CURVATURE AND RIEMANIAN SUBMERSIONS

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Abstract

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We study Riemannian submersions from positively curved manifolds and from Einstein manifolds. We first prove a diameter rigidity theorem for Riemannian submersions. Secondly we show that there is no nontrivial Riemannian submersion from positively curved four manifolds such that either the mean curvature vector field or the norm of the O’Neill tensor is basic. We also classify Riemannian submersions from compact four-dimensional Einstein manifolds with totally geodesic fibers.
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CHAPTER 1

INTRODUCTION

The central topic of this thesis is the study of curvature and Riemannian submersions. A smooth map \( \pi : (M, g) \rightarrow (N, h) \) is a Riemannian submersion if \( \pi \) is a submersion and satisfies the following property:

\[
g_p(v, w) = h_{\pi(p)}(\pi_*v, \pi_*w)
\]

for any \( v, w \) that are tangent vectors in \( TM_p \) and perpendicular to the kernel of \( \pi_* \). The main aim of this thesis is to study the following problem:

**Problem:** Explore the structure of \( \pi \) under additional curvature assumptions of \((M, g)\).

When \((M, g)\) has constant sectional curvature, we have the following classification results ([9], [23], [24]).

**Theorem 1.0.1.** Let \((M^m, g)\) be a compact \( m \)-dimensional Riemannian manifold with constant sectional curvature \( c \), and \( \pi : (M^m, g) \rightarrow (N, h) \) be a nontrivial Riemannian submersion (i.e. \( 0 < \dim N < \dim M \)) with connected fibers.

1. If \( c < 0 \), there is no such Riemannian submersion.

2. If \( c = 0 \), then locally \( \pi \) is the projection of a metric product onto one of its factors.

3. If \( c > 0 \) and \( M^m \) is simply connected, then \( \pi \) is metrically congruent to the
Hopf fibration, i.e., there exist isometries $f_1 : M^m \to \mathbb{S}^m$ and $f_2 : N \to \mathbb{P}(\mathbb{K})$ such that $pf_1 = f_2 \pi$, where $p$ is the standard projection from $\mathbb{S}^m$ to projective spaces $\mathbb{P}(\mathbb{K})$.

However, very little is known about the structure of $\pi$ if $(M, g)$ is not of constant curvature. In this thesis we will consider two different curvature conditions:

1. $(M, g)$ has positive sectional curvature.

2. $(M, g)$ is an Einstein manifold.

If $(M, g)$ is a Riemannian manifold with sectional curvature $\geq 1$, our first result is the following diameter rigidity theorem.

**Theorem 1.0.2.** Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion and $\text{sec}(g) \geq 1$. Suppose all fibers of $\pi$ are closed, connected and have positive dimensions. Then

1. $\text{diam}(N) \leq \frac{\pi}{2}$.

2. If $\text{diam}(M) = \text{diam}(N) = \frac{\pi}{2}$ and $M$ is simply connected, then $\pi$ is metrically congruent to $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, i.e., there exist isometries $f_1 : M \to \mathbb{C}P^{2n+1}$ and $f_2 : N \to \mathbb{H}P^n$ such that $pf_1 = f_2 \pi$, where $p$ is the standard projection from $\mathbb{C}P^{2n+1}$ to $\mathbb{H}P^n$.

**Remark 1.0.1.** In his thesis [16], A. Lytchak proved that for a submetry $\pi : (X, g) \to (Y, h)$, the radius of $Y$ is less or equal to $\frac{\pi}{2}$, where $(X, g)$ is an Aleksandrov space with $K \geq 1$.

**Remark 1.0.2.** If $\text{diam}(M) > \frac{\pi}{2}$ and $\text{diam}(N) = \frac{\pi}{2}$, we know that $M$ is homomorphic to a sphere [17]. However, it is not clear how to get more information in this case. For example, is it true that $(M, g)$ is isometric to the standard sphere?

Our second result is devoted to studying the following conjecture due to Fred Wilhelm.

**Conjecture 1** Let $\pi : (M, g) \to (N, h)$ be a nontrivial Riemannian submersion,
where \((M, g)\) is a compact Riemannian manifold with positive sectional curvature. Then \(\dim(F) < \dim(N)\), where \(F\) is the fiber of \(\pi\).

When \(\dim(M) = 4\), Conjecture 1 is equivalent to the following conjecture.

**Conjecture 2** There is no nontrivial Riemannian submersion from any compact four manifold \((M^4, g)\) with positive sectional curvature.

In fact, suppose there exists such a Riemannian submersion \(\pi : (M^4, g) \to (N, h)\). Then Conjecture 1 would imply \(\dim(N) = 3\). Hence the Euler number of \(M^4\) is zero. On the other hand, since \((M^4, g)\) has positive sectional curvature, \(H^1(M^4, \mathbb{R}) = 0\) by Bochner’s vanishing theorem ([18], page 208). By Poincaré duality, the Euler number of \(M^4\) is positive. Contradiction.

Let \(\pi : (M, g) \to (N, h)\) be a Riemannian submersion. We say that a function \(f\) defined on \(M\) is basic if \(f\) is constant along each fiber. A vector field \(X\) on \(M\) is basic if it is horizontal and is \(\pi\)-related to a vector field on \(N\). In other words, \(X\) is the horizontal lift of some vector field on \(N\). Let \(H\) be the mean curvature vector field of the fibers of \(\pi\) and \(A\) the O’Neill tensor of \(\pi\). We denote by \(|A|\) the norm of \(A\), i.e., \(|A|^2 = \sum_{i,j} \|A_{X_i}X_j\|^2\), where \(\{X_i\}\) is a local orthonormal basis of the horizontal distribution of \(\pi\). The next theorem gives a partial answer to Conjecture 2.

**Theorem 1.0.3.** There is no nontrivial Riemannian submersion from any compact four manifold with positive curvature such that either \(|A|\) or \(H\) is basic.

We emphasize that in Conjecture 1 the assumption that \((M, g)\) has positive sectional curvature can *not* be replaced by \((M, g)\) has positive sectional curvature *almost* everywhere, namely, \((M, g)\) has nonnegative sectional curvature everywhere and has positive sectional curvature on an open and dense subset of \(M\). Indeed, Let \(g\) be the metric on \(S^2 \times S^3\) constructed by B. Wilking which has positive sectional curvature *almost* everywhere [25]. Then by a theorem of K. Tapp [20], \(g\) can be extended to a
nonnegatively curved metric $\tilde{g}$ on $S^2 \times \mathbb{R}^4$ such that $(S^2 \times S^3, g)$ becomes the distance sphere of radius 1 about the soul. By Proposition 6.0.8, we get a Riemannian submersion $\pi : (S^2 \times S^3, g) \to (S^2, h)$, where $h$ is the induced metric on the soul $S^2$ from $\tilde{g}$. This example shows that in Conjecture 1 the assumption that $(M, g)$ has positive sectional curvature can not be replaced by $(M, g)$ has positive sectional curvature almost everywhere.

Riemannian submersions are also important in the study of compact Einstein manifolds, for example, see [3]. Our next theorem gives a complete classification of two-dimensional Riemannian submersions from compact four-dimensional Einstein manifolds with totally geodesic fibers.

**Theorem 1.0.4.** Suppose $\pi : (M^4, g) \to (N, h)$ is a Riemannian submersion, where $(M^4, g)$ is a compact four-dimensional Einstein manifold. If all fibers of $\pi$ are totally geodesic and have dimension 2, then locally $\pi$ is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors, where $B^2(c)$ is a two-dimensional compact manifold with constant curvature $c$.

**Remark 1.0.3.** If the dimension of the fibers of $\pi$ is 1 or 3 (all fibers are not necessarily totally geodesic), then the Euler number of $M^4$ is zero. By a theorem of Berger [2, 15], $(M^4, g)$ must be flat. Hence by a theorem of Walschap [23], locally $\pi$ is the projection of a metric product onto one of its factors.
CHAPTER 2

PRELIMINARIES

2.1 Basic definitions

In this section we recall some definitions and facts on Riemannian submersions. We refer to [17] for more details. A smooth map $\pi : (M, g) \rightarrow (N, h)$ is a Riemannian submersion if $\pi_*$ is surjective and satisfies the following property:

$$g_p(v, w) = h_{\pi(p)}(\pi_*v, \pi_*w)$$

for any $v, w$ that are tangent vectors in $TM_p$ and perpendicular to the kernel of $\pi_*$. Given such a Riemannian submersion, $\pi$ induces an orthogonal splitting $TM = H \oplus V$, where $V = ker(\pi_*)$ and $H$ is the orthogonal complement of $V$. $H / V$ are called the horizontal / vertical distributions of $TM$, respectively. We write $Z = Z^h + Z^v$ for the corresponding decomposition of $Z \in TM$. For any $p \in M$, $\pi^{-1}(\pi(p))$, called the fiber of $\pi$ at $p$, is a submanifold of $M$ with dimension $dim(M) - dim(N)$.

The O'Neill tensor $A$ is given by

$$A_XY = (\nabla_XY)^v = \frac{1}{2}([X,Y])^v,$$

where $X, Y \in H$ and are $\pi$-related to some vector field on $N$.

Fix $X \in H$, define $A^*_X$ by

$$A^*_X : V \rightarrow H$$

$$V \mapsto -(\nabla_XV)^h.$$
Then $A^*_X$ is the dual of $A_X$. By definition, it is not hard to see that $A^*_X V$ is perpendicular to $X$ for any $V \in \mathcal{V}$.

Define the second fundamental tensor $S$ of $\pi$ by

$$S_X V = -(\nabla_V X)^v,$$

where $X \in \mathcal{H}$ and $V \in \mathcal{V}$.

Define the mean curvature vector field $H$ of $\pi$ by

$$H = \sum_i (\nabla_{V_i} V_i)^h,$$

where $\{V_i\}_{i=1}^k$ is any orthonormal basis of $\mathcal{V}$ and $k = \text{dim} \mathcal{V}$.

Define the mean curvature form $\omega$ of $\pi$ by

$$\omega(Z) = g(H, Z),$$

where $Z \in TM$. It is clear that $i_V \omega = \omega(V) = 0$ for any $V \in \mathcal{V}$.

2.2 Basic cohomology of Riemannian submersions

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion. We say that a function $f$ defined on $M$ is basic if $f$ is constant along each fiber. A vector field $X$ on $M$ is basic if it is horizontal and is $\pi$-related to a vector field on $N$. In other words, $X$ is the horizontal lift of some vector field on $N$. A differential form $\alpha$ on $M$ is called basic if and only if $i_V \alpha = 0$ and $L_V \alpha = 0$ for any $V \in \mathcal{V}$, where $L_V \alpha$ is the Lie derivative of $\alpha$.

**Proposition 2.2.1.** $H$ is basic if and only if $\omega$ is basic, where $H / \omega$ are the mean curvature vector /form, respectively.
Proof. 1. \( H \) is basic \( \Rightarrow \) \( \omega \) is basic.

Let \( V, X \) be a vertical / basic vector field, respectively. Then \([V, X]\) is vertical.

On the other hand, since \( H \) is basic, \( g(H, X) \) is a basic function. Then

\[
\mathcal{L}_V \omega(X) = V(\omega(X)) - \omega([V, X]) = Vg(H, X) - g(H, [V, X]) = 0.
\]

On the other hand, let \( W \) be a vertical vector field, then

\[
\mathcal{L}_V \omega(W) = V(\omega(W)) - \omega([V, W]) = Vg(H, W) - g(H, [V, W]) = 0.
\]

Hence \( \mathcal{L}_V \omega = 0 \) and \( \omega \) is a basic form.

2. \( \omega \) is basic \( \Rightarrow \) \( H \) is basic.

Let \( \{X_i\}_{i=1}^n \) be basic vector fields forming an orthonormal basis of \( \mathcal{H} \), where \( \dim \mathcal{H} = n \). Then \( H = \sum_{i=1}^n f_i X_i \). If \( \omega \) is basic, for any vertical vector field \( V \), we see that

\[
0 = \mathcal{L}_V \omega(X_i) = Vg(H, X_i) - g(H, [V, X_i]) = Vf_i.
\]

Hence \( f_i \) is a basic function for each \( i \) and \( H \) is a basic vector field.

\[ \blacksquare \]

Proposition 2.2.2. Let \( \Omega_b(M) \) be the set of basic forms of \( M \). Then \( \Omega_b(M) \) is closed under the exterior derivative \( d \).

Proof. Let \( \alpha \) be a basic form. Then for any vertical vector field \( V \), we have

\[
0 = \mathcal{L}_V \alpha = i_V d\alpha + d(i_V \alpha) = i_V d\alpha.
\]

\[
\mathcal{L}_V d\alpha = i_V (d^2 \alpha) + d(i_V d\alpha) = 0.
\]

Hence \( d\alpha \) is also a basic form. So \( \Omega_b(M) \) is closed under the exterior derivative \( d \).
Thus the set of basic forms of $M$, denoted by $\Omega_b(M)$, constitutes a subcomplex

$$d : \Omega^r_b(M) \to \Omega^{r+1}_b(M)$$

of the De Rham complex $\Omega(M)$. The basic cohomology of $M$, denoted by $H^*_b(M)$, is defined to be the cohomology of $(\Omega^*_b(M), d)$.

**Proposition 2.2.3.** The inclusion map $i : \Omega_b(M) \to \Omega(M)$ induces an injective map

$$H^1_b(M) \to H^1_{DR}(M).$$

**Proof.** Let $\alpha$ be a basic one form and $[\alpha]$ represents a zero element in $H^1_{DR}(M)$. Then $\alpha = df$ for some smooth function on $M$. For any vertical vector field $V$, we have

$$Vf = df(V) = \alpha(V) = i_V \alpha = 0.$$  

Hence $f$ is a basic function and $[\alpha]$ represents a zero element in $H^1_b(M)$.

2.3 O’Neill’s formulas

In this section we collect several curvature formulas proved in [17].

**Theorem 2.3.1.** Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion with fiber $F$. Let $X, Y / U, V$ be basic /vertical vector fields, respectively. If $\|X\| = \|Y\| = \|U\| = \|V\| = 1$ and $g(X, Y) = g(U, V) = 0$, then

$$K_M(U, V) = K_F(U, V) + \|S_U V\|^2 - g(S_U U, S_V V);$$

$$K_M(X, V) = g((\nabla_X S)_V V, X) - \|S_V X\|^2 + \|A^*_X V\|^2;$$
\[ K_M(X,Y) = K_N(X,Y) - 3 \|A_XY\|^2, \]

where \( K_M/K_N/K_F \) are the sectional curvature of \( M/N/F \), respectively.

In particular, if all fibers of \( \pi \) are totally geodesic, then

\[ K_M(X,V) = \|A_X^*V\|^2 \geq 0. \]

Proof. See [17].

\[ \]

2.4 Critical point theory of distance functions

Let \((M,g)\) be a complete Riemannian manifold and \(K \subset M\) a compact subset. Then the distance function

\[ r(x) = d(x,K) = \min \{d(x,p) : p \in K\} \]

is not smooth everywhere in general. Grove and Shiohama [11] made the fundamental observation that there is a meaningful definition of "critical point" for distance functions, such that in the absence of critical points, the Isotopy Lemma of Morse Theory holds. They also observed that in the presence of a lower curvature bound, Toponogov’s comparison theorem [18] can be used to derive geometric information, from the existence of critical points.

**Definition 2.4.1.** The point \( q \) (\( q \) is not in \( K \)) is a critical point of \( d(\cdot,K) \) if for any \( v \) in the tangent space \( TM_q \), there is a minimal geodesic \( \gamma \) from \( q \) to \( K \) realizing \( d(q,K) \), making an angle \( \angle(v,\gamma'(0)) \leq \frac{\pi}{2} \).

The following useful lemma is due to Berger.
Lemma 2.4.2. If $q$ is a local maximum of the distance function $d(x, K)$, then $q$ is a critical point.

Proof. See [18], page 335-337.
In this chapter we prove theorem 1.0.2. We restate it here:

**Theorem 3.0.3.** Let \( \pi : (M, g) \to (N, h) \) be a Riemannian submersion and \( \sec(g) \geq 1 \). Suppose all fibers of \( \pi \) are closed, connected, oriented and have positive dimensions. Then

1. \( \text{diam}(N) \leq \frac{\pi}{2} \).

2. If \( \dim(M) = \text{diam}(N) = \frac{\pi}{2} \) and \( M \) is simply connected, then \( \pi \) is metrically congruent to \( S^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n \), i.e., there exist isometries \( f_1 : M \to \mathbb{CP}^{2n+1} \) and \( f_2 : N \to \mathbb{HP}^n \) such that \( pf_1 = f_2 \pi \), where \( p \) is the standard projection from \( \mathbb{CP}^{2n+1} \) to \( \mathbb{HP}^n \).

**Proof.** To see the first part, suppose that \( \text{diam}(N) > \frac{\pi}{2} \), we are going to derive a contradiction. Let \( p, q \in M \) such that \( d(\pi(p), \pi(q)) = \text{diam}(N) \) and \( F_p, F_q \) be the corresponding fibers passing through \( p \) and \( q \), respectively. Fix a point \( x \in F_q \), then \( d(x, F_p) = \max_{z \in M} d(z, F_p) \) since \( \pi \) is a Riemannian submersion (which forces \( d(z, F_p) = d(F_z, F_p) \)). Since \( \dim(F) > 0 \), we can choose a point \( y \in F_q \) such that \( y \neq x \). Join a minimal geodesic \( \gamma_1 \) from \( x \) to \( y \) (\( \gamma_1 \) is not necessary contained in \( F_q \)). By Berger’s Lemma 2.4.2 and critical point theory of distance functions, there is a minimal geodesic \( \gamma_2 \) realizing \( d(x, F_p) \) and the angle between \( \gamma_1 \) and \( \gamma_2 \) is less or equal to \( \frac{\pi}{2} \). Let \( z = \gamma_2 \cap F_p \).

Since \( \sec(g) \geq 1 \), by Toponogov’s comparison theorem, we see that \( d(y, z) < \)
$d(x, z)$. On the other hand, since $d(x, z) = d(x, F_p) = d(F_p, F_q)$, we see that $d(y, z) \geq d(x, z)$. Contradiction.

To prove the second part, we adapt the arguments in [8]. Define $A$ and $A'$ by

$$A = \{ x \mid d(x, F_p) = \frac{\pi}{2}, x \in M \}, \quad A' = \{ x \mid d(x, A) = \frac{\pi}{2}, x \in M \}.$$ 

Define $B$ and $B'$ by

$$B = \{ y \mid d(y, \pi(p)) = \frac{\pi}{2}, y \in N \}, \quad B' = \{ y \mid d(y, B) = \frac{\pi}{2}, y \in N \}.$$ 

A subset $K$ is called totally $\pi$ convex if for any pair $p_1$, $p_2$ in $K$ and any geodesic $\gamma : [0, l] \to M$ with $\gamma(0) = p_1, \gamma(l) = p_2$ and $l < \pi$, one has $\gamma([0, l]) \subset K$.

**Proposition 3.0.4.** $A$ and $A'$ are totally $\pi$-convex in $M$ and $B$ and $B'$ are totally $\pi$-convex in $N$.

**Proof.** Since $\sec(g) \geq 1$ and $\text{diam}(M) = \frac{\pi}{2}$, by Proposition 1.3 in [8], we see that $A$ and $A'$ are totally $\pi$-convex in $M$. On the other hand, by O'Neill’s formula ([17]), $\sec(h) \geq \sec(g) \geq 1$. Together with $\text{diam}(N) = \frac{\pi}{2}$, by Proposition 1.3 in [8] again, we see that $B$ and $B'$ are totally $\pi$-convex in $N$. \hfill \Box

**Proposition 3.0.5.** $A$ is not a point.

**Proof.** Since $\pi$ is a Riemannian submersion, any two fibers are equidistant. Hence $A$ contains at least one fiber of $\pi$. So $A$ is not a point. \hfill \Box

**Proposition 3.0.6.** $A$ has empty boundary.

**Proof.** Suppose that $A$ has nonempty boundary. We are going to derive a contradiction. Since $\pi$ is a Riemannian submersion, any two fibers are equidistant. Then it is direct to check that

$$A = \pi^{-1}(B).$$
Since $\pi : (M, g) \to (N, h)$ is a Riemannian submersion, $\pi$ is a locally trivial fibration by a theorem of Hermann \[14\]. Then $\pi : A \to B$ is also a locally trivial fibration.

If $A$ has nonempty boundary, then $B$ also has nonempty boundary. Since $B$ is convex, then by Cheeger and Gromoll’s soul theorem \[4\], $B$ is diffeomorphic to a disk. For the same reason, $A$ is also diffeomorphic to a disk. However, this is impossible since $\pi : A \to B$ is a locally trivial fibration with closed fibers. Hence $A$ has empty boundary.

Now by Propositions 3.0.4, 3.0.5, 3.0.6 and Theorem 4.3 in \[8\], Theorem 1 in \[24\], we see that $M$ is isometric to a compact rank one symmetric space. Moreover, $M$ can not be a sphere. Otherwise, $\pi$ is metrically congruent to the Hopf fibration \[9\] \[24\], which contradicts that $\text{diam}(M) = \text{diam}(N) = \frac{\pi}{2}$. Hence $M$ is isometric to a projective space or Cayley plane. Now by the classification results in \[5\], we see that $\pi$ is metrically congruent to $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$. 

\[\square\]
CHAPTER 4

NONEXISTENCE OF CERTAIN RIEMANNIAN SUBMERSIONS

4.1 Dimension restrictions for Riemannian submersions with totally geodesic fibers

This section is concerned with the study of Conjecture 1 for Riemannian submersions with totally geodesic fibers.

**Proposition 4.1.1.** Let \( \pi : (M, g) \to (N, h) \) be a nontrivial Riemannian submersion and \((M, g)\) has positive sectional curvature.

1. If at least two fibers of \( \pi \) are totally geodesic, then \( \dim(F) < \dim(N) \), where \( F \) is any fiber of \( \pi \);

2. If all fibers of \( \pi \) are totally geodesic, then \( \dim(F) < \rho(\dim(N)) + 1 \), where \( \rho(\dim(N)) = \rho(n) \) is the maximal number of linearly independent vector fields on \( S^{n-1}, n = \dim(N) \). Notice that we always have \( \rho(\dim(N)) + 1 \leq \dim(N) - 1 + 1 = \dim(N) \) and equality holds if and only \( \dim(N) = 2, 4 \) or 8.

**Remark 4.1.1.** Although not explicitly stated, Proposition 4.1.1 appears in [6].

**Proof.** To see the first part, assume that \( F_0 \) and \( F_1 \) are two totally geodesic fibers. Since \( F_0 \) and \( F_1 \) do not intersect with each other, Frankel’s theorem [7] implies that \( \dim(F_0) + \dim(F_1) < \dim(M) \). Hence \( \dim(F_0) = \dim(F_1) < \dim(N) \).

To see the second part, suppose all fibers of \( \pi \) are totally geodesic. Fix \( p \in M \) and choose \( X_p \) to be any point in the unit sphere of \( H_p \). Extend \( X_p \) to be a unit basic vector field \( X \). By O’Neill’s formula (Theorem 2.3.1), \( K(X, V) = \|A_X^V V\|^2 \)
for any unit vertical vector field $V$. Since $K(X,V) > 0$ by assumption, we see that $A_X^* V \neq 0$ for any $V \neq 0$. Let $v_1, v_2, \cdots, v_k$ be any orthonormal basis of $V_p$, where $k = \dim(F_p)$ and $F_p$ is the fiber passing through $p$. Then $A_X^*(v_1), A_X^*(v_2), \cdots A_X^*(v_k)$ are linearly independent and are perpendicular to $X_p$. Since $X_p$ could be any point in the unit sphere of $H_p$, we get $k$ linearly independent vector fields on the unit sphere of $H_p$. By the definition of $\rho(\dim N)$, we see that $\dim(F_p) = k \leq \rho(\dim(N)) < \rho(\dim(N)) + 1$.

**Remark 4.1.2.** It would be very interesting to know whether one can replace $\dim(F) < \dim(N)$ by $\dim(F) < \rho(\dim(N)) + 1$ in Conjecture 1. It would be the Riemannian analogue of Toponogov’s Conjecture (page 1727 in [19]) and would imply that $\dim(N)$ must be even (In fact, if $\dim(N)$ is odd, then $\rho(\dim(N)) = 0$. Hence $\dim(F) < \rho(\dim(N)) + 1$ implies $\dim(F) = 0$ and hence $\pi$ is trivial, contradiction). In particular, there would be no Riemannian submersion with one-dimensional fibers from even-dimensional manifolds with positive sectional curvature.

### 4.2 Nonexistence of certain Riemannian submersions

Let $(M^m, g)$ be an $m$-dimensional compact manifold with positive sectional curvature, $m \geq 4$ and $(N^2, h)$ be a 2-dimensional compact Riemannian manifold. We are going to prove the following theorem which implies Theorem 1.0.3.

**Theorem 4.2.1.** There is no Riemannian submersion $\pi : (M^m, g) \to (N^2, h)$ such that

1. the Euler numbers of the fibers are nonzero and
2. either $\|A\|$ or $H$ is basic.

**Remark 4.2.1.** If Conjecture 1 is true, then there would be no Riemannian submersion $\pi : (M^m, g) \to (N^2, h)$, where $(M^m, g)$ has positive sectional curvature and $m \geq 4$. 
Before we prove Theorem 4.2.1, we first show how to derive Theorem 1.0.3. The proof is by contradiction. Suppose there exists a nontrivial Riemannian submersion \( \pi : (M^4, g) \to (N, h) \) such that either \( \|A\| \) or \( H \) is basic, where \( (M^4, g) \) is a compact four manifold with positive sectional curvature. Since \( (M^4, g) \) has positive sectional curvature, \( H^1(M^4, \mathbb{R}) = 0 \) by Bochner’s vanishing theorem (\[18\], page 208). By Poincaré duality, \( \chi(M^4) = 2 + b_2(M^4) \) is positive. By a theorem of Hermann \[14\], \( \pi \) is a locally trivial fibration. Then \( \chi(M^4) = \chi(N)\chi(F) \), where \( F \) is any fiber of \( \pi \). It follows that \( \dim(N) = 2 \) and \( \chi(F) \) is nonzero (hence all fibers have nonzero Euler numbers), which is a contradiction by Theorem 4.2.1.

The proof of Theorem 4.2.1 is again by contradiction. By passing to its oriented double cover, we can assume that \( N^2 \) is oriented. The idea of the proof of Theorem 4.2.1 is to construct a nowhere vanishing vector field (or line field) on some fiber of \( \pi \), which will imply the Euler numbers of the fibers are zero. Contradiction.

Since \( (M, g) \) has positive sectional curvature, by a theorem of Walschap \[23\], \( \|A\| \) cannot be identical to zero on \( M \). Hence there exists \( p \in M \) such that \( \|A\|(p) \neq 0 \).

If \( \|A\| \) is basic, then \( \|A\| \neq 0 \) at any point on \( F_p \), where \( F_p \) is the fiber at \( p \). Let \( X, Y \) be any orthonormal oriented basic vector fields in some open neighborhood of \( F_p \). Then \( \|A_XY\|^2 = \|A\|^2 \neq 0 \) at any point on \( F_p \). Define a map \( s \) by

\[
\begin{align*}
s : F_p & \to TF_p \\
x & \mapsto \frac{A_XY}{\|A_XY\|}(x).
\end{align*}
\]

Let \( Z, W \) be another orthonormal oriented basic vector fields. Then \( Z = aX + bY \) and \( W = cX + dY, \; ad - bc > 0 \). Then

\[
A_ZW = (ad - bc)A_XY.
\]
Hence \( s \) does not depend on the choice of \( X, Y \). Then \( s \) is a nowhere vanishing vector field on \( F_p \). Thus the Euler number of \( F_p \) is zero. Contradiction.

If \( H \) is basic, the construction of such a nowhere vanishing vector field (or line field) is much more complicated. Under the assumption that \( H \) is basic, we first construct a metric \( \hat{g} \) on \( M^m \) such that \( \pi : (M^m, \hat{g}) \to (N^2, h) \) is still a Riemannian submersion and all fibers are minimal submanifolds with respect to \( \hat{g} \). Of course, in general \( \hat{g} \) can not have positive sectional curvature everywhere. However, the crucial point is that there exists some fiber \( F_0 \) such that \( \hat{g} \) has positive sectional curvature at all points on \( F_0 \). Pick any fiber \( F_1 \) which is close enough to \( F_0 \). Then using the classical variational argument, we construct a continuous codimension one distribution on \( F_1 \). Thus the Euler number of \( F_1 \) is zero. Contradiction.

Now we are going to explain the proof of Theorem 4.2.1 in details. We first need the following lemmas:

**Lemma 4.2.2.** Suppose \( \omega \) is the mean curvature form of a Riemannian submersion from compact Riemannian manifold. If \( \omega \) is a basic form, then it is a closed form.

*Proof.* See page 82 in [22] for a proof. \( \square \)

**Lemma 4.2.3.** Suppose \( \pi : (M^m, g) \to (N, h) \) is a Riemannian submersion such that \( H \) is basic, where \( (M^m, g) \) is a compact Riemannian manifold with positive sectional curvature. Then there exists a metric \( \hat{g} \) on \( M^m \) such that \( \pi : (M^m, \hat{g}) \to (N, h) \) is still a Riemannian submersion and all fibers are minimal submanifolds with respect to \( \hat{g} \). Furthermore, there exists some fiber \( F_0 \) such that \( \hat{g} \) has positive sectional curvature at all points on \( F_0 \).

*Proof.* The idea is to use partial conformal change of metrics along the fibers, see also page 82 in [22]. Let \( \omega \) be the mean curvature form of \( \pi \). Since \( H \) is basic, \( \omega \) is a basic form. Then \( \omega \) is closed by Lemma 4.2.2. So \([\omega]\) defines a cohomology
class in $H^1_b(M^m)$. Because $(M^m, g)$ has positive sectional curvature, $H^1_{DR}(M^m) = 0$ by Bochner’s vanishing theorem ([18], page 208). By Proposition 2.2.3, we see that $H^1_b(M^m) = 0$. Then there exists a basic function $f$ globally defined on $M^m$ such that $\omega = df$. Define $\hat{f} = f - \max_{p \in M^m} f(p)$. Then $\max_{p \in M^m} \hat{f}(p) = 0$ and $\omega = d\hat{f}$. Let $\lambda = e^\hat{f}$ and define

$$\hat{g} = (\lambda^{\hat{f}} g_v) \oplus g_h,$$

where $k = \dim(M^m) - \dim(N)$, $g_v / g_h$ are the vertical / horizontal components of $g$, respectively.

Since the horizontal components of $g$ remains unchanged, $\pi : (M^m, \hat{g}) \to (N, h)$ is still a Riemannian submersion. Now the mean curvature form $\hat{\omega}$ associated to $\hat{g}$ is computed to be

$$\hat{\omega} = \omega - dlog\lambda = 0.$$

Hence all fibers of $\pi$ are minimal submanifolds with respect to $\hat{g}$.

Let $\phi(p) = \lambda^{\hat{f}}(p), p \in M^m$. Then

$$\hat{g} = (\phi g_v) \oplus g_h.$$

Note for any $p \in M^m$, $0 < \phi(p) \leq 1$. Moreover, we have $\max_{p \in M^m} \phi(p) = 1$. Let $p_0 \in M^m$ such that $\phi(p_0) = 1$ and $F_0$ be the fiber of $\pi$ passing through $p_0$. Since $f$ is a basic function on $M^m$, $\phi$ is also basic. Then $\phi \equiv 1$ on $F_0$, which will play a crucial role in our argument below. Of course, in general $\hat{g}$ can not have positive sectional curvature everywhere. However, by Lemma [4.2.4] below, we will see that $\hat{g}$ still has positive sectional curvature at all points on $F_0$. (The reader should compare it to the following fact: Let $\hat{h} = e^{2f} h$ be a conformal change of $h$, where $h$ is a Riemannian metric with positive sectional curvature. Then $\hat{h}$ still has positive sectional curvature at those points where $f$ attains its maximum value.)
Indeed, by Lemma 4.2.4 below, for any basic vector fields $X, Y$ and vertical vector fields $V, W$, we have

\[
\hat{K}(X + V, Y + W)\|(X + V) \wedge (Y + W)\|^2 = \hat{R}(X + V, Y + W, Y + W, X + V)
\]

\[
= R(X + V, Y + W, Y + W, X + V) + (\phi - 1)P(\nabla \phi, \phi, X, Y, V, W)
\]

\[
+ Q(\nabla \phi, \phi, X, Y, V, W) + [-g(W, W)g(\nabla V \nabla \phi, X)
\]

\[
+ g(V, W)g(\nabla W \nabla \phi, X) + g(V, W)g(\nabla V \nabla \phi, Y)
\]

\[
- g(V, V)g(\nabla W \nabla \phi, Y)\] + \frac{1}{2}[-\text{Hess}(\phi)(X, X)g(W, W)
\]

\[
+ 2\text{Hess}(\phi)(X, Y)g(V, W) - \text{Hess}(\phi)(Y, Y)g(V, V),
\]

where $\hat{K}(X + V, Y + W)$ is the sectional curvature of the plane spanned by $X + V, Y + W$ with respect to $\hat{g}$ and

\[
\|(X + V) \wedge (Y + W)\|^2 = \hat{g}(X + V, X + V)\hat{g}(Y + W, Y + W) - [\hat{g}(X + V, Y + W)]^2.
\]

Moreover, $\nabla$ is the Levi-Civita connection and $\text{Hess}(\phi)$ is the Hessian of $\phi$ with respect to $g$. Also $\hat{R}/R$ are the Riemannian curvature tensors with respect to $\hat{g}/g$, respectively. Furthermore, $P(\nabla \phi, \phi, X, Y, V, W)$, $Q(\nabla \phi, \phi, X, Y, V, W)$ are two functions depending on $\nabla \phi, \phi, X, Y, V, W$ and $Q(0, \phi, X, Y, V, W) \equiv 0$ (which will be very important for our purpose).

Since $\phi \equiv 1 = \max_{p \in M} \phi(p)$ on $F_0$, we see that $\nabla \phi \equiv 0$ on $F_0$. Hence $Q(\nabla \phi, \phi, X, Y, V, W) \equiv Q(0, \phi, X, Y, V, W) \equiv 0$ and $\nabla V \nabla \phi \equiv 0, \nabla W \nabla \phi \equiv 0$ on $F_0$. Then at any point on $F_0$, we have

\[
\hat{R}(X + V, Y + W, Y + W, X + V) = R(X + V, Y + W, Y + W, X + V)
\]
\[ + \frac{1}{2} \left[ - \text{Hess}(\phi)(X,X)g(W,W) + 2\text{Hess}(\phi)(X,Y)g(V,W) \\
- \text{Hess}(\phi)(Y,Y)g(V,V) \right]. \] (4.2.1)

By letting
\[
A = \begin{pmatrix}
\text{Hess}(\phi)(X,X) & \text{Hess}(\phi)(X,Y) \\
\text{Hess}(\phi)(X,Y) & \text{Hess}(\phi)(Y,Y)
\end{pmatrix},
B = \begin{pmatrix}
g(W,W) & -g(V,W) \\
-g(V,W) & g(V,V)
\end{pmatrix},
\]
(4.2.1) can be written as
\[
\hat{R}(X + V, Y + W, Y + W, X + V) = R(X + V, Y + W, Y + W, X + V) + \frac{1}{2} tr(-AB).
\]

Since \( \phi \) attains its maximum at any point on \( F_0 \), we see that \(-A\) is nonnegative definite on \( F_0 \). It is easy to check that \( B \) is also nonnegative definite. Hence \( tr(-AB) \geq 0 \) (although \(-AB\) is not nonnegative definite if \( AB \neq BA \)). Since \( g \) has positive sectional curvature everywhere on \( M^m \) by assumption, then at any point on \( F_0 \), we see that
\[
\hat{R}(X + V, Y + W, Y + W, X + V) \geq R(X + V, Y + W, Y + W, X + V) > 0.
\]

Hence \( \hat{g} \) has positive sectional curvature at all points on \( F_0 \).

Lemma 4.2.4. Let \( \pi : (M^m, g) \to (N, h) \) be a Riemannian submersion and \( g = g_v \oplus g_h \), where \( g_v \) / \( g_h \) are the vertical / horizontal components of \( g \), respectively. Suppose \( \phi \) is a positive basic function defined on \( M^m \). Let \( \hat{g} = (\phi g_v) \oplus g_h \). Suppose \( \hat{\nabla} / \nabla \) are the Levi-Civita connections and \( \hat{R} / R \) are the Riemannian curvature tensors with respect to \( \hat{g} / g \), respectively. Moreover, let \( \text{Hess}(\phi) \) be the Hessian of \( \phi \) with respect to \( g \). Then for any horizontal vector fields \( X, Y \) (\( X,Y \) are not necessarily basic vector fields) and vertical vector fields \( V, W \), we have
\[
\hat{\nabla}_X Y = \nabla_X Y; \quad (4.2.2)
\]

\[
\hat{\nabla}_V W = \nabla_V W - \frac{g(V, W)}{2} \nabla \phi + (\phi - 1) (\nabla_V W)^h; \quad (4.2.3)
\]

\[
\hat{\nabla}_X V = \nabla_X V + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \epsilon_i], V) \epsilon_i; \quad (4.2.4)
\]

\[
\hat{\nabla}_V X = \nabla_V X + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \epsilon_i], V) \epsilon_i, \quad (4.2.5)
\]

where \(\{\epsilon_i\}_{i=1}^{n}\) is any orthonormal basis of the horizontal distribution with respect to \(g\) and \(n = \dim(N)\).

Moreover, if \(X, Y\) are basic vector fields and \(V, W\) are vertical vector fields, then

\[
\hat{R}(X + V, Y + W, Y + W, X + V) = R(X + V, Y + W, Y + W, X + V) \\
+ (\phi - 1) P(\nabla \phi, \phi, X, Y, V, W) + Q(\nabla \phi, \phi, X, Y, V, W) \\
+ [-g(W, W)g(\nabla_V \nabla \phi, X) + g(V, W)g(\nabla_W \nabla \phi, X) \\
+ g(V, W)g(\nabla_V \nabla \phi, Y) - g(V, V)g(\nabla_W \nabla \phi, Y)] \\
+ \frac{1}{2} [-Hess(\phi)(X, X)g(W, W) + 2Hess(\phi)(X, Y)g(V, W) \\
- Hess(\phi)(Y, Y)g(V, V)], \quad (4.2.6)
\]
where \( P(\nabla \phi, \phi, X, Y, V, W) \) and \( Q(\nabla \phi, \phi, X, Y, V, W) \) are two functions which depend on \( \nabla \phi, X, Y, V, W \) and \( Q(0, \phi, X, Y, V, W) \equiv 0 \).

**Remark 4.2.2.** To get 4.2.6 in Lemma 4.2.4, as we will see in the proof, it is important that 4.2.2, 4.2.3, 4.2.4, 4.2.5 in Lemma 4.2.4 hold without assuming that \( X, Y \) are basic vector fields.

**Proof.** The proof is based on a lengthy computation and the following Koszul’s formula:

\[
2\hat{g}(\hat{\nabla}_X Y, Z) = X\hat{g}(Y, Z) + Y\hat{g}(Z, X) - Z\hat{g}(X, Y)
+\hat{g}([X, Y], Z) - \hat{g}([Y, Z], X) - \hat{g}([X, Z], Y).
\]

In the computation below, we will use the following trick very often: If we encounter with anything like \( \phi X \), we will rewrite \( \phi X = X + (\phi - 1)X \). By rewriting it in this way, we can compare new curvature terms with odd terms. We will also use the fact that \( \phi \) is a basic function very often.

Now let \( X, Y \) be horizontal vector fields (not necessarily basic vector fields) and \( V, W \) be vertical vector fields. To see 4.2.2 note by Lemma 2 in [17],

\[
(\hat{\nabla}_X Y)^v = \frac{1}{2} [X, Y]^v = (\nabla_X Y)^v.
\]

On the other hand, for any horizontal vector field \( Z \), we have

\[
2g(\hat{\nabla}_X Y, Z) = 2\hat{g}(\hat{\nabla}_X Y, Z)
= X\hat{g}(Y, Z) + Y\hat{g}(Z, X) - Z\hat{g}(X, Y) + \hat{g}([X, Y], Z) - \hat{g}([Y, Z], X) - \hat{g}([X, Z], Y)
= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)
= 2g(\nabla_X Y, Z).
\]

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Hence

\[(\hat{\nabla} X Y)^h = (\nabla X Y)^h.\]

Now we prove 4.2.3. By Koszul’s formula, we see that

\[
2g(\nabla_V W, X) = Vg(W, X) + Wg(X, V) - Xg(V, W)
\]

\[
+ g([V, W], X) - g([W, X], V) - g([V, X], W)
\]

\[
= -Xg(V, W) - g([W, X], V) - g([V, X], W),
\]

and

\[
2g(\hat{\nabla}_V W, X) = 2\hat{g}(\hat{\nabla}_V W, X) = V\hat{g}(W, X) + W\hat{g}(X, V) - X\hat{g}(V, W)
\]

\[
+ \hat{g}([V, W], X) - \hat{g}([W, X], V) - \hat{g}([V, X], W)
\]

\[
= -X\hat{g}(V, W) - \hat{g}([W, X], V) - \hat{g}([V, X], W)
\]

\[
= -X(\phi g(V, W)) - \phi g([W, X], V) - \phi g([V, X], W)
\]

\[
= -\phi g(V, W) g(V, W) + \phi(-Xg(V, W) - g([W, X], V) - g([V, X], W))
\]

\[
= g(-g(V, W) \nabla \phi, X) + 2\phi g(\nabla_V W, X).
\]

Hence

\[(\hat{\nabla}_V W)^h = -\frac{g(V, W)}{2} \nabla \phi + \phi(\nabla_V W)^h.\]

On the other hand, for any vertical vector field \(U\), by Koszul’s formula and using the fact that \(\phi\) is a basic function,

\[
2\phi g(\hat{\nabla}_V W, U) = 2\hat{g}(\hat{\nabla}_V W, U)
\]

\[
= V\hat{g}(W, U) + W\hat{g}(U, V) - U\hat{g}(V, W) + \hat{g}([V, W], U) - \hat{g}([W, U], V) - \hat{g}([V, U], W)
\]
\[ \begin{align*}
&= V(\phi g(W, U)) + W(\phi g(U, V)) - U(\phi g(V, W)) + \phi g([V, W], U) - \phi g([W, U], V) - \phi g([V, U], W) \\
&= \phi [V g(W, U) + W g(U, V) - U g(V, W) + g([V, W], U) - g([W, U], V) - g([V, U], W)] \\
&= 2\phi g(\nabla_V W, U).
\end{align*} \]

Hence

\[ (\hat{\nabla}_V W)^v = (\nabla_V W)^v. \]

Thus

\[
\hat{\nabla}_V W = (\hat{\nabla}_V W)^h + (\hat{\nabla}_V W)^v \]

\[
= -\frac{g(V, W)}{2} \nabla \phi + \phi (\nabla_V W)^h + (\nabla_V W)^v \]

\[
= -\frac{g(V, W)}{2} \nabla \phi + (\phi - 1)(\nabla_V W)^h + (\nabla_V W)^v \]

\[
= \nabla_V W - \frac{g(V, W)}{2} \nabla \phi + (\phi - 1)(\nabla_V W)^h. \]

Now we prove \textbf{4.2.4}. Let \( \{\varepsilon_i\}_{i=1}^n \) be any orthonormal basis of the horizontal distribution with respect to \( g \). By \textit{Koszul's formula}, we see that

\[
2\hat{g}(\hat{\nabla}_X V, \varepsilon_i) = X\hat{g}(V, \varepsilon_i) + V\hat{g}(\varepsilon_i, X) - \varepsilon_i\hat{g}(X, V) \]

\[
+ \hat{g}([X, V], \varepsilon_i) - \hat{g}([V, \varepsilon_i], X) - \hat{g}([X, \varepsilon_i], V) \]

\[
= V\hat{g}(\varepsilon_i, X) + \hat{g}([X, V], \varepsilon_i) - \hat{g}([V, \varepsilon_i], X) - \hat{g}([X, \varepsilon_i], V). \]

\[
= V g(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - \phi g([X, \varepsilon_i], V) \]

\[
= V g(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - g([X, \varepsilon_i], V) + (1 - \phi) g([X, \varepsilon_i], V). \]

By \textit{Koszul's formula} again, we see that

\[
2g(\nabla_X V, \varepsilon_i) = V g(\varepsilon_i, X) + g([X, V], \varepsilon_i) - g([V, \varepsilon_i], X) - g([X, \varepsilon_i], V). \]
Thus
\[ 2g(\hat{\nabla}_X V, \varepsilon_i) = 2g(\nabla_X V, \varepsilon_i) + (1 - \phi)g([X, \varepsilon_i], V). \]

Hence
\[ (\hat{\nabla}_X V)^h = (\nabla_X V)^h + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \varepsilon_i], V)\varepsilon_i. \]

Note that \( \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \varepsilon_i], V)\varepsilon_i \) does not depend on the choice of \( \{\varepsilon_i\}_{i=1}^{n} \). By a similar argument as above, we see that
\[ (\hat{\nabla}_X V)^v = (\nabla_X V)^v + \frac{g(X, \nabla \phi)}{2\phi} V. \]

Hence
\[ \hat{\nabla}_X V = \nabla_X V + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \varepsilon_i], V)\varepsilon_i. \]

It follows that
\[ \hat{\nabla}_V X = \hat{\nabla}_X V + [V, X] \]
\[ = \nabla_X V + [V, X] + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \varepsilon_i], V)\varepsilon_i \]
\[ = \nabla_V X + \frac{g(X, \nabla \phi)}{2\phi} V + \frac{1 - \phi}{2} \sum_{i=1}^{n} g([X, \varepsilon_i], V)\varepsilon_i, \]
which finishes the proof of 4.2.5.

Now we are going to prove 4.2.6. In the following we always assume that \( X, Y \)
are basic vector fields. First of all, we have
\[ \hat{R}(X + V, Y + W, Y + W, X + V) = \hat{R}(X, Y, Y, X) + \hat{R}(V, W, W, V) \]
\[ + \hat{R}(X, W, W, X) + \hat{R}(Y, V, V, Y) + 2\hat{R}(X, Y, Y, V) + 2\hat{R}(Y, X, X, W) \]
\[ + 2\hat{R}(X, Y, W, V) + 2\hat{R}(X, W, Y, V) + 2\hat{R}(V, W, W, X) + 2\hat{R}(W, V, V, Y). \]
Since \( \hat{g}_h = g_h \), \((M^m, \hat{g}) \to (N, h)\) is still a Riemannian submersion. Then by O’Neill’s formula [17], we have

\[
I_0 = \hat{R}(X, Y, Y, X) = R_N(X, Y, Y, X) - \frac{3}{4} \hat{g}([X, Y]^v, [X, Y]^v)
\]

\[
= R_N(X, Y, Y, X) - \frac{3}{4} \hat{g}([X, Y]^v, [X, Y]^v) + \frac{3}{4} (1 - \phi) g([X, Y]^v, [X, Y]^v)
\]

\[
= R_N(X, Y, X) + \frac{3}{4} (1 - \phi) g([X, Y]^v, [X, Y]^v),
\]

where \( R_N \) is the Riemannian curvature tensor of \((N, h)\). On the other hand, by 4.2.2, 4.2.3, 4.2.4, 4.2.5,

\[
I_1 = \hat{R}(V, W, W, V) = \hat{g}((\hat{\nabla}_V \hat{\nabla}_W W - \hat{\nabla}_W \hat{\nabla}_V W - \hat{\nabla}_{[V, W]} W, V)
\]

\[
= \phi g(\hat{\nabla}_V [\nabla_W W - \frac{1}{2} g(W, W) \nabla \phi + (\phi - 1)(\nabla_W W)^h], V)
\]

\[
- \phi g(\hat{\nabla}_W [\nabla_V W - \frac{1}{2} g(V, W) \nabla \phi + (\phi - 1)(\nabla_V W)^h], V)
\]

\[
- \phi g(\nabla_{[V, W]} W - \frac{1}{2} g([V, W], W) \nabla \phi + (\phi - 1)(\nabla_{[V, W]} W)^h, V)
\]

\[
= \phi g(\nabla_V (\nabla_W W) - \nabla_W (\nabla_V W) - \nabla_{[V, W]} W, V)
\]

\[
- \frac{1}{2} \phi [g(W, W) g(\hat{\nabla}_V \nabla \phi, V) - g(V, W) g(\hat{\nabla}_W \nabla \phi, V)]
\]

\[
+ (\phi - 1) \tilde{P}_1(\nabla \phi, \phi, X, Y, V, W) + \tilde{Q}_1(\nabla \phi, \phi, X, Y, V, W).
\]

\[
= \phi g(\hat{\nabla}_V (\nabla_W W)^v + \hat{\nabla}_V (\nabla_W W)^h, V) - \phi g(\hat{\nabla}_W (\nabla_V W)^v + \hat{\nabla}_W (\nabla_V W)^h, V)
\]

\[
- \phi g(\nabla_{[V, W]} W, V) - \frac{1}{2} \phi [g(W, W) g(\nabla_V \nabla \phi, V) - g(V, W) g(\nabla_W \nabla \phi, V)]
\]

\[
+ (\phi - 1) \tilde{P}_1(\nabla \phi, \phi, X, Y, V, W) + \tilde{Q}_1(\nabla \phi, \phi, X, Y, V, W).
\]

Since \( \phi \) is a basic function, \( g(\nabla \phi, V) = V \phi = 0 \). Hence \( \nabla \phi \) is a horizontal vector.
field. Then by [4.2.2, 4.2.3, 4.2.4, 4.2.5] we see that

\[ g(\hat{\nabla}_W \nabla \phi, V) = -g(\nabla_W V, \nabla \phi) + \frac{g(\nabla \phi, \nabla \phi)}{2\phi} g(W, V). \]

On the other hand, since \((\nabla_W W)^h\) is a horizontal vector field (but not a basic vector field in general!), then by [4.2.2, 4.2.3, 4.2.4, 4.2.5] we get

\[ I_1 = \hat{R}(V, W, W, V) = \phi R(V, W, W, V) \]

\[ + (\phi - 1) \hat{P}_1(\nabla \phi, \phi, X, Y, V, W) + \hat{Q}_1(\nabla \phi, \phi, X, Y, V, W) \]

\[ = R(V, W, W, V) + (\phi - 1)R(V, W, W, V) \]

\[ + (\phi - 1) \hat{P}_1(\nabla \phi, \phi, X, Y, V, W) + \hat{Q}_1(\nabla \phi, \phi, X, Y, V, W) \]

\[ = R(V, W, W, V) + (\phi - 1)P_1(\nabla \phi, \phi, X, Y, V, W) + Q_1(\nabla \phi, \phi, X, Y, V, W), \]

where \(P_1(\nabla \phi, \phi, X, Y, V, W), Q_1(\nabla \phi, \phi, X, Y, V, W)\) are two functions depending on \(\nabla \phi, \phi, X, Y, V, W\) and \(Q_1(0, \phi, X, Y, V, W) \equiv 0.\)

Since \(X\) is a basic vector field, \([X, W]\) is vertical. Hence by [4.2.2, 4.2.3, 4.2.4, 4.2.5]

\[ I_2 = \hat{R}(X, W, W, X) = \hat{g}(\hat{\nabla}_X \hat{\nabla}_W W - \hat{\nabla}_W \hat{\nabla}_X W - \hat{\nabla}_{[X,W]} W, X) \]

\[ = g(\hat{\nabla}_X [\nabla_W W - \frac{1}{2} g(W, W) \nabla \phi + (\phi - 1)(\nabla_W W)^h], X) \]

\[ - g(\hat{\nabla}_W [\nabla_X W + \frac{g(X, \nabla \phi)}{2\phi} W + \frac{1 - \phi}{2} \sum_{i=1}^n g([X, \varepsilon_i], W) \varepsilon_i], X) \]

\[ - g(\nabla_{[X,W]} W - \frac{1}{2} g([X, W], W) \nabla \phi + (\phi - 1)(\nabla_{[X,W]} W)^h, X) \]

\[ = R(X, W, W, X) + (\phi - 1)P_2(\nabla \phi, \phi, X, Y, V, W) \]

\[ + Q_2(\nabla \phi, \phi, X, Y, V, W) - \frac{1}{2} \text{Hess}(\phi)(X, X) g(W, W), \]

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where \( P_2(\nabla \phi, \phi, X, Y, V, W), Q_2(\nabla \phi, \phi, X, Y, V, W) \) are two functions depending on \( \nabla \phi, \phi, X, Y, V, W \) and \( Q_2(0, \phi, X, Y, V, W) \equiv 0 \).

By the similar argument, we see that

\[
I_3 = \hat{R}(Y, V, V, Y) = R(Y, V, V, Y) + (\phi - 1)P_3(\nabla \phi, \phi, X, Y, V, W)
+ Q_3(\nabla \phi, \phi, X, Y, V, W) - \frac{1}{2} \text{Hess}(\phi)(Y, Y)g(V, V).
\]

\[
I_4 = \hat{R}(X, Y, Y, V) = R(X, Y, Y, V) + (\phi - 1)P_4(\nabla \phi, \phi, X, Y, V, W)
+ Q_4(\nabla \phi, \phi, X, Y, V, W).
\]

\[
I_5 = \hat{R}(Y, X, X, W) = R(Y, X, X, W) + (\phi - 1)P_5(\nabla \phi, \phi, X, Y, V, W)
+ Q_5(\nabla \phi, \phi, X, Y, V, W).
\]

\[
I_6 = \hat{R}(X, Y, W, V) = R(X, Y, W, V) + (\phi - 1)P_6(\nabla \phi, \phi, X, Y, V, W)
+ Q_6(\nabla \phi, \phi, X, Y, V, W).
\]

\[
I_7 = \hat{R}(X, W, Y, V) = R(X, W, Y, V) + (\phi - 1)P_7(\nabla \phi, \phi, X, Y, V, W)
+ Q_7(\nabla \phi, \phi, X, Y, V, W) + \frac{1}{2} \text{Hess}(\phi)(X, Y)g(V, W).
\]

\[
I_8 = \hat{R}(V, W, W, X) = R(V, W, W, X) + (\phi - 1)P_8(\nabla \phi, \phi, X, Y, V, W)
+ Q_8(\nabla \phi, \phi, X, Y, V, W) + \frac{1}{2}g(V, W)g(\nabla_W \nabla \phi, X) - \frac{1}{2}g(W, W)g(\nabla_V \nabla \phi, X).
\]

\[
I_9 = \hat{R}(W, V, Y, Y) = R(W, V, Y, Y) + (\phi - 1)P_9(\nabla \phi, \phi, X, Y, V, W)
+ Q_9(\nabla \phi, \phi, X, Y, V, W) + \frac{1}{2}g(V, W)g(\nabla_Y \nabla \phi, Y) - \frac{1}{2}g(V, V)g(\nabla_W \nabla \phi, Y),
\]

where \( P_i(\nabla \phi, \phi, X, Y, V, W), Q_i(\nabla \phi, \phi, X, Y, V, W) \) are two functions depending on
$\nabla \phi, \phi, X, Y, V, W$ and $Q_i(0, \phi, X, Y, V, W) \equiv 0$, $i = 3, 4, \ldots, 9$. Hence

$$
\hat{R}(X + V, Y + W, Y + W, X + V) = I_0 + I_1 + I_2 + I_3 + 2 \sum_{i=4}^{9} I_i
$$

$$
= R(X + V, Y + W, Y + W, X + V) + (\phi - 1) P(\nabla \phi, \phi, X, Y, V, W)
$$

$$
+ Q(\nabla \phi, \phi, X, Y, V, W) + [-g(W, W)g(\nabla V \nabla \phi, X) + g(V, W)g(\nabla W \nabla \phi, X)
$$

$$
+ g(V, W)g(\nabla V \nabla \phi, Y) - g(V, V)g(\nabla W \nabla \phi, Y)]
$$

$$
+ \frac{1}{2}[-\text{Hess}(\phi)(X, X)g(W, W) + 2\text{Hess}(\phi)(X, Y)g(V, W)
$$

$$
- \text{Hess}(\phi)(Y, Y)g(V, V)],
$$

where $P(\nabla \phi, \phi, X, Y, V, W), Q(\nabla \phi, \phi, X, Y, V, W)$ are two functions which depend on

$\nabla \phi, \phi, X, Y, V, W$ and $Q(0, \phi, X, Y, V, W) \equiv 0$. \hfill \square

Proof of Theorem \[4.2.1\]

Proof. We already proved it if $\|A\|$ is basic. Hence it suffices to show it if $H$ is basic. We prove it by contradiction. Let $\pi : (M^m, g) \to (N^2, h)$ be a Riemannian submersion such that $H$ is basic and the fibers have nonzero Euler numbers, where $(M^m, g)$ has positive sectional curvature and $m \geq 4$. By Lemma \[4.2.3\], there exists a metric $\hat{g}$ on $M^m$ such that $\pi : (M^m, \hat{g}) \to (N^2, h)$ is still a Riemannian submersion and all fibers of $\pi$ are minimal submanifolds with respect to $\hat{g}$. Furthermore, there exists some fiber $F_0$ such that $\hat{g}$ has positive sectional curvature at all points in $F_0$. Let $r$ be a fixed positive number such that the normal exponential map of $F_0$ is a diffeomorphism when restricted to the tubular neighborhood of $F_0$ with radius $r$. By continuity of sectional curvature, there exists $\epsilon$, $0 < \epsilon < r$ such that $\hat{g}$ has positive sectional curvature at the $\epsilon$ neighborhood of $F_0$. Choose another fiber $F_1$ such that $0 < \hat{d}(F_0, F_1) < \epsilon$, where $\hat{d}(F_0, F_1)$ is the distance between $F_0$ and $F_1$ with respect
to \( \hat{g} \). Since \( \pi : (M^m, \hat{g}) \to (N^2, h) \) is a Riemannian submersion, \( F_0 \) and \( F_1 \) are equidistant. On the other hand, since \( \hat{d}(F_0, F_1) < r \), then for any point \( q \in F_1 \), there is a unique point \( p \in F_0 \) such that \( \hat{d}(p, q) = \hat{d}(F_0, F_1) \). Let \( L = \hat{d}(p, q) \) and \( \gamma : [0, L] \to M^m, \gamma(0) = p, \gamma(L) = q \) the unique minimal geodesic with unit speed realizing the distance between \( p \) and \( q \). Let \( V \subseteq T_q(M^m) \) be the subspace of vectors \( v = X(L) \) where \( X \) is a parallel field along \( \gamma \) such that \( X(0) \in T_p(F_0) \). Then

\[
\dim(V \cap T_q(F_1)) = \dim(V) + \dim(T_q(F_1)) - \dim(V + T_q(F_1))
\]

\[
\geq (m - 2) + (m - 2) - (m - 1) = m - 3.
\]

We claim that \( \dim(V \cap T_q(F_1)) = m - 3 \). If not, then \( \dim(V \cap T_q(F_1)) = m - 2 \). Let \( X_i, i = 1, \ldots, m - 2 \), be orthonormal parallel fields along \( \gamma \) such that \( X_i(0) \in T_p(F_0), X_i(L) \in T_q(F_1) \). For each \( i \), choose a variation \( f_i(s, t) \) of \( \gamma \) such that \( f_i(s, 0) \in F_0, f_i(s, L) \in F_1 \) for small \( s \) and \( \frac{\partial f_i(0, t)}{\partial s} = X_i(t) \). By construction, \( \dot{X}_i(t) = \hat{\nabla}_\gamma X_i(t) = 0 \) for all \( t \), where \( \hat{\nabla} \) is the Levi-Civita connection with respect to \( \hat{g} \). By the second variation formula, for \( i = 1, \ldots, m - 2 \), we have

\[
\frac{1}{2} \frac{d^2 E_i(s)}{ds^2} \Big|_{s=0} = \int_0^L (\dot{\hat{g}}(\dot{X}_i, \dot{X}_i) - \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i))dt
\]

\[
+ \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0)
\]

\[
= - \int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i)dt + \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0),
\]

where \( E_i(s) = \int_0^L \hat{g}(\frac{\partial f_i(s, t)}{\partial t}, \frac{\partial f_i(s, t)}{\partial t})dt \), \( \hat{R} \) is the curvature tensor of \( \hat{g} \) and \( \hat{B}_j \) is the second fundamental form of \( F_j \) with respect to \( \hat{g} \), \( j = 0, 1 \).

Since \( F_0 \) and \( F_1 \) are minimal submanifolds in \( (M^m, \hat{g}) \), we have

\[
\sum_{i=1}^{m-2} \hat{B}_j(X_i, X_i) = 0, j = 0, 1.
\]
Then
\[
\frac{1}{2} \sum_{i=1}^{m-2} \frac{d^2 E_i(s)}{ds^2} \bigg|_{s=0} = - \sum_{i=1}^{m-2} \int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) dt.
\]
Since \( \hat{g} \) has positive sectional curvature at the \( \epsilon \) neighborhood of \( F_0 \) and \( 0 < \hat{d}(F_0, F_1) < \epsilon \), we see \( \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) < 0 \). Hence
\[
\frac{1}{2} \sum_{i=1}^{m-2} \frac{d^2 E_i(s)}{ds^2} \bigg|_{s=0} < 0.
\]
Then there exists some \( i_0 \) such that \( \frac{d^2 E_{i_0}(s)}{ds^2} \bigg|_{s=0} < 0 \), which contradicts that \( \gamma \) is a minimal geodesic realizing the distance between \( F_0 \) and \( F_1 \). So \( \dim(V \cap T_q(F_1)) = m - 3 \). Since \( \dim(T_q(F_1)) = m - 2 \), then \( V \cap T_q(F_1) \) is a codimension one subspace of \( T_q(F_1) \). Since \( q \) is arbitrary on \( F_1 \), by doing the same construction as above for any \( q \), then we get a continuous codimension one distribution on \( F_1 \). Thus the Euler number of \( F_1 \) is zero. Contradiction.
CHAPTER 5

RIEMANNIAN SUBMERSIONS FROM COMPACT FOUR-DIMENSIONAL
EINSTEIN MANIFOLDS

In this chapter we prove Theorem 1.0.4. Suppose $\pi : (M^4, g) \to (N^2, h)$ is a
Riemannian submersion with totally geodesic fibers, where $(M^4, g)$ is a compact
four-dimensional Einstein manifold. We are going to show that the $A$ tensor of $\pi$
vanishes and then locally $\pi$ is the projection of a metric product onto one of the
factors. We firstly need the following lemmas:

**Lemma 5.0.5.** Let $\pi$ be a Riemannian submersion from compact Riemannian man-
ifolds with totally geodesic fibers, then all fibers are isometric to each other.

**Proof.** See [14].

**Lemma 5.0.6.** Suppose $\pi : (M^4, g) \to (N^2, h)$ is a Riemannian submersion with to-
tally geodesic fibers, where $(M^4, g)$ is a compact four-dimensional Einstein manifold.
Let $c_1, c_2$ be the sectional curvature of $(F^2, g|_{F^2})$ and $(N^2, h)$, respectively, where $g|_{F^2}$
is the restriction of $g$ to the fibers $F^2$. Let $\text{Ric}(g) = \lambda g$ for some $\lambda$. Then

(i) $2c_1 + \|A\|^2 = 2\lambda$;

(ii) $2c_2 \circ \pi - 2\|A\|^2 = 2\lambda$;

(iii) $\|A\|^2 = \frac{2}{3}(c_2 \circ \pi - c_1)$,

where $\|A\|^2 = \|A_X^\ast U\|^2 + \|A_X^\ast V\|^2 + \|A_Y^\ast U\|^2 + \|A_Y^\ast V\|^2$. Here $X, Y/U, V$ is an
orthonormal basis of $\mathcal{H}/\mathcal{V}$, respectively.
Proof. See page 250, Corollary 9.62 in [3]. For completeness, we give a proof here.

Let \( U, V / X, Y \) are orthonormal basis of \( \mathcal{V} / \mathcal{H} \), respectively. Then by Theorem 2.3.1 we have

\[
\begin{align*}
\lambda &= \text{Ric}(U, U) = c_1 + \| A^*_X U \|^2 + \| A^*_Y U \|^2; \\
\lambda &= \text{Ric}(V, V) = c_1 + \| A^*_X V \|^2 + \| A^*_Y V \|^2; \\
\lambda &= \text{Ric}(X, X) = c_2 \circ \pi - 3 \| A_X Y \|^2 + \| A^*_X U \|^2 + \| A^*_Y V \|^2; \\
\lambda &= \text{Ric}(Y, Y) = c_2 \circ \pi - 3 \| A_X Y \|^2 + \| A^*_X U \|^2 + \| A^*_Y V \|^2.
\end{align*}
\]

On the other hand, by direct calculation, we see that \( 2 \| A_X Y \|^2 = \| A \|^2 \). Hence

\[
2c_1 + \| A \|^2 = 2\lambda;
\]

\[
2c_2 \circ \pi - 2 \| A \|^2 = 2\lambda;
\]

\[
\| A \|^2 = \frac{2}{3} (c_2 \circ \pi - c_1).
\]

By Lemmas 5.0.5 and 5.0.6 we see that \( c_1, \| A \| \) are constants on \( M^4 \) and \( c_2 \) is a constant on \( N^2 \).

Fix \( p \in M^4 \). Locally we can always choose basic vector fields \( X, Y \) such that \( X, Y \) is an orthonormal basis of the horizontal distribution. At point \( p \), since the image of \( A^*_X \) is perpendicular to \( X \) and \( \text{dim} \mathcal{V} = \text{dim} \mathcal{H} = 2 \), \( A^*_X \) must have nontrivial kernel. Then there exists some \( v \in \mathcal{V} \) such that \( \| v \| = 1 \) and \( A^*_X(v) = 0 \). Extend \( v \) to be a local unit vertical vector field \( V \) and choose \( U \) such that \( U, V \) is a local orthonormal basis of \( \mathcal{V} \).

Lemma 5.0.7.

\[
A^*_X V(p) = 0;
\]
\[ \lambda_Y V(p) = 0. \]

**Proof.** We already see \( \lambda_X V(p) = \lambda_{X,p}(v) = 0. \) On the other hand, at point \( p \), we have

\[
\begin{align*}
A_Y V &= g(A_Y V, X)X = -g(\nabla_Y V, X)X \\
&= g(V, \nabla_Y X)X = g(V, A_Y X)X \\
&= -g(V, A_X Y)X = -g(V, \nabla_X Y)X \\
&= g(\nabla_X V, Y)X = -g(A_X^* V, Y)X = 0.
\end{align*}
\]

\[ \square \]

Since all fibers of \( \pi \) are totally geodesic, we see that \( K(X, U) = \|A_X^* U\|^2 \) (Theorem 2.3.1). Because \((M^4, g)\) is Einstein, at point \( p \), we have

\[
\begin{align*}
\lambda &= \text{Ric}(U, U) = c_1 + \|A_X^* U\|^2 + \|A_Y^* U\|^2; \\
\lambda &= \text{Ric}(V, V) = c_1 + \|A_X^* V\|^2 + \|A_Y^* V\|^2;
\end{align*}
\]

Combined with Lemma 5.0.7, we see that \( \lambda = c_1 \) and \( \|A_X^* U\|^2(p) = 0, \|A_Y^* U\|^2(p) = 0. \) Then \( \|A\|^2(p) = 0. \) Hence \( \|A\|^2 \equiv 0 \) on \( M^4 \) and \( c_1 = c_2. \) Let \( c = c_1 = c_2. \) Then locally \( \pi \) is the projection of a metric product \( B^2(c) \times B^2(c) \) onto one of the factors, where \( B^2(c) \) is a two-dimensional compact manifold with constant curvature \( c. \)
CHAPTER 6

CONJECTURE 1 AND THE WEAK HOPF CONJECTURE

In this chapter we point out several interesting corollaries of Conjecture 1.

Suppose \((E, g)\) is a complete, open Riemannian manifold with nonnegative sectional curvature. By a well known theorem of Cheeger and Gromoll \([4]\), \(E\) contains a compact totally geodesic submanifold \(\Sigma\), called the soul, such that \(E\) is diffeomorphic to the normal bundle of \(\Sigma\). Let \(\Sigma_r\) be the distance sphere to \(\Sigma\) of radius \(r\). Then for small \(r > 0\), the induced metric on \(\Sigma_r\) has nonnegative sectional curvature by a theorem of Guijarro and Walschap \([12]\). In \([10]\), Gromoll and Tapp proposed the following conjecture:

**Weak Hopf Conjecture** Let \(k \geq 3\). Then for any complete metric with nonnegative sectional curvature on \(S^n \times \mathbb{R}^k\), the induced metric on the boundary of a small metric tube about the soul can not have positive sectional curvature.

The case \(n = 2, k = 3\) is of particular interest since the metric tube of the soul is diffeomorphic to \(S^2 \times S^2\).

Recall that a map between metric spaces \(\sigma : X \to Y\) is a submetry if for all \(x \in X\) and \(r \in [0, r(x)]\) we have that \(f(B(x, r)) = B(f(x), r)\), where \(B(p, r)\) denotes the open metric ball centered at \(p\) of radius \(x\) and \(r(x)\) is some positive continuous function. If both \(X\) and \(Y\) are Riemannian manifolds, then \(\sigma\) is a Riemannian submersion of class \(C^{1,1}\) by a theorem of Berestovskii and Guijarro \([1]\).

**Proposition 6.0.8.** Suppose \(\Sigma\) is a soul of \((E, g)\), where \((E, g)\) is a complete, open Riemannian manifold with nonnegative sectional curvature. If the induced metric on
\(\Sigma_r\) has positive sectional curvature at some point for some \(r > 0\), then there is a Riemannian submersion from \(\Sigma_r\) to \(\Sigma\) with fibers \(S^{l-1}\), where \(l = \text{dim}(E) - \text{dim}(\Sigma)\).

**Proof.** In fact, by a theorem of Guijarro and Walschap in [13], if \(\Sigma_r\) has positive sectional curvature at some point, the normal holonomy group of \(\Sigma\) acts transitively on \(\Sigma_r\). By Corollary 5 in [26], we get a submetry \(\pi : (E, g) \to \Sigma \times [0, +\infty)\) with fibers \(S^{l-1}\), where \(\Sigma \times [0, +\infty)\) is endowed with the product metric. Then \(\pi : (\pi^{-1}(\Sigma \times (0, +\infty)), g) \to \Sigma \times (0, +\infty)\) is also a submetry. By a theorem of Berestovskii and Guijarro in [1], \(\pi\) is a \(C^{1,1}\) Riemannian submersion. Then \(\Sigma_r = \pi^{-1}(\Sigma \times \{r\})\) and \(\pi : \Sigma_r \to \Sigma\) is also a \(C^{1,1}\) Riemannian submersion with fibers \(S^{l-1}\), where \(\Sigma_r\) is endowed with the induced metric from \((E, g)\).

\[\square\]

**Proposition 6.0.9.** When \(k > n\), Conjecture 1 implies the Weak Hopf Conjecture.

**Proof.** Suppose for some complete metric \(g\) on \(S^n \times \mathbb{R}^k\) with nonnegative sectional curvature, the induced metric on \(\Sigma_r\) has positive sectional curvature for some \(r > 0\), where \(\Sigma\) is a soul. Since \(S^n \times \mathbb{R}^k\) is diffeomorphic to the normal bundle of \(\Sigma\), we see that \(\Sigma\) is a homotopy sphere and \(\text{dim}(\Sigma) = n\). By Proposition 6.0.8, we get a Riemannian submersion from \(\Sigma_r\) to \(\Sigma\) with fibers \(S^{k-1}\), where \(\Sigma_r\) is endowed with the induced metric from \(g\) and hence has positive sectional curvature. Since \(k > n\), we see \(k - 1 \geq n\), which is impossible if Conjecture 1 is true for \(C^{1,1}\) Riemannian submersions.

\[\square\]

**Remark 6.0.3.** If Remark 4.1.2 is true, then by Proposition 6.0.8 again, any small metric tube about the soul can not have positive sectional curvature when the soul is odd-dimensional. This would give a solution to a question asked by K. Tapp in [21].


