A PRESSURE–POISSON BASED BOUSSINESQ–TYPE PHASE RESOLVING WAVE MODEL

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Abstract

by

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Understanding the hydraulic response during high energy events such as hurricanes and tsunamis is a problem of significant importance to coastal communities, as well as industries which have infrastructure along coastal waters. Due to the large range of temporal and spatial scales present during these events it isn’t possible to fully study them in a laboratory setting and thus computational models are the standard approach to simulating hurricanes and tsunamis. The current suite of ocean circulation models used in the coastal hydraulics community solve the hydrostatic depth–averaged shallow water equations (SWE) which are adept at representing the hydraulic response to the hurricane wind and pressure forces and the ensuing storm surge runup along the coast but are unable to capture the higher frequency gravity and infragravity waves that are generated nearshore. Through the use of Boussinesq scaling a model is developed for resolving non–hydrostatic pressure profiles in non-linear wave systems over varying bathymetry. In contrast to standard Boussinesq–type models, the model developed here focuses on solutions to the well known pressure–Poisson problem, with the advantage that the pressure–Poisson equations are a stand alone set of equations which do not exhibit mixed space/time derivatives. The result is a Boussinesq–type pressure–Poisson model (PPBOUSS) that can be solved in a separate module and then coupled back into the SWE model. This allows for
the straightforward modification of established SWE solvers to turn them into fully resolved nearshore models with very little structural change to the underlying SWE model and without the need to solve mixed space/time derivatives. Use of a Green–Naghdi type polynomial expansion for the pressure profile in the vertical axis reduces the dimensionality of the pressure–Poisson problem to only two dimensions, significantly reducing computational cost. The resulting model shows rapid convergence properties with increasing order of polynomial expansion which can be greatly improved through the application of asymptotic rearrangement. An optimum choice of basis functions in the Green–Naghdi expansion provides significant improvements in the dispersion, shoaling and nonlinear properties of the model, for example achieving fourth order dispersive accuracy in a formally second order model, and eighth order dispersive accuracy in a formally fourth order model. Various numerical approaches to solving and coupling the PPBOUSS model are discussed including application of finite difference and finite element methods. Demonstration of the improvement in nearshore accuracy through application of the PPBOUSS model is shown through coupling with the unstructured mesh Discontinuous Galerkin Shallow Water Equation Model (DGSWEM). A straightforward numerics based wave breaking algorithm is employed in the nearshore and wave runup is captured using a globally mass conservative wetting/drying algorithm specifically designed for discontinuous Galerkin finite element models. The model is verified and validated using analytical and experimental results for both the fully nonlinear $O(\mu^2)$ and weakly nonlinear $O(\mu^4)$ implementations.
To Enrica and Luca
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CHAPTER 1

INTRODUCTION AND MOTIVATION

Propagation of waves from deep water into the nearshore surf zone is a highly energetic process where waves break, shoal and dissipate energy. Resolution of these effects is integral to accurate simulations in the nearshore. In severe storms such as hurricanes these effects become even more pronounced through the combination of faster currents and increased water levels due to wind induced setup. The combination of wind and wave induced setup can directly impact the extent of inundation far inland and can effect the morphodynamics of the coastal region. For deep ocean islands waves this can account for a majority of the storm runup, ranging in scale from 1-10 meters [14, 36].

In the ocean modeling community the most commonly used approach for simulation of waves is to treat the wave field as an energy spectrum, using so called spectral wave models which apply the conservation of wave action density [28]. Spectral wave models are well-suited for simulation of wind wave generation, but assume small nonlinearity and slow evolution over wavelengths. Both of these assumptions are invalid in the surf zone and thus spectral models do a poor job of accurately predicting the details of wave dynamics such as wave runup and wave shoaling. An alternative approach is to institute phase-resolving wave models, which are capable of capturing the nearshore wave dynamics. Phase-resolving models can be developed based on solutions to the full Navier–Stokes equations, however despite advances in computational power these models are limited to relatively small domains. A more common approach is to use Boussinesq models, which are computationally more effi-
cient and have been shown to be very accurate for shallow water waves, such as those encountered within the surf zone. Although popular and still widely used today, Boussinesq models still have drawbacks, in particular higher–order solutions involve high–order differential equations, which include mixed space/time derivatives and are computationally expensive and complex to implement. Recently a third approach has gained in popularity, multi–layer non–hydrostatic pressure models. These models offer an alternative to the highly complex nature of higher–order Boussinesq models by achieving higher–order accuracy through multi–layer solutions, although an increase in the number of layers leads to an increase in computational cost.

1.1 Historical Context

In the seventeenth century Galileo attempted to describe the occurrence of tides as a result of the Earth’s rotation about the sun. Galileo’s theory had ultimately proven to be false but was representative of the fundamental desire to understand the motion of fluid in the ocean and along the coast. Pierre-Simon Laplace introduced partial differential equations, now known as the Laplace equations, in 1776, that would set the mathematical basis for understanding tides. The motion of tides in the ocean would be just one of the many fundamental fluid mechanics theories to be developed during the eighteenth and nineteenth centuries. In particular, the work on linear wave theory of George Biddell Airy in 1845 made it possible to understand the propagation of gravity waves for an inviscid, incompressible and irrotational flow. At around the same time Sir George Stokes, of the Navier–Stokes equations, was working on extensions of fluid mechanics theory into higher orders and a French mathematician, Joseph Boussinesq, worked on the theory of long waves.

The contributions of Boussinesq would prove to be fundamental to the development of the field of fluid mechanics and serve as the basis for what would later become Boussinesq–type models. The initial work of Boussinesq produced a model that ap-
proximated the motion of weakly non–linear water waves over a flat bathymetry and were thus limited in their application, however they provided a relatively accurate simplification of the Euler equations for fairly long waves.

The first operational wave forecast model was developed in preparation for the D–Day invasion in Normandy, France [74]. Following the war, the interest in properly understanding and simulating water waves both in the nearshore and in the deep ocean began to increase. Deep water applications were heavily funded by the petroleum industry as oil platforms began to be built and deployed. Interest in the nearshore focused on coastal erosion and the contribution of waves to storm surge and inundation during storm events. Early work focused on experimental analysis, but as computers became more widely used, numerical solutions to partial differential equations offered another option for predicting waves and their possible effects.

In 1967, Peregrine [57] introduced equations based on Boussinesq wave theory which considered a fluid of variable depth. Using non-dimensional analysis and assuming that the ratios of wave amplitude to fluid depth, \( \delta \), and the ratio of fluid depth to wave length, \( \mu \), were fairly small, Peregrine was able to develop a relatively simple model that described the propagation of long waves on a beach. This was achieved through the application of a perturbation analysis on the dependent variables with respect to \( \delta \) and \( \mu \). Truncation of the terms of order higher \( O(\mu^4, \delta \mu^2) \), based on the assumption that they were small, led to a weakly dispersive, weakly nonlinear wave model. This truncation also limited the derivatives in the conservation of momentum equations to third–order. This approach was capable of resolving the propagation of long waves over a varying bathymetry, and thus was able to solve problems in the nearshore where the bathymetry plays a large role in the fluid dynamics.

The seminal work of Peregrine wouldn’t be fully embraced by the wave modeling community until the early 1990’s with the work of Madsen et al. [55, 54], Witting [81] and Nwogu [56]. One of the major limitations of Peregrine’s original model was the
weak dispersion. The model was only applicable for relatively shallow water waves, $kh \leq \pi/2$, where $k$ is the wavenumber and $h$ is a reference depth. One approach to obtaining improved dispersion characteristics in the model was to retain higher order terms, $O(\mu^4, \delta \mu^2)$. However this led to greater than third order derivatives in the momentum equations, hence an increase in computational cost and model implementation difficulties. Alternatively Madsen [55] and his colleagues demonstrated that if they applied a linear operator to the momentum equation, $L(A) = (1 + B\mu^2 h^2 \nabla^2)A$, it was possible to obtain an extra contribution of the dispersive terms which were proportional to the free parameter $B$, while retaining only third–order derivatives. The proper choice of the $B$ parameter led to dispersive properties equivalent to a Padé [2,2] approximation of the linear Airy solution, which is fourth–order accurate, thus extending the applicability of the model with little change to the computational cost. Following this work they addressed the shoaling characteristics of the model and demonstrated that similar to the dispersion analysis it was possible to adjust the free-parameter in the model to optimize the accuracy of wave shoaling [54].

At the same time, Witting [81] and Nwogu [56] experimented with adjusting the reference velocity in the derivation of the governing equations as a way to include higher order dispersive terms without increasing the order of the derivatives in the conservation of momentum equations. In particular Nwogu considered a Taylor expansion of the horizontal velocity variables about an arbitrary vertical location in the water column, $z_\alpha$. He demonstrated that this new approach represented an entire class of second–order Boussinesq models, and that with the appropriate choice of $z_\alpha$, it was possible to reconstruct previously established Boussinesq–type models. As with the model developed by Madsen et al., a free parameter, in this case $z_\alpha$, could be adjusted to improve the dispersion properties of the model. Setting $z_\alpha = (\sqrt{5} - 5)/5h \approx -0.55h$ produced a dispersion relationship equivalent to a Padé [2,2] approximation of the linear Airy solution. Using numerical error minimization
methods Nwogu extended the accuracy of the dispersion relationship past that of a Padé [2,2] by choosing $z_\alpha = -0.53h$. The work of Madsen, Witting and Nwogu marked the beginning of a surge in research focused on improvement and applications of Boussinesq type models, which is still strong decades later. Following the improvement in dispersive properties, many researchers began to build models which incorporated more complex wave physics.

1.2 Development Of Boussinesq Models

One of the first advances in the physics of Boussinesq models was conducted by Hemming Schäffer [60] and his colleagues, who introduced a framework for the simulation of breaking waves. One of the assumptions of Boussinesq models is partially irrotational flow, as such the dissipation associated with breaking waves in the system has to be introduced through external equations. Building upon the concept of a “surface roller” introduced by Svendsen [68] and discussed in the PhD work of Madsen [48], Schäffer considered wave breaking as the formation and propagation of a turbulent bore of width $\delta$ riding at a wave celerity of $C$ on the front of the wave. This approach necessarily involved separation of the flow into two layers, the top layer describing the propagation of the surface roller and the bottom layer describing the rest of the fluid flow.

An alternative approach is to include an extra set of diffusive–type eddy viscosity terms in the momentum equation, which simulate the turbulent mixing and dissipation caused by wave breaking, see Kennedy et al. [33]. In this approach the eddy viscosity, $\nu = B\delta_b^2(h + \eta)\eta_t$, is dependent on the bathymetric depth, $h$, the free–surface displacement, $\eta$, the time rate of change of the free–surface, $\eta_t$, a mixing length coefficient, $\delta_b$, and a smoothly varying parameter $B$ which is defined such that the initiation of breaking does not occur sharply in the solution causing instabilities. Each of these approaches showed very good agreement with experimental results and
were relatively straightforward to implement numerically. Subsequent work on the incorporation of wave breaking and wave runup lead to significant improvements for both one-dimensional and two-dimensional horizontal models [51, 50, 12, 76, 44].

Although Madsen and Nwogu were able to extend the dispersive accuracy of the models to be fourth-order accurate, which extended the range of applicability for the models to the nominal shallow water limit, the models were still weakly nonlinear. Wei et al. [78] extended the approach of Nwogu in order to develop the fully nonlinear second-order Wei–Kirby–Grilli–Subramanya (WKGS) Boussinesq model. Kennedy et al. [35] further extended the nonlinear properties of the WKGS model by considering a time varying reference velocity, $z_a = z_a(t)$.

Gobbi et al. [24, 26] extended the WKGS to fourth-order accuracy and extensively tested the higher order model numerically [25]. Extensions of Madsen’s approach of using a linear operator to include higher-order dispersion terms also led to models of formal high-order, capable of accurately capturing dispersion and shoaling characteristics past the nominal deep water limit [2, 59]. The development of higher-order models led to more accurate solutions, however it came at the expense of higher-order spatial derivatives in the governing equations, which led to increased computational cost and complexity. As a result, almost all Boussinesq models currently used in applications are only second-order.

Another drawback to standard Boussinesq models is the assumption of irrationality in the flow which limits the physics that can be resolved in the surf zone. Building on the advances made in Serre–Green–Naghdi type modeling of water waves [5, 27, 61], Zhang et al. [84] were able to obtain Boussinesq-type models without the irrationality assumption, that had a formal high-order accuracy, using Green–Naghdi type polynomial expansions over the vertical domain for the velocities. Using asymptotic rearrangement on the free-parameters in the Green–Naghdi expansion functions, they were able to determine the set of basis functions that would optimize
dispersive, shoaling and nonlinear effects. The result was a Boussinesq–type model which was highly accurate, rotational and could be easily extended to higher–order without adding higher–order derivatives.

Many operational and open source models built on Boussinesq theory are currently in use for a variety of applications. Examples include; (1) FUNWAVE [37], a fully nonlinear, high–order hybrid finite–volume/finite–difference two–horizontal–dimensional model, (2) MIKE21 [77], a weakly nonlinear low–order finite element model in one–horizontal dimension and finite–difference model in two–horizontal dimensions, and many more. Boussinesq models find widespread use today in a variety of important applications, such as the simulation of ship motions and mooring forces for docked ships as a response to harbor waves [1], and simulation of tsunami propagation [11], to name a few.

1.3 Alternatives To Boussinesq Models

An alternative to complex higher–order Boussinesq–type models was to consider a multi–layer approach. Lynett et al. [43] proposed an approach to Boussinesq type modeling which involved solutions to the governing equations over multiple vertical layers. They were able to demonstrate that for the two–layer model this approach provided a greater number of free parameters which could be used to further optimize a second–order Boussinesq solution [40]. The work of Lynett and others was able to show that a multi–layer model may be a viable alternative to the prohibitive complexity of higher–order Boussinesq models.

A separate approach that has recently garnered more attention is the resolution of dispersive effects through direct solutions of the non–hydrostatic pressure term. In standard Boussinesq–type models the pressure gradient in the horizontal momentum equations is replaced by an approximate solution to the pressure derived through vertical integration of the vertical momentum equation. The result is that the dispersive
terms are represented by third–order or greater, mixed space/time derivatives of the horizontal velocities.

Casulli et al. \[9, 10\] and Stansby et al. \[66\] proposed that the pressure could be instead defined as the summation of the hydrostatic pressure, \(g(\eta - z)\), and an unknown non–hydrostatic term, \(\hat{p}\). Given the fluid velocity and the free–surface displacement it is possible to derive a pressure–Poisson equation from the conservation of momentum and continuity equations, which can then be used to determine \(\hat{p}\). A major advantage of this approach is the incorporation of dispersive effects without the need to solve complicated mixed space/time derivatives.

Following the theory laid out by Casulli et al. and Stansby et al., many highly accurate models have been developed and tested. Examples include the Simulating WAves till SHore (SWASH) model developed by Stelling et al. \[67\] and Zijlema et al. \[88, 87, 89\], the work of Yamazaki et al. \[82\] on a depth integrated non–hydrostatic model, the CCHE2D of Wei et al. \[79\] and the Non–Hydrostatic WAVE model (NHWAVE) of Ma et al. \[45\].

Increased dispersive, shoaling and nonlinear accuracy in these models is achieved through the use of multiple layers along the vertical axis, as opposed to retention of higher order terms as with classical Boussinesq modeling. It has been demonstrated that it is possible with just one or two layers to obtain the higher–order dispersive and shoaling characteristics present in Boussinesq models \[88\]. The single–layer model was shown to be accurate for only fairly shallow waves, while the more expensive two–layer model demonstrated a significant improvement with a dispersion relationship equivalent to a Padé [2,4] approximation to the linear Airy solution \[3\]. Furthermore, following the example of Nwogu, Bai et al. \[3\] derived a hybridized single–layer version of Yamazaki’s model which contains a free–parameter that can be adjusted to either optimize for dispersion or shoaling. The optimization of the dispersion relationship yielded a Padé [2,2] approximation. It is important to note
that each increase in the number of vertical layers is accompanied by an increase in computational cost. For this reason, in applications with a large spatial domains, these models are usually limited to one or two layers. However, the use of many vertical layers demonstrated the potential for models of this kind to be used for complex surf zone environments, such as the attenuation of waves in a vegetation canopy [46].

1.4 Motivation

This work is motivated by a need for an efficient integration of higher–order phase–resolving wave physics within the currently used ocean circulation models. We propose to develop a phase–resolving wave model which combines the advances made in both the study of Boussinesq models and multi–layer non–hydrostatic models. A Green–Naghdi type expansion along the vertical axis will be used to reduce the dimensionality of the problem from three dimensions to two dimensions. The Green–Naghdi approach offers a balance between full Navier–Stokes and Boussinesq. Green–Naghdi solutions can potentially be more accurate than Boussinesq solutions and are significantly less computationally expensive than Navier–Stokes. The proposed model will focus on solutions to the non–hydrostatic pressure profile and will be capable of being coupled with contemporary oceanographic circulation models in a straightforward manner. The result will be a highly accurate computationally efficient phase–resolving wave model capable of informing shallow water solvers in the nearshore, turning them into surf zone models.

Zhang et al. [84] used Green–Naghdi type vertical polynomial expansions of the dependent variables for the generation of higher–order Boussinesq–type models which do not contain higher–order derivatives. Building on this framework, the focus of this work is on the development of a model which focuses on solutions to the non–hydrostatic pressure profile. The result will be a model similar to the set of multi–layer non–hydrostatic models with the distinction that increased accuracy will
be obtained through retention of higher–order terms in the Green–Naghdi expansions. As with the multi–layer models it will be necessary to solve a pressure–Poisson problem to determine the pressure profile. It will be shown that it is possible to generate high–order models with a nominal increase in computational cost, avoiding the need to incorporate an increased number of vertical layers. It will be shown that using the present approach, the second–order model has the same computational cost and an equivalent dispersion relationship to the hybridized single–layer model of Bai et al. [3] while retaining an extra free parameter which can be used to optimize other model characteristics. The fourth–order model is computationally equivalent to the standard two–layer model with a more accurate Padé [4,4] dispersion relationship and higher order shoaling and nonlinear properties.

Taking advantage of asymptotic rearrangement, it will be possible to optimize the model for key model properties such as wave dispersion, shoaling and nonlinear interaction. In addition, the pressure model will be only loosely coupled to the equations of motion, providing the opportunity to design a stand alone model which can inform any hydrostatic shallow water equation solver to turn it into a phase–resolving model.

This thesis is organized as follows; chapter 2 details the development of a Boussinesq–type pressure–Poisson based phase resolving wave model, chapter 3 discusses the linear and nonlinear properties of the model, chapter 4 provides details regarding the numerical implementation of the model in one–horizontal dimension, chapter 5 discusses the verification and validation of the model for one–horizontal dimension, chapter 6 extends the model into two–horizontal dimensions and covers the coupling of the pressure–Poisson model to a discontinuous Galerkin based shallow water equation model, finally chapter 7 goes over conclusions and future work. There are also 4 appendices which go cover the mathematical derivations in greater detail. These are appendix A, which derives the governing equations, appendix B which derives
the linear properties of the model, appendix C which derives the nonlinear properties of the model, and finally appendix D which highlight publications and conference presentations based on this work.
2.1 Scaling

For the present study we will consider flow of a constant density inviscid fluid without bottom or surface shear stresses. We employ a Cartesian coordinate system \((x^*, y^*, z^*)\), where \(z^*\) represents the vertical axis centred on the still–water plane pointing upwards. The full vertical profile stretches from the bottom bathymetry at \(z^* = -h^*(x^*, y^*)\) to the free–surface \(z^* = \eta^*(x^*, y^*, t^*)\). The following nondimensional quantities are defined:

\[
\begin{align*}
(x, y) &= k_0(x^*, y^*), \\
\eta &= \frac{\eta^*}{a_0}, \\
w &= \frac{w^*}{a_0 k_0 \sqrt{g_0 h_0}}, \\
h &= \frac{h^*}{h_0}, \\
t &= k_0 \sqrt{g_0 h_0 t^*},
\end{align*}
\]

\[
(u, v) = \frac{h_0}{a_0 \sqrt{g_0 h_0}}(u^*, v^*), \\
z = \frac{z^*}{h_0}, \\
P = \frac{P^*}{\rho g_0 a_0}, \\
g = \frac{g^*}{g_0},
\]

where the superscript * denotes a dimensional variable. The variables \(u, v\) and \(w\) represent the velocities in the \(x, y\) and \(z\) directions respectively. The pressure, \(P\), is defined over the entire fluid domain and the free–surface elevation, \(\eta\), is defined over the horizontal axis \((x, y)\). The variable \(h\) represents the bathymetric depth, \(g\) is
the nondimensional gravitational constant and $t$ represents time. The parameters $k_0$, $h_0$, $g_0$, $a_0$ and $\rho$ stand for a reference wavenumber, reference bathymetric depth, the gravitational constant, a reference wave amplitude and the fluid density, respectively.

### 2.1.1 Governing Equations

Inserting the scaled parameters into the conservation of momentum equations, we obtain the following nondimensional forms of the conservation of momentum equations for incompressible inviscid fluid motion,

\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \delta \nabla \cdot (\mathbf{uu}^T) + \delta \frac{\partial}{\partial z}(w \mathbf{u}) + \nabla P &= 0, \quad -h \leq z \leq \delta \eta, \tag{2.2} \\
\mu^2 \frac{\partial w}{\partial t} + \delta \mu^2 \nabla \cdot (\mathbf{uw}) + \delta \mu^2 \frac{\partial}{\partial z}(w^2) + \frac{\partial P}{\partial z} + \frac{g}{\delta} &= 0, \quad -h \leq z \leq \delta \eta. \tag{2.3}
\end{align*}

where $\mu \equiv k_0 h_0$ is a measure of frequency dispersion, $\delta \equiv a_0/h_0$ is a measure of nonlinearity, $\nabla \equiv (\partial/\partial x, \partial/\partial y)^T$ is the two-dimensional gradient operator and $\mathbf{u} \equiv (u, v)^T$ is the horizontal velocity vector. The nondimensional forms of the continuity equation and the kinematic free–surface and bottom boundary conditions are given by,

\begin{align*}
\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} &= 0, \quad -h \leq z \leq \delta \eta, \tag{2.4} \\
w - \frac{\partial \eta}{\partial t} - \delta \mathbf{u} \cdot \nabla \eta &= 0, \quad z = \delta \eta, \tag{2.5} \\
w + \mathbf{u} \cdot \nabla h &= 0, \quad z = -h. \tag{2.6}
\end{align*}

Taking the divergence of the conservation of momentum equations, equations (2.2) and (2.3), and considering the continuity equation, Eq. (2.4), the pressure–Poisson equation is derived with the same scaling quantities,
\[
\mu^2 \nabla^2 P + \frac{\partial^2 P}{\partial z^2} = -\delta \mu^2 \nabla^{(3)} \cdot (u^{(3)} \cdot \nabla^{(3)} u^{(3)}), \quad -h \leq z \leq \delta \eta,
\]  

(2.7)

where \( \nabla^{(3)} \equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z) \) and \( u^{(3)} \equiv (u, v, w) \). Integrating Eq. (2.4) over the vertical domain from the bathymetric depth, \( z = -h \), to the free–surface, \( z = \delta \eta \) and applying the kinematic boundary conditions, equations (2.5) and (2.6), derives an expression for the conservation of mass dependent on the free–surface elevation and the fluid flux,

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot \int_{-h}^{\delta \eta} u \, dz = 0.
\]

(2.8)

Similarly, considering Eq. (2.3) at \( z = -h \) and applying the kinematic bottom boundary condition, Eq. (2.6), the bottom pressure boundary condition is obtained,

\[
\mu^2 (\nabla h) \cdot (\nabla P) + \frac{\partial P}{\partial z} + \frac{g}{\delta} = \delta \mu^2 u \cdot (u \cdot \nabla^2 h), \quad z = -h.
\]

(2.9)

See appendix A for more details regarding the derivation of these equations. Equations (2.2), (2.3), (2.7), (2.8) and (2.9) make up the complete set of governing equations that will be solved in this work. The pressure field for a particular velocity field \( u^{(3)} \), and free–surface displacement, \( \delta \eta \), at time \( t \) can be derived from equations (2.7) and (2.9). Once the pressure field has been determined, the time evolution of the velocity field \( u^{(3)} \) can be realized by substitution of the pressure field, \( P \), into equations (2.2) and (2.3). In addition, the time evolution of the free–surface is realized by substitution of the velocity field into Eq. (2.8). High–order models are constructed through application of Boussinesq–type approximations to the dependent variables; \( u, w \) and \( P \). The classic approach in Boussinesq models is to determine a solution for the pressure in terms of the horizontal and vertical velocity profiles through vertical integration of equation (2.3) and then to substitute this solution into the conservation
of momentum equations, \([2.2]\). This produces a system of equations based entirely on the free–surface and the velocities, but as mentioned previously this also requires solutions to mixed space/time derivatives.

The present work focuses on the derivation of an independent pressure model which does not include any mixed space/time derivatives. Similarly, the use of a single Poisson type model to resolve the dispersive terms will yield a model with fewer terms and a simpler form when compared with Boussinesq–type models of equal order.

2.1.2 Pressure Expansion

We begin by considering the standard approach to solving for the pressure profile in Boussinesq theory. Integrating equation \([2.3]\) from an arbitrary depth, \(z\), to the free–surface and applying the zero free–surface pressure condition, an expression for the vertical pressure profile in terms of the horizontal and vertical velocities is obtained,

\[
P = \frac{g}{\delta} (\delta \eta + h)(1 - q) + \mu^2 \int_z^{\delta \eta} \left( \frac{\partial w}{\partial t} + \delta \nabla \cdot (u w) + \delta \frac{\partial}{\partial \hat{z}} (w^2) \right) d\hat{z}, \quad (2.10)
\]

where the variable sigma coordinate \(q \in [0, 1]\), defined as

\[
q \equiv \frac{z + h}{\delta \eta + h}, \quad (2.11)
\]

has been introduced to simplify the equations. In this work we consider the Boussinesq approximation for the velocity expanded about the depth–averaged velocity, \(\bar{u}\). This was the approach first used by Peregrine \([57]\) and later by Madsen and Schäffer \([53]\), and others. The horizontal and vertical velocities are given by,
\[ u = \vec{u} + \mu^2 \left( \nabla (\nabla \cdot (h\vec{u})) - h\nabla (\nabla \cdot \vec{u}) \right) \left( \delta\eta + h \right) \left( \frac{1}{2} - q \right) + \ldots \]

\[ \mu^2 \left( \frac{1}{2} (\delta\eta + h)^2 \nabla (\nabla \cdot \vec{u}) \right) \left( \frac{1}{3} - q^2 \right) + \ldots \]

\[ \mu^4 u_3 \left( \frac{1}{4} - q^3 \right) + \mu^4 u_4 \left( \frac{1}{5} - q^4 \right) + O(\mu^6), \quad (2.12) \]

and,

\[ w = -\vec{u} \cdot \nabla h - (h + \delta\eta)(\nabla \cdot \vec{u})q + O(\mu^2), \quad (2.13) \]

where \( u_3 \) and \( u_4 \) are complicated differential functions of \( \vec{u} \), for more detail see Madsen and Schäffer [53]. Substitution of equations (2.12) and (2.13) into equation (2.10) provides an expression for the pressure profile, which includes the typical mixed space/time derivatives, i.e. \( \partial / \partial t (\nabla \cdot \vec{u}) \). With a simple rearrangement of terms it is possible to construct a general formula for the pressure profile which includes a Green–Nagdhi type polynomial expansion in the vertical axis,

\[ P = P_0 + \sum_{n=1}^{N} \mu^{\beta_n} P_n \phi_n + O(\mu^{N+2}), \quad (2.14) \]

where \( \beta_n = n + n_\text{mod}2 \) is equal to \( n \) when \( n \) is even and equal to \( n + 1 \) when \( n \) is odd, \( P_0 = P_0(x,t,q) \) is the zeroth–order component of the expansion, \( P_n = P_n(x,t) \) is the \( n^{th} \) term in the expansion, and the vertical basis functions \( \phi_n = \phi_n(q) \) are defined in such a way that the zero free–surface pressure condition is explicitly satisfied,

\[ \phi_n = \sum_{m=1}^{n} \hat{\phi}_{mn} (1 - q^m), \quad (2.15) \]
where the $\hat{\phi}_{mn}$ are arbitrary constants, and $\hat{\phi}_{nn} = 1$ can be assumed without loss of
generality.

It is noted that the use of polynomial expansions leads to a limited region of
accuracy in the dispersion and shoaling relationship [52]. This will be discussed
in further detail in chapter 3 where it will be demonstrated that the approximate
dispersion and shoaling relationships converge to the analytical solution of linear
theory as the number of terms in the polynomial expansion is increased, and that
with the appropriate choice of basis functions it is possible to expand the valid range
of the model to well beyond the region of interest even at low–order. It is noted that
in order to derive a pressure expansion with terms up to $q^N$ it is necessary to apply a
velocity expansion of an equivalent number of terms $q^N$. The theoretical framework
for obtaining higher order expansions has been outlined by Madsen and Schäffer [53],
among others.

In the following sections a strategy for determining each of the $P_n$ terms will be
developed. It will be shown that the solution for the pressure profile does not involve
any mixed space/time derivatives, has few degrees of freedom to solve for, and with
the appropriate choice of basis functions $\phi_n$ can produce highly accurate solutions.

2.2 A Boussinesq–Type Non–Hydrostatic Pressure–Poisson Model

Although expressions for the $P_n$ terms in the pressure expansion can be deter-
mined through substitution of Boussinesq approximations for the horizontal and ver-
tical velocities into the vertical conservation of momentum equation, see equation
(2.10), these expressions often involve higher order spatial derivatives and mixed
space/time derivatives [56]. Alternatively, we discuss an approach for determining
the pressure expansion terms where they are treated as unknowns which are found
as a solution to the pressure–Poisson equation and the bottom boundary condition
for pressure, equations (2.7) and (2.9). The weighted residual method is utilized in
order to generate a sufficient number of equations to match the order of the desired pressure expansion. The result is a set of equations equal in number to the order of the expansion which include spatial derivatives of at most second–order, and no mixed space/time derivatives.

The general form of the pressure solution follows equation (2.14). A direct integration of the vertical momentum equation provides an expansion about the hydrostatic pressure, which gives,

$$P = \frac{g}{\delta}(\delta \eta + h)(1 - q) + \sum_{n=1}^{N} \mu^{\beta_n} \phi_n P_n + O(\mu^{N+2}).$$  \hspace{1cm} (2.16)

Through the course of this work it was determined that the solution to the pressure profile was dependent on the choice of approximation for the zeroth–order term, \(P_0\). The solution to the pressure for this choice of \(P_0\) led to undesirable overall properties, which will be discussed in chapter 3. If the zeroth–order term in the expansion is taken to be the still water pressure term, as in Peregrine [57], the model exhibits more desirable properties in terms of dispersion characteristics. This particular expansion can be derived in the same manner through a simple rearrangement of terms. Considering Eq. (2.16), we rearrange the terms to obtain

$$P = \frac{g}{\delta} h(1 - q) + \left( g\eta + \mu^2 \tilde{P}_1 \right)(1 - q) + \sum_{n=2}^{N} \mu^{\beta_n} P_n \phi_n + O(\mu^{N+2}),$$  \hspace{1cm} (2.17)

where \(\frac{g}{\delta} h(1 - q)\) represents the pressure for a fluid at rest and \(\tilde{P}_1\) is an \(O(1)\) pressure correction term. Note that this expression is equivalent to the expansion given by Eq. (2.16) and that all the remaining terms in the pressure expansion remain unchanged. Dropping the \(\sim\) notation leads to the general solution for pressure,

$$P = \frac{g}{\delta} h(1 - q) + \tilde{P}_1 (1 - q) + \sum_{n=2}^{N} \mu^{\beta_n} P_n \phi_n + O(\mu^{N+2}).$$  \hspace{1cm} (2.18)
equation (2.18) defines the general $N^{th}$ order Boussinesq–type approximation to the pressure.

Given a desired order of accuracy in the model, an initial free–surface displacement, and set of horizontal velocities, the process for determining the non–hydrostatic pressure profile and updating the free–surface and velocities is as follows:

1. Determine the desired order of accuracy, $O(\mu^N)$.

2. Insert the velocity profile, equations (2.12) and (2.13), into equations (2.7) and (2.9) to generate a system of pressure–Poisson equations for the pressure.

3. Substitute the Boussinesq approximation to the pressure profile (2.18) and the free–surface elevation into the system of pressure–Poisson equations generated in step 2.

4. Discard any terms that are formally greater than the desired order.

5. Apply the weighted residual method over the vertical axis in order to generate a sufficient number of equations to match the number of unknown terms in the pressure expansion.

6. Solve the system of equations from step 5 for the unknown pressure expansion terms $P_n$, for $n = 1 \ldots N$.

7. Substitute the Boussinesq–type approximation for the pressure profile from step 6 and the velocities into the depth–integrated form of equation (2.2) and equation (2.8) to advance the horizontal velocities and free–surface forward in time. Substitution of the horizontal velocities into equation (2.13) then determines the vertical velocity, $w$.

2.2.1 Boussinesq System Of Equations With No Mixed Space–Time Derivatives

We consider a Boussinesq type expansion of the pressure field with scaling up to order $O(\mu^N)$. The horizontal velocities, $u$, and vertical velocity, $w$, are given by equations (2.12) and (2.13). The Boussinesq-type expansion for the pressure, $P$, is given by equation (2.18). Note that the basis functions for pressure, equation (2.15), are defined such that they disappear at the free–surface, $q = 1$, thus the zero pressure free–surface boundary condition is explicitly satisfied. Substituting equation (2.18) into equation (2.7) produces an expression for the pressure–Poisson equation,
\[
\sum_{n=1}^{N} \mu^n \beta_n \left[ \mu^2 \phi_n \nabla^2 P_n + 2 \mu^2 \phi'_n \nabla q \cdot \nabla P_n + \ldots \right. \\
\left. \left( \phi''_n \left( \mu^2 (\nabla q) \cdot (\nabla q) + \left( \frac{\partial q}{\partial z} \right)^2 \right) + \mu^2 \phi''_n \nabla^2 q \right) P_n \right] = \ldots \\
- \mu^2 \delta^2 \nabla^{(3)} \cdot (\mathbf{u}^{(3)} \cdot \nabla^{(3)} \mathbf{u}^{(3)}) - \ldots \\
\mu^2 \frac{g}{\delta} (\nabla^2 h (1 - q) - 2 (\nabla h) \cdot (\nabla q) - h \nabla^2 q) + O(\mu^{N+2}), \\
0 \leq q \leq 1. \quad (2.19)
\]

Similarly, substitution into equation (2.9) produces an expression for the pressure bottom boundary condition,

\[
\sum_{n=1}^{N} \mu^n \beta_n \left( \mu^2 (h + \delta \eta) \phi_n \nabla h \cdot \nabla P_n + \left( 1 + \mu^2 (\nabla h) \cdot (\nabla h) \right) \phi'_n P_n \right) = \ldots \\
\delta \mu^2 (h + \delta \eta) \mathbf{u} \cdot (\nabla \mathbf{u}) - g \left( 1 + \mu^2 (\nabla h) \cdot (\nabla h) \right) \eta + O(\mu^{N+2}), \\
q = 0. \quad (2.20)
\]

Since there are \( N \) unknowns, \( P_n \), and only two equations, a weighted residual in the vertical direction is used to solve the full system for the pressure–Poisson component of the equations. Arbitrary weighting functions were employed in order to determine if the choice of weighting function influenced the solution. As will be illustrated in the case for a two term expansion, i.e. \( N = 2 \), the choice of weighting functions has a direct impact on the solution accuracy and stability. The weighting functions are defined as follows,
\[ W_n = \sum_{m=0}^{n} \hat{W}_{mn} q^m, \quad \hat{W}_{nn} = 1, \quad (2.21) \]

where \( \hat{W}_{mn} \) are constants to be determined, and the set of weighting functions must be linearly independent. Each equation in the weighted residual method takes the form,

\[
\int_0^1 W_m \left( \sum_{n=1}^{N} \mu^2 \left[ \phi_n'' + \left( \frac{\partial q}{\partial z} \right)^2 + \mu^2 \phi_n'' q^2 \right] P_n \right) dq = \ldots \\
\int_0^1 W_m \left( -\mu^2 \delta^2 \nabla^2 (\bar{u}^{(3)} \cdot \nabla^2 h) \right) dq + O(\mu^{\hat{\beta}_n+2})
\]

for \( m = 1 \ldots N - 1 \). (2.22)

The system of equations comprised of equations (2.20) and (2.22) make up the Boussinesq–type pressure–Poisson model (PPBOUSS) and are solved to determine all \( P_n \), which in turn determines the pressure profile that can then be used with the conservation of momentum equations to update the velocities.

Substitution of the velocity expansion, equation (2.12) demonstrates that the conservation of mass, equation (2.8), is only dependent on the depth–averaged velocity,

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot \left( (\delta \eta + h) \bar{u} \right) = 0. \quad (2.23)
\]

In order to update the horizontal velocities we consider the depth integral of equation (2.2),
\[
\int_{-h}^{\delta \eta} \left[ \frac{\partial \mathbf{u}}{\partial t} + \delta \nabla \cdot (\mathbf{u}\mathbf{u}^T) + \delta \frac{\partial}{\partial z}(w\mathbf{u}) + \nabla P \right] dz = 0. \tag{2.24}
\]

Application of the Leibniz integration rule and the kinematic free–surface and bottom boundary conditions yields the following simplification of equation (2.24),

\[
\frac{\partial}{\partial t} \int_{-h}^{\delta \eta} \mathbf{u} dz + \delta \nabla \cdot \int_{-h}^{\delta \eta} (\mathbf{u}\mathbf{u}^T) dz + \int_{-h}^{\delta \eta} (\nabla P) dz = 0, \tag{2.25}
\]

see Nwogu equations (12) and (13) [56]. It is clear from the definition of the velocity profile expansion, equation (2.12), that at \(O(\mu^2)\) only the zeroth order velocity terms are retained in equation (2.25), thus,

\[
\frac{\partial}{\partial t} \left( (h + \delta \eta) \mathbf{u} \right) + \delta \nabla \cdot \left( (h + \delta \eta) \mathbf{u}\mathbf{u}^T \right) + \int_{-h}^{\delta \eta} (\nabla P) dz + O(\delta \mu^4) = 0. \tag{2.26}
\]

Substitution of the pressure expansion, equation (2.18), the conservation of mass, equation (2.23), and division by \((h + \delta \eta)\), yields the final form of the horizontal momentum equation in vector notation,

\[
\frac{\partial \mathbf{u}}{\partial t} + \delta \mathbf{u} \cdot \nabla \mathbf{u} + \sum_{n=0}^{N} \mu^{\beta_n} \left( G_n \nabla P_n - \ldots \right)
\]

\[
+ \frac{1}{h + \delta \eta} \left( \phi_n|_{q=0} \nabla h + R_n \nabla (h + \delta \eta) \right) P_n + O(\mu^{N+2}, \delta \mu^4) = 0, \tag{2.27}
\]

where \(G_n\) and \(R_n\) are given in Table 2.1 and evaluated at \(q = 1\). It is important to note that based on the definition of the depth averaged velocity, the first term in equation (2.27) is exact for all orders, while the convective term is exact up to \(O(\mu^2)\), with errors of \(O(\delta \mu^4)\). Given a pressure profile expansion of \(O(\mu^N)\), we have either a fully nonlinear \(O(\mu^2)\) or weakly nonlinear \(O(\mu^N), N > 2\), model which is
TABLE 2.1
DEFINITION OF INTEGRALS

<table>
<thead>
<tr>
<th>Integral</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Psi_{nm} )</td>
<td>( \int_0^q W_m \hat{q} \phi_n'' d\hat{q} )</td>
</tr>
<tr>
<td>( \Phi_{nm} )</td>
<td>( \int_0^q W_m \hat{q} \phi_n' d\hat{q} )</td>
</tr>
<tr>
<td>( \Omega_m )</td>
<td>( \int_0^q W_m d\hat{q} )</td>
</tr>
<tr>
<td>( G_n )</td>
<td>( \int_0^q \phi_n d\hat{q} )</td>
</tr>
<tr>
<td>( R_n )</td>
<td>( \int_0^q \hat{q} \phi_n' d\hat{q} )</td>
</tr>
<tr>
<td>( \xi_{nm} )</td>
<td>( \int_0^q W_m \phi_n d\hat{q} )</td>
</tr>
<tr>
<td>( \Gamma_{nm} )</td>
<td>( \int_0^q W_m \hat{q} \phi_n d\hat{q} )</td>
</tr>
<tr>
<td>( \chi_m )</td>
<td>( \int_0^q W_m \hat{q} d\hat{q} )</td>
</tr>
<tr>
<td>( S_{nm} )</td>
<td>( \int_0^q W_m \hat{q}^2 \phi_n'' d\hat{q} )</td>
</tr>
<tr>
<td>( \Theta_{nm} )</td>
<td>( \int_0^q W_m \phi_n' d\hat{q} )</td>
</tr>
<tr>
<td>( \Lambda_{nm} )</td>
<td>( \int_0^q W_m \phi_n'' d\hat{q} )</td>
</tr>
</tbody>
</table>

only dependent on the depth–averaged horizontal velocities.

Equations (2.20), (2.22), (2.23) and (2.27) make up the complete system of equations for the model. Recall that the solution for \( w \) is explicitly defined in terms of \( \bar{u} \) by equation (2.13). Having the pressure model only rely on the depth–integrated velocity is advantageous in that some ocean circulation models assume a depth–averaged velocity, thus all conclusions about increased accuracy from the inclusion of a non–hydrostatic pressure model can be extended to the general class of ocean circulation models. The equations for pressure are elliptic, while the equations for velocity and the free–surface are both hyperbolic. The strategy for solving this system of equations has two steps:

1. Given a specific velocity and free–surface profile, determine the pressure profile using equations (2.20) and (2.22).

2. Update the velocities and free-surface using equations (2.27) and (2.23) using the pressure profile found in the previous step. Recall that the Boussinesq expansion for the vertical velocity, \( w \), is represented in terms of the depth averaged horizontal velocity.

2.2.2 Fully Nonlinear \( O(\mu^2) \) Solution

Substituting the horizontal and vertical velocities, equations (2.12) and (2.13), into the governing equations for the pressure solution, equations (2.20) and (2.22),
and truncating all terms of order greater than $O(\mu^2)$ produces a system of equations to solve for the pressure expansion components, $P_1$ and $P_2$.

**Bottom Boundary Condition:**

\[
    a_{1,2}P_2 + a_{2,1} \cdot \nabla P_1 + a_{1,1}P_1 = a_0 + O(\mu^4), \tag{2.28}
\]

where,

\[
    a_{1,2} = \left( \mu^2 \hat{\phi}_{12} \right), \tag{2.29}
\]
\[
    a_{2,1} = -\left( \mu^2 (h + \delta \eta)(\nabla h) \right), \tag{2.30}
\]
\[
    a_{1,1} = \left( 1 + \mu^2 (\nabla h) \cdot (\nabla h) \right), \tag{2.31}
\]
\[
    a_0 = g \left( 1 + \mu^2 (\nabla h)^2 \right) \eta - \delta \mu^2 (h + \delta \eta) \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} \cdot \nabla^2 h). \tag{2.32}
\]

and

**Pressure–Poisson:**

\[
    b_{1,2,1}P_2 + b_{3,1,1} \nabla^2 P_1 + b_{2,1,1} \cdot \nabla P_1 + b_{1,1,1}P_1 = b_{0,1} + O(\mu^4), \tag{2.33}
\]
where,

\begin{align}
    b_{1,2,1} &= -\mu^2 \omega_1, \\
    b_{3,1,1} &= \frac{1}{6} \mu^2 \omega_2 (h + \delta \eta)^2, \\
    b_{2,1,1} &= \frac{1}{3} \mu^2 \left( (1 + \omega_2) \delta \nabla \eta - \omega_2 \nabla h \right) (h + \delta \eta), \\
    b_{1,1,1} &= -\frac{1}{6} \mu^2 \left[ \left( \omega_2 \nabla^2 h - \delta (1 + \omega_2) \nabla^2 \eta \right) (h + \delta \eta) + 2 \left( (1 + \omega_2) \delta \nabla \eta - \omega_2 \nabla h \right) \cdot \nabla (h + \delta \eta) \right], \\
    b_{0,1} &= -\frac{1}{2} \delta \mu^2 \omega_1 (h + \delta \eta)^2 \nabla \left( \tilde{u} \cdot \nabla \tilde{u} \right) - \frac{1}{6} g \mu^2 (1 + \omega_2) (h + \delta \eta) h \nabla^2 \eta \\
    &\quad - \frac{1}{3} g \mu^2 \left( \delta (1 + \omega_2) (\eta \nabla h - h \nabla \eta) + \omega_2 h \nabla h \right) \cdot \nabla \eta \\
    &\quad - \frac{1}{6} g \mu^2 \omega_2 \left( (h + \delta \eta) \nabla^2 h - 2 (\nabla h) \cdot (\nabla h) \right) \eta.
\end{align}

where \( \omega_1 = (1 + 2 \hat{W}_{01}) \) and \( \omega_2 = (1 + 3 \hat{W}_{01}) \) are defined in order to simplify the expressions and with \( W_1 = \hat{W}_{01} + q \) as the arbitrary weighting function used in the weighted residual approach. Simultaneous solutions of equations (2.28) and (2.33) determines the pressure profile components, \( P_1 \) and \( P_2 \). The pressure and velocity expansions are substituted into the horizontal velocity and conservation of mass equations, equations (2.27) and (2.23), to advance the velocities and free–surface forward in time as follows,

\begin{align}
\frac{\partial \tilde{u}}{\partial t} + \delta \tilde{u} \cdot \nabla \tilde{u} + \frac{1}{2} g \left( \frac{h (\nabla \eta) + (\nabla h) \eta}{h + \delta \eta} \right) + \frac{1}{6} \mu^2 (4 + 3 \hat{\phi}_{12}) \nabla P_2 + \ldots \\
+ \frac{1}{6} \mu^2 \left( \frac{\delta (4 + 3 \hat{\phi}_{12}) \nabla \eta - (2 + 3 \hat{\phi}_{12}) \nabla h}{h + \delta \eta} \right) P_2 + \ldots \\
+ \frac{1}{2} \nabla P_1 + \frac{1}{2} \left( \frac{\delta \nabla \eta - \nabla h}{h + \delta \eta} \right) P_1 + O(\mu^4) = 0, \quad (2.39)
\end{align}
and
\[
\frac{\partial \eta}{\partial t} + \nabla \left( (h + \delta \eta) \bar{u} \right) = 0. \tag{2.40}
\]

2.2.2.1 Reduction In The Degrees Of Freedom For The Fully Nonlinear \(O(\mu^2)\) Case

It is possible to reduce the degrees of freedom in the overall system. We first note that \(P_2\) can be solved for explicitly in terms of \(P_1\) and \(\nabla P_1\) through algebraic manipulation of equation (2.28),

\[
P_2 = \frac{1}{a_{1,2}} \left( a_0 - a_{2,1} \cdot \nabla P_1 - a_{1,1} P_1 \right) + O(\mu^4), \tag{2.41}
\]

provided \(a_{1,2} \neq 0\). Substituting equation (2.41) into equation (2.33), a single elliptic problem for \(P_1\) is obtained,

\[
b_{3,1,1} \nabla^2 P_1 + \left( b_{2,1,1} - \frac{b_{1,2,1}}{a_{1,2}} a_{2,1} \right) \cdot \nabla P_1 + \ldots
\]

\[
\left( b_{1,1,1} - \frac{b_{1,2,1}}{a_{1,2}} a_{1,1} \right) P_1 = \left( b_{0,1} - \frac{b_{1,2,1}}{a_{1,2}} a_0 \right) + O(\mu^4), \tag{2.42}
\]

where all the coefficients are defined as above. In one horizontal dimension equation (2.42) results in a tri–diagonal matrix to solve the linear system. Similar to standard Boussinesq models, extension into two horizontal dimensions will necessitate the use of iterative techniques to solve the linear system in a computationally efficient manner, however unlike standard Boussinesq theory, which has an elliptic problem associated with each of the horizontal velocities, the model has only one elliptic problem which is confined to solving for the first term in the pressure expansion, \(P_1\), while the velocity equations remain strictly hyperbolic. Given the solution for \(P_1\) it is straightforward to use equation (2.41) to determine \(P_2\). This reduction in degrees of freedom reduces the complexity of the model and should contribute to increased overall computational efficiency.
2.2.3 Weakly Nonlinear $O(\mu^N)$ Solution

It is possible to extend the $O(\mu^2)$ solution to arbitrary order where all $O(\mu^N)$ terms are retained. However the extension of the fully nonlinear model to higher order leads to very complex equations and numerical implementation becomes difficult. For this reason we explore a weakly nonlinear model for arbitrary $N \geq 4$ where all terms of $O(\delta \mu^4)$ or greater are truncated. The implementation of the fully nonlinear model will be left to future work. It is observed that solutions to the full set of equations follow a pattern. A pressure solution for an expansion of $O(\mu^N)$ can be found using the following expressions,

**Bottom Boundary Condition:**

\[
\sum_{n=1}^{N} \left( a_{2,n} \cdot \nabla P_n + a_{1,n} P_n \right) = a_0 + O(\mu^{\beta N+2}, \delta \mu^4),
\]

(2.43)

**Pressure Poisson:**

\[
\sum_{n=0}^{N} \left( b_{3,n,m} \Delta^2 P_n + b_{2,n,m} \cdot \nabla P_n + b_{1,n,m} P_n \right) = b_{0,m} + O(\mu^{\beta N+2}, \delta \mu^4),
\]

\[m = 1 \ldots N - 1, \quad (2.44)\]

where

\[
a_{2,n} = \mu^{\hat{\beta} n+2} (h + \delta \eta) (\nabla h) \phi_n \bigg|_{q=0}, \quad (2.45) \\
a_{1,n} = \left( \mu^{\hat{\beta} n+2} (\nabla h)^2 + \mu^{\hat{\beta} n} \right) \phi'_n \bigg|_{q=0}, \quad (2.46) \\
a_0 = -g \left( 1 + \mu^2 (\nabla h)^2 \right) \eta + \delta \mu^2 (h + \delta \eta) \bar{u} \cdot (\bar{u} \cdot \nabla^2 h), \quad (2.47)
\]
and

\[
\begin{align*}
\beta_{n,m} &= \mu\beta_n + 2(h + \delta\eta)^2 \xi_{nm} \bigg|_{q=1}, \\
\beta_{n,m} &= 2\mu\beta_n + 2(h + \delta\eta)\left(\Theta_{nm}(\nabla h) - \Phi_{nm}(\nabla h - \nabla \eta)\right) \bigg|_{q=1}, \\
\beta_{n,m} &= \mu\beta_n \left[\left(\mu^2(\nabla h)^2 + 1\right) \Lambda_{nm} + \mu^2 S_{nm} \left((\nabla h) + \delta(\nabla \eta)\right)^2 - \ldots \right] \bigg|_{q=1}, \\
\beta_{n,m} &= -2\delta\mu \left(\nabla \hat{u} \cdot \nabla \hat{u}\right) \bigg|_{q=1}, \\
\end{align*}
\]

where,

\[
\beta_n = \begin{cases} 
0 & \text{for } n = 0 \\
\beta_n & \text{otherwise}
\end{cases}
\]  

and the above integrals are defined in Table 2.1. It is important to note that, in order to construct a weakly nonlinear model, all terms that are of \(O(\mu^4)\) and higher are discarded in addition to all terms that are of \(O(\mu^{N+2})\). An advantage of considering the model in this framework is that all of the integrals as they are defined in Table 2.1 can be calculated once and substituted into the \(a_{i,j}\)'s and \(b_{i,j,k}\)'s. A versatile algorithm can then be designed such that the elliptic solver is based on arbitrary \(a_{i,j}\)'s and \(b_{i,j,k}\)'s.

The conservation of momentum equations follow a similar pattern and can be defined as,

\[
\frac{\partial \hat{u}}{\partial t} + \delta\hat{u} \cdot \nabla \hat{u} + \frac{1}{2} g \left( \frac{h\nabla \eta + \eta \nabla h}{h + \delta\eta} \right) + \sum_{n=1}^{N} \left( C_{2,n} \nabla P_n + C_{1,n} P_n \right) = 0, 
\]

where
where,

\[ C_{2,n} = \mu^{\hat{\beta}_n} G_n \big|_{q=1}, \quad (2.54) \]

\[ C_{1,n} = \mu^{\hat{\beta}_n} \left( \frac{(\phi_n - R_n) \nabla h - R_n \delta \nabla \eta}{h + \delta \eta} \right) \bigg|_{q=1}. \quad (2.55) \]

Finally the conservation of mass equation remains unchanged,

\[ \frac{\partial \eta}{\partial t} + \nabla \left( (h + \delta \eta) \bar{u} \right) = 0. \quad (2.56) \]

Equations (2.43), (2.44), (2.53) and (2.56) make up the complete system of equations for an arbitrary weakly nonlinear \( O(\mu^N) \) model.

2.2.3.1 Reduction In Degrees Of Freedom For Higher Order Models

The technique for reducing the degrees of freedom in the fully nonlinear \( O(\mu^2) \) case can be extended to the weakly nonlinear \( O(\mu^N) \) case. Considering equations (2.43) and (2.44), truncation of all terms of \( O(\mu^{\beta_N+2}) \) and nonlinear terms of order \( O(\delta \mu^4) \) in the coefficients \( a_{i,j} \) and \( b_{i,j,k} \) produces,

\[ a_{1,N} P_N + \sum_{n=1}^{N-1} \left( a_{2,n} \cdot \nabla P_n + a_{1,n} P_n \right) = a_0 + O(\mu^{\beta_N+2}, \delta \mu^4), \quad (2.57) \]

\[ b_{1,N,m} P_N + \sum_{n=1}^{N-1} \left( b_{3,n,m} \nabla^2 P_n + b_{2,n,m} \cdot \nabla P_n + b_{1,n,m} P_n \right) = \ldots \]

\[ b_{0,m} + O(\mu^{\beta_N+2}, \delta \mu^4), \]

\[ m = 1 \ldots N - 1, \quad (2.58) \]

where
\[ a_{1,N} = \mu^\beta \phi'_N \bigg|_{q=0} \in \mathbb{R}, \]  
(2.59)  
\[ b_{1,N,m} = \mu^\beta \Lambda_{N,m} \bigg|_{q=1} \in \mathbb{R}, \]  
(2.60)  

which implies that \( P_N \) can be explicitly written in terms of the \( P_n \) for \( n = 1 \ldots N - 1 \), assuming \( \phi'_N|_{q=0} \neq 0 \). This reduces the number of unknowns in the elliptic equation by one and gives the following system of equations:

\[
P_N = \frac{1}{a_{1,N}} \left( a_0 - \sum_{n=1}^{N-1} \left( a_{2,n} \cdot \nabla P_n + a_{1,n} P_n \right) \right),
\]
(2.61)  
\[
\sum_{n=1}^{N-1} \left( B_{3,n,m} \nabla^2 P_n + B_{2,n,m} \cdot \nabla P_n + B_{1,n,m} P_n \right) = B_{0,m},
\]
(2.62)  

where

\[ B_{3,n,m} = b_{3,n,m}, \]  
(2.63)  
\[ B_{2,n,m} = b_{2,n,m} - \left( \frac{b_{1,N,m}}{a_{1,N}} \right) a_{2,n}, \]  
(2.64)  
\[ B_{1,n,m} = b_{1,n,m} - \left( \frac{b_{1,N,m}}{a_{1,N}} \right) a_{1,n}, \]  
(2.65)  
\[ B_{0,m} = b_{0,m} - \left( \frac{b_{1,N,m}}{a_{1,N}} \right) a_0. \]  
(2.66)  

Furthermore it is possible to reduce the elliptic problem by an additional degree of freedom due to the fact that for \( N \geq 4 \) any term of order \( O(\mu^{\beta(N-1)+2}) \) will also be
truncated. For \( m = 1 \) equation (2.62) can be written as

\[
B_{1,N-1,1}P_{N-1} + \sum_{n=1}^{N-2} \left( B_{3,n,1} \nabla^2 P_n + B_{2,n,1} \cdot \nabla P_n + B_{1,n,1}P_n \right) = B_{0,1},
\]

(2.67)

where

\[
B_{1,N-1,1} = \mu^N \left( \Lambda_{(N-1)1} i_{q=1} - \Lambda N1 i_{q=1} \left( \frac{\phi'_N}{\phi_N} \right) \right).
\]

(2.68)

This implies that \( P_{N-1} \) can be written as an explicit combination of the \( P_n \) terms for \( n = 1 \ldots N - 2 \), assuming that \( B_{1,N-1,1} \neq 0 \). Thus

\[
P_N = \frac{1}{a_{1,N}} \left( a_0 - \sum_{n=1}^{N-1} \left( a_{2,n} \cdot \nabla P_n + a_{1,n}P_n \right) \right),
\]

(2.69)

\[
P_{N-1} = \frac{1}{B_{1,N-1,1}} \left( B_{0,1} - \sum_{n=1}^{N-2} \left( B_{3,n,1} \nabla^2 P_n + B_{2,n,1} \cdot \nabla P_n + B_{1,n,1}P_n \right) \right),
\]

(2.70)

\[
\sum_{n=1}^{N-2} \left( \hat{B}_{3,n,m} \nabla^2 P_n + \hat{B}_{2,n,m} \cdot \nabla P_n + \hat{B}_{1,n,m}P_n \right) = \hat{B}_{0,m},
\]

\[ m = 2 \ldots N - 1. \]  

(2.71)

where,

\[
\hat{B}_{3,n,m} = B_{3,n,m} - \left( \frac{B_{1,N-1,m}}{B_{1,N-1,1}} \right) B_{3,n,1},
\]

(2.72)

\[
\hat{B}_{2,n,m} = B_{2,n,m} - \left( \frac{B_{1,N-1,m}}{B_{1,N-1,1}} \right) B_{2,n,1},
\]

(2.73)

\[
\hat{B}_{1,n,m} = B_{1,n,m} - \left( \frac{B_{1,N-1,m}}{B_{1,N-1,1}} \right) B_{1,n,1},
\]

(2.74)

\[
\hat{B}_{0,m} = B_{0,m} - \left( \frac{B_{1,N-1,m}}{B_{1,N-1,1}} \right) B_{0,1}.
\]

(2.75)

No more reduction of degrees of freedom is possible in this manner. However this technique has reduced the degrees of freedom in the elliptic problem by two. This
reduction in the degrees of freedom will improve the computational cost of the method in the numerical implementation. For example, in one horizontal dimension the \( O(\mu^4) \) model has only six bands in the elliptic problem using a second order central finite difference method. Note that the set of basis functions and weighting functions is arbitrary, thus the set of constant coefficients in each of these sets can be chosen in order to enforce \( a_{1,N} \neq 0 \) and \( B_{1,N-1,1} \neq 0 \), making the reduction in the degrees of freedom always possible.

2.3 An Alternative Approach To The Coupled System

One of the main purposes of this work is in the development of a stand alone pressure module that can be coupled with an ocean circulation model in a simple and straightforward manner. Thus far we have developed the full model considering the primitive equations of motion and mass given by equations (2.2) and (2.8). Many of the standard ocean circulation models solve the shallow water equations (SWE) which can be derived from these equations but have a slightly different form, as follows,

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot Q = 0, \quad (2.76)
\]

\[
\frac{\partial Q}{\partial t} + \delta \nabla \cdot \left( Q \bar{u} + \frac{1}{2} g \eta (\eta + h) \right) + \int_{-h}^{\delta \eta} (\nabla P_d) dz = g \eta \nabla h - \tau Q + F, \quad (2.77)
\]

where \( Q = (h + \delta \eta) \bar{u} \) is the total volume flux, \( \tau \) is related to the bottom friction, \( P_d \) represents the dynamic pressure contribution and \( F \) represents a collection of source terms such as the Coriolis force, tidal potential forces and surface stresses to name a few, see Kubatko [38] for more details regarding the derivation of the SWE. Using this definition the total pressure is given by,

\[
P = P_h + P_d = g(\delta \eta - z) + P_d. \quad (2.78)
\]
Considering equations (2.18) and (2.78) a solution for $P_d$ can be found,

$$
P_d = (1 - q)(P_1 - g\eta) + \sum_{n=2}^{N} \mu^\beta_n \phi_n P_n. \tag{2.79}
$$

For practical application and simplicity in the notation we can combine the first two terms, $P_1$ and $g\eta$, into one single unknown,

$$
P_d = (1 - q)P_1 + \sum_{n=2}^{N} \mu^\beta_n \phi_n P_n. \tag{2.80}
$$

Substitution of equation (2.80) into equation (2.77) provides the final equation for the conservation of momentum in the SWE,

$$
\frac{\partial Q}{\partial t} + \delta \nabla \cdot \left( Q\bar{u}^T + \frac{1}{2}g\eta(\eta + h) \right) + \\
\sum_{n=1}^{N} \mu^\beta_n \left( (h + \delta\eta)G_n \nabla P_n - \phi_{n,q=0} \nabla h P_n - \nabla (h + \delta\eta) R_n P_n \right) = g\eta \nabla h - \tau Q + F, 
$$

(2.81)
As with classical Boussinesq theory, the linear and nonlinear properties of the pressure-Poisson model can be assessed through linear and nonlinear expansions. Results show that the model converges to the well known Stoke’s solution at high-order for linear properties. Increased accuracy can be achieved when arbitrary vertical basis functions for pressure are used and appropriate coefficients are chosen so as to optimize for linear and nonlinear properties. In this chapter, Stoke’s type analysis and multiple scales analysis are performed to explore the linear and nonlinear properties of the model and to determine the set of basis functions leading to optimal performance of the model.

3.1 Dispersion Properties

We consider the first order dispersion properties of a small amplitude wave train over a flatbed. The approximate dispersion relationship can then be compared with the well known Airy dispersion relationship \[ C_{Airy}^2/gh = \tanh(kh)/(kh) \] given by \( C_{Airy}^2/gh = \tanh(kh)/(kh) \). Details of the derivation are standard and can be found in appendix B. The order of accuracy for the approximate dispersion relationship is reliant on three important components of the model, (1) the order of approximation in the pressure solution \( N \), (2) the choice of basis functions \( \phi_n \), equation (2.15), and (3) for the \( N = 2 \) case the choice of weighting functions, \( w_m \), equation (2.21).

We begin with the simplest case, \( N = 2 \). We consider basis and weighting functions,
\[ \phi_1 = 1 - q, \quad (3.1) \]
\[ \phi_2 = \hat{\phi}_{12}(1 - q) + (1 - q^2), \quad (3.2) \]
\[ w_1 = \hat{W}_{01} + q, \quad (3.3) \]
\[ \frac{C^2}{gh} = 1 + B(kh)^2 \]
\[ \frac{1}{1 + (B + \frac{1}{3})(kh)^2}, \quad (3.5) \]

where,
\[ B = -\frac{2 + \hat{\phi}_{12} + 4\hat{W}_{01} + 3\hat{\phi}_{12}\hat{W}_{01}}{12\hat{W}_{01} + 6}. \]

Equation (3.5) is equivalent to the general dispersion relationship derived by Madsen \textit{et al.} [55]. As demonstrated by Madsen the appropriate choice of coefficients affects the accuracy of the dispersion relationship. If the basis functions and weighting functions are taken to be monomial, corresponding to a choice of \( \hat{\phi}_{12} = 0, \hat{W}_{01} = 0 \) and \( B = -\frac{1}{3} \), then the dispersion relationship becomes
\[ \frac{C^2}{gh} = 1 - \frac{1}{3}(kh)^2, \quad (3.7) \]
which is equivalent to the dispersion relationship for the model derived by Nwogu [56] expanded about the still water level, \( z_\alpha = 0 \). This dispersion relationship has a formal accuracy of \( O(\mu^2) \); however it produces a negative value for \( kh > \sqrt{3} \), setting a limit on the applicability of the model, as a negative dispersion relationship will lead to imaginary celerity in the solution. As a result, the use of monomial basis and weighting functions is not recommended. It is noted here that if the pressure
expansion is chosen to be given by equation (2.16) then the dispersion relationship for the model is given by equation (3.7), regardless of choice of basis or weighting functions. It is for this reason that an expansion about the still water pressure, equation (2.18) was considered for the model.

An optimal choice of coefficients in the weighting function can produce a more accurate model. The optimal choice,

$$\hat{W}_{01} = -\frac{15}{3} \hat{\phi}_{12} + 12$$

$$\hat{\phi}_{12} = 8 + \frac{1}{3} \hat{\phi}_{12}$$

(3.8)

corresponding to $B = \frac{1}{15}$, produces the well known Padé [2,2] approximation to the Airy dispersion relationship,

$$\frac{C^2}{gh} = \frac{1 + \frac{1}{15} (kh)^2}{1 + \frac{2}{5} (kh)^2}$$

(3.9)

which has a formal accuracy of $O(\mu^4)$. In addition, we still have one free parameter $\hat{\phi}_{12}$ which no longer affects the dispersion relationship, but as will be illustrated in the following section, will play a role in the accuracy of shoaling.

This technique can be extended to the $N = 4$ case with basis functions:

$$\phi_n = \sum_{m=1}^{n} \hat{\phi}_{mn} (1 - q^m), \quad \hat{\phi}_{nn} = 1, \quad n = 1 \ldots 4, \quad (3.10)$$

$$W_n = \sum_{m=0}^{n} \hat{W}_{mn} q^m, \quad \hat{W}_{nn} = 1, \quad n = 1 \ldots 3, \quad (3.11)$$

which in turn provides four independent free parameters, for simplicity we define
**TABLE 3.1**

CASES OF OPTIMIZED BASIS FUNCTIONS FOR THE FULLY NONLINEAR $O(\mu^2)$ MODEL.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Coefficients</th>
<th>Dispersion</th>
<th>Shoaling</th>
<th>Nonlinear</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\hat{\phi}<em>{12} = \frac{-6}{5} + \frac{2}{15}\sqrt{15}$, $\hat{W}</em>{01} = \frac{2}{3} - \sqrt{5/3}$</td>
<td>Padé [2,2]</td>
<td>$O(\mu^4)$</td>
<td>$O(\mu^2)$</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>$\hat{\phi}<em>{12} = -\frac{4}{3}$, $\hat{W}</em>{01} = -\frac{4}{3}$</td>
<td>Padé [2,2]</td>
<td>$O(\mu^2)$</td>
<td>$O(\mu^2)$</td>
<td>Bounded Shoaling Limit</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\hat{\phi}<em>{12} = -0.704720$, $\hat{W}</em>{01} = -0.631192$</td>
<td>Padé [2,2]</td>
<td>$O(\mu^2)$</td>
<td>$O(\mu^2)$</td>
<td>Minimized Shoaling Error over $kh \in [0, \pi]$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$\hat{\phi}<em>{12} = -\frac{353}{225}$, $\hat{W}</em>{01} = -\frac{187}{21}$</td>
<td>Padé [2,2]</td>
<td>$O(\mu^2)$</td>
<td>$O(\mu^4)$</td>
<td></td>
</tr>
</tbody>
</table>

*The case used for numerical validation is designated in bold.*
The general dispersion relation then becomes,

\[
\frac{C^2}{gh} = \frac{1 + A_1(kh)^2 + A_2(kh)^4 + A_3(kh)^6}{1 + A_1(kh)^2 + A_5(kh)^4},
\]

where

\[
A_1 = \frac{1}{12} \left( 2 - K_3 + 2K_4 \right),
\]
\[
A_2 = \frac{1}{360} (-12 + 5K_1 + 15K_2 + 10K_3 + 10K_4),
\]
\[
A_3 = \frac{1}{8640} (-40K_1 - 45K_2 - 48K_4),
\]
\[
A_4 = \frac{1}{12} \left( 6 - K_3 + 2K_4 \right),
\]
\[
A_5 = \frac{1}{72} (K_1 + 3K_2 + 6K_4).
\]

The four free parameters, \( K_1 - K_4 \), can be manipulated to improve the dispersion characteristics, both in the sense of order of accuracy, and in the sense of asymptotic behaviour. A choice of \( K_1 = -24/35 - 42/25K_4 \), \( K_2 = 64/105 - 64/25K_4 \) and \( K_3 = 2/3 + 2K_4 \) gives the well known Padé \([4,4]\) approximant to the dispersion relation,

\[
\frac{C^2}{gh} = \frac{1 + \frac{1}{6}(kh)^2 + \frac{1}{945}(kh)^4}{1 + \frac{1}{6}(kh)^2 + \frac{1}{63}(kh)^4},
\]
which is accurate up to $O(\mu^8)$. In order to get the $O(\mu^{10})$ accurate Padé \([6,4]\) approximant

$$
\frac{C^2}{gh} = \frac{1 + \frac{19}{165}(kh)^2 + \frac{2}{1485}(kh)^4 - \frac{1}{155925}(kh)^6}{1 + \frac{74}{165}(kh)^2 + \frac{26}{1485}(kh)^4},
$$

(3.23)
a choice of $K_1 = -21776/28875 + 42/25K_4$, $K_2 = 6464/9625 - 64/25K_4$ and $K_3 = 34/55 + 2K_4$ is required. It is noted that for both optimized dispersion cases there is a free variable, $K_4$. Just as with the $O(\mu^2)$ case this free-variable will be used to optimize either shoaling or nonlinear effects while retaining optimum dispersion properties. Both the Padé \([4,4]\) and Padé \([6,4]\) represent a significant improvement in accuracy for dispersion, however, the Padé \([6,4]\) approximant approaches $-\infty$ in the limit which will lead to imaginary celerity. Therefore the set of basis functions that lead to the Padé \([4,4]\) approximant are considered more advantageous. We note that the free coefficient $\hat{\phi}_{12}$ does not appear in the system, nor do any of the coefficients associated with the arbitrary weighting functions.

In general an optimum set of basis functions can be found to provide a Padé \([N,N]\) or Padé \([N+2,N]\) approximant solution to the dispersion for all $N \geq 4$, thus obtaining accuracy in the dispersion up to order $O(\mu^{2N})$ or $O(\mu^{2N+2})$ respectively. However the Padé \([N+2,N]\) solution is not recommended due to the fact that it will always lead to negative celerity for large dimensionless wavenumbers.

Figure 3.1 shows the ratio of the approximate dispersion for optimal basis functions. For the $N = 2$ case the optimal basis function solution is accurate to within 10% of the analytical dispersion up to the nominal deep water limit ($0 \leq kh \leq \pi$). For the $N = 4$ case the choice of optimal basis functions gives accurate dispersion to within 5% of the analytical dispersion past the deep water limit ($0 \leq kh \leq 2\pi$), which is well within the proposed application range of this model.
### Table 3.2

**Cases of Optimized Basis Functions for the Weakly Nonlinear $O(\mu^4)$ Model.**

<table>
<thead>
<tr>
<th>Case #</th>
<th>Coefficients</th>
<th>Dispersion / Shoaling / Nonlinear</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v)</td>
<td>$\hat{\phi}<em>{12} = -\frac{60812}{33915} \pm \frac{4}{101745\sqrt{1245}}, \hat{\phi}</em>{13} = \frac{35}{102} \mp \frac{5}{714\sqrt{1245}}$</td>
<td>Padé [4,4] / $O(\mu^6)$ / $O(\mu^4)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}<em>{23} = \frac{-41450}{57210} \pm \frac{229\sqrt{1245}}{57210}, \hat{\phi}</em>{14} = \frac{(2185 \pm 89\sqrt{1245})}{3570}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}_{24} = \frac{-118730}{57120} \mp \frac{2229}{\sqrt{1245}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(vi)</td>
<td>$\hat{\phi}<em>{12} = \frac{-204016}{113715} \pm \frac{16\sqrt{61}}{113715}, \hat{\phi}</em>{13} = \frac{10}{19} \pm \frac{10}{399\sqrt{61}}$</td>
<td>Padé [4,4] / $O(\mu^4)$ / $O(\mu^4)$</td>
<td>Bounded Shoaling Limit</td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}<em>{23} = \frac{29452}{15690} \mp \frac{3153\sqrt{61}}{15690}, \hat{\phi}</em>{14} = \frac{2(1261 \pm 89\sqrt{61})}{1995}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}_{24} = \frac{2012}{15960} \mp \frac{3953}{\sqrt{61}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(vii)</td>
<td>$\hat{\phi}<em>{12} = -1.79454, \hat{\phi}</em>{13} = 0.60440, \hat{\phi}_{23} = 0.605257$</td>
<td>Padé [4,4] / $O(\mu^4)$ / $O(\mu^4)$</td>
<td>Minimized Shoaling Error over $kh \in [0,\pi]$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}<em>{14} = -1.54214, \hat{\phi}</em>{24} = -1.27021, \hat{\phi}_{34} = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(viii)</td>
<td>$\hat{\phi}<em>{12} = -0.1206296, \hat{\phi}</em>{13} = -1.790467, \hat{\phi}_{23} = 1.45195$</td>
<td>Padé [4,4] / $O(\mu^4)$ / $O(\mu^4)$</td>
<td>Bounded Stokes’s Limit</td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}<em>{14} = 6.98359, \hat{\phi}</em>{24} = 4.36622, \hat{\phi}_{34} = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(ix)</td>
<td>$\hat{\phi}<em>{12} = -1.709480, \hat{\phi}</em>{13} = 0.60440, \hat{\phi}_{23} = 0.605257$,</td>
<td>Padé [4,4] / $O(\mu^4)$ / $O(\mu^2)$</td>
<td>Minimized Shoaling &amp; Stokes’s Error over $kh \in [0,\pi]$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\phi}<em>{14} = -1.54214, \hat{\phi}</em>{24} = -1.27021, \hat{\phi}_{34} = 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The case used for numerical validation is designated in bold.*
3.2 Shoaling Properties

Extension of the linear dispersion analysis to include a perturbation solution with a multiple scales expansion will generate an approximation to the linear shoaling gradient. We consider the case where the solution has two spatial scales, a slow and fast derivative, \( x \) and \( X \) respectively. The slow derivative is assumed to vary at a rate proportional to a parameter \( \epsilon \ll 1 \). In addition the bathymetry is considered to be slowly varying, i.e. \( \nabla h = \epsilon \frac{\partial h}{\partial X} \). Details of the shoaling gradient analysis can be found in appendix [3]. Solutions to the shoaling gradient can be compared with the well known solution derived by Madsen and Sørensen [49], given by,

\[
\gamma_h = \frac{h}{\tilde{\eta}^{(0)}} \frac{\partial \tilde{\eta}^{(0)}}{\partial X} \left( \frac{\partial h}{\partial X} \right)^{-1} = - \frac{2kh \sinh(2kh) + 2(kh)^2(1 - \cosh(2kh))}{(2kh + \sinh(2kh))^2}. \tag{3.24}
\]

An estimation of the error between the approximate shoaling gradient and the Stokes solution can be calculated using the cumulative shoaling error by integrating the difference between the analytic and approximate shoaling gradient from shallow to deep water following the formula used by Chen and Liu [13],

\[
\frac{A}{A_{st}} = \exp \left[ \int_0^{kh} \frac{\gamma_h(kh') - \gamma_B(kh')}{kh'} d(kh') \right], \tag{3.25}
\]

where \( \gamma_B \) is the approximate shoaling gradient and \( \gamma_h \) is the Stoke’s solution. As stated previously in section [3.1] after optimizing the basis functions for dispersion for the \( O(\mu^2) \) and \( O(\mu^4) \), one free–coefficient remained that could be used to optimize the shoaling gradient. After optimizing the dispersion to be equivalent to the Padé [2,2] approximant, the \( O(\mu^2) \) shoaling gradient becomes

\[
\gamma_h = \frac{1}{8} \frac{A_1(kh)^{10} + A_2(kh)^8 + A_3(kh)^6 + (kh)^4 + A_5(kh)^2 + A_6}{\phi_{12}(2(kh)^4 + 10(kh)^2 + 75)^2}, \tag{3.26}
\]

where,
Figure 3.1. Approximate linear dispersion relationships, $C$, compared with linear Airy dispersion, $C_{St}$, well past the nominal deep-water limit of $kh = \pi$; (i) optimum basis functions for $N = 2$, $O(\mu^4)$ Padé [2,2] accurate, (ii) optimum basis functions for $N = 4$, $O(\mu^8)$ Padé [4,4] accurate, and (iii) optimum basis functions for $N = 4$, $O(\mu^{10})$ Padé [6,4] accurate.

\[ A_1 = 60\hat{\phi}_{12}^2 + 128\hat{\phi}_{12} + 64, \]  
\[ A_2 = 450\hat{\phi}_{12}^2 + 992\hat{\phi}_{12} + 480, \]  
\[ A_3 = 3000\hat{\phi}_{12}^2 + 6420\hat{\phi}_{12} + 3200, \]  
\[ A_4 = 5625\hat{\phi}_{12}^2 + 13200\hat{\phi}_{12} + 6000, \]  
\[ A_5 = 8250\hat{\phi}_{12}, \]  
\[ A_6 = -11250\hat{\phi}_{12}. \]  

The approximate shoaling gradient, equation (3.26), has a Taylor expansion of,

\[ \gamma_h = -\frac{1}{4} + \frac{1}{4} (kh)^2 + \left( \frac{1}{360} \frac{45\hat{\phi}_{12}^2 + 88\hat{\phi}_{12} + 48}{\hat{\phi}_{12}} \right) (kh)^4 + O((kh)^6), \]  

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as compared with the Taylor expansion for the exact solution, equation (3.24)

\[ \gamma_h = -\frac{1}{4} + \frac{1}{4} (kh)^2 - \frac{1}{18} (kh)^4 - \frac{1}{540} (kh)^6 + O((kh)^8). \] (3.34)

There are several choices of \( \hat{\phi}_{12} \) which can optimize the shoaling gradient for different properties: (i) setting \( \hat{\phi}_{12} = -6/5 \pm 2/15 \sqrt{7(21)} \approx -1.2 \pm 0.61 \) will cause the solution to match the Taylor expansion up to \( O(\mu^4) \), (ii) a choice of \( \hat{\phi}_{12} \in (-4/3, -4/5) \) will cause the limit of the approximate shoaling gradient to be positive, indicating that waves travelling from deep water will decrease in amplitude as they enter shallow water, and (iii) A choice of \( \hat{\phi}_{12} = -0.704720 \) will minimize the cumulative error in the shoaling gradient, equation (3.25), up to the nominal deep water limit of \( kh = \pi \).

The details of these three cases can be found in Table 3.1.

An analysis of equation (3.26) demonstrates that setting \( A_1 = 0 \), corresponding to \( \hat{\phi}_{12} = -4/3 \) or \( \hat{\phi}_{12} = -4/5 \), will not only cause the limit to be positive but will also cause the deep water limit to be bounded, which matches the properties of the deep water limit of the Airy solution. It is important to note that the choices of \( \hat{\phi}_{12} \) for both case (i) and case (iii) are outside of the range for which the deep water limit of the shoaling gradient is positive, which can lead to stability issues. Thus these two cases are not recommended.

Figure 3.2 shows the results a comparison of cases (i)–(iii) for the \( O(\mu^2) \) case. Given that case (ii) represents a range of values, we take \( \hat{\phi}_{12} = -4/3 \) in order to take advantage of the agreement with the deep water limit. As can be seen from the figure both the optimization for the infinite limit (ii) and the Taylor expansion optimization (i) provide similar results in terms of accuracy, however the \( \hat{\phi}_{12} = -4/3 \) case also has better deep to shallow water properties. Thus it is recommended that when optimizing shoaling for the \( O(\mu^2) \) case the choice of \( \hat{\phi}_{12} = -4/3 \) be used.

A similar analysis is conducted on the \( O(\mu^4) \) case, although the approximate
shoaling gradient is more complex. For simplicity the formula is not given here, however the properties will be discussed. After choosing basis functions such that the dispersion is optimized to a Padé [4,4] approximant, the Taylor expansion of the approximate shoaling gradient becomes

\[ \gamma_h = -\frac{1}{4} + \frac{1}{4}(kh)^2 - \frac{1}{18}(kh)^4 - \ldots \]

\[ -\frac{1}{151200} \left( \frac{22491\hat{\phi}_{13}^2 - 9555\hat{\phi}_{13} - 125}{21\hat{\phi}_{13} - 5} \right) (kh)^6 + O(\mu^8). \quad (3.35) \]

As with the \( O(\mu^2) \) case it is possible to define basis functions such that certain properties of the shoaling gradient are optimized: (v) a choice of \( \hat{\phi}_{13} = 35/102 \pm 5/714\sqrt{1245} = 0.343 \pm 0.247 \) will cause the solution to be formally accurate up to \( O(\mu^6) \); (vi) the deep water limit of the shoaling gradient will be bounded with \( \hat{\phi}_{13} = 10/19 \pm 10/399\sqrt{61} = 0.526 \pm 0.196 \); and finally (vii) \( \hat{\phi}_{13} = 0.60440 \) will minimize the cumulative shoaling error, equation (3.25), over the interval \( kh \in [0, \pi] \). The details of these three cases are shown in Table 3.2. We note that if \( \hat{\phi}_{13} \in [0.3306, 0.7720] \) then the limit of the shoaling gradient will be positive, all three of the optimization cases shown here fall within these bounds. Figure 3.3 shows a comparison between the Stoke’s solution for the shoaling gradient and each of the three optimization cases. All three cases show good accuracy up to the nominal deep water limit, with cases (v) and (vii) showing the closest agreement to the exact solution.

3.3 Nonlinear Properties

A perturbation expansion of the dependent variables, \( \eta = \eta^{(0)} + \delta\eta^{(1)} + O(\delta^2) \), is substituted into the model. First–order linear terms, \( O(\delta) \), will match the first–order terms from the shoaling analysis and describe the dispersion properties of the model. The second–order linear terms combined with first–order nonlinear terms, \( O(\delta^2) \), will
Figure 3.2. Approximate shoaling gradient, $\gamma_h$, compared with linear theory well past the nominal deep-water limit shown for the $N = 2$ case, using optimum basis functions for cases (i)–(iii), see Table 3.1. The exact solution is shown as the solid black line. Direct comparison of shoaling gradients (top panel) and cumulative shoaling error (bottom panel).
Figure 3.3. Approximate shoaling gradient, $\gamma_h$, compared with linear theory well past the nominal deep–water limit shown for the $N = 4$ case, using optimum basis functions for cases (v)–(vii), see Table 3.2. Direct comparison of shoaling gradients (top panel) and cumulative shoaling error (bottom panel).
be different than those of the shoaling analysis and can be compared with the well
known Stoke’s second–order harmonic [17]. See appendix C for details regarding the
derivation of the second order harmonic term.

Recall that in the implementation, all models $O(\mu^4)$ or greater are weakly non-
linear. As a result all solutions for $N = 2$ will be formally fully nonlinear, while
solutions for $N \geq 4$ will be weakly nonlinear. Figure 3.4 shows a comparison of the
ratios between the approximate second harmonic, $\eta^{(1)}$ and the Stoke’s second har-
monic, $\eta_{St}$, for varying cases of optimized values. Details for each case can be found
in Tables 3.1 and 3.2, for the $O(\mu^2)$ and $O(\mu^4)$ models respectively.

As with the case of shoaling, only one free–parameter, $\hat{\phi}_{12}$, is available at $O(\mu^2)$ to
optimize nonlinear properties. Although there are several approaches to optimizing
the nonlinear properties of the model, for brevity only one case is examined: (iv)
choosing $\hat{\phi}_{12} = -353/225$ optimizes the nonlinear accuracy making the formal ac-
curacy of the second harmonic $O(\mu^4)$. The top panel of figure 3.4 shows the ratios
of the approximate solution and the Stoke’s solution for the $O(\mu^2)$ implementation.
The first three cases, (i)–(iii), represent the three optimized shoaling cases previously
discussed. For these cases the formal accuracy of the approximate second harmonic
is $O(\mu^2)$. As can be seen in the figure all four cases have reasonable accuracy for very
shallow waves, but cases (i) and (iii) begin to diverge after $kh = 0.5$. Cases (ii) and
(iv) are able to maintain good agreement with the analytical solution throughout the
shallow water region. For all of these cases the dispersion has been optimized to a
Padé [2,2] approximant.

For the $O(\mu^4)$ case there are two free–parameters that allow for the optimization
of the solution. The first is $\hat{\phi}_{13}$, which can also be used to optimize shoaling, and
the second is $\hat{\phi}_{12}$, which only appears in the equation for the nonlinear solution. For
brevity, two possible approaches for optimizing the nonlinear properties are examined.
We first consider the optimal shoaling cases (v)–(vii) with $\hat{\phi}_{12} = -2144/1197 -
8/1425σ13 chosen such that the second harmonic is formally accurate up to $O(\mu^4)$. Case (viii) optimizes the nonlinear terms to $O(\mu^4)$ accuracy as well as makes the deep water limit close to zero with a choice of $\hat{\phi}_{31} = -1.790467$ and $\hat{\phi}_{12} = -0.1206296$. Lastly, case (ix) minimizes the shoaling and nonlinear error over the nominal shallow water range, $kh \in [0, \pi]$ with a choice of $\hat{\phi}_{31} = 0.60440$ and $\hat{\phi}_{12} = -1.709480$.

The bottom panel of Figure 3.4 show the ratios for the $O(\mu^4)$ implementation. Each of the five cases demonstrate good agreement with the Stoke’s solution in the shallow water regime, with cases (viii) and (ix) showing the highest accuracy at the nominal shallow water limit. All five cases remained stable for the numerical validation cases and showed excellent agreement in comparison with experimental/analytical results. For brevity only the results of simulations from case (viii) are included in chapter 5.
Figure 3.4. Ratio of approximate second harmonic, $\eta$, to Stoke’s second harmonic, $\eta_{St}$, for $N = 2$ case (top panel) and $N = 4$ case (bottom panel) using optimized dispersion, shoaling and nonlinearity, nonlinear cases (i)–(ix), see Tables 3.1 and 3.2.
CHAPTER 4

NUMERICAL DISCRETIZATION FOR 1-D MODEL

Two numerical formulations of the model are examined in this chapter. These approaches are as follows:

1. Finite difference approximations for the non–conservative SWE, as well as a finite difference method for the PPBOUSS equations.

2. Discontinuous Galerkin finite element approximation of the conservative SWE coupled with a finite difference approximation of the PPBOUSS equations.

Each of these methods provides certain strengths and challenges which will be discussed in greater detail in this chapter. Validations of each numerical approach are shown in chapter 5.

4.1 Fully Finite Difference Approach

Second–order central finite differences are used for the spatial derivatives and the time dependent hyperbolic component of the model is solved using a second order leap–frog scheme. Solutions to the pressure profile are found using a tri–diagonal solver for the elliptic problem for $O(\mu^2)$ and an LU solver for $O(\mu^4)$. In order to avoid the generation of high frequency spurious oscillations in the solution a one-dimensional Arakawa staggered C grid in space was used such that the pressure, $P$, and free–surface elevation, $\eta$, are defined at each node and the velocity, $u$, is defined at each half–node. Following the example of Zijlema et al. [89] the temporal domain is also staggered such that the velocity solution is offset in time from the free–surface and pressure solution by half a time step.
4.2 Mixed Finite Element/Finite Difference Approach

The governing equations are solved using a mixed finite element/finite difference scheme in space and an explicit third order Strong–Stability–Preserving Runge Kutta method in time. The modified SWE’s are solved using a second order DG finite element discretization with linear nodal basis functions. The elliptic pressure–Poisson problem is solved using second order central finite differences in space.

4.2.1 Computational Grid

The spatial domain is partitioned into \( N \) elemental domains,

\[
\Omega = \{ \Omega_i : \Omega_i \in [x_{i-1/2}, x_{i+1/2}], i = 1 \ldots N \},
\]

where the nodal values, \( x_i \), are located at each elemental center and the element face values, \( x_{i\pm1/2} \), are located at the elemental interfaces. In order to facilitate a simpler coupling paradigm between the DG finite element solution and the finite difference pressure–Poisson problem, the finite elements are taken to be structured. The finite difference nodes are defined at the elemental centers, \( x_i \) for \( i = 1 \ldots N \), such that the computational domain makes up an Arakawa staggered C grid in space.

4.2.2 Dynamic Pressure

For the \( O(\mu^2) \) model the system reduces to an equation of one variable, \( P_1 \). Each of the terms in the expansion of equation (2.80), can be found through the following pressure–Poisson problem,

\[
\hat{b}_{3,1,1} \nabla^2 P_1 + \hat{b}_{2,1,1} \cdot \nabla P_1 + \hat{b}_{1,1,1} P_1 = \hat{b}_{0,1}.
\]

Considering the optimum basis functions and weighting function as discussed in chapter 3, the coefficients become,
\[ \hat{b}_{3,1,1} = -\mu^2 \frac{1}{2} (h + \delta \eta)^2, \] (4.3)

\[ \hat{b}_{2,1,1} = -\mu^2 \frac{1}{12} (h + \delta \eta)(3\nabla h + 8\delta \nabla \eta), \] (4.4)

\[ \hat{b}_{1,1,1} = -\frac{1}{12} \left( -15 - 3\mu^2 (\nabla h) \cdot (\nabla h) - 6\mu^2 \delta \eta \nabla^2 h + 4\mu^2 \delta (\nabla h) \cdot (\nabla \eta) \\
- 8\mu^2 \delta^2 (\nabla \eta) \cdot (\nabla \eta) + 4\mu^2 \delta^2 \nabla^2 \eta + 2\mu^2 h(2\delta \nabla^2 \eta - 3\nabla^2 h) \right), \] (4.5)

\[ \hat{b}_{0,1} = \frac{1}{12} g \left[ 6\delta \mu^2 \eta^2 \nabla^2 h + 4\mu^2 h \left( 3(\nabla h) \cdot (\nabla \eta) - 2\delta (\nabla \eta) \cdot (\nabla \eta) + h \nabla^2 \eta \right) \\
\eta \left( 15 + 3\mu^2 (\nabla h) \cdot (\nabla h) + 8\delta \mu^2 (\nabla h) \cdot (\nabla \eta) + 2\mu^2 h(3\nabla^2 h + 2\delta \nabla^2 \eta) \right) \right]. \] (4.6)

Once \( P_1 \) has been determined then \( P_2 \) can be found explicitly with the following expression,

\[ P_2 = \frac{3}{4} \left[ g \left( 1 + (\nabla h) \cdot (\nabla h) \right) \eta - (h + \delta \eta)(\nabla h) \cdot (\nabla P_1) + \left( 1 + (\nabla h) \cdot (\nabla h) \right) P_1 \right]. \] (4.7)

Equations (4.2)–(4.7) are solved on a regular structured mesh, as described above, using a second order central difference method in space. Once \( P_1 \) and \( P_2 \) have been obtained they are used to inform the conservation of momentum, equation (2.81).

A similar technique is applied to determine the solution to the dynamic pressure forcing for the weakly nonlinear \( O(\mu^4) \) case, however for brevity, details are not shown here but may be found in Donahue et al. [20] or in section 2.2.3.

### 4.2.3 Nonlinear Shallow Water Equations

A DG finite element solution is used to discretize the spatial derivatives in the conservation of mass and momentum equations. A focus of this work is to generate a simple coupling paradigm between these equations and the pressure–Poisson
based dynamic pressure solution for application in coastal ocean circulation models. Many of the most popular coastal ocean circulation models employ some discontinuous framework for solving the conservation of mass and momentum equations. For example the Discontinuous Galerkin Shallow Water Equation Model (DG–SWEM) \[16, 39\] and the Finite Volume Coastal Ocean Model (FVCOM) \[11\] both solve the SWE’s on an unstructured triangle based grid using the DG finite element method and finite volume method respectively.

Recall that the conservation of mass and conservation of momentum are

\[
\begin{align*}
\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial Q}{\partial t} + \delta \nabla \cdot \left( Qu + \frac{1}{2} g \eta (\eta + h) \right) - \frac{1}{2} g \left( (h + \delta \eta) \nabla \eta - \nabla (h - \delta \eta) \eta \right) + \sum_{n=1}^{N} \mu \beta_n \left( (h + \delta \eta) G_n \nabla P_n - \phi_{n,q=0} P_n - \nabla (h + \delta \eta) R_n P_n \right) = \frac{g n \nabla h}{H^{1/3}},
\end{align*}
\]

where \( Q = u H \) is the flow discharge, \( H = (h + \delta \eta) \) is the total water column height, the bottom friction term is approximated by a Mannings \( n \) formulation and the other force terms \( F \) are neglected.

A finite element space \( V_h^k = \{ v : v|_{\Omega_i} \in P^k(\Omega_i), i = 1 \ldots N \} \) is defined, where \( P^k(\Omega_i) \) is the space of piecewise polynomials of degree of at most \( k \) defined over the element \( \Omega_i \). In this work, we only consider the case where \( k = 1 \). A weak formulation of the equations is found through multiplication of equations (4.8) and (4.9) by a test function \( w_h|_{\Omega_i} \) and integrating over the local elemental domain. Application of the integration by parts rule leads to the following local problem,
\[
\left( \frac{\partial H}{\partial t}, w_h \right)_\Omega - \left( Q, \frac{d w_h}{d x} \right)_\Omega + \left\langle Q, w_h \right\rangle_{\partial \Omega_i} = 0, 
\]
\[
\left( \frac{\partial Q}{\partial t}, w_h \right)_\Omega - \left( \delta r, \frac{d w_h}{d x} \right)_\Omega - \left( s, w_h \right)_\Omega + \left\langle r, w_h \right\rangle_{\partial \Omega_i} = 0, 
\]

where

\[
\begin{align*}
    r &= \delta u^2 H + \frac{1}{2} g \delta \eta^2 + gh \eta, \\
    s &= g \eta \frac{\partial h}{\partial x} - \mu \frac{g n^2 Q |Q|}{H^{1/3}} - \sum_{n=1}^{N} \mu \beta_n ((h + \delta \eta) G_n \frac{\partial P_n}{\partial x} - \phi_{n,q=0} P_n - \frac{\partial}{\partial x} (h + \delta \eta) R_n P_n). 
\end{align*}
\]

The exact solutions for \( H \) and \( Q \) are approximated by \( H_h, Q_h \in V_h^k \). Furthermore, the solutions are allowed to be discontinuous across the element edges located at \( x_{i-1/2} \) and \( x_{i+1/2} \). Thus the flux terms in each of the equations, \( \left\langle Q, w_h \right\rangle_{\partial \Omega_i} \) and \( \left\langle r, w_h \right\rangle_{\partial \Omega_i} \) can be multi-valued at the boundary. This problem is addressed through the introduction of single valued, monotone, numerical fluxes, \( \hat{Q} = \hat{Q}(Q_h^+, Q_h^-) \) and \( \hat{r} = \hat{r}(r_h^+, r_h^-) \), where,

\[
\begin{align*}
    (*)^+ &= \lim_{x \to x_{i-1/2}^+} (*), \\
    (*)^- &= \lim_{x \to x_{i-1/2}^-} (*), \\
    (*) &= \eta, u. 
\end{align*}
\]

In this work the local Lax–Friedrichs (LLF) flux was chosen,

\[
\hat{f}(a, b) = \frac{1}{2} \left( f(a) + f(b) - C(b-a) \right), 
\]

where \( C \) is chosen as the maximum eigenvalue of the flux Jacobian at the element
interface. Considering this definition of the value at each flux interface equations (4.10) and (4.11) can be rewritten as,

\[
\left( \frac{\partial H_h}{\partial t}, w_h \right)_{\partial \Omega_i} - \left( Q_h, \frac{dw_h}{dx} \right)_{\Omega_i} + \left\langle \dot{Q}, w_h \right\rangle_{\partial \Omega_i} = 0, \tag{4.17}
\]

\[
\left( \frac{\partial Q_h}{\partial t}, w_h \right)_{\Omega_i} - \left( \delta r_h, \frac{dw_h}{dx} \right)_{\Omega_i} - \left( s_h, w_h \right)_{\Omega_i} + \left\langle \hat{r}, w_h \right\rangle_{\partial \Omega_i} = 0. \tag{4.18}
\]

For the work here the DG basis functions and weighting functions were chosen to be Lagrange polynomials. All integrations were performed using Gauss–Legendre quadrature methods.

4.2.4 Finite Difference(PP)/Finite Element(SWE) Coupling

The finite difference mesh is constructed through the center points, \( x_i \), of each element in the finite element mesh. Coupling from the DG finite element solution to the finite difference pressure–Poisson solution is achieved through a projection of the free surface and velocity solutions onto the center point of each element.

\[
(\eta, u)_i = \sum_{j=1}^{k} \psi_j(x_i) (\hat{\eta}, \hat{u})_j. \tag{4.19}
\]

These element centered values are then used in the finite difference solution to the PP problem.

In the reverse direction, the finite difference solution to the PP equation is projected onto a nodal finite element framework through interpolation of the cell centered solutions onto the element edges and then treating these values as the prospective weights of a linear nodal finite element solution.
4.2.5 Discretization In Time

Following the spatial discretization of the non–pressure terms, a set of ordinary differential equations in time must be solved. In the FD/FE model we use a third order Strong–Stability–Preserving Runge Kutta method [63, 64]. The dynamic pressure profile is found for each stage of the Runge Kutta method and used to inform the solution at that stage.

4.3 Initial And Boundary Conditions

In cases where the initial conditions involve the generation of waves, the generating–absorbing sponge later technique developed by Zhang et al. [85] is used. The computational domain is broken up into three distinct sections. The first section represents a wave generation region of length $L_1$, the second section is of length $L_2$ and represents a computational zone, the third section of length $L_3$ represents a wave absorption region where the incoming waves are gradually damped. Figure 5.1a shows an example setup where the wave generation region is of length $L_1 = 5$, the computational area is of length $L_2 = 32$ and the wave absorption region is of length $L_3 = 10$.

In all cases the boundary conditions are taken to be reflective, thus the gradient of the pressure and free surface is zero and the normal velocity along the boundary is set to zero.

4.4 Treatment Of Nearshore Wave Physics

Now that the structure of the pressure–Poisson based Boussinesq–type model has been developed the next step is to consider the treatment of nearshore wave physics such as how the model captures the effect of a moving shoreline or the complex physics of breaking waves.
4.4.1 Treatment Of Wave Breaking

The free-surface solution is single valued and as such is incapable of representing the multi-valued surface that would be expected during a wave breaking event. As a result a significant amount of research has been conducted on approaches to handling breaking waves within the framework of Boussinesq-type models. One approach is to include a viscosity term in the conservation of momentum equations. The early work of Zelt [83] focused on detection of the wave breaking front through the gradient of the velocity which then triggered an artificial viscosity term in the momentum equations. Other approaches considered an eddy viscosity type term which, such as the work of Karambas et al. [31] and Kennedy et al. [33]. For example, in the approach of Kennedy et al. the breaking wave front was detected by considering the time rate of change in the free-surface, which then determines the location and magnitude of the eddy viscosity term.

A separate approach applied by Shäffer et al. [60] and extended to two dimensions by Madsen et al. [51] was built upon the detection of surface rollers in the solution through analysis of the steepness in the wave. The effect of the excess momentum due to a non-uniform velocity distribution in the presence of a breaking wave is represented by the inclusion of extra terms in the momentum equations.

More recently an approach that has become popular is to treat the breaking waves as nondispersive bores which are ideally solved by the SWE. This is achieved through temporarily suspending the dispersive terms in the momentum equations in the presence of a breaking wave. Tonelli et al. [75] proposed a relatively simple mechanism for detecting a breaking wave through an approximation to the Froude number based on the ratio of the wave height to water depth. Tissier et al. [73] extended the detection of the wave breaking criteria to consider the dissipation of energy across the wave front. Building on the work of Kennedy et al. [33], Smit et al. [65] determined the location of a breaking wave front based on the spatial
gradient of the free–surface. Kazolea et al. [32] combined the approaches of Kennedy et al. with the approach of Schäffer et al. [60] to construct a hybrid wave breaking detection mechanism. An advantage of these approaches is the relative simplicity of the implementation as well as the lack of a requirement to track a travelling wave front.

Treatment of the bottom stress term, $\tau_{zx}$, is approximated by the Manning’s $n$ formulation, where $n$ represents the Manning’s roughness coefficient. This work utilizes a relatively simple wave breaking detection method which takes advantage of the DG framework. Wave breaking is measured through analysis of the discontinuity across the elemental interface of free–surface solution. This approach is described in detail by Duran et al. [21]. A wave breaking detector is used, $T_j$, defined as follows,

$$T_j = \frac{\Delta^-|H_{j-1/2}^+ - H_{j-1/2}^-| + \Delta^+|H_{j+1/2}^+ - H_{j+1/2}^-|}{|\Omega_j||H_j|_\infty},$$

(4.20)

where,

$$\Delta^- = \begin{cases} 1, & \text{if } Q_{j-1/2}^+ \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad \Delta^+ = \begin{cases} 1, & \text{if } Q_{j+1/2}^- \leq 0, \\ 0, & \text{otherwise,} \end{cases}$$

(4.21)

and $|H_j|_\infty$ is the absolute maximum water column value over the element $j$. An element is considered to contain a breaking wave, if $T_j > 1$. In this case, the dispersive terms in the element are turned off, i.e. $\hat{P}_d = 0$ and $P_b = 0$. Furthermore, to avoid spurious oscillations due to sharp transitions from dispersive to non-dispersive elements a stencil of five elements to the left and to the right of the troubled element is chosen to create an eleven element range where dispersive terms are turned off.
4.4.2 Moving Shoreline

The moving shoreline algorithm follows the work of Bunya et al. [7]. This wetting and drying scheme was specifically designed to work well with DG finite element schemes and takes advantage of their local mass conservation properties. As such the scheme is capable of tracking a moving shoreline while preserving local and global mass conservation properties.

A minimum water level, $H_0$, is set to ensure that a thin layer of water exists throughout the element. Elements that do not have a water level higher than this threshold are considered dry. The minimum water level in a dry element is set to be parallel to the bathymetry and the gravitational constant is set to zero to eliminate nonphysical drainage of the element. Furthermore, the dynamic pressure within these elements is set to zero. The reader is directed to Bunya et al. [7] for details.
In this chapter the one-dimensional algorithms for the PPBOUSS model from the previous chapter are validated against analytical and experimental results. There are nine total validation test cases, the first four test cases use the finite difference/finite difference algorithm discussed in section 4.1. These are the linear wave propagation, regular wave generation accuracy, propagation of a solitary wave over a shelf and wavetrain travelling over a submerged shoal tests. The final five test cases use the mixed finite element/finite difference algorithm discussed in section 4.2. These are the solitary wave runup on a simple beach, laboratory comparison for runup, runup of a solitary wave on a sloping beach, plunging and spilling breakers and solitary waves travelling over a breakwater test cases.

All solitary wave initial conditions are imposed on the model at $t = 0$ and follow the equation for a solitary wave given by,

\begin{align}
\eta_{t=0} &= A \text{sech}^2(\kappa(x - x_0)), \\
{u}_{t=0} &= c \left( 1 - \frac{h_b}{\eta_{t=0} + h_b} \right),
\end{align}

(5.1)
(5.2)
(5.3)
where,

\[ \kappa = \frac{\sqrt{3A}}{2hb\sqrt{A + hb}}, \quad (5.4) \]

\[ c = \sqrt{g(h_b + A)}, \quad (5.5) \]

and \( A \) represents the solitary wave height, \( x_0 \) is the \( x \)-coordinate of the solitary wave peak and \( h_b \) is the depth of the water at the solitary wave location.

5.1 Linear Wave Propagation

A simple case of the propagation of a small amplitude monochromatic wavetrain is examined. Waves of wavelength \( L = 5h \), where \( h \) is the depth, are generated in the source region and allowed to propagate downstream. The spatial domain setup follows the diagram shown in Figure 5.1a. As seen in Figure 5.1b, the waves initially generated in the source region are accurately propagated downstream. Upon reaching the sponge layer region the wave-absorption is seen as the waves gradually decrease in amplitude until eventually reaching an amplitude of nearly zero. It should be noted that in this case the boundary conditions are treated as reflective. However, given the strength of the wave generation–absorption method, any reflections that occur at the boundaries are damped by the sponge layers.

5.2 Regular Wave Generation Accuracy

In order to evaluate the accuracy of the wave generation-absorption a set of model runs were conducted for various degrees of nonlinearity. The model domain is the same as the test case described in section 5.1. A wave of length \( L = 5h \) is generated in a channel of depth \( h \), corresponding to \( kh = 1.25664 \) and \( T \sqrt{g/h} = 6.07898 \). In all cases the integrated sponge strength is \( \tilde{\omega}_1 = 10\sqrt{gh} \) and the sponge is quadratically varying, see Zhang et al. [85] for more details regarding the sponge layer setup. The
wavetrain is allowed to propagate for 15 wave periods before a record is made of the solution in a range of length $20h$ located at a distance of $5h$ from the edge of the wave generation region and $7h$ from the edge of the wave absorption region. At each spatial domain point the wave heights are averaged over five wave periods. The maximum and minimum wave heights over the recorded region are then compared with the imposed wave height from the wave generation region. Figure 5.2 shows the results of this analysis, for both the fully nonlinear $O(\mu^2)$ model, hereafter referred to as the PPBOUSS2 model, and the weakly nonlinear $O(\mu^4)$ model, hereafter referred to as the PPBOUSS4 model.

The measured wave heights compare very well with the imposed wave heights for small amplitude waves. As the amplitude is increased, nonlinear effects become more prominent and the error in the measured wave heights increases. Given the higher order of accuracy inherent in the PPBOUSS4 model, the error observed, even
at strongly nonlinear waves, is relatively low. The imposed wave in the generation layer was of first order and we would expect the errors to increase at a quadratic rate and this is observed in these results.

It was found through the course of these tests that the size of the generation and absorption layer had an impact on the level of accuracy in the downstream propagated wave. Figure 5.3 shows the results of testing sponge layers of different lengths, $S$, using the PPBOUSS2 model. Three wave amplitudes were examined, a small wave amplitude, $H_{imp} = 0.01h$, serves as a reference, since it would be expected that such small amplitude waves would be accurately captured. It can be seen in the figure that for the smaller of the sponge lengths the small wave amplitude is accurately captured to within 0.5% and as the sponge length increases the solution approaches a steady solution that is very accurate. As the imposed wave amplitude is increased to $H_{imp} = 0.1h$ and $H_{imp} = 0.2h$ respectively a similar pattern is observed. The smaller sponge lengths are capable of capturing the solution to within 2% of the imposed amplitude, but as the sponge lengths are increased the accuracy of the solution increases to within 0.2%. A similar phenomenon was observed by Dommermuth [19] for the gradual introduction of nonlinear interactions over time. Due to the nature of the sponge layer generation technique, the larger the sponge length the greater the distance over which the linear imposed wave dominates, allowing for a gradual introduction of the nonlinear interactions into the system.

Clear improvement for all three test cases is observed simply by increasing the length of the sponge layer from $S/h = 5$ to $S/h = 10$. In the case of an imposed wave height of $H_{imp} = 0.2$ the accuracy of the downstream wave improves from a maximum error of 1.8% to a maximum error of 0.6%, a similar observation is made for the minimum measured wave heights.
Figure 5.2. Maximum and minimum wave heights recorded over a 20m sample area for varying imposed wave heights, $H_{imp}$. Results are shown for the fully nonlinear $O(\mu^2)$ solution, solid red line with circles, and for the weakly nonlinear $O(\mu^4)$ solution, dashed blue line with triangles.

Figure 5.3. Maximum and minimum wave heights recorded over a 20m sample area for varying sponge lengths, $S$. All simulations shown were conducted using the fully nonlinear $O(\mu^2)$ model. Three imposed wave heights are shown, $H_{imp}/h = 0.01$: red solid line with circles, $H_{imp}/h = 0.1$: blue dashed line with triangles, and $H_{imp}/h = 0.2$: magenta dash-dot line with squares.
5.3 Propagation Of A Solitary Wave Over A Shelf

As a first test of the model’s ability to predict nonlinear wave transformation, shoaling of a solitary wave is examined. The domain is broken into three regions; an initial level of depth $h_s$ and length $25h_s$, a plane slope of length $10h_s$ which reaches a shelf depth of $h_t$, and a final level of depth $h_t$ and length $215h_s$. A solitary wave of height $0.12h_s$ propagates from the left hand boundary towards the shelf. Figure 5.4a shows a schematic of the system. All numerical simulations were conducted with a spatial resolution of $\Delta x = 0.01$. A CFL number of 0.5 was chosen to maintain a low truncation error in the temporal solution while ensuring relatively good computational efficiency. The solution was allowed to propagate a sufficient amount of time to ensure that the leading soliton would reach a steady state near the right edge of the domain.

Johnson [30] proved that according to KdV theory the solitary wave will break up into a finite number of solitons upon traveling over the shoal and that the number of solitons can be predicted by the ratio of the shelf depth to the initial depth, $h_t/h_s$. The results for the numerical model given three separate shelf depths are explored here; $h_t/h_s = 0.6137$, $h_t/h_s = 0.5$ and $h_t/h_s = 0.4510$. The first and third of these shelf depths are eigendepths of the KdV equation and according to theory the solitary wave should break up into two distinct solitons for the former case and three distinct solitons for the latter case. The second case was chosen to facilitate comparison with the Boussinesq-type solution of Madsen and Mei [47]. A time evolution of the results for each of the three cases is shown in Figures 5.4b–5.4d. In Figure 5.4b at the final time step it is observed that the solitary wave has fissioned into two solitons that have propagated far downstream of the shelf, in agreement with the theory. Similarly, in Figure 5.4d it is observed that the solitary wave breaks down into three solitons. It is noted that in all cases an oscillating wave is observed trailing the primary soliton train, this is a result of a deviation of the model from KdV theory. The amplitude of
TABLE 5.1

COMPARISON OF WAVE HEIGHT RESULTS FOR THE CASE OF A
SOLITARY WAVE PROPAGATING OVER A SHELF.

<table>
<thead>
<tr>
<th>$h_t/h_s$</th>
<th>PPBOUSS2</th>
<th>PPBOUSS4</th>
<th>LPA ($n = 7$)</th>
<th>GN</th>
<th>KdV</th>
<th>Bouss.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6137</td>
<td>0.1779</td>
<td>0.1772</td>
<td>0.1745</td>
<td>0.168</td>
<td>0.181</td>
<td>–</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2029</td>
<td>0.2010</td>
<td>0.1988</td>
<td>0.184</td>
<td>0.207</td>
<td>0.20</td>
</tr>
<tr>
<td>0.4510</td>
<td>0.2167</td>
<td>0.2137</td>
<td>0.2120</td>
<td>0.190</td>
<td>0.220</td>
<td>–</td>
</tr>
</tbody>
</table>

this wave is relatively small when compared with the amplitudes of the main soliton train.

Verification of the numerical results for both the PPBOUSS2 model and the PP-BOUSS4 model is conducted through comparison of the leading soliton height with the heights predicted by KdV theory as well as the numerical results reported for the LPA model [34], Green–Naghdi (GN) restricted Theory 1 [22] and the previously mentioned numerical Boussinesq–type (Bouss.) solution [47], these results can be found in Table 5.1. The Green–Naghdi, KdV and Boussinesq models reported here have a relatively low level of approximation and are not expected to perform as well when the waves become highly nonlinear. The LPA model has been demonstrated to be very accurate for cases of highly nonlinear waves and is considered to be the most accurate prediction for the purposes of comparison. Both the PPBOUSS2 and PPBOUSS4 results compare well with the LPA model, and as expected the higher order accuracy in the PPBOUSS4 model provides results more closely aligned with the LPA results.
Figure 5.4. Evolution of a solitary wave travelling over a shelf. The final time solution is shown with a bold line, the dotted line designates the height of the leading soliton.
5.4 Wavetrain Travelling Over A Submerged Shoal

Validation of dispersive accuracy of the model is conducted through comparison of numerical results with experimental results collected by Beji and Battjes [4] on the effect of a wave train travelling over a submerged shoal. These experimental results have been used in many cases to validate a range of Boussinesq-type models [15, 25, 58, 82, 84]. Figure 5.5 illustrates the experimental setup. A wave train is travelling from the left of the domain to the right in a channel of 0.4m depth. The incoming wavetrain has a wave height of $H_0 = 0.020m$ and a wave period of $T = 2.02s$, resulting in $kh = 0.67$. The upstream slope of the submerged shoal is 1 : 20 and begins 6m into the domain, the downstream slope of the shoal is 1 : 10 and begins 14m into the domain. The shoal has an overall height of 0.3m and length of 11m, see Dingeman [18] for more details regarding the experimental setup. The experimental data set is composed of time histories for free-surface elevation at ten locations throughout the domain.

A quantitative assessment of the agreement between the model and the experimental results is conducted using the index of agreement, $d$, proposed by Wilmott [80] and later used by Gobbi et al. [25] to assess the accuracy of WKGS, WN4 and FN4 models. The formula for $d$ is as follows,

$$d = 1 - \frac{\sum_{j=n_1}^{n_2} [y(j) - y_d(j)]^2}{\sum_{j=n_1}^{n_2} [|y(j) - \bar{y}_d| + |y_d(j) - \bar{y}_d|]^2}$$

(5.6)

where the points $n_1$ and $n_2$ are the bounds covering one full wave period, $y_d(j)$ represents measured data used for comparison, $y(j)$ represents predicted values from the model and $\bar{y}_d$ is the average value of the observed data over one wave period. All simulations were conducted on a mesh spanning from $-15m$ to $30m$, with a grid spacing of $\Delta x = 0.025m$ and $\Delta t = 0.00025s$. A small $\Delta t$ value was chosen in order to facilitate a better match with the time history of the experimental data. Due to
the sharp corners in the bathymetry, which cause discontinuities in the bathymetric derivatives, a five-point moving average filter was used with ten iterations. The smoothing was relatively minor at this grid scale and did not significantly alter the bathymetry.

Comparison of the numerical and experimental results are illustrated in Figures 5.6 and 5.7 for the PPBOUSS2 and PPBOUSS4 models respectively. A comparison of the index of agreement is given Table 5.2 along with the values reported by Gobbi et al. [25] for the fully nonlinear $O(\mu^2)$ WKGS model and the two $O(\mu^4)$ models, WN4 (weakly nonlinear) and FN4 (fully nonlinear) and values for the fully nonlinear $O(\mu^2)$ and weakly nonlinear $O(\mu^4)$ fully rotational Green–Naghdi Boussinesq–type model developed by Zhang et al. [84], Z2 and Z4 respectively.

Upstream of the shoal the wave propagation is relatively monochromatic and serves as a verification that the sponge layer wave generation boundary condition is producing the appropriate wave train, subfigures (a) and (b). As the wave train approaches the shoal the wave height begins to increase, a reflection of the shoaling dynamics. In addition the peaks become sharper indicating the separation of the wave energy into higher frequency waves, as seen in subfigures (c)–(e). Agreement between both the PPBOUSS2 and PPBOUSS4 models and the experimental data is very good in this region and on the same order as the models developed by Gobbi and Zhang. Once the wave train has passed over the shoal the dynamics of the problem become considerably more complex and the need for higher–order dispersion becomes more pronounced. The system now exhibits waves of many frequencies travelling at varying speeds, in addition the nonlinear interactions between the various wave frequencies becomes more prominent, as seen in subfigures (f)–(j). It is in this region that the benefit of added accuracy for dispersion, shoaling and nonlinear effects can be observed. The PPBOUSS2 model does a poor job capturing the downstream solution, however the PPBOUSS4 model does remarkably well. The index of agreement
TABLE 5.2

COMPARISON OF $d$ VALUES*

<table>
<thead>
<tr>
<th>Gauge Location (m)</th>
<th>PPBOUSS2</th>
<th>WKGS</th>
<th>Z2</th>
<th>PPBOUSS4</th>
<th>FN4</th>
<th>WN4</th>
<th>Z4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream of the shoal</td>
<td>2</td>
<td>0.998</td>
<td>0.998</td>
<td>0.988</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.992</td>
<td>0.996</td>
<td>0.973</td>
<td>0.992</td>
<td>0.996</td>
<td>0.996</td>
</tr>
<tr>
<td>On the shoal</td>
<td>10.5</td>
<td>0.995</td>
<td>0.995</td>
<td>0.983</td>
<td>0.995</td>
<td>0.995</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>12.5</td>
<td>0.996</td>
<td>0.999</td>
<td>0.983</td>
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<td>0.992</td>
<td>0.992</td>
<td>0.996</td>
<td>0.980</td>
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<tr>
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<td>0.972</td>
<td>0.994</td>
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<td>0.972</td>
</tr>
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<td>0.973</td>
<td>0.964</td>
<td>0.995</td>
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<td>0.927</td>
<td>0.902</td>
<td>0.997</td>
<td>0.993</td>
<td>0.962</td>
</tr>
</tbody>
</table>

*See Equation (5.6) for stations downstream of the shoal for the PPBOUSS4 model matches the fully nonlinear FN4 model and surpasses the weakly nonlinear WN4 and Z4 models.

5.5 Solitary Wave Runup On A Simple Beach

One of the simplest test problems for the verification of wetting and drying is the simulation of the interaction between a solitary wave and a simple beach with a slope angle of $\alpha$ (Figure 5.8a).

The analytical solution of a non–breaking solitary wave on a sloping beach was explored by Synolakis et al. [70, 69] who built upon the theory of Carrier and Greenspan [8] for analytical solutions to the SWE’s for non–breaking waves on a sloping beach.
Figure 5.5. Experimental setup for wave transformation over a submerged shoal, gauge locations are listed along the top axis, all units are in (m).

The Carrier–Greenspan solution provides an excellent test case to assess the accuracy of a model’s implementation of wetting and drying. The model is compared with the analytical Carrier–Greenspan solution for a solitary wave of initial amplitude to depth ratio of $A/h = 0.0185$ propagating towards a beach with a slope of $\cot(\alpha) = 1/19.85$.

The computational domain is broken into elements with a length of $\Delta x = 0.05$ and the simulation was run with a CFL condition of 0.1 to ensure that spatial errors dominated. The analytical solution is derived without considering bottom friction, thus for this simulation the Manning’s $n$ value is set to zero, the wetting/drying criteria was set to $H_0 = 0.00001$.

Figure 5.9 shows the results of the simulation for the entire domain at seven times. The figure demonstrates that the model performs very well at capturing the propagation of the solitary wave and the wetting and drying front. Figure 5.10 shows the time history of the model and the analytical solution for two locations in the domain, $x/h = 9.95$ and $x/h = 0.25$. Note that the location at $x/h = 0.25$ is initially wet and then undergoes a transition from wet to dry as the solitary wave propagates up the beach and then recedes. This is accurately represented in the model results.
Figure 5.6. Comparison for PPBOUSS2 simulation on Beji and Battjes [4] Case (A), between the observed data (circles) and modeled data (line), \( d \) represents the index of agreement, see equation (5.6).
Figure 5.7. Comparison for the PPBOUSS4 simulation on Beji and Battjes [4] Case (A), between the observed data (circles) and modeled data (line), \( d \) represents the index of agreement, see equation (5.6). Numerical implementation using case (viii), see Table 3.2.
supporting the argument that the model captures both the propagation of the solitary wave and the wetting/drying front. The comparison of the analytical solution and model results is nearly perfect.

5.6 Laboratory Comparison For Runup Of Solitary Wave On A Simple Beach

A set of two experiments conducted by Synolakis [70, 69] in a basin of dimension (31.73m long, 0.6096m deep and 0.3937m wide) provide the basis for validation of the model for the propagation of both breaking and nonbreaking waves. Two solitary waves propagate towards a sloping beach with a slope of \( \cot(\alpha) = 1/19.85 \). The first experiment considered a solitary wave with an amplitude of \( a_0/h = 0.0185 \), which is not expected to break on the approach to the beach. The second experiment considered a solitary wave with a considerably higher amplitude of \( a_0/h = 0.30 \), which would be expected to break at the beach. This test case further validates the ability of the model to simulate the propagation of solitary waves in the nearshore and capture both wave breaking and wave runup.

The computational domain is discretized into elements with a length of \( \Delta x = 0.05 \) and the simulation was conducted using a CFL condition of 0.1 to ensure that spatial errors dominate. A Manning’s \( n \) value of 0.004 was used for each simulation to approximate the bottom friction of the experimental setup, with a wetting/drying criteria of \( H_0 = 0.0005 \).

Figures 5.11 and 5.12 show a comparison between the model and the data for the nonbreaking and breaking waves respectively. In both cases the model performs very well, capturing both the breaking and wetting/drying dynamics. A minor discrepancy can be seen in Figure 5.11 at \( t = 70 \) during the drying process. The model appears to dry more quickly than the measurements show, however the difference is relatively minor and the overall dynamics of the system are accurately captured.

Similarly the breaking solitary wave case is well represented by the model. There is
Figure 5.8. Computational and Experimental Domain Setups

(a) Solitary wave on a simple beach. Station locations are at a) $x/h = 9.95$ and b) $x/h = 0.25$.

(b) Ting and Kirby breaking waves experiments. Measurement locations are shown by stations $s_1$–$s_7$.

(c) Hsiao et al. experimental setup, stations $s_1$–$s_{11}$ are shown with dashed lines.

slight phase lag in the initial wetting front, see Figure 5.12 at $t = 25$. However, in the next panel, for $t = 30$, it is observed that wetting front represents the data very well. These discrepancies can be explained by the implementation of the wetting/drying algorithm due to the sensitivity of the choice of wetting threshold $H_0$. 
Figure 5.9. Comparison of model solution with the Carrier–Greenspan analytical solution for solitary wave runup on a simple beach [70, 69]. Each panel shows the spatial comparison of the two solutions at a different timesnap. Gray circles: analytical, o; Solid red line: $O(\mu^2)$ model, -; Blue dashed line: $O(\mu^4)$ model, - -.
Figure 5.10. Comparison of model solution with Carrier–Greenspan analytical solution for solitary wave runup on a simple beach [70, 69]. The top panel shows the time history for a location near the beginning of the beach, $x/h = 9.95$. The bottom panel shows the time history for a location near the still water beach front, $x/h = 0.25$. The flat region in the bottom panel between $t \sqrt{g/h} = 60$ and $t \sqrt{g/h} = 80$ represents a period when the location was dry. Colors are the same as for Figure 5.9.
Importantly, both the $O(\mu^2)$ and $O(\mu^4)$ cases produce fairly similar results. Outside of the breaking zone this demonstrates how accurate the $O(\mu^2)$ model is at capturing the propagation of solitary waves. Within the breaking zone nearshore the dispersive terms have been turned off as per the breaking wave criteria in the model and both simulations are solving the same shallow water equations.

5.7 Runup Of Solitary Wave On A Sloping Beach

In addition to the three cases previously discussed, Synolakis et al. [70, 69] tested the theory behind the maximum runup of solitary waves on a sloping beach for numerous breaking and non-breaking cases. Synolakis experimentally tested 77 cases for varying ratios of wave height to water depth ($a_0/h$). Each experiment was run for still water levels of varying depths. For each case the ratio of maximum level of runup to depth was recorded ($R/h$). In order to test the ability of the model to simulate the runup of solitary waves for both breaking and non-breaking cases, all 77 cases were simulated for a beach of slope $\cot(\alpha) = 1/19.85$. The computational domain divided into elements of size $\Delta x/h = 0.1$ with a CFL condition of 0.2 to ensure that spatial errors dominate. A Manning’s $n$ value of 0.004 was chosen to approximate the bottom friction associated with the experimental setup. Considering that each of the 77 cases represented solitary waves of varying amplitude, the wetting/drying criteria was set to 0.0001 ($m$).

Figure 5.13 shows a comparison of the recorded runup level versus the simulated runup level. According to the analysis conducted by Synolakis et al. [69], solitary waves with an initial amplitude to depth ratio of $a_0/h > 0.055$ became breaking waves upon approaching the shore. The comparison of the maximum runup for the 77 cases shows good agreement between the model and the recorded data, with a good correlation coefficient of $R^2 = 0.8495$ for the $O(\mu^2)$ case and a significant improvement for the $O(\mu^4)$ case with $R^2 = 0.9006$ determined by calculating the
Figure 5.11. Comparison of model and experimental results for runup of a non-breaking solitary wave on a simple beach [70, 69]. Gray circles: data, o; Solid red line: $O(\mu^2)$ model, -; Blue dashed line: $O(\mu^4)$ model, - -.
Figure 5.12. Comparison of model and experimental results for runup of a breaking solitary wave on a simple beach [70, 69]. Gray circles: data, o; Solid red line: $O(\mu^2)$ model, -; Blue dashed line: $O(\mu^4)$ model, - -.
correlation between the measured and modelled data. The modelled runup is accurate for both the breaking and nonbreaking regime, however it is clearly less accurate when the waves are expected to break. This is illustrated in greater detail by the observation that for just the waves in the nonbreaking zone the correlations for the $O(\mu^2)$ and $O(\mu^4)$ are $R^2 = 0.9724$ and $R^2 = 0.9764$ respectively. In the breaking regime the correlations drop to $R^2 = 0.5613$ and $R^2 = 0.7107$ for the $O(\mu^2)$ and $O(\mu^4)$ models respectively. This can be attributed to the approximate nature of the wave breaking criteria. Although a travelling bore representation of a breaking wave is fairly accurate, it does not capture all of the three-dimensional dynamics in the breaking zone.

5.8 Plunging And Spilling Breakers

To further test the model’s ability to capture the dynamics of breaking wave in the nearshore, we compare model results to experiments conducted by Ting and Kirby [71, 72]. In the previous cases only a single solitary wave is present, in this test case a train of regular nonlinear waves is generated on the left hand side of the domain which propagate towards the shore. This testcase allows us to test not only the ability for the model to resolve complex nonlinear waves, but to also simulate the train of breaking waves as well as the interaction between the wave train and wave energy reflected off the beach.

Ting and Kirby ran a series of experiments in a basin with dimensions of (40 m long, 0.6 m wide and 1.0 m deep) to examine regular waves breaking on plane slopes. Both plunging [71] and spilling [72] breakers were considered and data was collected at a series of stations throughout the domain. Regular nonlinear waves were generated in a water depth of 0.4m and propagated downstream to a plane beach with slope $\cot(\alpha) = 1/35$, see Figure 5.8b. In the present model, the wave train is generated using the sponge layer generation/absorption technique of Zhang.
Figure 5.13. Comparison of depth normalized wave runup for seventy-seven solitary wave runup experiments conducted by Synolakis et al. [70, 69]. The breaking threshold is set to $A/h = 0.055$. Model predicted runup (dashed black or red line) for $O(\mu^2)$ and $O(\mu^4)$ respectively, experimental measurements (circles).
et al. [85]. A Fourier decomposition of the regular nonlinear waves is calculated using the techniques of Fenton [23]. In the present work an approximation of the regular nonlinear wave train uses 32 Fourier modes.

The details for each case are as follows,

1. Plunging Breaker, $H/h = 0.32$, $T/\sqrt{gh} = 2.524$
2. Spilling Breaker, $H/h = 0.3125$, $T/\sqrt{gh} = 1.01$

where $T$ is the period of the wave in seconds, $h$ is the offshore still water depth and $H$ is the overall wave amplitude. The computational domain was broken into elements of length $\Delta x = 0.05$ to ensure that higher frequency modes in the Fourier decomposition were accurately represented. A CFL condition of 0.2 was chosen to ensure that spatial errors dominate. Considering the amplitudes of the incident waves, the wetting/drying threshold was set to $H_0 = 0.00001$ and a Manning’s $n$ value of 0.01 was chosen to represent the bottom friction of the experiment.

At each measurement location, see Figure 5.8b, Ting and Kirby performed the experiments several times in order to measure the velocity at varying depths. In addition, for each experiment, a free–surface displacement was similarly recorded. Due to this, the experimental data at measurement location has a set of 20–30 distinct time histories for the free–surface. Here, we compare the mean water level of the numerical simulations with the mean water level of each dataset. Given the volume of datasets for each measurement location the numerical simulation is compared with the minimum and maximum mean water levels at each data point. The model is compared against the mean water surface displacement over all experiments for the $O(\mu^2)$ and $O(\mu^4)$ simulations.

Figures 5.14 and 5.15 show the model timeseries comparison for the spilling type breaker and Figures 5.16 and 5.17 show the model timeseries comparison for the plunging type breaker. Figures 5.18a and 5.18b show a comparison at each station.
for the average wave crest, mean water level and average wave trough elevations for
the $O(\mu^2)$ and $O(\mu^4)$ simulations respectively.

As can be seen from the figures, the model is capable of capturing the propagation
of the regular nonlinear waves very well, and in each case is able to represent
the breaking dynamics, including the high frequency, low amplitude waves that are
present due to reflection of the wave off the sloped beach. For the direct timeseries
comparison of the spilling breaker case as shown in Figures 5.14 and 5.15, the model
results are nearly identical to the mean water level over all experiments. There are
minor discrepancies at wave crests and troughs which can be explained by the nu-
merical technique to capture wave breaking. A more diffusive breaking model would
provide a smoother breaking front. This reasoning can also be used to explain the
discrepancies observed in the wave peaks between $x = 6 m$ and $x = 8 m$, see Figures
5.18a and 5.18b.

5.9 Solitary Waves Travelling Over A Breakwater

Hsiao et al. [29] performed a series of experiments to simulate the interaction
of solitary type waves with an impermeable seawall. The experiments took place at
the Tainan Hydraulics Laboratory (THL) at the National Cheng Kung University,
Taiwan. Solitary waves were generated in a two–dimensional wave flume with di-
mensons of (22 m long, 0.5 m wide and 0.75 m deep). A simple sloping beach made
of aluminum, with a Plexiglas surface and a slope of $\cot(\alpha) = 1/20$ was situated at a
distance of 10 m from the wave generation paddle. A separate Plexiglas trapezoidal
section with a seaward slope of 1 : 4 and a landward slope of 1 : 1.8 was placed on
the beach to represent a seawall. See Figure 5.8c for a schematic of the initial setup.

This test case is well suited for determining how well this model can simulate
the overtopping of nearshore structures. The computational domain was divided into
elements of length $\Delta x = 0.025$, the simulation was run using a CFL condition of 0.1,
Figure 5.14. Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free-surface displacement to experimental measurements for a spilling breaker regular nonlinear wave train on a simple beach at station locations $s_1$–$s_4$ [72]. The shaded area represents the data range for all experiments performed.
Figure 5.15. Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free-surface displacement to experimental measurements for a spilling breaker regular nonlinear wave train on a simple beach at station locations $s_5$–$s_7$ [72]. The shaded area represents the data range for all experiments performed.
Figure 5.16. Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free–surface displacement to experimental measurements for a plunging breaker regular nonlinear wave train on a simple beach at station locations $s_1$–$s_4$ [71]. The shaded area represents the data range for all experiments performed.
Figure 5.17. Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free-surface displacement to experimental measurements for a plunging breaker regular nonlinear wave train on a simple beach at station locations $s_5$–$s_7$ [71]. The shaded area represents the data range for all experiments performed.
Figure 5.18. Comparison of mean water level (magenta circles), mean peak wave height (blue triangles) and mean trough depth (red squares) for the spilling and plunging breaker data and model simulations. Model simulations are shown using a solid line and dashed line for the $O(\mu^2)$ and $O(\mu^4)$ models respectively.
a Manning’s $n$ value of 0.01 and a wetting/drying criteria of $H_0 = 0.00001$.

Three solitary type waves were tested, as shown in Table 5.3. Figures 5.19 through 5.24 show the results of the laboratory measurements compared with model predictions at stations with locations given in Table 5.4. For stations on the seaward side of the seawall and on top of the seawall the model performs very well, predicting the advancing breaking wave as well as the appropriate wetting and drying front, stations $s_1$–$s_7$. The solution fidelity breaks down on the landward side of the seawall, stations $s_8$–$s_{11}$. The model accurately predicts overtopping and inundation of the landward side. Similarly it does a good job of capturing the overall landward side flooding state after some time has passed. The model is unable, however, to capture the full dynamics of the wave overtopping. This can be attributed to the fact that the model is not fully three-dimensional, and thus not capable of capturing the multi–valued surface that occurs when the wave crashes into the seawall, causing water to spray forward.

Although the model is incapable of capturing all of the fine scale processes that occur when a wave interacts with a barrier, it is noted that the model does accurately capture the overall dynamics of wave breaking and inundation of the beach on the landward side of the seawall.
Figure 5.19. Comparison of modeled free–surface displacement and experimental measurements for case 1 of solitary waves interacting with an impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
Figure 5.20. Comparison of modeled free–surface displacement and experimental measurements for case 1 of solitary waves interacting with a impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
Figure 5.21. Comparison of modeled free-surface displacement and experimental measurements for case 2 of solitary waves interacting with a impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
Figure 5.22. Comparison of modeled free–surface displacement and experimental measurements for case 2 of solitary waves interacting with an impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
Figure 5.23. Comparison of modeled free–surface displacement and experimental measurements for case 3 of solitary waves interacting with a impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
Figure 5.24. Comparison of modeled free–surface displacement and experimental measurements for case 3 of solitary waves interacting with a impermeable seawall, see Table 5.4. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data.
### TABLE 5.3

**EXPERIMENTAL SETUP FOR THE THREE SOLITARY WAVES IN HSIAO ET AL. [29].**

<table>
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<tr>
<th>Case</th>
<th>$h_0$ (m)</th>
<th>$H_0$ (m)</th>
<th>$\epsilon = H_0/h_0$</th>
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<td>0.35</td>
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<td>2</td>
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### TABLE 5.4

**STATION LOCATION FOR HSIAO ET AL. [29].**

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<th>Location</th>
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<tr>
<td></td>
<td>2</td>
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<td>Seaward of seawall</td>
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<td>10</td>
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<tr>
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<tr>
<td></td>
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<td>11.57</td>
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</table>
The main goal of this work is to develop a Boussinesq–type model which is capable of coupling with an ocean circulation model in a straightforward manner with a minimal amount of manipulation of the underlying code. To this point we have developed the framework of the model and validated it in one horizontal dimension. This chapter will discuss the coupling paradigm for incorporating the PPBOUSS model within the framework of the Discontinuous Galerkin Shallow Water Equation Model (DGSWEM) \[16\]. DGSWEM is a discontinuous–Galerkin based finite element model which solves the shallow water equations in conservative form on an unstructured triangular mesh.

6.1 Continuous Galerkin PPBOUSS

In order to take advantage of the unstructured mesh used in the DGSWEM code, a finite element solution to the PPBOUSS equations is developed. The pressure–Poisson component of the PPBOUSS model is an elliptic equation and thus a continuous–Galerkin (CG) finite element approach is ideally suited to solve these equations. In this section we will develop the framework for solving the PPBOUSS system using a CG finite element approach. In this current work we only focus on the \(O(\mu^2)\) system, leaving the \(O(\mu^4)\) model to future work. Throughout this work the optimum basis
functions are used such that,

\[
\phi_2 = -\frac{4}{3}(1 - q) + (1 - q^2) \tag{6.1}
\]

\[
W_1 = -\frac{4}{3} + q. \tag{6.2}
\]

Recall that the Green–Naghdi type approximation to the pressure can be written as follows,

\[
P = g(\eta - z) + (1 - q)P_1 + \mu^2 \left( -\frac{4}{3}(1 - q) + (1 - q^2) \right) P_2. \tag{6.3}
\]

We consider the governing equations for the \(O(\mu^2)\) PPBOUSS system,

**Bottom Boundary Condition:**

\[
\mu^2(\nabla h) \cdot (\nabla P) + \frac{\partial P}{\partial z} + g = \Gamma_{bbc}, \tag{6.4}
\]

\[
\Gamma_{bbc} = \delta \mu^2 u \cdot (u \cdot \nabla^2 h), \tag{6.5}
\]

**Pressure Poisson Equation:**

\[
\mu^2 \nabla^2 P + \frac{\partial^2 P}{\partial z^2} = \Gamma_{pp}, \tag{6.6}
\]

\[
\Gamma_{pp} = -2\delta \mu^2 \left[ \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + \left( \frac{\partial^2 v}{\partial y^2} \right)^2 + \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y \partial x} \right]. \tag{6.7}
\]

Substituting equation (6.3) into equation (6.4) and solving for \(P_2\) provides the following,

\[
P_2 = -\frac{3}{4} \left[ \mu^2 H(\nabla h) \cdot (\nabla P_1) - (1 + \mu^2(\nabla h) \cdot (\nabla h)) \right] P_1 + \left( 1 + \mu^2 g H(\nabla h) \cdot (\nabla \eta) + \mu^2 H \Gamma_{bbc} \right) . \tag{6.8}
\]

This leaves only the \(P_1\) term left to determine. This is accomplished by substituting equation (6.8) into equation (6.6) and solving for \(P_1\). Recall that a weighted residual approach over the vertical is used in conjunction with the pressure–Poisson equation.
to generate a two-dimensional elliptic problem, as follows,

\[
\int_{-h}^{\delta \eta} \left( -\frac{4}{3} + q \right) \left( \mu^2 \nabla^2 P + \frac{\partial^2 P}{\partial z^2} \right) \, dz = \int_{-h}^{\delta \eta} \Gamma_{pp} \left( -\frac{4}{3} + q \right) \, dz. \tag{6.9}
\]

Substituting equations (6.3) and (6.8) into equation (6.9) and simplifying provides,

\[
\alpha_1 \nabla^2 P_1 + \alpha_2 \nabla^2 \eta + \alpha_3 \nabla^2 h + \alpha_4 \cdot \nabla P_1 + \alpha_5 P_1 = \alpha_0, \tag{6.10}
\]

where,

\[
\alpha_1 = -\frac{1}{2} \mu^2 H^2, \tag{6.11}
\]

\[
\alpha_2 = -\frac{1}{3} \mu^2 H P_1 - \frac{5}{6} \mu^2 g H^2, \tag{6.12}
\]

\[
\alpha_3 = \frac{1}{2} \mu^2 H P_1, \tag{6.13}
\]

\[
\alpha_4 = -\frac{1}{12} \mu^2 H \left( 8 \nabla \eta + 3 \nabla h \right), \tag{6.14}
\]

\[
\alpha_5 = \frac{1}{12} \left( 15 + 8 (\nabla \eta) \cdot (\nabla \eta) + 3 (\nabla h) \cdot (\nabla h) - 4 (\nabla \eta) \cdot (\nabla h) \right), \tag{6.15}
\]

\[
\alpha_0 = \frac{5}{4} \left( 1 + \mu^2 g H (\nabla h) \cdot (\nabla \eta) \right) + \frac{5}{4} \mu^2 H \Gamma_{bcc} + \hat{\Gamma}_{pp}, \tag{6.16}
\]

\[
\hat{\Gamma}_{pp} = \int_{-h}^{\delta \eta} \Gamma_{pp} \left( -\frac{4}{3} + q \right) \, dz. \tag{6.17}
\]

The solution to (6.10) will provide the solution for \( P_1 \) as well as \( P_2 \) by equation (6.8). It is noted here that the \( \nabla^2 \eta \) and \( \nabla^2 h \) terms are not well defined in terms of the finite element solution, due to the fact that they are only of a Sobelov space of first order, \( W^{(1)} \). In equation (6.10) they have been isolated along with the \( \nabla^2 P_1 \) term to better facilitate a CG solution in the next step. A CG solution to equation (6.10) will now be developed. It is assumed that each of the dependent variables can be
approximated by the finite element space,

\[ P_1 = \sum_{i=0}^{p} \phi_i P_{1,i}, \quad (6.18) \]

\[ \eta = \sum_{i=0}^{p} \phi_i \eta_i, \quad (6.19) \]

\[ h = \sum_{i=0}^{p} \phi_i h_i, \quad (6.20) \]

\[ (u, v) = \sum_{i=0}^{p} \phi_i (u, v)_i, \quad (6.21) \]

where \( \phi_i \) are the nodal shape functions on the element, \( p \) is the order of the approximation and the \( P_{1,i}, \eta_i \) and \( h_i \) are the weights for the nodal basis functions with respect to \( P_1, \eta \) and \( h \). Multiplying equation \((6.10)\) by a test function \( \phi_j \) and integrating over each element we get,

\[ \int \phi_j \left[ \alpha_1 \nabla^2 P_1 + \alpha_2 \nabla^2 \eta + \alpha_3 \nabla^2 h + \alpha_4 \cdot \nabla P_1 + \alpha_5 P_1 - \alpha_0 \right] d\Omega = 0, \quad j = 1 \ldots p. \quad (6.22) \]

Application of integration by parts on all \( \nabla^2 \) terms provides,

\[ \int \left[ -\nabla (\phi_j \alpha_1) \cdot \nabla P_1 - \nabla (\phi_j \alpha_2) \cdot \nabla \eta - \nabla (\phi_j \alpha_3) \cdot \nabla h + \phi_j \alpha_4 \cdot \nabla P_1 + \right. \]

\[ \left. \phi_j \alpha_5 P_1 - \phi_j \alpha_0 \right] d\Omega + [\phi_j (\alpha_1 \nabla P_1 + \alpha_2 \nabla \eta + \alpha_3 \nabla h) \cdot \mathbf{n}]_{\partial \Omega} = 0. \quad (6.23) \]

Note that integration by parts has been applied to the \( \nabla^2 P_1 \) term as well as the \( \nabla^2 \eta \) and \( \nabla^2 h \) terms. This allows for a full reduction of the system to only first–order
derivatives. Simplification provides a CG model for determining \( P_1 \),

\[
\int \left[ \hat{\alpha}_1 \cdot \nabla P_1 + \hat{\alpha}_2 P_1 - \hat{\alpha}_0 \right] d\Omega + \left[ \phi_j \left( (\hat{\alpha}_1 \nabla P_1 + \hat{\alpha}_2 P_1 - \hat{\alpha}_0) \right) \right]_{\partial \Omega} = 0,
\]

(6.24)

\[
\hat{\alpha}_1 = \frac{1}{12} H \left( 8 \nabla \eta + 3 \nabla h \right) \phi_j + \frac{1}{2} H^2 \nabla \phi_j
\]

(6.25)

\[
\hat{\alpha}_2 = \frac{1}{4} \left( 5 + 5 \mu^2 (\nabla \eta) \cdot (\nabla \eta) - \mu^2 (\nabla H) \cdot (\nabla H) \right) \phi_j + \frac{1}{6} \mu^2 H (2 \nabla \eta - 3 \nabla h) \cdot \nabla \phi_j,
\]

(6.26)

\[
\hat{\alpha}_0 = -\frac{5}{12} \mu^2 g H \left[ (\nabla h + 4 \nabla \eta) \phi_j + 2 H \nabla \phi_j \right] \cdot \nabla \eta + \frac{5}{4} \left( 1 + \mu^2 H \Gamma_{\mu \rho \epsilon} \right) \phi_j + \hat{\Gamma}_{pp} \phi_j,
\]

(6.27)

\[
\tilde{\alpha}_1 = -\frac{1}{2} \mu^2 H^2,
\]

(6.28)

\[
\tilde{\alpha}_2 = \frac{1}{6} \mu^2 H \left( 3 \nabla h - 2 \nabla \eta \right),
\]

(6.29)

\[
\tilde{\alpha}_0 = \frac{5}{6} \mu^2 g H^2 \nabla \eta.
\]

(6.30)

### 6.2 Coupling Paradigm

In this section a strategy for integrating the dynamic pressure terms will be discussed based on the solution for \( P_1 \) found from solving equation (6.24).

#### 6.2.1 DGSWEM To PPBOUSS

In equation (6.24) the variables \( \eta, h, u \) and \( v \) are considered known values that do not need to be solved for. These are to be provided by the ocean circulation model, in this case the DGSWEM model. In the case of the DGSWEM model the \( \eta, u \) and \( v \) variables are provided within a discontinuous–Galerkin framework, however one of the assumptions in the development of a CG model for the PPBOUSS system is that all of the variables can be represented as continuous finite element solutions. Thus, before the \( \eta, u \) and \( v \) variables can be used in PPBOUSS they must first be projected onto a continuous finite element solution through an \( L_2 \) projection.
6.2.2 PPBOUSS To DGSWEM

The DGSWEM model solves the SWE in conservative form as follows,

\[
\begin{align*}
\frac{\partial H}{\partial t} + \nabla \cdot Q &= 0, \quad (6.31) \\
\frac{\partial Q}{\partial t} + \nabla \cdot \left( \frac{QQ}{H} \right) + \nabla \left( \int_{-h}^{\eta} Pdz \right) &= (P_{z=-h}) \nabla h - \tau Q + F, \quad (6.32)
\end{align*}
\]

where \(H = \eta + h\) is the total water column height, \(Q = uH\) is the flux, \(P\) is the pressure and \(F\) represents any forcing terms. Substitution of equation (6.3) into equation (6.32) provides,

\[
\begin{align*}
\frac{\partial H}{\partial t} + \nabla \cdot Q &= 0, \quad (6.33) \\
\frac{\partial Q}{\partial t} + \nabla \cdot \left( \frac{QQ}{H} \right) + \nabla \left( \frac{1}{2}gH^2 + \int_{-h}^{\eta} Pdz \right) &= (gH + P_{d,z=-h}) \nabla h - \tau Q + F. \quad (6.34)
\end{align*}
\]

Note that if only the hydrostatic pressure is considered then \(P_d = 0\) and the original SWE are recovered. Substituting equations (6.3) and (6.8) into equation (6.34) and simplifying provides the following,

\[
\begin{align*}
\frac{\partial Q}{\partial t} + \nabla \cdot \left( \frac{QQ}{H} \right) + \nabla \left( \frac{1}{2}gH^2 \right) &= (gH + P_b) \nabla h - \frac{1}{2} (H \nabla P_1 + \nabla HP_1) - \tau Q + F, \quad (6.35)
\end{align*}
\]

where,

\[
P_b = \frac{1}{4} \left[ 1 + \mu^2 H (\nabla h) \cdot (\nabla P_1) + \left( 3 - \mu^2 (\nabla h) \cdot (\nabla h) \right) P_1 \right.
\]

\[
\quad + \mu^2 g H (\nabla h) \cdot (\nabla \eta) + \mu^2 H \Gamma_{bbb} \right]. \quad (6.36)
\]
6.2.3 DGSWEM+PPBOUSS

The strategy for solving the DGSWEM and CG–PPBOUSS together is straightforward, for each time step, or Runge–Kutta stage if applicable, the steps are as follows;

1. Given a free–surface and velocity state from the DGSWEM model the $L_2$ projection of the $\eta$, $u$ and $v$ variables is transmitted into the CG–PPBOUSS module.

2. Using the free–surface and velocity state equation (6.24) is solved to determine $P_1$.

3. The CG solution for $P_1$ is then used to provide an extra dynamic pressure forcing term that is added to the right hand side of the equations during the time step.

We note here that DGSWEM is exclusively an explicit method. A slightly different strategy would be necessary for a model which solves the SWE implicitly in time.

6.3 Validation Of The DGSWEM+PPBOUSS Model

In this section we discuss the results of two testcases which will be used to validate the DGSWEM+PPBOUSS model as well as illustrate the regimes in which the inclusion of the PPBOUSS model provides a significant impact on the accuracy of the simulation. For comparison, the results using the well known $O(\mu^2)$ Boussinesq model FUNWAVE are also provided \[62, 37\]. For each of the validation testcases the boundaries are set to be walls with a slip condition. All solitary wave initial conditions are imposed on the model at $t = 0$ and follow the equation for a solitary...
wave given by,

\[ \eta_{t=0} = A \text{sech}^2 \left( \kappa (x - x_0) \right), \]  
\[ u_{t=0} = c \left( 1 - \frac{h_b}{\eta_{t=0} + h_b} \right), \]  
\[ v_{t=0} = 0, \]  

where,

\[ \kappa = \frac{\sqrt{3A}}{2h_b\sqrt{A + h_b}}, \]  
\[ c = \sqrt{g(h_b + A)}, \]

and \( A \) represents the solitary wave height, \( x_0 \) is the \( x \)-coordinate of the solitary wave peak and \( h_b \) is the depth of the water at the solitary wave location.

6.3.1 Solitary Wave Propagation On A Circular Island

A standard validation test for run–up in wave models is based on the results of a large–scale experiment conducted by Briggs et al. [6] in 1995 [82, 86]. They sought to measure wave run–up on a circular island due to a solitary wave. A conical island was placed in the center of a wave tank measuring 30m wide and 25m long. The island was constructed to be a right angled truncated circular cone with a base diameter of 7.2m and a top diameter of 2.2m. The center of the island was located at \((x, y) = (12.96, 13.80)m\). Figure 6.1a shows the domain of the experiment and the location of the circular island. Experiments were conducted for waters of 0.32m and 0.42m depth. Here we will only focus on the experiments conducted in a depth of 0.32m. In total 27 wave gauges were placed in the experiment domain to measure data, we will focus on the results of 4 of the stations. The locations of these stations are given in Table 6.1 and shown in Figure 6.1c.
For the experiment solitary waves of heights $A/h = 0.05, 0.10$ and $0.20$ were generated, however Yamazaki et al. [82] found that for numerical simulation solitary waves of $A/h = 0.045, 0.096$ and $0.181$, respectively, better matched the laboratory results at the incident wave gauge, thus in these simulations solitary waves of these amplitudes are used. A quadratic friction coefficient of 0.01 was chosen to represent the smooth concrete floor of the experiment, and an average horizontal eddy viscosity of 0.01 was used. The computational domain consisted of an unstructured mesh with 15,663 nodes and 31,044 elements, see Figure 6.1b. The initial solitary wave is placed within a 10$m$ range of the domain which is extended seaward from the island. Taking advantage of the unstructured mesh the majority of the mesh resolution is concentrated at the circular island, and the element edges have been placed so as to follow the bathymetric contours of the island. An absorption sponge layer is placed 5$m$ from the far edge of the domain in the $x$–direction to absorb the waves and minimize reflections, see Zhang et al. [85] for details.

TABLE 6.1


<table>
<thead>
<tr>
<th>Gauge</th>
<th>$x$ (m)</th>
<th>$y$ (m)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>9.36</td>
<td>13.80</td>
<td>Front of island</td>
</tr>
<tr>
<td>9</td>
<td>10.36</td>
<td>13.80</td>
<td>Seaware shoreline of island</td>
</tr>
<tr>
<td>16</td>
<td>12.96</td>
<td>11.22</td>
<td>90° south shoreline of island</td>
</tr>
<tr>
<td>22</td>
<td>15.56</td>
<td>13.80</td>
<td>Leeward shoreline of island</td>
</tr>
</tbody>
</table>
Figures 6.2–6.7 show the results of each of the 3 experiments for the DGSWEM, the DGSWEM+PPBOUSS and the FUNWAVE models. The simulations for the lowest amplitude solitary wave, Figures 6.2 and 6.3, show very little difference between the three models. In Figures 6.2e and 6.2f there is a slight difference in that the DGSWEM+PPBOUSS simulation has a slightly more defined wake after the island while the DGSWEM model alone demonstrates more diffusion. All three are able to capture the propagation and ensuing runup of the solitary wave, capturing the peak at each gauge in both amplitude and phase.

When the waves become larger the difference between the DGSWEM model and the DGSWEM+PPBOUSS and FUNWAVE models becomes clear. For the middle solitary wave, Figures 6.4 and 6.5, the DGSWEM model is still capable of capturing the overall dynamics of the runup, and remain fairly accurate in terms of the amplitude of the wave, however the phase is shifted and the DGSWEM model consistently over predicts the timing of the peak. This is demonstrated when looking at the timeseries of each of the gauge locations in Figure 6.5. The non–hydrostatic DGSWEM+PPBOUSS and FUNWAVE models have near perfect agreement in capturing the peak wave at each gauge, however the DGSWEM peak arrives consistently early. In terms of the comparison between the DGSWEM+PPBOUSS and FUNWAVE simulations they are fairly similar. There is slight difference between the two solutions at gauge 22 where the FUNWAVE model slightly over predicts the peak water level.

In addition the shape of the DGSWEM wave does not match the data as well as the non–hydrostatic models. This difference between the models can also be observed when looking at the individual timesnaps in Figure 6.4. At \( t = 32s \) the DGSWEM simulation has nearly finished passing the island while the DGSWEM+PPBOUSS simulation is still half way along the island. Once again, as with the lower amplitude wave, the DGSWEM model appears to have more diffusion in the solution in the
Figure 6.1. Domain setup for circular island test case [6]
final timesnap.

For the largest solitary wave experiment the differences between the DGSWEM model and the Boussinesq-type models, DGSWEM+PPBOUSS and FUNWAVE, are much clearer, see Figures 6.6 and 6.7. A shock begins to form in the DGSWEM simulation almost immediately. This has two major effects on the accuracy of the solution, first the shock arrives at the island before the actual solitary wave does. This is most clear for gauge 6, Figure 6.7a. The second effect is a dissipation of the wave amplitude, thus the DGSWEM consistently under predicts the maximum wave amplitudes.

In contrast, both the DGSWEM+PPBOUSS and FUNWAVE solutions capture the timing of the peak nearly perfectly. Similarly both of these models do not exhibit the dissipation in the solution and thus also capture the peak amplitude very well. There is a slight phase lag for the DGSWEM+PPBOUSS model at gauge 22. This gauge is located at the far side of the island opposite the direction of incidence. This phase lag may be due to a sensitivity in the implementation of wetting and drying or an inaccurate choice in bottom friction.

This discrepancy between the DGSWEM model and the other two models can be explained by examining the dispersive properties of the models. The DGSWEM model, which is solving just the SWE, has poor dispersive accuracy. The propagation speed of the wave is based on the overall water column height, causing the peak of the solitary wave to travel faster than the base. For the low amplitude solitary wave the difference between the base and peak of the wave is small enough that this inaccuracy does not cause a significant degradation of the solution. However, as the solitary wave amplitude is increased, the waves begin to break much earlier and a shock front is formed. In contrast, the DGSWEM+PPBOUSS and FUNWAVE models have greater dispersive accuracy and thus are able to capture the propagation of the solitary wave much better.
6.3.2 Solitary Wave Runup On A Shelf With A Conical Island

Lynett et al. [42] performed an experiment to examine what occurs when a solitary wave encounters a conical island that is placed on a shelf. Similar to the circular island test case of Briggs et al. [6], this experiment provides an excellent test case for validating wave refraction, wave runup and wave shoaling in the model. The experiment was conducted in a 48.8m long and 26.5m wide wave basin. The offshore water depth of the experiment was 0.78m. A three-dimensional bathymetry was constructed consisting of a triangular shaped sloped beach with a conical island located at the toe of the beach, see Figure 6.8a. For the experiment a solitary wave with an amplitude of \( A = 0.39m \) was generated offshore.

For this validation study a mesh with 10893 nodes and 21460 elements was used. Taking advantage of the unstructured domain finer resolution was focused on the main features of the bathymetry such as the conical island and the triangular shelf. In addition, the elements were constructed so that the edges of each element aligned well with the contours of the bathymetry, see Figure 6.8b. The numerical domain was extended 5m away from the shore to facilitate the solitary wave initial condition. In order to compare the model results with the measured data the free–surface displacement at 9 wave gauges will be examined, see table 6.2 for the exact locations of each gauge and see Figure 6.8c for a spatial representation of the gauge placement.

This test case does an excellent job of demonstrating the importance of incorporating higher order physics into the model. The difference between the DGSWEM model alone and the simulation using the DGSWEM+PPBOUSS model are immediately clear. Figure 6.9 shows the temporal progression of the solution for both the DGSWEM and DGSWEM+PPBOUSS simulations. We note that for a solitary wave with such a high amplitude the DGSWEM model generates a shock front fairly quickly. This translates into a degradation of the solitary wave profile and a decrease
TABLE 6.2

GAUGE LOCATIONS FOR THE SHELF WITH CONICAL ISLAND EXPERIMENT, LYNETT ET AL. [42].

<table>
<thead>
<tr>
<th>Gauge</th>
<th>x (m)</th>
<th>y (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.3</td>
<td>12.0</td>
</tr>
<tr>
<td>2</td>
<td>17.8</td>
<td>13.0</td>
</tr>
<tr>
<td>3</td>
<td>25.8</td>
<td>13.0</td>
</tr>
<tr>
<td>4</td>
<td>12.3</td>
<td>18.0</td>
</tr>
<tr>
<td>5</td>
<td>17.8</td>
<td>18.0</td>
</tr>
<tr>
<td>6</td>
<td>25.8</td>
<td>18.0</td>
</tr>
<tr>
<td>7</td>
<td>29.8</td>
<td>13.0</td>
</tr>
<tr>
<td>8</td>
<td>29.8</td>
<td>18.0</td>
</tr>
<tr>
<td>9</td>
<td>29.8</td>
<td>23.0</td>
</tr>
</tbody>
</table>

in the wave amplitude, see Figures 6.9a and 6.9b respectively.

As the wave passes over the conical island, see Figures 6.9c and 6.9d, the differences between the two solutions can be clearly seen. The DGSWEM model has formed a relatively consistent shock front parallel to the shore, while the DGSWEM+PPBOUSS exhibits the more complex features one would anticipate given the complexity of the bathymetry. In particular, the wave front parallel to the shore has varying amplitude based on the effect of the shelf and the conical island. After the wave front has passed over the conical island and begins to inundate the beach on the shoreward side the two solutions look fairly similar, however the DGSWEM+PPBOUSS simulation exhibits more distinct wave refraction and defraction patterns in the wake of the conical island, see Figures 6.9e and 6.9f.

This difference between the DGSWEM and DGSWEM+PPBOUSS models is
clear when we consider the comparison between the models and the data. Figures 6.10 and 6.11 show the results for these two models as well as the FUNWAVE model along with the data for the 9 wave gauges. The main observation from these figures is that the FUNWAVE and DGSWEM+PPBOUSS model perform fairly similarly and are capable of capturing both the magnitude and timing of the peak wave heights for nearly all of the stations. The DGSWEM model suffers from the generation of a shock front early in the simulation. This causes the wave to arrive far earlier than the data represents and with a lower peak than recorded. Once again, this experiment does an excellent job representing the importance of higher order physics when modelling wave propagation in the nearshore, especially for complex bathymetries.

An interesting observation is seen for gauge 3, see Figure 6.10c. The FUNWAVE model predicts a much higher peak amplitude which arrives a few seconds later in the model than the measured data. The DGSWEM+PPBOUSS and DGSWEM model both capture the arrival of the peak, and furthermore the DGSWEM+PPBOUSS model captures the subsequent decrease in water level after the wave has passed the gauge. Similarly, for gauge 7, see Figure 6.11b all three models accurately capture the peak amplitude, however only the DGSWEM+PPBOUSS model also accurately captures the timing of the peak amplitude at this gauge location. Note that gauge 3 and 7 are placed nearby each other along the propagation path of the wave. Finally, all three models similarly over predict the water levels for gauge 6, see Figure 6.11a. This could be due to an error in the data itself, this is supported when we consider the data at gauges nearby, such as gauges 8 and 9, which show the arrival of a sharp front with a long tail, whereas gauge 6 does not appear to show a similar phenomenon.
Figure 6.2. Comparison of DGSWEM (left column) and DGSWEM+PPBOUSS (right column) for the circular island test case A.
Figure 6.3. Comparison of experimental data (red circles) with DGSWEM (magenta dashed line), DGSWEM+PPBOUSS (blue line) and FUNWAVE (green line) for the circular island test case A.
Figure 6.4. Comparison of DGSWEM (left column) and DGSWEM+PPBOUSS (right column) for the circular island test case B.
Figure 6.5. Comparison of experimental data (red circles) with DGSWEM (magenta dashed line), DGSWEM+PPBOUSS (blue line) and FUNWAVE (green line) for the circular island test case B.
Figure 6.6. Comparison of DGSWEM (left column) and DGSWEM+PPBOUSS (right column) for the circular island test case C.
Figure 6.7. Comparison of experimental data (red circles) with DGSWEM (magenta dashed line), DGSWEM+PPBOUSS (blue line) and FUNWAVE (green line) for the circular island test case C
Figure 6.8. Domain setup for conical island on a shelf test case [42]
Figure 6.9. Comparison of solutions for DGSWEM (left column) and DGSWEM+PPBOUSS (right column) for the conical island on a shelf test case.
Figure 6.10. Comparison of experimental data (red circles) at gauges 1–5 with the DGSWEM (magenta dashed line), DGSWEM+PPBOUSS (blue line) and FUNWAVE (green line) models for the shelf with a conical island test case.
Figure 6.11. Comparison of experimental data (red circles) at gauges 6–9 with the DGSWEM (magenta dashed line), DGSWEM+PPBOUSS (blue line) and FUNWAVE (green line) models for the shelf with a conical island test case.
7.1 Conclusions

The framework for a Boussinesq–type non–hydrostatic pressure model with a Green–Nagdhi type expansion along the vertical axis has been developed. The resulting model is capable of resolving high–order dispersion, shoaling and nonlinear effects and when coupled with a SWE model produces a highly accurate solution for waves in both shallow and intermediate water. The vertical dimension in the model is handled through vertical modes in the pressure expansion, thus reducing the unknowns of the model to those only resolving the horizontal dimensions. Through the application of asymptotic rearrangement, optimal vertical basis functions can be found which produce the highest order dispersion accuracy as well as improved accuracy in the shoaling and nonlinear characteristics. The result is a relatively simple model to implement that is highly accurate.

In the surf zone, dispersive and shoaling accuracy play a major role in capturing the nearshore wave dynamics. The nondispersive properties of the shallow water equations (SWE) for higher frequency waves makes them inapplicable in this regime. In order to capture these higher order physics in the nearshore a separate model is often required. The present work has developed a Boussinesq–type system that is capable of seamlessly extending SWE models, thereby introducing a framework for extending the applicability of standard SWE instead of replacing them in the nearshore.
In classical Boussinesq–type models the higher order dispersive terms are resolved through the inclusion of higher–order mixed space/time derivatives for the velocity. These mixed space/time derivatives can be difficult and inefficient to implement numerically. Similarly, extension of the model to two horizontal dimensions will involve mixed spatial derivatives, such as $\partial^3 u/(\partial x \partial y \partial t)$ and $\partial^3 v/(\partial x \partial y \partial t)$, which add an extra layer of computational complexity. While it is true that these issues have been largely addressed in the literature, in two horizontal dimensions there still remains a set of two elliptic equations to solve, one for each of the horizontal velocities, respectively. In contrast, the model developed in this work resolves the dispersive terms through the non–hydrostatic pressure component, which at $O(\mu^2)$ involves only a single equation to be solved, even for two–horizontal dimensions. Furthermore, the final forms of the model are simple and have fewer terms than many other comparable order equations.

Use of a Green–Naghdi type expansion of the pressure profile leads to a stand alone set of equations for the pressure built on the pressure–Poisson equation and the bottom boundary equation for pressure. The pressure–Poisson approximation allows for the implementation of both $O(\mu^2)$ and $O(\mu^4)$ models without the need to solve any mixed space/time derivatives or to resolve third order or higher spatial derivatives. Given the prohibitive nature of solving the $O(\mu^4)$ equations in mixed space–time Boussinesq systems the present work offers an advantage for solving problems where higher–order dispersive accuracy is needed. Numerical implementation is straightforward and it is possible to take advantage of many of the well known techniques for solving elliptic problems. Through algebraic manipulation of the governing equations it is possible to reduce the degrees of freedom in the problem, which in turn improves the computational efficiency of the model.

In comparison with equivalent order Boussinesq–type models the accuracy of the present model is similar, however the present work is better suited for coupling with
current SWE based coastal ocean circulation models. In addition, the set of equations solved in order to resolve the dispersive terms is a scalar equation and, contrary to standard Boussinesq approaches, does not involve solutions to mixed space/time derivatives.

Three numerical approaches to solving the pressure–Poisson based equations, PPBOUSS, were developed and validated against a series of well known test cases and experiments. It has been demonstrated that the PPBOUSS model performs well using both structured and unstructured meshes and can be easily coupled with both finite difference and finite element shallow water equation models. In particular, a coupling paradigm for integrating the PPBOUSS model with the well known Discontinuous Galerkin Shallow Water Equation Model (DGSWEM) has been developed and tested showing a significant improvement in the nearshore accuracy of the simulations.

The advantage of using a stand alone set of equations such as the PPBOUSS model has been shown through the relative ease of integrating the PPBOUSS model with the DGSWEM model. A separate module was constructed to solve for the dynamic pressure terms and coupling between the two models is achieved through a relatively small amount of change to each code. In the case of the PPBOUSS module, the discontinuous finite element solution from DGSWEM must be first projected onto a continuous Galerkin finite element solution before it can be passed to the PPBOUSS module. In turn the PPBOUSS module passes two finite element variables to the right hand source term in the momentum equations in the DGSWEM model.

It was demonstrated that the model is capable of capturing the relatively simple phenomenon of low amplitude wave propagation as well the more complicated physics involved with a wavetrain travelling over a submerged shoal as well as the considerably more complex phenomenon of nonlinear wave runup or the interaction of a solitary wave with a shelf and island. In all test cases the numerical results of the model performed well in comparison to analytic and experimental data.
In the case of the wavetrain travelling over a submerged shoal the results from the current model were compared with the results of the fully rotational Boussinesq model of Zhang et al. [84], and the results computed by Gobbi et al. [25], for fully nonlinear second–order models (Z2, WKGS) and weakly and fully nonlinear fourth–order models (Z4, WN4, FN4). The fully nonlinear second–order model demonstrated comparable performance with the models of Gobbi and Zhang in front of and upon the submerged shoal but showed decreased accuracy behind the shoal, demonstrating the need for higher order accuracy in the dispersive properties. However the current weakly nonlinear fourth order model was shown to be very accurate and to be comparable in accuracy to the fully nonlinear fourth–order model of Gobbi (FN4) including the region behind the shoal.

Verification of the model with the well known Carrier–Greenspan solution showed excellent agreement, demonstrating accuracy in the wetting/drying algorithm. Validation of the model for both breaking and nonbreaking wave propagation over both a simple sloping beach topography as well as an impermeable seawall demonstrated the model’s ability to capture the nearshore breaking and shoaling wave dynamics and inundation on the landward side of a barrier. Although the model is limited to single–valued free–surfaces and thus is unable to accurately capture the small scale vertical structures that form during fluid–structure interactions or wave overtopping, the overall dynamics of the system are accurately represented.

Comparison between the standard DGSWEM model and the DGSWEM+PPBOUSS model for experimental test cases involving complex two dimensional bathymetries further demonstrated the importance of including the higher order physics provided by the inclusion of the dynamic pressure terms. The SWE alone were unable to accurately represent the propagation of solitary waves and the ensuing shoaling and nonlinear response when the solitary waves reached a beach or an island due to the lack of accuracy both in dispersion, as well as shoaling and nonlinear wave inter-
actions. This was especially true for solitary waves of high amplitudes. When the DGSWEM model was used without the enhancement of the PPBOUSS model, these solitary waves generated a shock front fairly quickly, which led to artificial breaking of the wave and an overall decrease in wave amplitude which was not reflected in the data.

Inclusion of the PPBOUSS terms with the DGSWEM model showed significant improvement. The DGSWEM+PPBOUSS model was capable of capturing the propagation of all of the solitary waves used in the experimental test cases, including those of relatively high amplitude. As a result, both the timing and the magnitude of the peak wave heights were very accurate when compared with the data. As expected the DGSWEM+PPBOUSS model performed comparable to, and in a few cases even better than, the well known Boussinesq model FUNWAVE.

In this work the basis for a Green–Naghdi Boussinesq–type pressure–Poisson wave model has been developed and validated against a number of analytical and experimental results. The model is robust, stable and very accurate. Use of the PPBOUSS model with SWE solvers will extend the applicability of these models to the representation of higher frequency gravity and infragravity wave events in the nearshore. This will have a significant impact on the ability of the coastal modelling community to produce large scale simulations that capture a range of temporal scale events ranging from storm surge flooding to wave impacts on structures and levee systems. The improved physics of the model will also allow these models to be used to represent the propagation of tsunamis, extending the range of problems that can be addressed.

7.2 Future Work

Extension of the DGSWEM model to include the phase–resolving properties of the PPBOUSS has extended the range of applicable problems which can be solved using the DGSWEM architecture. Following this work there are many steps that
can be taken to further improve the model and test the range of applicability. The following is a list of a few of the most important items of future work to consider.

- In the current work, two classic validation cases for testing phase resolving wave models has been discussed. Both involve the interaction of solitary waves with complex bathymetries. In order to validate the model against experiments which involve monochromatic and irregular waves a proper wave generation method must be implemented in the model. The current implementation of DGSWEM is capable of generating waves at a boundary edge in order to generate a tidal signal. However, this approach is infeasible for generating the higher frequency gravity waves that are present in these experimental cases. To address this, current ongoing work is focusing on implementing and testing the sponge layer generation technique of Zhang et al. [85] within the DGSWEM architecture.

- The current implementation of the DGSWEM+PPBOUSS model focuses only on the $O(\mu^2)$ implementation of the pressure–Poisson model. As discussed and demonstrated in this work, the model is capable of extension to higher order physics such as the $O(\mu^4)$ set of PPBOUSS equations. In the validation cases for one–horizontal dimension, use of the higher order PPBOUSS model provided more accurate solutions. Extension of the DGSWEM+PPBOUSS model to include the $O(\mu^4)$ PPBOUSS equations would improve the nearshore accuracy of the coupled system.

- Parallelization of the CG–PPBOUSS module would extend the applicability of the model to larger and finer resolution domains. The current serial implementation of the CG–PPBOUSS model utilizes an open source sparse matrix library, SPARSKIT, to solve the global matrices. Parallelization of the PP–BOUSS model would involve using similar libraries that have been optimized for use with MPI.

- The CG–PPBOUSS implementation removes some of the advantages of using a DG framework in the DGSWEM model. The choice to use a CG method with the PPBOUSS model was motivated by the fact that the pressure–Poisson problem is elliptic. A reformulation of the PPBOUSS equations to take advantage of either local discontinuous Galerkin (LDG) or hybridized discontinuous Galerkin (HDG) approaches would allow the coupled model to be a fully discontinuous finite element model. Similarly, the current implementation is limited to only linear elements for the CG solution to the PPBOUSS model, as a result the improved convergence from the choice of higher order basis functions is lost. Extension of the CG solution to include higher order nodal basis functions, or a reformulation to an LDG or HDG method would address this issue.

- As discussed in this work, the near shore accuracy of the DGSWEM model
alone is limited. The DGSWEM model is ill-suited to handle gravity waves and is incapable of generating infragravity waves. The addition of the PPBOUSS module has extended the range of frequencies that the model can capture. This in turn has extended the applicability of the model in the near shore to problems where higher frequency waves play a more significant role. Such as in island topographies, levee over topping, wave/structure interaction and the generation of infragravity waves during high energy storm events, to name a few. Future work should focus on the application of the DGSWEM+PPBOUSS model to these sorts of problems.

- In this work the coupling of the PPBOUSS model to the DGSWEM model was developed. This has demonstrated how straightforward the coupling paradigm with the PPBOUSS model is. Following this work, modules for coupling the PPBOUSS model to other hydrostatic ocean circulation models could be developed, thereby extending the range of applications for these models as well.
APPENDIX A

DERIVATION OF GOVERNING EQUATIONS

In this appendix we briefly cover the derivation of the conservation of mass, pressure–Poisson equation and the bottom boundary condition on the pressure. We begin by considering the non–dimensional conservation of momentum and continuity equations, written in tensor notation for simplicity,

\[ \delta \frac{\partial u_i}{\partial t} + \delta^2 u_j \frac{\partial u_i}{\partial x_j} + \delta^2 w \frac{\partial u_i}{\partial z} + \frac{\partial P}{\partial x_i} = 0, \quad -h \leq z \leq \delta \eta, \quad (A.1) \]

\[ \delta \mu^2 \frac{\partial w}{\partial t} + \delta^2 \mu^2 u_j \frac{\partial w}{\partial x_j} + \delta^2 \mu^2 w \frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} + g = 0, \quad -h \leq z \leq \delta \eta. \quad (A.2) \]

\[ \delta \frac{\partial u_i}{\partial x_i} + \delta \frac{\partial w}{\partial z} = 0, \quad -h \leq z \leq \delta \eta. \quad (A.3) \]

We also consider the kinematic and free–surface bottom boundary conditions,

\[ \delta w + \delta u_i \frac{\partial h}{\partial x_i} = 0, \quad z = -h, \quad (A.4) \]

\[ \delta w - \delta \frac{\partial \eta}{\partial t} - \delta^2 u_i \frac{\partial \eta}{\partial x_i} = 0 \quad z = \delta \eta. \quad (A.5) \]

A.1 Conservation Of Mass

Derivation of the conservation of mass equation used in this work begins by considering the integral of the continuity equation, equation (A.3), over the vertical axis,

\[ \delta \int_{z=-h}^{\delta \eta} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial w}{\partial z} \right) = 0, \quad (A.6) \]
which provides,
$$\delta \int_{z=-h}^{\delta \eta} \frac{\partial u_i}{\partial x_i} dz + \delta w_{z=\delta \eta} - \delta w_{z=-h} = 0. \quad (A.7)$$

Substituting equations (A.4) and (A.5) into equation (A.7) provides the following,
$$\delta \int_{z=-h}^{\delta \eta} \frac{\partial u_i}{\partial x_i} dz + \delta \left( \frac{\partial \eta}{\partial t} + \delta u_i \frac{\partial \eta}{\partial x_i} \right)_{z=\delta \eta} - \delta \left( \delta u_i \frac{\partial h}{\partial x_i} \right)_{z=-h} = 0. \quad (A.8)$$

By the Leibniz integration rule the first term in equation (A.8) can be rewritten as follows,
$$\delta \int_{-h}^{\delta \eta} \frac{\partial u_i}{\partial x_i} dz = \delta \frac{\partial}{\partial x_i} \int_{-h}^{\delta \eta} u_i dz - \delta^2 (u_i)_{z=\delta \eta} \frac{\partial \eta}{\partial x_i} + \delta (u_i)_{z=-h} \frac{\partial h}{\partial x_i}. \quad (A.9)$$

Substitution of equation (A.9) into equation (A.8) provides,
$$\delta \frac{\partial}{\partial x_i} \int_{-h}^{\delta \eta} u_i dz - \delta^2 (u_i)_{z=\delta \eta} \frac{\partial \eta}{\partial x_i} + \delta (u_i)_{z=-h} \frac{\partial h}{\partial x_i} + \delta \left( \frac{\partial \eta}{\partial t} + \delta u_i \frac{\partial \eta}{\partial x_i} \right)_{z=\delta \eta} - \delta \left( \delta u_i \frac{\partial h}{\partial x_i} \right)_{z=-h} = 0. \quad (A.10)$$

Simplification provides the conservation of mass equation,
$$\delta \frac{\partial \eta}{\partial t} + \delta \frac{\partial}{\partial x_i} \int_{-h}^{\delta \eta} u_i dz = 0, \quad (A.11)$$

which in vector notation takes the form,
$$\delta \frac{\partial \eta}{\partial t} + \delta \nabla \cdot \int_{-h}^{\delta \eta} \mathbf{u} dz = 0 \quad (A.12)$$

A.2 Bottom Boundary Condition On Pressure

Derivation of the bottom boundary condition on pressure begins by considering the vertical conservation of momentum equation, (A.2), taken at \( z = -h \) and applying
the kinematic bottom boundary condition. Solving equation (A.4) for \( w \) provides,

\[
\delta w = -\delta u_i \frac{\partial h}{\partial x_i}. \tag{A.13}
\]

Thus, if we consider each term in the vertical conservation of momentum equation evaluated at \( z = -h \) we get,

\[
\delta \mu^2 \frac{\partial w}{\partial t} = -\delta \mu^2 \frac{\partial h \partial u_i}{\partial x_i \partial t} \tag{A.14}
\]

\[
\delta^2 \mu^2 u_j \frac{\partial w}{\partial x_j} = -\delta^2 \mu^2 \frac{\partial h}{\partial x_i} \left( u_j \frac{\partial u_i}{\partial x_j} \right) - \delta^2 \mu^2 u_j u_i \frac{\partial^2 h}{\partial x_i \partial x_j} \tag{A.15}
\]

\[
\delta^2 \mu^2 w \frac{\partial w}{\partial z} = -\delta^2 \mu^2 \frac{\partial h}{\partial x_i} \left( w \frac{\partial u_i}{\partial z} \right) \tag{A.16}
\]

\[
\delta \frac{\partial P}{\partial z} + g = \delta \frac{\partial P}{\partial z} + g \tag{A.17}
\]

Combining these together we get,

\[
- \mu^2 \frac{\partial h}{\partial x_i} \left( \delta \frac{\partial u_i}{\partial t} + \delta^2 u_j \frac{\partial u_i}{\partial x_j} + \delta^2 w \frac{\partial u_i}{\partial z} \right) + \delta \frac{\partial P}{\partial z} + g - \delta^2 \mu^2 u_j u_i \frac{\partial^2 h}{\partial x_i \partial x_j} = 0. \tag{A.18}
\]

It is clear from equation (A.1) that the first term in equation (A.18) can be replaced with the gradient of the pressure, \( P \), as follows,

\[
\delta \mu^2 \frac{\partial h}{\partial x_i} \frac{\partial P}{\partial x_i} + \delta \frac{\partial P}{\partial z} + g - \delta^2 \mu^2 u_j u_i \frac{\partial^2 h}{\partial x_i \partial x_j} = 0. \tag{A.19}
\]

Converting equation (A.19) back to vector notation we get the final form for the bottom boundary condition,

\[
\delta \mu^2 (\nabla h) \cdot (\nabla P) + \delta \frac{\partial P}{\partial z} + g = \delta^2 \mu^2 \mathbf{u} \cdot (\mathbf{u} \cdot \nabla^2 h). \tag{A.20}
\]
A.3 Pressure–Poisson Equation

Derivation of the pressure–Poisson equation begins by taking the gradient of the conservation of momentum equations as follows,

\[
\begin{align*}
\mu^2 \frac{\partial}{\partial x_i} \left( \delta \frac{\partial u_i}{\partial t} + \delta^2 w \frac{\partial u_i}{\partial z} + \delta \frac{\partial P}{\partial x_i} \right) &= 0, & -h \leq z \leq \delta \eta, \quad (A.21) \\
\frac{\partial}{\partial z} \left( \delta \mu^2 \frac{\partial w}{\partial t} + \delta^2 \mu^2 u_j \frac{\partial w}{\partial x_j} + \delta^2 \mu^2 w \frac{\partial w}{\partial z} + \delta \frac{\partial P}{\partial z} + g \right) &= 0. & -h \leq z \leq \delta \eta. \quad (A.22)
\end{align*}
\]

For simplicity we consider each term of equation (A.21) individually,

\[
\begin{align*}
\mu^2 \frac{\partial}{\partial x_i} \left( \delta \frac{\partial u_i}{\partial t} \right) &= \delta \mu^2 \frac{\partial}{\partial t} \left( \frac{\partial u_i}{\partial x_i} \right), \quad (A.23) \\
\mu^2 \frac{\partial}{\partial x_i} \left( \delta^2 u_j \frac{\partial u_i}{\partial x_j} \right) &= \delta^2 \mu^2 \left( \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_i} \right) \right), \quad (A.24) \\
\mu^2 \frac{\partial}{\partial x_i} \left( \delta^2 w \frac{\partial u_i}{\partial z} \right) &= \delta^2 \mu^2 \left( \frac{\partial w}{\partial z} \frac{\partial u_i}{\partial x_i} + w \frac{\partial}{\partial z} \left( \frac{\partial u_i}{\partial x_i} \right) \right), \quad (A.25) \\
\mu^2 \frac{\partial}{\partial x_i} \left( \delta \frac{\partial P}{\partial x_i} \right) &= \delta \mu^2 \frac{\partial^2 P}{\partial x_i \partial x_i}, \quad (A.26)
\end{align*}
\]

and the each term of equation (A.22),

\[
\begin{align*}
\frac{\partial}{\partial z} \left( \delta \mu^2 \frac{\partial w}{\partial t} \right) &= \delta \mu^2 \frac{\partial}{\partial t} \left( \frac{\partial w}{\partial z} \right), \quad (A.27) \\
\frac{\partial}{\partial z} \left( \delta^2 \mu^2 u_j \frac{\partial w}{\partial x_j} \right) &= \delta^2 \mu^2 \left( \frac{\partial u_j}{\partial z} \frac{\partial w}{\partial x_j} + u_j \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial x_j} \right) \right), \quad (A.28) \\
\frac{\partial}{\partial z} \left( \delta^2 \mu^2 w \frac{\partial w}{\partial z} \right) &= \delta^2 \mu^2 \left( \left( \frac{\partial w}{\partial z} \right)^2 + w \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial z} \right) \right), \quad (A.29) \\
\frac{\partial}{\partial z} \left( \delta \frac{\partial P}{\partial z} + g \right) &= \delta \frac{\partial^2 P}{\partial z^2}. \quad (A.30)
\end{align*}
\]

Combining all of these terms and reorganizing certain terms we get the following
expression,

\[ \delta^2 \mu^2 \frac{\partial^2 P}{\partial x_i \partial x_i} + \delta \frac{\partial^2 P}{\partial z^2} + \delta^2 \mu^2 \frac{\partial}{\partial t} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial w}{\partial z} \right) + \delta^2 \mu^2 u_j \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial w}{\partial z} \right) + \delta^2 \mu^2 w \frac{\partial}{\partial z} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial w}{\partial z} \right) + \delta^2 \mu^2 \left( \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial w}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial w}{\partial z} \frac{\partial u_i}{\partial x_j} + \left( \frac{\partial w}{\partial z} \right)^2 \right) = 0. \quad (A.31) \]

Equation (A.31) can be further simplified by substituting equation (A.3). Thus the final equation for the pressure–Poisson takes the form,

\[ \delta^2 \mu^2 \frac{\partial^2 P}{\partial x_i \partial x_i} + \delta \frac{\partial^2 P}{\partial z^2} + \delta^2 \mu^2 \left( \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial w}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial w}{\partial z} \frac{\partial u_i}{\partial x_j} + \left( \frac{\partial w}{\partial z} \right)^2 \right) = 0, \quad -h \leq z \leq \delta \eta. \quad (A.32) \]

Furthermore, by equation (A.3),

\[ \frac{\partial w}{\partial z} = -\frac{\partial u_i}{\partial x_i}, \quad (A.33) \]

thus,

\[ \delta^2 \mu^2 \frac{\partial^2 P}{\partial x_i \partial x_i} + \delta \frac{\partial^2 P}{\partial z^2} + \delta^2 \mu^2 \left( \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial w}{\partial x_i} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial z} \frac{\partial w}{\partial x_j} + \left( \frac{\partial u_i}{\partial x_i} \right)^2 \right) = 0, \quad -h \leq z \leq \delta \eta. \quad (A.34) \]
APPENDIX B

LINEAR PROPERTIES

In this appendix the linear properties of the pressure–Poisson based model are detailed. In particular, approximate solutions to the dispersion relationship and the shoaling coefficient are examined.

B.1 Setting Up The Equations

The process of examining the linear properties of the model requires the following steps,

1. Isolation of only the linear terms, i.e. terms of $O(\delta)$,
2. We assume a multiple scales expansion with both a slow and fast spatial scale,
3. We assume a perturbation expansion of the dependent variables,
4. We assume the solutions to each component of the perturbation expansion follow a general solution format, and
5. Finally, for simplicity of the solution we also consider so–called transition functions.
B.1.1 Linear Equations

We first consider the governing equations in one spatial dimension with where all terms of $O(\delta^2)$ have been truncated.

$$\frac{\partial \eta}{\partial t} + h \frac{\partial \bar{u}}{\partial x} + \frac{\partial h}{\partial x} \bar{u} = 0$$  \hspace{1cm} (B.1)

$$h \frac{\partial \bar{u}}{\partial t} + g h \frac{\partial \eta}{\partial x} + \sum_{n=1}^{N} \mu^{\beta_n} \left( h G_n \frac{\partial P_n}{\partial x} - (\phi_{n,q=0} + R_n) \frac{\partial h}{\partial x} P_n \right) = 0$$  \hspace{1cm} (B.2)

$$\mu^2 g h \frac{\partial h}{\partial x} \frac{\partial \eta}{\partial x} - h \frac{\partial^2 h}{\partial x^2} u^2 + \sum_{n=1}^{N} \mu^{\beta_n} \left( 1 + \mu^2 \frac{\partial h}{\partial x} \right)^2 \phi'_{n,q=0} P_n + \mu^2 h \frac{\partial h}{\partial x} \phi_{n,q=0} \frac{\partial P_n}{\partial x} \right) = 0$$  \hspace{1cm} (B.3)

$$\mu^2 g^2 h^2 \Omega_m \frac{\partial^2 \eta}{\partial x^2} + \sum_{n=1}^{N} \mu^{\beta_n} \left( \mu^2 h^2 \xi_{nm} \frac{\partial^2 P_n}{\partial x^2} + 2 \mu^2 h \frac{\partial h}{\partial x} (\Theta_{nm} - \Phi_{nm}) \frac{\partial P_n}{\partial x} + \left( \Lambda_{nm} + \mu^2 (S_{nm} - 2 \Theta_{nm} + \Lambda_{nm} + 2 \Phi_{nm} - 2 \Psi_{nm}) \frac{\partial h}{\partial x} \right)^2 + \mu^2 (\Theta_{nm} - \Phi_{nm}) h \frac{\partial^2 h}{\partial x^2} \right) P_n \right) = 0$$  \hspace{1cm} (B.4)

B.1.2 Multiple Scales

In this step we consider a multiple scales expansion with both a slow and fast spatial scale. In particular, all spatial derivatives are replaced with the following expressions,

$$\frac{\partial h}{\partial x} = \epsilon \frac{\partial h}{\partial X}$$  \hspace{1cm} (B.5)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}$$  \hspace{1cm} (B.6)

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2} + \epsilon \left( \frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial X \partial x} \right)$$  \hspace{1cm} (B.7)

Substitution of equations (B.5)-(B.7) into equations (B.1)-(B.4) and simplifying provides the following expressions,
\[
\left( \frac{\partial \eta}{\partial t} + h \frac{\partial \bar{u}}{\partial x} \right) + \epsilon \left( h \frac{\partial \bar{u}}{\partial X} + h \frac{\partial h}{\partial X} \bar{u} \right) + O(\epsilon^2) = 0 \quad (B.8)
\]
\[
\left( h \frac{\partial \bar{u}}{\partial t} + gh \frac{\partial \eta}{\partial x} + h \sum_{n=1}^{N} \mu \hat{\beta}_n G_n \frac{\partial P_n}{\partial x} \right) + \\
\epsilon \left( gh \frac{\partial \eta}{\partial X} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( hG_n \frac{\partial P_n}{\partial X} - (\phi_{n,q=0} + R_n) \frac{\partial h}{\partial X} P_n \right) \right) + O(\epsilon^2) = 0 \quad (B.9)
\]
\[
\sum_{n=1}^{N} \phi'_{n,q=0} P_n \right) + \epsilon \left( \mu^2 gh \frac{\partial h}{\partial X} \frac{\partial \eta}{\partial x} + h \frac{\partial h}{\partial X} \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \phi_{n,q=0} \frac{\partial P_n}{\partial x} \right) + O(\epsilon^2) = 0 \quad (B.10)
\]
\[
\left( \mu^2 gh^2 \Omega_m \frac{\partial^2 \eta}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \Lambda_{nm} P_n + \mu^2 h^2 \xi_{nm} \frac{\partial^2 P_n}{\partial x^2} \right) \right) + \\
\epsilon \left( \mu^2 gh^2 \left( \frac{\partial^2 \eta}{\partial x \partial X} + \frac{\partial^2 \eta}{\partial X \partial x} \right) + \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \left( 2h \frac{\partial h}{\partial X} (\Theta_{nm} - \Phi_{nm}) \frac{\partial P_n}{\partial x} + \\
h^2 \xi_{nm} \left( \frac{\partial^2 P_n}{\partial x \partial X} + \frac{\partial^2 P_n}{\partial X \partial x} \right) \right) \right) + O(\epsilon^2) = 0 \quad (B.11)
\]

### B.1.3 Perturbation Expansion

In this section we consider a perturbation expansion of the dependent variables with respects to \(\epsilon\),

\[
(*) = (*)^{(0)} + \epsilon(*)^{(1)} + O(\epsilon^2), \quad (B.12)
\]

where \((*)\) represents either \(\eta, \bar{u}\) or \(P_n\). Substitution of equation \((B.12)\) into equations \((B.8)-(B.11)\) provides the following expressions,
\[ \left( \frac{\partial \eta^{(0)}}{\partial t} + h \frac{\partial \bar{u}^{(0)}}{\partial x} \right) + \epsilon \left( \frac{\partial \eta^{(1)}}{\partial t} + h \frac{\partial \bar{u}^{(1)}}{\partial x} + \frac{\partial h}{\partial X} \bar{u}^{(0)} \right) + O(\epsilon^2) = 0 \quad (B.13) \]
\[ \left( h \frac{\partial \bar{u}^{(0)}}{\partial t} + g \frac{\partial \eta^{(0)}}{\partial x} + h \sum_{n=1}^{N} \mu^{\hat{\beta}_n} G_n \frac{\partial P_n^{(0)}}{\partial x} \right) + \epsilon \left( h \frac{\partial \bar{u}^{(1)}}{\partial t} + g \frac{\partial \eta^{(1)}}{\partial x} + g h \frac{\partial \eta^{(0)}}{\partial X} \right) \]
\[ + \epsilon \frac{h}{2} \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( G_n \frac{\partial P_n^{(1)}}{\partial x} + h G_n \frac{\partial P_n^{(0)}}{\partial X} - \left( \phi_{n,q=0} + R_n \frac{\partial h}{\partial X} P_n^{(0)} \right) \right) \]
\[ + O(\epsilon^2) = 0 \quad (B.14) \]
\[ \left( \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \phi_{n,q=0}^{'} P_n^{(0)} \right) + \epsilon \left( \mu^2 g h^{1/2} \frac{\partial h}{\partial X} \frac{\partial \eta^{(0)}}{\partial x} + \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \phi_{n,q=0}^{'} P_n^{(1)} \right) \]
\[ + \epsilon \left( \mu^2 g h^{1/2} \frac{\partial h}{\partial X} \frac{\partial \eta^{(1)}}{\partial x} + \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \phi_{n,q=0}^{'} P_n^{(1)} \right) \]
\[ + \mu^2 g h^{1/2} \frac{\partial \eta^{(0)}}{\partial X} + \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( \Lambda_{nm} P_n^{(0)} + \mu^2 h h \frac{\partial P_n^{(0)}}{\partial x} \right) \]
\[ + \epsilon \left( \mu^2 g h^{1/2} \frac{\partial \eta^{(1)}}{\partial X} + \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( \Lambda_{nm} P_n^{(1)} + \mu^2 h h \frac{\partial P_n^{(1)}}{\partial x} \right) \right) \]
\[ + \mu^2 g h^{1/2} \frac{\partial \eta^{(0)}}{\partial X} + \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( \Lambda_{nm} P_n^{(0)} + \mu^2 h h \frac{\partial P_n^{(0)}}{\partial x} \right) \]
\[ + \frac{\partial^2 \eta^{(0)}}{\partial X \partial x} + \sum_{n=1}^{N} \sum_{n=1}^{N} \mu^{\hat{\beta}_n + 2} \left( 2 h \frac{\partial h}{\partial X} \left( \Theta_{nm} - \Phi_{nm} \right) - \frac{\partial P_n^{(0)}}{\partial x} \right) \]
\[ + h^2 \xi_{nm} \left( \frac{\partial^2 P_n^{(0)}}{\partial X \partial x} + \frac{\partial^2 P_n^{(0)}}{\partial X \partial x} \right) \]
\[ + O(\epsilon^2) = 0 \quad (B.16) \]

The equations above are linearly independent in terms of \( \epsilon \) and thus we can organize the governing equations into two sets of equations of both \( O(1) \) and \( O(\epsilon) \) as follows,
\( O(1) : \)

\[
\frac{\partial \eta^{(0)}}{\partial t} + h \frac{\partial \bar{u}^{(0)}}{\partial x} = 0 \tag{B.17}
\]

\[
h \frac{\partial \bar{u}^{(0)}}{\partial t} + g h \frac{\partial \eta^{(0)}}{\partial x} + h \sum_{n=1}^{N} \mu \hat{\beta}_n G_n \frac{\partial P_n^{(0)}}{\partial x} = 0 \tag{B.18}
\]

\[
\sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0}^{'} P_n^{(0)} = 0 \tag{B.19}
\]

\[
\mu^2 g h^2 \Omega_m \frac{\partial^2 \eta^{(0)}}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \Lambda_{nm} P_n^{(0)} + \mu^2 h^2 \xi_{nm} \frac{\partial^2 P_n^{(0)}}{\partial x^2} \right) = 0 \tag{B.20}
\]

\( O(\epsilon) : \)

\[
\frac{\partial \eta^{(1)}}{\partial t} + h \frac{\partial \bar{u}^{(1)}}{\partial x} = 0 \tag{B.21}
\]

\[
h \frac{\partial \bar{u}^{(1)}}{\partial t} + g h \frac{\partial \eta^{(1)}}{\partial x} + g h \frac{\partial \eta^{(0)}}{\partial \bar{X}} + h \sum_{n=1}^{N} \mu \hat{\beta}_n \left( G_n \frac{\partial P_n^{(1)}}{\partial \bar{X}} + h G_n \frac{\partial P_n^{(0)}}{\partial \bar{X}} - (\phi_{n,q=0}^{'} + R_n) \frac{\partial \bar{h}}{\partial \bar{X}} P_n^{(0)} \right) = 0 \tag{B.22}
\]

\[
\mu^2 g h \frac{\partial \eta^{(0)}}{\partial \bar{X}} + \sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0}^{'} P_n^{(1)} + h \frac{\partial \bar{h}}{\partial \bar{X}} \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \phi_{n,q=0}^{'} \frac{\partial P_n^{(0)}}{\partial x} = 0 \tag{B.23}
\]

\[
\mu^2 g h^2 \Omega_m \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \Lambda_{nm} P_n^{(1)} + \mu^2 h^2 \xi_{nm} \frac{\partial^2 P_n^{(1)}}{\partial x^2} \right) +
\]

\[
\mu^2 g h^2 \Omega_m \left( \frac{\partial^2 \eta^{(0)}}{\partial \bar{X} \partial x} + \frac{\partial^2 \eta^{(0)}}{\partial x \partial \bar{X}} \right) + \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \left( 2 h \frac{\partial \bar{h}}{\partial \bar{X}} (\Theta_{nm} - \Phi_{nm}) \frac{\partial P_n^{(0)}}{\partial \bar{X}} +
\]

\[
h^2 \xi_{nm} \left( \frac{\partial^2 P_n^{(0)}}{\partial \bar{X} \partial x} + \frac{\partial^2 P_n^{(0)}}{\partial x \partial \bar{X}} \right) \right) = 0 \tag{B.24}
\]

B.1.4 General Solution

In this section we assume that each of the dependent variables has a general solution of the form,

\[
(*)^{(0,1)} = (\tilde{*})^{(0,1)} e^{i\psi}, \tag{B.25}
\]
such that,

\[
\frac{\partial(\ast)^{(0,1)}}{\partial t} = -i\sigma(\ast)^{(0,1)} e^{i\psi}, \quad \frac{\partial(\ast)^{(0,1)}}{\partial x} = ik(\ast)^{(0,1)} e^{i\psi},
\]

\[
\frac{\partial^2(\ast)^{(0,1)}}{\partial x^2} = -k^2(\ast)^{(0,1)} e^{i\psi}, \quad \frac{\partial(\ast)^{(0,1)}}{\partial X} = \frac{\partial(\ast)^{(0,1)}}{\partial X} e^{i\psi},
\]

\[
\frac{\partial^2(\ast)^{(0,1)}}{\partial X \partial x} = ik \frac{\partial(\ast)^{(0,1)}}{\partial X} e^{i\psi}, \quad \frac{\partial^2(\ast)^{(0,1)}}{\partial X \partial x} = ik \frac{\partial(\ast)^{(0,1)}}{\partial X} e^{i\psi} + i\frac{\partial k}{\partial X} (\ast)^{(0,1)} e^{i\psi}.
\]

Substitution of (B.26) into equations (B.17)–(B.20) and (B.21)–(B.24) provides the following set of equations for the dependent variables,
\( O(1) : \)

\[
(k h \tilde{u}^{(0)} - \sigma \tilde{\eta}^{(0)}) i e^{i \psi} = 0 \tag{B.27}
\]

\[
\left( g k \tilde{\eta}^{(0)} - \sigma \tilde{u}^{(0)} + \sum_{n=1}^{N} \mu^{\beta_n} k G_n \tilde{P}_n^{(0)} \right) i h e^{i \psi} = 0 \tag{B.28}
\]

\[
\left( \sum_{n=1}^{N} \mu^{\beta_n} \phi_{n,q=0}^{'} \tilde{P}_n^{(0)} \right) e^{i \psi} = 0 \tag{B.29}
\]

\[
\left( -\mu^2 g (kh)^2 \Omega_m \tilde{\eta}^{(0)} + \sum_{n=1}^{N} \mu^{\beta_n} \left( \Lambda_{nm} - \mu^2 (kh)^2 \xi_{nm} \right) \tilde{P}_n^{(0)} \right) e^{i \psi} = 0 \tag{B.30}
\]

\( O(\epsilon) : \)

\[
\left( i k h \tilde{u}^{(1)} - i \sigma \tilde{\eta}^{(1)} + h \frac{\partial \tilde{u}^{(0)}}{\partial X} + \frac{\partial h}{\partial X} \tilde{u}^{(0)} \right) e^{i \psi} = 0 \tag{B.31}
\]

\[
\left( i g k h \tilde{\eta}^{(1)} - i h \sigma \tilde{u}^{(1)} + \sum_{n=1}^{N} \mu^{\beta_n} \left( i k h G_n \tilde{P}_n^{(1)} + h G_n \frac{\partial \tilde{P}_n^{(0)}}{\partial X} - (\phi_{n,q=0} + R_n) \frac{\partial h}{\partial X} \tilde{P}_n^{(0)} \right) \right) e^{i \psi} = 0 \tag{B.32}
\]

\[
(i \mu^2 g k h \frac{\partial h}{\partial X} \tilde{\eta}^{(0)} + \sum_{n=1}^{N} \mu^{\beta_n} \phi_{n,q=0}^{'} \tilde{P}_n^{(1)} + i k h \frac{\partial h}{\partial X} \sum_{n=1}^{N} \mu^{\beta_n+2} \phi_{n,q=0} \tilde{P}_n^{(0)} \right) e^{i \psi} = 0 \tag{B.33}
\]

\[
\left( -\mu^2 g (kh)^2 \Omega_m \tilde{\eta}^{(1)} + \sum_{n=1}^{N} \mu^{\beta_n} \left( \Lambda_{nm} - \mu^2 (kh)^2 \xi_{nm} \right) \tilde{P}_n^{(1)} + i \mu^2 g h^2 \left( \frac{\partial k}{\partial X} \tilde{\eta}^{(0)} + 2 k \frac{\partial \tilde{\eta}^{(0)}}{\partial X} \right) + \sum_{n=1}^{N} \mu^{\beta_n+2} \left( i h \left( \frac{\partial k}{\partial X} \xi_{nm} + 2 k \frac{\partial h}{\partial X} (\Theta_{nm} - \Phi_{nm}) \right) \tilde{P}_n^{(0)} + 2 i k h^2 \xi_{nm} \frac{\partial \tilde{P}_n^{(0)}}{\partial X} \right) \right) e^{i \psi} = 0 \tag{B.34}
\]

B.1.5 Transition Functions

In this section we construct transition functions based on the assumption that both \( \tilde{u}^{(0,1)} \) and \( \tilde{P}_n^{(0,1)} \) are proportional to \( \tilde{\eta}^{(0)} \). In addition we neglect \( \tilde{\eta}^{(1)} \). The basic transition functions are as follows,
\[ \tilde{u}^{(0)} = \sigma S_0 \tilde{\eta}^{(0)}, \quad \tilde{u}^{(1)} = i \sigma \frac{\partial h}{\partial X} S_1 \tilde{\eta}^{(0)}, \]
\[ \tilde{P}_n^{(0)} = g T_n^{(0)} \tilde{\eta}^{(0)}, \quad \tilde{P}_n^{(1)} = i g \frac{\partial h}{\partial X} T_n^{(1)} \tilde{\eta}^{(0)}. \]  \hspace{1cm} (B.35)

We also note that equations (B.31)–(B.34) also contain slow derivatives of the dependent variables. In order to fully simplify the governing equations we must determine a set of transition functions for these terms as well. We first consider an application of the chain rule such that,
\[ \frac{\partial (\ast)}{\partial X} = \frac{\partial (\ast)}{\partial (kh)} \frac{\partial (kh)}{\partial X}. \]  \hspace{1cm} (B.36)

Furthermore, by the product rule,
\[ \frac{\partial (kh)}{\partial X} = h \frac{\partial k}{\partial X} + k \frac{\partial h}{\partial X}. \]  \hspace{1cm} (B.37)

In order to find an equation for the slow derivative of \( k \) we first consider a function \( Q \) which is dependent on the nondimensional number \( kh \) and is related to the dispersion relationship,
\[ Q(kh) = \frac{\sigma^2}{gk}. \]  \hspace{1cm} (B.38)

We then consider the slow spatial derivative of \( Q \) and apply the chain rule as follows,
\[
\frac{\partial Q}{\partial X} = \frac{\partial Q}{\partial (kh)} \frac{\partial (kh)}{\partial X},
\]
\[
= \frac{\partial Q}{\partial (kh)} \left( h \frac{\partial k}{\partial X} + k \frac{\partial h}{\partial X} \right),
\]
\[
= h \frac{\partial Q}{\partial (kh)} \frac{\partial k}{\partial X} + k \frac{\partial h}{\partial X} \frac{\partial Q}{\partial (kh)},
\]
(B.39)

and

\[
\frac{\partial Q}{\partial X} = \frac{\partial Q}{\partial k} \frac{\partial k}{\partial X},
\]
\[
= -\frac{\sigma^2}{g k^2} \frac{\partial k}{\partial X}. 
\]
(B.40)

Considering that equations (B.39) and (B.40) are equivalent we can solve for \( \frac{\partial k}{\partial X} \) as follows,

\[
\frac{\partial k}{\partial X} = -\frac{k}{h} \frac{\partial h}{\partial X} \hat{Q},
\]
(B.41)

where,

\[
\hat{Q} = \left( \frac{k h \frac{\partial Q}{\partial (kh)}}{k h \frac{\partial Q}{\partial (kh)} + Q} \right).
\]
(B.42)

Substituting equations (B.41) and (B.37) into equation (B.36) we get an expression for the slow derivatives of the dependent variables,

\[
\frac{\partial \bar{u}^{(0)}}{\partial X} = \sigma \left( k \frac{\partial h}{\partial X} (1 - \hat{Q}) \frac{\partial S_0}{\partial (kh)} \tilde{\eta}^{(0)} + S_0 \frac{\partial \tilde{\eta}^{(0)}}{\partial X} \right),
\]
(B.43)

\[
\frac{\partial \bar{P}_n^{(0)}}{\partial X} = g \left( k \frac{\partial h}{\partial X} (1 - \hat{Q}) \frac{\partial T_n^{(0)}}{\partial X} \tilde{\eta}^{(0)} + T_n^{(0)} \frac{\partial \tilde{\eta}^{(0)}}{\partial X} \right).
\]
(B.44)
Finally, before simplifying the equations we consider that the shoaling coefficient $\gamma_h$ is given by the following relationship,

$$\frac{\partial \tilde{\eta}^{(0)}}{\partial X} = \left( \frac{\partial h}{\partial X} \right) \tilde{\eta}^{(0)} \gamma_h$$  \hspace{1cm} (B.45)

B.1.6 System Of Equations

Substitution of equations (B.35), (B.43), (B.44) and (B.45) into equations (B.27)–(B.34) provides the full system of equations for the linear properties of the model;
O(1):

$$\left( khS_0 - 1 \right) i\sigma \bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.46)}$$

$$\left( gk - \sigma^2 S_0 + gk \sum_{n=1}^{N} \mu^{\beta_n} G_n T_n^{(0)} \right) i\eta^{(0)} e^{i\psi} = 0 \quad \text{(B.47)}$$

$$\left( \sum_{n=1}^{N} \mu^{\beta_n} \phi_{n,q=0} T_n^{(0)} \right) g\bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.48)}$$

$$\left( \mu^2 (kh)^2 \Omega_m - \sum_{n=1}^{N} \mu^{\beta_n} \left( \Lambda_{nm} - \mu^2 (kh)^2 \xi_{nm} \right) T_n^{(0)} \right) g\bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.49)}$$

O(\epsilon):

$$\left( S_0 \gamma_h - khS_1 + S_0 - kh(1 - \hat{Q}) \frac{\partial S_0}{\partial (kh)} \right) \frac{\partial h}{\partial X} \sigma \bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.50)}$$

$$\left[ \left( 1 + \sum_{n=1}^{N} \mu^{\beta_n} G_n T_n^{(0)} \right) \gamma_h + khQS_1 - kh \sum_{n=1}^{N} \mu^{\beta_n} G_n T_n^{(1)} \right] \frac{\partial h}{\partial X} \bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.51)}$$

$$\left( \mu^2 kh + \sum_{n=1}^{N} \mu^{\beta_n} \phi_{n,q=0} T_n^{(1)} + kh \sum_{n=1}^{N} \mu^{\beta_n+2} \phi_{n,q=0} T_n^{(0)} \right) ig \frac{\partial h}{\partial X} \bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.52)}$$

$$\left( 2kh \left( 2\mu^2 \Omega_m + \sum_{n=1}^{N} \mu^{\beta_n+2} \xi_{nm} T_n^{(0)} \right) \gamma_h + \sum_{n=1}^{N} \mu^{\beta_n} \left( \Lambda_{nm} - \mu^2 (kh)^2 \xi_{nm} \right) T_n^{(1)} - \mu^2 kh \Omega_m \hat{Q} + kh \sum_{n=1}^{N} \mu^{\beta_n+2} \left( 2\Theta_{nm} - 2\Phi_{nm} - \hat{Q} \xi_{nm} \right) T_n^{(0)} \right) \frac{\partial ^2 T_n^{(0)}}{\partial (kh)} \frac{\partial h}{\partial X} \bar{\eta}^{(0)} e^{i\psi} = 0 \quad \text{(B.53)}$$

B.2 Dispersion Relationship: \(\sigma/k\)

To determine the dispersion relationship we first solve equation (B.46) for \(S_0^{(0)}\):

$$S_0^{(0)} = \frac{1}{kh} \quad \text{(B.54)}$$
We then substitute equation (B.54) into equation (B.47) and solve for \(\sigma\):

\[
gk - \sigma^2 S_0 + gk \sum_{n=1}^{N} \mu^\beta_n G_n T_n^{(0)} = 0
\]

\[
\Rightarrow \frac{\sigma^2}{kh} = gk \left( 1 + \sum_{n=1}^{N} \mu^\beta_n G_n T_n^{(0)} \right)
\]

\[
\Rightarrow \frac{\sigma^2}{gk^2h} = 1 + \sum_{n=1}^{N} \mu^\beta_n G_n T_n^{(0)} \quad (B.55)
\]

Furthermore from this expression we find that,

\[
Q = kh \left( 1 + \sum_{n=1}^{N} \mu^\beta_n G_n T_n^{(0)} \right). \quad (B.56)
\]

We note that if only hydrostatic pressure is taken, i.e. \(T_n^{(0)} = 0, \forall n\), then the well known dispersion relationship for the shallow water equations is obtained,

\[
\frac{\sigma}{k} = \sqrt{gh}. \quad (B.57)
\]

Furthermore we note that the dispersion relationship is directly related to the dynamic pressure through the solutions to the \(T_n^{(0)}\) terms. These can be found by solving the system of equations given by equations (B.48) and (B.49).

Once the approximate dispersion relationship for this system is determined it can be compared with the Airy solution for the dispersion relationship [17], given by,

\[
\frac{\sigma}{k} = \sqrt{\frac{gh}{kh}} \tanh(kh) \quad (B.58)
\]

B.3 Shoaling Relationship: \(\gamma_h\)

Solving for the shoaling coefficient, \(\gamma_h\), is considerably more difficult in that the linear system is more complicated. In order to find the \(O(\epsilon)\) terms we must first have
solved the $O(1)$ expressions to find expressions for $\sigma$, $S_0$ and the $T_n^{(0)}$s. From these we can determine $Q$ and $\hat{Q}$. We then setup a linear system of equations for $\gamma_h$, $S_1$ and the $T_n^{(1)}$s from equations (B.50) – (B.53). This system takes the form,

$$
\begin{bmatrix}
S_0 & -kh & 0 & \cdots & 0 \\
A_0 & khQ & -kh\mu \hat{\beta}_1 G_1 & \cdots & -kh\mu \hat{\beta}_N G_N \\
0 & 0 & \mu \hat{\beta}_1 \phi_{1,l=0} & \cdots & \mu \hat{\beta}_N \phi_{N,l=0} \\
A_1 & 0 & B_{1,1} & \cdots & B_{N,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{N-1} & 0 & B_{1,N-1} & \cdots & B_{N,N-1}
\end{bmatrix}
\begin{bmatrix}
\gamma_h \\
S_1 \\
T_1^{(1)} \\
\vdots \\
T_{N}^{(1)}
\end{bmatrix}
= \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
D_{N-1}
\end{bmatrix}
$$

(B.59)

where,

$$A_0 = \left(1 + \sum_{n=1}^{N} \mu \hat{\beta}_n G_n T_n^{(0)}\right)$$

(B.60)

$$A_m = 2kh \left(\mu^2 \Omega_m + \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \xi_{nm} T_n^{(0)}\right)$$

(B.61)

$$B_{n,m} = \mu \hat{\beta}_i (A_{nm} - \mu \mu^2 (kh)^2 \xi_{nm})$$

(B.62)

$$C_1 = kh(1 - \hat{Q}) \frac{\partial S_0}{\partial (kh)} - S_0$$

(B.63)

$$C_2 = \sum_{n=1}^{N} \mu \hat{\beta}_n \left((\phi_{n,q=0} + R_n)T_n^{(0)} + kh(\hat{Q} - 1)G_n \frac{\partial T_n^{(0)}}{\partial (kh)}\right)$$

(B.64)

$$C_3 = -kh \left(\mu^2 + \sum_{n=1}^{N} \mu \hat{\beta}_n \left((\phi_{n,q=0} + R_n)T_n^{(0)} + kh(\hat{Q} - 1)G_n \frac{\partial T_n^{(0)}}{\partial (kh)}\right)\right)$$

(B.65)

$$D_m = \mu^2 kh \Omega_m \hat{Q} -$$

$$kh \sum_{n=1}^{N} \mu \hat{\beta}_n + 2 \left((2\Theta_{nm} - 2\Phi_{nm} - \hat{Q} \xi_{nm})T_n^{(0)} + 2kh(1 - \hat{Q} \xi_{nm}) \frac{\partial T_n^{(0)}}{\partial (kh)}\right)$$

(B.66)

The linear system of equations given by (B.59) can be solved to determine the approximate shoaling coefficient, $\gamma_h$, for this model, which can then be compared
with the *Airy* solution for the shoaling coefficient given by,

\[
\gamma_h = -\frac{2kh \sinh(2kh) + 2(kh)^2(1 - \cosh(2kh))}{(2kh + \sinh(2kh))^2}
\]  
(B.67)
APPENDIX C

NONLINEAR PROPERTIES

In this appendix the nonlinear properties of the pressure–Poisson based model are detailed. In particular, approximate solutions to the second Stokes harmonic are examined.

C.1 Setting Up The Equations

The process for determining the approximate second harmonic of this model is as follows,

1. We only examine the special case of the equations for a flat bed topography, i.e. \( h \) is constant,

2. We assume a perturbation expansion of the dependent variables with respects to \( \delta \),

3. We assume a general solution to each of the terms in the perturbation expansion,

4. Finally, for simplicity we define so-called transition functions to simplify the system of equations.
C.1.1 Flatbed Equations

We first consider the set of governing equations for the flatbed case, i.e. \( h \) is constant and thus all gradients of \( h \) go to zero,

\[
\delta \left( \frac{\partial \eta}{\partial t} + (h + \delta \eta) \frac{\partial \bar{u}}{\partial x} + \delta \frac{\partial \eta}{\partial x} \bar{u} \right) = 0, \tag{C.1}
\]

\[
\delta \left( (h + \delta \eta) \frac{\partial \bar{u}}{\partial t} + \delta(h + \delta \eta) \bar{u} \frac{\partial \bar{u}}{\partial x} + g(h + \delta \eta) \frac{\partial \eta}{\partial x} + \right.
\]

\[
\sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( (h + \delta \eta)G_n \frac{\partial P_n}{\partial x} - \delta R_n \frac{\partial \eta}{\partial x} P_n \right) \bigg) = 0, \tag{C.2}
\]

\[
\delta \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \phi'_{n,q} = 0. \tag{C.3}
\]

\[
\delta \mu^2 g(h + \delta \eta)^2 \Omega_m \frac{\partial^2 \eta}{\partial x^2} + 2 \delta^2 \mu^2(h + \delta \eta)^2 \Omega_m \left( \frac{\partial \bar{u}}{\partial x} \right)^2 +
\]

\[
\delta \sum_{n=1}^{N} \mu^{\hat{\beta}_n} \left( \mu^2(h + \delta \eta)^2 \xi_{nm} \frac{\partial^2 P_n}{\partial x^2} - 2 \delta \mu^2(h + \delta \eta) \Phi_{nm} \frac{\partial \eta}{\partial x} \frac{\partial P_n}{\partial x} \right.
\]

\[
\left. + \left( \Lambda_{nm} + \delta^2 \mu^2(2 \Phi_{nm} + \Psi_{nm}) \left( \frac{\partial \eta}{\partial x} \right)^2 - \delta \mu^2(h + \delta \eta) \Phi_{nm} \frac{\partial^2 \eta}{\partial x^2} \right) P_n \right) = 0,
\]

\[
m = 1, \ldots, N - 1. \tag{C.4}
\]

C.1.2 Perturbation Expansion

Similar to the approach in the appendix on the linear properties, appendix B, we consider a perturbation expansion of the dependent variables, \( \eta, \bar{u} \) and \( P_n \), such that,

\[
(*) = (\ast)^{(0)} + \delta(\ast)^{(1)} + O(\delta^2), \tag{C.5}
\]
where (∗) represents one of the dependent variables. Substituting equation \((C.5)\) into equations \((C.1)–(C.4)\) and retaining only those terms up to \(O(\delta^2)\) we get,

\[
\delta \left( \frac{\partial \eta(0)}{\partial t} + h \frac{\partial u(0)}{\partial x} \right) + \\
\delta^2 \left( \frac{\partial \eta(1)}{\partial t} + h \frac{\partial u(1)}{\partial x} + \eta(0) \frac{\partial u(0)}{\partial x} + \frac{\partial \eta(0)}{\partial x} u(0) \right) + O(\delta^3) = 0, \tag{C.6}
\]

\[
\delta \left( h \frac{\partial u(0)}{\partial t} + g \frac{\partial \eta(0)}{\partial x} + \sum_{n=1}^{N} \mu_n \beta_n G_n \frac{\partial P_n(0)}{\partial x} \right) + \\
\delta^2 \left( h \frac{\partial u(1)}{\partial t} + g \frac{\partial \eta(1)}{\partial x} + \sum_{n=1}^{N} \mu_n \beta_n G_n \frac{\partial P_n(0)}{\partial x} + \eta(0) \frac{\partial u(0)}{\partial t} + h u(0) \frac{\partial u(0)}{\partial x} + \\
g \eta(0) \frac{\partial \eta(0)}{\partial x} + \sum_{n=1}^{N} \mu_n \beta_n \left( \eta(0) G_n \frac{\partial P_n(0)}{\partial x} - R_n \frac{\partial \eta(0)}{\partial x} P_n(0) \right) \right) + O(\delta^3) = 0, \tag{C.7}
\]

\[
\delta \left( \sum_{n=1}^{N} \mu_n \beta_n \phi_{n,q=0} P_n(0) \right) + \delta^2 \left( \sum_{n=1}^{N} \mu_n \beta_n \phi_{n,q=0} P_n(1) \right) + O(\delta^3) = 0, \tag{C.8}
\]

\[
\delta \left( \mu^2 g h^2 \Omega_m \frac{\partial^2 \eta(0)}{\partial x^2} + \sum_{n=1}^{N} \mu_n \beta_n \left( \Lambda_{nm} P_n(0) + \mu^2 h^2 \xi_{nm} \frac{\partial P_n(0)}{\partial x} \right) \right) + \\
\delta^2 \left( \mu^2 g h^2 \Omega_m \frac{\partial^2 \eta(1)}{\partial x^2} + \sum_{n=1}^{N} \mu_n \beta_n \left( \Lambda_{nm} P_n(1) + \mu^2 h^2 \xi_{nm} \frac{\partial P_n(0)}{\partial x} \right) + \\
2 \mu^2 h^2 \Omega_m \left( \frac{\partial u(0)}{\partial x} \right)^2 + 2 \mu^2 g h \Omega_m \eta(0) \frac{\partial^2 \eta(0)}{\partial x^2} + \\
\sum_{n=1}^{N} \mu_n \beta_n + 2 \left( 2 \xi_{nm} \eta(0) \frac{\partial P_n(0)}{\partial x^2} - 2 \Phi_{nm} \frac{\partial \eta(0)}{\partial x} \frac{\partial P_n(0)}{\partial x} - \Phi_{nm} \frac{\partial^2 \eta(0)}{\partial x^2} P_n(0) \right) \right) + \\
O(\delta^3) = 0, \quad m = 1, \ldots, N - 1. \tag{C.9}
\]

The equations above are linearly independent with respect to \(\delta\) and thus we can organize the governing equations into two sets of equations of both \(O(\delta)\) and \(O(\delta^2)\)
as follows,

\[O(\delta) :\]
\[
\left( \frac{\partial \eta^{(0)}}{\partial t} + h \frac{\partial u^{(0)}}{\partial x} \right) = 0, \quad \text{(C.10)}
\]
\[
\left( h \frac{\partial u^{(0)}}{\partial t} + g \frac{\partial \eta^{(0)}}{\partial x} + \sum_{n=1}^{N} \mu \hat{\beta}_n G_n \frac{\partial P_n^{(0)}}{\partial x} \right) = 0, \quad \text{(C.11)}
\]
\[
\left( \sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0} \frac{P_n^{(0)}}{\partial x} \right) = 0, \quad \text{(C.12)}
\]
\[
\left( \mu^2 g h^2 \Omega_m \frac{\partial^2 \eta^{(0)}}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \Lambda_{nm} P_n^{(0)} + \mu^2 h^2 \xi_{nm} \frac{\partial P_n^{(0)}}{\partial x^2} \right) \right) = 0, \quad \text{for } m = 1, \ldots, N - 1, \quad \text{(C.13)}
\]

\[O(\delta^2) :\]
\[
\left( \frac{\partial \eta^{(1)}}{\partial t} + h \frac{\partial u^{(1)}}{\partial x} + \eta^{(0)} \frac{\partial u^{(0)}}{\partial x} + \frac{\partial \eta^{(0)}}{\partial x} u^{(0)} \right) = 0, \quad \text{(C.14)}
\]
\[
\left( h \frac{\partial u^{(1)}}{\partial t} + g \frac{\partial \eta^{(1)}}{\partial x} + \sum_{n=1}^{N} \mu \hat{\beta}_n \frac{\partial P_n^{(1)}}{\partial x} + \frac{\partial \eta^{(0)}}{\partial t} u^{(0)} + h u^{(0)} \frac{\partial u^{(0)}}{\partial x} + g \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial x} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \eta^{(0)} G_n \frac{\partial P_n^{(0)}}{\partial x} - R_n \frac{\partial \eta^{(0)}}{\partial x} \frac{P_n^{(0)}}{\partial x} \right) \right) = 0, \quad \text{(C.15)}
\]
\[
\left( \sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0} \frac{P_n^{(1)}}{\partial x} \right) = 0, \quad \text{(C.16)}
\]
\[
\left( \mu^2 g h^2 \Omega_m \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \Lambda_{nm} P_n^{(1)} + \mu^2 h^2 \xi_{nm} \frac{\partial P_n^{(1)}}{\partial x^2} \right) \right) + 2 \mu^2 h^2 \Omega_m \left( \frac{\partial u^{(0)}}{\partial x} \right)^2 + 2 \mu^2 g h \Omega_m \eta^{(0)} \frac{\partial^2 \eta^{(0)}}{\partial x^2} + \sum_{n=1}^{N} \mu \hat{\beta}_n^2 \left( 2 \xi_{nm} \eta^{(0)} \frac{\partial^2 P_n^{(0)}}{\partial x^2} - 2 \Phi_{nm} \frac{\partial \eta^{(0)}}{\partial x} \frac{\partial P_n^{(0)}}{\partial x} - \phi_{nm} \frac{\partial \eta^{(0)}}{\partial x^2} \frac{P_n^{(0)}}{\partial x} \right) = 0, \quad \text{for } m = 1, \ldots, N - 1. \quad \text{(C.17)}
\]
C.1.3 General Solutions

Once again, similar to the approach in the appendix on linear properties we consider a general solution for each component of the perturbation expansion,

\[ (*)^{(0)} = (\tilde{(*)})^{(0)} e^{i\psi}, \quad (C.18) \]
\[ (*)^{(1)} = (\tilde{(*)})^{(1)} e^{2i\psi}, \quad (C.19) \]

such that

\[ \frac{\partial (*)^{(0)}}{\partial t} = -i\sigma (\tilde{(*)})^{(0)} e^{i\psi}, \quad \frac{\partial (*)^{(0)}}{\partial x} = ik (\tilde{(*)})^{(0)} e^{i\psi}, \quad \frac{\partial^2 (*)^{(0)}}{\partial x^2} = -k^2 (\tilde{(*)})^{(0)} e^{i\psi}, \quad (C.20) \]
\[ \frac{\partial (*)^{(1)}}{\partial t} = -2i\sigma (\tilde{(*)})^{(1)} e^{2i\psi}, \quad \frac{\partial (*)^{(1)}}{\partial x} = 2ik (\tilde{(*)})^{(1)} e^{2i\psi}, \quad \frac{\partial^2 (*)^{(1)}}{\partial x^2} = -4k^2 (\tilde{(*)})^{(1)} e^{2i\psi}. \quad (C.21) \]

Substitution of (C.20) and (C.21) into equations (C.10)–(C.17) provides the fol-
following equations for the dependent variables,

$$O(\delta) :$$

$$\left( \sigma \tilde{\eta}^{(0)} - kh \tilde{u}^{(0)} \right) e^{i\psi} = 0,$$

$$\left( h \sigma \tilde{u}^{(0)} - gkh \tilde{\eta}^{(0)} - kh \sum_{n=1}^{N} \mu \hat{\beta} \eta_n \tilde{P}_n^{(0)} \right) e^{i\psi} = 0,$$

$$\left( \sum_{n=1}^{N} \mu \hat{\beta} \phi_{n,q} \tilde{P}_n^{(0)} \right) e^{i\psi} = 0,$$

$$\left( \mu^2 g(kh)^2 \tilde{\eta}^{(0)} + \sum_{n=1}^{N} \mu \hat{\beta} \left( \mu^2 (kh)^2 \xi_{nm} - \Lambda_{nm} \right) \tilde{P}_n^{(0)} \right) e^{i\psi} = 0,$$

$$m = 1, \ldots, N - 1. \quad (C.25)$$

$$O(\delta^2) :$$

$$\left( 2\sigma \tilde{\eta}^{(1)} - 2kh \tilde{u}^{(1)} - k\tilde{\eta}^{(0)} \tilde{u}^{(0)} \right) e^{2i\psi} = 0,$$

$$\frac{1}{2} \left( 4h \sigma \tilde{u}^{(1)} - 4gkh \tilde{\eta}^{(1)} - 4kh \sum_{n=1}^{N} \mu \hat{\beta} \eta_n \tilde{P}_n^{(1)} + \sigma \tilde{\eta}^{(0)} \tilde{u}^{(0)} \right.\right.$$

\hspace{2cm}$$- kh \left( \tilde{u}^{(0)} \right)^2 - gk \left( \tilde{\eta}^{(0)} \right)^2 - k \sum_{n=1}^{N} \mu \hat{\beta} \left( G_n - R_n \right) \tilde{\eta}^{(0)} \tilde{P}_n^{(0)} \left. \right) e^{2\psi} = 0,$$

$$\left( \sum_{n=1}^{N} \mu \hat{\beta} \phi_{n,q} \tilde{P}_n^{(1)} \right) e^{2i\psi} = 0,$$

$$\left( 4\mu^2 g(kh)^2 \tilde{\eta}^{(1)} + \sum_{n=1}^{N} \mu \hat{\beta} \left( 4\mu^2 (kh)^2 \xi_{nm} - \Lambda_{nm} \right) \tilde{P}_n^{(1)} + 2\mu^2 gk^2 h \Omega_m \left( \tilde{\eta}^{(0)} \right)^2 \right.$$\hspace{2cm}$$+ 2\mu^2 (kh)^2 \Omega_m \left( \tilde{u}^{(0)} \right)^2 + k^2 h \sum_{n=1}^{N} \mu \hat{\beta} \eta_n^2 \left( 2\xi_{nm} - 3\Phi_{nm} \right) \tilde{\eta}^{(0)} \tilde{P}_n^{(0)} \left. \right) e^{2i\psi} = 0,$$

$$m = 1, \ldots, N - 1. \quad (C.29)$$

C.1.4 Transition Functions

We define a set of so-called transition functions for $\tilde{\eta}^{(1)}, \tilde{P}_n^{(0,1)}$ and $\tilde{u}^{(0,1)}$ that will allow us simplify the equations. We assume that these dependent variables are
proportional to the free-surface displacement $\tilde{\eta}^{(0)}$ as follows,

\[
\tilde{\eta}^{(1)} = \frac{1}{h} \left( \tilde{\eta}^{(0)} \right)^2 \tilde{A},
\]

\[
\tilde{P}_n^{(0)} = g\tilde{\eta}^{(0)} T_n^{(0)},
\]

\[
\tilde{P}_n^{(1)} = \frac{g}{h} \left( \tilde{\eta}^{(0)} \right)^2 T_n^{(1)},
\]

(C.30)

\[
\tilde{\bar{u}}^{(0)} = \sigma \tilde{\eta}^{(0)} S_0,
\]

\[
\tilde{\bar{u}}^{(1)} = \frac{\sigma}{h} \left( \tilde{\eta}^{(0)} \right)^2 S_1.
\]
C.1.5  System Of Equations

Substitution of (C.30) into equations (C.22)–(C.29) provides a system of equations,

\[ O(\delta) : \]
\[ (khS_0 - 1) \sigma \tilde{\eta}^{(0)} e^{i\psi} = 0, \quad \text{(C.31)} \]
\[ \left( \sigma^2 S_0 - gk - gk \sum_{n=1}^{N} \mu \hat{\beta}_n G_n T_n^{(0)} \right) h \tilde{\eta}^{(0)} e^{i\psi} = 0, \quad \text{(C.32)} \]
\[ \left( \sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0} T_n^{(0)} \right) g \tilde{\eta}^{(0)} e^{i\psi} = 0, \quad \text{(C.33)} \]
\[ \left( \mu^2 (kh)^2 \Omega_m + \sum_{n=1}^{N} \mu \hat{\beta}_n \left( \mu^2 (kh) \xi_{nm} - \Lambda_{nm} \right) T_n^{(0)} \right) g \tilde{\eta}^{(0)} e^{i\psi} = 0, \quad m = 1, \ldots, N - 1, \quad \text{(C.34)} \]

\[ O(\delta^2) : \]
\[ \left( \tilde{A} - khS_1 - khS_0 \right) \sigma \left( \tilde{\eta}^{(0)} \right)^2 e^{2i\psi} = 0, \quad \text{(C.35)} \]
\[ \left( 2gk \tilde{A} - 2\sigma^2 S_1 + 2gk \sum_{n=1}^{N} \mu \hat{\beta}_n G_n T_n^{(1)} \right) \quad \text{(C.36)} \]
\[ \left( \sum_{n=1}^{N} \mu \hat{\beta}_n \phi_{n,q=0} T_n^{(1)} \right) g \left( \tilde{\eta}^{(0)} \right)^2 e^{2i\psi} = 0, \quad \text{(C.37)} \]
\[ \left( 4\mu^2 g(kh)^2 \Omega_m \tilde{A} + g \sum_{n=1}^{N} \mu \hat{\beta}_n \left( 4\mu^2 (kh)^2 \xi_{nm} - \Lambda_{nm} \right) T_n^{(1)} + 2\mu^2 g(kh)^2 \Omega_m + 2\mu^2 \sigma^2 (kh)^2 h \Omega_m S_0^2 + g(kh)^2 \sum_{n=1}^{N} \mu \hat{\beta}_n^{n+2} (2\xi_{nm} - 3\Phi_{nm}) T_n^{(0)} \right) \left( \tilde{\eta}^{(0)} \right)^2 e^{2i\psi} = 0, \quad m = 1, \ldots, N - 1. \quad \text{(C.38)} \]

Here we note that the \( O(\delta) \) system of equations, (C.31)–(C.34), are exactly the same as the \( O(1) \) system of equations from appendix B, equations (B.46)–(B.49). As
with the $O(1)$ equations from the linear properties we can use the $O(\delta)$ system of equations to find an expression for the dispersion relationship, as well as $S_0$ and the $T_n^{(0)}$. For brevity this process will not be discussed here, instead see section B.2.

C.2 Stokes Second Harmonic

Solving for the approximate second harmonic requires solving the system of equations given by equations (C.35)–(C.38), which can be written as,

$$
\begin{pmatrix}
1 & -kh & 0 & \cdots & 0 \\
2 & -2\frac{\sigma^2}{gk} & 2G_1 & \cdots & 2\mu^2 G_N \\
0 & 0 & \phi'_{1,q=0} & \cdots & \mu^2 N \phi'_{N,q=0} \\
4\mu^2(kh)^2\Omega_1 & 0 & B_{1,1} & \cdots & B_{N,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
4\mu^2(kh)^2\Omega_{N-1} & 0 & B_{1,N-1} & \cdots & B_{N,N-1}
\end{pmatrix}
\begin{pmatrix}
\tilde{A} \\
\tilde{T}_1^{(1)} \\
\tilde{T}_N^{(1)}
\end{pmatrix}
= 
\begin{pmatrix}
khS_0 \\
S_1 \\
C_2 \\
D_1 \\
D_{N-1}
\end{pmatrix},
$$

(C.39)

where,

$$
B_{n,m} = (4\mu^2(kh)^2\xi_{nm} - \Lambda_{nm}),
$$

(C.40)

$$
C_2 = -(khS_0 - 1)\frac{\sigma^2}{gk}S_0 - 1 - (G_1 - R_1)T_1^{(0)} - (G_2 - R_2)T_2^{(0)},
$$

(C.41)

$$
D_m = -2\mu^2(kh)^2\Omega_m - 2\mu^2(kh)^3\frac{\sigma^2}{gk}S_0^2\Omega_m - (kh)^2(2\xi_{1m} - 3\Phi_{1m})T_1^{(0)}.
$$

(C.42)

Once the system given by (C.39) has been solved the approximate second harmonic for this model is given by the solution for $\tilde{A}$ which can then be compared with the Airy solution for Stokes second harmonic [17],

$$
A_{Stokes} = \left(\frac{kh \cosh(kh)}{4(\sinh(kh))^2}\right) (2 + \cosh(2kh)).
$$

(C.43)
It is also important to point out at this point that for this particular model it is weakly nonlinear at order higher than $O(\mu^2)$, thus all terms in the original equations of $O(\delta^2 \mu^4)$ will be truncated. This is reflected in the right hand side terms of the system of equations, (C.41) and (C.42), where all $O(\mu^4)$ terms have been truncated since the right hand side terms represent components that were of $O(\delta^2)$ in the original equations.
APPENDIX D

PUBLICATIONS AND CONFERENCE PRESENTATIONS

D.1 Publications


2. A.S. Donahue, A.B. Kennedy, J.J. Westerink, Y. Zhang, C. Dawson, Simulation of wave phenomena in the nearshore through application of $O(\mu^2)$ and $O(\mu^4)$ pressure–Poisson Boussinesq type models, Coastal Engineering, Under Revision

D.2 Conference Presentations


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