PATHS, TREES, AND THE COMPUTATIONAL STRENGTH
OF SOME RAMSEY-TYPE THEOREMS

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PATHS, TREES, AND THE COMPUTATIONAL STRENGTH
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Abstract
by
Stephen Flood

An industry has arisen dedicated to the study of the interplay between combinatorial principles and computational strength. In particular, much work has been done on theorems similar to Ramsey’s Theorem and to Weak König’s Lemma. We study two related principles, which are interesting both for their combinatorial form and for their computational content.

We begin by studying the computational strength of a version of Ramsey’s Theorem that combines features of finite and infinite Ramsey theory. Paul Erdős and Fred Galvin proved that for each coloring \( f \), there is an infinite set that is “packed together” which is given “a small number” of colors by \( f \). We show that this theorem is close in computational strength to standard Ramsey’s Theorem, giving arithmetical bounds for solutions to computable instances. In reverse mathematics, we show that this Packed Ramsey’s Theorem is equivalent to Ramsey’s Theorem for exponents \( n \neq 2 \). When \( n = 2 \), we show that it implies Ramsey’s Theorem, and that it does not imply \( \text{ACA}_0 \).

We next introduce a new combinatorial principle, called \( \text{RKL} \), which combines features of Weak König’s Lemma and Ramsey’s Theorem. We show that this principle is strictly weaker than both \( \text{WKL}_0 \) and \( \text{RT}^2_2 \), and that it is strictly stronger
than RCA$_0$. We also consider two generalizations of this principle. We obtain the surprising result that these stronger principles are closer in strength to RT$_2$ than they are to WKL$_0$. 
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CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction

We study the interplay between combinatorics and computation. Together, computability theory and reverse mathematics provide a robust set of tools for studying the “computational strength” of theorems in infinite combinatorics. Two of these theorems have particularly interesting computational strength: Ramsey’s Theorem and Weak König’s Lemma.

Weak König’s Lemma is the statement “every infinite binary tree has an infinite path.” Weak König’s Lemma is essentially the statement “$2^\mathbb{N}$ is compact,” and is equivalent in reverse mathematics to many theorems about compactness (see [22]). In computability theory, the Low Basis Theorem of [18] says that every computable infinite binary tree has a low infinite path.

Ramsey’s Theorem is a generalization of the pigeonhole principle. The infinite form of Ramsey’s Theorem says that for any coloring $f$ of $n$ element subsets of $\mathbb{N}$ with $k$ colors, there is an infinite set which is given a single color by $f$. The computational strength of this theorem has been studied extensively, particularly in [1, 2, 16, 19, 21].

By studying variants of Ramsey’s Theorem and Weak König’s Lemma, we can deepen our understanding of the relationship between specific combinatorial fea-

\footnote{The previous two paragraphs appear in [10] in a slightly modified form.}
tures and computational strength. We begin in Chapter 2 by studying a theorem of Erdős and Galvin that blends features of finite and infinite Ramsey’s Theorem. We continue in Chapter 3 by studying a new combinatorial principle that blends features of Ramsey’s Theorem and Weak König’s Lemma.

We will present the motivation, combinatorial background, and a detailed summary of results at the beginning of each chapter. In the remainder of this chapter, we will give background on the two notions of “computational strength” that we study: computability theory and reverse mathematics. In Section 1.2, we discuss reverse mathematics. Chapter 3 will make frequent use of the material discussed in Section 1.2.2. In Section 1.3, we discuss two aspects of computability theory. Section 1.3.1 surveys some results on the computational complexity of trees, which we will be used during the proofs of Chapter 2. We close in Section 1.3.2 by surveying some major results concerning the strength of infinite Ramsey’s Theorem.

1.2 Reverse mathematics

The standard introduction to, and reference for, reverse mathematics is [22]. Our goal here to highlight the connections between reverse mathematics and computability theory. A key intuition of reverse mathematics is that the statement “$T_1$ implies $T_2$ over RCA$_0$” essentially means that “$T_2$ is proved using $T_1$, computable constructions, and computable verifications.” We will return to this intuition with greater detail in Section 1.2.3.

1.2.1 Second order arithmetic

In reverse mathematics, we formalize theorems of different areas of mathematics using a single language: second-order arithmetic. Because it is arithmetic, we
have symbols for 0, 1, +, ×, and <. Because it is second order, we can use two sorts of variables: one for natural numbers and one for sets of natural numbers. We also use the symbol ∈, and we have equality for number variables.

A model of this language has the form \( M = (\mathbb{N}, S(M), 0, 1, +, \times, <) \). Here \( \mathbb{N} \) denotes the first order part of \( M \) (the collection of numbers which exist in the model), and \( S(M) \) denotes the second order part (the collection of sets which exist in the model). A model of second order arithmetic is often abbreviated \( M = (\mathbb{N}, S(M)) \).

Note that in reverse mathematics, \( \mathbb{N} \) refers to the first order part of some (often unspecified, possibly non-standard) model of arithmetic. We will write \( \omega \) to refer to the first order part of the true natural numbers, when it is important to distinguish between the two.

We also include (in our language) additional symbols for set parameters and number parameters, and use a natural deduction system as the background logical axioms. More on natural deduction systems, which are equivalent in strength to the other standard logical systems, can be found in \([24]\), (particularly Chapters 1 and 2).

When presented with an existential statement, we generally work with an “arbitrary witness” for this statement. Arbitrary set parameters give a formal analog of these witnesses in our formal system. When our proof is complete, this system contains logical rules which eliminate the use of the set parameter by an appeal to the original existential assumption. Thus, set parameters will not appear in the theorems studied. Instead, they are used within individual proofs, and axiom schemes are stated to cover all possible sentences obtained by assigning different parameters to free variables.
We define levels of formula complexity as usual, with the one important remark that the presence of set parameters does not influence a formula’s complexity.

**Definition 1.2.1.** A formula is $\Delta_0^0 = \Sigma_0^0 = \Pi_0^0$ if it has only bounded number quantifiers, and no set quantifiers. A formula is $\Sigma_{n+1}^0$ if it has the form $(\exists x)\psi(x)$ for $\psi$ a $\Pi_n^0$ formula, and a formula is $\Pi_{n+1}^0$ if its negation is $\Sigma_{n+1}^0$. A formula $\theta(x)$ is $\Delta_n^0$ if there is a $\Pi_n^0$ formula $\psi(x)$ and a $\Sigma_n^0$ formula $\phi(x)$ such that $(\forall x)[\theta(x) \equiv \psi(x) \equiv \phi(x)]$. A formula is *arithmetical* if it is $\Delta_n^0$ for any $n$.

A formula is $\Pi_1^1$ if it has the form $(\forall X)\psi(X)$, where $\psi$ is arithmetical. A formula is $\Sigma_n^1$ if it is the negation of a $\Pi_n^1$ formula. A formula is $\Pi_{n+1}^1$ if it has the form $(\forall X)\psi(X)$, where $\psi$ is $\Sigma_n^1$. A formula $\theta(x)$ is $\Delta_n^1$ if there is a $\Pi_n^1$ formula $\psi(x)$ and a $\Sigma_n^1$ formula $\phi(x)$ such that $(\forall x)[\theta(x) \equiv \psi(x) \equiv \phi(x)]$.

### 1.2.2 The base system $\text{RCA}_0$

In order to compare the strength of theorems in second order arithmetic, we first require a set of background axioms. Let $\Gamma$ be a collection of formulas. We say that $T_1$ implies $T_2$ over $\Gamma$, if there is a proof of $T_2$ from $\Gamma \cup \{T_1\}$. We will say that we *work over* $\Gamma$, or that $\Gamma$ is our *base system*, to say that the implications being discussed hold over $\Gamma$. We will work almost exclusively with the base system $\text{RCA}_0$, which roughly corresponds to computable mathematics.

**Statement 1.2.2.** $\text{RCA}_0$ is the axiom scheme that consists of:

1. The ordered semi-ring axioms (the usual rules for $0, 1, +, \times, <$), denoted $\mathbb{Q}$.

2. $\Delta_1^0$ comprehension, denoted $\Delta_1^0$-CA. This asserts that each $\Delta_1^0$ definable set exists. More formally, $\Delta_1^0$-CA is the axiom scheme which, for each $\Pi_1^0$
formula $\phi(x, X)$ and each $\Sigma^0_1$ formula $\psi(x, X)$, asserts

$$(\forall X)[(\forall x)[\psi(x, X) \leftrightarrow \phi(x, X)]] \implies (\exists Y)(\forall x)[x \in Y \leftrightarrow \psi(x, X)],$$

3. $\Sigma^0_1$ induction, written $I\Sigma^0_1$. This is the axiom scheme asserting, for each $\Sigma^0_1$ formula $\phi(x)$ with free variable $x$, that

$$\phi(0) \land (\forall x)[\phi(x) \rightarrow \phi(x + 1)] \rightarrow (\forall x)\phi(x)$$

Inside $\textsf{RCA}_0$, we have access to a number of other first order principles, which we freely use in our verifications. We will discuss the relationship between these principles in more generality in Section 1.2.5.

Remark 1.2.3. In $\textsf{RCA}_0$, the following are provable:

1. $\Pi^0_1$ induction, written $I\Pi^0_1$.

2. $\Sigma^0_1$ least-element principle, written $L\Sigma^0_1$. This is the axiom scheme asserting, for each $\Sigma^0_1$ formula $\phi(x)$ with free variable $x$, that

$$ (\exists x)\phi(x) \rightarrow (\exists x)[\phi(x) \land (\forall y < x) \neg \phi(y)] $$

3. $\Pi^0_1$ least-element principle, written $L\Pi^0_1$.

4. $\Sigma^0_1$ bounding, written $B\Sigma^0_1$. This is the axiom scheme asserting, for each $\Sigma^0_1$ formula $\phi(x, y)$ with free variables $x, y$, that

$$ (\forall a)[(\forall x < a)(\exists y)\phi(x, y) \rightarrow (\exists b)(\forall x < a)(\exists y < b)\phi(x, y)] $$
5. It is possible to iterate an arbitrary function of the second order part an arbitrary finite number of times. This formal axiom, denoted PREC, states:

\[(\forall m)(\forall f \text{ a total function})(\exists g \text{ a total function})\]

\[g(0) = m \land (\forall z)[g(z + 1) = f(g(z))].\]

In fact, as Hirschfeld and Shore show in [12], RCA₀ can be stated in an equivalent form by replacing IΣ₁⁰ by the combination of PREC and induction for all sets in \(S(M)\). This gives additional support for the suggestion in [22] that RCA₀ corresponds roughly to “computable mathematics.”

**Remark 1.2.4.** It is also important to note that a number of standard first order proof techniques cannot be proved from RCA₀. In particular, BΠ₀¹, BΣ₀², IΔ₀², and LΔ₀² are all independant of RCA₀.

An important intuition is that RCA₀ lacks the tools required to perform verifications which reference functions of high enough arithmetical complexity. For example, verifications that use \(\emptyset'\)-computable functions often fail in RCA₀, or at least require additional subtlety.

### 1.2.3 Proofs over RCA₀

Given a \(\Pi^1_2\) axiom \(T_1\), we often wish to give a proof of some \(\Pi^1_2\) sentence \(T_2\) over RCA₀. Working over RCA₀ gives us access to three types of steps (beyond the usual logical rules) that we can use in our proof:

1. Describe how to compute a new set from a given set parameter,

2. Apply \(T_1\) to a set parameter \(X\) (obtained previously), obtaining a new set parameter \(Y\) (representing a solution of \(T_1\) for the instance \(X\)), and

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3. Prove an arithmetical fact using $\Sigma^0_1$ and/or the semi-ring axioms.

Intuitively, (1) and (2) correspond to *constructions*, because they give us the ability to begin with one set $X$ and to obtain a new set $Y$ with a certain desired property. Intuitively, (3) is used in the *verification* that the constructions behave as desired (or expected).

To be more precise, note that most $\Pi^1_2$ theorems $T_1$ have the form

$$(\forall X)[X \text{ has property } P \implies (\exists Y)Q(X,Y)].$$

Thus to apply $T_1$, and obtain the desired instance $Y$, we must first verify that $X$ does indeed have property $P$. Often, $P$ is an arithmetical statement about $X$. Furthermore, if we obtained $X$ using $\Delta^0_1$-CA, we simply know that $X$ is a set. Because $\Delta^0_1$-CA does not say that $X$ has any specific property, we must verify statements about $X$ ourselves.

It is important to note that many arithmetical statements cannot be proved using computable methods alone, and certainly not with $\Sigma^0_1$ induction alone. Therefore, proofs in reverse mathematics must be careful when verifying that $X$ truly has property $P$.

1.2.4 Subsystems of second order arithmetic

There are five central systems in reverse mathematics, sometimes called the “big five.” In order of (strictly) increasing strength, they are: $\text{RCA}_0$, $\text{WKL}_0$, $\text{ACA}_0$, $\text{ATR}_0$, and $\Pi^1_1$-CA. Each includes $\text{RCA}_0$, together with a more powerful set comprehension axiom or axiom scheme. The more powerful the comprehension principle, the more complicated the sets that can be “built” in proofs. In this thesis, we will only work with the first three of these systems: $\text{RCA}_0$, $\text{WKL}_0$ and...
ACA$_0$.

**Statement 1.2.5.** WKL$_0$ consists of RCA$_0$ together with the statement:

$$(\forall T)(\exists P)[T \text{ is an infinite binary tree } \implies P \text{ is a path through } T]$$

Weak König’s Lemma is equivalent to many theorems about compactness.

**Statement 1.2.6.** ACA$_0$ consists of RCA$_0$ together with the statement

$$(\forall X)(\exists Y)[x \in Y \iff \phi(x, X)]$$

for each arithmetical formula $\phi(x, X)$.

It is well known that ACA$_0$ is equivalent to the existence of the jump. ACA$_0$ is often equivalent to theorems asserting the existence of limits.

For any set $Z$ and any arithmetical $\theta$ with a single free set and number variable, we can define the set $\theta^Z = \{x : \theta(x, Z)\}$. Note that ACA$_0$ essentially says that we can iterate $Z \mapsto \theta^Z$ a fixed, (standard) finite number of times.

**Statement 1.2.7.** ATR$_0$ consists of RCA$_0$ together with the axiom scheme which, for each arithmetical formula $\theta$, asserts “for any well ordering $X$, the set defined by iterating $Z \mapsto \theta^Z$ along $X$ exists.”

ATR$_0$ is the weakest subsystem of arithmetic where it is possible to develop the basic theory of ordinal arithmetic [13].

**Statement 1.2.8.** $\Pi^1_1$-CA consists of RCA$_0$ together with statement

$$(\forall X)(\exists Y)[x \in Y \iff \phi(x, X)]$$
for each $\Pi^1_1$ formula $\phi(x, X)$.

For more on these systems, including an overview of the strength of many different theorems, see [22].

1.2.5 First-order subsystems of second order arithmetic

In this section, we consider the first order principles of induction, least element, and bounding discussed in Section 1.2.2, now stated for formulas of various complexity. To define these principles, simply replace $\Sigma^0_1$ with $\Gamma \in \{\Pi^0_n, \Sigma^0_n, \Delta^0_n\}$ in the both the name and the definition of the $\Sigma^0_1$ analog.

We begin by comparing the strength of these first order principles. We write $\text{exp}$ to denote the formula asserting that the exponential function is total. Note that $\text{exp}$ is provable from $I\Sigma^0_1$. Working over $Q + I\Sigma^0_1 + \text{exp}$, these principles form a linear hierarchy of strength.

**Theorem 1.2.9** (Gandy, Kirby, Paris, and Slaman). *Over $Q + I\Sigma^0_1 + \text{exp}$,*

1. $I\Sigma^0_n \leftrightarrow \Pi^0_n \leftrightarrow L\Sigma^0_n \leftrightarrow L\Pi^0_n$,

2. $B\Pi^0_n \leftrightarrow B\Sigma^0_{n+1} \leftrightarrow I\Delta^0_{n+1} \leftrightarrow L\Delta^0_{n+1}$,

3. $I\Sigma^0_{n+1} \Rightarrow B\Sigma^0_{n+1} \Rightarrow I\Sigma^0_n$.

While interesting in their own right, induction principles are not immediately informative from the perspective of “computable mathematics.” Hirschfeldt and Shore introduced the principle $PREC_n$, which roughly states that every $\Pi^0_{n-1}$-definable function can be iterated an arbitrary finite number of times. For each $n$, Hirschfeldt and Shore show that $PREC_n$ is equivalent to $I\Sigma^0_n$ over a relatively weak induction scheme.
**Definition 1.2.10** (Hirschfeldt and Shore [12]). \( \text{PREC}_n \) is the axiom scheme which asserts for each \( \Pi^0_{n-1} \) formula \( \phi(x, y) \): if \( \phi \) defines a total function, then

\[
(\forall z)(\forall m)(\exists \sigma)[|\sigma| = z \land \sigma(0) = m \land (\forall i < z)[\phi(\sigma(i), \sigma(i + 1))]]
\]

**Proposition 1.2.11** (Hirschfeldt and Shore [12]). For each \( n \in \omega \),
\( \text{PREC}_n \) is equivalent to \( \Sigma^0_n \) over \( Q + \Delta^0_1-\text{CA} + I\Delta^0_1 \).

1.3 Computability theory

For an introduction to computability theory, see of Part A of [23]. Our goal here is to provide background, and to introduce some results which will be used in Chapter 2.

1.3.1 Computability theory and trees

In Chapter 2 we will frequently define functions via initial segments. We will use trees to organize these definitions, leading us to identify elements of \( k^{<\omega} \) with initial segments of functions \( g : \mathbb{N} \rightarrow \{1, \ldots, k\} \).

We begin with the following basic definitions to fix our choice of notation. These definitions also provide a point of comparison for the more unusual trees introduced in Section 2.4.1 (used to prove Packed Ramsey’s Theorem for exponent \( n \)).

**Definition 1.3.1.** Let \( k^{<\omega} \) denote the set of all functions \( \tau \) such that for some \( w \in \mathbb{N}, \ \tau : \{1, \ldots, w\} \rightarrow \{1, \ldots, k\} \). If \( \text{dom}(\tau) = \{1, \ldots, w\} \), we will call \( w = |\tau| \) the length of \( \tau \). Given \( \tau, \rho \in k^{<\omega} \), we say that \( \tau \preceq \rho \) if and only if \( |\tau| \leq |\rho| \) and \( \tau(x) = \rho(x) \) for each \( x \in \{1, \ldots, |\tau|\} \).
**Definition 1.3.2.** A set \( T \subseteq k^{<\mathbb{N}} \) is a tree if it is closed downward under \( \preceq \). Let \([T]\) denote the set of infinite paths through \( T \subseteq k^{<\mathbb{N}} \). Then each \( g \in [T] \) is a function \( g : \mathbb{N} \to \{1, \ldots, k\} \).

A set \( X \) is low if \( X' \equiv_T \emptyset' \), and \( X \) is low\(_n\) if \( X^{(n)} \equiv_T \emptyset^{(n)} \). We say that \( a \) is a lowness index of \( X \) if \( X' = \Phi^a_{\emptyset'} \). Working relative to a set \( B \subseteq \mathbb{N} \), \( X \) is low\(_n^B\) if \( (X \oplus B)^{(n)} \leq_T B^{(n)} \).

The next theorem is often called the Low Basis Theorem.

**Theorem 1.3.3** (Jockusch and Soare [18]). For any computable infinite binary tree \( T \subseteq k^{<\mathbb{N}} \), there is a low infinite path \( g \in [T] \).

In Section 2.3 when we construct low\(_2\) solutions to the \( n = 2 \) case of Packed Ramsey’s Theorem, we will work with low trees. Cholak, Jockusch, and Slaman note in [1] that the proof of the Low Basis Theorem gives the following:

**Remark 1.3.4.** There is a \( \emptyset'\)-computable uniform procedure that takes any lowness index for an infinite low tree \( T \) and returns a lowness index for a path through \( T \).

We will also use the following standard fact:

**Remark 1.3.5.** If \( L \) is low, any statement \( S(X) \) that is \( \Pi^0_2 \) is actually \( \Pi^0_2 \).

**Definition 1.3.6.** Given \( X, P \subseteq \mathbb{N} \), we say that \( P \) is PA over \( X \), written \( P \gg X \), if \( P \) is able to compute a path through each infinite \( X \)-computable binary tree.

A tree \( T \subseteq \mathbb{N}^{<\mathbb{N}} \) is \( X \)-computably bounded if \( T \) is \( X \)-computable and if there is an \( X \)-computable function \( l : \mathbb{N} \to \mathbb{N} \) such that for each \( w \), \( l(w) \) bounds the codes for all strings in \( T \) of length \( w \).

In fact, any \( P \gg X \) can compute a path through each infinite \( X \)-computably bounded tree \( T \). In particular, any \( P \gg X \) is able to compute a path through
every infinite \( X \)-computable tree \( T \subseteq k^{<\mathbb{N}} \). For the remainder of the paper, we restrict our attention to computably bounded infinite trees.

**Lemma 1.3.7** (Lemma 4.2 of [1]). Suppose that \( P \gg \emptyset \) and that \((\gamma_e,0,\gamma_e,1)_{e<\omega}\) is an effective enumeration of all ordered pairs of \( \Pi^0_2 \) sentences of first order arithmetic. Then there is a \( P \)-computable \( \{0,1\}\)-valued (total) function \( f \) such that \( \gamma_{e,f(e)} \) is true whenever \( \gamma_{e,0} \lor \gamma_{e,1} \) is true.

Fix any \( P \gg \emptyset \) and any \( k \in \mathbb{N} \). Whenever we are given \( k \)-many \( \Pi^0_2 \) formulas \( \gamma_1, \ldots, \gamma_k \) such that \( \gamma_1 \lor \cdots \lor \gamma_k \) is true, we can use \( P \) to uniformly find \( c \in \{1, \ldots, k\} \) s.t. \( \gamma_c \) is true. Simply query the function \( f \) from the above lemma \( k - 1 \) times.

We will make frequent use of the following corollary of Lemma 1.3.7.

**Lemma 1.3.8.** Any \( P \gg \emptyset \) can compute a path through each \( \Pi^0_2 \) definable tree.

Lemma 1.3.8 is also a consequence of the following lemma, which will be useful when considering Packed Ramsey’s Theorem for \( n \)-tuples when \( n > 2 \). The proof is taken from the first half of Proposition 12 of [10].

**Lemma 1.3.9.** For any \( \Pi^0_2^{\mathcal{X}} \) tree \( T \), there is a \( \Sigma^0_1^{\mathcal{X}} \) tree \( S \) s.t. \([T] = [S] \).

**Proof.** Fix a \( \Pi^0_2^{\mathcal{X}} \)-definable tree \( T \). Then there is a formula \( \phi \) which is \( \Delta^0_1 \) such that \( \tau \in T \iff (\forall y)(\exists z) \phi(\tau, y, z) \). Using the \( \Delta^0_1 \) formula

\[
\psi(\tau, \hat{z}) = \text{def} (\forall x, y \leq |\tau|)(\exists z < \hat{z}) \phi(\tau \restriction x, y, z),
\]

we can define a \( \Sigma^0_1 \) tree \( S \) by \( \tau \in S \iff (\exists \hat{z}) \psi(\tau, \hat{z}) \). Then

\[
[S] = \{ f : (\forall w)(\exists \hat{z}) \psi(f \restriction w, \hat{z}) \} = \{ f : (\forall x)(\forall y)(\exists z) \phi(f \restriction x, y, z) \} = [T]. \]

\[\square\]
1.3.2 Computability theory and Ramsey’s Theorem

The study of the computational strength of Ramsey’s Theorem has woven together the two themes of computability theory and reverse mathematics.

**Definition 1.3.10.** Given \( X \subseteq \mathbb{N} \) and \( n \in \mathbb{N} \), we write \([X]^n = \{Z \subseteq X : |Z| = n\}\).

**Theorem 1.3.11** (Ramsey [20]). For each \( n, k \in \mathbb{N} \), and each coloring \( f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\} \), there is an infinite set \( H \) given a single color by \( f \).

We say that the one colored set \( H \) is *homogeneous* for \( f \). We write \( \text{RT}_k^n \) to refer to Ramsey’s Theorem (for fixed \( n \) and \( k \) as above) formalized in the context of second order arithmetic.

Jockusch fully classified the arithmetical complexity of homogeneous solutions to computable instances of Ramsey’s Theorem.

**Theorem 1.3.12** (Jockusch [16]). Fix \( n, k \in \mathbb{N} \) with \( n \geq 2 \).

1. Each computable \( f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\} \) has a \( \Pi_0^n \) definable homogeneous set.

2. There is a computable \( f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\} \) with no \( \Sigma_0^n \) definable homogeneous set.

A similar proof formalized in second order arithmetic, and working over \( \text{RCA}_0 \), gives a reverse mathematical analog.

**Corollary 1.3.13** (Simpson [22]). For each \( n \in \omega \) with \( n \geq 3 \), \( \text{RT}_2^n \) is equivalent to \( \text{ACA}_0 \) over \( \text{RCA}_0 \).

This completes our review of the computational strength of \( \text{RT}_k^n \) for \( n \geq 3 \). For colorings \( f \) of singletons, it is clear that every computable \( f \) has a computable infinite homogeneous set. The reverse mathematical strength of \( \text{RT}_1^k \) was fully characterized by Hirst in his thesis.
Theorem 1.3.14 (Hirst [14]). For each \( k \in \omega \), \( \text{RCA}_0 \vdash \text{RT}^1_k \). On the other hand, working over \( \text{RCA}_0 \), \( (\forall k)[\text{RT}^1_k] \) is equivalent to \( \text{B}\Sigma^0_2 \).

The reverse mathematical strength of \( \text{RT}^2_k \) has taken longer to fully characterize. Work has generally proceeded by giving an increasingly refined analysis of the computational strength of homogeneous sets, which is then translated into reverse mathematical results.

Theorem 1.3.15 (Seetapun). For each computable \( f : [\mathbb{N}]^2 \to \{1, \ldots, k\} \), there is an infinite \( f \)-homogeneous set \( H \) such that \( H \) does not compute \( \emptyset' \).

Using this theorem, Seetapun proved that \( \text{RT}^2_2 \) does not imply \( \text{ACA}_0 \) over \( \text{RCA}_0 \). In fact, he showed the following stronger result.

Corollary 1.3.16. \( \text{WKL}_0 + \text{RT}^2_2 \) does not imply \( \text{ACA}_0 \) over \( \text{RCA}_0 \).

Earlier, Hirst had separated \( \text{WKL}_0 \) and \( \text{RT}^2_2 \) by showing that \( \text{WKL}_0 \) is not able to prove \( \text{RT}^2_2 \).

Theorem 1.3.17 (Hirst [14]). \( \text{WKL}_0 \) does not imply \( \text{RT}^2_2 \) over \( \text{RCA}_0 \).

Hirst gives two different proofs of this separation. The second, more powerful proof produces an \( \omega \)-model of \( \text{WKL}_0 \) where \( \text{RT}^2_2 \) fails. The first proof of this separation uses the fact \( \text{RT}^2_2 \) implies \( (\forall k)[\text{RT}^1_k] \), and that \( \text{WKL}_0 \) does not. Harrington showed in more generality that \( \text{WKL}_0 \) does not have first order strength beyond \( \text{RCA}_0 \).

Theorem 1.3.18 (Harrington). \( \text{WKL}_0 \) is \( \Pi^1_1 \) conservative over \( \text{RCA}_0 \).

Further research on the computational strength of Ramsey’s Theorem for pairs yielded a conservativity result for \( \text{RT}^2_2 \). Cholak, Jockusch, and Slaman showed the following.
Theorem 1.3.19 (Cholak, Jockusch, Slaman [1]). For each computable coloring $f : \mathbb{N}^2 \to \{1, \ldots, k\}$, there is a an infinite low$_2$ set that is homogeneous for $f$.

Adapting the proof of the above theorem, Cholak, Jockusch, and Slaman showed that $RT_2^2$ is $\Pi^1_1$ conservative over $RCA_0 + I\Sigma^0_2$.

For many years, it was unknown if $RT_2^2$ implied $WKL_0$. This was answered recently by Liu, who produced an $\omega$-model of $RT_2^2$ where $WKL_0$ fails.

Theorem 1.3.20 (Liu [19]). For each $f : \mathbb{N}^2 \to \{1, \ldots, k\}$ s.t. $f \not\gg \emptyset$, there is an infinite $f$-homogeneous set $H$ such that $f \oplus H \gg \emptyset$.

Corollary 1.3.21 (Liu [19]). $RT_2^2$ does not imply $WKL_0$ over $RCA_0$.

In other words, there is no way to use $RT_2^2$, together with computable constructions and verifications, to prove $WKL_0$. 

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Figure 1.1. The big 5 subsystems and $\text{RT}_2^2$. 

\begin{center}
\begin{tikzcd}
\Pi_1^1\text{-CA} \arrow{d} \\
\text{ATR}_0 \arrow{d} \\
\text{RCA}_0 \arrow{d} \arrow{dd} \\
\text{RT}_2^2 \arrow{dd} & \text{WKL}_0 \arrow{dd} \\
\text{RCA}_0 \\
\end{tikzcd}
\end{center}
2.1 Introduction

The pigeonhole principle says that for any \( w \) and \( k \in \mathbb{N} \), if \( w \)-many pigeons try to roost in \( k \) holes, at least one hole will contain at least \( \lceil \frac{w}{k} \rceil \) pigeons.

More generally, fix \( n, k, \) and \( m \in \mathbb{N} \). Instead of acting alone, suppose that our pigeons are team players – where each team has \( n \) members. Any set of \( n \) pigeons is a possible team. When an \( n \) pigeon team is chosen, it selects some hole and all members of this team roost in this hole. Different teams (even ones that differ by a single pigeon) may chose very different roosts. In this case, finite Ramsey’s Theorem says that if the number of pigeons \( w \) is large enough, then there will be collection of \( m \) pigeons and a hole \( c \) such that each \( n \) pigeon team chosen from this collection will roost in hole \( c \). Along these lines, we make the following definitions:

**Definition 2.1.1.** Fix any \( X \subseteq \mathbb{N} \) and \( n, k \in \mathbb{N} \). We write \([X]^n\) to refer the set of \( n \)-element subsets of \( X \). That is, \([X]^n = \{ Z \subseteq X : |Z| = n \}\).

Given a coloring \( f : [X]^n \to \{1, \ldots, k\} \), we say \( H \) is *homogeneous for \( f \)* if \( f \) assigns a single color to \([H]^n\).

**Theorem 2.1.2** (Finite Ramsey’s Theorem). Fix any \( n, k, m \in \mathbb{N} \). There is some \( w \in \mathbb{N} \) such that for any coloring \( f : \{1, \ldots, w\}^n \to \{1, \ldots, k\} \), there is a size \( m \) homogeneous set.
The following notation is very useful when working with finite Ramsey’s Theorem.

**Definition 2.1.3.** Given $w, m, n, k$, we say that $w \rightarrow (m)^n_k$ if for each $X \subseteq \mathbb{N}$ such that $|X| = w$ and each coloring $f : [X]^n \rightarrow \{1, \ldots, k\}$, there is a homogeneous $H \subseteq X$ with $|H| = m$.

In arrow notation, finite Ramsey’s Theorem says that for any $n, k, m \in \mathbb{N}$, there is a $w \in \mathbb{N}$ such that $w \rightarrow (m)^n_k$.

**Theorem 2.1.4 (Infinite Ramsey’s Theorem).** Fix any $n, k \in \mathbb{N}$. For each $f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\}$, there is an infinite $H \subseteq \mathbb{N}$ which is homogeneous for $f$.

In arrow notation, infinite Ramsey’s Theorem simply asserts that $\mathbb{N} \rightarrow (\mathbb{N})^n_k$ for each $n, k \in \mathbb{N}$. The following notation is the standard way to refer to Ramsey’s Theorem when it is formalized in second-order arithmetic:

**Statement 2.1.5.** $\text{RT}_k^n$ is the assertion that

“$(\forall f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\})(\exists H \subseteq \mathbb{N})[H \text{ is infinite and is homogeneous for } f].$”

The infinite form of Ramsey’s Theorem is particularly nice because it tells you that a very big (infinite) homogeneous set always exists. However, even though infinite sets are big, they can be arbitrarily spread out. Erdős and Galvin use the following notion to say how spread out an infinite set is:

**Definition 2.1.6.** Fix some $\phi : \mathbb{N} \rightarrow \mathbb{N}$. We say that $A \subseteq \mathbb{N}$ is packed for $\phi$ if $|A \cap \{1, \ldots, w\}| \geq \phi(w)$ for infinitely many $w$. We say that $A \subseteq \mathbb{N}$ is sparse for $\phi$ if it is not packed for $\phi$. 

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This notion is only interesting for $\phi$ such that $\liminf_w \phi(w) = \infty$. Otherwise, any large enough finite set is packed for $\phi$. Unfortunately, there is no interesting function $\phi : \mathbb{N} \to \mathbb{N}$ such that each coloring has a homogeneous set that is packed for $\phi$. The following is essentially Theorem 2.3 of [8]:

**Theorem 2.1.7** (Erdős and Galvin [8]). Fix any $\phi : \mathbb{N} \to \mathbb{N}$ with $\liminf_w \phi(w) = \infty$ and any $n \in \mathbb{N}$, $n \geq 1$. Then there is a function $g : [\mathbb{N}]^n \to 2^{n-1}$ such that for any set $A$, either

1. $|\{g(Z) : Z \in [A]^n\}| = 2^{n-1}$ or

2. $A$ is sparse for $\phi$.

This motivates the following weakening of homogeneity:

**Definition 2.1.8.** Fix $n \in \mathbb{N}$. A set $A$ is semi-homogeneous for a coloring $f : [\mathbb{N}]^n \to \{1, \ldots, k\}$ if $A$ is given at most $2^{n-1}$ colors by $f$. That is, $A$ is semi-homogeneous if $|\{f(Z) : Z \in [A]^n\}| \leq 2^{n-1}$.

**Remark 2.1.9.** For colorings of singletons ‘semi-homogeneous’ means ‘homogeneous,’ but this changes for larger exponents. Any set that is semi-homogeneous for $f : [\mathbb{N}]^2 \to \{1, \ldots, k\}$ can be assigned up to 2 colors by $f$. For a coloring of triples, a semi-homogeneous set can be assigned up to 4 colors.
Using this weakening of homogeneity, Erdős and Galvin proved the following infinitary version of Ramsey’s Theorem, which has a finite-Ramsey flavor:

**Theorem 2.1.10 (Erdős and Galvin [8]).** Fix $n, k \in \mathbb{N}$, and any $\phi : \mathbb{N} \to \mathbb{N}$ such that $w \to (\phi(w))_{k+1}^n$ for all (big enough) $w$. For any $f : [\mathbb{N}]^n \to \{1, \ldots, k\}$, there is a set $A$ which is packed for $\phi$ and semi-homogeneous for $f$.

**Statement 2.1.11.** $\text{PRT}^n_k$ is the assertion that

$$\left( \forall \phi : \mathbb{N} \to \mathbb{N} \text{ s.t. } (\forall w)[w \to (\phi(w))_{k+1}^n]\right) \left( \forall f : [\mathbb{N}]^n \to \{1, \ldots, k\}\right) \left( \exists H \subseteq \mathbb{N} \right)[H \text{ is packed for } \phi \text{ and semi-homogeneous for } f].$$

In second-order arithmetic (when we are working over $\text{RCA}_0$), $\text{PRT}^n_k$ will refer to this $\Pi^1_2$ formula. When we are not working over $\text{RCA}_0$, we will sometimes abuse this notation and write $\text{PRT}^n_k$ to refer to Theorem 2.1.10 itself.

Because $\text{PRT}^n_k$ is trivial when $\lim \inf w \phi(w) < \infty$, our proofs of $\text{PRT}^n_k$ will always assume that $\lim \inf w \phi(w) = \infty$. In this case, any set $A$ which is packed for $\phi$ is automatically infinite.

Some asymptotic lower bounds are known for the fastest growing $\phi$ which satisfies $w \to (\phi(w))_{k+1}^n$. For $n \in \mathbb{N}$, $\log_{n-1}$ denotes the $n - 1$-iterated logarithm.

**Theorem 2.1.12** (Theorem 26.6 of [9]). For any integers $n, k \geq 2$, there is a constant $\hat{c}_{n,k} > 0$ depending only on $n$ and $k$ such that for all large enough $w$, $w \to (\hat{c}_{n,k} \cdot \log_{n-1} w)^n_k$.

In other words, for each $n, k \geq 2$, the function $\phi(w) = \hat{c}_{n,k+1} \cdot \log_{n-1} w$ satisfies the conditions of $\text{PRT}^n_k$ for all large enough $w$. The next theorem gives upper bounds on the rate of growth of $\phi$. 
Theorem 2.1.13 (Theorem 26.3 of [9]). For $n \geq 3$, there are constants $c_n > 0$ and $d_n > 0$ depending only on $n$ such that for all large enough $w$,

$$w \not\rightarrow (c_n \cdot \sqrt{\log n - 2 w})^n_2,$$

$$w \not\rightarrow (d_n \cdot \log n - 1 w)^n_4$$

Thus for each $n, k \geq 3$ and each $\phi$ as in $\text{PRT}^n_k$, we have $\phi(w) < d_n \cdot \log n - 1 w$ for all large enough $w$.

2.1.1 Summary of results

Definition 2.1.14. Given $n, k \in \mathbb{N}$, a computable instance of $\text{PRT}^n_k$ is computable coloring $f : \mathbb{N}^n \rightarrow \{1, \ldots, k\}$ and a computable $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $w \rightarrow (\phi(w))^{n}_{k+1}$ for all $w$. We say that $A \subseteq \mathbb{N}$ is a solution to a computable instance of $\text{PRT}^n_k$ if $A$ is packed for the appropriate $\phi$, and semi-homogeneous for the appropriate $f$.

From the perspective of computability theory, we show:

Theorem 2.1.15. Fix $n, k \in \mathbb{N}$.

1. For any $P \gg \emptyset^{(n-1)}$, each computable instance of $\text{PRT}^n_k$ has a $P$-computable solution. Hence, there is always a $\Delta^0_{n+1}$ solution.

2. If $n \geq 2$, there is a computable instance of $\text{PRT}^n_k$ with no $\Sigma^0_n$ solution.

3. Any computable instance of $\text{PRT}^2_k$ has a low$_2$ solution.

4. Each computable instance of $\text{PRT}^1_k$ has a computable solution.
Proof. (1) is Theorem 2.5.1 and (2) is Theorem 2.6.2. (3) is Theorem 2.3.1 in the case where $B$ is computable. (4) is Corollary 2.1.20. □

From the perspective of reverse mathematics, we show:

**Theorem 2.1.16.** Over $\text{RCA}_0$,

1. $\text{PRT}_k^n$ is equivalent to $\text{RT}_k^n$ for each $n \in \omega$ such that $n \neq 2$ and each $k \in \mathbb{N}$ such that $k > 2^{n-1}$,

2. $\text{PRT}_k^2$ implies $\text{RT}_k^2$ for each $k \in \mathbb{N}$, and

3. $\text{PRT}_k^2$ does not imply $\text{ACA}_0$ for any $k \in \mathbb{N}$.

4. $(\forall k) \text{PRT}_k^1$ is equivalent to $B\Sigma^0_2$.

**Proof.** (2) is Theorem 2.6.1 for exponent $n = 2$, and (3) is Corollary 2.3.2. For (4), recall that $(\forall k) \text{RT}_k^1$ is equivalent to $B\Sigma^0_2$ by Theorem 1.3.14.

To prove (1), we work over $\text{RCA}_0$. First, consider $n = 1$. For any $k \in \mathbb{N}$, $\text{RT}_k^1$ implies $\text{PRT}_k^1$ by Theorem 2.1.18 and $\text{PRT}_k^1$ implies $\text{RT}_k^1$ by Theorem 2.6.1.

Next, consider any $n \in \omega$ with $n \geq 3$ and any $k \in \mathbb{N}$ with $k > 2^{n-1}$. Because $n \geq 3$, $\text{RT}_k^n$ is equivalent to both $\text{RT}_2^n$ and $\text{ACA}_0$. By Theorem 2.6.1, $\text{PRT}_k^n$ implies $\text{RT}_2^n$, so $\text{PRT}_k^n$ also implies $\text{RT}_k^n$. By Theorem 2.5.1, $\text{ACA}_0$ implies $\text{PRT}_k^n$, so $\text{RT}_k^n$ also implies $\text{PRT}_k^n$. □

2.1.2 Outline

We begin in Section 2.1.3 with a proof of $\text{PRT}_k^1$ from $\text{RCA}_0 + \text{RT}_k^1$. In Section 2.2, we prove $\text{PRT}_k^2$ using paths through a $\Pi^0_2$-definable tree. In Section 2.3, we adapt this proof to produce low$^2$ solutions to computable instances of $\text{PRT}_k^2$. 22
In Sections 2.4.1 and 2.4.2 we present the combinatorial tools which we use to prove $\text{PRT}_n^k$. In Section 2.5 we show that each computable instance of $\text{PRT}_n^k$ has a solution which can be computed using any path through a certain $\Pi_0^n$-definable tree.

These proofs share a common proof method, and the intuition from the earlier proofs is helpful in the later proofs. In fact, Section 2.2 is exactly the $n = 2$ case of the proof of $\text{PRT}_n^k$ in Section 2.5. It is given separately to illuminate both the $\text{low}_2$ proof of $\text{PRT}_k^2$ and the general proof of $\text{PRT}_k^n$.

**Definition 2.1.17.** Suppose we have fixed $f$ and $\phi$ as in $\text{PRT}_n^k$ for some $n, k$. A finite set $Y \subseteq \mathbb{N}$ is a *block* if it is $f$-homogeneous and there is $w \in \mathbb{N}$ such that $Y \subseteq \{1, \ldots, w\}$ and $|Y| \geq \phi(w)$. We say that a sequence of blocks $\{Y_i\}_{i \in I}$ is an *increasing sequence of blocks* if $\max(Y_i) < \min(Y_{i+1})$ for each $i$.

The central idea of these proofs is (1) to define paths through certain trees, then (2) to use these paths to define an increasing sequence of blocks, and finally (3) to refine this sequence to obtain the desired set.

In Section 2.6 we prove lower bounds on the complexity of $\text{PRT}_n^k$. We begin by using $\text{PRT}_n^{2n-1-1+k}$ to prove to $\text{RT}_k^n$ over $\text{RCA}_0$. Adapting this argument, we show that there is a computable instance of $\text{PRT}_n^{2n-1+1}$ that has no $\Sigma^0_n$ solution.

### 2.1.3 A proof of $\text{PRT}_k^1$

We begin by showing:

**Theorem 2.1.18 (RCA$_0$).** For each $k \in \mathbb{N}$, $\text{RT}_k^1$ implies $\text{PRT}_k^1$.

In this proof, no tree is needed: we simply select and refine a sequence of blocks. We will use the following fact to obtain the sequence of finite sets:
Claim 2.1.19 (RCA₀). Suppose φ : N → N satisfies w → (φ(w))ₖ₊₁ for all w. Then (∀m)(∃w > m)[w − m → (φ(w))ₖ].

Proof. Given m, take w large enough so that φ(w) > m. Fix any A ⊆ N with |A| = w − m, and any coloring f : A → {1, ..., k}. We must obtain a φ(w)-element homogeneous set. First, select any X ⊆ N of size w such that A ⊆ X. Next, define ˆf : X → {1, ..., k, k₊₁} by setting ˆf(x) = f(x) if x ∈ A, and setting ˆf(x) = k₊₁ if x /∈ A.

Let Y be a ˆf-homogeneous subset of X of size φ(w). Then Y is ˆf homogeneous with color c ∈ {1, ..., k} since the color k₊₁ was assigned to m < φ(w) numbers. It follows that Y ⊆ A is the desired f homogeneous set of size φ(w).

Proof of Theorem 2.1.18. Fix f : [N]¹ → {1, ..., k} and φ as in PRTₖ¹. We produce a set A which is packed for φ and semi-homogeneous for f. Because n = 1, ‘semi-homogeneous’ means ‘homogeneous.’

Inductively define an increasing sequence w₀ < w₁ < ... by setting w₀ = 1 and wᵢ₊₁ to be the least w > wᵢ such that w − wᵢ → (φ(w))ₖ. By Claim 2.1.19, wᵢ₊₁ exists whenever wᵢ exists. Notice that wᵢ is defined by iterating a total ∆₀₁ function i many times. It follows that i → wᵢ is total by Σ₀₁ induction (and Proposition 6.5 of [12]). Furthermore, {wᵢ : i ∈ N} is unbounded by Σ₀₁ induction.

For each i, let Yᵢ ⊆ (wᵢ, wᵢ₊₁] be the f-homogeneous subset of size φ(wᵢ₊₁) with least index as a finite set. For each i, Yᵢ exists because wᵢ₊₁ − wᵢ → (φ(wᵢ₊₁))ₖ. This sequence has a ∆₀₁ definition.

The sequence {Yᵢ} induces a coloring g : N → {1, ..., k} such that g(i) is the color given to any/all x ∈ Yᵢ by f. Then g is ∆₀₁ because f and the Yᵢ are both ∆₀₁, and g is well defined because each Yᵢ is f-homogeneous. By ∆₀₁ comprehension, g exists as a set in the model. By RTₖ¹, there is a c ∈ {1, ..., k} and an infinite
Let $A = \bigcup_{i \in H} Y_i$. Clearly, $A$ is $f$-homogeneous. Furthermore, $Y_i \subseteq A$ for all $i \in H$, hence $|A \cap \{1, \ldots, w_i\}| \geq \phi(w_i)$ for each $i \in H$. Recall that $H$ is infinite. In other words, $A$ is packed for $\phi$ and homogeneous for $f$.

Recall that $\text{RCA}_0$ is true in the the $\omega$-model of second order arithmetic whose second order part is the collection of all computable sets (that is, $(\omega, \text{REC}) \models \text{RCA}_0$). Thus $\text{PRT}^1_k$ is true in $(\omega, \text{REC})$, and we obtain the following corollary.

**Corollary 2.1.20.** For each computable $f : \mathbb{N} \to \{1, \ldots, k\}$ and each computable $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(w) \leq \lceil \frac{w}{k+1} \rceil$ for all $w$, there is a computable set $A$ which is packed for $\phi$ and homogeneous for $f$.

2.2 A tree proof of $\text{PRT}^2_k$

We begin by proving:

**Theorem 2.2.1.** Fix any $P \gg \emptyset$. For any computable $f : [\mathbb{N}]^2 \to \{1, \ldots, k\}$ and any computable $\phi$ as in $\text{PRT}^2_k$, there is a set $A$ computable from $P$ which is packed for $\phi$ and semi-homogeneous for $f$.

Both the statement and the proof of Theorem 2.2.1 are the $n = 2$ case of Theorem 2.5.1. In Section 2.3, we will adapt this proof to obtain the low$_2$ proof of $\text{PRT}^2_k$, and in Section 2.5 we will generalize it to prove $\text{PRT}^n_k$.

For the remainder of this section, fix a computable instance of $\text{PRT}^2_k$. That is, fix a computable coloring $f : [\mathbb{N}]^2 \to \{1, \ldots, k\}$ and a computable total function $\phi : \mathbb{N} \to \mathbb{N}$ such that $w \to (\phi(w))_{k+1}^2$ for all $w$. Recall that we assume, without loss of generality, that $\liminf_w \phi(w) = \infty$. 

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2.2.1 Outline

In proofs of Ramsey’s Theorem, one usually builds homogeneous sets by adding one number at each step.

We will build packed sets by adding a block set \( Y \) at each step\(^7\) Our goal is to build a semi-homogeneous (2-colored) set where each pair \( x, y \) in the same block is given one single fixed color, and each pair \( x, y \) in different blocks is given another (possibly different) fixed color. We ensure that we can always pick the next block by keeping the set of potential future numbers sufficiently “large.”

During the construction, each block will be \( f \) homogeneous, and all elements of our finite set \( Y \) will be given a single color with all elements of future blocks. At the end of the construction, we will apply the infinite pigeonhole principle twice: once to ensure that all blocks are homogeneous with the same color, and once to ensure that there is a single color given to pairs in different blocks.

2.2.2 Largeness for exponent 2

We will use a helper coloring \( g : \mathbb{N} \to \{1, \ldots, k\} \) to define this sequence of blocks. When we select any block \( Y \), we will commit to choosing all future blocks inside \( \{ y : (\forall x \in Y)[f(x,y) = g(x)] \} \). By choosing each \( Y \) to be \( g \)-homogeneous, we ensure that the elements of each \( Y \) are given a single color with all future blocks.

The notion of “largeness” given by Erdős and Galvin is \( \Pi^1_1 \) (quantifying over all possible \( g : \mathbb{N} \to \{1, \ldots, p\} \)), and corresponds to our Lemma 2.2.5. To make the construction computable relative to some \( P \gg \emptyset' \), we work with the following \( \Pi^0_2 \) definition of largeness:

\(^7\)Recall that we call a finite set \( Y \) a block if it is \( f \)-homogeneous and there is \( w \in \mathbb{N} \) such that \( Y \subseteq \{1, \ldots, w\} \) and \( |Y| \geq \phi(w) \).
**Definition 2.2.2** (Largeness for exponent 2). A set \(X \subseteq \mathbb{N}\) is large if

\[
(\forall m)(\forall p)(\exists w)(\forall \rho \in p^w)[\exists Y \subseteq (m, w] \cap X \text{ s.t.} \\
|Y| \geq \phi(w), \\
Y \text{ is homogeneous for } f, \text{ and} \\
Y \text{ is homogeneous for } \rho.]
\]

Here, we are thinking of \(\rho\) as a partial function with domain \(\{1, \ldots, w\}\). We say \(X\) is small if \(X\) is not large. Note that “\(X\) is large” is a \(\Pi^0_2, X\) statement.

We begin with the analog of the \(n = 2\) case of Claim 1 of [8].

**Lemma 2.2.3.** \(\mathbb{N}\) is large.

*Proof.* Given \(m\) and \(p\), define \(w\) large enough so that \((\phi(w) - m) \to (2)^1_p\). For any \(\rho \in p^w\), we must obtain \(Y \subseteq (m, w]\) as in the definition of largeness.

First, define a coloring \(F : [\{1, \ldots, w\}]^2 \to \{1, \ldots, k, k + 1\}\) as follows:

For \(Z \in [\{1, \ldots, w\}]^2\), set \(F(Z) = f(Z)\) if \(Z \subseteq (m, w]\) and \(Z\) is \(\rho\)-homogeneous. Otherwise, set \(F(Z) = k + 1\).

Because \(w \to (\phi(w))^2\), there is a set \(Y \subseteq \{1, \ldots, w\}\) such that \(|Y| \geq \phi(w)\) and \(Y\) is \(F\) homogeneous for some \(i \in \{1, \ldots, k, k + 1\}\).

We show that \(i \neq k + 1\): Because \(\phi(w) - m \to (2)^1_p\), there is an 2-element subset \(Z \subset Y \cap (m, w]\) which is \(\rho\)-homogeneous. By definition of \(F\), \(F(Z) \neq k + 1\). Because \(Y\) is \(F\) homogeneous, we see that \(i = F(Z) \neq k + 1\).

Consequently, \(Y\) is \(f\)-homogeneous, \(Y \subseteq (m, w]\), and \(|Y| \geq \phi(w)\). To see that \(Y\) is \(\rho\)-homogeneous, notice that any 2-element subset of \(Y\) is \(\rho\)-homogeneous. \(\square\)

Next, we give the analog of the \(n = 2\) case of Claim 2 of [8].

\[\text{This is where we use the assumption in PRT}^2_k \text{ that } w \to (\phi(w))^2_{k+1} \text{ for all } w.\]
Lemma 2.2.4. The union of two small sets is small. Therefore, for any partition \( L = L_1 \cup \cdots \cup L_k \) of a large set \( L \), one of the \( L_i \) is large.

Proof. Given \( S_1 \) and \( S_2 \) small, fix \( m_i, p_i, \) and \( w \mapsto \rho_{i,w} \in p^w \) witnessing the smallness of \( S_i \). Define \( m = \max\{m_1, m_2\} \) and \( p = p_1 \cdot p_2 \cdot 2 \). Define \( s : \mathbb{N} \to \{1, 2\} \) by \( s(x) = 1 \) if \( x \in S_1 \), and \( s(x) = 2 \) otherwise. Given \( w \), define \( \hat{\rho}_w(x) = \langle \rho_{1,w}(x), \rho_{2,w}(x), s(x) \rangle \) for each \( x \leq w \).

Suppose toward a contradiction that \( S_1 \cup S_2 \) is large. Then there is some \( \hat{w} \) witnessing that \( S_1 \cup S_2 \) is large for \( p \) and \( m \) defined as above. Obtain the set \( \hat{Y} \subseteq S_1 \cup S_2 \) promised by the definition of largeness applied to \( m, p, \hat{w}, \) and \( \hat{\rho}_w \). Note that \( \hat{Y} \) is homogeneous for \( s \), so \( \hat{Y} \subseteq S_i \) for some \( i \). In either case, \( \hat{Y} \) is contained in the interval \( (m_i, \hat{w}] \), is homogeneous for \( f \) and \( \rho_{i,\hat{w}} \), and has size \( |\hat{Y}| \geq \phi(\hat{w}) \). This contradicts our choice of witnesses of the smallness of \( S_i \).

2.2.3 The construction

To prove \( \text{PRT}_k^2 \), we first show that if \( X \) is large in the sense of Definition 2.2.2, it is large in the sense used by Erdős and Galvin:

Lemma 2.2.5 (The inductive step). If \( X \) is large and \( g : \mathbb{N} \to \{1, \ldots, p\} \), then for each \( m \in \mathbb{N} \) there is a \( w \in \mathbb{N} \) and \( Y \subseteq (m, w] \cap X \) such that \( |Y| \geq \phi(w) \) and \( Y \) is homogeneous for \( f \) and \( g \).

Proof. Fix \( X \) large, \( g : \mathbb{N} \to \{1, \ldots, p\} \), and any \( m \in \mathbb{N} \). Find \( w \in \mathbb{N} \) as in the definition of largeness. Then \( g \upharpoonright w \in p^w \) so there is a set \( Y \subseteq (m, w] \) with \( |Y| \geq \phi(w) \) homogeneous for \( g \upharpoonright w \) and \( f \). Hence, \( Y \) is homogeneous for \( g \). □

We will use a single well-chosen helper coloring \( g : \mathbb{N} \to \{1, \ldots, k\} \) to build a packed semi-homogeneous set.

\(^3\)Note that \( p > p_i \). This is why Definition 2.4.4 quantifies over all possible choices of \( p \).
Lemma 2.2.6 (Obtaining a guide). Given any $P \gg \emptyset'$, there is a $P$-computable coloring $g : \mathbb{N} \to \{1, \ldots, k\}$ such that for each $w \in \mathbb{N}$, the set $\{y > w : (\forall x \leq w)[f(x, y) = g(x)]\}$ is large.

**Proof.** We begin by defining a $\Pi_2^0$ tree $T$. For each $\tau \in k^{<\mathbb{N}}$,

$$\tau \in T \iff \{y > |\tau| : (\forall x \leq |\tau|)[\tau(x) = f(x, y)]\} \text{ is large}.$$

We show that $T$ is infinite by induction on $|\tau|$. The empty string is an element of $T$, by Lemma 2.2.3. Suppose $\tau \in T$. Then $\{y > |\tau| + 1 : f(|\tau| + 1, y) = i \land (\forall x \leq |\tau|)[\tau(x) = f(x, y)]\}$ is large for some $i \in \{1, \ldots, k\}$ by Lemma 2.2.4. It follows that $\tau \dashv i \in T$ for some $i \in \{1, \ldots, k\}$.

By Lemma 1.3.8, any $P \gg \emptyset'$ is able to compute a path $g \in [T]$. By the definition of $T$, the set $\{y > w : (\forall x < w)[g(x) = f(x, y)]\}$ is large for any $w$. In other words, $g$ is the desired helper coloring. \hfill \square

Lemma 2.2.7 (Building a packed semi-homogeneous set). Fix $g : \mathbb{N} \to \{1, \ldots, k\}$ such that for each $w \in \mathbb{N}$, the set $\{y > w : (\forall x \leq w)[f(x, y) = g(x)]\}$ is large. Then there is a set $A$ computable from $g$ which is packed and semi-homogeneous.

**Proof.** We begin by computing an increasing sequence of blocks $\{Y_i\}$ and an infinite set $\{w_0 < w_1 < \ldots\}$ such that for each $i$, two properties hold: (1) $Y_i \subseteq (w_{i-1}, w_i]$ with $|Y_i| \geq \phi(w_i)$ and (2) there is a color $c_i$ such that for each $j > i$, each $x \in Y_i$, and each $y \in Y_j$, we have $f(x, y) = c_i = g(x)$.

We proceed by induction on $s$. Let $w_1$ and $Y_1$ be the number and set obtained by applying Lemma 2.2.5 to the large set $\mathbb{N}$ with $m = w_0 = 1$ and $p = k$.

---

4Here, $\tau \dashv i$ denotes the string obtained by adding the character $i$ to the end of the string $\tau$. 29
For the inductive step, suppose \( Y_1, \ldots, Y_s \) has been defined. By our choice of \( g \), \( X = \{ y > w_s : (\forall x \leq w_s)[f(x, y) = g(x)] \} \) is large. Let \( w_{s+1} \) and \( Y_{s+1} \) be the number and set obtained by applying Lemma 2.2.5 with \( m = w_s \) and \( p = k \) to the large set \( X \).

The set \( Y_{s+1} \) is homogeneous for \( f \) and \( g \), and \( Y_{s+1} \subseteq (w_s, w_{s+1}] \) with \( |Y_{s+1}| \geq \phi(w_{s+1}) \). In other words, \( Y_{s+1} \) is a block and property (1) holds for \( i = s + 1 \). We must show that property (2) holds for \( j = s + 1 \). Let \( i \leq s \) and let \( c_i = g(\min(Y_i)) \). Because \( Y_i \) is homogeneous for \( g \), and because

\[
Y_{s+1} \subseteq X \subseteq \{ y : (\forall i \leq s)(\forall x \in Y_i)[f(x, y) = g(x)] \},
\]

we see that \( f(x, y) = c_i = g(x) \) for each \( x \in Y_i \) and each \( y \in Y_{s+1} \). In other words, property (2) continues to hold and our construction produces the desired increasing sequence of blocks \( \{Y_i\} \).

We will now extract an infinite semi-homogeneous subsequence of blocks. By property (2), there is a total function \( d : \mathbb{N} \to \{1, \ldots, k\} \) given by \( d(i) = f(x, y) \) for any/all \( x \in Y_i \) and \( y \in Y_j \) for \( j > i \). Because each \( Y_i \) is homogeneous for \( f \), there is also a total function \( s : \mathbb{N} \to \{1, \ldots, k\} \) given by \( s(i) = f(x, y) \) for any/all \( x < y \in Y_i \). Note that \( d \) and \( s \) are computable from \( f \).

Applying the infinite pigeonhole principle twice, we obtain \( I \subseteq \mathbb{N} \) infinite and homogeneous for \( d \) and \( s \). Furthermore, we can (non-uniformly) compute \( I \) from \( g \). Let \( A = \bigcup_{i \in I} Y_i \).

To see that \( A \) is semi-homogeneous, note that the color given by \( f \) to pairs in a single \( Y_i \) is the given by \( s \), and the color given by \( f \) to pairs in different blocks is given by \( d \). Because \( I \) is homogeneous for both colorings, \( \bigcup_{i \in I} Y_i \) is given at most two colors by \( f \). Because \( I \) is infinite and because each \( Y_i \) is a block, \( A \) is packed.
Because this procedure was uniform in any $P \gg \emptyset$, the set $A = \bigcup_{i \in I} Y_i$ is the desired $g$-computable packed semi-homogeneous set.

In summary:

*Proof of Theorem 2.2.1.* Suppose we have fixed $P$, $f$, and $\phi$. By Lemma 2.2.6, we obtain a $P$-computable function $g : \mathbb{N} \to \{1, \ldots, k\}$ such that for each $w \in \mathbb{N}$, the set $\{y > w : (\forall x \leq w)[f(x, y) = g(x)]\}$ is large. By Lemma 2.2.7, there is a $P$-computable set $A$ which is packed for $\phi$ and semi-homogeneous for $f$, as desired.

2.3 A $\text{low}_2$ proof of PRT$_k$

Modifying the above construction, we prove:

**Theorem 2.3.1.** Fix any set $B \subseteq \mathbb{N}$. For any $B$-computable $f : [\mathbb{N}]^2 \to \{1, \ldots, k\}$ and any $B$-computable $\phi$ as in PRT$_k^2$, there is a $\text{low}_2^B$ set $A$ which is packed for $\phi$ and semi-homogeneous for $f$.

Iterating this result, we obtain an $\omega$-model of PRT$_k^2$ with only $\text{low}_2$ sets.

**Corollary 2.3.2.** There is an $\omega$-model of $\text{RCA}_0 + \text{PRT}_k^2$ that is not a model of ACA$_0$.

In the previous section, we built a sequence of blocks $\{Y_i\}$ such that each element of $Y_i$ was given color $g(\min Y_i)$ with each element of *every* later block. In this section, we define a sequence of blocks $\{Y_i\}$ with a weaker property: each element of $Y_i$ will be given color $g(\min Y_i)$ with each element of *almost every* later block. This will allow us to use Mathias forcing (in the style of [1]) to build the sequence of blocks such that $\bigoplus_i Y_i$ is $\text{low}_2$. This induces a $\text{low}_2$ coloring of
pairs: \( d(i, j) = f(\min Y_i, \min Y_j) \). Applying the following result, we will obtain an infinite low\(_2\) semi-homogeneous sequence of blocks.

**Theorem 2.3.3** (Cholak, Jockusch, and Slaman [1]). For each computable coloring \( f : [\mathbb{N}]^2 \to \{1, \ldots, k\} \), there is an infinite low\(_2\) homogeneous set.

A coloring \( f : [\mathbb{N}]^2 \to \{1, \ldots, k\} \) is stable if \( \lim_y f(x, y) \) exists for each \( x \in \mathbb{N} \). Note that \( \lim_y d(x, y) \) will exist for each \( x \), so we will only apply the above theorem to stable colorings.

For simplicity we present the proof of 2.3.1 when \( B = \emptyset \). We leave the straightforward process of relativizing this proof to an arbitrary set \( B \) to the reader.

2.3.1 The strategy

Fix a computable coloring \( f : [\mathbb{N}]^2 \to \{1, \ldots, k\} \) and a computable function \( \phi : \mathbb{N} \to \mathbb{N} \) such that \( w \to (\phi(w))^2_{k+1} \) for all \( w \).

In this subsection, we give an overview of the construction. Beginning in 2.3.2, we will define our conditions and give the individual steps of the construction. Recall that \( \{Y_i\} \) is an increasing sequence of blocks if for each \( i \), \( Y_i \subseteq \mathbb{N} \) is \( f \)-homogeneous, \( \max(Y_i) < \min(Y_{i+1}) \), and \( |Y_i \cap \{1, \ldots, w\}| \geq \phi(w) \) for some \( w \).

**Definition 2.3.4.** For any \( I \subseteq \mathbb{N} \), an increasing sequence of blocks \( \{Y_i\}_{i \in I} \) is **pre-semi-homogeneous for** \( f \) if for each \( i, j \in I \) with \( i < j \), there is a single color \( c \) such that \( f(x, y) = c \) for any \( x \in Y_i \), and any \( y \in Y_j \).

Note that any infinite sequence of blocks \( \{Y_i\} \) that is pre-semi-homogeneous induces a coloring \( d : [\mathbb{N}]^2 \to \{1, \ldots, k\} \), where \( d(i, j) = f(\min Y_i, \min Y_j) \) for each \( \{i, j\} \in [\mathbb{N}]^2 \). We will build an infinite pre-semi-homogeneous sequence of blocks \( \{Y_i\}_{i \in \mathbb{N}} \). Furthermore, we will ensure that for each \( i \), there is a color
$c_i \in \{1, \ldots, k\}$ such that for each $x \in Y_i$, \( \lim_{y \in \bigcup Y_j} f(x, y) = c_i \). In other words, the induced coloring $d$ is stable.

In the following construction, we will build a \textit{low}$_2$ set $X$ which is the union of the desired pre-semi-homogeneous sequence of blocks $\{Y_i\}$. We will also define an infinite set $W_X = \{w_0 < w_1 < \ldots\}$ such that $Y_i = X \cap (w_{i-1}, w_i]$, and we will force $X \oplus W_X$ to be \textit{low}$_2$. Intuitively, $W_X$ records boundaries between the blocks making up $X$. In this way, we will ensure that we can uniformly recover $\{Y_i\}$ from the \textit{low}$_2$ sets we build. Let $C\{Y_i\}$ denote $X \oplus W_X$ (the code for $\{Y_i\}$).

The induced stable coloring $d$ is computable from $C\{Y_i\}$. By the stable case of Theorem 2.3.3 relativized to $C\{Y_i\}$, we obtain a homogeneous set $H$ such that $(H \oplus C\{Y_i\})'' \leq_T (C\{Y_i\})''$. Note that $\bigcup_{i \in H} Y_i$ is a $H \oplus C\{Y_i\}$-computable packed set, with a single color that is assigned by $f$ to any pair which is not contained in a single block.

For each $i \in H$, let $s(i)$ be the color assigned by $f$ to each/any pair of elements in $Y_i$. Applying the infinite pigeonhole principle to $s$, we obtain an $H \oplus C\{Y_i\}$-computable infinite set $I \subseteq H$ such that $A = \bigcup_{i \in I} Y_i$ is packed and semi-homogeneous. Note that $A$ is $H \oplus C\{Y_i\}$-computable. Because $C\{Y_i\}$ is \textit{low}$_2$ and because $(H \oplus C\{Y_i\})'' \leq_T (C\{Y_i\})''$, we see that $A'' \leq_T \emptyset''$. Thus $A$ is \textit{low}$_2$, as desired.

### 2.3.2 Conditions

The construction proceeds by Mathias forcing. For convenience, we define ‘pre-conditions’ (which have computable definitions) and ‘conditions’ (which are pre-conditions with \textit{low} sets that satisfy a certain $\Pi^0_2$ property).

We use a string $\tau$ to keep track of the colors we are committed to assigning
elements of $X$ with all large enough numbers. All the blocks will be chosen to be $f$ and $\tau$ homogeneous, but different blocks may be given different colors.

To ensure that $X \oplus W$ is low$_2$, our construction is computable by any degree PA over $\emptyset'$, and it forces the $e^{th}$ jump at stage $2e$. At stage $2e+1$, it forces $X \oplus W$ to be infinite by adding another block to the initial segment built so far.

2.3.2.1 Pre-conditions

Our pre-conditions have the form $(\tau, D, W, L)$ where $\tau \in k^{<\mathbb{N}}$ is a string, $D$ and $W$ are finite sets, and $L$ is a (possibly infinite) set.

**Definition 2.3.5.** Consider $(\tau, D, W, L)$ with $W = \{w_0 < w_1 < \ldots < w_l\}$. For each $i$ such that $1 \leq i \leq l$, set $Y_i := D \cap (w_{i-1}, w_i]$.

We say that $(\tau, D, W, L)$ is a pre-condition if (1) $D = \bigcup_{i \leq l} Y_i$, (2) $\{Y_i\}_{i \leq l}$ is pre-semi-homogeneous for $f$, (3) $|Y_i| \geq \phi(w_i)$, (4) $w_l \leq |\tau| < \min(L)$ (hence each $Y_i$ is in the domain of $\tau \in k^{<\mathbb{N}}$), and (5) each $Y_i$ is $f$- and $\tau$-homogeneous.

In the construction, we will first choose a finite extension of $\tau$, then pick a finite number of blocks $Y_i$ which are homogeneous for $\tau$. This is a key difference between the proof in this section and that in the previous section: here we interleave extending the initial segment of the helper function and extending to the sequence of blocks $\{Y_i\}$.

**Definition 2.3.6.** We say that a pre-condition $(\hat{\tau}, \hat{D}, \hat{W}, \hat{L})$ extends the pre-condition $(\tau, D, W, L)$ if the following hold: $\tau \preceq \hat{\tau}$, $D \subseteq \hat{D} \subset D \cup L$ and $W \subseteq \hat{W} \subset W \cup L$, and $\hat{L} \subseteq L$.

We wish to ensure that elements of each block in $D$ will be given a single fixed color with elements of all sufficiently large blocks.
In fact, we will ensure that each element $x$ of a block that is added to $D$ at stage $i$ will be given the color $\tau(x)$ with each element of any block which is added to $D$ at any stage $j > i$.

2.3.2.2 The tree

We use $\tau(x)$ to keep track of the color we have committed to assigning all large enough numbers with $x$. When $f$ is not a stable coloring, there may be many possible choices of $\tau$. We use a tree to organize our options for extending $\tau$. When we extend $(\tau, D, W_D, L)$, we consider the tree $T^L$ defined by

$$\sigma \in T^L \iff \{ y \in L : y > |\sigma| \land (\forall x \leq |\sigma|)[f(x, y) = \sigma(x)] \} \text{ is large.}$$

Clearly $T^L$ is $\Pi^0_2$-definable. If $L$ is large, then $\lambda \in T^L$ and Lemma 2.2.4 implies that $T^L$ has no dead ends. Note that $T^L$ has a fixed $\Pi^0_2$ definition (it does not depend on $L$ or $\tau$).

When we extend any condition $(\tau, D, W_D, L)$, we will always extend $\tau$ to a string $\hat{\tau} \in T^L$ such that $\hat{\tau} \succeq \tau$, and we will always choose $\hat{L}$ such that $\hat{\tau} \in T^L$.

**Definition 2.3.7 (Conditions).** A pre-condition $(\tau, D, W_D, L)$ is a condition if $L$ is large, $L$ is low, and $L \subseteq \{ y > |\tau| : (\forall x \leq |\tau|)[f(x, y) = \tau(x)] \}$.

In particular, each condition will have $\tau \in T^L$.

2.3.3 The module for even stages

At stage $2e$, our goal is to force $\Phi^X_e(e)$. Let $(\tau, D, W_D, L)$ be the current condition, and select $t$ such that $D = Y_1 \cup \cdots \cup Y_t$. 

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Definition 2.3.8 (The Π^{0,L}_1 class). If \( g \in k^N \), then \( g \in \mathcal{U} \) if and only if \( g \succ \tau \) and

\[
(\forall l \geq 1)(\forall w_l > \cdots > w_1 > w_0 = |\tau|)(\forall Y_{t+1}, \ldots, Y_{t+l} \text{ s.t. } Y_{t+i} \subseteq (w_{i-1}, w_i] \cap L)
\]

\[
[\text{If } \{Y_{i+i}\}_{i \leq l} \text{ is pre-semi-homogeneous for } f, \text{ and for each } i \leq l, |Y_{i+i}| \geq \phi(w_i) \text{ and } Y_{i+i} \text{ is } g \text{ homogeneous, then } \Phi_{e}^{(D, Y_{i+i} \cup \cdots \cup Y_{i+l}) \oplus (W_D \cup \{w_1, \ldots, w_l\})}(e) \uparrow].
\]

2.3.3.1 Forcing divergence

Suppose \( \mathcal{U} \neq \emptyset \). By the Low Basis Theorem (Theorem 1.3.3), there is some \( g \in \mathcal{U} \) which is low over \( L \). Because \( L \) is low, \( L \oplus g \) is low. The sets \( L \cap g^{-1}(c) \) for \( c \in \{1, \ldots, k\} \) partition the large set \( L \). For each such \( c \), \( L \cap g^{-1}(c) \) is computable from \( L \oplus g \), so is low.

By Lemma 2.2.4, there is a \( c \) such that \( L \cap g^{-1}(c) \) is large. This statement is \( \Pi^{0,L \oplus g}_2 \). Because \( L \oplus g \) is low, this statement is \( \Pi^0_2 \). Therefore a degree which is PA over \( \emptyset' \) can select one of these sets which is large.

Let \( \hat{L} = L \cap g^{-1}(c) \) for the \( c \) selected above. By our choice of \( c \), \( \hat{L} \subseteq \{y > |\tau| : (\forall x \leq |\tau|)[f(x, y) = \tau(x)]\} \), and \( \hat{L} \) is large and low. Let \( \hat{\tau} = \tau \), \( \hat{D} = D \), and \( \hat{W}_D = W_D \). Then \( (\hat{\tau}, \hat{D}, \hat{W}_D, \hat{L}) \) is a condition extending \( (\tau, D, W_D, L) \).

The definition of \( g \in \mathcal{U} \) and the \( g \)-homogeneity of \( \hat{L} \) ensures that no future initial segment of \( X \oplus W \) will cause \( \Phi_{e}^{X \oplus W}(e) \) to converge. We have thus forced that \( \Phi_{e}^{X \oplus W}(e) \uparrow \).

2.3.3.2 Forcing convergence

Suppose \( \mathcal{U} = \emptyset \). Recall that \( T^L \) has no dead ends, and that \( \tau \in T^L \) because \( (\tau, D, W_D, L) \) is a condition. In particular, \( T^L \) is infinite, and no path through \( T^L \)
is in $\mathcal{U}$.  

Note that $T^L$ is $\Pi^0_2$ because $T^L$ is $\Pi^0_2$ and because $L$ is low. Uniformly in any $P \gg \emptyset'$, we can compute longer and longer (comparable) strings in $T^L$ which extend $\tau$. Because $\mathcal{U}$ is empty, we will eventually compute a string $\hat{\tau} \in T^L$, a pre-semi-homogeneous sequence of blocks $Y_{t+1}, \ldots, Y_{t+l} \subseteq \mathbb{N}$, and dividers $w_1 < \cdots < w_l \leq |\hat{\tau}|$ which witness $\Phi^{(\cdots)\oplus(\cdots)}(e) \downarrow$. Set $\hat{D} = D \cup Y_{t+1} \cup \cdots \cup Y_{t+l}$ and $\hat{W}_D = W_D \cup \{w_1, \ldots, w_l\}$. Let $u$ be larger than all numbers appearing so far (including the use of the computation and $|\hat{\tau}|$), and set $\hat{L} = L \cap \{y \geq u : (\forall x \leq |\hat{\tau}|)[\hat{\tau}(x) = f(x, y)]\}$.  

$\hat{L}$ remains low because $\{y \geq u : (\forall x \leq |\hat{\tau}|)[\hat{\tau}(x) = f(x, y)]\}$ is a computable set. $\hat{L}$ remains large because $\hat{L} =^* L \cap \{y > |\hat{\tau}| : (\forall x \leq |\hat{\tau}|)[\hat{\tau}(x) = f(x, y)]\}$, which is large by definition of $\hat{\tau} \in T^L$. In short, $(\hat{\tau}, \hat{D}, \hat{W}_D, \hat{L})$ is a condition extending $(\tau, D, W_D, L)$.

We have made progress toward our pre-semi-homogeneous packed set, and we have forced that $\Phi^{D \oplus W_D}(e) \downarrow$.

2.3.4 The module for odd stages

At stage $2e + 1$, our goal is to ensure that $X$ is made up of at least $e + 1$ many blocks by adding another block on the end.

By the induction hypothesis $L$ is large. Applying the definition of largeness with $p = k$ and $m = |\tau|$, gives a $w$ such that for any $\rho \in k^w$ there is a block $Y \subseteq (m, w] \cap L$ with $|Y| \geq \phi(w)$ which is homogeneous for $\rho$ and $f$.

Because $T^L$ contains $\tau$ and has no dead ends, it contains a string $\hat{\tau} \succeq \tau$ of length $w$. Take $Y \subseteq (m, w] \cap L$ to be the block with $|Y| \geq \phi(w)$ that is homogeneous for $\hat{\tau}$ and $f$.

Define $\hat{L} = L \cap \{y > |\hat{\tau}| : (\forall x \leq |\hat{\tau}|)[\hat{\tau}(x) = f(x, y)]\}$. This set is large by the
definition of $\hat{\tau} \in T^L$. Note that $\hat{L}$ is low because it computable from $L$. Define $\hat{D} = D \cup Y$ and $\hat{W_D} = W_D \cup \{w\}$. Then $(\hat{\tau}, \hat{D}, \hat{W_D}, \hat{L})$ is a condition extending $(\tau, D, W_D, L)$.

2.3.5 Putting it all together

We now complete the proof Theorem 2.3.1.

Proof of Theorem 2.3.1. The construction above relativizes to any set $B \subseteq \mathbb{N}$. Fix any $P$ that is PA over $B'$ and is low over $B'$. We claim that the construction is $P$-uniform. On even stages, deciding which case to enter requires asking if a $\Pi^0_1$ class is nonempty. This can be rephrased as a $\Pi^0_1$ question, which can be answered uniformly by $P$.

Forcing divergence required selecting some $g \in U$ that is low over $L$, and finding a correct color $c$. By the second half of Theorem 1.3.3, $g$ can be found $P$-uniformly, together with an index witnessing that $g$ is low^B. As noted in the construction, $c$ can also be found $P$-uniformly.

Odd stages and forcing convergence both require finding longer and longer $\tau \in T^L$. Because $T^L$ is $\Pi^0_2$, $P$ has a uniform procedure for computing arbitrarily long initial segments of a path through $T^L$. Finally, we can computably find an index witnessing the lowness of $\hat{L}$ over $B$ using the computable reduction of the appropriate $\hat{L}$ to $L$ together with an index witnessing the lowness of $L$ over $B$.

This gives a $P$-uniform sequence of conditions $(\tau_i, D_i, W_{D_i}, L_i)$. From these conditions, we can $P$-uniformly recover a code $C\{Y_i\}$ for a sequence $\{Y_i\}$. Furthermore, the relativized construction ensures that $P$ can compute the jump of the $B \oplus C\{Y_i\}$. Because $P$ is low over $B'$, it follows that $B''$ can compute the double jump of $B \oplus C\{Y_i\}$. In other words, $C\{Y_i\}$ is low^2_B.
Refining this sequence as described in Section 2.3.1 produces a low$_2^B$ set that is packed for $\phi$ and semi-homogeneous for $f$, as desired.

2.4 Tools for proving $\text{PRT}_k^n$

2.4.1 Trees and colorings of $n$-tuples

As in Section 2.2, we will use helper colorings to prove $\text{PRT}_k^n$. When $n > 2$ we will need helper colorings which assign colors to $[\mathbb{N}]^a$ for $a \in \{1, \ldots, n-1\}$. As before, we will define these helper colorings via initial segments, which we will identify with elements of $k^{[<\mathbb{N}]}^a$.

**Definition 2.4.1.** Let $k^{[<\mathbb{N}]}^a$ denote the set of all partial functions $\tau$ such that $\tau : \{1, \ldots, w\}^a \rightarrow \{1, \ldots, k\}$ for some $w \in \mathbb{N}$. If $\tau : \{1, \ldots, w\}^a \rightarrow \{1, \ldots, k\}$, we will call $w = |\tau|$ the length of $\tau$.

Given $\tau, \rho \in k^{[<\mathbb{N}]}^a$, we say that $\tau \preceq \rho$ if and only if (1) $|\tau| \leq |\rho|$ and (2) $\tau(Z) = \rho(Z)$ for each $Z \in \{\{1, \ldots, |\tau|\}\}^a$.

**Remark 2.4.2.** We will sometimes refer to a string $\tau \in k^{[1, \ldots, w]}^a$ when $w < a$. In this case, $\text{dom}(\tau) = \emptyset$. This has the strange, but not serious, consequence that the empty string in $k^{[1, \ldots, w]}^a$ has length 0, 1, …, and $a - 1$.

While $k^{[<\mathbb{N}]}^a$ is not a $k$-ary tree, or even a $k$-branching tree, there is a computable function that bounds the strings of any given length $w$. In fact, the set $\{\sigma \in k^{[<\mathbb{N}]}^a : |\sigma| = w\}$ is computable for each $w \in \mathbb{N}$.

**Remark 2.4.3.** There are $\binom{w+1}{a} - \binom{w}{a}$ many strings in $[\{1, \ldots, w+1\}]^a$ that are not in $[\{1, \ldots, w\}]^a$. Therefore, each string in $k^{[1, \ldots, w]}^a$ has exactly $k^{(w+1)} - \binom{w}{a}$ immediate successors in $k^{[1, \ldots, w, w+1]}^a$. 

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Our motivation for working with subtrees of $k^{<\mathbb{N}^a}$ is the natural correspondence between colorings $g : [\mathbb{N}]^a \rightarrow \{1, \ldots, k\}$ and elements of $k^{[\mathbb{N}]^a}$. If $\tau \in k^{<\mathbb{N}^a}$ and $g \in k^{[\mathbb{N}]^a}$, we say that $\tau \prec g$ if $\tau(Z) = g(Z)$ for each $Z \in \{|1, \ldots, |\tau||\}^a$.

2.4.2 Largeness for exponent $n$

2.4.2.1 Motivation

We will build a sequence of blocks $\{Y_i\}$ so that the color of $Z \in \bigcup Y_i^n$ depends only on how $Z$ is partitioned by the $Y_i$. When $n = 2$, we built this sequence with the aid of a single helper function (which assigned $x$ the color it would be given with all big enough $y$).

For $n > 2$, we will need more than one helper function. In fact, when $f : [\mathbb{N}]^n \rightarrow \{1, \ldots, k\}$, we will need $2^n - 1$ helper colorings. When we select $Y$, we will need to ensure that $Y$ is homogeneous for each of the helper colorings $g_1, \ldots, g_{2^n - 1}$.

When $n = 2$, we defined what it meant for a subset of $\mathbb{N}$ to be large. For each $n > 2$, we now define what it means for a subset of $[\mathbb{N}]^{n-1}$ to be large. Before, the helper function was a map of numbers and large sets were sets of numbers. Now, the helper functions will be maps of (up to) $n - 1$-element sets and our large sets will be subsets $[\mathbb{N}]^{n-1}$. In fact, each $a \in \{1, \ldots, n - 1\}$ will be the exponent of at least one helper coloring.

2.4.2.2 Definitions and lemmas

In the construction, we will define a helper coloring of exponent $r_1$ for each ordered tuple $(r_1, \ldots, r_j)$ such that $r_1 + \cdots + r_j = n$ and $j > 1$. Fix some enumeration of these $2^{n-1} - 1$-many tuples.
For clarity, we will write \( \hat{l} = 2^{n-1} - 1 \) for the number of helper colorings. We will write \( a_i \) to refer to 1st component of the \( i \)th tuple in our enumeration (which will be the exponent of the \( i \)th helper coloring). We can define \( a_1, \ldots, a_{\hat{l}} \) using any listing of the tuples that define the helper colorings.

The earlier discussion suggests a \( \Pi^1_1 \) notion of largeness (quantifying over possible choices of the \( g_i \)). To make our constructions as effective as possible, we work with the following \( \Pi^0_2 \) notion of largeness:

**Definition 2.4.4** (Largeness for exponent \( n \)). A set \( L \subseteq [N]^{n-1} \) is *large* if

\[
(\forall m)(\forall p_1, \ldots, p_{\hat{l}} \in N)
(\exists w)(\forall p_1, \ldots, p_{\hat{l}} \text{ s.t. } \rho_i \in p_i[[1, \ldots, w]^{a_i}})
[\exists Y \subseteq (m, w) \text{ with } [Y]^{n-1} \subseteq L \text{ s.t. } |Y| \geq \phi(w), \text{ and } Y \text{ is homogeneous for } f, \text{ and } Y \text{ is homogeneous for each } \rho_i.]
\]

Here, we are thinking of each \( \rho_i \) as a partial function with domain \([\{1, \ldots, w\}]^{a_i}\). We say \( L \) is *small* if \( L \) is not large. Note that "\( L \) is large" is a \( \Pi^0_2 \) statement.

Once we have defined our helper functions, the following lemma will allow us to extract a sequence of blocks.

**Lemma 2.4.5** (The inductive step). Fix \( f : [N]^n \to \{1, \ldots, k\} \) and \( \hat{l} \)-many colorings \( g_i : [N]^{a_i} \to \{1, \ldots, p_i\} \). Suppose that \( L \subseteq [N]^{n-1} \) is large and \( m \in N \). Then there exists \( w \in N \) and \( Y \subseteq (m, w) \) such that \([Y]^{n-1} \subseteq L, |Y| \geq \phi(w)\), and \( Y \) is \( f \)- and \( g_i \)-homogeneous for each \( i \).

---

\(^5\)In this section, our definition of largeness appears to diverge from the one used by Erdős and Galvin in [8]. They give a \( \Pi^1_1 \) definition which quantifies over a *single* coloring \( g : [N]^{n-1} \to \{1, \ldots, p\} \). In fact, the proofs in this section are very similar to their analogs in [8].
Proof. Given $m$ and the $p_i$’s, let $w$ be as in the definition of largeness. Setting $\rho_i = g \upharpoonright w$ for each $i$, we obtain the desired set $Y$. \hfill \square

In the next two lemmas, we verify that Definition 2.4.4 satisfies two key properties of “largeness”: (1) the set of all $n-1$-element sets is large, and (2) any finite partition of a large set contains at least one large set.

We begin with the analog of Claim 1 in [8]:

**Lemma 2.4.6.** $[N]^{n-1}$ is large.

**Proof.** Fix $m, p_1, \ldots, p_{\widehat{l}} \in \mathbb{N}$. First we must select $w \in \mathbb{N}$. To help define $w$, we define numbers $w_1, \ldots, w_{\widehat{l}}$ by induction from $\widehat{l}$ down to 1. Let $w_{\widehat{l}} \in \mathbb{N}$ be large enough such that $w_{\widehat{l}} \rightarrow (n)^{a_{\widehat{l}}}_{p_{\widehat{l}}}$.

Beginning with $i = \widehat{l} - 1$, and counting down until $i = 1$, let $w_i \in \mathbb{N}$ be large enough such that $w_i \rightarrow (w_{i+1})^{a_i}_{p_i}$. Finally, let $w \in \mathbb{N}$ be large enough such that $\phi(w) - m \geq w_1$.

Given any $\rho_1, \ldots, \rho_{\widehat{l}}$ such that $\rho_i \in p_i^{[1, \ldots, w]}$, we will obtain the desired set $Y \subseteq (m, w)$. Toward this end, we define an auxiliary coloring $F : [N]^n \rightarrow \{1, \ldots, k, k+1\}$ as follows: We set $F(Z) = f(Z)$ if $Z$ is homogeneous for each $\rho_i$, and $Z \subseteq (m, w)$. Otherwise, we set $F(Z) = k + 1$.

Take any $F$-homogeneous subset $Y \subseteq \{1, \ldots, w\}$ with $|Y| \geq \phi(w)$. Such a set $Y$ exists because $w \rightarrow ((\phi(w))^{a}_{k+1})$. We will next argue that $Y$ is homogeneous for $F$ with some color $i \in \{1, \ldots, k\}$, and is therefore the desired set.

Because $|Y| = \phi(w)$, it is clear that $|Y \cap (m, w)| \geq \phi(w) - m \geq w_1$. Beginning with $i = 1$, and counting up until $i = \widehat{l} - 1$, we see that there is a $w_{i+1}$-element subset of $Y \cap (m, w)$ which is homogeneous for $\rho_1, \ldots, \rho_i$. Finally, there is an $n$-element subset $Z$ of $Y \cap (m, w)$ which is homogeneous for $\rho_1, \ldots, \rho_{\widehat{l}-1}, \rho_{\widehat{l}}$.

---

6 This is where we use the assumption in PRT that $w \rightarrow ((\phi(w))^{a}_{k+1}$ for all $w$. 42
Note that by the definition of $F$, that $F(Z) = f(Z) \in \{1, \ldots, k\}$. Because $Z \in [Y]^n$, and because $Y$ is $F$-homogeneous, $Y$ is given color $c \neq k + 1$ by $F$. It follows that $Y \subset (m, w]$ and that $Y$ is $f$ homogeneous. It also follows that each $V \in [Y]^n$ is homogeneous for $\rho_1, \ldots, \rho_{\ell}$. Because the exponent of each of these maps is less than $n$, $Y$ itself is homogeneous for each $\rho_i$. Clearly $[Y]^{n-1} \subseteq [N]^{n-1}$, and $|Y| \geq \phi(w)$. In other words, $Y$ is the desired set. \hfill \Box

The next lemma is the analog of Claim 2 in [S]:

**Lemma 2.4.7.** The union of any two small sets of $[N]^{n-1}$ is small. In particular, for any partition $L = L_1 \cup \cdots \cup L_s$ of a large set $L$, one of the $L_i$ is large.

**Proof.** Suppose that $S_1, S_2 \subset [N]^{n-1}$ are small. We show that $S_1 \cup S_2$ is small.

Let $m_1, p_1, \ldots, p_\ell \in \mathbb{N}$ and $w \mapsto \rho_i^w$ be chosen to witness the smallness of $S_1$ (the strings $\rho_i^w \in p_i^{[\{1, \ldots, w\}]^{a_i}}$ demonstrate the failure of $w \in \mathbb{N}$ to satisfy the definition of largeness). Let $m_2, q_1, \ldots, q_l \in \mathbb{N}$ and $w \mapsto \sigma_i^w$ (such that $\sigma_i^w \in q_i^{[\{1, \ldots, w\}]^{a_i}}$) witness the smallness of $S_2$. Recall that by our choice of the $a_i$, $a_t = n - 1$ for some $t \leq \ell$.

To apply the definition of largeness to $S_1 \cup S_2$, we must define $m$ (the lower bound on $Y$) and the $p_i$ (the number of colors assigned by the $\rho_i$). Define $m = \max\{m_1, m_2\}$, define $\hat{p}_t = p_t \cdot q_t \cdot 2$, and define $\hat{p}_i = p_i \cdot q_i$ for $i \neq t$.

We want to define $\hat{\rho}_i$ such that any set $Y$ which is homogeneous for each $\hat{\rho}_i$ has $[Y]^{n-1} \subset S_c$ for $c = 1$ or 2. Recall that $S_c \subseteq [N]^{n-1}$. As a first step, define $s : [N]^{n-1} \to \{1, 2\}$ by $s(U) = 1$ if $U \in S_1$, and $s(U) = 2$ otherwise.

Given $w$ we define $\hat{\rho}_i^w(U) = \langle \rho_i^w(U), \sigma_i^w(U), s(U) \rangle$ for each $U \in [\{1, \ldots, w\}]^{n-1}$. For each $i \neq t$ and for each $U \in [\{1, \ldots, w\}]^{a_i}$, we define $\hat{\rho}_i^w(U) = \langle \rho_i^w(U), \sigma_i^w(U) \rangle$.

\footnote{Note that $\hat{p}_i > p_i$. This is why Definition 2.4.4 quantifies over all possible choices of $p_i$.}
Toward a contradiction, suppose that $S_1 \cup S_2$ is large. Fix $\hat{w}$ witnessing that $S_1 \cup S_2$ is large with the $m$ and $\hat{p}_i$ defined as above. Obtain $\hat{Y}$ as in the definition of largeness. Then $[\hat{Y}]^{n-1} \subseteq S_1 \cup S_2$ and $Y$ is homogeneous for the $\hat{p}_i^{\hat{w}}$ defined above. Note that $\hat{Y}$ is homogeneous for $s$ (because it is homogeneous for $\hat{p}_i^{\hat{w}}$) so $[\hat{Y}]^{n-1} \subseteq S_j$ for some $j \in \{1, 2\}$. In either case, $\hat{Y} \subseteq (m_j, \hat{w}]$ and $|\hat{Y}| \geq \phi(\hat{w})$. Furthermore, $\hat{Y}$ is homogeneous for $f$, each $\rho_i^{\hat{w}}$, and each $\sigma_i^{\hat{w}}$. This contradicts our choice of parameters to witness of the smallness of both $S_1$ and $S_2$.

Our last largeness lemma comes from the proof of Claim 4 of [8]. Essentially, it says that for any coloring $h$ of exponent less than $n$, most elements of a large set are $h$-homogeneous.

**Lemma 2.4.8.** Suppose that $L \subseteq [N]^{n-1}$ is large and $p \leq n-1$. For any coloring $h : [N]^p \rightarrow \{1, \ldots, s\}$, the set $\{Z \in L : Z$ is $h$-homogeneous$\}$ is large.

**Proof.** Let $E = \{Z \in L : (\exists D_1, D_2 \in [Z]^p)[h(D_1) \neq h(D_2)]\}$. Then $L$ is the union of $E$ and $\{Z \in L : Z$ is $h$ homogeneous$\}$, so one of these is large by Lemma 2.4.7.

Suppose toward a contradiction that $E$ is large. Because $\lim \inf_x \phi(x) = \infty$, and by the definition of large, there are arbitrarily large finite sets $Y$ such that $[Y]^{n-1} \subseteq E$. Take $Y$ such that $|Y| \rightarrow (n-1)^n_s$. Then there is some $Z \in [Y]^{n-1}$ which is $h$-homogeneous. But then $Z \in E$ by our choice of $Y$, contradicting the definition of $E$.

2.5 A tree proof of $\text{PRT}_k^n$

We now prove:

**Theorem 2.5.1.** Given $n \in \omega$, fix any $P \gg \emptyset^{(n-1)}$. Each computable instance of $\text{PRT}_k^n$ has a $P$-computable solution.
For each \( n \in \omega \), there is a \( \Delta^0_{n+1} \)-definable set \( P \gg \emptyset^{(n-1)} \). Because each set computable from a \( \Delta^0_{n+1} \)-definable set is itself \( \Delta^0_{n+1} \)-definable, we obtain the following corollary.

**Corollary 2.5.2.** Fix \( n \in \omega \). Each computable instance of \( \text{PRT}^n_k \) has a \( \Delta^0_{n+1} \)-definable solution.

For the rest of this section, fix a computable instance of \( \text{PRT}^n_k \). That is, fix a computable coloring \( f : [\mathbb{N}]^n \to \{1, \ldots, k\} \) and a computable function \( \phi : \mathbb{N} \to \mathbb{N} \) with unbounded range such that \( w \to (\phi(w))_{k+1}^n \) for all \( w \).

### 2.5.1 The strategy

**Definition 2.5.3.** Let \( S \) be the set of all ways of partitioning \( n \) numbers into disjoint intervals. In other words, \( S = \{(r_1, \ldots, r_l) : r_1 + \cdots + r_l = n\} \) where each \( r_i > 0 \). We say that \((r_1, \ldots, r_l)\) has length \( l \). Note that each \((r_1, \ldots, r_l) \in S\) has length \( l \leq n \).

As before, our goal is to define a sequence of blocks \( \{Y_i\} \) such that the color of any \( Z \in [\bigcup_i Y_i]^n \) depends only on how the \( \{Y_i\} \) partition \( Z \). More precisely, suppose that we are given an increasing sequence of blocks \( \{Y_i\} \). For any \( Z \subset \bigcup Y_i \), we say that \((r_1, \ldots, r_s)\) is the *partition type of \( Z \)* if there are \( i_1 < \cdots < i_s \) such that \(|Z \cap Y_{i_j}| = r_j\) for each \( j \leq s \), and if \( Z = \bigcup_{j \leq s} Y_{i_j} \). Note that there are \( 2^{n-1} \) elements in \( S \), including a single length 1 partition type \((n)\). If we can ensure that the color of an \( n \)-tuple depends only on its partition type, we will have ensured that \( X \) is semi-homogeneous.

The main work of the proof is to obtain a sequence of blocks \( \{Y_i\} \) such that the color of any \( Z \in [\bigcup Y_i]^n \) depends on *two* things: (1) how it is partitioned by
the \{Y_i\} and (2) the \(i \in \mathbb{N}\) such that \(\min(Z) \in Y_i\). The desired sequence of blocks is then obtained by \(2^{n-1}\) applications of the infinite pigeonhole principle.

The first step in building this sequence of blocks is to define a collection of helper colorings. That is, we must define a collection of \(2^{n-1}\) colorings \(f_{r_1,\ldots,r_l} : \mathbb{N}^{r_1} \to \{1,\ldots,k\}\) — one for each valid partition \((r_1,\ldots,r_l)\).

Each helper coloring makes a promise. To be more precise, we need the following:

**Definition 2.5.4.** Given finite \(U,Z \subset \mathbb{N}\), we say that \(Z\) extends \(U\) if \(U = Z \cap \{1,\ldots,\max(U)\}\). That is, \(Z\) extends \(U\) if \(U\) is an initial segment of \(Z\).

For each \(r_1\) element set \(U \in [\mathbb{N}]^{r_1}, f_{r_1,\ldots,r_l}(U)\) is the color that we promise to give any \(n\) element set \(Z \subset \bigcup Y_i\) with partition type \((r_1,\ldots,r_l)\) that extends \(U\).

We will proceed by induction on \(l\), using the coloring \(f_{r_1+r_2,r_3,\ldots,r_l}\) to define the coloring \(f_{r_1,r_2,\ldots,r_l}\).

Recall that \((n)\) is the unique partition type of length \(l = 1\). It is easy to see that we will want \(f_n = f\). That is, we should commit to give each \(n\)-element set \(Z \in [\mathbb{N}]^n\) the color that we actually give it.

When \(l > 1\), we must be more careful. We will define the colorings without reference to any sequence \{\(Y_j\)\} in Section 2.5.2. In Section 2.5.3, we will use the colorings to obtain the desired sequence of blocks \{\(Y_i\)\}.

We will later show that it suffices to define the collection of helper colorings \{\(f_{r_1,r_2,\ldots,r_l}\)\} and the infinite sequence \{\(Y_i\)\} so that for any \(U \in \bigcup Y_i^{r_1}, f_{r_1,r_2,\ldots,r_l}(U) = f_{r_1+r_2,r_3,\ldots,r_l}(U \cup V)\)

whenever \(V \in [Y_j]^{r_2}\) is taken from a block \(Y_j\) with \(\min(Y_j) > \max(U)\). In words,
it is enough for us to ensure that the color promised to each extension $Z$ of $U$ — where $Z \setminus U$ has partition type $(r_2, \ldots, r_3)$ — is the same as the color promised to each extension $\hat{Z}$ of $U \cup V$ — where $\hat{Z} \setminus U \cup V$ has partition type $(r_3, \ldots, r_1)$.

By choosing each $Y_i$ to be homogeneous for each $f_{r_1, \ldots, r_l}$, we will obtain a sequence of blocks $\{Y_i\}$ such that the color of $Z \in [\bigcup Y_i]^n$ depends only on two things: (1) how it is partitioned by the $\{Y_i\}$ and (2) the $i \in \mathbb{N}$ such that $\min(Z) \in Y_i$.

### 2.5.2 Obtaining the helper colorings

We will use the notion of ‘largeness’ to define helper colorings without reference to any sequence of blocks. Recall that for exponent $n$, largeness and smallness is defined for $S \subseteq [\mathbb{N}]^{n-1}$. First, we define:

**Definition 2.5.5.** Suppose we have fixed a collection of helper colorings. For any finite set $W \subset \mathbb{N}$ and any $Z \in [\mathbb{N} \setminus W]^{n-1}$, we say that $Z$ is good with $W$ if:

$$(\forall (r_1, \ldots, r_l) \in S)(\forall U \in [W]^{r_1})(\forall V \in [Z]^{r_2})[f_{r_1, r_2, \ldots, r_l}(U) = f_{r_1 + r_2, r_3, \ldots, r_l}(U \cup V)].$$

We must define our helper colorings so that for each finite set $W$, there is a large set of $Z$ which is good with $W$ (i.e. $Z$ is compatible with all of the promises about subsets of $W$). More formally:

**Definition 2.5.6.** A collection of helper colorings is made up of *compatible* helper colorings if $\{Z : Z$ is good with $\{1, \ldots, w\}\}$ is large for each $w \in \mathbb{N}$.

The goal of this subsection is to prove the following lemma:

**Lemma 2.5.7 (Obtaining guides).** For any $P \gg \emptyset^{(n-1)}$, there is a $P$-computable collection of compatible helper colorings.
Before proving this lemma (with its computability bounds), we first describe
the general construction (without computability bounds). For each \( l \), let \( S_l = \{(r_1, \ldots, r_l) \in S : t = l\} \). That is, \( S_l \) is the set of partition types of length \( l \).

Each helper coloring will be a path through a certain tree. We define these
trees by induction on \( l \), using colorings of the form \( f_{s_1, \ldots, s_{l-1}} \) to define the tree that
we use to select \( f_{r_1, \ldots, r_l} \). Let \( T_1 = \{ \sigma : \sigma < f \} \). We will take \( f_n \in [T_1] \). That is,
we will define \( f_n = f \).

If \( l > 1 \), suppose we have defined \( f_{s_1, \ldots, s_{l-1}} \) for each \( (s_1, \ldots, s_{l-1}) \in S_{l-1} \). We
must define the function \( f_{r_1, \ldots, r_l} \) for each \( (r_1, \ldots, r_l) \in S_l \). Intuitively: we will
define a tree \( T_{r_1, \ldots, r_l} \subseteq k^{<\aleph_1} \), and \( f_{r_1, \ldots, r_l} : [\aleph_1]^{\aleph_1} \rightarrow \{1, \ldots, k\} \) will be some path
through this tree.

To ensure that the functions are compatible, we define these trees simultane-
ously. More precisely, we define a single tree \( T_l \) whose elements are a direct sum
of strings. For each \( (r_1, \ldots, r_l) \in S_l \), the \( (r_1, \ldots, r_l)^{th} \) component of \( \tau \in T_l \) is a
potential initial segment for \( f_{r_1, \ldots, r_l} \). Consider some string

\[
\tau = \left( \bigoplus_{(r_1, \ldots, r_l) \in S_l} \tau_{r_1, \ldots, r_l} \right) \in \left( \bigoplus_{(r_1, \ldots, r_l) \in S_l} k^{[\{1, \ldots, w\}]^{\aleph_1}} \right).
\]

We say this string has length \( w \) because each \( \tau_{r_1, \ldots, r_l} \) is defined on all subsets of
\( \{1, \ldots, w\} \). We define \( T_l \) as follows:

\[
\left( \bigoplus_{(r_1, \ldots, r_l) \in S_l} \tau_{r_1, \ldots, r_l} \right) \in T_l
\]
\[\text{iff} \]
\[
\{Z \in [\aleph_1 \setminus \{1, \ldots, w\}]^{\aleph_1} : (\forall (r_1, \ldots, r_l) \in S_l)[r_1 \geq w \Rightarrow
\]

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(∀U ∈ [{1, . . . , w}]r1)(∀V ∈ [Z]r2)[τr1,r2,...,r1(U) = f_{r1+r2,r3,...,r1}(U ∪ V)]} is large.

In words, a collection of length w strings τr1,...,r1 makes progress toward a set of compatible functions f_{r1,...,r1} if there is a large number of n − 1-element sets Z, such that for each U ∈ [{1, . . . , w}]r1 and each V ∈ [Z]r2, the promise made by τr1,...,r1(U ∪ V) agrees with the promise made by f_{r1+r2,r3,...,r1}(U).

Claim 2.5.8. Fix l ≥ 2. If T_{l−1} is infinite and if ⊕ f ∈ [T_{l−1}] is used to define T_l, then T_l is infinite.

Proof. Fix any w ∈ N. We will show that ρ ∈ T_l for some ρ of length w. Consider

ρ = ⊕_{(r1,...,r_l) ∈ S_l} ρ_{r1,...,r_l} ∈ ⊕_{(r1,...,r_l) ∈ S_l} k^{|{1,...,w}|r1}.

By definition, ρ ∈ T_l if and only if there is a large set of Z ∈ [N \{1, . . . , w}\]n−1 which respects the promises that ρ makes about finite subsets of {1, . . . , w}. Unfortunately, for any given Z, there may be some (r1, . . . , r_l) ∈ S_l and some U ∈ [{1, . . . , w}]r1 such that Z is not homogeneous for V ↦ f_{r1+r2,r3,...,r1}(U ∪ V).

In this case, Z does not respect the promises made by any string ρ.

We claim that the set

{Z ∈ [N \{1, . . . , w}\]n−1 : Z respects the promises made by some ρ with |ρ| = w}

is large. Because S_l and {1, . . . , w} are finite, there are finitely many functions V ↦ f_{r1+r2,...,r_l}(U ∪ V). Iterating Lemma 2.4.8 (once for each function) yields a large set of Z such that for each (r1, . . . , r_l) ∈ S_l and each U ∈ [{1, . . . , w}]r1, there is a color c such that (∀V ∈ [Z]r2)[f_{r1+r2,r3,...,r_l}(U ∪ V) = c]. Letting ρ_{r1,r2,r3,...,r_l}(U) be the corresponding c, we see that Z respects the promises made by ρ, as desired.
The set of all $\rho$ of length $w$ induces a partition of this large set into the finitely many sets \{ $Z : Z$ respects the promises made by $\rho$ \}. By Lemma 2.4.7, one of the \{ $Z : Z$ respects the promises made by $\rho$ \} is large; hence, the associated string $\rho$ is an element of $T_l$. Because $w$ was arbitrary, we have shown that $T_l$ contains a string of each length $w$. Thus, $T_l$ is infinite.

Fix any path $p \in [T_l]$. Then $p$ will have the form

$$\bigoplus_{(r_1, \ldots, r_l) \in S_l} f_{r_1, \ldots, r_l}.$$ 

In other words, the $(r_1, \ldots, r_l)_{th}$ component of $p$ will be the desired helper function $f_{r_1, \ldots, r_l}$. Because the definition of “large” is $\Pi_0^2$, the tree $T_l$ is $\Pi_0^2$ relative to the parameter

$$\bigoplus_{(r_1, \ldots, r_{l-1}) \in S_{l-1}} f_{r_1, \ldots, r_{l-1}}.$$

We can now prove the desired lemma:

**Proof of Lemma 2.5.7.** We first define $T_l$ for each $l < n$ by induction, ensuring $T_l$ has a path which is $low^{\theta(l-1)}$. Recall that $p_1 = f_n$ is computable because $f$ is computable and $[T_1] = \{f\}$. Trivially, it follows that $p_1$ is $low$.

Suppose $l$ satisfies $n > l \geq 2$, and that we have chosen $p_{l-1} \in [T_{l-1}]$ to be $low^{\theta(l-2)}$. Define $T_l$ using $p_{l-1}$ as above, and note that $T_l$ is infinite by Claim 2.5.8. Because $T_l$ is $\Pi_2^0, p_{l-1}$, Lemma 1.3.9 gives a $\Sigma_1^0, p_{l-1}$ tree $S_l$ such that $[T_l] = [S_l]$. Because $p_{l-1}$ is $low^{\theta(l-2)}$, $S_l$ is $\theta^{(l-1)}$ computable and there is a $low^{\theta(l-1)}$ path $p_l \in [S_l] = [T_l]$.

Finally, suppose $l = n$, and that $p_{n-1} \in [T_{n-1}]$ is $low^{\theta(n-2)}$. Define $T_n$ using $p_{n-1}$, as above, and note that $T_n$ is infinite, by Claim 2.5.8. Because $T_n$ is $\Pi_2^0, p_{n-1}$,
Lemma 1.3.9 gives a $\Sigma^0_{p_n-1}$ tree $S_n$ such that $[T_n] = [S_n]$. Because $p_{n-1}$ is low$(n-2)$, the tree $S_n$ is $\emptyset^{(n-1)}$-computable, and $P$ computes some path $p_n \in [S_n] = [T_n]$.

Define the desired collection of helper colorings to be the set of the components of the functions $p_1, \ldots, p_n$. Considering the definition of the trees $T_i$, we see that the set $\{Z \in [N \setminus \{1, \ldots, w\}]^{n-1} : Z \text{ is good with } \{1, \ldots, w\}\}$ is large for each $w \in \mathbb{N}$, as desired.

2.5.3 Selecting a sequence of blocks

**Lemma 2.5.9.** Fix any $P$-computable collection of compatible helper colorings. Then $P$ computes an infinite sequence of blocks $\{Y_i\}$ such that the color of any $Z \in [\{Y_i\}]^n$ depends only on two things: (1) the smallest block that contains an element of $Z$ and (2) the partition type of $Z$.

More precisely, consider some $Z \in [\bigcup_i Y_i]^n$. If $(r_1, \ldots, r_l)$ is the partition type of $Z$ and if $Z_1$ is the $r_1$ smallest elements of $Z$, then $f(Z) = f_{r_1, \ldots, r_l}(Z_1)$.

**Proof.** We begin by giving a $P$-uniform definition of the sequence $\{Y_i\}$. We proceed by induction on $i$.

Suppose that we have defined $Y_1, \ldots, Y_i$ and $w_0, w_1, \ldots, w_i$. The set of $Z \in \mathbb{N}^{n-1}$ that are good with $Y_1 \cup \cdots \cup Y_i$ is clearly large because $Y_1 \cup \cdots \cup Y_i \subseteq \{1, \ldots, w_i\}$ and because $\{Z \in [N \setminus W]^{n-1} : Z \text{ is good with } \{1, \ldots, w_i\}\}$ is large.

Look for some finite set $Y_{i+1}$ and $w_{i+1} \in \mathbb{N}$ such that

- $Y_{i+1} \subseteq (w_i, w_{i+1}]$ and each $Z \in [Y_{i+1}]^{n-1}$ is good with $Y_1 \cup \cdots \cup Y_i$,
- $|Y_{i+1}| \geq \phi(w_{i+1})$,
- $Y$ is homogeneous for $f$, and
- $Y$ is homogeneous for $f_{r_1, \ldots, r_l}$ for each $(r_1, \ldots, r_l) \in S$.  

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By the Largeness Lemma 2.4.5, we will eventually find $Y_{i+1}$ and $w_{i+1}$. Note that we can $P$-uniformly determine if a given finite set satisfies this property. This completes our definition of $\{Y_i\}$.

Now consider any $Z \in \{\{Y_i\}\}^n$. For $i \leq l$, let $Z_i \subseteq Z$ be the smallest $r_1 + \cdots + r_i$ elements of $Z$. By the construction of $\{Y_i\}$ and the definition of “good with $W$”, if $i < l$ then

$$f_{r_1 + \cdots + r_i, r_{i+1}, \ldots, r_l}(Z_i) = f_{r_1 + \cdots + r_i + r_{i+1}, \ldots, r_l}(Z_{i+1}).$$

Inductively, we see that $f_{r_1, \ldots, r_l}(Z_1) = f(Z)$, as desired. \hfill \Box

2.5.4 Putting it all together

We now complete the proof of Theorem 2.5.1. Recall that the lemmas above were proved for an arbitrary computable instance of $\text{PRT}_k^n$.

Proof of Theorem 2.5.1 Fix $P \gg \emptyset^{(n-1)}$. Let $f, \phi$ be a computable instance of $\text{PRT}_k^n$. By Lemma 2.5.7 there is a $P$-computable collection of compatible helper colorings.

Let $\{Y_i\}$ be the sequence of blocks obtained by applying Lemma 2.5.9 to these functions. Note that this sequence of blocks induces $2^{n-1}$ colorings (one coloring for each partition type). More precisely, define $h_{r_1, \ldots, r_l} : \mathbb{N} \to \{1, \ldots, k\}$ by setting $h_{r_1, \ldots, r_l}(i) = f_{r_1, \ldots, r_l}(Z)$ for any/all $Z \in [Y_i]^{r_1}$. Iterating the infinite pigeonhole principle, once for each of the $2^{n-1}$-many induced colorings, we get an infinite set $I$ homogeneous for each $h_{r_1, \ldots, r_l}$. Note that we can (non-uniformly) compute $I$ from $P$.

Define $A = \bigcup_{i \in I} Y_i$. Clearly $A \leq_T P$. Because $A$ is the union of infinitely many
blocks, \( A \) is packed for \( \phi \). Note that the color given to any \( Z \in [A]^n \) is completely determined by the way that \( Z \) is partitioned by \( \{Y_i\}_{i \in I} \). In other words, \( A \) is the desired packed semi-homogeneous set.

### 2.6 Lower bounds and reversals

We conclude by giving lower bounds on the strength of \( \text{PRT}^n_k \). We first prove:

**Theorem 2.6.1** (\( \text{RCA}_0 \)). \( \text{PRT}^n_{2^n-1+k} \) implies \( \text{RT}^n_k \) for each \( n \in \omega \) and \( k \in \mathbb{N} \).

Adapting this argument, we also prove:

**Theorem 2.6.2.** If \( n \geq 2 \) and \( k > 2^{n-1} \), there is a computable instance of \( \text{PRT}^n_k \) such that no \( \Sigma^0_n \)-definable set is both packed for \( \phi \) and semi-homogeneous for \( f \).

#### 2.6.1 Sharpness of number of colors

Erdős and Galvin describe Theorem 2.1.7 (Theorem 2.3 of [8]) as showing that 
"2^{n-1} is best possible" in Theorem 2.1.10.

We will use this sharpness of colors to prove \( \text{RT}^n \) from \( \text{PRT}^n \). The first step is to state a version of Theorem 2.1.7 appropriate for reverse mathematics. The second step is to use this result to construct a helper coloring. The final step is to apply \( \text{PRT}^n \) to this helper coloring and refine the packed semi-homogeneous set, obtaining a set homogeneous for the original coloring.

**Definition 2.6.3.** We say that \( \phi : \mathbb{N} \to \mathbb{N} \) is an order function if \( \phi \) is total, non-decreasing, and has unbounded range.

The most natural choice for \( \phi \) in \( \text{PRT} \) is an order function.

**Remark 2.6.4** (\( \text{RCA}_0 \)). For each \( w \), let \( \phi_{\text{max}}(w) \) be the largest \( m \) such that \( w \to (m)^n_{k+1} \). Then \( \phi_{\text{max}} \) is a total, \( \Delta^0_1 \) definable order function.
Proof. Clearly $\phi_{\text{max}}$ is total, $\Delta^0_1$ definable, and non-decreasing. Finite Ramsey’s Theorem, which is provable in $\text{RCA}_0$, implies that $\phi_{\text{max}}$ has unbounded range. □

Recall that for exponent $n$, we write $S$ for the set of all finite sequences $(r_1, \ldots, r_l)$ such that $r_1 + \cdots + r_l = n$. The elements of $S$ represent all possible ways of partitioning an $n$-element set into disjoint intervals. We will write $\bar{T}_1$ to refer to the partition type where $r_i = 1$ for each $i$ (that is, $\bar{T}_1 = (1, \ldots, 1)$).

Given an increasing sequence $\{w_i\}$ and a set $X \in [\mathbb{N}]^n$, we say that $(r_1, \ldots, r_l)$ is the partition type of $X$ with respect to $\{(w_i, w_{i+1}]\}$ if there are $j_1 < \cdots < j_l$ such that $|X \cap (w_{j_i}, w_{j_i+1}]| = r_i$ for each $i \leq l$.

The following lemma is an adaptation of the proof of Theorem 2.1.7:

**Lemma 2.6.5 (RCA$_0$).** Fix $n \in \omega$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be any order function. There is a coloring $g : [\mathbb{N}]^n \to S$ and a strictly increasing function $i \mapsto w_i$ such that

- $g(X)$ is the partition type of $X \in [\mathbb{N}]^n$ with respect to $\{(w_i, w_{i+1}]\}$, and
- for any infinite $A \subseteq \mathbb{N}$, either $A$ is sparse for $\phi$ or $\{g(X) : X \in [A]^n\} = S$.

**Proof.** We define $w_i$ by induction on $i$. Let $w_1 = 1$. For $i > 1$, define $w_i$ to be the least element of $\{w > w_{i-1} : \phi(w) \geq n \cdot i\}$. This set is nonempty because $\phi$ has unbounded range, and has a least element by $\Delta^0_1$ induction. We have defined $i \mapsto w_i$ by iterating a total $\Delta^0_1$ function, so the map is total by $\Sigma^0_1$ induction (and Proposition 6.5 of [12]).

Define $g(X) : [\mathbb{N}]^n \to S$ to be the partition type of $X$ with respect to the sequence $\{(w_i, w_{i+1}]\}$. Then $g$ and $i \mapsto w_i$ have $\Delta^0_1$ definitions, so exist by $\Delta^0_1$ comprehension.

We next verify that $g$ assigns all colors to any infinite $A = \{a_1 < a_2 < \ldots\}$ such that $|A \cap \{1, \ldots, w\}| \geq \phi(w)$ for infinitely many $w$. If there are $n$ values of $i$
such that $|A \cap (w_i, w_{i+1}]| \geq n$, then $\{g(X) : X \in [A]^n\} = S$, and we are done.

Suppose instead that there is $i$ such that $(\forall i \geq \hat{i})[|A \cap (w_i, w_{i+1}]| < n]$. We will define $i_0 \geq \hat{i}$ such that for all $i \geq i_0$, $|A \cap \{1, \ldots, w_i+1\}| < n \cdot i$. Simply set $\hat{m} = \min(|A \cap \{1, \ldots, w_i\}| - n \cdot \hat{i}, 0)$, and set $i_0 = \hat{i} + \hat{m}$. Furthermore, there is some $j_0$ such that for each $j \geq j_0$, there is some $i \geq i_0$ such that $a_j \in (w_i, w_{i+1}]$.

Fix any $j \geq j_0$. Then $n \cdot i > |A \cap \{1, \ldots, w_{i+1}\}|$ because $i \geq i_0$, and $a_j \in A \cap \{1, \ldots, w_{i+1}\}$. Recall that we defined $w_i$ so that $\phi(w_i) \geq n \cdot i$. Putting it all together, because $\phi$ is non-decreasing and $a_j \geq w_i$ we see that

$$\phi(a_j) \geq \phi(w_i) \geq n \cdot i > |A \cap \{1, \ldots, w_{i+1}\}| \geq |A \cap \{1, \ldots, a_j\}|.$$

Because $\phi(a_j) > |A \cap \{1, \ldots, a_j\}|$ for all but finitely many $j$, and because $\phi$ is non-decreasing, it follows that $A$ is sparse for $\phi$. \qed

2.6.2 Using PRT$^n$ to prove RT$^n$ over RCA$^0$

*Proof of Theorem 2.6.1.* Suppose PRT$^n_{2n-1+k}$ holds. Given a function $f : [N]^n \to \{1, \ldots, k\}$, we must produce a set $H$ homogeneous for $f$.

Recall that $\phi_{max}(w) = \max m[w \to (m)^n_{k+1}]$ is an order function. Using $\phi_{max}$, define $g : [N]^n \to S$ as in Lemma 2.6.5. We can assume that $S \cap \{1, \ldots, k\} = \emptyset$. Define a helper coloring $h : [N]^n \to (S\{1\}) \cup \{1, \ldots, k\}$ as follows:

$$h(Z) = \begin{cases} f(Z) & \text{if } g(Z) = 1, \\ g(Z) & \text{otherwise.} \end{cases}$$

Claim 2.6.6 (RCA$^0$). Let $A$ be semi-homogeneous for $h$ and packed for $\phi_{max}$. Then there is a unique color $\hat{c} \in \{1 \ldots, k\}$ such that for any $X \in [A]^n$, if $g(X) = 1$,
then \( f(X) = \hat{c} \)

Proof. We first examine the colors assigned to \( A \) by the coloring \( g : [N]^n \to S \). Recall that \( g \) was obtained from Lemma 2.6.5. Because \( A \) is packed for \( \phi_{max} \), it follows that \( g \) assigns all possible colors to subsets of \( A \). For each \( c \in S \setminus \{\overline{1}\} \), select some \( X_c \in [A]^n \) such that \( g(X_c) = c \).

We now examine the colors assigned to \( A \) by the helper coloring \( h : [N]^n \to \{1, \ldots, k\} \sqcup S \setminus \{\overline{1}\} \). For each \( c \in S \setminus \{\overline{1}\} \), our definition of \( h \) implies that \( h(X_c) = g(X_c) = c \). Recall that \( |S \setminus \{\overline{1}\}| = 2^{n-1} - 1 \). Also by our definition of \( h \), we know that \( h(X) \in \{1, \ldots, k\} \) for each \( X \) with \( g(X) = \overline{1} \). Because \( A \) is semi-homogeneous for \( h \), and because \( S \cap \{1, \ldots, k\} = \emptyset \), it follows that there is a unique color \( \hat{c} \in \{1, \ldots, k\} \) such that \( h(X) = \hat{c} \) for each \( X \in [A]^n \) with \( g(X) = \overline{1} \).

Examining our definition of \( h \) one last time, we see that whenever \( g(X) = \overline{1} \), we have \( h(X) = f(X) \). Consequently, we have shown that there is a unique color \( \hat{c} \in \{1, \ldots, k\} \) such that for each \( X \in [A]^n \) with \( g(X) = \overline{1} \), \( f(X) = \hat{c} \).

Note that \( h \) is a coloring of \([N]^n\) into \( 2^{n-1} - 1 + k \) colors. Furthermore, \( h \) has a \( \Delta^0_1 \) definition, so exists by \( \Delta^0_1 \) comprehension. By \( \text{PRT}^n_{2^{n-1} - 1 + k} \), there is a set \( A \) that is semi-homogeneous for \( h \) and packed for \( \phi_{max} \).

Let \( H = \bigcup_{i \in \mathbb{N}} \{\min(A \cap (w_i, w_{i+1}])\} \). For each \( i \), \( A \cap (w_i, w_{i+1}] \) is \( \Delta^0_1 \) with parameters, so it has a least element. Because \( A \) is infinite and each interval is finite, \( H \) is infinite. Clearly, \( H \) is \( \Delta^0_1 \) definable from \( A \), so exists by \( \Delta^0_1 \) comprehension.

Suppose \( X \in [H]^n \). By definition of \( H \), at most 1 element of \( X \) is in any interval \( (w_i, w_{i+1}] \). Then \( X \) has partition type \( \overline{1} \), so \( g(X) = \overline{1} \). Let \( \hat{c} \) be the unique color in Claim 2.6.6. Because \( H \subseteq A \), it follows that \( f(X) = \hat{c} \).

In summary, \( H \) is infinite and \( f \)-homogeneous with color \( \hat{c} \), as desired. \( \square \)
2.6.3 $\Sigma^0_n$ sets and $\text{PRT}_k^n$

We conclude by showing that for $n > 1$ and $k > 2^{n-1}$, there is a computable instance of $\text{PRT}_k^n$ with no $\Sigma^0_n$ solution. We use the following result:

**Theorem 2.6.7** (Jockusch [16]). For each $n \geq 2$, there exists a computable coloring $f : [\mathbb{N}]^n \to \{1, 2\}$ such that no $\Sigma^0_n$ set is homogeneous for $f$.

We now prove Theorem 2.6.2.

**Proof of Theorem 2.6.2.** Suppose toward a contradiction that for each appropriate computable coloring and $\phi$, there is a $\Sigma^0_n$ definable set which is both packed and semi-homogeneous.

Let $f : [\mathbb{N}]^n \to \{1, \ldots, k\}$ be a coloring with no $\Sigma^0_n$-definable homogeneous set, which exists by Theorem 2.6.7. Define $h$ as in the proof of Theorem 2.6.1. Note that the function $g$ from Lemma 2.6.5 is computable. Then $h$ is computable because it is computable from $f$, $g$, and $\phi_{\text{max}}$.

Suppose that $A$ is a $\Sigma^0_n$ set that is packed for $\phi_{\text{max}}$ and semi-homogeneous for $h$. Let $\theta$ be a $\Delta^0_n$ formula such that $x \in A \iff (\exists y)[\theta(x, y)]$. Let

$$x \in H \iff (\exists y)(\exists i)[\theta(x, y) \land (x \in (w_i, w_{i+1}]) \land (\forall z \in (w_i, x))(\forall t \leq y)[\neg \theta(z, t)]].$$

Note that this is a $\Sigma^0_n$ definition for $H$.

Because $A$ is infinite, and because each interval $(w_i, w_{i+1}]$ is finite, we see that $H$ is infinite. Because each element of $[H]^n$ has partition type $\overline{1}$ with respect to the sequence $\{(w_i, w_{i+1}]\}$, it follows that $H$ is $g$-homogeneous with color $\overline{1}$. By Claim 2.6.6, there is a unique $\hat{c}$ such that $f(X) = \hat{c}$ for each $X \in [H]^n$. In short, $H$ is an infinite $f$ homogeneous set that is $\Sigma^0_n$, contradicting our choice of $f$. \qed
CHAPTER 3

REVERSE MATHEMATICS AND A RAMSEY-TYPE KÖNIG’S LEmMA

3.1 Introduction

Weak König’s Lemma and Ramsey’s Theorem can be thought of as asserting the existence of different types of regularity. Viewed topologically, Weak König’s Lemma is essentially the statement “2^\mathbb{N} is compact.” This carries over to reverse mathematics, where WKL_0 is equivalent to many theorems about compactness (see [22]). For its part, Ramsey’s Theorem says that no matter how badly behaved a coloring is, it always has a sizable homogeneous set. In the words of T.S. Motzkin, absolute disorder is impossible.

In this chapter, we study the computational and reverse mathematical strength of a regularity principle that combines features of Weak König’s Lemma and Ramsey’s Theorem. We will refer to the statement “for each infinite binary tree T, there is an infinite set H homogeneous for a path through T” as a Ramsey-type König’s Lemma. In statement [3.1.2] we give a formal definition of our Ramsey-type König’s Lemma, denoted RKL, in terms of finite strings. It is the novelty of this principle, rather than the complexity of the proofs, that is the main innovation of this chapter.

This chapter is adapted from a paper which will appear in the Journal of Symbolic Logic as [10].
We begin by showing that $\text{RKL}$ is a nontrivial weakening of $\text{WKL}_0$ and of $\text{RT}_2^2$. More formally, we show that $\text{SRT}_2^2$ implies $\text{RKL}$, that $\text{WKL}_0$ implies $\text{RKL}$, and that $\text{RKL}$ implies $\text{DNR}$ (unless specified, we always work over $\text{RCA}_0$). Applying results of [1] and [19], we conclude that $\text{RKL}$ is strictly weaker than $\text{WKL}_0$ and $\text{SRT}_2^2$.

In the remaining sections, we state analogs of $\text{RKL}$ for trees generated by infinite sets of strings ($\text{RKL}^{(1)}$) and for arithmetically-definable trees ($\text{RKL}^{(<\omega)}$). We then study the strength of each principle, and obtain the surprising result that these stronger principles are more closely related to $\text{RT}_2^2$ than to $\text{WKL}_0$.

We show that $\text{RT}_2^2$ implies $\text{RKL}^{(1)}$, and that $\text{RKL}^{(1)}$ implies $\text{SRT}_2^2$. By the main results of [5] and [19], it follows that $\text{RKL}^{(1)}$ and $\text{WKL}_0$ are incomparable. We also show that $\text{RT}_2^2$ does not imply $\text{RKL}^{(<\omega)}$ and, by using a result of [19], we show that $\text{RKL}^{(<\omega)}$ does not imply $\text{WKL}_0$. It is open whether $\text{RKL}^{(<\omega)}$ implies $\text{RT}_2^2$. We summarize our results in figure 3.1.

### 3.1.1 Working in second order arithmetic

Some care is required to formalize $\text{RKL}$ in $\text{RCA}_0$. Our goal here is to study the computational complexity of the homogeneous set $H \subseteq \mathbb{N}$, not of the path $p$. While there are computable trees $T$ such that each path through $T$ is reasonably complicated, there are paths $p$ with relatively simple homogeneous sets $H$. We begin with definitions that allow us to say that $H$ is homogeneous for some path through $T$ without explicit reference to a path $p$.

**Definition 3.1.1** ($\text{RCA}_0$). A set of natural numbers $H$ is homogeneous for $\sigma \in 2^{\mathbb{N}}$ with color $c \in \{0, 1\}$ if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. $H$ is homogeneous for a path through $T$ if $\exists c \in \{0, 1\}$ s.t. $H$ is homogeneous for $\sigma$ with color $c$ for $\text{RT}_2^2$ and $\text{DNR}$ will be defined below.
arbitrarily long $\sigma \in T$.

The idea of Definition 3.1.1 is this: if $H$ is homogeneous for arbitrarily long strings in $T$, we obtain an infinite subtree of $T$ by taking the downward closure of these strings. By compactness, $H$ is homogeneous for some path through this induced subtree, but it is possible that neither the induced subtree nor any of its paths exist in $S(M)$.

**Statement 3.1.2 (RCA$_0$).** RKL asserts that “for each infinite binary tree $T$, there is an infinite set $H$ which is homogeneous for a path through $T$.”

Unless stated otherwise, all strings and trees we consider will be binary ($\{0, 1\}$-valued). Given $\tau, \sigma \in 2^{<\mathbb{N}}$ we write $\tau \leq \sigma$ if $\tau$ is an initial segment of $\sigma$. We write $\sigma \upharpoonright t$ (or $p \upharpoonright t$) to denote the initial segment of $\sigma \in 2^{<\mathbb{N}}$ (or $p \in 2^\mathbb{N}$) of length $t$.

### 3.2 Reverse mathematics of RKL

**Theorem 3.2.1 (RCA$_0$).** WKL$_0$ implies RKL.

*Proof.* Given an infinite binary tree $T$, let $p$ be an infinite path through $T$. Note that $p : \mathbb{N} \to \{0, 1\}$ maps singletons into two colors. Applying RT$^1_2$, which is provable in RCA$_0$, we obtain a set $H$ that is homogeneous for $p$. In particular, $p \upharpoonright t \in T$ and $H$ is homogeneous for $p \upharpoonright t$ for each $t \in \mathbb{N}$. Thus, $H$ satisfies Definition 3.1.1 as desired. \qed

**Definition 3.2.2 (RCA$_0$).** A coloring $f : [\mathbb{N}]^2 \to \{0, 1\}$ is stable if for each $x$, there is some $t > x$ s.t. $(\forall y > t)[f(x, y) = f(x, t)]$. SRT$^2_2$ is the theorem “every stable coloring of pairs into two colors has an infinite homogeneous set.”

**Theorem 3.2.3 (RCA$_0$).** SRT$^2_2$ implies RKL.
Proof. Given an infinite tree $T$, we define a coloring $f : [\mathbb{N}]^2 \to \{0, 1\}$ as follows. For each $y$, let $\sigma_y$ be the lexicographically least element of $T$ of length $y$. For each $x < y$, define $f(x, y) = \sigma_y(x)$.

We show that $f$ is a stable coloring. Fix $x$, and let $T_{ext}$ denote the strings in $T$ that are extended by arbitrarily long strings in $T$. For each $\tau \in T_{ext}$ of length $x + 1$, there is a bound on the length of strings in $T$ extending $\tau$, so there is a least such bound $s_\tau$. Note that $s_\tau$ is $\Delta^0_1$ definable (with parameters) from $\tau$. By $\Sigma^0_1$ induction, there is a uniform upper bound $t$ for $\{s_\tau : \tau \in 2^{x+1} \land \tau \in T - T_{ext}\}$.

By $\Pi^0_1$ induction, there is a lexicographically least element $\tau_{x+1} \in T_{ext}$ of length $x + 1$. Then for each $y > t$, $\sigma_y \upharpoonright (x+1) = \tau_{x+1}$ hence $f(x, y) = \tau_{x+1}(x)$. In general, for each $x$, $(\exists t)(\forall y > t)[f(x, y) = \tau_{x+1}(x)]$. In other words, $f$ is a stable coloring.

Suppose that $H$ is homogeneous for $f$ with color $c \in \{0, 1\}$. We now show that $H$ is homogeneous for a path through $T$. Fix $t \in \mathbb{N}$. Because $H$ is an infinite set, there is an element $y \in H$ with $y \geq t$. By the definition of $f$, $(\forall x < y)[\sigma_y(x) = f(x, y)]$. Because $H$ is homogeneous for $f$ with color $c$ and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. Then $H$ is homogeneous for $\sigma_y \in T$ with color $c$. Since $t$ is arbitrary and $|\sigma_y| \geq t$, we have satisfied Definition 3.1.1.

Corollary 3.2.4 (RCA$_0$). RKL does not imply SRT$_2^2$ or WKL$_0$.

Proof. By the main result of [19], SRT$_2^2$ does not imply WKL$_0$. Because SRT$_2^2$ implies RKL, RKL cannot imply WKL$_0$. Similarly, RKL cannot imply SRT$_2^2$ over RCA$_0$ because WKL$_0$ does not imply SRT$_2^2$ (by Theorem 3.3 of [21] and Theorem 10.5 of [11]).

We conclude our analysis of RKL by showing that RKL is not provable in RCA$_0$, and that RKL implies DNR. When $T \subseteq 2^{<\mathbb{N}}$ is a tree, $[T] \subseteq 2^{\mathbb{N}}$ will be the set of
infinite paths through $T$. The next lemma follows from the proof of Lemma 2 in [17].

**Lemma 3.2.5** (Jockusch [17]). There is an infinite computable tree $T$ such that for any $p \in [T]$ and for any $e \in \mathbb{N}$, if $|W_e| \geq e + 3$ then $W_e$ is not homogeneous for $p$. In fact, for each $e \in \mathbb{N}$, there is a $t \in \mathbb{N}$ such that if $|W_e| \geq e + 3$ then $W_e$ is not homogeneous for any string in $T$ of length greater than $t$.

A simple corollary is that $\text{RCA}_0 \nvdash \text{RKL}$, via an $\omega$-model. Adapting the proof of Theorem 2.3 from [11], we can obtain a slightly stronger result. We say that a function $f$ is **diagonally non-computable** relative to $X$ if $f(e) \neq \Phi^X_e(e)$ for each $e$ such that $\Phi^X_e(e) \downarrow$. The principle DNR asserts that for any set parameter $X$, there is a function that is diagonally non-computable relative to $X$.

**Theorem 3.2.6** ($\text{RCA}_0$). RKL implies DNR.

**Proof.** We work relative to a set parameter $X$. The proof of Lemma 2 of [17] (Lemma 3.2.5 above) can be carried out in $\text{RCA}_0$. Let $T$ be the tree defined in this proof. By RKL, there is a set $H$ homogeneous for a path through $T$. Note that there is a $\Delta^0_1$ definable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $W_{g(e)}$ consists of the least $e + 3$ elements of $H$ (in the $\mathbb{N}$ order).

We now show that $g$ is a fixed point free function. If $|W_e| < 2^{e+1}$, then $|W_{g(e)}| \neq |W_e|$. Suppose that $|W_e| \geq 2^{e+1}$. By the lemma above, there is some $t$ such that $W_e$ is not homogeneous for any $\sigma \in T$ of length greater than $t$. Because $W_{g(e)} \subset H$ and because $H$ is homogeneous for a path through $T$, $W_{g(e)}$ is homogeneous for arbitrarily long $\sigma \in T$. In particular $W_e \neq W_{g(e)}$. In other words, $g$ is fixed point free, so it can be used to give a $\Delta^0_1$ definition for a DNR function (to see this, we may formalize V.5.8 of [23] in $\text{RCA}_0$).
Question 3.2.7. Does DNR imply RKL?

A number of principles are known to be stronger than DNR, such as ASRAM and ASRT₂ from [7]. Proving that one of these principles does not imply RKL would separate DNR from RKL.

3.3 Trees generated by sets of strings

Definition 3.3.1. Given an infinite set of strings Σ, let $T_Σ$ denote the downward closure of Σ. More formally, $T_Σ = \{\tau : (\exists \sigma \in \Sigma)[\tau \preceq \sigma]\}$.

Statement 3.3.2 ($\text{RCA}_0$). RKL$^{(1)}$ asserts that “for each infinite set of strings Σ, there is an infinite set $H$ which is homogeneous for a path through $T_Σ$.”

Note that if Σ is an infinite computable set of strings, $T_Σ$ is an infinite c.e. tree. In [8], Downey and Jockusch note that each infinite $\Pi^0_1\emptyset'$-class can be generated by a c.e. tree. We extend this slightly, to further motivate our definition of RKL$^{(1)}$.

Proposition 3.3.3. If $T$ is an infinite $\Pi^0_2$ tree, then there is an infinite computable set of strings Σ such that $[T] = [T_Σ]$. Furthermore, Σ can be taken to contain exactly one string of each length.

Proof. It suffices to consider $\Sigma^0_1$ trees. To see this, suppose that $T$ is $\Pi^0_2$. Then there is a formula $\phi$ which is $\Delta^0_1$ such that $\tau \in T \iff (\forall y)(\exists z)\phi(\tau, y, z)$. Using the $\Delta^0_1$ formula $\psi(\tau, \hat{z}) =_{def} (\forall x, y \leq |\tau|)(\exists z < \hat{z})\phi(\tau \upharpoonright x, y, z)$, we can define a $\Sigma^0_1$ tree $S$ by $\tau \in S \iff (\exists \hat{z})\psi(\tau, \hat{z})$. Then

$$[S] = \{f : (\forall w)(\exists \hat{z})\psi(f \upharpoonright w, \hat{z})\} = \{f : (\forall x)(\forall y)(\exists z)\phi(f \upharpoonright x, y, z)\} = [T],$$

so we may work with $S$ instead.
Given a $\Sigma^0_1$ tree $T$, fix a computable enumeration $\{T_s\}$ of $T$. If necessary, we computably modify the enumeration to ensure that no $\tau$ enters $T_s$ until $s \geq |\tau|$ and that exactly one string enters $T$ at each stage. We computably enumerate the elements of $\Sigma$ in increasing order. At stage $s > 0$, find $\tau \in T_s - T_{s-1}$, take one $\sigma \succeq \tau$ with $|\sigma| = s$ (the specific choice is not important), and put $\sigma$ into $\Sigma$. It is not difficult to show that $T \subseteq T_\Sigma$ and that $T^\text{ext}_\Sigma \subseteq T^\text{ext}$. It follows that $[T] = [T_\Sigma]$.

We now examine the strength of $\text{RKL}^{(1)}$.

**Theorem 3.3.4 (RCA$_0$).** $\text{RT}^2_2$ implies $\text{RKL}^{(1)}$.

**Proof.** Fix an infinite set of strings $\Sigma$. For each $y$, let $l \geq y$ be the length of the shortest string in $\Sigma$ of length at least $y$. Let $\sigma_y$ be the lexicographically least string in $\Sigma$ of length $l$. We now define a coloring $f : [N]^2 \rightarrow \{0, 1\}$ as before. For each $x < y \in N$, set $f(x, y) = \sigma_y(x)$. Note that $f$ is $\Delta^0_1$-definable.

By $\text{RT}^2_2$, there is an infinite set $H$ homogeneous for $f$ with color $c \in \{0, 1\}$. We claim that $H$ is homogeneous for a path through $T_\Sigma$.

Fix $t \in N$. Because $H$ is infinite, there is some $y \in H$ with $y \geq t$. By the definition of $f$, $(\forall x < y)[f(x, y) = \sigma_y(x)]$. Because $H$ is homogeneous for $f$ with color $c$ and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. In other words, $H$ is homogeneous for $\sigma_y \upharpoonright y$ with color $c$. Because $\sigma_y \upharpoonright y \in T_\Sigma$ and because $y \geq t$ with $t$ arbitrary, $H$ is homogeneous for a path through $T_\Sigma$ (in the sense of Definition 3.1.1).

**Remark 3.3.5.** The coloring defined in the proof of Theorem 3.3.4 is not (necessarily) stable because it is defined in terms of $\Sigma$, and not in terms of $T$. For example, suppose that $\sigma(0) = 0$ for even length strings $\sigma \in \Sigma$, and that $\sigma(0) = 1$ for odd length strings. Then $\lim_y f(0, y)$ does not exist.
There is a natural correspondence between computable colorings $f : [N] \to \{0, 1\}$ and computable sets $\Sigma \subset 2^{<\mathbb{N}}$ that contain exactly one string of each length. Given $f$, simply define $\Sigma = \{\sigma_y : \sigma_y \in 2^y \land (\forall x < y)[\sigma_y(x) = f(x, y)]\}$. Using the induced tree $T_\Sigma$, it is not difficult to show that $\text{RKL}^{(1)}$ implies $\text{SRT}_2^2$ over $\text{RCA}_0 + B\Sigma_2^0$. Yokoyama was able to eliminate the use of $B\Sigma_2^0$ by introducing the following principle.

**Statement 3.3.6 (RCA$_0$).** $P_2^2$ asserts that “for any $\Pi_2^0$-definable subsets $A_0, A_1$ of $\mathbb{N}$ s.t. $A_0 \cup A_1 = \mathbb{N}$, there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A_0$ or $H \subseteq A_1$.”

The principle $P_2^2$ is particularly useful because it implies the better understood principle $D_2^2$. This is a special instance of the principle $D_n^2$, which we will return to in the next section.

**Statement 3.3.7 (RCA$_0$).** For each $n \in \omega$, $D_n^2$ asserts that “for each $\Delta_n^0$-definable subset $A$ of $\mathbb{N}$, there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A$ or $H \subseteq \overline{A}$.”

**Theorem 3.3.8** (Cholak, Chong, Jockush, Lempp, Slaman, Yang [1, 4]). Over $\text{RCA}_0$, $D_2^2$ is equivalent to $\text{SRT}_2^2$.

**Theorem 3.3.9** (Yokoyama [25]). $\text{RKL}^{(1)}$ implies $P_2^2$, and hence $\text{SRT}_2^2$, over $\text{RCA}_0$.

**Proof.** Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \text{RCA}_0 + \text{RKL}^{(1)}$ and suppose that $A_0, A_1$ are $\Pi_2^0$-definable subsets of $\mathbb{N}$ such that $A_0 \cup A_1 = \mathbb{N}$. We will define a $\Delta_1^0$ function $f : [\mathbb{N}]^2 \to \{0, 1\}$ such that if $f(x, y) = i$ for infinitely many $y$, then $x \in A_i$.

Fix two $\Sigma_0^0$ formulas $\theta_i(x, m, n)$ such that $x \in A_i \iff (\forall m)(\exists n)\theta_i(x, m, n)$. Using these formulas, we define a helper function:

$$h(x, y) = (\mu z)[(\forall m < y)(\exists n < z)[\theta_0(x, m, n)] \lor (\forall m < y)(\exists n < z)[\theta_1(x, m, n)]]$$

$$h(x, y) = (\mu z)[(\forall m < y)(\exists n < z)[\theta_0(x, m, n)] \lor (\forall m < y)(\exists n < z)[\theta_1(x, m, n)]]$$

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Clearly, $h$ is a $\Delta^0_1$ function.

We must verify in $\text{RCA}_0$ that $h$ is total. Fix $x \in \mathbb{N}$ arbitrary. Then $x \in \mathbb{N} = A_0 \cup A_1$, so $x \in A_0$ or $x \in A_1$. So $(\forall m)(\exists n)\theta_0(x, m, n)$ or $(\forall m)(\exists n)\theta_1(x, m, n)$. Let $y \in \mathbb{N}$ be arbitrary. Suppose $x \in A_i$. Then $(\forall m)(\exists n)\theta_i(x, m, n)$, so clearly $(\forall m < y)(\exists n)\theta_i(x, m, n)$. By $\text{BS}_1$, there is a $z_i$ such that $(\forall m < y)(\exists n < z_i)\theta_i(x, m, n)$. Thus, $h$ will find a least $z$ such that the desired condition holds.

Define $f(x, y) = 0$ if $(\forall m < y)(\exists n < h(x, y))[\theta_0(x, m, n)]$, and $f(x, y) = 1$ otherwise. Clearly, $f$ is total since $h$ is total, and $f$ is $\Delta^0_1$ since $h$ is total and $\Delta^0_1$.

If $f(x, y) = i$ for infinitely many $y$, then our definition of $h(x, y)$ confirms that $x \in A_i$.

Using $f$, let $\Sigma = \{\sigma_y : \sigma_y \in 2^y \land (\forall x < y)[\sigma_y(x) = f(x, y)]\}$ and define $T_\Sigma$ as before. By $\text{RKL}^{(1)}$, there is an infinite set $H \in S(\mathcal{M})$ which is homogeneous for a path through $T_\Sigma$ with some color $c \in \{0, 1\}$.

Let $x \in H$ be arbitrary. By the definition of “homogeneous for a path through $T_\Sigma$ with color $c$,” there are infinitely many $y$ such that $f(x, y) = \sigma_y(x) = c$. By our choice of $f$, this means that $x \in A_c$. In other words, $H \subseteq A_c$ is the desired infinite set. \hfill $\Box$

**Question 3.3.10** (Yokoyama [25]). Does $D^2_2$ imply $P^2_2$? Does $P^2_2$ imply $\text{RKL}^{(1)}$?

**Corollary 3.3.11.** $\text{RKL}^{(1)}$ is incomparable with $\text{WKL}_0$ over $\text{RCA}_0$

**Proof.** Because $\text{WKL}_0$ does not imply $\text{SRT}^2_2$ (Theorem 3.3 of [21] and Theorem 10.5 of [1]), $\text{WKL}_0$ does not imply $\text{RKL}^{(1)}$. By the main result of [19], $\text{RT}^2_2$ does not imply $\text{WKL}_0$, so $\text{RKL}^{(1)}$ cannot imply $\text{WKL}_0$. \hfill $\Box$

**Remark 3.3.12.** Using the arguments above, we can rephrase $\text{RT}^2_2$ as the statement “for each $\Sigma$ that contains exactly one string of each length, there is an infinite $H$ which is homogeneous (with fixed color $c$) for each $\sigma \in \Sigma$ s.t. $|\sigma| \in H$.”

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Question 3.3.13. Does $RKL^{(1)}$ imply COH, CAC, ADS or $RT^2_2$? One implication holds if and only if all implications hold. Does $SRT^2_2$ imply $RKL^{(1)}$?

3.4 Arithmetically-definable trees

Statement 3.4.1 ($RCA_0$). $RKL^{(<\omega)}$ is the axiom scheme which, for each arithmetic formula $\phi$, asserts that “if $\phi$ defines a tree $T$ containing arbitrarily long strings, there is an infinite set $H$ which is homogeneous for a path through $T$.”

Theorem 3.4.2. Over $RCA_0$, we have the following strict implications: $ACA_0 \implies RKL^{(<\omega)} \implies RKL^{(1)} \implies RKL$.

The implications are immediate. We have already seen that the third implication is strict. We now show that the first two implications are also strict. We first use the following result from [19] to separate $RKL^{(<\omega)}$ from $ACA_0$.

Theorem 3.4.3 (Liu [19]). For every $C \succ \emptyset$ and every coloring $p : \mathbb{N} \to \{0, 1\}$, there exists an infinite set $H$ homogeneous for $p$ such that $H \oplus C \succ \emptyset$.

Corollary 3.4.4. There is an $\omega$-model of $RKL^{(<\omega)}$ where $WKL_0$ fails.

Proof. To build an $\omega$-model $\mathcal{M} = (\omega, S(\mathcal{M}))$ of $RKL^{(<\omega)}$, we begin with $S(\mathcal{M}) = REC$ and add sets to $S(\mathcal{M})$.

The general strategy for creating a model of $RKL^{(<\omega)}$ uses a list of the infinite trees which are arithmetically-definable from any set $X \in S(\mathcal{M})$. For each $i \in \mathbb{N}$, we must ensure that there is some finite stage $s$ where we select a path $p$ through the $i^{th}$ tree $T$, where we select an infinite set $H_s$ homogeneous for $p$, and where we add $H_s$ to $S(\mathcal{M})$ and close downward under $\leq_T$. To ensure that $\mathcal{M} \not\models WKL_0$, we use Theorem 3.4.3 to select $H_s$ such that $H_s \oplus \bigoplus_{j \leq s-1} H_j \succ \emptyset$. 

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It is possible that adding the set \( H_s \) to \( S(\mathcal{M}) \) causes new sets to become arithmetically-definable with parameters from \( S(\mathcal{M}) \). Therefore, each time we add \( H_s \) to \( S(\mathcal{M}) \), we create a new list containing the trees arithmetically-definable from \( \bigoplus_{i \leq s} H_i \). We dovetail the lists, eventually running the general strategy for each tree in each list. In the limit, we obtain \( \mathcal{M} \models RKL^{(<\omega)} + \neg WKL_0. \)

\[ \square \]

**Corollary 3.4.5** (RCA\(_0\)). \( RKL^{(<\omega)} \) does not imply \( WKL_0 \).

We separate \( RKL^{(<\omega)} \) from \( RKL^{(1)} \) with an \( \omega \)-model by the following observation.

**Lemma 3.4.6.** For each \( n \), no model of \( RKL^{(<\omega)} \) is bounded by \( \emptyset^n \).

*Proof.* By the proof of Lemma \[3.2.5\] relativized to \( X = \emptyset^n \), we obtain an \( \emptyset^n \)-computable infinite tree \( T \) such that no infinite set \( W_e^{\emptyset^n} \) is homogeneous for a path through \( T \). Since each \( \emptyset^n \)-computable set is \( W_e^{\emptyset^n} \) for some \( e \), it follows that no infinite \( \emptyset^n \)-computable set is homogeneous for a path through \( T \). \( \square \)

**Proposition 3.4.7.** \( \text{RT}_2^2 \) does not imply \( RKL^{(<\omega)} \) over RCA\(_0\). Hence \( RKL^{(1)} \) does not imply \( RKL^{(<\omega)} \) over RCA\(_0\).

*Proof.* By the previous lemma, there is no model of \( RKL^{(<\omega)} \) which is bounded by \( \emptyset^2 \). By Theorem 3.1 of \[1\], there is an \( \omega \)-model of \( \text{RT}_2^2 \) consisting of only low\(_2\) sets. This model is bounded by \( \emptyset^2 \) so is not a model of \( RKL^{(<\omega)} \). \( \square \)

**Question 3.4.8.** Does \( RKL^{(<\omega)} \) imply \( \text{COH} \) over RCA\(_0\)? That is, does \( RKL^{(<\omega)} \) imply \( \text{RT}_2^2 \) over RCA\(_0\)?

3.4.1 Subsets, co-subsets, and trees

There is a close relationship between finding subsets/co-subsets of a fixed set, and finding sets that are homogeneous for a path through a fixed tree.
**Statement 3.4.9** (RCA$_0$). We define $D_2^{<\omega}$ to be the axiom scheme which asserts $D_2^n$ for each $n \in \omega$.

**Proposition 3.4.10** (RCA$_0$). RKL$^{(<\omega)}$ implies $D_2^{<\omega}$.

*Proof.* Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \text{RCA}_0 + \text{RKL}^{(<\omega)}$ and suppose that $A$ is a $\Delta^0_n$-definable subset of $\mathbb{N}$. We give a $\Pi^0_n$ definition for a tree $T$ as follows. Given $\tau \in 2^{<\mathbb{N}}$, we say that $\tau \in T$ if and only if $(\forall x < |\tau|)[\tau(x) = 1$ if and only if $x \in A]$.

By RKL$^{(<\omega)}$, there is a set $H \in S(\mathcal{M})$ which is homogeneous for arbitrarily long strings in $T$ with color $c \in \{0, 1\}$. Note that the only strings in $T$ are initial segments of $\chi_A$, so $H$ is homogeneous for $\chi_A$ with color $c$. Then $H \subseteq A$ if $c = 1$, and $H \subseteq \overline{A}$ if $c = 0$, as desired. 

**Remark 3.4.11.** For $\omega$-models, the reverse implication also holds.

**Question 3.4.12.** Does $D_2^{<\omega}$ imply RKL$^{(<\omega)}$ over RCA$_0$?

By results of [1], SRT$^2_2$ implies B$\Sigma^0_2$.

**Question 3.4.13.** Are there first order consequences of RKL$^{(<\omega)}$ beyond B$\Sigma^0_2$?

Chong, Slaman, and Yang have recently announced a proof that $D_2^2$ does not imply COH over RCA$_0$ [3].

**Question 3.4.14.** Does $D_2^n$ imply COH for any $n \in \omega$?

Theorem 2.1 of [15] gives another way to state this question for $\omega$-models.

**Question 3.4.15.** Is there any arithmetically-definable $f : \mathbb{N} \rightarrow \{0, 1\}$ such that any set $H$ homogeneous for $f$ satisfies $H' \gg \emptyset'$?
Figure 3.1. The reverse mathematical strength of RKL.
Figure 3.2. The reverse mathematical strength of RKL and PRT.
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