if and only if there is a global $M$-definable type $p(\overline{x}) \in S(U)$ such that $\varphi(\overline{x}, \overline{c}) \in p(\overline{x})$.

Proof. Given a non-forking formula $\varphi(\overline{x}, \overline{c})$, let $p(\overline{x})$ be any global non-forking type containing $\varphi(\overline{x}, \overline{c})$. By Theorem [5.1.1], $p(\overline{x})$ must be definable over $M$. \qed

Although o-minimal theories are unstable, in [8] Dolich showed that the same characterization as in Theorem [5.1.2] holds in many o-minimal theories, and in fact holds for forking over arbitrary sets. In the following theorem, nice is a technical condition satisfied by many o-minimal theories; for example, any o-minimal expansion of a field is nice.

Theorem 5.1.3. Suppose $T$ is a nice o-minimal theory; let $A \subseteq U$ be any set. Then for any formula $\varphi(\overline{x}, \overline{c})$ the following are equivalent:

(1): $\varphi(\overline{x}, \overline{c})$ does not fork over $A$;

(2): the set $\{ \varphi(\overline{x}, \overline{c}') : \overline{c}' \equiv_A \overline{c} \}$ is consistent;

(3): there is a global $A$-definable type $p \in S_{\overline{x}}(U)$ containing $\varphi(\overline{x}, \overline{c})$.

5.2 Main result

In this section, we demonstrate that the characterization of forking given in Theorems [5.1.2] and [5.1.3] holds with respect to forking over models in VC-minimal theories with unpackable generating directed family $\Psi$.

Theorem 5.2.1. Suppose that $T$ is VC-minimal, with generating directed family $\Psi$, and suppose $\Psi$ is unpackable. Let $M \preceq U$ be a small model of $T$. Let $\varphi(\overline{x}, \overline{y})$ be any formula, and $\overline{c}$ any parameter tuple from $U$. Then the following are equivalent:
(I) $\varphi(\overline{x}, \overline{c})$ does not fork over $M$;

(II) there is a global type $p \in S_{\overline{x}}(U)$ which is definable over $M$ with $\varphi(\overline{x}, \overline{c}) \in p$.

Proof. (II) $\Rightarrow$ (I): This is straightforward, and follows from the definitions of forking and definable types.

(I) $\Rightarrow$ (II): We proceed by induction on $l(\overline{x})$.

Base Case: $x$ is a single variable.

Suppose that $\varphi(x, \overline{c})$ does not fork over $M$.

Consider the Swiss cheese decomposition of $\varphi(x, \overline{c})$. One of the Swiss cheeses must not divide over $M$, and thus does not fork over $M$. Replace $\varphi(x, \overline{c})$ with this Swiss cheese; any type containing it will also contain the original $\varphi(x, \overline{c})$.

We will henceforth assume that $\varphi(x, \overline{c})$ is a single Swiss cheese; it follows that

$\{\varphi(x, \overline{c}') : \overline{c}' \equiv_M \overline{c}, \overline{c}' \in U}\}$

is consistent by Corollary 4.1.10. In particular, note that the set $\{\varphi(x, \overline{c}') : \overline{c}' \equiv_M \overline{c}, \overline{c}' \in U}\}$ is 2-consistent.

Letting $q(\overline{y}) = tp(\overline{c}/M)$, we have that $q(\overline{y}_1) \cup q(\overline{y}_2) \vdash \exists x [\varphi(x, \overline{y}_1) \land \varphi(x, \overline{y}_2)]$. By Compactness, there must be a single formula $\theta(\overline{y}) \in q(\overline{y})$ with parameters in $M$ such that $\theta(\overline{y}_1) \land \theta(\overline{y}_2) \vdash \exists x [\varphi(x, \overline{y}_1) \land \varphi(x, \overline{y}_2)]$. Modifying $\theta(\overline{y})$ if necessary, we may also assume it implies that $\varphi(x, \overline{y})$ is a single Swiss cheese.

Using $\theta$, we will build the desired type $p$. Informally, it describes points which are generic in the set

$\bigcap_{\varphi(x, \overline{c}') : \overline{c}' \equiv_M \overline{c}, \overline{c}' \in U} \varphi(x, \overline{c}')$.

First, build a complete $\Psi$-type $p^*(x)$ over $U$, such that a $U$-definable ball $B$ will be in $p^*$ if $\varphi(x, \overline{c}') \subseteq B$ for some $\overline{c}' \in \theta(U)$; otherwise, the complement of $B$ will be in $p^*$. Then $p^*$ is consistent: if not, by Compactness there is some finite part of $p^*$, $\{\psi_i(x, \overline{d}_i) : i = 1, \ldots, n\} \cup \{-\psi_i(x, \overline{d}_i) : i = n+1, \ldots, m\}$ which is inconsistent. Let $\overline{c}_1, \ldots, \overline{c}_n \in \theta(U)$ witness that $\psi_1(x, \overline{d}_1), \ldots, \psi_n(x, \overline{d}_n)$ are in $p^*$: $\varphi(x, \overline{c}_i) \subseteq \psi_i(x, \overline{d}_i)$ for each $i = 1, \ldots, n$. As $\{\varphi(x, \overline{c}_i) : i = 1, \ldots, n\}$ is a 2-consistent set of Swiss
cheeses, their outer balls must form a chain; let $B$ be the smallest of the outer balls. It follows that the ball $B$ would be covered by finitely many other balls (the holes of each $\varphi(x, c_i)$ and the balls $\{\psi_i(x, d_i) : i = n + 1, \ldots, m\}$), contradicting unpackability.

We also get that $p^*$ is definable over $M$: for any $\psi(x, z) \in \Psi$ and any $b \in U$, $\psi(x, b) \in p^*$ if and only if $U \models \exists \overline{y}[\theta(\overline{y}) \land \varphi(x, \overline{y}) \subseteq \psi(x, b)]$.

By Proposition 2.1.6, $p^*$ extends uniquely to a complete type $p \in S_1(U)$, which is also definable over $M$. Finally, according to the definition of $p^*$, the outer ball of $\varphi(x, \overline{c})$ is in $p^*$, but, since $\{\varphi(x, \overline{c'}) : \overline{c'} \in \theta(U)\}$ is 2-consistent, each of the holes of $\varphi(x, \overline{c})$ is not; thus, $\varphi(x, \overline{c}) \in p$.

**Inductive step:** Assume that $(I) \Rightarrow (II)$ holds for all formulae where $l(\overline{x}) = n$.

We prove it here for formulae in which $l(\overline{x}) = n + 1$.

Let $\varphi(\overline{x}, \overline{c}) = \varphi(x_0, x_1, \ldots, x_n, \overline{c})$, and suppose $\varphi(\overline{x}, \overline{c})$ does not fork over $M$. By Corollary 4.1.10, the family

$$\{\varphi(\overline{x}, \overline{c'}) : \overline{c'} \equiv_M \overline{c}, \overline{c'} \in U\}$$

is consistent. In particular, this family is $2(n + 1)$-consistent.

Take $\theta(\overline{y}) \in \text{tp}(\overline{x}/M)$ to be a formula guaranteeing $2(n + 1)$-consistency:

$$U \models \bigwedge_{i=1}^{2(n+1)} \theta(\overline{y}_i) \rightarrow \exists x_0 \exists x_1 \ldots \exists x_n \left[ \bigwedge_{i=1}^{2(n+1)} \varphi(\overline{x}, \overline{y}_i) \right].$$

Letting $\varphi^*(x_1, \ldots, x_n, \overline{y}_1, \overline{y}_2)$ be the formula

$$\exists x_0 [\varphi(x_0, x_1, \ldots, x_n, \overline{y}_1) \land \varphi(x_0, x_1, \ldots, x_n, \overline{y}_2)],$$
the set \( \{ \varphi^*(x_1, \ldots, x_n, \bar{c}_1, \bar{c}_2) : \bar{c}_1, \bar{c}_2 \in \theta(U) \} \) must be \((n+1)\)-consistent. It follows that for any choice of tuples \( \bar{c}_1, \bar{c}_2 \in \theta(U) \), the family

\[
\{ \varphi^*(x_1, \ldots, x_n, \bar{c}_1', \bar{c}_2') : \bar{c}_1' \equiv_M \bar{c}_1, \bar{c}_2' \equiv_M \bar{c}_2, \bar{c}_1', \bar{c}_2' \in U \}
\]

is \((n+1)\)-consistent. Therefore, by Corollary 4.1.10, the formula \( \varphi^*(x_1, \ldots, x_n, \bar{c}_1, \bar{c}_2) \) does not fork over \( M \) for each \( \bar{c}_1, \bar{c}_2 \in \theta(U) \).

Applying the inductive hypothesis, we get that for each pair \( \bar{c}_1, \bar{c}_2 \in \theta(U) \), there is a type \( q_{\bar{c}_1, \bar{c}_2} \in S_n(U) \) which contains \( \varphi^*(x_1, \ldots, x_n, \bar{c}_1, \bar{c}_2) \) and which is definable over \( M \).

**Claim 5.2.2.** There are finitely many \( M \)-definable types \( q_1, \ldots, q_k \in S_n(U) \) such that for every \( \bar{c}_1, \bar{c}_2 \in \theta(U) \), the formula \( \varphi^*(x_1, \ldots, x_n, \bar{c}_1, \bar{c}_2) \) is contained in one of the types \( q_1, \ldots, q_k \).

**Proof.** Since each type \( q_{\bar{c}_1, \bar{c}_2} \) is \( M \)-definable, for each \( \bar{c}_1, \bar{c}_2 \in \theta(U) \) there is a formula \( d_{q_{\bar{c}_1, \bar{c}_2}}(\bar{y}_1, \bar{y}_2) \) over \( M \) which defines \( \varphi^* \) for the type \( q_{\bar{c}_1, \bar{c}_2} \). Any \( \bar{c}_1, \bar{c}_2 \) taken from \( \theta(U) \) must satisfy \( d_{q_{\bar{c}_1, \bar{c}_2}}(\bar{y}_1, \bar{y}_2) \), and so the set of formulae

\[
\{ \theta(\bar{y}_1) \land \theta(\bar{y}_2) \} \cup \{ \neg d_{q_{\bar{c}_1, \bar{c}_2}}(\bar{y}_1, \bar{y}_2) : \bar{c}_1, \bar{c}_2 \in \theta(U) \}
\]

has no realization in \( U \).

Since every formula \( d_{q_{\bar{c}_1, \bar{c}_2}}(\bar{y}_1, \bar{y}_2) \) is over \( M \), by Compactness, there are finitely many types \( q_1, \ldots, q_k \) taken from among the types \( \{ q_{\bar{c}_1, \bar{c}_2} : \bar{c}_1, \bar{c}_2 \in \theta(U) \} \) such that

\[
\{ \theta(\bar{y}_1) \land \theta(\bar{y}_2), \neg d_{q_1}(\bar{y}_1, \bar{y}_2), \ldots, \neg d_{q_k}(\bar{y}_1, \bar{y}_2) \}
\]

is inconsistent, and so \( U \models \left[ \theta(\bar{y}_1) \land \theta(\bar{y}_2) \right] \rightarrow \bigvee_{i=1}^k d_{q_i}(\bar{y}_1, \bar{y}_2) \). \( \Box \)
Henceforth, we will work in a $|\mathbb{U}|^+$-saturated elementary extension $\mathbb{V} \models \mathbb{U}$.

Let $\overline{X}$ denote the tuple of variables $x_1 x_2 \ldots x_n$, and let $q_1(\overline{X}), \ldots, q_k(\overline{X})$ be $M$-definable types from $S_n(\mathbb{U})$ as in Claim 5.2.2. Within $\mathbb{V}$, choose realizations of $q_1, \ldots, q_k$ as follows: let $\overline{\beta}_1 \models q_1$. We know $q_2$ is an $M$-definable type; use its definition scheme to extend $q_2$ to a type over $\mathbb{U}\overline{\beta}_1$, and take $\overline{\beta}_2$ to realize the extended type $q_2|\mathbb{U}\overline{\beta}_1$. Similarly, let $\overline{\beta}_3 \models q_3|\mathbb{U}\overline{\beta}_1\overline{\beta}_2$, and so on, up through $\overline{\beta}_k \models q_k|\mathbb{U}\overline{\beta}_1 \ldots \overline{\beta}_{k-1}$.

Finally, let $\overline{\beta} \in \mathbb{V}$ be the tuple $\overline{\beta}_1 \ldots \overline{\beta}_k$. It follows that $\text{tp}(\overline{\beta}/\mathbb{U})$ is $M$-definable. As in Remark 4.2.2, note that for any $d, d' \in \mathbb{U}$, we have that $\overline{d} \equiv_M \overline{d}'$ if and only if $\overline{d} \equiv_M \overline{\beta} \overline{d}'$.

Let $\overline{X}_1, \ldots, \overline{X}_k$ be $k$ disjoint copies of the tuple $\overline{X}$, and let $\varphi(x_0, \overline{X}_1, \ldots, \overline{X}_k, \overline{y})$ be the formula $\bigvee_{i=1}^k \varphi(x_0, \overline{X}_i, \overline{y})$. Consider the family $\{ \varphi(x_0, \overline{\beta}, \overline{c}') : \overline{c}' \in \theta(\mathbb{U}) \}$. For any pair of tuples $\overline{c}_1, \overline{c}_2 \in \theta(\mathbb{U})$, there must be some index $j$ such that $\varphi^*(\overline{X}, \overline{c}_1, \overline{c}_2) \in q_j$, and so $\mathbb{V} \models \exists x_0[\varphi(x_0, \overline{\beta}_j, \overline{c}_1) \land \varphi(x_0, \overline{\beta}_j, \overline{c}_2)]$. As a result, we know that $\mathbb{V} \models \exists x_0[\bigvee_{i=1}^k \varphi(x_0, \overline{\beta}_i, \overline{c}_1) \land \bigvee_{i=1}^k \varphi(x_0, \overline{\beta}_i, \overline{c}_2)]$, which gives us that $\mathbb{V} \models \exists x_0[\varphi(x_0, \overline{\beta}, \overline{c}_1) \land \varphi(x_0, \overline{\beta}, \overline{c}_2)]$.

So, the family $\{ \varphi(x_0, \overline{\beta}, \overline{c}') : \overline{c}' \in \theta(\mathbb{U}) \}$ is 2-consistent. Note that because we only consider tuples $\overline{c}'$ in $\mathbb{U}$, and the tuple $\overline{\beta}$ is not in $\mathbb{U}$, this is not a definable family.

Remark 5.2.3. Our goal now is to construct a type $r(x) \in S_1(\mathbb{U}\overline{\beta})$ which contains $\varphi(x, \overline{\beta}, \overline{c})$ and which is definable over $M\overline{\beta}$ — i.e. for every formula $\sigma(x, \overline{v}, \overline{z})$, there is a formula $d_r(\overline{v}, \overline{z})$ over $M$ such that for any $\overline{c} \in \mathbb{U}$, we have that $\sigma(x, \overline{\beta}, \overline{c}) \in r(x)$ if and only if $\mathbb{V} \models d_r(\overline{\beta}, \overline{c})$. This type, combined with the $M$-definable type $\text{tp}(\overline{\beta}/\mathbb{U})$, will give us the desired $M$-definable type in $S_{n+1}(\mathbb{U})$. Our overall strategy is similar to that used in the Base Case. However, since the set $\mathbb{U}\overline{\beta}$ is not a model, we cannot necessarily decompose $\varphi(x, \overline{\beta}, \overline{c})$ into Swiss cheeses which are defined over
Instead, we will use \( U \beta \)-irreducible ucheeses. We will also work with \( \beta \)-forking and \( \beta \)-dividing.

First, note that the formula \( \hat{\varphi}(x, \beta, c) \) does not \( \beta \)-divide over \( M \): if the sequence \( I = \langle c_0 = c, c_1, c_2, \ldots \rangle \subseteq U \) is indiscernible over \( M \beta \) by Remark 4.2.2. Since we know \( \{ \hat{\varphi}(x, \beta, c_i) : i \in \omega \} \) is 2-consistent, by Proposition 4.1.9 it must be consistent. Hence, \( \hat{\varphi}(x, \beta, c) \) must not \( \beta \)-divide over \( M \), and so by Corollary 4.2.8 it does not \( \beta \)-fork over \( M \).

As \( \hat{\varphi}(x, \beta, \bar{c}) \) is a formula over \( U \beta \), by Theorem 2.2.14 it decomposes into finitely many disjoint \( U \beta \)-irreducible ucheeses. One of these ucheeses must not \( \beta \)-fork over \( M \); let \( \tau(x, \beta, d) \) be the formula defining this \( U \beta \)-irreducible ucheese.

Notice that \( V \models \tau(x, \beta, d) \rightarrow \hat{\varphi}(x, \beta, c) \).

**Claim 5.2.4.** The family \( \{ \tau(x, \beta, d) : d \equiv_M \bar{d}, \bar{d} \in U \} \) is 2-consistent.

**Proof.** This proof is modeled on that of Proposition 2.1 in [15].

Since \( \tau(x, \beta, d) \) does not \( \beta \)-fork over \( M \), there is a complete type \( \Gamma(x) \in S_1(U \beta) \) that does not \( \beta \)-fork over \( M \), i.e. every formula in \( \Gamma(x) \) does not \( \beta \)-fork.

Let \( \bar{d} \in U \) with \( \bar{d} \equiv_M \bar{d} \). It suffices to show that \( \tau(x, \beta, \bar{d}) \) is in \( \Gamma \) for any such \( \bar{d} \).

Suppose not: assume that \( \neg \tau(x, \beta, \bar{d}) \) is in \( \Gamma \). Let \( q = \text{tp}(\bar{d}/M) \), and let \( q^* \in S(U) \) be a coheir of \( q \). As a coheir, the type \( q^* \) is \( M \)-invariant.

Let \( I = \langle \bar{d}_1, \bar{d}_2, \bar{d}_3, \ldots \rangle \subseteq U \) be a Morley sequence in \( q^* \) over \( M \bar{d} \bar{d}' \), i.e. for all \( i, \bar{d}_{i+1} \models q^* \upharpoonright M \bar{d} \bar{d}' \bar{d}_1 \ldots \bar{d}_i \). Note that the sequences \( \bar{d}I \) and \( \bar{d}'I \) are also Morley sequences in \( q^* \) over \( M \), and are therefore both \( M \)-indiscernible.

Now, consider \( \tau(x, \beta, \bar{d}_1) \).

**Case 1:** \( \tau(x, \beta, \bar{d}_1) \) is in \( \Gamma \).

Let \( \bar{d}_0 = \bar{d} \). Since \( \bar{d}I \) is \( M \)-indiscernible, so is \( J = \langle \bar{d}_0 \bar{d}_1, \bar{d}_2 \bar{d}_3, \bar{d}_4 \bar{d}_5, \ldots \rangle \).

Because \( \neg \tau(x, \beta, \bar{d}_0) \land \tau(x, \beta, \bar{d}_1) \) is in \( \Gamma \), it does not \( \beta \)-divide over \( M \), and so the
set \( \{ \neg \tau(x, \beta, d_{2i}) \land \tau(x, \beta, d_{2i+1}) : i \in \omega \} \) is consistent. Let \( a \in V \) realize it.

Then \( \tau(a, \beta, d_i) \) is true for all odd values of \( i \), and false for all even values of \( i \), and so \( \tau(x, v, w) \) has infinite alternation number, violating NIP.

**Case 2:** \( \tau(x, \beta, d_1) \notin \Gamma \). Setting \( d_0 = d \) leads to a similar contradiction. \( \square \)

So for any \( d_1, d_2 \in U \), if \( d_1 \equiv_M d_2 \equiv_M d \), then \( V \models \exists x \left[ \tau(x, \beta, d_1) \land \tau(x, \beta, d_2) \right] \), and so this formula is in \( \text{tp}(\beta/U) \). Since this type is definable over \( M \), there is a formula \( \delta(w_1, w_2) \) over \( M \) such that for any \( d_1, d_2 \in U \), we have that \( U \models \delta(d_1, d_2) \) if and only if \( V \models \exists x \left[ \tau(x, \beta, d_1) \land \tau(x, \beta, d_2) \right] \). Again letting \( q(w) = \text{tp}(\beta/M) \), for \( d_1, d_2 \in U \) we have that \( d_1, d_2 \models q \) implies \( U \models \delta(d_1, d_2) \). By Compactness, there is a formula \( \chi(w) \in q(w) \) such that \( U \models \chi(w_1) \land \chi(w_2) \rightarrow \delta(w_1, w_2) \).

Therefore, for any pair of tuples \( d_1, d_2 \in U \), we know that if \( U \models \chi(d_1) \land \chi(d_2) \) then \( V \models \exists x \left[ \tau(x, \beta, d_1) \land \tau(x, \beta, d_2) \right] \).

Further, let \( m \) be the number of outer balls in \( \tau(x, \beta, d) \). Let \( \rho(v, w) \) be a formula over \( \emptyset \) which says that the number of outer balls in \( \tau(x, v, w) \) is \( m \). Since \( \text{tp}(\beta/U) \) is definable over \( M \), there is some formula \( \rho'(w) \) over \( M \) such that for any \( v \in U \) we have that \( U \models \rho'(v) \) iff \( \tau(x, \beta, v) \) has \( m \) outer balls. Adjusting the formula \( \chi(v) \) if necessary, we may assume that for any \( v \in \chi(U) \), the set \( \tau(x, \beta, v) \) is a ucheese with \( m \) outer balls.

We need one final claim before using \( \chi \) to construct the desired type \( r(x) \).

**Claim 5.2.5.** For any \( v \in \chi(U) \), the set defined by \( \tau(x, \beta, d) \cap \tau(x, \beta, v) \) is a \( U\beta \)-irreducible ucheese, and its outer uball contains either \( \tau(x, \beta, d) \) or \( \tau(x, \beta, v) \).

**Proof.** Certainly \( \tau(x, \beta, v) \) is a ucheese with \( m \) outer balls, and by our choice of \( \chi \) we know that \( \tau(x, \beta, d) \cap \tau(x, \beta, v) \neq \emptyset \). Write \( \tau(x, \beta, d) = B \setminus B_1 \) and \( \tau(x, \beta, v) = C \setminus C_1 \), where \( B, C, B_1, \) and \( C_1 \) are \( U\beta \)-definable uballs, \( B \) and \( C \) have \( m \) balls each, and \( B \) is \( U\beta \)-irreducible.
Since \((B \setminus B_1) \cap (C \setminus C_1) \neq \emptyset\), some ball of \(B\) must intersect some ball of \(C\). If some ball of \(B\) does not intersect \(C\), then the union of the balls of \(B\) which intersect \(C\) and the union of those which do not will be disjoint non-empty \(U\beta\)-definable uballs whose union is \(B\), contradicting the irreducibility of \(B\).

So, every ball of \(B\) intersects some ball of \(C\). There are two possibilities:

- every ball of \(B\) is contained in some ball of \(C\);
- every ball of \(B\) contains some ball of \(C\).

(If it were a mix of the two, we would again violate the irreducibility of \(B\).)

If every ball of \(B\) is contained in some ball of \(C\), then \(B \subseteq C\), and thus \((B \setminus B_1) \cap (C \setminus C_1) = (B \cap C) \setminus (B_1 \cup C_1) = B \setminus (B_1 \cup C_1)\) is a \(U\beta\)-irreducible ucheese.

If every ball of \(B\) contains some ball of \(C\), then \(C \subseteq B\). Further, \(C\) must be \(U\beta\)-irreducible: if \(D\) and \(E\) are disjoint \(U\beta\)-definable uballs and \(D \cup E = C\), then “the balls of \(B\) which intersect \(D\)” and “the balls of \(B\) which intersect \(E\)” are \(U\beta\)-definable uballs; their union is \(B\), since every ball of \(B\) contains some ball of \(C\); and they are disjoint, since \(B\) and \(C\) have the same number of balls. So, \((B \setminus B_1) \cap (C \setminus C_1) = (B \cap C) \setminus (B_1 \cup C_1) = C \setminus (B_1 \cup C_1)\) is a \(U\beta\)-irreducible ucheese.

At this point, we begin constructing an \(M\overline{\beta}\)-definable type \(r(x) \in S_1(U\overline{\beta})\) which contains \(\tau(x, \overline{\beta}, \overline{d})\). Informally, it describes points that are generic in the intersection \(\bigcap_{\tau \in \chi(U)} \tau(x, \overline{\beta}, \overline{e})\).

We can begin by building a partial type \(r^*(x)\), consisting of \(U\overline{\beta}\)-irreducible uballs and their complements. Build a set of formulae \(r^*(x)\) over \(U\overline{\beta}\) as follows: if the formula \(\sigma(x, \overline{\beta}, \overline{f})\) defines a \(U\overline{\beta}\)-irreducible uball, where \(\overline{f} \in U\), then:
\[ \begin{align*}
\text{• if for some } e \in \chi(U) \text{ we have } \tau(x, \beta, e) \subseteq \sigma(x, \beta, f), \text{ then } \sigma(x, \beta, f) \in r^*; \\
\text{• otherwise, } \neg \sigma(x, \beta, f) \in r^*. 
\end{align*} \]

**Claim 5.2.6.** The set \( r^*(x) \) is consistent.

*Proof.* Suppose not: then, applying Compactness, there must exist finitely many \( U^\beta \)-irreducible uballs \( A_1, \ldots, A_{m_1}, B_1, \ldots, B_{m_2} \) such that for each \( A_i \), the formula \( x \in A_i \) is in \( r^* \); for each \( B_i \), the formula \( x \notin B_i \) is in \( r^* \); and the set of formulae 
\[ \{ x \in A_i : i = 1, \ldots, m_1 \} \cup \{ x \notin B_i : i = 1, \ldots, m_2 \} \]
is inconsistent.

For each \( A_i \), there is some \( e_i \in \chi(U) \) such that \( \tau(x, \beta, e_i) \subseteq A_i \); it follows from Claim 5.2.4 that the sets \( A_1, \ldots, A_{m_1} \) are 2-consistent. By Proposition 2.2.15 then, the \( A_i \)s must form a chain: after renumbering, \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{m_1} \), and so the set \( \{ x \in A_1 \} \cup \{ x \notin B_i : i = 1, \ldots, m_2 \} \) must be inconsistent.

For each \( B_i \), we know that \( \tau(x, \beta, e_1) \not\subseteq B_i \), so \( A_1 \not\subseteq B_i \); it follows from Proposition 2.2.15 that either \( B_i \not\subseteq A_1 \) or \( A_1 \cap B_i = \emptyset \). Let \( B_1, \ldots, B_n \) be a list of the sets \( B_i \) which are properly contained in \( A_1 \).

Then it must be that \( A_1 = \bigcup_{i=1}^n B_i \), and so the sets \( B_1, \ldots, B_n \) are finitely many \( U^\beta \)-definable uballs whose union is \( A_1 \), contradicting that \( A_1 \) is \( U^\beta \)-irreducible. \[ \square \]

By definition the partial type \( r^* \) is complete for \( U^\beta \)-irreducible uballs. By Theorem 2.2.14 \( r^* \) generates a unique complete type \( r \in S_1(U^\beta) \).

**Claim 5.2.7.** A \( U^\beta \)-definable uball \( B \) is in \( r \) iff \( \tau(x, \beta, e) \subseteq B \) for some \( e \in \chi(U) \).

*Proof.* Write \( B = B_1 \cup \cdots \cup B_k \), where \( B_1, \ldots, B_k \) are \( U^\beta \)-irreducible uballs, as per Proposition 2.2.13.

\( \Rightarrow \): Since \( r \) is generated by \( r^* \), the uball \( B \) is in \( r \) iff \( B_i \in r^* \) for some \( i \), in which case there is some \( e \in \chi(U) \) with \( \tau(x, \beta, e) \subseteq B_i \subseteq B \).
Assume there is some $\bar{e} \in \chi(\mathbb{U})$ such that $\tau(x, \bar{\beta}, \bar{e}) \subseteq B$. It follows that $\tau(x, \bar{\beta}, \bar{e}) \cap \tau(x, \bar{\beta}, \bar{d}) \subseteq B$. By Claim 5.2.5, $\tau(x, \bar{\beta}, \bar{e}) \cap \tau(x, \bar{\beta}, \bar{d})$ is a $\mathbb{U}\bar{\beta}$-irreducible ucheese; let $C$ be its outer uball.

For some $i$, the intersection $C \cap B_i$ is non-empty. Since $B_i$ and $C$ are both $\mathbb{U}\bar{\beta}$-irreducible uballs, by Proposition 2.2.15, either $C \subseteq B_i$ or $B_i \subsetneq C$.

If $B_i \subsetneq C$, then since $C \subseteq B$, the set $C$ intersects some of the other sets $B_j$. Let $B_1, \ldots, B_{k'}$ be a list of the sets $B_j$ which have non-trivial intersection with $C$. Then these sets are finitely many disjoint $\mathbb{U}\bar{\beta}$-definable uballs whose union is $C$, contradicting the irreducibility of $C$.

So, it must be that $C \subseteq B_i$, and since $C$ contains either $\tau(x, \bar{\beta}, \bar{e})$ or $\tau(x, \bar{\beta}, \bar{d})$ by Claim 5.2.5, the uball $B_i$ is in $r^*$, and so $B \in r$. 

We also get several other basic results about $r$, making use of the notation for layers from Theorem 2.2.7:

- if $B$ and $C$ are uballs, and each ball of $C$ is properly contained in a ball of $B$, then the ucheese $B \setminus C$ is in $r$ if and only if $B \in r$ and $C \notin r$;
- a formula $\sigma(x, \bar{\beta}, \bar{a})$ is in $r$ if and only if one of its layers $L_i^\sigma(x, \bar{\beta}, \bar{a})$ is in $r$.

Since each layer is a ucheese, we can easily define whether or not it is in $r$.

Recall that the outer and inner uballs of a layer $L_i(x, \bar{\beta}, \bar{a})$ are defined by $\lambda_i^+(x, \bar{\beta}, \bar{a})$ and $\lambda_i^-(x, \bar{\beta}, \bar{a})$, respectively. Fixing any formula $\sigma(x, \bar{\beta}, \bar{z})$, we will make use of the fact that $\text{tp}(\bar{\beta}/\mathbb{U})$ is definable over $M$. For each layer $L_i(x, \bar{\beta}, \bar{z})$, let $\delta_i^+(\bar{w}, \bar{z})$ define whether or not the formula $\chi(\bar{w}) \land \forall x \left[ \tau(x, \bar{\beta}, \bar{w}) \rightarrow \lambda_i^+(x, \bar{\beta}, \bar{z}) \right]$ is in $\text{tp}(\bar{\beta}/\mathbb{U})$.

That is, for parameters $\bar{e}, \bar{a} \in \mathbb{U}$, we have that $\mathbb{U} \models \delta_i^+(\bar{e}, \bar{a})$ if and only if $\mathbb{V} \models \chi(\bar{e}) \land (\tau(x, \bar{\beta}, \bar{e}) \rightarrow \lambda_i^+(x, \bar{\beta}, \bar{a}))$. Similarly, let $\delta_i^-(\bar{w}, \bar{z})$ define whether or not $\chi(\bar{w}) \land \forall x \left[ \tau(x, \bar{\beta}, \bar{w}) \rightarrow \lambda_i^-(x, \bar{\beta}, \bar{z}) \right]$ is in $\text{tp}(\bar{\beta}/\mathbb{U})$. 

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Then the layer $L_i(x, \overline{\beta}, \overline{a})$ will be in $r$ if and only if its outer uball $\lambda^+_i(x, \overline{\beta}, \overline{a})$ is in $r$, i.e. $U \models \exists \overline{w} \left[ \delta^+_i(\overline{w}, \overline{a}) \right]$, and its inner uball $\lambda^-_i(x, \overline{\beta}, \overline{a})$ is not in $r$, i.e. $U \models \neg \exists \overline{w} \left[ \delta^-_i(\overline{w}, \overline{a}) \right]$. It follows that the entire type $r$ is definable over $M\overline{\beta}$. We also get that $\tau(x, \overline{\beta}, \overline{d}) \in r$, since its outer uball must be in $r^*$ but its inner uball must not be by Claim 5.2.4.

Let $a$ realize $r$. Then, since $\tau(x, \overline{\beta}, \overline{d}) \vdash \varphi(x, \overline{\beta}, \overline{c})$, we have that $V \models \varphi(a, \overline{\beta}, \overline{c})$, i.e. $V \models \bigvee_{i=1}^k \varphi(a, \overline{\beta}_i, \overline{c})$. Fix a value of $i$ whose disjunct is true.

Let $p(x_0, x_1, \ldots, x_n) = \text{tp}(a\overline{\beta}_i/U)$. The type $p$ is in $S_{n+1}(U)$.

Since $\text{tp}(\overline{\beta}/U)$ is definable over $M$, and $r = \text{tp}(a/\overline{\beta})$ is definable over $M\overline{\beta}$, it follows that $\text{tp}(a\overline{\beta}/U)$ is definable over $M$; as $p$ is a subtype of $\text{tp}(a\overline{\beta}/U)$, we get that $p$ is also definable over $M$.

Finally, since $V \models \varphi(a, \overline{\beta}_i, \overline{c})$, it must be that $\varphi(x_0, x_1, \ldots, x_n, \overline{c}) \in p$. \qed
6.1 Almost unpackability

In this section, we will modify the proof of Theorem 5.2.1 to apply when $\Psi$ is merely almost unpackable, rather than unpackable. The chief difficulties in this are, firstly, that unbreakability, which is no longer guaranteed for every ball, is not definable in parameters; and secondly, that the sets of formulae constructed in the proof of Theorem 5.2.1 to generate types need not be consistent if not every ball is unbreakable. We can maneuver around the first issue; however, the second requires major adjustments to the original proof.

**Theorem 6.1.1.** Suppose that $T$ is VC-minimal, with generating directed family $\Psi$, and suppose that $\Psi$ is almost unpackable. Let $U$ be a large saturated model of $T$, and let $M \preceq U$ be a small model. Let $\varphi(x, y)$ be any formula, and $c$ any parameter tuple from $U$. Then the following are equivalent:

(I) $\varphi(x, c)$ does not fork over $M$;

(II) there is a global type $p \in S_x(U)$ which is definable over $M$ with $\varphi(x, c) \in p$.

**Proof.** (II) $\Rightarrow$ (I): Straightforward.

(I) $\Rightarrow$ (II): Proceed by induction on $l(\bar{x})$.

**Base Case:** $x$ is a single variable.
Suppose that $\varphi(x, \bar{c})$ does not fork over $M$. Let $\{S_1, \ldots, S_n\}$ be a set of disjoint unbreakable Swiss cheeses whose union is $\varphi(x, \bar{c})$. As in Theorem 5.2.1, we replace $\varphi(x, \bar{c})$ by a formula defining one of these Swiss cheeses which does not fork over $M$.

So, assume that $\varphi(x, \bar{c})$ is of the form $\psi_0(x, \bar{b}_0) \setminus (\psi_1(x, \bar{b}_1) \cup \cdots \cup \psi_k(x, \bar{b}_k))$, where $\psi_0(x, \bar{b}_0)$ is unbreakable. For any $\bar{c}' \equiv_M \bar{c}$, the set $\varphi(x, \bar{c}')$ is also an unbreakable Swiss cheese of this form. Thus $\{\varphi(x, \bar{c}') : \bar{c}' \equiv_M \bar{c}, \bar{c}' \in U\}$ is a family of unbreakable Swiss cheeses; it is 2-consistent by Proposition 4.1.10 and thus is consistent by Proposition 2.2.17. Our goal going forward will be to build an $M$-definable type describing points that are generic in the set $\bigcap_{\bar{c}' \equiv_M \bar{c}} \varphi(x, \bar{c}')$.

By Compactness, there is a formula $\theta(\bar{y}) \in \text{tp}(\bar{c}/M)$ such that:

- $U \models \theta(\bar{y}_1) \land \theta(\bar{y}_2) \rightarrow \exists x [\varphi(x, \bar{y}_1) \land \varphi(x, \bar{y}_2)]$, and
- $U \models \theta(\bar{y}) \rightarrow \exists z_0 \cdots \exists z_k [\varphi(x, \bar{y}) = \psi_0(x, z_0) \setminus (\psi_1(x, z_1) \cup \cdots \cup \psi_k(x, z_k))].$

It follows that $\{\varphi(x, \bar{d}) : \bar{d} \in \theta(U)\}$ is a 2-consistent family of Swiss cheeses, though we do not know that it is consistent - it is possible that $\varphi(x, \bar{d})$ might be breakable for some $\bar{d} \in \theta(U)$.

Build a set $p^*$ of formulae, consisting of $U$-definable balls and their complements, as follows:

- $\psi(x, \bar{b}) \in p^*$ if $U \models \exists \bar{y} [\theta(\bar{y}) \land \varphi(x, \bar{y}) \subseteq \psi(x, \bar{z})]$;
- $\neg\psi(x, \bar{b}) \in p^*$ otherwise.

If $p^*$ is consistent, then it is a complete $M$-definable $\Psi$-type, and thus by Proposition 2.1.6 generates a complete $M$-definable type $p \in S(U)$; as the outer ball of $\varphi(x, \bar{c})$ must be in $p^*$, and (by 2-consistency) each of its holes must not, we also get that $\varphi(x, \bar{c}) \in p$, as desired.

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If $p^*$ is not consistent, then by Compactness there are balls $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$ such that $\{x \in B_1, \ldots, x \in B_m\} \cup \{x \notin C_1, \ldots, x \notin C_n\} \subseteq p^*$ is inconsistent. Since each $B_i$ contains $\varphi(x, \bar{c}_i)$ for some $\bar{c}_i \in \theta(U)$, and these sets are 2-consistent, the sets $B_i$ must be 2-consistent; since each $B_i$ is a ball, it follows that they form a chain, and so for one of them (say, $B_1$) we have that $B_1 = \bigcap_i B_i$. It follows that $B_1$ is covered by finitely many balls $C_i$ which are not in $p^*$.

So, for each finite $\Pi \subseteq p^*$ witnessing inconsistency, there is a least $B \in p^*$ represented in $\Pi$, which must be the union of finitely many balls not in $p^*$.

Let $\mathcal{B} = \{B : B \text{ is a ball in } p^*, B \text{ is finitely covered by balls not in } p^*\}$. Again, $\mathcal{B}$ must be 2-consistent, so for any $B_1, B_2 \in \mathcal{B}$, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

**Claim 6.1.2.** There is $B \in \mathcal{B}$ which is $\subseteq$-minimal (and so $B = \bigcap \mathcal{B}$).

**Proof.** Suppose not. There there is a sequence $B_0, B_1, B_2, \ldots$ of balls in $\mathcal{B}$ such that $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$.

For some $n$, we have that $B_0 = \bigcup_{i=1}^n C_i$, where the sets $C_i$ are disjoint balls not contained in $p^*$. It follows that $B_1$ must be the union of some strictly fewer than $n$ of the sets $C_i$: we know that $B_1 \subset \bigcup_i C_i$, and because $B_1$ is in $p^*$ but each $C_i$ is not, each $C_i$ is either a proper subset of $B_1$ or disjoint from $B_1$. Similarly, $B_2$ is the union of even fewer of the sets $C_i$; continuing in this manner, there cannot be a strictly decreasing chain in $\mathcal{B}$ beginning with $B_0$ of length greater than $n$, a contradiction. $\square$

So, there is a least $B \in \mathcal{B}$. We know that for every $\bar{y} \models \theta$, the set $\varphi(x, \bar{y})$ can be written as a single Swiss cheese whose outer ball is an instance of $\psi_0$. This least $B$ contains $\varphi(x, \bar{d})$ for some $\bar{d} \in \theta(U)$; as the outer ball of this Swiss cheese must be in $\mathcal{B}$, if follows that $B$ must be the outer ball. So, $B$ is defined by $\psi_0(x, \bar{b})$ for some parameters $\bar{b}$.
The type $tp(\bar{b}/M)$ includes formulae which state that:

- for some $\bar{y} \models \theta$, the Swiss cheese $\varphi(x, \bar{y})$ has outer ball $\psi_0(x, \bar{b})$; and
- for every $\bar{y} \models \theta$, the outer ball $\psi_0(x, \bar{z})$ of $\varphi(x, \bar{y})$ contains $\psi_0(x, \bar{b})$.

Consider any $\sigma \in \text{Aut}(U/M)$; the image of $B$ is $\sigma(B) = \psi_0(x, \sigma(\bar{b}))$. Since $\sigma(\bar{b}) \equiv_M \bar{b}$, the tuple $\sigma(\bar{b})$ also satisfies the two properties given above; as these suffice to identify the set $B$, it follows that $B$ is fixed by $\sigma$.

Write $B$ as the disjoint union of finitely many unbreakable balls $C_1, \ldots, C_n$. Since any $\sigma \in \text{Aut}(U/M)$ fixes $B$, and maps unbreakable balls to unbreakable balls, it must permute $C_1, \ldots, C_n$; thus, by Lemma III.2.3 of [22], each of the sets $C_1, \ldots, C_n$ is definable over $M$.

The set $B$ is the outer ball of $\varphi(x, \bar{d})$ for some $\bar{d} \in \theta(U)$; as we know that $\varphi(x, \bar{c}) \cap \varphi(x, \bar{d})$ is not empty, one of the sets $C_i$, call it $C$, must contain points from $\varphi(x, \bar{c})$. Consider the generic type $p_C$, as in Proposition 2.1.7 The outer ball of $\varphi(x, \bar{c})$ contains $B$, which contains $C$, so it is in $p_C$. Further, if any hole of $\varphi(x, \bar{c})$ were in $p_C$, it would contain $C$, contradicting that $C$ contains points from $\varphi(x, \bar{c})$. So $p_C$ is a complete $M$-definable type, and must contain $\varphi(x, \bar{c})$.  

**Inductive step:** Assume that (I) $\Rightarrow$ (II) holds for formulae in which $l(\pi) = n$.

Let $\varphi(\bar{x}, \bar{c}) = \varphi(x_0, x_1, \ldots, x_n, \bar{c})$, and suppose $\varphi(\bar{x}, \bar{c})$ does not fork over $M$.

By Corollary 4.1.10 it follows that the family $\{ \varphi(\bar{x}, \bar{c}') : \bar{c}' \equiv_M \bar{c}, \bar{c}' \in U \}$ is consistent, and in particular $2(n+1)$-consistent. As in Theorem 5.2.1 we can extract a formula $\theta(\bar{y}) \in tp(\bar{c}/M)$ guaranteeing $2(n+1)$-consistency. Moving from $\varphi(\bar{x}, \bar{y})$ to the new formula

$$\varphi^*(x_1, \ldots, x_n, \bar{y}_1, \bar{y}_2) = \exists x_0 [\varphi(x_0, x_1, \ldots, x_n, \bar{y}_1) \land \varphi(x_0, x_1, \ldots, x_n, \bar{y}_2)],$$

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we get that the set \( \{ \varphi^*(x_1, \ldots, x_n, \overline{c}_1, \overline{c}_2) : \overline{c}_1, \overline{c}_2 \in \theta(U) \} \) is \((n+1)\)-consistent.

So, for every pair of tuples \( \overline{c}_1, \overline{c}_2 \in \theta(U) \), the set of formulae

\[
\{ \varphi^*(x_1, \ldots, x_n, \overline{c}_1', \overline{c}_2') : \overline{c}_1', \overline{c}_2' \in U, \overline{c}_1' \overline{c}_2' \equiv_M \overline{c}_1 \overline{c}_2 \}
\]

is \((n+1)\)-consistent, and by Corollary 4.1.10 the formula \( \varphi^*(x_1, \ldots, x_n, \overline{c}_1, \overline{c}_2) \) does not fork over \( M \). Thus, by inductive hypothesis, for every \( \overline{c}_1, \overline{c}_2 \in \theta(U) \) there is an \( M \)-definable type \( q_{\overline{c}_1, \overline{c}_2} \in S_n(U) \) which contains \( \varphi^*(x_1, \ldots, x_n, \overline{c}_1, \overline{c}_2) \).

A brief Compactness argument (see Claim 5.2.2) shows that there are finitely many of these types, \( q_1(X), \ldots, q_k(X) \in S_n(U) \), such that for every \( \overline{c}_1, \overline{c}_2 \in \theta(U) \), the formula \( \varphi^*(x_1, \ldots, x_n, \overline{c}_1, \overline{c}_2) \) is in \( q_i \) for some \( i \).

Let \( V \) be a \(|U|^+\)-saturated extension of \( U \). Choose a type \( q \in S_{n,k}(U) \) as follows: let \( \overline{\beta}_1 \in V \) realize \( q_1 \). Let \( \overline{\beta}_2 \) realize the definable type \( q_2 \) extended to a type over \( U \overline{\beta}_1 \). Let \( \overline{\beta}_3 \) realize \( q_3 \) extended to a type over \( U \overline{\beta}_1 \overline{\beta}_2 \), and so on; finally, let \( \overline{\beta} \in V \) be the tuple \( \overline{\beta}_1 \overline{\beta}_2 \ldots \overline{\beta}_k \), and let \( q = \text{tp}(\overline{\beta}/U) \). Note that the type \( q \) is also definable over \( M \).

Let \( \widehat{\varphi}(x_0, X_1, \ldots, X_k, \overline{y}) \) be the formula \( \bigvee_{i=1}^k \varphi(x_0, X_i, \overline{y}) \). By choice of \( \widehat{\varphi} \) and \( q \), for any \( \overline{c}_1, \overline{c}_2 \in \theta(U) \) we get that \( V \models \exists x_0 [\widehat{\varphi}(x_0, \overline{\beta}, \overline{c}_1) \land \widehat{\varphi}(x_0, \overline{\beta}, \overline{c}_2)] \).

So, the family \( \{ \widehat{\varphi}(x, \overline{\beta}, \overline{c}') : \overline{c}' \in \theta(U) \} \) is \( 2 \)-consistent. We henceforth work to build an \( M\overline{\beta} \)-definable type \( r(x) \in S(U \overline{\beta}) \) which contains the formula \( \widehat{\varphi}(x, \overline{\beta}, \overline{c}) \).

As per Proposition 2.2.14 the set \( \widehat{\varphi}(x, \overline{\beta}, \overline{c}) \) can be written as the union of some finitely many ucheeses which are irreducible over \( U \overline{\beta} \). As in Claim 5.2.4 one of these \( U \overline{\beta} \)-irreducible ucheeses, defined by a formula \( \tau(x, \overline{\beta}, \overline{d}) \) where \( \overline{d} \in U \), must have the property that the set \( \{ \tau(x, \overline{\beta}, \overline{d}') : \overline{d}' \in U, \overline{d}' \equiv_M \overline{d} \} \) is \( 2 \)-consistent. Also as in Theorem 5.2.1 using the fact that \( q = \text{tp}(\overline{\beta}/U) \) is definable over \( M \), we can extract a formula \( \chi(\overline{w}) \in \text{tp}(\overline{d}/M) \) such that for any \( \overline{d}_1, \overline{d}_2 \in U \), if
\[ \mathbb{U} \models \chi(\bar{d}_1) \land \chi(\bar{d}_2), \text{ then } \mathbb{V} \models \exists x [\tau(x, \bar{\beta}, \bar{d}_1) \land \tau(x, \bar{\beta}, \bar{d}_2)]. \]

Since \( \tau(x, \bar{\beta}, \bar{d}) \) is a ucheese, we can write it as \( \tau^+(x, \bar{\beta}, \bar{d}) \setminus \tau^-(x, \bar{\beta}, \bar{d}) \), where \( \tau^+ \) and \( \tau^- \) define its outer and inner uballs, respectively, and the uball \( \tau^+(x, \bar{\beta}, \bar{d}) \) is \( \mathbb{U}\beta \)-irreducible. Let \( k \) be the number of unbreakable balls in \( \tau^+(x, \bar{\beta}, \bar{d}) \); we can write \( \tau^+(x, \bar{\beta}, \bar{d}) \) as \( D_1 \cup \cdots \cup D_k \). Although each ball \( D_i \) need not be definable over \( \mathbb{U}\beta \), we have that \( \mathbb{V} \models \exists W_1 \cdots \exists W_k [\tau^+(x, \bar{\beta}, \bar{d}) = \psi_1(x, W_1) \cup \cdots \cup \psi_k(x, W_k)]; \) thus, we can adjust \( \chi \) so that for every \( \bar{e} \in \chi(\mathbb{U}) \), the uball \( \tau^+(x, \bar{\beta}, \bar{e}) \) can be written as the union of \( k \) balls.

Next, define a set \( r^*(x) \) of formulae, such that whenever \( \rho(x, \bar{\beta}, \bar{e}) \) is a formula over \( \mathbb{U}\beta \) defining a uball:

- \( \rho(x, \bar{\beta}, \bar{e}) \in r^*(x) \) iff there is some \( \bar{d}' \in \chi(\mathbb{U}) \) such that \( \tau(x, \bar{\beta}, \bar{d}') \subseteq \rho(x, \bar{\beta}, \bar{e}); \)
- otherwise, \( \neg \rho(x, \bar{\beta}, \bar{e}) \in r^*(x). \)

If \( r^*(x) \) is consistent, then \( r^+ \) is a partial type over \( \mathbb{U}\bar{\beta} \) which is definable over \( M\bar{\beta} \); it generates a unique complete type \( r(x) \in S_1(\mathbb{U}\bar{\beta}) \), which by Theorem 2.2.7 is definable over \( M\bar{\beta} \).

If \( r^*(x) \) is not consistent, then as in the Base Case, we will use witnesses to inconsistency to choose a replacement type. First, we need two claims.

**Claim 6.1.3.** If \( \bar{e} \in \chi(\mathbb{U}) \), then either:

- \( \tau^+(x, \bar{\beta}, \bar{d}) \subseteq \tau^+(x, \bar{\beta}, \bar{e}); \) or
- \( \tau^+(x, \bar{\beta}, \bar{e}) \subseteq \tau^+(x, \bar{\beta}, \bar{d}), \) and further, \( \tau^+(x, \bar{\beta}, \bar{e}) \) can be written as the union of \( k \) balls, exactly one of which is contained in each of the \( k \) unbreakable balls making up \( \tau^+(x, \bar{\beta}, \bar{d}) \).

**Proof.** For notational simplicity, we denote by \( \tau^+(\bar{z}) \) the set defined by \( \tau^+(x, \bar{\beta}, \bar{z}). \) We know we can write \( \tau^+(\bar{e}) \) as the union of \( k \) balls: \( \tau^+(\bar{e}) = E_1 \cup \cdots \cup E_k. \)
Additionally, we can write $\tau^+(\overline{e})$ as the union of some number of $\mathbb{U}B$-irreducible uballs: $\tau^+(\overline{e}) = B_1 \cup \cdots \cup B_j$. Because $\overline{e} \in \chi(\mathbb{U})$, there must be some $i$ such that $B_i \cap \tau^+(\overline{d}) \neq \emptyset$; without loss of generality, $B_1 \cap \tau^+(\overline{d}) \neq \emptyset$. As $B_1$ and $\tau^+(\overline{d})$ are both $\mathbb{U}B$-irreducible uballs, one must be contained in the other.

If $\tau^+(\overline{d}) \subseteq B_1$, then $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e})$.

If $B_1 \subsetneq \tau^+(\overline{d})$: the uball $B_1$ must intersect some of the balls $E_i$; assume without loss of generality that $E_1 \cap B_1 \neq \emptyset$. Since $B_1 \subseteq \tau^+(\overline{d}) = D_1 \cup \cdots \cup D_k$, there must be some $i$ such that $E_1 \cap D_i \neq \emptyset$. Since $E_i$ and $D_i$ are balls, one must contain the other.

If $D_i \subseteq E_1$, then by the irreducibility of $\tau^+(\overline{d})$, each of $D_1, \ldots, D_k$ must be contained in one of the balls $E_1, \ldots, E_k$; it follows that $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e})$, and so $\tau^+(\overline{d})$ must be properly covered by $B_1, \ldots, B_j$, a contradiction. So, $E_1 \subseteq D_i$. By irreducibility, each $D_i$ must contain one of the balls $E_i$, and so $\tau^+(\overline{e}) \subseteq \tau^+(\overline{d})$, as desired.

\hfill \Box

**Claim 6.1.4.** For any $A_1, A_2 \in r^*$, the uball $A_1 \cap A_2$ is non-empty and also in $r^*$.

**Proof.** Since $A_1$ and $A_2$ are in $r^*$, there exist $\overline{e}_1, \overline{e}_2 \in \chi(\mathbb{U})$ such that $\tau^+(\overline{e}_1) \subseteq A_1$ and $\tau^+(\overline{e}_2) \subseteq A_2$. Each of $\tau^+(\overline{e}_1)$ and $\tau^+(\overline{e}_2)$ must satisfy one of the two conditions in Claim 6.1.3.

- If $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_1)$ and $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_2)$: then we get that $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_1) \cap \tau^+(\overline{e}_2) \subseteq A_1 \cap A_2$.

- If $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_1)$ and $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_2)$: then $\tau^+(\overline{e}_2) \subseteq A_1 \cap A_2$.

- If $\tau^+(\overline{e}_1) \subseteq \tau^+(\overline{d})$ and $\tau^+(\overline{d}) \subseteq \tau^+(\overline{e}_2)$: then $\tau^+(\overline{e}_1) \subseteq A_1 \cap A_2$. 

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• If $\tau^+(\bar{e}_1) \subseteq \tau^+(\bar{d})$ and $\tau^+(\bar{e}_2) \subseteq \tau^+(\bar{d})$: each of the sets $\tau^+(\bar{e}_1)$ and $\tau^+(\bar{e}_2)$ is the union of $k$ balls, one inside each $D_i$. Since $\tau^+(\bar{e}_1)$ and $\tau^+(\bar{e}_2)$ must intersect, by the irreducibility of $\tau^+(\bar{d}) = D_1 \cup \cdots \cup D_k$, the two sets must have non-empty intersection inside each ball $D_i$. So, for all $i$, we have that $(\tau^+(\bar{e}_1) \cap D_i) \cap (\tau^+(\bar{e}_2) \cap D_i) \neq \emptyset$. Since the sets $\tau^+(\bar{e}_1) \cap D_i$ and $\tau^+(\bar{e}_2) \cap D_i$ are both balls, one must be contained in the other. If $\tau^+(\bar{e}_1) \cap D_i$ were the larger ball for some but not all $i$, we would violate that $\tau^+(\bar{d})$ is irreducible; thus, either $\tau^+(\bar{e}_1) \cap D_i$ or $\tau^+(\bar{e}_2) \cap D_i$ is larger for all $i$, and so either $\tau^+(\bar{e}_1) \subseteq \tau^+(\bar{e}_2)$ or $\tau^+(\bar{e}_2) \subseteq \tau^+(\bar{e}_1)$. It follows that $A_1 \cap A_2$ contains either $\tau^+(\bar{e}_1)$ or $\tau^+(\bar{e}_2)$.

In each case, $A_1 \cap A_2$ contains $\tau^+(\bar{e})$ for some $\bar{e} \in \chi(\bar{U})$, and so $A_1 \cap A_2 \in r^*$. □

So if $\{x \in A_i : i = 1, \ldots, m_1\} \cup \{x \notin B_i : i = 1, \ldots, m_2\}$ witnesses the inconsistency of $r^*$, then $\bigcap_{i=1}^{m_1} A_i$ is in $r^*$, and is finitely covered by the uballs $B_1, \ldots, B_{m_2}$. It follows that if we write the $U\beta$-definable uball $\bigcap A_i$ as the union of $U\beta$-irreducible uballs $C_1, \ldots, C_j$, then each of the uballs $C_i$ is not in $r^*$.

Therefore, there exists some $U\beta$-definable uball $A$ which is in $r^*$, but whose $U\beta$-irreducible uballs are not in $r^*$. Let $\sigma(x, \bar{\beta}, \bar{e})$ be a formula defining $A$, where $\bar{e} \in U$, and let $\sigma_1(x, \bar{\beta}, \bar{e}), \ldots, \sigma_j(x, \bar{\beta}, \bar{e})$ define the irreducible uballs of $A$.

Some definable properties of $\bar{e}$:

• the sets $\sigma(x, \bar{\beta}, \bar{e}), \sigma_1(x, \bar{\beta}, \bar{e}), \ldots, \sigma_j(x, \bar{\beta}, \bar{e})$ are all uballs;

• $\sigma(x, \bar{\beta}, \bar{e})$ is in $r^*$, i.e. there exists $\bar{d}' \in \chi(\bar{U})$ such that $\tau(x, \bar{\beta}, \bar{d}') \subseteq \sigma(x, \bar{\beta}, \bar{e})$;

• the sets $\sigma_i(x, \bar{\beta}, \bar{e})$ partition $\sigma(x, \bar{\beta}, \bar{e})$;

• none of the sets $\sigma_i(x, \bar{\beta}, \bar{e})$ is in $r^*$.
Each of these pieces of information is contained in the $M$-definable $\text{tp}(\beta/U)$, so each is definable using parameters from $M$. Let $\Phi(z)$ be a formula over $M$ expressing all of these; note that $U \models \exists z[\Phi(z)]$. Since $M \preceq U$, it follows that $M \models \exists z[\Phi(z)]$, too.

Thus, there is an $M\beta$-definable uball $\sigma(x, \bar{\beta}, \bar{m})$ in $r^*$, covered by uballs not in $r^*$. Let $B_1, \ldots, B_j'$ be the $U\beta$-irreducible uballs making up $\sigma(x, \bar{\beta}, \bar{m})$; it follows that $B_1, \ldots, B_j'$ are $M\beta$-definable.

Since $\sigma(x, \bar{\beta}, \bar{m})$ is in $r^*$, there is $\vec{d}' \in \chi(U)$ such that $\tau(x, \bar{\beta}, \vec{d}') \subseteq \sigma(x, \bar{\beta}, \bar{m})$; since $\tau(x, \bar{\beta}, \vec{d}')$ must intersect $\tau(x, \bar{\beta}, \vec{d})$, it follows that one of the sets $B_i$ must contain points from $\tau(x, \bar{\beta}, \vec{d})$. Call it $B$. It will follow by Proposition 2.2.18 that the generic type $p_B$ is an $M\beta$-definable type over $U\beta$, and must contain $\tau(x, \bar{\beta}, \vec{d})$.

Let $r(x) = p_B(x)$.

Then letting $a$ realize $r(x)$, we get that $V \models \varphi(a, \bar{\beta}_i, \bar{c})$ for some $i$. Let $p(x_0, x_1, \ldots, x_n) = \text{tp}(a\bar{\beta}_i/U)$. Since $q = \text{tp}(\beta/U)$ is definable over $M$ and $r = \text{tp}(a/U\beta)$ is definable over $M\beta$, the type $\text{tp}(a\beta/U)$ must be definable over $M$; since $p \subseteq \text{tp}(a\beta/U)$, the type $p$ is also definable over $M$.

Finally, $\varphi(x_0, x_1, \ldots, x_n, \bar{c}) \in p$. \hfill \Box

This result extends the forking characterization found in stable theories to a wide variety of VC-minimal theories, including (among others) strongly minimal theories, o-minimal theories, and the theory of any ACVF. On the other end of the spectrum, in Example 27 of [7], Chernikov and Simon present an NIP theory in which there is a formula which does not fork over a model $M$, but which is not contained in any global $M$-definable type. Consequently, the equivalence “for any formula $\varphi(\bar{x}, \bar{c})$ and any model $M$, $\varphi(\bar{x}, \bar{c})$ does not fork over $M$ if $\varphi(\bar{x}, \bar{c})$ is contained in a global $M$-definable type” does not hold for NIP theories in general.
The status of this equivalence for general dp-minimal theories, or for VC-minimal theories which do not admit almost unpackable generating families, is currently unknown.

6.2 Quasi-VC-minimality

As inspiration for this section, consider the notion of quasi-o-minimality, as defined in [4].

**Definition 6.2.1.** A complete theory $T$ in a language $\mathcal{L}$ including a linear order $<$ is quasi-o-minimal if for every model $M$ of $T$, every definable subset of $M$ is a finite boolean combination of intervals and $\emptyset$-definable sets.

As discussed in [4], many results previously known for o-minimal theories also hold for quasi-o-minimal theories, and a number of linearly ordered structures which fail to be o-minimal are quasi-o-minimal.

In this section, we define quasi-VC-minimality, a strict weakening of the notion of VC-minimality. We show that several results from Chapter 2 have analogues in the quasi-VC-minimal setting, and outline a proof that the forking result from Chapter 5 also holds in quasi-VC-minimal theories satisfying a version of almost unpackability.

**Definition 6.2.2.** A complete theory $T$ is quasi-VC-minimal if there exists a family $\Psi = \{\psi_i(x, \overline{y}_i) : i \in I\}$ of formulae, each in a single variable $x$ and some finite tuple of variables $\overline{y}_i$, such that for any model $M$ of $T$:

- **(i)** the set of instances $\{\psi(x, \bar{b}) : \psi \in \Psi, \bar{b} \in M\}$ is a directed family;

- **(ii)** any definable subset of $M$ is a finite boolean combination of instances of $\Psi$ and $\emptyset$-definable sets.
This $\Psi$ is called a generating directed family for $T$.

Clearly any VC-minimal theory is quasi-VC-minimal, as is any quasi-o-minimal theory. Every quasi-VC-minimal theory is also dp-minimal. There are also quasi-VC-minimal theories which are not VC-minimal:

**Example 6.2.3.** Presburger Arithmetic, $\text{Th}(\mathbb{Z}, +, <)$, is quasi-VC-minimal: by Example 2 of [4], it is actually quasi-o-minimal. In this case, we can take the set $\Psi = \{x > y, x \geq y\}$ as a witness to quasi-VC-minimality.

As before, we call a set defined by an instance of $\Psi$ a ball, and a set of the form $A \setminus (B_1 \cup \cdots \cup B_n)$, where $A, B_1, \ldots, B_n$ are balls, a Swiss cheese. We define $\text{uball}, \text{ucheese}$, and $\text{irreducible}$ exactly as we did in Definitions 2.2.5 and 2.2.11.

**Remark 6.2.4.** For any $\emptyset$-definable set $Z$, the family $\Psi \cap Z = \{B \cap Z : B \text{ a ball}\}$ is also a directed family.

**Proposition 6.2.5.** Suppose that $T$ is quasi-VC-minimal. Then for any formula $\varphi(x, \overline{y})$ and any parameters $\overline{a}$:

- the set $\varphi(x, \overline{a})$ is a finite boolean combination of sets of the form $B \cap Z$, where $B$ is a ball and $Z$ is $\emptyset$-definable; and

- the set $\varphi(x, \overline{a})$ is the disjoint union of finitely many sets of the form $S \cap Z$, where $S$ is a Swiss cheese and $Z$ is $\emptyset$-definable.

**Proposition 6.2.6.** Suppose that $T$ is quasi-VC-minimal; let $M \models T$, and let $A \subseteq M$ be any set. Let $p^* \in S^\Psi(M)$ be a complete $A$-definable $\Psi$-type, and let $Z$ be $\emptyset$-definable, defined by a formula $\xi(x)$, such that $p^*(x) \cup \{\xi(x)\}$ is consistent. Then there is a complete $A$-definable type $p \in S(M)$ containing $p^*(x)$ and $\xi(x)$. 
Proof. Build a \((\Psi \cap Z)\)-type \(q\), consisting of the sets \(C \cap Z\) for every ball \(C\) which is in \(p^*\), and \(\neg C \cap Z\) for every ball \(C\) whose complement is in \(p^*\). This \(q\) will be consistent and definable over \(A\).

Let \(p\) be any extension of \(q\) to a complete type over \(M\); this \(p\) will also be an extension of \(p^*\). Take any formula \(\psi(x, \overline{y}) \in \Psi\) and any formula \(\chi(x)\) over \(\emptyset\). Then for any parameters \(\overline{a} \in M\), we get that \(\psi(x, \overline{a}) \cap \chi(x) \in p\) iff \(\psi(x, \overline{a}) \in p\) and \(\chi(x) \in p\). Whether or not \(\psi(x, \overline{a}) \in p\) depends only on whether or not \(\psi(x, \overline{a}) \in p^*\), which is defined by a formula over \(A\); whether or not \(\chi(x) \in p\) does not depend on \(\overline{a}\) at all, and so is defined by either \(\overline{y} = \overline{y}\) or \(\overline{y} \neq \overline{y}\). Therefore, the type \(p\) is definable for sets of the form \(C \cap Y\), where \(C\) is a ball and \(Y\) is \(\emptyset\)-definable. As these sets generate all definable sets by Proposition 6.2.5 it follows that all of \(p\) is \(A\)-definable by Lemma 2.3.1 of [14].

\(\quad\)

**Definition 6.2.7.** Suppose \(T\) is quasi-VC-minimal, with generating family \(\Psi\).

1. A ball is *unbreakable* if it is not the union of finitely many smaller balls.

2. \(\Psi\) is *almost unpackable* if for every \(\emptyset\)-definable set \(Z\), the directed family \(\Psi \cap Z\) is almost unpackable.

**Proposition 6.2.8.** Suppose \(T\) is quasi-VC-minimal, and let \(M \models T\). Let \(B\) be an unbreakable ball, which is definable over a set \(A \subseteq M\), and let \(Z\) be a \(\emptyset\)-definable set such that \(B \cap Z \neq \emptyset\). Then there is an \(A\)-definable type \(p_{B \cap Z} \in S_1(M)\) such that for any ball \(C \subseteq M\), the formula defining \(C \cap Z\) is in \(p_{B \cap Z}\) iff \(B \subseteq C\).

Proof. Begin by building an \(A\)-definable \(\Psi\)-type \(p^*\): for every \(\psi \in \Psi\) and every tuple \(\overline{a} \in M\), let \(\psi(x, \overline{a}) \in p^*\) iff \(B \subseteq \psi(x, \overline{a})\). The type \(p^*\) is consistent by the unbreakability of \(B\), so by Proposition 6.2.6 a complete \(A\)-definable type \(p_{B \cap Z}\) with the desired properties exists. \(\square\)
As in VC-minimal theories, we also have a decomposition of definable sets in terms of uballs.

**Proposition 6.2.9.** Suppose that $T$ is quasi-VC-minimal. For any formula $\varphi(x, \bar{y})$, there exist formulae $\lambda_0^+(x, \bar{y}), \lambda_0^-(x, \bar{y}), \ldots, \lambda_0^+(x, \bar{y}), \lambda_0^-(x, \bar{y})$ over $\emptyset$ such that for any parameters $\bar{a}$:

- for each $i$, $\lambda_i^+(x, \bar{a}) = \sigma_i^+(x, \bar{a}) \land \xi_i(x)$ and $\lambda_i^-(x, \bar{a}) = \sigma_i^-(x, \bar{a}) \land \xi_i(x)$;
- for each $i$, $\sigma_i^+(x, \bar{a})$ and $\sigma_i^-(x, \bar{a})$ are uballs, $\sigma_i^-(x, \bar{a}) \subsetneq \sigma_i^+(x, \bar{a})$, and $\xi_i(x)$ is $\emptyset$-definable;
- if for some $i \neq j$ we have that $\xi_i(x) = \xi_j(x)$, then either $\lambda_i^+(x, \bar{a}) \subsetneq \lambda_j^-(x, \bar{a})$ or $\lambda_j^+(x, \bar{a}) \subsetneq \lambda_i^-(x, \bar{a})$;
- $\varphi(x, \bar{a})$ is the disjoint union of the sets $\lambda_i^+(x, \bar{a}) \setminus \lambda_i^-(x, \bar{a})$.

**Proof.** For any instance $\varphi(x, \bar{a})$, we know that $\varphi(x, \bar{a})$ is the disjoint union of sets $S_i \cap Z_i$ for $i = 1, \ldots, n$, where each $S_i$ is a Swiss cheese and each $Z_i$ is $\emptyset$-definable. We may assume that for any $i \neq j$, either $Z_i = Z_j$ or $Z_i \cap Z_j = \emptyset$.

For each $i$, the set $\varphi(x, \bar{a}) \cap Z_i$ is constructible from the directed family $\Psi \cap Z_i$; thus, by Theorem 1.2.8, we can write $\varphi(x, \bar{a}) \cap Z_i$ as the union of finitely many ucheeses intersected with $Z_i$. As these ucheeses are fixed under automorphisms fixing $\bar{a}$, each is definable over $\bar{a}$: letting $\xi_i(x)$ define the set $Z_i$, we get formulae $\sigma_1^+(x, \bar{a}), \sigma_1^-(x, \bar{a}), \ldots, \sigma_k^+(x, \bar{a}), \sigma_k^-(x, \bar{a})$ defining the inner and outer uballs of these ucheeses, with $\sigma_k^-(x, \bar{a}) \subsetneq \sigma_k^+(x, \bar{a}) \subsetneq \cdots \subsetneq \sigma_1^+(x, \bar{a})$. Set $\lambda_1^+(x, \bar{a}) = \sigma_1^+(x, \bar{a}) \cap \xi_i(x)$, $\lambda_1^-(x, \bar{a}) = \sigma_1^-(x, \bar{a}) \cap \xi_i(x)$, and so on.

Letting $\lambda_1^+(x, \bar{a}), \lambda_0^-(x, \bar{a}), \ldots, \lambda_s^+(x, \bar{a}), \lambda_s^-(x, \bar{a})$ be a list of these formulae as determined by $\varphi(x, \bar{a}) \cap Z_i$ for all $i$, the sets defined by these formulae will have the desired properties.
Finally, by Compactness, as the parameters $\bar{a}$ vary, we use only finitely many formulae $\lambda^+_i(x, y)$ and $\lambda^-_i(x, y)$. Combining these formulae as in Theorem 2.2.7, we get a uniform set of formulae which work for all instances of $\varphi(x, \bar{y})$.

**Corollary 6.2.10.** Suppose that $T$ is quasi-VC-minimal and $\Psi$ is almost unpackable. Let $A$ be any set, $\varphi(x, \bar{y})$ any formula, and $\bar{a} \in A$ any parameters. Then the set $\varphi(x, \bar{a})$ is the disjoint union of finitely many sets of the form $D_i \cap Z_i$, where each $D_i$ is an $A$-irreducible ucheese and each $Z_i$ is $\emptyset$-definable.

*Proof.* By Proposition 6.2.9, $\varphi(x, \bar{a})$ is the union of finitely many sets $C_i \cap Z_i$, where each $Z_i$ is $\emptyset$-definable and each $C_i$ is an $A$-definable ucheese. By almost unpackability, each $C_i$ is the union of finitely many $A$-irreducible ucheeses $D_j$. $\square$

Finally, as in VC-minimal theories, we have results allowing us to generate definable types based on uballs.

**Proposition 6.2.11.** Suppose that $T$ is quasi-VC-minimal and $\Psi$ is almost unpackable. Let $B \subseteq A$ be sets, and let $Z$ be a $\emptyset$-definable set, defined by a formula $\xi(x)$. Let $p^*$ be a $B$-definable partial type over $A$ which is complete for $A$-definable uballs, such that $p^*(x) \cup \{\xi(x)\}$ is consistent. Then there is a complete $B$-definable type $p \in S(A)$ which contains $p^*$ and $\xi(x)$.

*Proof.* First, build the $B$-definable type $p^* \cap Z = \{\varphi(x, \bar{a}) \land \xi(x) : \varphi(x, \bar{a}) \in p^*\}$. Take $p$ to be any complete type over $A$ extending $p^* \cap Z$. Then $p$ is definable for sets of the form $C \cap Y$, where $C$ is an $A$-definable uball and $Y$ is $\emptyset$-definable, and so by Proposition 6.2.9 all of $p$ is definable. $\square$

**Proposition 6.2.12.** Suppose that $T$ is quasi-VC-minimal and $\Psi$ is almost unpackable. Let $B \subseteq A$ be sets, let $C$ be a uball which is definable over $B$ and irreducible over $A$, and let $Z$ be a $\emptyset$-definable set such that $C \cap Z \neq \emptyset$. Then there
is a type $p_{C \cap Z} \in S_1(A)$ which is definable over $B$ such that for any $A$-definable uball $D$, the set $D \cap Z$ is in $p_{C \cap Z}$ iff $C \subseteq D$.

Proof. Build a partial type $p^*$ consisting of $A$-definable uballs and their complements, such that any $A$-definable uball $D$ is in $p^*$ iff $C \subseteq D$. $p^*$ is definable over $B$, and is consistent by irreducibility. Apply Proposition 6.2.11 to $p^*$ and $Z$. □

Theorem 6.2.13. Suppose that $T$ is quasi-VC-minimal, and suppose that $\Psi$ is almost unpackable. Let $U$ be a large saturated model of $T$, and let $M \preceq U$ be a small model. Let $\varphi(\overline{x}, \overline{y})$ be any formula, and $\overline{c}$ any parameter tuple from $U$. Then the following are equivalent:

(I) $\varphi(\overline{x}, \overline{c})$ does not fork over $M$;

(II) there is a global type $p \in S_{\varphi}(U)$ which is definable over $M$ with $\varphi(\overline{x}, \overline{c}) \in p$.

Proof. Most of this proof is identical to that of Theorem 6.1.1; we outline the changes needed for that proof to work in a quasi-VC-minimal theory.

Base Case: Suppose $\varphi(x, \overline{c})$ does not fork over $M$.

Without loss of generality, assume that $\varphi(x, \overline{c})$ is of the form $S \cap Z$, where $S$ is an unbreakable Swiss cheese and $Z$ is $\emptyset$-definable. Let $\mu(x, \overline{c})$ define $S$, and let $\xi(x)$ define $Z$. For each $\overline{c'} \equiv_0 \overline{c}$, the set $\mu(x, \overline{c'})$ is an unbreakable Swiss cheese.

Choose $\theta(\overline{y}) \in \text{tp}(\overline{c}/M)$ such that $\{\varphi(x, \overline{c'}) : \overline{c'} \in \theta(U)\}$ is 2-consistent and $\{\mu(x, \overline{c'}) : \overline{c'} \in \theta(U)\}$ is a 2-consistent set of Swiss cheeses. Build an $M$-definable $\Psi$-type $p^*$: if $B$ is any ball, then the formula $x \in B$ is in $p^*$ iff $\exists \overline{y}[\theta(\overline{y}) \land \mu(x, \overline{y}) \subseteq B]$; otherwise, $x \notin B$ is in $p^*$.

If $p^*$ is consistent, then applying Proposition 6.2.6 to $p^*$ and $\xi(x)$, there is a complete $M$-definable type $p \in S(U)$ containing $\varphi(x, \overline{c})$. 68
If \( p^* \) is not consistent, then as in Theorem \[6.1.1\] there is a ball \( B \) in \( p^* \) such that the unbreakable balls \( C_1, \ldots, C_n \) of \( B \) are \( M \)-definable and not in \( p^* \), and \( B \) is the outer ball of some instance \( \mu(x, \overline{d}) \), where \( \overline{d} \models \theta \). Since \( \varphi(x, \overline{c}) \cap \varphi(x, \overline{d}) \neq \emptyset \), and \( \mu(x, \overline{d}) \) is partitioned by \( C_1, \ldots, C_n \), for some \( i \) the \( M \)-definable set \( C_i \cap \xi(x) \) contains points of \( \varphi(x, \overline{c}) \). It follows that the \( M \)-definable type \( \rho_{C_i \cap \xi(x)} \) as in Proposition \[6.2.8\] is definable over \( M \), and must contain \( \varphi(x, \overline{c}) \).

**Inductive Step:** As in Theorem \[6.1.1\] we arrive at a formula \( \hat{\varphi}(x, \overline{\beta}, \overline{c}) \), where the tuple \( \overline{\beta} \in \mathbb{V} \triangleleft U \) realizes an \( M \)-definable type \( q \in S(U) \), such that the family \( \{ \hat{\varphi}(x, \overline{\beta}, \overline{c}') : \overline{c}' \in U, \overline{c}' \equiv_M \overline{c} \} \) is 2-consistent. We need to build an \( M\overline{\beta} \)-definable type \( r \in S_1(U\overline{\beta}) \) containing the formula \( \hat{\varphi}(x, \overline{\beta}, \overline{c}) \).

As per Corollary \[6.2.10\] write \( \hat{\varphi}(x, \overline{\beta}, \overline{c}) \) as the union of finitely many sets \( D_i \cap Z_i \), where each \( Z_i \) is \( \emptyset \)-definable and each \( D_i \) is a \( U\overline{\beta} \)-irreducible ucheese. Fix some \( i \) such that \( D_i \cap Z_i \) does not \( \overline{\beta} \)-fork over \( M \). Let \( \tau^+(x, \overline{\beta}, \overline{d}) \) define the outer uball of \( D_i \), let \( \tau^-(x, \overline{\beta}, \overline{d}) \) define the inner uball, let \( \xi(x) \) define \( Z_i \), and let \( \tau(x, \overline{\beta}, \overline{d}) = (\tau^+(x, \overline{\beta}, \overline{d}) \setminus \tau^-(x, \overline{\beta}, \overline{d})) \cap \xi(x) \) define \( D_i \cap Z_i \). It follows that \( \{ \tau(x, \overline{\beta}, \overline{d}') : \overline{d}' \in U, \overline{d}' \equiv_M \overline{d} \} \) is 2-consistent.

Fix \( k \) such that \( \tau^+(x, \overline{\beta}, \overline{d}) \) is the union of \( k \) unbreakable balls. Choose a formula \( \chi(\overline{w}) \in tp(\overline{d}/M) \) guaranteeing 2-consistency, such that \( \overline{d} \in \chi(\mathbb{U}) \) implies that \( \tau^+(x, \overline{\beta}, \overline{d}') \) is also the union of \( k \) balls. Define a set of formulae \( r^*(x) \) as follows: for any formula \( \rho(x, \overline{\beta}, \overline{c}) \) over \( U\overline{\beta} \) defining a uball, let \( \rho(x, \overline{\beta}, \overline{c}) \in r^*(x) \) if \( \tau^+(x, \overline{\beta}, \overline{d}) \subseteq \rho(x, \overline{\beta}, \overline{c}) \) for some \( \overline{d} \in \chi(\mathbb{U}) \); otherwise, let \( \neg \rho(x, \overline{\beta}, \overline{c}) \in r^*(x) \).

If \( r^* \) is consistent, then applying Proposition \[6.2.11\] to \( r^* \) and \( \xi(x) \), we get a complete \( M\overline{\beta} \)-definable type \( r \in S_1(U\overline{\beta}) \), which must contain \( \tau(x, \overline{\beta}, \overline{d}) \) and thus \( \hat{\varphi}(x, \overline{\beta}, \overline{c}) \). Combining \( r \) with the \( M \)-definable type \( tp(\overline{\beta}/U) \), we get the desired \( M \)-definable type \( p \in S_{n+1}(U) \).
If $r^*$ is not consistent: Claims 6.1.3 and 6.1.4 still hold, so there must be an $M\bar{\beta}$-definable uball $B$ in $r^*$ which is covered by finitely many uballs $C_1, \ldots, C_j$, each of which is definable over $M\bar{\beta}$, irreducible over $U\bar{\beta}$, and not in $r^*$. For some $i$, the set $C_i \cap \xi(x)$ must contain points from $\tau(x, \bar{\beta}, \bar{d})$. Letting $r(x) \in S_1(U\bar{\beta})$ be the type $p_{C_i \cap \xi(x)}$ as in Proposition 6.2.12, which is $M\bar{\beta}$-definable, it follows that $\hat{\varphi}(x, \beta, c) \in r(x)$. Combine $r$ with $tp(\beta/U)$ to get the desired $M$-definable type $p \in S_{n+1}(U)$.

Finally, we have the following corollary, which applies to any VC-minimal or quasi-VC-minimal theory whose generating directed family is unpackable or almost unpackable.

**Corollary 6.2.14.** Suppose that $T$ is quasi-VC-minimal, and that the generating family $\Psi$ is almost unpackable. Let $U$ be a large saturated model of $T$, and let $M \subseteq U$ be a small submodel. Let $\varphi(\bar{x}, \bar{y})$ be a formula over $\emptyset$ and $\theta(\bar{y})$ a formula over $M$ such that $\{\varphi(\bar{x}, \bar{c}) : \bar{c} \in \theta(U)\}$ is $(l(x) + 1)$-consistent. Then there exist finitely many formulae $\theta_1(\bar{y}), \ldots, \theta_k(\bar{y})$ over $M$ such that:

- $\theta(\bar{y}) \rightarrow \bigvee_{i=1}^k \theta_i(\bar{y})$; and
- for each $i$, the set $\{\varphi(\bar{x}, \bar{c}) : \bar{c} \in \theta_i(U)\}$ is consistent.

**Proof.** For any $\bar{c} \in \theta(U)$, the set of formulae $\{\varphi(\bar{x}, \bar{c}') : \bar{c}' \in U, \bar{c}' \equiv_M \bar{c}\}$ is also $(l(\bar{x}) + 1)$-consistent, and so $\varphi(\bar{x}, \bar{c})$ does not fork over $M$ by Corollary 4.1.10. Thus, by Theorem 6.2.13, for each $\bar{c} \in \theta(U)$ there is an $M$-definable type $p_{\tau}(\bar{x})$ containing the formula $\varphi(\bar{x}, \bar{c})$. Let $d_{\tau}(\bar{y})$ be the formula over $M$ defining $\varphi$ for the type $p_{\tau}$. Then the set $\{\theta(\bar{y})\} \cup \{\neg d_{\tau}(\bar{y}) : \bar{c} \in \theta(U)\}$ is inconsistent. By Compactness, there are $\bar{c}_1, \ldots, \bar{c}_k \in \theta(U)$ such that $\{\theta(\bar{y})\} \cup \{\neg d_{\tau_1}(\bar{y}), \ldots, \neg d_{\tau_k}(\bar{y})\}$ is inconsistent. Set $\theta_i(\bar{y}) = d_{\tau_i}(\bar{y})$ for each $i = 1, \ldots, k$. \qed
BIBLIOGRAPHY


