ORIENTED ONE-DIMENSIONAL SUPERSYMMETRIC EUCLIDEAN FIELD THEORIES AND K-THEORY

A Dissertation

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This dissertation defines a simplicial set $1|1$–$EFT$ of certain functorial quantum field theories and determines that it has the homotopy type of a representing space for $K$-theory. There have been other spaces of 1-dimensional supersymmetric Euclidean field theories related to the $K$-theory spectrum; novel aspects of the approach in this dissertation are the use of non-projective modules in the algebraic codomain of the functors defining $1|1$–$EFT$ and the use of an oriented bordism category whose space of connected endomorphisms of the super-point is connected.
This work is dedicated to my wife.
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CHAPTER 1
INTRODUCTION

This dissertation shows that a space arising from physical ideas related to quantum field theory has a homotopy type interesting to mathematicians. Although the ideas in the definition of the space have a physical inspiration, everything beyond this introduction is purely mathematical.

1.1 Functorial formulation of quantum field theory

A functorial quantum field theory is a symmetric monoidal functor from a category of bordisms to some algebraic category. Such a functor is to encode desired properties of a not necessarily well-defined path integral.

Atiyah, Kontsevich, Segal, and Witten formalized quantum field theories as functors in the 1980s.

1.2 Non-linear $\sigma$-model and the Atiyah-Singer Index Theorem

One perspective on the functors studied in this dissertation is given by the following example of a supersymmetric non-linear $\sigma$-model with target a spin manifold. The quantization of this $\sigma$-model gives supersymmetric quantum mechanics. The discussion here follows [34]; see also [1], [7].

Consider a $\sigma$-model of maps $\gamma : \mathbb{R}^{1|1} \to X$. Here $X$ is an even-dimensional Riemannian manifold with spin structure.

Witten observes that the index of the Dirac operator $D$ on $X$ can be computed as the partition function of the theory. More precisely, he notes that $str(e^{-tD^2})$ is
independent of $t$. (Here $\text{str}$ denotes the super-trace, which is defined for parity-preserving endomorphisms of $\mathbb{Z}/2$ graded vector spaces as the ordinary trace on the even summand minus the ordinary trace on the odd summand.) In the limit of large $t$ one obtains the analytic index of $\mathcal{D}$. For small $t$ one expands perturbatively about the classical solutions and obtains a term giving the topological index.

In summary, from a spin manifold one obtains a supersymmetric 1-dimensional Euclidean field theory whose partition function gives the index of the Dirac operator. A family of spin manifolds should yield a family of Euclidean field theories whose partition function gives the family index, an element in the $K$-theory of the base. This is the goal of this dissertation — to establish that families of supersymmetric 1-dimensional Euclidean field theories yield $K$-theory elements compatibly with the case of the example just given.

1.3 Plan of this work

After this introduction, there is a chapter which develops the essential ideas for the definition of the simplicial set $1|1-EFT$ and enumerates its relevant properties.

Next is the chapter containing the main theorem. We state the main theorem and sketch its proof in the final portion of this introduction.

The final chapter collects peripheral results and presents full proofs of auxiliary statements.

Finally, there are appendices providing foundational definitions and examples for the constituent elements involved in defining $1|1-EFT$. The three appendices treat categories, topological vector spaces, and geometries for supermanifolds.
1.4 Context and relation to previous work

1.4.1 Context

One wishes to have geometric cocycles for the cohomology theory of topological modular forms. Segal [26] suggested that 2-dimensional conformal field theories would provide such a model. His ideas were developed by Stolz and Teichner [30] who propose 2|1-Euclidean field theories as the model for TMF. One result in this direction is the theorem that the partition function of a 2|1-Euclidean field theory is a holomorphic modular function [28].

The 2|1-dimensional case is an area of continuing work; results exist which relate 0|1 and 1|1-Euclidean field theories to other cohomology theories.

In [14] it is shown that concordance classes of 0|1-Euclidean (or topological) field theories yield de Rham cohomology, either with a $\mathbb{Z}/2$ grading or with the ordinary grading.

There are a variety of works relating 1|1-field theories and $K$-theory. In [30], field theories are identified with super semigroups of self-adjoint Clifford linear Hilbert-Schmidt operators on a Hilbert space. In [21] it is shown that one can obtain connective $ko$-theory by proceeding in a similar manner but considering right and left linearity with respect to two Clifford algebra actions. Finally, [13] shows that a space of 1|1-Euclidean field theories is a representing space for $KO$ by relating field theories to self-adjoint trace-class semigroups of operators on a Hilbert space.

1.4.2 Distinctive aspects of the present work

A principal distinction between the approach taken in this dissertation and that of previous work is that the algebraic object assigned to the super point is more general. Specifically, rather than requiring that an $S$-family of field theories is a functor whose algebraic target is essentially a category with a single object $S \times H$
(\(H\) an infinite dimensional Hilbert space), we allow that the algebraic target contains a broad class of modules over \(C^\infty(S)\), not only the projective ones. Recall that 1-dimensional topological field theories correspond to finite dimensional vector spaces. Working in families (i.e. in \(C^\infty(S)\)-modules), we get that \(S\)-families of 1-dimensional topological field theories correspond to dualizable modules. Such dualizable modules are finitely generated projective, hence vector bundles by Serre-Swan.

We would like to construct a model of 1|1-Euclidean field theories that incorporates topological theories as particular examples. On the other hand, in order to obtain the right homotopy type, we incorporate geometry and supersymmetry in the bordism category. One consequence of these additions is that it is possible to obtain non-dualizable objects. Concretely, the vector space \(\{0\}\) and the sum of an even and odd line both represent the same element in \(KO^0(pt)\). Our construction of Euclidean field theories means we can write down a family of field theories, parameterized by \(\mathbb{R}\), which restricts to these two vector spaces at \(0 \in \mathbb{R}\) and \(1 \in \mathbb{R}\), respectively. In order to have a single algebraic object parameterized by \(\mathbb{R}\) that restricts in this way, we necessarily use non-projective modules.

In previous models of 1|1-Euclidean field theories which assign infinite-dimensional Hilbert spaces to the (super) point, one must use a (necessarily finite rank) projection operator to view topological field theories as giving Euclidean ones, and the translation requires that there be two endomorphisms of the point - one which is the identity, and one which gets assigned the projection under the field theory functor. In the current approach one can directly produce a 1|1-Euclidean field theory from a topological one.

Another distinction of this dissertation is the use of a category of oriented bordisms. Previous work used a category of unoriented bordisms. A consequence of the choice of unoriented bordisms is that the operators assigned to intervals are symmetric with respect to the field theory’s pairing. (The pairing is the linear map arising
as the image of a bordism from a disjoint union of points to the empty set.) In the oriented case, one has instead of a symmetric operator simply a pairing exhibiting duality between the vector spaces assigned to the positively and negatively oriented super points. This makes the operators somewhat less well-behaved. The effort is worth it, though, for the following reason.

A 1-dimensional unoriented topological field theory corresponds to a finite dimensional vector space equipped with a non-degenerate symmetric pairing. While all such pairings are equivalent over \( \mathbb{C} \), over \( \mathbb{R} \) they are distinguished by their signature. Thus the path components of the space of 1-dimensional unoriented topological field theories are the submonoid of \( \mathbb{Z} \oplus \mathbb{Z} \) given by \((a, b)\) with \( a \in \mathbb{N} \cup \{0\} \) and \( b \) satisfying \(|b| \leq a\). The group completion of this monoid is \( \mathbb{Z} \oplus \mathbb{Z} \). This shows that unoriented field theories are not the correct framework for realizing the group completion defining \( KO \)-theory as the map on classifying spaces of field theories induced by a forgetful functor among bordism categories, and that the oriented bordism category is a more natural setting.

1.5 Statement of main theorem

We now state the main theorem, which will be proved in chapter 3.

**Theorem 1.1.** There are spaces of functors \( 1|1-\text{EFT} \) from a bordism category \( 1|1-\text{EBord} \) to an algebraic target of (families of) vector spaces \( TV(H) \) which represent real or complex \( K \)-theory. The domain \( 1|1-\text{EBord} \) is a category of oriented bordisms in which the moduli space of connected endomorphisms of a super point is connected.

A brief sketch of the method of proof is as follows. Quillen’s \( S^{-1}S \) construction constructs from a symmetric monoidal category a symmetric monoidal category whose classifying space is the group completion of the classifying space of the origi-
nal category. Applying this in the case of vector spaces, endowed with the monoidal structure of direct sum, we obtain a space representing real or complex $K$-theory, according to whether the input category consists of vector spaces over the real or complex numbers. We use a certain form of these spaces and denote it $\mathcal{N}Q\text{Vect}$. We then establish a homotopy equivalence between $\mathcal{N}Q\text{Vect}$ and $1|1\text{–EFT}$. In order to do this, we give a map $\mathcal{N}Q\text{Vect} \to 1|1\text{–EFT}$, and then establish that maps into $1|1\text{–EFT}$ may be subdivided, just as one does barycentric subdivision when proving excision, into a collection of maps that are essentially in the image of the chosen map $\mathcal{N}Q\text{Vect} \to 1|1\text{–EFT}$. Showing that this is so relies on a lemma about the continuous variation of the spectrum of a family of operators, which is proven by evaluating the field theory on a circle, which is also known as taking the partition function. With the lemma in hand, the family of field theories can be shown to be, locally, a sum of vector bundles — which are the sorts of things that come from $\mathcal{N}Q\text{Vect}$ — plus a strange summand that is not a vector bundle but that can be eliminated through reparameterizing the time parameter of the semigroup of endomorphisms.
CHAPTER 2

ESSENTIAL ELEMENTS

In this chapter we quickly introduce the essential elements necessary for the main theorem of the next chapter, deferring greater detail and basic proofs to the auxiliary chapter and the appendices. The appendices additionally provide references to definitions and exposition in the mathematical literature.

2.1 \(1|1\)-EFT as a generalized manifold and simplicial set

This section introduces an object \(1|1\)-EFT of \(1|1\)-Euclidean field theories. This object will be viewed as a presheaf of sets on the category of smooth manifolds and, by restriction, as a simplicial set. Our \(1|1\)-Euclidean field theories are functors between certain categories which we introduce here.

2.1.1 The form of the constituent categories

Both the domain and codomain of the Euclidean field theories are categories fibered over the site of supermanifolds. The categories are moreover equipped with a flip, which is an involutory endofunctor. The categories are also endowed with symmetric monoidal structure.

Generalizing from the notion of open covers of a topological space, one can define a topology on a category called a Grothendieck topology. See A.5 for more details. A category with such a topology is called a site. The category of ordinary smooth manifolds has a Grothendieck topology arising from open covers. The category of
supermanifolds (see definition at C.1) is made a site in a similar way. Covers are
collections (indexed here by \(i\)) of maps of supermanifolds \(U_i \to S\) such that the
maps on reduced manifolds constitute an ordinary open cover. We further require
that the maps of supermanifolds \(U_i \to S|_{U_i}\) are isomorphisms. Here \(S|_{U_i}\) denotes
the restriction of the structure sheaf of the supermanifold \(S\) to the open subset \(U_i\).
We do not attend closely to this topology on the category of supermanifolds in what
follows. The significance is that the target category of the field theory functors is a
stack, so that it suffices to present the bordism category by considering only families
on contractible open sets. Stacks are, loosely speaking, sheaves of categories, and we
review them in A.2.

A supermanifold is a sheaf of super-algebras on a topological space. The structure
sheaf endows the underlying space with the structure of a smooth manifold. We refer
to this underlying smooth manifold as the reduced manifold of the supermanifold,
and for a supermanifold \(S\) will denote the reduced manifold by \(S_{\text{red}}\).

2.1.2 Euclidean bordisms

This subsection recalls relevant supergeometry and has as its goal the presentation
of the \(1|1\)-Euclidean bordism category given in Theorem 2.7. See Appendix C for a
review of the \(S\)-point formalism, Euclidean structures, and useful references. We will
use \(S\text{Man}\) to denote the category of supermanifolds. The algebra of global sections
of the structure sheaf of a supermanifold \(S\) is denoted \(C^\infty(S)\).

The supermanifold \(\mathbb{R}^{p|q}\) is indeed the (categorical) product of \(p\) even and \(q\) odd
lines.

\[
\mathbb{R}^{p|q} \cong \mathbb{R} \times \ldots \times \mathbb{R} \times \mathbb{R}^{0|1} \times \ldots \times \mathbb{R}^{0|1}
\]

The universal property of products then means that a map \(S \to \mathbb{R}^{p|q}\) corresponds
to \(p\) maps \(S \to \mathbb{R}\) and \(q\) maps \(S \to \mathbb{R}^{0|1}\). A map \(S \to \mathbb{R}\) corresponds to an even
global section of \( C^\infty(S) \), and a map \( S \rightarrow \mathbb{R}^{0|1} \) corresponds to an odd global section of \( C^\infty(S) \). See the discussion at C.2.

In particular, for \( \mathbb{R}^{1|1} \), a map \( S \rightarrow \mathbb{R}^{1|1} \) corresponds to a pair \((s, \theta)\), where \( s \) is an even section of the structure sheaf of \( S \) and \( \theta \) is an odd section. Using such expressions we now define a super Lie group structure on \( \mathbb{R}^{1|1} \) by the following formula, in which \((s, \theta)\) and \((t, \eta)\) correspond to maps \( S \rightarrow \mathbb{R}^{1|1} \).

\[
(s, \theta), (t, \eta) \mapsto (s + t + \theta \eta, \theta + \eta)
\]

The formula gives a way to produce from two maps \( S \rightarrow \mathbb{R}^{1|1} \) a single such map. The expression is natural in \( S \), thus by the Yoneda lemma we have determined a map of supermanifolds \( \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \rightarrow \mathbb{R}^{1|1} \). This super Lie group structure extends ordinary addition on \( \mathbb{R} \) along the inclusion \( \mathbb{R} \rightarrow \mathbb{R}^{1|1} \) of the reduced manifold.

There is an action of \( \mathbb{Z}/2 \) on \( \mathbb{R}^{1|1} \) where the generator of \( \mathbb{Z}/2 \) acts by negating the odd coordinate. Such negation is in fact a group homomorphism. This means that we can define a semi-direct product. The super Lie group \( Iso(\mathbb{R}^{1|1}) \) of Euclidean isometries of \( \mathbb{R}^{1|1} \) is defined as \( \mathbb{R}^{1|1} \ltimes \mathbb{Z}/2 \).

Having established the relevant isometry group we may now define the objects and morphisms of the bordism category \( 1|1-EBord \). The category we define will be closely modeled on the bordism category of [13]. As was noted in the introduction, two important differences between this bordism category and that of [13] are, loosely speaking, that we add orientations and that we do not require that the reduced part of a family of bordisms forms a fiber bundle over the parameter space.

First, we define what it means to have a family of oriented Euclidean supermanifolds.

**Definition 2.1.** A family of oriented Euclidean \( 1|1 \)-manifolds over a supermanifold \( S \) is
• A map of supermanifolds \( p : Y \to S \)

• An atlas of open sets \( U_i \) on \( Y_{\text{red}} \) and isomorphisms
  
  \[ Y_{|U_i} \cong V_i \subset S \times \mathbb{R}^{1|1} \]

  compatible with the projection map, such that the transition maps are given by maps \( p_1(V_{ij}) \to Iso(\mathbb{R}^{1|1}) \)

• An orientation on the reduced fibers of \( p : Y \to S \)

Now we define an object of \( 1|1-\text{EBord} \) over a supermanifold \( S \).

**Definition 2.2.** An object of \( 1|1-\text{EBord} \) over the supermanifold \( S \) is

• A family of oriented \( 1|1 \)-Euclidean supermanifolds \( Y \) over \( S \)

• A family of \( 0|1 \)-Euclidean supermanifolds \( Y^c \) over \( S \), referred to as the core

• An isometric embedding \( Y^c \to Y \) over \( S \)

• A decomposition \( Y^\pm \) of \( Y_{\text{red}} \setminus Y^c_{\text{red}} \) into the union of two open subsets, both containing \( Y^c_{\text{red}} \) in their closure

  These data must satisfy the condition that the map of reduced manifolds \( Y^c_{\text{red}} \to S_{\text{red}} \) is proper, where the map is the one induced by \( p : Y \to S \) and the embedding \( Y^c \to Y \).

A morphism of \( 1|1-\text{EBord} \) over a supermanifold \( S \) is defined as an equivalence class. Representatives have the following form.

**Definition 2.3.** A morphism of \( 1|1-\text{EBord} \) over the supermanifold \( S \) is represented by

• A family of oriented \( 1|1 \)-Euclidean supermanifolds \( \Sigma \) over \( S \)

• A pair of objects \( Y_0 \) and \( Y_1 \) over \( S \) (the source and target, respectively)

• Neighborhoods \( W_0 \) and \( W_1 \) of the cores \( Y^c_0 \) and \( Y^c_1 \)

• Isometric embeddings \( W_0 \to \Sigma \) and \( W_1 \to \Sigma \) over \( S \)
The data must satisfy three conditions. First, the core of \( \Sigma \), denoted \( \Sigma^c \) and defined as \( \Sigma^c \), \( \Sigma^c = \Sigma_{red} \setminus \{W_{0,red}^+ \cup W_{1,red}^-\} \) is such that the map \( \Sigma^c \to S_{red} \) is proper. Here \( W_{i,red}^\pm \) refers to the decomposition of \( W_{i,red} \setminus Y_{i,red}^c \) induced by the decomposition \( Y_i^\pm \) associated with \( Y_i \). Second, the image of the reduced part of the map \( W_1 \to \Sigma \) is contained in the core \( \Sigma^c \). Third, the maps \( W_i \to \Sigma \) preserve orientations.

The import of the second condition is best understood by considering Figure 2.2 associated with the presentation of the bordism category given in Theorem 2.7. Loosely speaking, there are four ways of viewing an oriented interval as a bordism. Two of the ways are as an endomorphism, of either a positively or negatively oriented point. The other two ways are as a map from a union of two points to the empty set, or as a map from the empty set to the union of two points. The second condition in the definition above means that the way of viewing an interval as a map from the union to the empty set can shrink down to have length 0. On the other hand, the other morphism, the one from the empty set to the union of two points, must have strictly positive length. The fact that this second morphism can not shrink down to have length zero means that we are not constrained to have dualizable objects assigned to the super-point when considering functors out of the bordism category.

Because we wish to have an ordinary category and not a higher category, we must consider equivalence classes of bordisms. The relation is as follows.

**Definition 2.4.** Two representatives \( \Sigma_1 \) and \( \Sigma_2 \) of morphisms as in Definition 2.3 represent the same morphism in \( 1\vert 1-\text{EBord} \) when there are open neighborhoods \( U_i \) of the cores \( \Sigma^c_i \), restrictions of the neighborhoods \( W_i \), and an orientation preserving isometric isomorphism \( U_0 \to U_1 \) over \( S \) which induces isomorphisms on the restrictions of the \( W_i \).

We can use the super Lie group structure of \( \mathbb{R}^{1\ vert 1} \) to define a basic example of a family of bordisms parameterized by the supermanifold \( \mathbb{R}^{1\ vert 1} > 0 \). We refer to this as the
universal family of super-intervals. Note that $\mathbb{R}_{>0}^{1|1}$ denotes the restriction of $\mathbb{R}^{1|1}$ to the open subset of $\mathbb{R}$ consisting of the positive real numbers.

The (positively oriented) super point, which we denote by $spt$, is the object of $1|1-\text{EBord}$ given by $\mathbb{R}^{1|1}$ with the core inclusion $\mathbb{R}^{0|1} \hookrightarrow \mathbb{R}^{1|1}$ and the decomposition of $\mathbb{R} \setminus \{0\}$ into the positive and negative axes, and the standard orientation.

**Definition 2.5.** The following is the definition of the universal family of intervals. Pull $spt$ back to $\mathbb{R}_{>0}^{1|1}$ by the unique map $\mathbb{R}_{>0}^{1|1} \to pt$, and continue to refer to this object as $spt$. The universal family of intervals, a bordism over $\mathbb{R}_{>0}^{1|1}$ which is an endomorphism of $spt$, is defined as $\mathbb{R}_{>0}^{1|1} \times \mathbb{R}^{1|1}$ with standard orientation. The inclusion of the source is the identity, and the inclusion of the target is given by

$$(s, \theta)(t, \eta) \mapsto (s, \theta)(s + t + \theta \eta, \theta + \eta)$$

In words, we use the super Lie group structure of $\mathbb{R}^{1|1}$ to translate the super point when including it as the codomain.

The negatively oriented super point is like the positive one, except for the choice of the other orientation of the real line. We denote the negatively oriented super point by $\overline{spt}$.

Having defined the objects and morphisms of the category, we now combine them to define $1|1-\text{EBord}$.

**Definition 2.6.** The category $1|1-\text{EBord}$ is defined as follows. The objects of $1|1-\text{EBord}$ are families of Euclidean $0|1$-manifolds. The morphisms are equivalence classes of families of Euclidean $1|1$-manifolds together with inclusions of source and target. Composition of bordisms is by gluing. The fibration $1|1-\text{EBord} \to S\text{Man}$ maps a family to its parameter space. The flip is negation of odd coordinates. The symmetric monoidal structure is by taking disjoint unions of families of supermanifolds.
Theorem 2.7. The category 1|1–EBord admits the following presentation as a symmetric monoidal category with flip. The following objects and bordisms are the generators.

- $spt$ and $\overline{spt}$ the positively and negatively oriented superpoints, which are families over $pt \in S\text{Man}$
- $\mathcal{L} : \overline{spt} \coprod spt \to \emptyset$ a bordism over $pt$
- $\mathcal{R} : \emptyset \to \overline{spt} \coprod spt$ a family of bordisms over $\mathbb{R}_{>0}^{1|1}$, with the inclusion of one of the points defined by translation using the group structure on $\mathbb{R}^{1|1}$ (as in the universal family of intervals discussed in 2.5)

These data are subject to two relations, spelled out in detail below. One relation expresses that the family $\mathcal{R}$ composes with itself along $\mathcal{L}$ in a way determined by the Lie group structure of $\mathbb{R}^{1|1}$. The other relation is that the identity is a limit of the family of endomorphisms given by $\mathcal{R} \circ \mathcal{L}$.

See [13] for a further discussion of the presentation of a closely related bordism category as well as a sketch of the proof.

Figure 2.1 depicts the reduced manifold part of the generating objects. In that figure, the solid portion of the interval is the part $Y^+$ that enables gluing of bordisms along shared objects and which is involved in one of the conditions for bordisms; namely, it is the part that must be embedded in the core of the bordism when the object is part of the codomain. The arrow denotes the orientation on the ambient 1-manifold.

Figure 2.2 depicts the reduced manifold part of the generating bordisms. In the figure the source of a bordism is on the right, and the target on the left, to relate
Figure 2.2. Generating morphisms of the bordism category

The inclusions of the objects are implied by the arrow which denotes the orientation.

We use the convention that $L$ is a morphism from $\overline{spt} \bigsqcup spt$ to the empty set, and that $R$ is a morphism from the empty set to $spt \bigsqcup \overline{spt}$.

We now describe the two relations.

The first is a super-semigroup relation. Given maps $f, g : S \to \mathbb{R}^{1|1}$, one can construct a family of bordisms over $S$ by pulling back $R$ along $f$ and $g$ and then composing them via $L$. One can also use the super Lie group structure of $\mathbb{R}^{1|1}$ so that $f$ and $g$ determine a single map

$$\mu \circ f \times g : S \to \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$$

to $\mathbb{R}^{1|1}$ and then pullback the family $R$ along this map. The relation is that these two constructions produce isomorphic families of bordisms over $S$. The relation is depicted in Figure 2.3, with the vertically stacked bordisms on the right glued along $\mathcal{L}$ to produce the bordism on the left.

The second relation is that there is a family connecting the identity with the intervals of positive length. The map $(x^2, \theta) : \mathbb{R}^{1|1} \setminus \{0\} \to \mathbb{R}^{1|1}_{>0}$ defines a family of bordisms by pulling back $R$ to the complement of $0 \in \mathbb{R}^{1|1}$. Composing with $L$, we obtain endomorphisms of $spt$ or $\overline{spt}$. We observe that the families thus defined (which have parameter space $\mathbb{R}^{1|1} \setminus \{0\}$) extend as the identity at $0 \in \mathbb{R}^{1|1}$. The relation is
Figure 2.3. Semigroup relation

depicted in Figure 2.4. In that figure, we show that the composition of $R$ and $L$ on the right is the restriction of a family connected to the identity, namely the family on the far left, in which the vertical direction is to depict the reduced length of the interval $I(s^2, \theta)$.

**Remark 2.8.** The second relation constrains the type of linear map that can arise as the image of a super interval under a field theory; the semigroup property of the previous relation together with this family imply that such a linear map cannot have a kernel.

**Remark 2.9.** The associativity of composition of bordisms (which holds exactly since we take equivalence classes of bordisms) implies that taking an $R$ and gluing an $L$, then gluing an $L$ to the other side, is the same as gluing on the $L$s in the opposite order. This simple observation will later be useful when characterizing the pairing associated to $L$ by a field theory, which we do in Lemma 2.25.

General families of 1|1-Euclidean manifolds over a parameter space $S$ are neither connected nor are they necessarily pulled back along maps $S \to \mathbb{R}_{>0}^{1|1}$. Nonetheless, when working with symmetric monoidal functors out of the bordism category with codomain a stack, it suffices to define the functor on the generating objects and
bordisms above, since general families are assembled from those (and their progeny via the monoidal structure) over a sufficiently fine cover.

2.1.3 The algebraic target

Before introducing the algebraic target we remark that for a suitably large class of topological vector spaces $V$, the space $C^\infty(S) \otimes V$, where $S$ is a smooth manifold, may be identified with smooth $V$-valued functions on $S$. Here the symbol $\otimes$ denotes the completed projective tensor product; this is the algebraic tensor product of two topological vector spaces, given the finest topology such that the canonical bilinear map is continuous, which is subsequently completed by adding limits of Cauchy nets. See Theorem B.5 for more on smooth $V$-valued functions and a reference, and more generally the review of topological vector spaces in Appendix B.

We now define the algebraic target for our field theories. The objects of the category over a supermanifold $S$ are $C^\infty(S)$-submodules of $C^\infty(S) \otimes H$, where $H$ is a fixed infinite-dimensional super Hilbert space, and morphisms are suitable module maps. At present, we restrict ourselves to modules lying inside some finite rank subbundle.
of \( S_{\text{red}} \times H \). Future work should involve more general modules involving infinite dimensional subspaces of \( H \), such as would arise from a family of spin manifolds by taking smooth spinors on the total space of the family and considering this as a module over smooth functions on the base.

**Definition 2.10.** The category \( \text{TV}(H) \) consists of the following.

- **Objects** are \( C^\infty(S) \)-submodules \( V \subset C^\infty(S) \otimes H \) which lie in some finite rank subbundle of \( S_{\text{red}} \times H \).

- **Morphisms** \( V \to W \) covering \( f : T \to S \) in \( S\text{Man} \) are (even, i.e. grading preserving) module maps \( V \to C^\infty(T) \otimes_{C^\infty(S)} W \).

- The flip is induced by the grading involution on \( H \).

- The monoidal structure given by tensor product (having fixed an isomorphism \( H \otimes H \cong H \), so that a tensor product of concrete submodules is again a concrete submodule). The braiding is induced by braiding isomorphism \( H \otimes H \cong H \otimes H \) in the category of super vector spaces; i.e. it is the one that introduces a sign.

The action of \( C^\infty(S) \) is required to be continuous, and morphisms must be continuous module maps. Note that the modules have induced topologies from the topology (projective) topology on the tensor product \( C^\infty(S) \otimes H \). Once again, further details are available in Appendix B.

**Remark 2.11.** The category \( \text{TV}(H) \) is a stack on \( S\text{Man} \). This is an essential feature that we use when defining functors into it from the bordism category. For the relation between fibered functors and stacks seen as weak functors to the 2-category of categories, see A.2.1.

### 2.1.4 Field theory functors

Both the category of bordisms and the category of vector spaces have been given involutory endofunctors called flips. We require that functors between them respect these.
**Definition 2.12.** A functor between categories with flip is said to satisfy the spin-statistics relation if it commutes with the flips.

The fibered categories $1|1$–\textit{EBord} and TV(H) are now defined, and with these definitions in place we can introduce the main object of study in this dissertation.

**Definition 2.13.** A $1|1$-Euclidean field theory is a symmetric monoidal spin-statistics functor

$$1|1$–\textit{EBord} \to TV(H)$$

of categories with flip fibered over supermanifolds.

The definition just given leads to a set (or category, when including natural transformations) of field theories. We wish to obtain a space. It is possible (see A.4 of the appendix), given two fibered categories, to define a new fibered category consisting of fibered functors between them. We now consider the associated functor $S\text{Man}^{\text{op}} \to \text{Set}$ assigning to a supermanifold the set of fibered functors $1|1$–\textit{EBord} \to TV(H) in the fiber over $S \in S\text{Man}$.

We introduce some notation to clarify this approach. For a supermanifold $S$, we denote by $TV(H)_S$ the category whose objects and morphisms over $T \in S\text{Man}$ are objects and morphisms in $TV(H)$ over $S \times T$ which cover the identity of $S$.

Then the fiber over $S \in S\text{Man}$ of the fibered category of fibered functors

$$1|1$–\textit{EBord} \to TV(H)$$

consists of fibered functors

$$1|1$–\textit{EBord} \to TV(H)_S$$

We will only be concerned with field theories parameterized by ordinary manifolds, via $\text{Man}^{\text{op}} \to S\text{Man}^{\text{op}} \to \text{Set}$. We will ignore the morphisms in the category of
fibered functors. See Remark 2.16 for a justification, as well as the fuller treatment of this idea in the case of topological field theories in 4.2.

The definition just given of Euclidean field theories will lead to a space representing complex $K$-theory. For real $K$-theory, we must introduce additional structure to the algebraic target and impose a condition on the functors. We now let our Hilbert space $H$, used to define the category $TV(H)$, be the complexification of a real super Hilbert space $H_{\mathbb{R}}$. Concretely, we have that $H := H_{\mathbb{R}} \times H_{\mathbb{R}}$ as real vector spaces, and multiplication by $i$ is implemented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The $\mathbb{Z}/2$ grading is induced by that on the real Hilbert space. The complex anti-linear involution giving $H$ a real structure is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**Definition 2.14.** An $S$-family of real 1|1-Euclidean field theories is a symmetric monoidal spin-statistics functor

$$1|1-E\text{Bord} \to TV(H)_S$$

subject to the condition that the universal family of intervals leads to a family of real endomorphisms, meaning linear maps commuting with the real structure.

**Remark 2.15.** In [13], the authors deduce that the endomorphisms must be real by working with complex supermanifolds and making a definition of real field theories associated with a real structure on the category of bordisms. Here, we impose a condition which is simpler to state; this condition could be derived from phrasing things as those authors do and similarly requiring our field theories to be ‘real.’

**Remark 2.16.** The omission of natural transformations between field theories, so that $1|1-E\text{FT}$ is a simplicial set rather than a simplicial groupoid, is justified. Consider, for instance, the case of the singular set of a topological space $X$. In this case the $k$-simplices are continuous maps $|\Delta^k| \to X$, where $|\Delta^k|$ denotes the standard $k$-simplex. There is a natural topology on the set of continuous maps $|\Delta^k| \to X$, namely
the compact-open topology. Nonetheless this topology is safely ignored when one defines the singular set of a space as a simplicial set (rather than a simplicial space). Higher dimensional simplices suitably encode the topology of the spaces of maps of lower dimensional simplices. Similarly in the case of families of field theories, an isomorphism of field theories can be realized as a path in the space of field theories, so that there is no need to worry about morphisms.

**Remark 2.17.** In the future, in order to define twisted field theories, it will be useful to formulate 1|1-Euclidean field theories as functors between internal categories. We defer further discussion of this framework to the appendix, at A.3.

2.1.5 Identifying field theories as quadruples

Since TV(H) is a stack, symmetric monoidal spin-statistics fibered functors

\[
1|1-\text{EBord} \to \text{TV}(H)_S
\]

are determined by the images of the generating objects and bordisms in the presentation given in Theorem 2.7. Thus from this point on we consider functors \(1|1-\text{EBord} \to \text{TV}(H)_S\) to be given by collections of data \((V, W, L, R)\) parameterized by \(S\). Thus \(V\) and \(W\), the respective images of \(spt\) and \(spl\), are \(C^\infty(S)\)-modules in TV(H), i.e. submodules of the set of smooth \(H\)-valued functions on \(S\). The element \(L\), the image of \(\mathcal{L}\), is a \(C^\infty(S)\)-linear map of the following form.

\[
L : W \otimes_{C^\infty(S)} V \to C^\infty(S)
\]

The codomain \(C^\infty(S)\) is indeed the monoidal unit in the fiber of TV(H) over \(S\).

The morphism \(R\), the image of \(\mathcal{R}\), is a \(C^\infty(S)\)-linear map of the following form.
\[ R : C^\infty(S) \to C^\infty(\mathbb{R}_{>0}^{1|1}) \otimes V \otimes W \]

In other words, \( R \) is an element of the codomain, so that we can consider it in the following way.

\[ R \in C^\infty(\mathbb{R}_{>0}^{1|1}) \otimes V \otimes W \]

The morphisms \( R \) and \( L \) satisfy the relations implied by Theorem 2.7. The algebraic consequences of those relations will be spelled out later.

2.1.6 Passage to simplicial sets

We have just considered the functor \( SMan^{op} \to Set \) that assigns to a supermanifold \( S \) the set of quadruples \((V, W, L, R)\), and we consider this functor as being the presheaf of sets associated to the fibered category of fibered functors

\[ 1|1 - EBord \to TV(H)_S \]

over supermanifolds. We can restrict this functor to smooth manifolds, which are a full subcategory of the category of supermanifolds. We now produce from this restricted functor \( Man^{op} \to Set \) a simplicial set. The simplicial set we produce will be denoted \( 1|1 - EFT \), and it is the purpose of this dissertation to determine the homotopy type of this simplicial set. Some basic ideas regarding simplicial sets are reviewed in [A.5].

Here and elsewhere, we refer to functors \( Man^{op} \to Set \) as generalized manifolds. The Yoneda embedding gives ordinary manifolds as particular generalized manifolds, but there are many more generalized manifolds. For example, the smooth maps \( Man(M, N) \) between smooth manifolds \( M \) and \( N \) can be viewed as a generalized
manifold by the following assignment.

\[ S \mapsto \text{Man}(M \times S, N) \]

The construction of a simplicial set from a generalized manifold proceeds in two steps. First, we produce an embedding of the ordinal category in the category of smooth manifolds. This step already gives us a simplicial set, which we refer to here as intermediate. In the second step of the construction, we introduce an equivalence relation on the \( k \)-simplices of the intermediate simplicial set so that we essentially work with the compact standard simplices, and not the non-compact ones of the intermediary set.

Regarding the first step, we first note that it is simpler to work with supermanifolds without boundary. As a result the standard simplices, which are smooth manifolds with corners, are not supermanifolds in the base category of the fibrations defining the bordism and vector space categories. Hence we cannot simply evaluate our generalized manifold on the standard simplices. As a result we seek an embedding \( \Delta \hookrightarrow \text{Man} \) which assigns smooth manifolds without boundary to finite ordinals in a suitable way. We sketch here the idea of the construction, and give precise maps in A.5.3. Essentially, the functor \( \Delta \hookrightarrow \text{Man} \) assigns \([k]\) to Euclidean space \( \mathbb{R}^k \) of dimension \( k \). The co-face maps \([k] \rightarrow [k + 1]\) are sent to affine linear inclusions of hyperplanes \( \mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1} \), and the co-degeneracy maps \([k] \rightarrow [k - 1]\) are mapped to orthogonal projections onto the relevant hyperplanes (orthogonality meaning with respect to the standard inner product on \( \mathbb{R}^k \)). Once we have this embedding \( \Delta \hookrightarrow \text{Man} \), we can produce a simplicial set from a generalized manifold by restriction.

Summarizing the previous paragraph; starting with a functor \( S\text{Man}^{\text{op}} \rightarrow \text{Set} \), we restricted to smooth manifolds, and then restricted along the embedding \( \Delta \hookrightarrow \text{Man} \), to produce an intermediate simplicial set. We now make the second step and define
an equivalence relation on the $k$-simplices of the intermediate simplicial set.

It is important to note for the following definition that we have identified the standard $k$-simplices $|\Delta^k|$ as subsets of $\mathbb{R}^k$ in a way compatible with the coface and codegeneracy maps among the $\mathbb{R}^k$ determined by the embedding $\Delta \rightarrow \text{Man}$; again, see [A.5.3] for precise detail.

**Definition 2.18.** Given a generalized manifold $\mathcal{G}$, two elements $E_0$ and $E_1$ of the set $\mathcal{G}(\mathbb{R}^k)$ are deemed equivalent if, for every $f : S \rightarrow \mathbb{R}^k$ whose image lies in the standard $k$-simplex, we have that $f^*E_0 = f^*E_1$ as elements of $\mathcal{G}(S)$.

Observe that in the previous definition we use equality, not isomorphism. The definition is for generalized manifolds, not stacks. We only consider sets of families of field theories, not categories. This approach was justified above in Remark 2.16.

Observe as well that the $k$-simplices of $\mathcal{G}$ are essentially elements of $\mathcal{G}$ evaluated on $|\Delta^k|$ (though strictly speaking this is not possible) which admit a smooth extension to some open neighborhood $U \supset |\Delta^k|$. We do not deal with germs, in which case one is concerned with the extension; rather, the particular smooth extension is irrelevant. This generalizes the notion of a smooth function on a closed subset of a smooth manifold. In fact, given a representable generalized manifold $\text{Man}(\mathcal{G}, M)$, the preceding construction of a simplicial set from a generalized manifold produces the subobject of the singular set of $M$ consisting of its smooth simplices.

The embedding $\Delta \hookrightarrow \text{Man}$ and the equivalence relation of Definition 2.18 now combine in the definition of 1|1−EFT.

**Definition 2.19.** The simplicial set 1|1−EFT is defined as the simplicial set which assigns to $[k] \in \Delta$ the set of equivalence classes of quadruples $(V, W, L, R)$ parameterized by $\mathbb{R}^k$, where $(V, W, L, R)$ are as in 2.1.5, meaning they are collections of objects and morphisms allowing us to construct from them a symmetric monoidal spin-statistics fibered functor $1|1−\text{EBord} \rightarrow \text{TV}(\mathcal{H})_{\mathbb{R}^k}$.

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2.2 Key properties of families of field theories

This section highlights properties of families of 1|1-Euclidean field theories that are useful in establishing the homotopy type of 1|1–EFT.

2.2.1 The universal family of endomorphisms

By composing $\mathcal{R}$ and $\mathcal{L}$ along $\overline{\text{sp}t}$, we obtain a family of endomorphisms of $\text{sp}t$ parameterized by $\mathbb{R}_{>0}^{1|1}$ which we refer to as the universal family of intervals, as in Definition 2.5. The image of such a family under a field theory functor yields a map $\mathbb{R}_{>0}^{1|1} \rightarrow \text{End}(V)$, where $\text{End}(V)$ denotes $C^\infty(S)$-linear endomorphisms of the module $V$. The relation regarding composition of the bordisms $\mathcal{R}$ and $\mathcal{L}$ and the group operation on $\mathbb{R}^{1|1}$ implies the following condition.

**Lemma 2.20.** The map $\mathbb{R}_{>0}^{1|1} \rightarrow \text{End}(V)$ determined by the universal family of intervals is a homomorphism of semigroup objects in the category of functors $\text{SMan}^{op} \rightarrow \text{Set}$.

There is also the corresponding statement for endomorphisms of $W$. Note that $\text{End}(V)$ is a semigroup in the following way. Given a supermanifold $T$, the set of $C^\infty(S)$-linear maps

$$V \rightarrow C^\infty(T) \otimes V$$

can be given a semigroup structure using the fact that $C^\infty(T)$ is an algebra. In other words, given two such maps $M$ and $N$, we compose them as follows,

$$V \xrightarrow{M} C^\infty(T) \otimes V \xrightarrow{id \otimes N} C^\infty(T) \otimes C^\infty(T) \otimes V \xrightarrow{\text{id} \otimes \text{id}} C^\infty(T) \otimes V$$

with the last map arising from the algebra structure on $C^\infty(T)$.

A point of terminology: for $G$ a semigroup in the category of generalized supermanifold, we will refer to a semigroup morphism $\mathbb{R}_{>0}^{1|1} \rightarrow G$ as a super-semigroup.
In the case that $V$ is a finite-dimensional vector space (in which case we are considering a family of field theories parameterized by a 0-manifold), super-semigroups $\mathbb{R}_{>0}^{1|1} \to \text{End}(V)$ correspond bijectively with odd endomorphisms of $V$ via

$$\mathcal{D} \mapsto e^{-t\psi^2}(1 + \theta \mathcal{D})$$

This statement, justified at B.9, admits the following generalization.

**Lemma 2.21.** Let $S$ be an ordinary smooth manifold. Suppose that an $S$-family of Euclidean field theories $E$ consisting of $(V, W, L, R)$ is such that the $C^\infty(S)$-modules $V$ and $W$ are finitely-generated and projective (i.e. arise as sections of finite-rank vector bundles over $S$). Then the super-semigroup which is the image of the universal family of intervals has an infinitesimal generator which is an odd bundle endomorphism of $V$. Also, any odd endomorphism of $V$ will lead to an $S$-family of field theories.

**Proof.** By restricting the family $I_{t^2, \theta}$ (whose existence is the second relation of the presentation in Theorem 2.7) of super-intervals along $\mathbb{R}^{0|1} \to \mathbb{R}^{1|1}$, we obtain from a field theory functor a map

$$V \to C^\infty(\mathbb{R}^{0|1}) \otimes V$$

which corresponds to a single odd endomorphism of $V$. At every point of $S$ the restriction of such an endomorphism determines the necessary super-semigroup, and linearity of the endomorphism over $C^\infty(S)$ means that we have a bundle map. \qed

We do not have such a clear statement in the case that $V$ is infinite dimensional, or when the rank of $V$ varies over the parameter space. It is true that an infinitesimal generator exists by the same argument as above. What is not clear is the conditions the odd endomorphism must satisfy in order to determine a super-semigroup participating in a family of field theories. In any case, it will be useful to have the following description of super-semigroups of endomorphisms of a vector space $V$, here taken to
be finite dimensional.

**Lemma 2.22.** A super-semigroup $\mathbb{R}_{>0}^{1|1} \rightarrow \text{End}(V)$ determines and is determined by a pair of maps $A : \mathbb{R}_{>0} \rightarrow \text{End}(V)^0$ and $B : \mathbb{R}_{>0} \rightarrow \text{End}(V)^1$ subject to

- $A(s + t) = A(s)A(t)$
- $B(s + t) = A(s)B(t)$
- $A'(s + t) = B(s)B(t)$

This statement may be found as Lemma 3.2.14 of [30] and as Proposition 5.9 in [13]. We sketch the idea here. Before doing so, we remark that the reader may find in C.1.1 a description of how to postcompose $S$-points with smooth functions using the Taylor expansion, as we do in the following computation.

A homomorphism is an even element $\Phi$ of $C^\infty(\mathbb{R}_{>0}^{1|1}) \otimes \text{End}(V)$. Since $C^\infty(\mathbb{R}_{>0}^{1|1}) \cong C^\infty(\mathbb{R}_{>0})[\theta]$, we can identify $\Phi$ with a pair $A \in C^\infty(\mathbb{R}_{>0}) \otimes \text{End}(V)^0$ and $B \in C^\infty(\mathbb{R}_{>0}) \otimes \text{End}(V)^1$, where $\text{End}(V)^i$ denotes, according to the parity of $i$, even or odd endomorphisms of the super vector space $V$. The map $\Phi$ is to be compatible with the semigroup structure on $\mathbb{R}_{>0}^{1|1}$ and $\text{End}(V)$. This means we must have that $\Phi(s + t + \theta \eta, \theta + \eta) = \Phi(s, \theta)\Phi(t, \eta)$. When we expand $\Phi$ in terms of $A$ and $B$, we get the following.

\[
\Phi(s + t + \theta \eta, \theta + \eta) = A(s + t) + A'(s + t)\theta \eta + (\theta + \eta)[B(s + t) + \theta \eta B'(s + t)]
\]
\[
\Phi(s, \theta)\Phi(t, \eta) = (A(s) + \theta A'(s) + \theta B(s))(A(t) + \eta A'(t) + \eta B(t))
\]

Expanding both right hand sides, using the fact that $\theta^2 = \eta^2 = 0$, and comparing coefficients when viewing the two expressions as polynomials in $\theta$ and $\eta$, we obtain the relations which constitute Lemma 2.22.

In what follows, we will focus primarily on the even family of endomorphisms $A$. We will use the operator $A(1)$ to decompose and manipulate the module $V$. 

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2.2.2 The form taken by the constituent module sheaves

Let $S$ be an ordinary smooth manifold. An $S$-family of field theories involves two modules sheaves, the images of $spt$ and $\overline{spt}$, which we denote by $V$ and $W$, respectively.

Since field theories are a generalized manifold, we can restrict the family along the inclusion of any point $x$ of $S$ in order to obtain a pair of vector spaces $V_x$, $W_x$ and the requisite linear maps involving those spaces. We will later prove, in Lemma 2.30, a statement which implies the following. Note that by the rank of $V$ we mean the dimension of $V_x$.

**Lemma 2.23.** The rank of $V$ at a point determines a continuous map $S \to \mathbb{N}$ where the codomain is given the order topology.

The same conclusion holds for $W$ as well.

We conclude, for example, that skyscraper sheaves do not arise as the image of an object of $1|1$–EBord under a field theory.

**Remark 2.24.** We emphasize that although at various points in this work we do rely on the concreteness of the algebraic target, the proof of Lemma 2.30 in no way relies on the details of the target category. All that is necessary is that field theories form a generalized manifold. Thus the conclusion of the previous lemma holds true even if the algebraic target consists of abstract modules over $C^\infty(S)$ (and not just the concrete submodules of $H$-valued functions we consider here). This means that the previous lemma really is a (very) limited characterization of the type of object that can be the image of $spt$ under a general symmetric monoidal functor.

2.2.3 Perfect Pairing

Denote by $\bar{\mathcal{A}}$ the family of operators that arises from composing $\mathcal{L}$ and $\mathcal{R}$ along $spt$ rather than $\overline{spt}$, and let $V_\lambda$ (or $W_\lambda$) denote the generalized eigenspace corresponding
to the eigenvalue \( \lambda \) of \( A(1) \) (or \( \bar{A}(1) \)).

**Lemma 2.25.** The spectra of \( A(1) \) and of \( \bar{A}(1) \) coincide, and \( L \) induces an isomorphism \( W_\lambda \cong V_\lambda^\vee \).

**Proof.** It is helpful to consider Figure 2.5. Note that, by the associativity of composition of bordisms, we obtain the same element whether we compose the bordisms depicted there first along \( \text{spt} \), or instead first along \( \bar{\text{spt}} \). For notational simplicity in the proof we let \( A \) denote \( A(1) \), and similarly for \( \bar{A} \).

The algebraic consequence of the relation depicted in Figure 2.5 is as follows, for any \( v \in V \) and \( w \in W \).

\[
L(\bar{A}w, v) = L(w, Av)
\]

Suppose that \( Av = \lambda v \) and \( \bar{A}w = \mu w \). Then the equation

\[
\mu L(w, v) = L(\bar{A}w, v) = L(w, Av) = \lambda L(w, v)
\]

implies that \( L \) can pair eigenvectors non-trivially only if \( \lambda = \mu \).

Now suppose that \( w \) is an eigenvector and \( v \) is a generalized eigenvector; i.e.
\[ \bar{A}w = \mu w, \text{ and } v \in \ker(A - \lambda)^n. \] Then we have the following.

\[ 0 = L(w, (A - \lambda)^n v) = L((\bar{A} - \lambda)^n w, v) = (\mu - \lambda)^n L(w, v) \]

Hence \( L(w, v) \) is non-zero only if \( \mu = \lambda \). The argument is exactly the same when we consider generalized eigenvectors of \( W \) and eigenvectors of \( V \).

Now proceed by induction. Suppose that \( \lambda \neq \mu, m \in \mathbb{N} \), that for all \( n \geq m \) \( L \) pairs \( \ker(\bar{A} - \mu)^m \) and \( \ker(A - \lambda)^n \) trivially, and that it pairs \( \ker(\bar{A} - \mu)^n \) and \( \ker(A - \lambda)^m \) trivially. Then this holds for \( m + 1 \) as well, which we see in the following way.

Suppose that \( w \in \ker(\bar{A} - \mu)^{m+1} \) and \( v \in \ker(A - \lambda)^{m+1} \). Then we have the following two expressions.

\[ 0 = L((\bar{A} - \mu)w, v) = L(\bar{A}w, v) - \mu L(w, v) \]
\[ 0 = L(w, (A - \lambda)v) = L(w, Av) - \lambda L(w, v) \]

Since \( L(\bar{A}w, v) = L(w, Av) \), we see that \( L(w, v) = 0 \). Hence we have the triviality of the pairing for \( m + 1 \) and \( n = m + 1 \). The same argument proves that the case \( n = m + k \) implies the statement for \( n = m + k + 1 \).

To see that \( L \) induces the stated isomorphism, note that every vector \( v \in V \) must pair non-trivially with some \( w \in W \) since \( A(t) \) approaches the identity of \( V \) as \( t \) approaches 0. Thus \( W_{\lambda} \to V_{\lambda}^\vee \) is surjective. Applying the same argument with the roles of \( V \) and \( W \) exchanged implies that \( V_{\lambda} \) and \( W_{\lambda} \) have the same dimension. \( \square \)

### 2.2.4 Trace and super-trace as partition functions

Consider a single Euclidean field theory (i.e. a point-family) consisting of the quadruple \( (V, W, L, R) \). Note that here \( V \) and \( W \) are just finite-dimensional vector spaces. Since \( L \) provides an isomorphism \( W \cong V^\vee \), we are able to conclude that
evaluation on various super-circles gives information about the operators associated
to the universal family of intervals.

It is a theorem (Batchelor’s theorem, C.3) that all supermanifolds come from
vector bundles. This is done by producing from the bundle $E \rightarrow S$ the supermanifold
whose reduced manifold is $S$ and whose structure sheaf is sections of the exterior
algebra bundle $\Lambda^* E$ graded by parity. For the purpose of the discussion in this
subsection we will think of a vector bundle associated to a supermanifold, even though
this association is not canonical.

By Batchelor’s theorem, a 1|1-dimensional supermanifold whose reduced manifold
is a circle (we will call such supermanifolds here super-circles) comes from one of
the two isomorphism classes of line bundles on the circle, either the trivial or the
nontrivial one. What this means in the bordism category $1|1$–$EBord$ is that when
we compose the generating morphisms $\mathcal{L}$ and $\mathcal{R}$ of the bordism category as they
are given in Theorem 2.7, we get a super-circle coming from a trivial line bundle.
There is another way we can compose the generators, however. We can apply the
flip to the super-point (which means multiplication by $-1$ on the associated line)
before composing the left and right elbows. This gives a super-circle coming from a
non-trivial line bundle.

Evaluating a field theory on a bordism with empty boundary produces an endo-
morphism of the ground field of the algebraic target of vector spaces, since a monoidal
functor carries the monoidal unit of the bordism category (the empty set) to (the
isomorphism class of) the monoidal unit of the algebraic target category. Thus eval-
uation on super-circles gives complex numbers, since the space of linear maps $\mathbb{C} \rightarrow \mathbb{C}$
is naturally identified with $\mathbb{C}$. This number is called the partition function and de-
PENDS on the chosen bordism. In the event that bordisms come in a smooth family,
the partition function is a smooth function on the moduli space.

Before stating the following lemma, we recall that the family $\mathcal{R}$ comes in a family
parameterized by $\mathbb{R}^1_{>0}$. Because the pairing arising from the left elbow is even, the partition functions depend only on the restriction of the family along $\mathbb{R}_{>0} \hookrightarrow \mathbb{R}^1$. Thus, for the purpose of the following lemma, the moduli space of circles we consider is $\mathbb{R}_{>0}$, where the real number $t \in \mathbb{R}_{>0}$ indicates the circumference of the circle (which, as a Euclidean circle, has a Riemannian metric). Also, we recall that (see 2.21 and B.9) this family of field theories comes from an infinitesimal generator which is an odd endomorphism of $V$ that we denote by $\partial$.

**Lemma 2.26.** The partition function corresponding to gluing $\mathcal{L}$ and $\mathcal{R}$ without using the flip produces the smooth function $\text{str}(e^{-t\partial^2})$, where $\text{str}$ indicates the super-trace. The partition function corresponding to gluing with a flip on one super-point leads to the function $\text{tr}(e^{-t\partial^2})$.

*Proof.* The proof is essentially given in Figure 2.6. In that figure, we compose the generating bordisms $\mathcal{R}$ and $\mathcal{L}$, and to do so we must use the braiding isomorphism, depicted by the crossed lines. The vertical line with the parenthesized word ‘flip’ below it indicates the possibility of gluing the bordisms with or without using the flip on $spt$. We comment now on the algebraic implications of the compositions depicted in the figure. The braiding isomorphism for super vector spaces is the one which introduces a negative sign on the odd summand. As a result, the composition without flip has a single negative sign, leading to the super-trace.

When composing the braiding with grading involution on one factor, as we do in the composition including the flip of the super-point, we obtain a second sign which cancels the one due to the braiding and leads to the ordinary trace.\[\square\]

**Remark 2.27.** A Euclidean structure as used in this work endows the underlying manifold with a Riemannian metric and a spin structure. One may find discussion of this fact in Example 6.16 of [13]. Seen in this light, to distinguish among various
super-circles corresponds to considering their associated spin structures. The zero-bordant spin structure is referred to as the anti-periodic spin structure. The other spin structure is called periodic. The super-circle with anti-periodic spin structure comes from a non-trivial line bundle on the circle. The periodic super-circle comes from a trivial line bundle.

Lemma 2.26 is proved in the case of infinite-dimensional topological vector spaces in [29].

2.2.5 Producing a field theory from an infinitesimal generator

We record here a construction which will be used in the main theorem.

**Lemma 2.28.** Let $(V, \mathcal{D})$ be a pair consisting of a $C^\infty(S)$-module and an odd endomorphism. Suppose that the endomorphism $e^{-t\mathcal{D}^2}$ has smooth trace and super-trace. Then $(V, \mathcal{D})$ may be completed to a field theory.

**Proof.** The second constituent module, which we have denoted $W$, is defined as $W := V$. Choose an isomorphism $\Phi : H \cong H^\vee$. Then the pairing $L : W \otimes V \to C^\infty(S)$ is defined by $L(f \otimes g) := \Phi(f)(g)$; i.e. we use the isomorphism $\Phi$ and the evaluation pairing. The family $R$, an element of $C^\infty(\mathbb{R}_{>0}) \otimes V \otimes W$, is determined by $L$ together with the requirement that the super-semigroup of operators induced by composing $L$ and $R$ has the form $(1 + \theta \mathcal{D})e^{-t\mathcal{D}^2}$. 

Figure 2.6. Partition functions
In this way we satisfy the relations implied by the presentation of Theorem 2.7. The final thing to check is that the trace and super-trace gives smooth functions on the parameter space, but that was assumed.

2.3 Perturbation theory in the non-projective setting

This section demonstrates that, given a finite dimensional inner product space $E$ and a domain $U \in \mathbb{R}^n$, semigroups of endomorphisms of submodules of $C^\infty(U) \otimes E$ have a continuously varying spectrum if we require that the trace varies continuously. In the Euclidean field-theoretic setting, we do in fact have such a continuously varying trace by evaluating the field theory on super-circles constructed by using the flip, as in 2.26.

2.3.1 Some introductory examples

First, it is worth noting that by a continuously varying spectrum in the case of general submodules, we mean a continuously varying family of configurations of points in $\mathbb{C} \setminus \{0\}$ labeled by natural numbers, with the topology such that points can combine (in which case their labels add), and additional points can come in from 0 or disappear at 0. Note that points lie in the complement of the origin in $\mathbb{C}$, since our families of operators form semigroups which approach the identity as the time parameter becomes small. If an operator had a kernel this would not be possible.

Second, observe that we obtain the spectrum of $\mathcal{A}(1)$ at a point $x$ in the parameter space $S$ by considering it as an endomorphism of the fiber of $V$ at that point by restriction. By the fiber of $V$ at a point $x$ we mean the span of all vectors $f(x)$, where $f$ is an element of $V$.

We begin by considering an example that shows that linearity over $C^\infty(U)$ does not suffice to produce a continuously varying spectrum. Let $V$ denote the ideal in $C^\infty(\mathbb{R})$ consisting of functions whose support lies in the non-negative reals. Consider
the identity on $V$ as a (trivial) semigroup of endomorphisms; one whose infinitesimal generator is identically zero. The map $id_V$ is clearly continuous and linear over smooth functions. Observe, however, that the spectrum of $id_V$ does not vary continuously. For positive $x$ the spectrum consists of 1, whereas the spectrum is empty for non-positive $x \in \mathbb{R}$ (since there are no non-zero vectors that could possibly be eigenvectors). We can identify this discontinuity in the spectrum by considering the trace of the operator (of course the trace is, in this trivial case, the same as the operator itself). The trace is discontinuous; we have $tr(id_V) = 1$ for $x > 0$ and $tr(id_V) = 0$ for $x \leq 0$.

A slight modification of the previous example will serve to demonstrate the need to consider the whole semigroup in order to have a continuously varying spectrum. Consider the direct sum of two copies of $V$ as above, and define an endomorphism which is the identity on one summand and the negative of the identity on the other summand. This is clearly a continuous linear endomorphism. Moreover, its trace is a smooth function of the parameter space, namely the constant function taking the value zero. The spectrum evidently does not vary in a continuous manner, since it is empty for $\mathbb{R}_{\leq 0}$ and $\{-1, 1\}$ for $\mathbb{R}_{> 0}$. We can identify this discontinuity, though, by considering the trace of the square of the operator, which is discontinuous at $0 \in \mathbb{R}$, where it jumps from 0 to 2.

2.3.2 Continuity of spectrum

We now prove that the spectrum of the operator $A(1)$ arising from a family of field theories varies in a continuous manner.

The following is a precise formulation of the fact that the roots of a polynomial depend continuously on the coefficients. It is proved in [12] and is Theorem B in that work. As has been the case throughout, we work over the complex numbers.
Lemma 2.29. Let the polynomial \( f(z) \) be such that

\[
f(z) = z^n + a_1 z^{n-1} + \ldots + a_n = (z - \lambda_1)^{\mu_1}(z - \lambda_2)^{\mu_2}\ldots(z - \lambda_k)^{\mu_k}
\]

with the roots \( \lambda_i \) in the right hand expression all distinct, so that \( \mu_i \) denotes the multiplicity of the root \( \lambda_i \). For any \( \epsilon > 0 \) such that balls of radius \( \epsilon \) about \( \lambda_i \) and \( \lambda_j \) are disjoint for \( i \neq j \), there is a \( \delta > 0 \) so that if \( \{b_i\}_{i=1}^n \) satisfy \( |b_i - a_i| < \delta \) for all \( i \), then the polynomial

\[
g(z) = z^n + b_1 z^{n-1} + \ldots + b_n
\]

has exactly \( \mu_j \) roots in the ball of radius \( \epsilon \) centered at \( \lambda_j \).

Recall that the elementary symmetric polynomials \( S_k \) are defined in the following way.

\[
S_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}
\]

The polynomials \( S_1, S_2, \ldots, S_n \) generate the ring of symmetric polynomials in \( n \) variables. The power sum polynomials \( P_k \) are another generating set, and have the following form.

\[
P_k(x_1, x_2, \ldots, x_n) := x_1^k + x_2^k + \ldots + x_n^k
\]

The Newton identities, or Newton-Girard identities, express the relationship between these two generating sets. The general form of the identities is as follows, see [33].

\[
(-1)^k k S_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} (-1)^{k+j+1} P_j(x_1, x_2, \ldots, x_n) S_{k-j}(x_1, x_2, \ldots, x_n)
\]
Particular cases for small $k$ include the following.

\[ S_1 = P_1 \]
\[ 2S_2 = S_1P_1 - P_2 \]

Recall that the coefficients of a monic polynomial are the elementary symmetric polynomials evaluated on the roots of the polynomial.

The relation between symmetric and power sum polynomials will allow us to prove the continuity of the spectrum, which is the purpose of this section.

Before giving details, we outline the idea of the proof. We use the fact that the traces of $A(1)^k_x$ and $A(1)^k_y$ are similar when $y$ is near $x$ to infer that the power sums of the roots of the characteristic polynomials of $A(1)_x$ and $A(1)_y$ are similar. The power sums being approximately the same means that the elementary symmetric polynomials are too, and as closely as we wish near $x$. Then Lemma 2.29 establishes that the roots are close to each other.

We now formulate and prove the necessary lemma.

**Lemma 2.30.** Let $V$ be an element of $TV(H)$ parameterized by a smooth manifold $S$, and let $A : V \to C^\infty(\mathbb{R}_{>0}) \otimes V$ be a smooth semigroup of endomorphisms which has smooth trace and extends as the identity at $t = 0$ (where $t$ is the coordinate on $\mathbb{R}$). Then the spectrum of $A(1)$ varies continuously as a function of $S$.

**Proof.** The essential idea was already sketched above, in the observation that the traces of powers of an operator determine its spectrum. To complete the proof, we must address the complication present in our case of non-projective modules. This discussion will also highlight the role of the fact that $A(0) = id_V$. Recall that we have required that the module $V$ is a submodule of some finite-rank vector bundle. As a consequence, there is some finite maximal possible dimension of $V_y$ for all points $y$ in the parameter space. Suppose that $m$ is the dimension of $V_x$, and that $n$ is the
rank of the ambient vector bundle containing $V$. When applying Lemma 2.29, we apply it not only to the characteristic polynomial of $A(1)_x$ but rather to the following collection of polynomials. Denote the characteristic polynomial of $A(1)_x$ by $\alpha(z)$. We then consider these polynomials.

$$\alpha(z), z\alpha(z), z^2\alpha(z), \ldots, z^{n-m}\alpha(z)$$

Since $A(t)_x$ is invertible for all $t$, the smallest eigenvalue of $A(1)_x$ is some distance $\nu$ from $0 \in \mathbb{C}$. Now anytime we choose $\epsilon < \frac{\nu}{2}$, we can find a $\delta$ (as in Lemma 2.29, governing the extent to which coefficients vary) which simultaneously satisfies the statement about the nearness of roots for all of the polynomials $\alpha(z), z\alpha(z), \ldots, z^{n-m}\alpha(z)$.

We then find a neighborhood of $x$ such that the coefficients of the characteristic polynomial of $A(1)_y$ are within $\delta$ of $z^k\alpha(z)$, where $k$ is defined as $k := dimV_y - dimV_x$. The existence of this neighborhood is guaranteed by the previous discussion relating power sums and elementary symmetric polynomials. We then conclude that the extra $k$ eigenvalues of $A(1)_y$ are all within a distance $\epsilon$ of $0 \in \mathbb{C}$. Recall that $\epsilon$ is less than half the norm of the least eigenvalue of $A(1)_x$. This means that in a neighborhood of $x$ we distinguish the eigenvalues coming from $A(1)_x$ and those which appear at the origin.

2.4 Spaces representing $K^0$ and $KO^0$

This section defines and studies a simplicial set $\mathcal{N}Q\text{Vect}$ to which $1|1$–EFT will be compared in the next chapter in order to establish its homotopy type.

2.4.1 Quillen’s group completion for symmetric monoidal categories

This subsection closely follows [13]. See also [11, 25, 27].

Given a monoidal category $\mathcal{C}$, the monoidal product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ endows the
classifying space of \( \mathcal{C} \) with the structure of an H-space. If \( \mathcal{C} \) is a symmetric monoidal category this H-space is homotopy commutative and associative.

Given an H-space \( X \), the map \( X \to \Omega BX \) (where \( \Omega \) denotes based loops and \( B \) the classifying space) is a group completion. This means that if the path components of \( X \) were already a group the map is a homotopy equivalence, and otherwise the map induces isomorphisms on homology after inverting path components in \( H_*(X) \).

See [23] as well as further references there.

Quillen defined a construction of a symmetric monoidal category \( Q\mathcal{C} \) from a symmetric monoidal category \( \mathcal{C} \), together with a functor \( \mathcal{C} \to Q\mathcal{C} \), such that the induced map on classifying spaces is group completion.

The construction proceeds as follows. The category \( Q\mathcal{C} \) has

- **Objects**: Pairs \((C^+, C^-)\) of objects in \( \mathcal{C} \)
- **Morphisms**: An object \( A \) of \( \mathcal{C} \) together with isomorphisms \( \alpha^\pm : C^\pm \oplus A \to D^\pm \)

Introduce an equivalence relation so that two morphisms \((A, \alpha^\pm)\) and \((B, \beta^\pm)\) are considered the same if there is an isomorphism \( A \cong B \) compatible with the maps \( \alpha^\pm \) and \( \beta^\pm \). The category \( Q\mathcal{C} \) defined in this way does in fact have a classifying space with the right homotopy type.

2.4.2 Specific model of \( NQ\text{Vect} \)

The topological category \( \text{Vect} \) of finite dimensional vector spaces and invertible linear maps (endowed with their natural topology) has classifying space \( \coprod BO(k) \), hence applying \( Q \) to \( \text{Vect} \) yields a category whose classifying space has the homotopy type of \( BO \times \mathbb{Z} \). See the discussion at Q.7 of [8].

For the purposes of this dissertation it will be useful to treat only vector bundles which are subbundles of a fixed trivial bundle with fiber an infinite-dimensional, separable, super Hilbert space. In order to define a monoidal structure on such a
topological category, we fix an isomorphism $H \oplus H \cong H$. Then the sum of (possibly intersecting) subspaces is again a subspace of the correct dimension.

We now define a topological category $QVect$. 

Let $QVect_0$ denote the Grassmannian of finite-dimensional super subspaces of the super Hilbert space $H$ endowed with its ordinary topology.

Let $QVect_1$ denote the space whose points are inclusions of super subspaces of $H$, $V_0 \subset V_1 \subset H$, together with an odd involution $\alpha$ on some complementary subspace $V'_1$ of $V_0$ in $V_1$. In other words we have $V_1 = V_0 + V'_1$, and an odd involution on the second summand.

Via orthogonal projection onto a subspace we can view the involution as an element of $B(H)$. Now give $QVect_1$ a topology by considering it as a subspace of $Gr(H) \times Gr(H) \times B(H)$.

This is a slightly different, though equivalent, formulation of Quillen’s construction applied to the monoidal category of subspaces of $H$ endowed with sum as monoidal product. Note that morphisms are not linear maps $V \to V'$ in general, but rather only inclusions of subspaces. We related our specific model, which will be useful for the field theories we consider, to another model involving orthogonal subspaces and orthogonal involutions, in Section 4.1.

The nerve of $QVect$ is a simplicial space. Taking the level-wise singular set gives a bisimplicial set, whose $(k,l)$-bisimplices are continuous maps of the following form.

$$
\Delta^k \to QVect_1 \times_{Gr(H)} QVect_1 \times_{Gr(H)} \cdots \times_{Gr(H)} QVect_1
$$

A continuous map $\Delta^k \to QVect_1$ classifies a pair of nested vector bundles $V_0 \subset V_1$ on $\Delta^k$ and a continuous involution on a complement of $V_0$ in $V_1$. A collection of $l$ such maps is composable (i.e. in the nerve) when the larger bundle of one pair classified by $\Delta^k \to QVect_1$ is the smaller bundle of the next pair. Thus, a $(k,l)$-bisimplex is
a collection of $l + 1$ bundles $V_0 \subset V_1 \subset \ldots \subset V_l$ on $\Delta^k$ together with involutions of complements of $V_i$ in $V_{i+1}$.

**Theorem 2.31.** The classifying space of the topological category $QVec$ is a representing space for real or complex $K$-theory.

This is a well-known result proven, for instance, in [13] in a slightly different form than the one we use here. Again, see Section 4.1 for the comparison of that result with our model.

We now make the following definition which possesses the smoothness necessary to relate to Euclidean field theories.

**Definition 2.32.** The simplicial set $NQVect$ is the subobject of the diagonal of the levelwise singular set of the nerve of $QVect_\bullet$, consisting of smooth bundles and smooth families of involutions.

By the equivalence of the simplicial sets determined by continuous and smooth simplices in a smooth manifold, we conclude the following.

**Lemma 2.33.** The inclusion $NQVect \to d(Sing(N(QVect_\bullet)))$ is a weak equivalence.

**Remark 2.34.** The lemma holds for $Vect$ (and the ambient Hilbert space $H$) defined to be real or complex vector spaces.

**Corollary 2.35.** The geometric realization of the simplicial set $NQVect$ represents $K^0$ or $KO^0$, depending on whether complex or real vector spaces are chosen.
CHAPTER 3

MAIN THEOREM

This chapter defines a map of simplicial sets $F : \mathcal{N}QVect \to 1|1\text{-EFT}$ and proves it is a weak equivalence.

3.1 The map

Fix a decreasing smooth function $\rho(x)$ which is $\frac{1}{x^2}$ for $x$ near 0 and is identically 0 for $x \geq 1$. This function will be used in the definition of the family of operators in the family of field theories determined by a $k$-simplex of $\mathcal{N}QVect$.

Recall that we have fixed a super Hilbert space $H$, and that all vector bundles over a parameter space $S$ are subbundles of $S \times H$.

In 2.32 we defined the simplicial set $\mathcal{N}QVect$, essentially the nerve of a topological category. A $k$-simplex of $\mathcal{N}QVect$ consists of smooth vector bundles

$$V_0 \subset V_1 \subset \ldots \subset V_k$$

on the standard $k$-simplex $|\Delta^k|$, together with odd involutions $\alpha_i : V'_i \to V'_i$ (where $V'_i$ is some complement of $V_{i-1}$ in $V_i$). We denote this $k$-simplex by $(V_\bullet, \alpha_\bullet)$. Associated to these data we must produce a family of $1|1\text{-Euclidean}$ field theories parameterized by the $k$-simplex. See 2.19 for the definition of the simplicial set of field theories, which we denote $1|1\text{-EFT}$. Without repeating the definition; we can loosely think of a $k$-simplex of field theories as a submodule $V$ of $H$-valued functions on $\mathbb{R}^k$, together with a super semigroup $\Phi : \mathbb{R}^{|1|1}_{>0} \to \text{End}(V)$ of endomorphisms of $V$. These data are...
subject to conditions, such as that there is a sensible notion of trace and super trace of $\Phi$. The trace and super trace are smooth functions on $\mathbb{R}^k$. The semigroup has an infinitesimal generator, which is an odd endomorphism of $V$.

To define $F : \mathcal{NQVect} \to 1|1-EFT$ it suffices to assign to $(V_\bullet, \alpha_\bullet)$ a module $V$ over $C^\infty(U)$, where $U$ is some open neighborhood of $\Delta^k \subset \mathbb{R}^k$, and an odd endomorphism $\hat{D}$ of this module with the property that the trace and super-trace of $e^{-t\hat{D}^2}$ are smooth functions on $U$ for all $t > 0$. We can then obtain the data of a field theory by using dualization and the evaluation pairing. Concretely, we produce the quadruple $(V, V, ev, R)$, with the pairing $ev$ defined by the choice of an isomorphism $H \cong H^\vee$ and $R$ defined by coevaluation and the semigroup. See 2.28 for further details.

Pick some smooth extensions of the bundles $V_i$ and involutions $\alpha_i$ to $U$. Denote by $V_i(x_0, \ldots, x_k)$ fiber of $V_i$ at $(x_0, \ldots, x_k) \in U$. It is a finite-dimensional linear subspace of the ambient Hilbert space $H$. The module $V$ associated to the $k$-simplex $(V_\bullet, \alpha_\bullet)$ is defined as follows. Elements of $V$ are smooth functions $U \to H$ with that property that

$$f(x_0, \ldots, x_k) \in V_k(x_0, \ldots, x_k)$$

for all $(x_0 \ldots, x_k)$ and that, moreover, satisfy

$$f(x_0, \ldots, x_k) \in V_i(x_0, \ldots, x_k)$$

whenever $x_i, \ldots, x_k = 0$.

We further require that the composition of $f$ with the projection onto $V_i'$ gives a function vanishing to all orders at $x_i = x_{i+1} = \ldots = x_k = 0$.

The topology on the module is induced by $H$ and the parameter space $U$; it is the topology if uniform convergence on compact sets of functions and their derivatives, restricted to the (closed) subspace of functions having the prescribed vanishing behavior.
Express $V_k$ as a direct sum $V_0 \oplus V'_1 \oplus V'_2 \oplus \ldots \oplus V'_k$. With respect to this decomposition, the odd endomorphism $\mathcal{D}$ is given as follows.

$$\mathcal{D} := 0 \oplus \rho \left( \sum_{i=1}^{k} x_i \right) \alpha_1 \oplus \rho \left( \sum_{i=2}^{k} x_i \right) \alpha_2 \oplus \ldots \oplus \rho \left( \sum_{i=j}^{k} x_i \right) \alpha_j \oplus \ldots \oplus \rho(x_k)\alpha_k$$

The definition uses homogeneous coordinates on $\mathbb{R}^k$, i.e. the $x_i$ are a collection of $k + 1$ real numbers whose sum is 1; this is discussed in the context of the definition of our embedding $\Delta \to \text{Man}$ given in A.5.3. That $\mathcal{D}$ defined as above does in fact give a continuous endomorphism of the module $V$ follows from the fact that when it becomes unbounded it does so as sections are required to vanish to all orders. More precisely, we have the following lemma.

**Lemma 3.1.** Multiplication by $\frac{1}{x^2}$ is a continuous automorphism (linear over $C^\infty(\mathbb{R})$) of the space of functions

$$V := \{ f \in C^\infty(\mathbb{R}) | f(x) = 0 \text{ for } x \leq 0 \}$$

endowed with the topology induced by $V \subset C^\infty(\mathbb{R})$, where the latter space has its standard Fréchet topology involving norms of derivatives on compact subsets of $\mathbb{R}$.

**Proof.** The space of functions $V$ is a closed subspace of a Fréchet space, and hence itself a Fréchet space, and in particular complete and metrizable, so that the open mapping theorem B.6 may be applied. We must then simply show that multiplication by $x^2$ is a continuous linear bijection. It is continuous, since the seminorms defining the topology of $V$ involve functions and their derivatives on compact subsets, and the function $x^2$ is bounded on such subsets, so that a norm $|x^2 f|_{C,k}$ is bounded by $B|f|_{C,k}$ for some $B$. To establish surjectivity note that by L’Hopital, a smooth function $f$ which vanishes to all orders at 0 may be divided by $x^2$ in order to produce another such function.
Now we apply the open mapping theorem and conclude that $\frac{1}{x^2}$ is a continuous automorphism.

Returning to the definition of $F : \mathcal{N}Q\text{Vect} \to 1|1-\text{EFT}$, and in particular the definition of the infinitesimal generator $\mathcal{D}$; the trace and supertrace of $e^{-t \mathcal{D}^2}$ are clearly smooth functions of the parameter space and of $t \in \mathbb{R}_{>0}$. By omitting parameters $x_i$ or inserting values equal to zero, and reindexing, one sees that the map defined above is in fact a map of simplicial sets.

Finally note that the construction is independent of the choice of smooth extensions of the bundles and involutions to $U \supset \Delta k$, since $k$-simplices of $1|1-\text{EFT}$ are equivalence classes (recall Definition 2.19) of families parameterized by $\mathbb{R}^k$. Two such families represent the same simplex, by Definition 2.18, when they agree on their restrictions to $|\Delta^k| \subset \mathbb{R}^k$.

### 3.2 Constructing preimages

This section defines a construction of (subdivided) simplices of $\mathcal{N}Q\text{Vect}$ from simplices of $1|1-\text{EFT}$ in such a way that postcomposition with $F : \mathcal{N}Q\text{Vect} \to 1|1-\text{EFT}$ gives a subdivided simplex in $1|1-\text{EFT}$ equivalent to the (suitably subdivided) original simplex.

#### 3.2.1 Algebraic and geometric subdivision

We now briefly introduce subdivision as an endofunctor of the category of simplicial sets. This is further discussed in 4.4. An excellent reference, which we follow closely here, is III.4 of [10].

The simplicial set $sd\Delta^k$ may be defined as the nerve of the poset of non-empty subsets of $\{0, 1, \ldots, k\}$, partially ordered by inclusion. We refer to $sd\Delta^k$ as the subdivided $k$-simplex. There is a last vertex map $h : sd\Delta^k \to \Delta^k$. This map is
induced by the map of posets which sends a subset of \( \{0, 1, \ldots, k\} \) to its largest element.

Every simplicial set is a canonical colimit of its simplices, and thus we may extend subdivision to an endofunctor of simplicial sets. The functor \( Ex \) is the right adjoint of the subdivision functor.

By means of the last vertex map \( sd \Delta^k \to \Delta^k \), we can produce from a \( k \)-simplex of any simplicial set \( X \) a map \( sd\Delta^k \to X \) by precomposition. We refer to this as the algebraic subdivision of the simplex.

**Definition 3.2.** The algebraic subdivision of a \( k \)-simplex of a simplicial set \( X \) is the composition

\[
sd\Delta^k \to \Delta^k \to X
\]

in which the first map is the last vertex map.

This definition makes sense for any simplicial set.

Recall that we use the phrase generalized manifold (see 2.1.6) to refer to a functor \( Man^{op} \to Set \). Ordinary smooth manifolds are generalized manifolds by the Yoneda embedding. As was discussed in 2.1.6 and will be further described in \[A.5.3\], there is an embedding \( \Delta \hookrightarrow Man \) by which a generalized manifold can be restricted to produce a simplicial set. Moreover, we have fixed embeddings of standard simplices \( |\Delta^k| \subset \mathbb{R}^k \) such that the coface and codegeneracy maps induced by \( \Delta \hookrightarrow Man \) restrict along these embeddings to the affine linear maps between the standard simplices which are the geometric realizations of the coface and codegeneracy maps of \( \Delta \).

Since we work with simplicial sets that arise in this manner from generalized manifolds (i.e. by precomposition with \( \Delta \to Man \)), we are able to produce a different subdivision of a simplex, which we refer to as geometric subdivision.

According to the power set description given earlier, a non-degenerate \( k \)-simplex of the simplicial set \( sd\Delta^k \) consists of a strictly increasing chain of inclusions of subsets
of the following form.

\[
\{m_0\} \subset \{m_0, m_1\} \subset \ldots \subset \{m_0, m_1, \ldots, m_k\} = \{0, 1, \ldots, k\}
\] (3.1)

Let \(v_i\) denote the \(i\)th vertex of \(|\Delta^k|\). There is a unique affine linear extension of the map sending \(v_i\) to \(\frac{1}{i+1} (v_{m_0} + v_{m_1} + \ldots + v_{m_i})\). This extension gives a smooth map \(\mathbb{R}^k \to \mathbb{R}^k\).

From the \((k+1)!\) distinct increasing chains of subsets of \(\{0, 1, \ldots, k\}\) (as in 3.1) we thus obtain \((k+1)!\) smooth maps \(\mathbb{R}^k \to \mathbb{R}^k\). These maps allow us to define now the geometric subdivision of a \(k\)-simplex of a generalized manifold.

**Definition 3.3.** Given a \(k\)-simplex \(g\) of a generalized manifold \(\mathcal{G}\), the geometric subdivision of \(g\) is defined as the map (of simplicial sets) \(sd\Delta^k \to \mathcal{G}\) determined by pulling back \(g\) along the \((k+1)!\) smooth maps \(\mathbb{R}^k \to \mathbb{R}^k\) (henceforth referred to as geometric subdivision maps) determined by \(v_i \mapsto \frac{1}{i+1} (v_{m_0} + v_{m_1} + \ldots + v_{m_i})\) for the nondegenerate simplices given by chains of subsets as in Equation 3.1.

The definition of the geometric subdivision maps ensures that this collection of simplices in fact constitute a single map \(sd\Delta^k \to \mathcal{G}\).

The interplay between generalized manifolds and their associated simplicial sets is potentially confusing. Let us then emphasize a few key points to summarize before moving on. The simplicial set \(sd\Delta^k\) is just a simplicial set. It has no connection with generalized manifolds. Its purely algebraic formulation leads to a map \(sd\Delta^k \to \Delta^k\), referred to as the last vertex map. The last vertex map is the algebraic gadget that we need for working with the functor \(Ex\) used to define the fibrant replacement \(Ex^\infty X\) of a simplicial set \(X\) which might not be Kan. From any \(k\)-simplex of a simplicial set \(X\) we can produce a map \(sd\Delta^k \to X\) simply by precomposing with the last vertex map, and we refer to this as the algebraic subdivision of the simplex. Now we discuss the other subdivision, the geometric one. If the simplicial set we work with is the
restriction of a generalized manifold \( \mathcal{G} \), there is a way of producing a simplicial set map \( sd\Delta^k \to \mathcal{G} \) other than the algebraic method of precomposition with the last vertex map. This other way exists because there are many more maps \( \mathbb{R}^k \to \mathbb{R}^k \) in the category of smooth manifolds than there are maps \([k] \to [k]\) in the category \( \Delta \).

When the simplicial set we work with is the restriction of a generalized manifold, it makes sense to pull simplices back along the maps \( \mathbb{R}^k \to \mathbb{R}^k \) which do not arise from the ordinal category. At the end of the day, as long as we obtain \((k + 1)!\) \( k \)-simplices whose faces agree in the requisite way, we can produce a subdivided simplex. Therefore what we need is a collection of maps \( \Delta^k \to \mathcal{G} \) which are compatible in the sense that the factorization in the following diagram exists.

\[
\begin{array}{ccc}
\bigsqcup \Delta^k & \longrightarrow & \mathcal{G} \\
\downarrow & & \\
sd\Delta^k & \end{array}
\]

Such a collection of simplices is indeed provided when we pull back simplices along the geometric subdivision maps described earlier \[3.3\].

**Remark 3.4.** A more categorical, but equivalent, way of formulating the previous discussion is as follows. Every presheaf of sets is a canonical colimit of representables. We can postcompose the diagram \( \mathcal{J} \to \Delta \) defining \( sd\Delta^k \) with \( \Delta \to \text{Man} \) in order to obtain a diagram of smooth manifolds whose colimit (a generalized manifold) we abusively refer to as \( sd\Delta^k \). There is a map of generalized manifolds \( sd\Delta^k \to \Delta^k \) induced by the last vertex map, i.e. arising from the embedding \( \Delta \hookrightarrow \text{Man} \). There is another map \( sd\Delta^k \to \Delta^k \) involving barycentric subdivision, whose specific constituent maps were given above. Finally, a generalized manifold \( \mathcal{G} \) may be evaluated on the colimit of the diagram defining \( sd\Delta^k \) to obtain a set. Refer to the composition \( \mathcal{J} \to \Delta \to \text{Man} \to \text{Fun}(\text{Man}^{op},\text{Set}) \) as \( Z \). Then the following expression holds, in which \( j \) denotes an object of the diagram category \( \mathcal{J} \).
\[
\text{Hom}(\text{colim}_{\mathcal{J}} \mathcal{Z}, \mathcal{G}) \cong \text{lim} \text{Hom}(F(j), \mathcal{G}) \cong \text{lim} \mathcal{G}_{F(j)}
\]

We do not emphasize the generalized manifold approach in the text, since the proof is fundamentally simplicial. Nonetheless it is another way of thinking about the meaning of the geometric subdivision.

**Remark 3.5.** The simplicial set \(\mathcal{NQVect}\) is not the restriction of a generalized manifold. Rather, it arises from a topological category, as the diagonal of the level-wise singular set of the nerve. This is a subtle point which is easily overlooked, especially since there is a relatively straightforward way of defining a geometric subdivision for simplices of \(\mathcal{NQVect}\). Again, this does not arise by considering \(\mathcal{NQVect}\) as a generalized manifold, since it is not such an object. We provide details in Section 4.4; in short, everything works out as we wish it too.

3.2.2 Subdividing families of field theories

In the previous chapter, in Lemma 2.30, we showed that given a field theory \((V, W, L, R)\) parameterized by a smooth manifold \(S\), the spectrum of the endomorphism \(A(1)\) of \(V\) (which we obtain by composing \(R\) and \(L\) along the negatively oriented super point and evaluating at time \(t = 1\)) depends continuously on the parameter space \(S\). The idea of the construction given in this section is that, because of the continuous variation of the spectrum, we can locally view the module \(V\) as a vector bundle together with an unusual module consisting of functions having some required vanishing behavior on a closed subset. As we move about the parameter space \(S\), we might have to deal with changes to the rank of the vector bundle; i.e. we do not get a globally defined vector bundle which is a submodule of \(V\). We can think of regions of \(S\) on which the bundle rank increases as being something like the 1-simplices \(V_0 \subset V_1\) of \(\mathcal{NQVect}\) in which a lower rank bundle is included into a higher
A complication arises from the fact that the simplices of $\mathcal{N}Q\text{Vect}$ do not involve arbitrarily ordered collections of bundles; rather, the bundles must go in order of increasing rank. The point of defining a sufficiently fine subdivision, as we do below in Definition 3.10 of a family of field theories is to deal exactly with this complication.

We will define the following sequence of data associated to a family of field theories. First is the resolvent collection, which has to do with the spectrum of $A(1)$. Next is the intermediate subdivision, which says which portions of the spectrum of $A(1)$ are considered the relevant ones for particular regions of the parameter space. Finally, the sufficiently fine subdivision, which guarantees that the rank-increasing constraint of simplices of $\mathcal{N}Q\text{Vect}$ can be met.

The reader interested in going straight to a definition may proceed to Definition 3.6.

Before making the definitions, consider the following example as a motivation. For simplicity we ignore many aspects of field theories, like supersymmetry, the pairing, etc., giving an incomplete picture whose purpose is to inspire the definitions to come.

Fix some vector space $E$. Suppose we have a family of field theories parameterized by the real line, with the property that the operator $A(1)$ is 0 in a small neighborhood of 0 and $id_E$ in a neighborhood of 1 (e.g. we obtain $A(1)$ by multiplying $id_E$ by some smooth step function). Note that this implies that the module $V$ must consist of $E$-valued functions on $\mathbb{R}$ which vanish on some neighborhood of $0 \in \mathbb{R}$. The module $V$ is not a vector bundle. This family of field theories should be thought of as a path in the space of field theories between the trivial field theory (whose vector space is just $\{0\}$) and the topological field theory whose vector space is $E$.

At first glance we might think this could define a 1-simplex of $\mathcal{N}Q\text{Vect}$. For such a 1-simplex we need a pair of vector bundles on the interval and an involution on the complement. The smaller of the vector bundles will be $\{0\}$. A problem we now face
is that a larger bundle does not naturally exist over the whole interval. Thus, we will need to do some sort of subdivision.

If we restrict the family of field theories to \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\) we can define suitable vector bundles. On \([0, \frac{1}{2}]\), we just take our bundles to be \(\{0\}\). On \([\frac{1}{2}, 1]\), we have \(\{0\} \subset E\). The problem we run into now (we are avoiding talking about the involution, though that is also a problem) is that we cannot use these data to produce a map \(sd\Delta^1 \to \mathcal{NQVect}\) going from \(\{0\}\) to \(E\). The reason for this is that the 1-simplices of \(\mathcal{NQVect}\) go in the direction of increasing rank. Thus, the 1-simplex we would define using the data of the field theory restricted to \([\frac{1}{2}, 1]\) will have the ‘wrong’ faces. Since the face \(d_1\) of a 1-simplex of \(\mathcal{NQVect}\) gives the smaller bundle (restricted to the left endpoint of the interval), the map we produce \(sd\Delta^1 \to \mathcal{NQVect}\) just goes from \(\{0\}\) to \(\{0\}\). A further subdivision is needed so that we can both define the necessary bundles as well as ensure that the simplices go in the right direction. This concludes the inspirational sketch.

We recall, for the notational convenience of the definition to come, that from the data \((V, W, L, R)\) of a family of field theories parameterized by \(S\), we compose \(L\) and \(R\) to obtain a family of endomorphisms of \(V\). We denote the even part of this family \(A(t)\), where \(t\) is a positive real number. Since field theories are a generalized manifold, we can pull them back along inclusions of points of the parameter space. Restricting \(A(t)\) along such an inclusion \(x \mapsto S\) gives the endomorphism we denote \(A(1)_x\). It is just an endomorphism of the finite dimensional vector space \(V_x\), with \(V_x\) the restriction of \(V\) along the inclusion of \(x\).

A resolvent collection, which we now define, determines open subsets of the parameter space on which the module \(V\), restricted to such an open set, has a submodule which corresponds to a vector bundle.

**Definition 3.6.** A resolvent collection \(\{(U_i, \mu_i)\}\) for a \(k\)-simplex of 1|1-Euclidean field theories is a finite set of pairs \((U_i, \mu_i)\) where \(U_i\) is an open set in \(|\Delta^k| \subset \mathbb{R}^k\) and
\( \mu_i \) is a positive real number less than 1. These data are subject to the condition that the collection of open subsets \( U_i \) constitute an open cover of the standard \( k \)-simplex. Furthermore, the data are required to satisfy the following resolvent condition. For every point \( x \in U_i \), the circle of radius \( \mu_i \) does not intersect the spectrum of the endomorphism \( A(1)_x \); in other words, all complex numbers of norm \( \mu_i \) are in the resolvent of \( A(1)_x \).

We now show that any \( k \)-simplex of field theories can be given such a collection, because the spectrum of \( A(1) \) depends continuously on the parameter space.

**Lemma 3.7.** Every \( k \)-simplex of 1|1–EFT admits a resolvent collection \( \{(U_i, \mu_i)\} \).

**Proof.** The spectrum of \( A(1) \) varies continuously as a function of the parameter space \( \mathbb{R}^k \), as was shown in Lemma \[2.30\]. This implies that resolvent elements at a point continue to be in the resolvent in a neighborhood of that point. Since the \( U_i \) must only cover the compact standard \( k \)-simplex, it is possible to find a finite collection of them. \( \square \)

The next definition establishes which resolvent element should apply to points in intersections \( U_{ij} \) of open subsets in a resolvent collection.

**Definition 3.8.** An intermediate subdivision of a \( k \)-simplex of 1|1–EFT is a resolvent collection \( \{(U_i, \mu_i)\} \) together with a natural number \( p \) together with a certain set map \( f \) to be described momentarily. The natural number \( p \) must be such that the image of the standard \( k \)-simplex under every

\[
m_\alpha := \Delta^k \subset \mathbb{R}^k \to \mathbb{R}^k \to \ldots \to \mathbb{R}^k = \mathbb{R}^k
\]

is contained in some \( U_i \). The unspecified maps \( \mathbb{R}^k \to \mathbb{R}^k \) in the composition above are any collection of \( p \) maps \( \mathbb{R}^k \to \mathbb{R}^k \) which arise in the geometric subdivision, as at \[3.3\]. The assignment which is the other part of an intermediate subdivision is a set
map of the form \( m_\alpha \mapsto i \), subject to the condition that the image of \( \Delta^k \) under \( m_\alpha \) lies in \( U_i \).

In the previous definition, note that when constructing the composition of geometric subdivision maps, there are at each of the \( p \) steps \((k + 1)!\) choices of maps \( \mathbb{R}^k \to \mathbb{R}^k \). Hence the indices \( \alpha \) are from an index set of size \(((k + 1)!)^p\). This is also the number of non-degenerate \( k \)-simplices in \( sd^p \Delta^k \).

**Lemma 3.9.** Every \( k \)-simplex of field theories admits an intermediate subdivision.

**Proof.** This follows from the Lebegue Number Lemma. A suitably large \( p \) will be such that all the images of \( \Delta^k \) under maps \( m_\alpha \) are small, so that \( m_\alpha(\Delta^k) \subset U_i \) for some \( i \). The assignment \( m_\alpha \to i \) can be defined any number of ways, by choosing some \( i \) satisfying the condition. For what follows it will not be important to ensure that the assignment involves making, say, the largest or smallest (speaking in terms of \( \mu_i \)) possible choice. \( \square \)

One might hope that a \( k \)-simplex of field theories with a resolvent collection and intermediate subdivision would suffice to construct a suitable map \( sd^p \to \mathcal{N}\text{QVect} \). This is not so, as we described in the sketch earlier. The simplices in \( \mathcal{N}\text{QVect} \) have an orientation implied by the fact that an inclusion of vector bundles only goes one way, if it is not an equality. The simplices in \( 1\text{|1\text{-EFT}} \) do not have this constraint. Thus we must take an extra step in order to ensure that we can coherently define simplices in \( \mathcal{N}\text{QVect} \) that will assemble to a map \( sd^p \Delta^k \to \mathcal{N}\text{QVect} \).

Recall that given a natural number \( p \), we have \(((k + 1)!)^p\) different maps \( m_\alpha : \Delta^k \to \mathbb{R}^k \) defined by composing geometric subdivision maps. In the definition that follows we will also consider a second collection of maps denoted by \( n_\beta \), which are the \(((k + 1)!)^q\) maps arising from the composition of \( q \) subdivision maps.
Definition 3.10. A sufficiently fine subdivision of a $k$-simplex of $1|1$-Euclidean field theories is an intermediate subdivision $\{(U_i, \mu)\}$ together with a natural number $q$, $q > p$, subject to the condition that if $m_\alpha(\Delta^k) \cap n_\beta(\Delta^k) \neq \emptyset$, then $n_\beta(\Delta^k) \subset U_f(m_\alpha)$.

We now describe the motivation for the condition. We know in advance that $n_\beta(\Delta^k)$ lies in some $U_i$, since the image of $n_\beta$ is contained in the image of some $m_\alpha$ which was already in one of the $U_i$. The number $q$ is a sufficiently fine subdivision when, loosely speaking, those little simplices given by $n_\beta$ which are at the boundary of the intermediate simplices $m_\alpha$ are so small that they lie not just in the $U_i$ corresponding to $m_\alpha$, but also in the $U_j$ corresponding to adjacent intermediate simplices $m_\alpha'$. In this way, not only the resolvent element $\mu_i$, but also the elements $\mu_j$ corresponding to adjacent intermediate simplices, extend over the whole little simplex $n_\beta$. These extra resolvent elements will allow us to take a field theory over the big simplex and define a collection of vector bundles, together with involutions on the bundles, on the little simplex, in such a way that we can produce a map $sd_\nu\Delta^k \to \mathcal{N}\text{QVect}$.

Lemma 3.11. Every $k$-simplex of field theories admits a sufficiently fine subdivision.

Proof. The $k$-simplex admits an intermediate subdivision, by Lemma 3.9. What remains is to show that another use of the Lebesgue Number Lemma gives the desired conclusion. Observe that if $q$ gives a sufficiently fine subdivision, so does $q+1$. Thus it suffices to find, for each $m_\alpha$, a $q'$ that is sufficiently fine. We then take the maximum of the $q'$ associated to the various intermediate simplices.

Thus our situation is the following one. We have a $k$-simplex of field theories, which we have obtained by pulling back some other simplex along a map $m_\alpha$. The intermediate subdivision gives us the additional data of some resolvent element $\mu_i$ persisting over the whole of the simplex. There are also resolvent elements $\mu_j$ associated to simplices which share some subface of the given one, but which do not in general persist over the whole simplex.
Let \( \iota : \mathbb{R}^j \rightarrow \mathbb{R}^k \) be some composition of coface maps. In other words, pulling back along it induces some composition of face maps. Suppose that the image of \( \Delta^j \) under \( m_\alpha \circ \iota \) intersects the image of \( \Delta^k \) under some \( m_\alpha' \). The intermediate subdivision assigns to \( m_\alpha' \) an open subset \( U_{f(m_\alpha')} \) containing the image of \( \Delta^k \) under \( m_\alpha' \). Define the open subset \( V_\iota \) associated to the subface \( \iota \) to be the intersection of all preimages \( m_\iota^{-1}U_{f(m_\alpha')} \) and the complement of the image of \( \partial \Delta^j \) under \( \iota \). (We intersect with the complement of \( \partial \Delta^j \) to ensure that \( n_\beta \) which contain a vertex of \( m_\alpha \) are contained in all \( U_i \) corresponding to simplices \( m'_\alpha \) also containing that same vertex.) Then the collection \( V_\iota \) as \( \iota \) ranges over the injections \([j] \hookrightarrow [k]\) for \( j \) ranging from 0 to \( k-1 \), together with the interior of \( \Delta^k \), constitute an open cover of \( \Delta^k \subset \mathbb{R}^k \). We can apply the Lebesgue Lemma and obtain a natural number \( q' \) such that a composition of \( q' \) subdivision maps will have image contained either in one of the \( V_\iota \) or the interior of \( \Delta^k \).

By construction, every composition of \( q' \) geometric subdivision maps

\[
\Delta^k \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \rightarrow \ldots \rightarrow \mathbb{R}^k = \mathbb{R}^k
\]

will have image lying in all relevant \( U_i \). Thus, a suitable subdivision parameter \( q' \) exists for the simplex indexed by \( m_\alpha \). Repeating this procedure for each \( m_\alpha \), we find a potentially larger \( q'' \) that works for all the simplices. This \( q'' \) is a sufficiently fine subdivision for the \( k \)-simplex with intermediate subdivision.

The preceding definitions of resolvent collection, intermediate subdivision, and sufficiently fine subdivision were all given in terms of single \( k \)-simplices. We require analogous statements regarding resolvents and subdivisions for general maps \( sd^n \Delta^k \rightarrow 1|1-\text{EFT} \). The definitions already given were phrased in terms of single simplices for clarity. Rather than repeat everything for subdivided simplices, we bundle all necessary statements into the following definition and lemma.
**Definition 3.12.** A compatible sufficiently fine subdivision of a map

$$sd^n \Delta^k \to 1|1-\text{EFT}$$

is a sufficiently fine subdivision on each non-degenerate $k$-simplex

$$\Delta^k \to sd^n \Delta^k \to 1|1-\text{EFT}$$

subject to the condition that the induced sufficiently fine subdivisions on common faces coincide.

Now we give the existence lemma for such things.

**Lemma 3.13.** A map $sd^n \Delta^k \to 1|1-\text{EFT}$ admits a compatible sufficiently fine subdivision.

**Proof.** Start at any $k$-simplex of $sd^n \Delta^k$ and choose a resolvent collection, which according to Lemma 3.7 is always possible. Observe that this collection will impose conditions on resolvent collections chosen for simplices sharing a face with the given one. It will always be possible to construct a resolvent collection for a neighboring simplex which will restrict to the given one on the shared face; the continuity of the spectrum of $A(1)$ guarantees that this is so.

Since extra subdivisions do not cause a problem when establishing an intermediate subdivision, we can find a subdivision parameter which simultaneously works for all simplices, and define the various set maps $m_a \to i$ (part of the data of an intermediate subdivision, see 3.8) compatibly on faces.

Then, we take the maximum of the sufficiently fine subdivisions of the constituent $k$-simplices of $sd^n \Delta^k \to 1|1-\text{EFT}$ in order to produce a single parameter which works for all. We have thereby produced a compatible sufficiently fine subdivision. $\square$

We now return to treating only individual $k$-simplices explicitly. The fact that we
can produce sufficiently fine subdivisions which are compatible, as was just shown, takes care of the general case.

The sufficiently fine subdivision was so defined as to give us the data necessary for the construction of a map $sd^n\Delta^k \to NQVect$. We now spell out the construction.

**Lemma 3.14.** From a $k$-simplex of $1|1$-EFT with sufficiently fine subdivision

\[\{(U_i, \mu_i), p, f, q\}\]

we can construct a map of simplicial sets $sd^n\Delta^k \to NQVect$.

**Proof.** Recall the notation $m_\alpha$ and $n_\beta$, where $m_\alpha$ denotes a map $\Delta^k \to \mathbb{R}^k$ arising from $p$ composed geometric subdivision maps, and $n_\beta$ denotes such a map arising from the composition of $q$ geometric subdivision maps. The function $f$ assigns a resolvent radius $\mu_i$ to each map $m_\alpha$.

Non-degenerate $k$-simplices $\Delta^k \to sd^n\Delta^k$ have a natural bijective correspondence with compositions of $q$ geometric subdivision maps $\mathbb{R}^k \to \mathbb{R}^k$; these maps are as in 3.3. Let $\gamma : \Delta^k \to sd^n\Delta^k$ be any such simplex. We will now use the data of a sufficiently fine subdivision to produce a $k$-simplex of $NQVect$. After doing this we will then verify that as $\gamma$ ranges over the simplices of $sd^n\Delta^k$, the simplices we define assemble to produce a map $sd^n\Delta^k \to NQVect$.

Recall that $k$-simplices of $NQVect$ are given by inclusions of $k+1$ vector bundles and involutions on complements; the definition is at 2.32. We define the sequence of vector bundles for $\gamma$ in the following way. As $i$ ranges from 0 to $k$, define a collection of positive real numbers $\lambda_i$ as follows.

\[\lambda_i := \max\{\mu_f(m_\alpha) | \text{im}(\gamma) \cap \text{im}(m_\alpha) \neq \emptyset\}\]

Observe that $i \mapsto \lambda_i$ is a decreasing function. Loosely speaking, this follows from
the fact that for any geometric subdivision map \(|\Delta^k| \rightarrow |\Delta^k|\), as the vertices \(i\) increase they are carried into the interior of increasingly higher dimensional faces of \(|\Delta^k|\).

Denote by \(\lambda_i V\) the submodule of \(V\) consisting of functions lying in (sums of) eigenspaces of \(A(1)\) whose eigenvalues have norm greater than \(\lambda_i\). (Note that the notation \(\lambda_i V\) is to be a mnemonic device; it is the part of \(V\) in some sense larger than \(\lambda_i\), hence appearing to the right of \(\lambda_i\).) Recall that \(\gamma\) corresponds to some map \(n_\beta : \mathbb{R}^k \rightarrow \mathbb{R}^k\). The sufficiently fine subdivision condition implies that \(n_\beta^*(\lambda_i V)\) is a vector bundle over \(|\Delta^k|\) which extends smoothly to a neighborhood of \(|\Delta^k|\).

Note the following additional data provided by a field theory. The operator \(A(1)\) gives an eigenspace decomposition \(\lambda_{i+1} V = \lambda_{i+1} V_{\lambda_i^+} \oplus \lambda_{i+1} V_{\lambda_i^-}\). Here \(\lambda_{i+1} V_{\lambda_i}\) denotes the sum of eigenspaces whose eigenvalues have norm larger than \(\lambda_{i+1}\) and smaller than \(\lambda_i\). The infinitesimal generator (see 2.21) of the family of field theories is diagonal with respect to this decomposition. On \(\lambda_{i+1} V_{\lambda_i}\) the infinitesimal generator \(\mathcal{D}\) has no purely imaginary eigenvalues, since \(\lambda_i < 1\) and \(A(1) = e^{-\mathcal{D}^2}\). As a result, we can decompose \(\lambda_{i+1} V_{\lambda_i}\) further into

\[
\lambda_{i+1} V_{\lambda_i} = \lambda_{i+1} V_{\lambda_i^+} \oplus \lambda_{i+1} V_{\lambda_i^-}
\]

with the superscripts \(+\) and \(−\) denoting the sum of eigenspaces of the infinitesimal generator whose eigenvalues have positive (respectively negative) real part. Observe that they are of the same dimension since the infinitesimal generator is odd. With respect to this decomposition define \(\alpha_{i+1} := \text{Id} \oplus -\text{Id}\).

We can now define a \(k\)-simplex of \(\mathcal{NQVect}\) as the following sequence of vector bundles

\[
n_\beta^*(\lambda_0 V) \subset n_\beta^*(\lambda_1 V) \subset \ldots \subset n_\beta^*(\lambda_k V)
\]

together with the involutions \(\alpha_i \circ n_\beta\).

This defines the \(k\)-simplex of \(\mathcal{NQVect}\) assigned to the map \(\gamma : \Delta^k \rightarrow \text{sd}^q \Delta^k\).
As $\gamma$ ranges over all (nondegenerate) $k$-simplices of $sd^q\Delta^k$, the simplices of $\mathcal{N}\text{QVect}$ constructed as above will in fact give a map $sd^q\Delta^k \to \mathcal{N}\text{QVect}$. This is so because the simplices are defined naturally in terms of pulling back the bundles along the maps $n_\beta$, so that when two such maps have shared faces, they will lead to the same collection of bundles and involutions along that face. Finally, when two non-degenerate $k$-simplices in $sd\Delta^k$ share some $(k - 1)$-simplex as a face, then it is the $i$th face of both of them for some common $i$, so that the same bundle is omitted from the sequence given above.

We now make a simple definition that will be useful in talking about induced maps on homotopy groups in the next section. Here we speak of a single field theory, not of a family depending on an interesting parameter space.

**Definition 3.15.** A $1|1$-Euclidean field theory is said to be topological if the infinitesimal generator is 0.

A topological $1|1$-Euclidean field theory is thus essentially determined by the vector space assigned to the super-point. This vector space is some super vector space which is a subspace of the ambient space $H$.

Now we state the lemma relevant for later considerations regarding basepoints. By topologically based map, we mean a map of pairs to $(1|1-EFT, E)$, where $E$ is a topological $1|1$-Euclidean field theory.

**Lemma 3.16.** Applying the construction of Lemma 3.14 to a topologically based map

$$sd^q(\Delta^k, \partial\Delta^k) \to (1|1-EFT, E)$$

with a compatible sufficiently fine subdivision produces a map of pairs

$$sd^q(\Delta^k, \partial\Delta^k) \to (\mathcal{N}\text{QVect}, E)$$
where $E$ is considered as a vertex of $\mathcal{N}\text{QVect}$ in the natural way.

Proof. The statement is proven by the following observation. Let $\Delta^k \rightarrow 1|1\text{-EFT}$ be a $k$-simplex with subface $\Delta^j \rightarrow \Delta^k \rightarrow 1|1\text{-EFT}$ determining a $j$-fold degeneracy of a topological $1|1$-Euclidean field theory. Then for any sufficiently fine subdivision of $\Delta^k \rightarrow 1|1\text{-EFT}$, the bundles defined in Lemma 3.14 along the collection of little simplices lying at the subface $\Delta^j \rightarrow \Delta^k$ will necessarily restrict to the trivial bundle with fiber $E$ at that face. It is not possible for any other bundles to be defined using resolvent elements, since there is no way of dividing $E$ up according to the spectrum of $A(1)$, since the field theory is topological. \qed

3.3 Proof of weak equivalence

We are now ready for the proof of the main theorem of this work. This section reviews simplicial homotopy groups and the functor $Ex^\infty$, then establishes that the map

$$Ex^\infty F : Ex^\infty \mathcal{N}\text{QVect} \rightarrow Ex^\infty 1|1\text{-EFT}$$

is a bijection on path components and an isomorphism on all homotopy groups for all basepoints.

3.3.1 Review of simplicial homotopy

We briefly recall simplicial homotopy groups here; further detail is at A.5. A reference for this material is I.6 and I.7 of [10].

We presume some familiarity with simplicial sets. By $\Delta^k$ we mean the simplicial set represented by $[k] \in \Delta$. A simplicial homotopy from $f$ to $g$, where $f$ and $g$ are maps of simplicial sets $X \rightarrow Y$, is a map $h : X \times \Delta^1 \rightarrow Y$ such that $h \circ (id_X \times d^1)$ is $f$ and $h \circ (id_X \times d^0)$ is $g$. Here $d^i$ is the coface map $[0] \rightarrow [1]$ omitting $i$, which becomes a morphism of
simplicial sets by the Yoneda embedding.

Homotopy is not an equivalence relation for maps to a general simplicial set. For example, consider the two inclusions of a vertex of $\Delta^1$.

Homotopy is an equivalence relation for maps to a simplicial set which is Kan. The definition of Kan is in [A.5] as an idea to keep in mind here, a simplicial set which is the singular set of a topological space is Kan, and the reason for this is that there is a deformation retraction of the standard $k$-simplex onto any union of $k$ of its faces.

In 3.2.1 we introduced the simplicial sets $sd\Delta^k$ and the last vertex map $h : sd\Delta^k \to \Delta^k$. Subdivision extends to an endofunctor of simplicial sets; the functor $Ex$ is its right adjoint. In other words a $k$-simplex of $ExX$, which is a map $\Delta^k \to ExX$, corresponds to a map $sd\Delta^k \to X$, so that $k$-simplices of $ExX$ are exactly the subdivided simplices of $X$. Precomposition with the last vertex map gives a natural map $X \to ExX$. The colimit of

$$X \to ExX \to Ex^2X \ldots \to Ex^nX \to \ldots$$

is denoted $Ex^\infty X$.

The natural map $X \to Ex^\infty X$ is a weak equivalence of simplicial sets. Moreover, the simplicial set $Ex^\infty X$ is Kan. These statements are found in Theorem III.4.8 of [10].

We give here a very rough idea of how the application of $Ex$ leads to a Kan complex. First, note that a $k$-simplex of $Ex^\infty X$ factors through some finite $Ex^nX$. Thus the $k$-simplices of $Ex^\infty X$ are maps $sd^n\Delta^k \to X$ for some natural number $n$, and two such maps represent the same $k$-simplex of $Ex^\infty$ if they become the same after suitable compositions with the last vertex map.

Recall that homotopy does not give an equivalence relation for inclusions of vertices into a general simplicial set $X$. Note that maps $sd\Delta^1 \to X$ represent 1-simplices
of $Ex^\infty X$.

Consider this more explicit description of $sd\Delta^1$. The nerve of the poset arising from non-empty elements of the power set of $\{0,1\}$ is $\{0\} \hookrightarrow \{0,1\} \twoheadleftarrow \{1\}$. We see then that $sd\Delta^1$ consists of an ordered pair of 1-simplices $(a,b)$ subject to the condition that $d_0a = d_0b$. Speaking colloquially, we have taken an arrow that went in one direction and replaced it with two arrows, one going in either direction. Suppose $X$ has two vertices $v_0$ and $v_1$ with the property that there is a 1-simplex $\gamma$ satisfying $d_0\gamma = v_1$ and $d_1\gamma = v_0$, but there is no 1-simplex connecting the vertices in the opposite direction. Observe that in $ExX$ there are in fact 1-simplices going both ways between $v_0$ and $v_1$. The first such simplex is given by $(\gamma, s_0(v_1))$ (where $s_0$ is the degeneracy map). This map $sd\Delta^1 \to X$ is in fact just $\gamma$ precomposed with the last vertex map. There is a second map, given by $(s_0(v_1), \gamma)$. This ordered pair represents a 1-simplex in $ExX$ which has $v_0$ as its 0th face, and $v_1$ as its 1st face. Thus by applying $Ex$ we have that homotopy now gives an equivalence relation on vertices. This concludes the rough outline of how $Ex$ leads to Kan complexes.

Given a simplicial set $X$ which is Kan, and a vertex $v \in X$, we define the simplicial homotopy groups of $X$ as homotopy classes of maps $(\Delta^k, \partial\Delta^k) \to (X,v)$. If $X$ were not Kan, we can apply $Ex^\infty$. Then we can compute homotopy groups of $X$ as the simplicial homotopy groups of $Ex^\infty X$. In this case representatives of $\pi_k(X,v)$ have the form $sd^n(\Delta^k, \partial\Delta^k) \to (X,v)$, and two such elements represent the same element of $\pi_k$ if they are homotopic relative boundary, possibly after some precomposition with the last vertex map. We emphasize this point here because later it will be important to relate certain maps $sd^n\Delta^k \to \mathcal{N}\text{QVect}$ to those which arise by precomposition with $h$, i.e. by postcomposing with $Ex^n\mathcal{N}\text{QVect} \to Ex^{n+1}\mathcal{N}\text{QVect} \to \ldots \to Ex^{n+m}\mathcal{N}\text{QVect}$.

The simplicial set $\mathcal{N}\text{QVect}$ is not Kan (since 1-simplices only go in the direction of increasing rank), and it is unclear whether $1|1$–EFT is Kan. Thus the homotopy
groups of the geometric realizations cannot be directly computed simplicially. As a result, we now prove that $F : \mathcal{N}Q\text{Vect} \to 1|1-\text{EFT}$ is a weak equivalence by showing that it is one after applying $Ex^\infty$.

### 3.3.2 Bijection on path components

So far in dealing with families of field theories we have only used the ordinary trace, and not the supertrace, of the family of endomorphisms, even though a field theory determines both. We used the trace in verifying that the spectrum of the family of operators varies continuously \[2.30\] Now, we use the supertrace to show that the map $F : \mathcal{N}Q\text{Vect} \to 1|1-\text{EFT}$ defined in Section 3.1 is bijective on $\pi_0$.

**Lemma 3.17.** The map $F : \mathcal{N}Q\text{Vect} \to 1|1-\text{EFT}$ induces a bijection on $\pi_0$.

**Proof.** If $(V_0, V_1, \alpha_1)$ is a 1-simplex of $\mathcal{N}Q\text{Vect}$, then $V_0$ and $V_1$ have the same superdimension, since $\alpha$ is an odd involution on the complement. Distinct vector spaces of the same superdimension will both include into some larger one, as in $V_0 \subset V_1 \supset V_0'$, and thus will be connected by a path in $Ex^\infty \mathcal{N}Q\text{Vect}$ (in fact already in $Ex\mathcal{N}Q\text{Vect}$). Thus the path components of $\mathcal{N}Q\text{Vect}$ correspond to the integers, with the map $\mathcal{N}Q\text{Vect} \to \mathbb{Z}$ given on vertices by taking the superdimension.

Evaluating a field theory on the circle which comes from gluing $L$ and $R(1)$ and using the braiding isomorphism (see \[2.26\]) gives the supertrace of $A(1)$. Since $A(1)$ is generated by the square of an odd endomorphism (see Lemma \[2.21\] and following), eigenspaces corresponding to eigenvalues other than 1 have superdimension 0. We can see this by noting that the infinitesimal generator is an odd automorphism of the eigenspace corresponding to some eigenvalue $\lambda \neq 1$. A super vector space which admits an odd automorphism has superdimension 0. Thus the supertrace of $A(1)$ is the superdimension of the eigenspace with eigenvalue 1. This value is locally constant, since the spectrum of $A(1)$ varies continuously (by Lemma \[2.30\]), and portions of the
spectrum which move away from $1 \in \mathbb{C}$ must have superdimension 0, since $A(1)$ has an odd generator as we just used.

3.3.3 Injectivity

The construction given in Lemma 3.14 in fact brings us very near a proof of the injectivity of the induced map $F_*$ on $\pi_k$. Given some map

$$D : sd^n(\Delta^k, \partial \Delta^k) \to (\mathcal{NQVect}, E)$$

we postcompose with $F$ in order to obtain a topologically based map (recall Definition 3.15 and following)

$$F \circ D : sd^n(\Delta^k, \partial \Delta^k) \to (1|1-EFT, E)$$

Suppose that this element represents 0 in $\pi_k(1|1-EFT, E)$. This means there is a (highly subdivided) $k + 1$ simplex of $1|1-EFT$ exhibiting that $F \circ D$ represents 0. We apply choose a compatible sufficiently fine subdivision of this subdivided simplex and apply Lemma 3.16 in order to get a subdivided $k + 1$-simplex of $\mathcal{NQVect}$ that is the sort of thing we would use to show that $D$ itself represents 0. The problem we must face in carrying out this strategy is to establish that $D$ is homotopic relative boundary to a face of the subdivided $k + 1$-simplex we obtain by applying $F$ and then the construction of Lemma 3.14. We establish the requisite homotopy for any choice of a compatible sufficiently fine subdivision.

**Lemma 3.18.** The induced map $F_* : \pi_k(\mathcal{NQVect}, E) \to \pi_k(1|1-EFT, E)$ is injective for all basepoints $E \in \mathcal{NQVect}_0$. 

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Proof. Given a representative

\[ D : \text{sd}^m(\Delta^k, \partial \Delta^k) \to NQVect \]

suppose that there is an \( n \geq 0 \) and an \( E \) as below such that

\[ E : \text{sd}^n(\Delta^{k+1}, \Lambda_{k+1}^k) \to 1|1-EFT \]

has \((k + 1)\)st face equal to \( F \circ D \circ h^{n-m} \). Recall that \( h \) is the last vertex map \( \text{sd} \Delta^k \to \Delta^k \) (see the discussion near \[3.2\]). The existence of such an \( E \) means that \( D \) represents an element in the kernel of \( F_* \). Then applying construction (3.14) to \( E \) yields a map as below.

\[ \hat{D} : \text{sd}^p(\Delta^{k+1}, \Lambda_{k+1}^k) \to NQVect \]

Thus the whole of the current lemma is the statement that for any compatible sufficiently fine subdivision of

\[ F \circ D : \text{sd}^n(\Delta^k, \partial \Delta^k) \to 1|1-EFT \]

the map to \( NQVect \) produced by 3.14 is homotopic relative boundary to \( D \). Note that the maps are to be considered as maps \((\Delta^k, \partial \Delta^k) \to (Ex^\infty NQVect, E)\).

The proof is by means of the claim that for \( D \) as above, and any compatible sufficiently fine subdivision of \( F \circ D \), the map \( \hat{D} : \text{sd}^p(\Delta^k, \partial \Delta^k) \to (NQVect, E) \), restricted to its \((k + 1)\)st face, is a subordinate geometric subdivision of \( D \). The definition of subordinate geometric subdivision is given in Definition 4.12 and loosely speaking means that it looks like a thing involving restricting the bundles defining \( D \) and perhaps only choosing some of the lower rank ones, with redundancy. Once
this is known, we can apply Lemma 4.14, which gives a homotopy relative boundary between \( d_{k+1}(\hat{D}) \) and \( D \) (as maps to \( \text{Ex}^\infty \mathcal{N}\text{QVect} \)).

We have placed Lemma 4.14 and its prerequisites elsewhere in order to streamline the current proof. Without repeating what is said in 4.4.3, we prove the necessary claim here. The essential idea is contained in the discussion which immediately follows the conclusion of this proof, and it is best to look to that first.

To prove the claim that any compatible sufficiently fine subdivision leads to a subordinate geometric subdivision, we essentially repeat the observation used to prove Lemma 3.16. For little simplices of \( s d^k \Delta^k \) which lie at the \((k+1)\)st face of the subdivided \( F \circ D \), the only possible choices of resolvent elements \( \mu_i \) in a resolvent collection will be those which either include all the (extensions of) bundles defining \( D \), or those which exclude some of them. No matter how the resolvent elements are chosen, the little \((k+1)\)-simplices near the \((k+1)\)st face produced by Lemma 3.14 will only involve extensions to \( |\Delta^{k+1}| \) of restrictions (by geometric subdivision maps) of bundles defining \( D \). It is not possible for other bundles to be involved until one moves away from the \((k+1)\)st face, by the continuity of the spectrum of \( A(1) \). Thus the simplices produced, when restricted to their faces which constitute \( d_{k+1}(s d^k \Delta^{k+1}) \), satisfy the subordination condition of Definition 4.12.

Now we give the spirit of the preceding proof. Let us ignore details regarding subdivision and basepoints and just take a single 1-simplex of \( \mathcal{N}\text{QVect} \). Such a simplex consists of \( V_0 \) and \( V_1 \), super vector bundles on an interval, and an odd involution \( \alpha \) on a complement of \( V_0 \subset V_1 \). The algebraic subdivision (see Definition 3.2) of this 1-simplex is the map

\[
\text{sd} \Delta^1 \to \mathcal{N}\text{QVect}
\]

consisting of the following two simplices (recall the concrete description of \( \text{sd} \Delta^1 \) from
The first is simply \((V_0, V_1, \alpha)\), i.e. the simplex we started with. Denote by \(i\) the inclusion \(\{1\} \hookrightarrow [0, 1]\) and by \(p\) the map \([0, 1] \rightarrow \{1\}\). Then the second 1-simplex is \(((i \circ p)^*V_1, (i \circ p)^*V_1, 0)\). In other words, it is the degeneracy of the 0-simplex obtained by restricting \(V_1\) to the point \(1 \in [0, 1]\).

We contrast the previous map \(sd\Delta^1 \rightarrow \mathcal{N}QVect\) obtained by algebraic subdivision with the sort of map that arises by the construction of Lemma \[3.14\]. In the latter case, we will have something like the following. Denote by \(j_{[0, \frac{1}{2}]}\) the map \([0, 1] \hookrightarrow [0, 1]\) defined by \(x \mapsto \frac{x}{2}\), and by \(j_{[\frac{1}{2}, 1]}\) the map \([0, 1] \hookrightarrow [0, 1]\) defined by \(x \mapsto \frac{x + 1}{2}\) (and note that these are what we called geometric subdivision maps in \[3.3\]). The first 1-simplex will be \((j_{[0, \frac{1}{2}]}^*V_0, j_{[0, \frac{1}{2}]}^*V_1, j_{[0, \frac{1}{2}]}^*\alpha)\). The second will be \((j_{[\frac{1}{2}, 1]}^*V_1, j_{[\frac{1}{2}, 1]}^*V_1, 0)\).

Let us emphasize here that we are concerned with the particular (strictly associative) pullbacks of vector bundles (which are concrete subbundles of a trivialized bundle with fiber \(H\)), and not just abstract categorical limits determined only up to unique isomorphism. The reason for this is that we are working with honest simplicial sets, not just some sort of weak functor to sets, or a simplicial groupoid. Hence it is important to distinguish between a particular subbundle of \([0, 1] \times H\) and its restriction to \([0, \frac{1}{2}]\) (or, more precisely, its pullback along \(j_{[0, \frac{1}{2}]}\). They lead in general to distinct 1-simplices of the simplicial set \(\mathcal{N}QVect\).

Returning to the two maps \(sd\Delta^1 \rightarrow \mathcal{N}QVect\) we just described, the first being the algebraic subdivision, and the second a sort of geometric subdivision, note that in the first case the second 1-simplex is genuinely degenerate. In the second case, however, we do not have a true degenerate simplex. Instead, the bundle \(j_{[\frac{1}{2}, 1]}^*V_1\) can have a fiber which is not the same subspace of \(H\) over the whole interval. Thus the simplex \((j_{[\frac{1}{2}, 1]}^*V_1, j_{[\frac{1}{2}, 1]}^*V_1, 0)\) is not degenerate, even though it could seem like it should be (since the inclusion of bundles is just the identity).

The point of Lemma \[4.14\] which implies the lemma we just proved, is to show that maps \(sd\mu\Delta^k \rightarrow \mathcal{N}QVect\) which arise from Lemma \[3.14\] (which constructs simplices of
\( \mathcal{NQVect} \) from simplices of \( 1|1-\text{EFT} \) applied to simplices in the image of \( F \), are just like the example above, where we have something which is almost like the algebraic subdivision of the simplex, except with ‘pseudo-degenerate’ simplices rather than true degeneracies. With this picture in mind it should appear reasonable that the necessary homotopy relative boundary exists between the algebraic subdivision and the closely related products of Lemma 3.14.

3.3.4 Surjectivity

Given a family of field theories parameterized by \( \Delta^k \), we must show that after producing a map \( sd^m \Delta^k \to \mathcal{NQVect} \) as above, and then postcomposing with the map \( F : \mathcal{NQVect} \to 1|1-\text{EFT} \), we obtain something which is homotopic to the original, when the two are considered as maps to \( Ex^\infty 1|1-\text{EFT} \). Field theories in the image of \( F \) have families \( A(1) \) of endomorphisms with nice behavior. All eigenvalues of the infinitesimal generator move in from infinity in a uniform fashion. A general family of field theories, on the other hand, will have a spectrum that does not behave in any standard fashion. As a result an essential element of the surjectivity proof is showing that there is a way to standardize the infinitesimal generator, and to do so in a way that is compatible on faces. The other necessary part of the proof is to throw away the non-vector bundle portion of the module; we do this by reparameterizing the families of operators \( A(t) \) and \( B(t) \).

**Lemma 3.19.** The induced map \( F_* : \pi_k(\mathcal{NQVect}, E) \to \pi_k(1|1-\text{EFT}, E) \) is surjective for all basepoints \( E \).

**Proof.** Given some topologically based representative

\[
M : sd^m(\Delta^k, \partial\Delta^k) \to (1|1-\text{EFT}, E)
\]

we may produce a compatible sufficiently fine subdivision (see Lemma 3.13) which
leads by Lemma 3.16 to a map $N : sd^n(\Delta^k, \partial\Delta^k) \to (\mathcal{NQVect}, E)$, where $n$ is some natural number larger than $m$. Post-composition with $F$ yields a map

$$sd^n(\Delta^k, \partial\Delta^k) \to (1|1-EFT, E)$$

We prove surjectivity of the induced map $F_*$ by establishing a simplicial homotopy between the $(n - m)$-fold algebraic subdivision (defined at Definition 3.2) of the original map $M$, and the map $F \circ N$. This will prove that $M$ and $F \circ N$ represent the same element in $\pi_k(Ex^{\infty}1|1-EFT, E)$. For a review of why this is so, see the earlier discussion at 3.3.1.

We work one simplex at a time, and then verify that the maps we define together constitute a map from the colimit $sd^n\Delta^k$. Consider the $k$-simplex of field theories determined by

$$\Delta^k \to sd^n\Delta^k \to sd^m\Delta^k \to 1|1-EFT$$

where the first map is some non-degenerate $k$-simplex of $\sigma$ of $sd^n\Delta^k$, the middle map arises from composing $(n - m)$ geometric subdivision maps, and the final map is $M$. The compatible sufficiently fine subdivision of $M$ means that this $k$-simplex (incorrectly, but conveniently, denoted $M \circ \sigma$) comes with a collection $\{\lambda_i\}_{i=0}^k$ of elements in the resolvent set of $A(1)$. We then decompose the module $V$ as

$$V = V_{\lambda_k} \bigoplus \lambda_k V_{\lambda_{k-1}} \bigoplus \lambda_{k-1} V_{\lambda_{k-2}} \bigoplus \cdots \bigoplus \lambda_1 V_{\lambda_0} \bigoplus \lambda_0 V$$

with the notation for the modules as before, meaning that $\lambda_j V_{\lambda_{j-1}}$ is the subbundle of $V$ given by generalized eigenspaces of $A(1)$ corresponding to eigenvalues with norm between $\lambda_j$ and $\lambda_{j-1}$. Each of the summands is a vector bundle on $|\Delta^k|$ except, possibly, for $V_{\lambda_k}$.

Our task is to define a simplicial homotopy between $M \circ \sigma$ and $F \circ N \circ \sigma$. The
homotopy is defined by producing a family of field theories parameterized by $|\Delta^k| \times |\Delta^1|$, and then producing from this a simplicial homotopy. This essentially just means cutting up the prism $|\Delta^k| \times |\Delta^1|$ along hyperplanes into copies of $|\Delta^{k+1}|$; see 4.7. In what follows we do not repeatedly state the existence of smooth extensions and the independence of constructions from the choice of smooth extension, nonetheless the form of the constructions will make such independence immediate.

Let $s$ be the homotopy/concordance parameter, i.e. the coordinate on $|\Delta^1|$. First, we define the sheaf of modules for the family of field theories. It consists of functions on $|\Delta^k| \times |\Delta^1|$ which lie in $\lambda_k V_{\lambda_k - 1} \oplus \lambda_{k-1} V_{\lambda_{k-2}} \oplus \ldots \oplus \lambda_1 V_{\lambda_0} \oplus \lambda_0 V$

for $s \leq 0$, and which lie in $V$ for $s \geq 0$.

We also require that for any function in $V$, the component which lies in $\lambda_i V_{\lambda_{i-1}}$ vanishes to all orders at points $x_i = x_{i+1} = x_{i+2} = \ldots = x_k = s = 0$.

Note that in order for the module to restrict correctly at $s = 0$ to something in the image of $F$, we see that the prescribed vanishing must occur at $s = 0$. We do not impose the condition on functions at all $s \leq 0$ because we wish to be able extend the family we define here to a neighborhood of $|\Delta^k| \times |\Delta^1| \subset \mathbb{R}^k \times \mathbb{R}^1$. This point will be clarified when discussing the infinitesimal generator shortly.

We now define the family of endomorphisms on each summand separately.

On $\lambda_0 V$ we define $A_s := A(s^4t)$ and $B_s := s^2 B(s^4t)$. It is clear that these definitions give continuous endomorphisms. Moreover, they satisfy the super semigroup relations (see Lemma 2.22) for all $s$.

On $V_{\lambda_k}$ we define $A_s := A(s^4) \left( \frac{1}{s^4} \right)$ and $B_s := \frac{1}{s^2} B(s^4) \left( \frac{1}{s^4} \right)$. In words, we speed up time using the homotopy parameter, and thereby wash out the non-vector bundle summand of $V$. 
Lemma 3.20. This gives a continuous endomorphism, and satisfies the super semigroup relations.

Proof. The only place we need to check continuity is at \( s = 0 \), since otherwise we simply invoke the continuity of the original family \( A(t) \). Since \( |e^{-D^2}| < \lambda_k < 1 \) we see that the operator is near zero as \( s \) gets small.

It is straightforward to check that the super semigroup relations are satisfied for all \( s > 0 \).

Finally, we treat \( \lambda_i V_{\lambda_{i-1}} \). On this summand, the difficulty in defining the endomorphisms is at the subface consisting of those points for which \( x_i = x_{i+1} = x_{i+2} = \ldots = x_k = 0 \). On this subface, as \( s \) approaches 0, the infinitesimal generator of the semigroup must become infinite. Because this summand is a vector bundle in the module defining the original family of field theories, we may work directly with the infinitesimal generator of the semigroup, thanks to Lemma [2.21].

Denote by \( \mathcal{D} \) this infinitesimal generator, keeping in mind its dependence on the parameter space. Our goal is to define a family of infinitesimal generators \( \mathcal{D}_s \) with \( \mathcal{D}_1 = \mathcal{D} \), and \( \mathcal{D}_0 \) being the generator which comes from the map \( F \).

In order for the supertrace \( stre^{-tD^2} \) (see Lemma [2.26]) to give a smooth function on the parameter space, it is necessary that \( \mathcal{D}_s \) becomes infinite when summands of the module \( V \) are ‘turned off.’ Moreover, the operator must become infinite in a constrained way. It is not enough for the spectrum to move to \( \infty \) in the Riemann sphere. Instead, because the spectrum of \( A(1) \) varies continuously (by Lemma [2.30]), we see that the spectrum of \( \mathcal{D}_s \) must grow large while remaining in the portion of the complex plane given by \( |Im(z)| < |Re(z)| \); i.e. the portion consisting of complex numbers whose squares have positive real part.

In other words, we seek an operator \( \mathcal{D}_s^{-1} \) which vanishes at \( x_i = x_{i+1} = x_{i+2} = \ldots = x_k = s = 0 \) and which, near that subset, has a spectrum whose elements when
squared have positive real part.

Before defining the operator, we introduce a function and recall an operator and a function. The function we introduce is denoted $\phi(s)$, and is a smooth step function satisfying $\phi(0) = 1$ and $\phi(1) = 0$. We recall the operator $\alpha_i$, which arose in Lemma 3.14. This is the odd involution defined by the fact that the spectrum of $\partial$ has no imaginary eigenvalues, since we are working only with $\lambda_i V_{\lambda_{i-1}}$, on which $A(1) = e^{-\partial^2}$ has eigenvalues of norm less than $\lambda_{i-1} < 1$. Specifically, $\alpha_i$ is defined as multiplication by $\text{sign}(\text{Re}(\mu))$ on generalized eigenvectors $v$ with eigenvalue $\mu$. Recall also that the function of a single real variable $\rho(x)$, used in the definition of the map $F$ (see Section 3.1) is $\frac{1}{x^2}$ for $x$ near 0 and identically 0 for $x$ near 1.

We also recall that we use homogeneous coordinate $(x_0, x_1, \ldots, x_k)$ on $|\Delta^k|$ which sum to 1.

Now we define $\partial_s$ by defining its inverse and checking that it satisfies the spectral requirements enumerated earlier.

$$\partial_s^{-1} := s^2 \partial^{-1} + \phi(s) \frac{\alpha_i}{\rho(x_i + x_{i+1} + \ldots + x_k)} \quad (3.2)$$

Observe that this operator vanishes exactly at $x_i = x_{i+1} = x_{i+2} = \ldots = x_k = s = 0$. It certainly vanishes there, since both terms in the sum do. That it vanishes nowhere else can be seen in the following way. For any positive real numbers $a$ and $b$, the linear combination $a\partial^{-1} + b\alpha_i$ is invertible. This is so since, if $\lambda$ is a complex number with positive real part, then $a\lambda + b$ is non-zero, and similarly if $\lambda$ has negative real part, then $a\lambda - b$ is non-zero. According to the description of the relation between $\alpha_i$ and the spectrum of $\partial$, this proves the requisite invertibility. Finally, similar considerations show that $\partial_s^{-1}$ has spectrum whose elements square to have positive real part.

Before continuing, observe that the operator is defined for $s < 0$ as well as $x_i < 0$, $x_{i+1} < 0$, etc and is invertible there. This means that we can extend the family,
defined at first only on the compact standard \( k \)-simplex, to a neighborhood in \( \mathbb{R}^k \). It is necessary that we be able to do this, since according to Definitions 2.18 and 2.19 \( k \)-simplices of 1|1–EFT are represented by families defined in some neighborhood of \( \Delta^k \subset \mathbb{R}^k \).

The expression \[3.2\] clearly takes the correct form at \( s = 0, 1 \) in a neighborhood of \( x_i = x_{i+1} = \ldots = x_k = 0 \), so that it is a candidate for producing the necessary concordance between the initial family of field theories and the family in the image of \( F \). The operator \( \mathcal{D}_s \) is in fact a continuous endomorphism of the module sheaf \( V \) on \( \Delta^k \times \mathbb{R} \) defined above. The proof is as in Lemma 3.1, where we apply the open mapping theorem.

**Lemma 3.21.** The family of operators \( \mathcal{D}_s \) defines a continuous \( C^\infty(\Delta^k) \)-linear automorphism of \( V \).

**Proof.** We check that \( \mathcal{D}_s^{-1} \) is surjective. It is evidently continuous, linear, and injective, so by proving surjectivity we can apply the open mapping theorem, as we did previously in 3.1. To prove surjectivity we check that \( \mathcal{D}_s \), applied to an element of \( V \), yields another such element. So, choosing any function \( f \), we must confirm the continuity and vanishing of \( g := \mathcal{D}_s f \) and all its derivatives at \( S_i := \{(x_0, x_1, \ldots, x_k, s) \in |\Delta^k| \times |\Delta^1| |x_i = x_{i+1} = x_{i+2} = \ldots = x_k = s = 0\} \). Observe that \( g \) is a smooth function on the complement of \( S_i \), since there \( \mathcal{D}_s \) is just an invertible bundle endomorphism.

First, we note that \( g \) extends continuously as \( 0 \) to \( S_i \). The operator norm of \( \mathcal{D} \) is bounded above by some \( L > 0 \). This means we can establish the following bound.

\[ |g| \leq \left| \frac{1}{\frac{s^2}{L} + (x_i + x_{i+1} + \ldots + x_k)^2} f \right| \]

The right-hand side is the norm of a continuous (smooth, even) function vanishing on \( S_i \), so \( g \) extends continuously as \( 0 \) to \( S_i \).
A similar estimate then lets us show that the first derivatives of $g$ extend continuously as 0 to $S_i$. To see that $g$ is continuously differentiable at $S_i$, produce a bound of $|\partial_x, \mathcal{D}|$ and proceed as above.

Iteratively, then, all derivatives of $g$ exist and vanish on $S_i$. □

We now have a smooth infinitesimal generator defined on $\Delta^k \times \{0,1\} \cup U$, where $U$ is a neighborhood of the subface on which the definition of Equation 3.2 restricts to the correct operators at $s = 0, 1$. This map can be smoothly extended to the whole of $|\Delta^k| \times |\Delta^1|$. Choosing a trivialization of the vector bundle, we identify a family of endomorphisms with a smooth map to $M_{n \times n}(\mathbb{C})$. The partially-defined smooth infinitesimal generator admits a smooth extension by Whitney’s Approximation Theorem. See e.g. 6.26 of [17].

When defining the homotopy separately on each simplex of $sd^m \Delta^k$ we have not ensured that the families of infinitesimal generators are equal when restricted to shared faces. In order to construct a homotopy of maps $sd^m \Delta^k \to 1|1-\mbox{EFT}$ we need to ensure that the homotopies we construct agree on these shared faces. This is always possible to do. There are two things two consider. The first, which is compatibility when $\mathcal{D}_s$ becomes infinite, is true by the form of 3.2. The second thing is that the extensions to the whole prism are compatible. We can always ensure this since when choosing the smooth extension of the infinitesimal generator it is always also possible to do so with additional smooth boundary conditions (which arise from having previously defined the homotopy for an adjacent simplex). We emphasize that by 2.21 we only need to define a smooth map to a contractible space (of odd endomorphisms) so there is no extension problem.

The homotopy constructed here is not precisely one between $(V,W,L,R)$ and their modification by the construction of a simplex in $\mathcal{N}\mbox{QVect}$, since such a simplex forgets about $W$. By 2.25 we know that $L$ gives an isomorphism $W \cong V^\vee$, and according to 4.7 an isomorphism of families of field theories can be implemented by
a smooth path of families of field theories.

That the homotopy constructed here is a homotopy relative boundary follows from the fact that the modification of the semigroups used to define the homotopy leaves field theories that are already topological unchanged. Since they have $A(t) = \text{id}$ and $B(t) = 0$ for all $t$, nothing happens to them during the reparameterization using $s$.

The homotopy just defined was between a geometric subdivision (recall Definition 3.3) of $M$ and $F \circ N$. According to Lemma 4.15, the geometric subdivision of $M$ is homotopic to the algebraic one, which is the relevant subdivision to consider when comparing simplices of $Ex^\infty(1|1\text{-EFT})$. Moreover, we may conclude that this is a homotopy relative boundary, by the naturality of the homotopies defined in Lemma 4.15 and the fact that $M$ maps the boundary of $sd\Delta^k$ constantly to $E$.

This establishes the requisite homotopy relative boundary between $M$ and $F \circ N$ as maps $(\Delta^k, \partial \Delta^k) \to (1|1\text{-EFT}, E)$.

\begin{align*}
3.3.5 \text{ Conclusion} & \\
From the previous lemmas we conclude the following, which together with 2.35 establishes the homotopy type of $1|1\text{-EFT}$.
\end{align*}

**Theorem 3.22.** $F : \mathcal{N}Q\text{Vect} \to 1|1\text{-EFT}$ is a weak equivalence of simplicial sets.

**Remark 3.23.** The equivalence 3.22 holds for both real and complex $K$-theory. The real case is exactly the same, with the following added nuance. The difference is that in Lemma 3.14 we must produce real bundles instead of complex ones. To do this, note that if the endomorphisms of $V$ and $W$ are real, then the eigenspaces $\lambda_{i+1} V_{\lambda_i}$ obtain a real structure by restriction of that on $V$. Denote by $\sigma$ the real structure on $V$. Suppose that $A(1)v = \mu v$. Then

\[ A(1)\sigma v = \sigma A(1)v = \sigma(\mu v) = \bar{\mu}\sigma(v) \]
so that $\sigma(v)$ is an eigenvector with eigenvalue $\bar{\mu}$. Since $\mu$ and $\bar{\mu}$ have the same norm, $\sigma$ restricts to an involution of $\lambda_{i+1}V_{\lambda_i}$. Finally, the odd involution defined as $\pm \text{Id}$ depending on whether an eigenvalue has positive or negative real part is compatible with this induced real structure, since complex conjugation does not change the sign of the real part. As a result, the odd family of involutions on the complex bundle $\lambda_{i+1}V_{\lambda_i}$ induces one on the underlying real bundle, and the construction of simplices of $\mathcal{NQVect}$ proceeds as before.
CHAPTER 4

AUXILIARY RESULTS

4.1 Comparison of classifying spaces

Here we prove that our simplicial set $\mathcal{N}Q\text{Vect}$ in fact has the correct homotopy type. In Theorem 7.1 of [13] it is shown that a topological category $\mathcal{C}$, which consists of objects super subspaces of a (real) super Hilbert space $H$ (like in our case) and morphisms $V_0 \to V_1$ given by inclusions $V_0 \hookrightarrow V_1$ together with odd orthogonal involutions on the orthogonal complement of $V_0$ in $V_1$, has the homotopy type of a space representing $KO^0$. There is also the analogous statement for $K^0$ involving a complex Hilbert space and unitary odd involutions.

Now we must compare our model involving not necessarily orthogonal involutions to that one. Before doing so it is worth noting here how the more general topological category arises in our case. Since we work with oriented field theories, we do not end up with a single vector space with a pairing together with a family of endomorphisms of that space which are self-adjoint with respect to that pairing. (Recall Figure 2.5, and consider how it changes when objects and morphisms do not have orientations.) Because [13] treats unoriented field theories and assumes certain things about the pairing induced by the generating bordism $\mathcal{L}$ (see Figure 2.2), they can relate their field theories more directly to the topological category $\mathcal{C}$, defined above, which involves the inner product on the Hilbert space.

We now prove the lemma that our space $\mathcal{N}Q\text{Vect}$ retains the homotopy type of $\mathcal{C}$ even when allowing non-orthogonality in the morphisms.
Lemma 4.1. The inclusion of the smooth simplices of the nerve of $\mathcal{C}$ into $\mathcal{N}Q\text{Vect}$ is a weak equivalence.

Proof. Observe that an odd involution $\alpha$ on some complement of $V_0$ in $V_1$ induces an odd involution on the orthogonal complement. Denoting by $p$ the orthogonal projection onto $V_0^\perp$, the map $p : V_1' \to V_0^\perp$ is an isomorphism. Then $p \alpha p^{-1}$ is an odd involution on the orthogonal complement. Thus the problem amounts to replacing an odd involution by an odd orthogonal involution. Writing $V_1'$ as a sum according to the grading involution, the odd involution (induced by) $\alpha$ has the form

$$\alpha = \begin{pmatrix} 0 & T \\ T^{-1} & 0 \end{pmatrix}$$

where $T$ is an isomorphism from the odd to the even summand. Writing $T$ according to its polar decomposition $O \sqrt{T^*T}$ where $O$ is orthogonal, we replace the given involution with

$$\beta = \begin{pmatrix} 0 & O \\ O^T & 0 \end{pmatrix}$$

which is an orthogonal involution.

The argument concludes with the observation that this replacement can be done smoothly in the operator $\alpha$. \qed

4.2 Simplicial and bisimplicial sets as models of field theories

Field theories are functors, hence it is natural when working with such functors to account as well for natural transformations. Earlier, making a comparison with the singular set of a topological space, we stated that treating field theories as simplicial sets rather than simplicial groupoids does not alter the homotopy type. Here we demonstrate this in detail in the case of topological field theories.
Remark 4.2. In [20] Madsen and Weiss use what they call ‘graphic’ maps in order to ensure that they get a strictly associative pullbacks. They also work with sets rather than groupoids. What we do here is analogous, using the ambient Hilbert space $H$ to obtain natural strict pullbacks, and omitting isomorphisms.

First, consider the simplicial set $1\text{--TFT}$ which assigns to $[k]$ the set of functors $1\text{--Bord} \to C^\infty(|\Delta^k|) – Mod$ (where the target consists of concrete modules, i.e. submodules of $|\Delta^k| \times H$ for a fixed Hilbert space $H$).

Recall the presentation of the Euclidean bordism category given in [2.7]. The category of topological bordisms admits a similar presentation, except that there is only a single bordism from the empty set to the positively and negatively oriented points, rather than a whole family of them. Consider this presentation in the following lemma.

Lemma 4.3. The simplicial set $1\text{--TFT}$ of functors $1\text{--Bord} \to C^\infty(|\Delta^k|) – Mod$ is equivalent to the one which restricts to generating data (i.e. that assigns to a parameter space a quadruple $(V,W,L,R)$ satisfying the evident relations).

Proof. The forgetful map from fully-defined functors to their generating data is an acyclic fibration. Given a $\Delta^k$-family of generating data, one simply assigns the same $\Delta^k$-family to every one-point space (connected object in $1\text{--Bord}$).

Henceforth we denote by $1\text{--TFT}$ the generalized manifold obtained by restriction of a functor to generating data.

Lemma 4.4. The simplicial set $1\text{--TFT}$ (of symmetric monoidal functors subject to the spin statistics condition) is equivalent to the functor which assigns to $\Delta^k$ the set of finite rank smooth subbundles of $|\Delta^k| \times H$ (where $H$ is ungraded, i.e. the vector space is purely even).
Proof. The flip is the identity for smooth manifolds. A spin statistics functor must take this to the grading involution of $V = E(pt)$, yet this is the identity. So the space is purely even.

A family $E \in 1\text{--TFT}_{\Delta^k}$ consists (by 4.3) of a quadruple $(V, W, L, R)$ subject to the conditions that $L : W \times V \to C^\infty(\mathbb{R}^k)$ is a point-wise perfect pairing and that $R$, a section of $V \times W$, leads via suitable compositions with $L$ to the identity bundle endomorphisms of $V$ and $W$.

There is a forgetful map $(V, W, L, R) \mapsto V$. We show this to be an equivalence by using A.16. Consider the lifting problem.

$$
\begin{array}{ccc}
\partial \Delta^k & \rightarrow & 1\text{--TFT} \\
\downarrow^i & & \downarrow^{\text{forget}} \\
\Delta^k & \rightarrow & \text{Vect}
\end{array}
$$

We have a smooth concrete bundle $W$ on the boundary of the $k$-simplex. Since $V$ extends over the $k$-simplex, it admits a trivialization. This trivialization, together with the isomorphism $W \rightarrow V^\vee$ provided by $L$, give a map $\partial|\Delta^k| \rightarrow \text{Emb}(\mathbb{C}^n, H)$ to the space of embeddings of $\mathbb{C}^n$ into $H$. This space is contractible, so that the map extends to the $k$-simplex. Hence we simultaneously produce an extension of $W$ as a subbundle of $|\Delta^k| \times H$ as well as an extension of the isomorphism induced by $L$ between $W$ and $V^\vee$. Then the evaluation and coevaluation maps involving $V$ and $V^\vee$ may be transferred to $W$ and $V$ compatibly with the given isomorphism, so that we have the requisite extensions of $L$ and $R$.

The forgetful map thus satisfies the desired lifting property.

Lemma 4.5. The functor $\text{Man}^{op} \rightarrow \text{Set}$ which assigns to $S$ the collection of finite rank smooth subbundles of $S \times H$ is equivalent (as a simplicial set) to the smooth simplices of $\text{Gr}(H)$.

Proof. Without restricting to $\Delta \hookrightarrow \text{Man}$, we can simply observe that assigning to $S \rightarrow \text{Gr}(H)$ the subbundle of $S \times H$ it classifies gives a natural bijection between
we may then conclude that $1$–$\text{TFT}$ has the homotopy type of the Grassmanian of $H$. This is true because smooth simplices give the same homotopy type as continuous ones for ordinary smooth manifolds, and the argument applies as well to $Gr(H)$ as a colimit of smooth manifolds.

We now consider a more elaborate version of $1$–$\text{TFT}$; this is the one that takes morphisms of the functor category into account. It is, in what immediately follows, a sheaf of groupoids on smooth manifolds, assigning to $S$ the groupoid of (concrete) vector bundles on $S$. Restrict the stack to the extended simplices in order to obtain a simplicial groupoid. Since vector bundles over these (contractible) spaces are trivializable, there is only one isomorphism class of objects for each bundle rank. Thus, the simplicial groupoid is equivalent to the one which assigns to $|\Delta^k|$ the group (groupoid with one object) $C^\infty(|\Delta^k|, GL_n(\mathbb{R}))$ (with the discrete topology). This simplicial groupoid leads, via the level-wise nerve, to a bisimplicial set. The $(k,l)$ bisimplices of the simplicial set are $\coprod_n C^\infty(|\Delta^k|, GL_n(\mathbb{R})^l)$. In other words, this simplicial set determines the homotopy type of the fancy version of $1$–$\text{TFT}$.

On the other hand, we can consider $\coprod_n GL_n(\mathbb{R})$ as a simplicial space by taking its nerve when we view it as a topological groupoid. This simplicial space leads to a bisimplicial set by taking the level-wise singular set. The $(k,l)$ bisimplices are $C^0(\Delta^k, GL_n(\mathbb{R})^l)$. We now apply Proposition IV.1.9 of [10], which states that a point-wise weak equivalence of bisimplicial sets induces a weak equivalence on diagonals. Since $GL_n(\mathbb{R})^l$ is a finite-dimensional smooth manifold, its continuous simplices are equivalent to its smooth simplices via the inclusion of the latter into the former. Fixing an $l$ (the number of composable morphisms, i.e. the simplicial level in the nerve of the groupoid) we obtain the necessary point-wise equivalence.

We thus conclude that the sheaf-of-groupoids model of $1$–$\text{TFT}$, like the sheaf-of-sets model, has the homotopy type of $Gr(H)$. 

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4.3 Implementing isomorphisms as concordances

In this section we define concordance for generalized manifolds and show that from a concordance we can produce a simplicial homotopy. We then show that an isomorphism of field theories leads to a concordance. Thus we have simplicial homotopies between isomorphic field theories.

4.3.1 Preliminary remarks on concordance for generalized manifolds

**Definition 4.6.** A concordance between elements of \( a, b \in \mathcal{G}_S \) of a generalized manifold \( \mathcal{G} \) evaluated on a smooth manifold \( S \) is an element of \( C \in \mathcal{G}_{S \times \mathbb{R}} \) with the property that there is an \( \epsilon > 0 \) such that restricting \( C \) along \( (-\infty, \epsilon) \subset \mathbb{R} \) gives \( p^*a \), where \( p \) is the projection \( S \times (-\infty, \epsilon) \rightarrow S \), and restricting \( C \) along \( (1-\epsilon, \infty) \subset \mathbb{R} \) gives \( q^*b \), with \( q \) the projection \( S \times (1-\epsilon, \infty) \rightarrow S \).

**Lemma 4.7.** A concordance between \( k \)-simplices of a generalized manifold yields a simplicial homotopy.

*Proof.* There are many ways, including the straightforward affine linear one, of viewing \( |\Delta^k| \times |\Delta^1| \) (a smooth manifold with corners) as a union of copies of \( |\Delta^{k+1}| \). Picking any of them, and pulling back the concordance along the various inclusions \( |\Delta^{k+1}| \rightarrow |\Delta^k| \times |\Delta^1| \), gives the \((k+1)\)-simplices of the generalized manifold which constitute the simplicial homotopy. \( \square \)

4.3.2 From isomorphisms to concordances

Here we show something alluded to in the main theorem; namely, that isomorphisms of field theories can be implemented by concordances (defined at Definition 4.6). For families of field theories whose module sheaves come from smooth vector bundles on the parameter space, this can be thought of as similar to the fact that
isomorphic vector bundles (which are subbundles of some fixed trivial bundle) have smoothly homotopic classifying maps.

Suppose that \( f : V \to V' \) and \( g : W \to W' \) give an isomorphism of \( S \)-families of field theories \((V, W, L, R) \to (V', W', L', R')\).

The idea of the proof is the following. We construct a map \( \Phi : S \to GL(H) \) taking \( V \) to \( V' \). Then, by a homotopy between \( \Phi \) and the constant map at the identity, we make a path of field theories interpolating between the two. We rely on Kuiper’s Theorem on the contractibility of the general linear group of an infinite dimensional separable Hilbert space; see [16].

For brevity, we will only write of \( V \); the same conclusions holding for \( W \), \textit{mutatis mutandis}.

The module \( V \) determines a filtration of \( S \) by closed subsets; see Lemma 2.23. Thus we can express \( S \) as the following union, in which the index \( i \) in the expression \( S_i \) refers to the maximal rank of \( V \) (or equivalently, \( W \)) when restricted to \( S_i \).

\[
S_0 \subset S_1 \subset \ldots \subset S_{n-1} \subset S_n = S
\]

Restricted to the open subset \( U_1 := S \setminus S_{n-1} \), the module \( V \) is a vector bundle, and a subbundle of the trivial bundle \( U_1 \times H \).

The bundle \( V|_{U_1} \) has an orthogonal complement \((V|_{U_1})^\perp\), which is a trivializable bundle with fiber an infinite dimensional separable super Hilbert space. The same is true of \( V'|_{U_1} \). We may pick some isomorphism \( f_1 : (V|_{U_1})^\perp \to (V'|_{U_1})^\perp \). Then \( f \oplus f_1 \) is a map \( U_1 \to GL(H) \).

There is a neighborhood \( K \) of \( \partial S_{n-1} \) in \( S_n \setminus \hat{S}_{n-1} \) on which \( V \) may be written as a sum \( V_{\infty} \oplus V_{n-1} \), where \( V_{n-1} \) is a vector bundle extending to \( U_2 := S_{n-1} \setminus S_{n-2} \) and \( V_{\infty} \) is, loosely speaking, the part of \( V \) that gets added as we move up from \( S_{n-1} \) to \( S_n \). Note that on \( U_2 \) we have that \( f \) restricts to an isomorphism \( V_{n-1} \to V'_{n-1} \).
The map $f \oplus f_1$, defined on $K$, restricts to an isomorphism $(V_{n-1})^\perp \to (V'_{n-1})^\perp$. This may, by Kuiper's Theorem, be extended over $U_2$. Call this map $f_2$.

Iterating this procedure, we produce maps $f \oplus f_k$ on neighborhoods of $S \setminus S_{n-k}$ which are compatible with the maps $f \oplus f_{k-1}$. Putting all the maps together, we get a smooth map $\Phi : S \to GL(H)$ which implements the isomorphism $V \to V'$ by construction.

There is a homotopy from the constant map at $id_H$ to $\Phi$. In other words, we have a map $\Psi : S \times I \to GL(H)$. Using this homotopy, we produce at each $S \times \{t\}$ a module by transporting $V$. This means the derivatives of functions in $S$ directions are, at each $t \in I$, constrained in the same way that they were in the original module $V$. There are no constraints in the $t$ direction. In this way we produce a family of modules on $S$ parameterized by $I$, and we use the homotopy $\Psi$ to define the various operators $A(t), B(t)$.

The preceding discussion has proved the following.

**Lemma 4.8.** If two $S$-families of 1|1-Euclidean field theories are isomorphic, then they are concordant.

4.4 Geometric subdivision for $NQVect$

4.4.1 Preliminary combinatorial remarks

Let $k$ be a natural number. The non-empty subsets of the set $\{0, 1, \ldots, k\}$ form a poset when ordered by inclusion. The nerve of this poset is denoted $sd\Delta^k$. The map of posets sending a subset $S \subset \{0, 1, \ldots, k\}$ to its largest element $i \in \{0, 1, \ldots, k\}$ induces a map $sd\Delta^k \to \Delta^k$ called the last-vertex map.

The non-degenerate $k$-simplices of $sd\Delta^k$ can be enumerated by bijections $\sigma : \{0, 1, \ldots, k\} \to \{0, 1, \ldots, k\}$ in the following way. The non-degenerate $k$-simplices of $sd\Delta^k$ are determined by increasing sequences of subsets $\{m_0\} \subset \{m_0, m_1\} \subset \ldots \subset$
\{m_0, m_1, \ldots, m_k\}. Assign to such a sequence the permutation \(\sigma : i \mapsto m_i\). This establishes a bijection between the non-degenerate \(k\)-simplices of \(sd\Delta^k\) and elements of the group of permutations on \(k + 1\) letters.

**Remark 4.9.** According to the bijection determined above, two \(k\)-simplices which correspond to permutations \(\sigma\) and \(\tau\) share a non-degenerate \((k - 1)\)-simplex as a face in \(sd\Delta^k\) exactly when they differ by a single transposition. If \(\sigma = \tau \circ (ii + 1)\), then this shared face is the \(i\)th face of both simplices.

**Definition 4.10.** For a permutation \(\sigma \in S_{k+1}\), we define an increasing function \(\hat{\sigma}\) by the following formula.

\[
\hat{\sigma} : i \mapsto \max\{\sigma(j) | j \leq i\}
\]

Note the following characterization of the last vertex map \(sd\Delta^k \to \Delta^k\). A vertex \(\Delta^0 \xrightarrow{i} \Delta^k \xrightarrow{b} sd\Delta^k\) is sent to \(\hat{\sigma}(i)\), where \(i\) denotes the \(i\)-th vertex of the simplex determined by \(\sigma\). One can see this by associating to \(\sigma\) its chain of subset of \(\{0, 1, \ldots, k\}\) as in the discussion above, and then taking the largest element from each set in turn.

4.4.1.1 Simplicial homotopies

A simplicial homotopy is a map \(X \times \Delta^1 \to Y\) of simplicial sets. It will be useful later to have the following explicit description of a simplicial homotopy.

The simplicial set \(\Delta^k \times \Delta^1\) is a union of \(k + 1\) non-degenerate \((k + 1)\)-simplices, which we denote \(\{H_j\}_{j=0}^k\). Here \(H_j\) is the \(k + 1\) simplex which contains the 1-simplex \((s_0(j), id_{\Delta^1})\). These simplices are subject to \(d_{j+1}H_j = d_{j+1}H_{j+1}\), and all other relations are derived from these by the simplicial identities. The subset \(\Delta^{k-1} \times \Delta^1 \to \Delta^k \times \Delta^1\) corresponding to the inclusion of the \(i\)-th face of \(\Delta^k\) consists of the following faces of elements \(H_j\).
\[
\begin{cases}
d_{i+1}H_j, & j < i \\
nH_j, & j > i
\end{cases}
\]

4.4.2 Geometric subdivision

Let \( A^k \) denote the hyperplane of points in \( \mathbb{R}^{k+1} \) whose coordinates sum to 1. Let \( v_i \) denote the \( i \)-th element of the standard ordered basis of \( \mathbb{R}^{k+1} \). We are repeating here something discussed in A.5.3.

Having fixed a permutation \( \sigma \in S_{k+1} \), we define an affine linear map \( f_\sigma : A^k \to A^k \).

\[
f_\sigma : v_i \mapsto \frac{1}{i+1} \left( v_{\sigma(0)} + v_{\sigma(1)} + v_{\sigma(2)} + \ldots + v_{\sigma(i)} \right)
\]

We use the embedding \( \Delta \to \text{Man} \) sending \([k] \to A^k\) as in A.5.3, and \( d^i \) in the following remark should be interpreted accordingly.

**Remark 4.11.** Given permutations \( \sigma \) and \( \tau = \sigma \circ (i \, i+1) \), note that \( f_\sigma \circ d^i = f_\tau \circ d^i \).

In the definition that follows, we abusively use \( f_\sigma \) to denote the restriction of \( f_\sigma \) to \(|\Delta^k| \subset A^k\), with the embedding of the standard simplex \(|\Delta^k|\) arising from identifying the \( i \)-th vertex of \( \Delta^k \) with \( v_i \) as above and extending affine linearly.

**Definition 4.12.** A subordinate geometric subdivision of a \( k \)-simplex \( \Delta^k \to NQVect \) (whose constituents are \((V_\bullet, \alpha_\bullet)\)) is a map \( sd\Delta^k \to NQVect \) which satisfies the following condition. For each \( \sigma \in S_{k+1} \), there is an increasing function \( \tau \) satisfying \( \tau \leq \hat{\sigma} \) such that the \( k \)-simplex determined by \( \sigma \) consists of the sequence of bundles

\[ f_\sigma^*V_{\tau(0)} \subset f_\sigma^*V_{\tau(1)} \subset \ldots \subset f_\sigma^*V_{\tau(k)} \]

**Remark** together with the relevant involutions \( \alpha_{\tau(i)} \circ f_\sigma \).
The word subordinate indicates that the ranks of the bundles on the simplex determined by \( \sigma \) are smaller than or equal to the ranks of the bundles in the algebraic subdivision, which arises via precomposition with the last vertex map and hence has its sequence of bundles determined by \( \hat{\sigma} \). The collection of subordinate geometric subdivisions of a \( k \)-simplex is non-empty. One may define the map \( sd\Delta^k \to \mathcal{NQVect} \) whose \( k \)-simplex corresponding to the permutation \( \sigma \) is given by

\[
f_\sigma^*V_{\hat{\sigma}(0)} \subset f_\sigma^*V_{\hat{\sigma}(1)} \subset \cdots \subset f_\sigma^*V_{\hat{\sigma}(k)}
\]

together with the involutions \( \alpha_{\hat{\sigma}(i)} \circ f_\sigma \). The definitions of \( \hat{\sigma} \) and \( f_\sigma \) (see the Remark 4.11) ensure that this collection of \( k \)-simplices of \( \mathcal{NQVect} \) together constitute a map \( sd\Delta^k \to \mathcal{NQVect} \). This is what we might consider the prototypical geometric subdivision, most directly related to the algebraic one.

4.4.3 Homotopy between algebraic and geometric subdivisions

In this subsection we give a homotopy from any subordinate geometric subdivision of a \( k \)-simplex to its algebraic subdivision. The homotopy is constructed so that its definition on a face depends only on that face of the original simplex, which means that the homotopies can be used with maps \( sd^{hm}\Delta^k \to \mathcal{NQVect} \).

We define the homotopy on each \( k \)-simplex separately, then check that they are compatible. Suppose that \((V_\bullet, \alpha_\bullet)\) define the initial \( k \)-simplex. The algebraic subdivision, restricted to the \( k \)-simplex \( \sigma \), consists of

\[
\hat{\sigma}^*V_{\sigma(0)} \subset \hat{\sigma}^*V_{\sigma(1)} \subset \cdots \subset \hat{\sigma}^*V_{\sigma(k)}
\]

together with the relevant involutions pulled back by \( \hat{\sigma} \). In the expression above \( \hat{\sigma} \) is abusively used to denote the map \( |\Delta^k| \subset \mathbb{A}^k \to \mathbb{A}^k \) defined by \( v_i \mapsto v_{\hat{\sigma}(i)} \).

Recall that for a subordinate geometric subdivision, there is for each \( \sigma \) a function
\[ \tau \leq \hat{\sigma} \] which determines the bundles defining the simplex corresponding to \( \sigma \).

Define for each \( j \) the function \( f_{\sigma,j} : A^{k+1} \to A^k \) by the following.

\[
 f_{\sigma,j}(v_i) = \begin{cases} 
 f_\sigma(v_i), & i \leq j \\
 v_{\hat{\sigma}(i)}, & i > j 
\end{cases} \quad (4.1)
\]

The homotopy on the \( k \)-simplex given by \( \sigma \) is defined to consist of this sequence of bundles on \( H_j \).

\[
 f_{\sigma,j}^* V_\tau(0) \subset f_{\sigma,j}^* V_\tau(1) \subset \ldots \subset f_{\sigma,j}^* V_{\hat{\sigma}(j)} \subset \ldots \subset f_{\sigma,j}^* V_k \quad (4.2)
\]

Here we see why it is necessary for the function \( \tau \) to satisfy \( \tau \leq \hat{\sigma} \), since otherwise the simplex given above would not make sense for some \( j \). The subordination condition guarantees that the homotopy always can be defined from any geometric to the algebraic subdivision.

**Lemma 4.13.** The collection of sequences of bundles defines a homotopy \( \Delta^k \times \Delta^1 \to NQVect \) from a subordinate geometric subdivision to the algebraic subdivision.

**Proof.** According to the description of the simplicial homotopy (4.4.1.1), we must verify the relations \( d_{j+1} H_j = d_{j+1} H_{j+1} \). Inspecting (4.2) we see that applying \( d_{j+1} \) to \( H_j \) and \( H_{j+1} \) means omitting \( V_{\hat{\sigma}(j)} \) (from \( H_j \)) or \( V_{\tau(j+1)} \) (from \( H_{j+1} \)). Having removed those bundles from the respective sequences, what remains is identical.

Finally, to see that the map is a homotopy from the geometric to the algebraic subdivision, note that \( d_0 H_0 \) yields the algebraic subdivision, and \( d_{k+1} H_k \) gives the geometric subdivision.

We must ensure that the homotopy given for the simplex corresponding to \( \sigma \) will be compatible with adjacent simplices.
Lemma 4.14. The homotopies assemble to yield a homotopy \( sd\Delta^k \times \Delta^1 \to \mathcal{N}Q\text{Vect} \) from the subordinate geometric subdivision to the algebraic subdivision.

Proof. Recall that adjacent simplices corresponding to \( \sigma \) and \( \tau \) share their \( i \)-th face when \( \sigma = \tau \circ (i, i+1) \), and that \( (4.11) \) the restrictions of \( f_\sigma \) and \( f_\tau \) by \( d^i \) agree. This means that when restricting the relevant bundles to the \( i \)-th face, they are pulled back by the same map.

Now, we confirm that the relevant faces of the homotopy \((k+1)\)-simplices \( H_j \) restrict to the same homotopy along the face. This is the case, since the bundle sequences \( (4.2) \) for the homotopies depend, when restricted to the faces, only on the faces, and by the explicit description of the faces of the homotopy simplices \( H_j \) of 4.4.1.1.

4.4.4 Homotopy of subdivisions for generalized manifolds

Having constructed a homotopy between algebraic and geometric subdivisions for \( \mathcal{N}Q\text{Vect} \), we note here that the work we just did allows us to prove a similar statement for any generalized manifold \( \mathcal{G} \). Recall that by generalized manifold we mean any functor \( \text{Man}^{op} \to \text{Set} \). The simplicial set \( \mathcal{N}Q\text{Vect} \) did not arise as the restriction of a generalized manifold, and thus we had to keep track of which bundles went where. In the case of a generalized manifold the situation is much simpler. The direct application of the functions of Equation 4.1 leads to the following.

Lemma 4.15. For any generalized manifold \( \mathcal{G} \), there are naturally defined homotopies between the algebraic and geometric subdivisions of a \( k \)-simplex. Naturally defined means that the homotopies induced on faces agree with those defined for a \( (k-1) \) simplex.

Proof. Pulling back the element in \( \mathcal{G}_k \) along the maps of Equation 4.1 gives \((k+1)\)-simplices which assemble to give a simplicial homotopy, just as in the previous section.
The maps are so defined that the naturality condition is satisfied. \qed

4.5 Some examples

This section collects some examples of families of field theories to illustrate behavior that might not be immediately evident.

4.5.1 The filtration of the parameter space

As mentioned earlier in Lemma 2.23, a family of 1|1-Euclidean field theories determines a collection of closed subsets whose union is the parameter space. We show here that any finite filtration \( C_0 \subset C_1 \subset \ldots \subset C_N = S \) of a smooth parameter space by closed sets \( C_i \) arises from a family of field theories parameterized by \( S \).

Suppose \( S \) is a smooth manifold, and \( f \) is a smooth real-valued function on \( S \). Let \( C^2 \) be graded by \((\frac{1}{2}, \frac{1}{2})\). We may define a \( S \)-family of field theories as follows. Let the infinitesimal generator \( \mathcal{D} \) be defined as

\[
\mathcal{D} := \begin{pmatrix} 0 & \frac{1}{f} \\ \frac{1}{f} & 0 \end{pmatrix}
\]

The module sheaf is the one generated by \( (e^{-\frac{1}{f}}, 0) \) and \( (0, e^{-\frac{1}{f}}) \).

Verifying that this gives a smooth family of field theories amounts to checking that \( \frac{1}{f} e^{-\frac{1}{f}} \) is a smooth function whenever \( f \) is.

The example just given demonstrates that the module sheaves obtained by evaluating a family of field theories on the super point can be pretty strange. Take for example the function on \( \mathbb{R} \) given by \( e^{-\frac{1}{f}} \cos(\frac{1}{2x}) \). The \( \mathbb{R} \)-family of field theories determined by this function includes a module sheaf whose fiber dimension changes infinitely often in every neighborhood of \( 0 \in \mathbb{R} \). More generally, any closed set of \( \mathbb{R}^n \) is the zero locus of some smooth real-valued function. Moreover, choosing complementary 2-planes of \( H \), we can add field theories defined as above by adding the
vector spaces and linear maps. So very wild families of field theories exist on any parameter space. Note, though, that the theories constructed in this fashion are all trivial. They are all concordant to the trivial field theory whose underlying vector space is 0. This construction is given in order to emphasize that we must be very careful about the subtleties that can arise because we are not working with sheaves that are locally free.

4.5.2 Non-finite generation

In general we obtain not only non-projective but also non finitely generated modules from families of field theories. We do not spell this out fully but instead simply observe the following. The continuity of the spectrum of $A(1)$, as in Lemma 2.30, implies that the infinitesimal generator of the family must become infinite at points at which some portion of the vector space ‘turns off.’ The fact that the positive length intervals are connected with the identity leads (see the proof of Lemma 2.21, whose ideas apply here too) to the condition that the infinitesimal generator must itself be an endomorphism of the module sheaf. Since this generator is unbounded near points at which the rank of the module changes, the module cannot consist simply of functions which vanish, or which vanish to finite order, at the given closed subset. If the (necessarily unbounded) infinitesimal generator is to be an endomorphism of the module, then the functions constituting the module must vanish to all orders. Such a module will not be finitely generated.
APPENDIX A

CATEGORICAL PRELIMINARIES

We presume familiarity with ordinary categories; the excellent standard reference is [18].

A.1 Fibered Categories

Throughout this dissertation we work with families of objects parameterized by smooth manifolds or by supermanifolds. Fibered categories are a useful way of encoding the idea that objects are parameterized by some space, and fibered functors capture the notion that a functor between categories of families of objects respects the parameterizations. We follow [32] closely in what follows, and it is an excellent reference for this material.

We begin with the notion of a Cartesian morphism. A concrete example to keep in mind is the map $f^{*}E \to E$ from the total space of a pullback bundle into the total space of the original vector bundle. We fix a functor $p : \mathcal{C} \to \mathcal{B}$, where for now $\mathcal{C}$ and $\mathcal{B}$ are just ordinary categories.

**Definition A.1.** A morphism $f : C_1 \to C_0$ in $\mathcal{C}$ is Cartesian if it satisfies the universal property indicated in the following diagram.

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\alpha} & C_1 \\
p & & f \\
p(C_2) & & C_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
p(C_1) & \xrightarrow{g} & p(C_0) \\
p & & p \\
\end{array}
\]

\[
\begin{array}{ccc}
p(C_1) & \xrightarrow{p(f)} & p(C_0) \\
p & & p \\
\end{array}
\]

\[
\begin{array}{ccc}
p(C_1) & \xrightarrow{p(f)} & p(C_0) \\
p & & p \\
\end{array}
\]

\[
\begin{array}{ccc}
p(C_1) & \xrightarrow{p(f)} & p(C_0) \\
p & & p \\
\end{array}
\]

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Note that this is not a commutative diagram. The diagram depicts two compositions of morphisms; the top row in \( C \), and the bottom row in \( B \). The diagram says that the image of the upper composition is the lower one under the functor \( p : C \to B \).

In words, a Cartesian morphism is the universal way of mapping into a family in \( C \) parameterized by some \( B \in B \) in a way compatible with a given map into \( B \).

Consider again the case of a category of smooth vector bundles on smooth manifolds, whose morphisms are smooth bundle maps. This category has a forgetful functor to smooth manifolds by remembering only the base of the vector bundle and the smooth map induced by the one on total spaces. Observe that given any map \( T \to S \) in \( \text{Man} \) and a vector bundle \( E \to S \) we can construct a pullback vector bundle \( f^*E \to T \), and the map \( f^*E \to E \) covering \( f \) is Cartesian. Smooth vector bundles (with the forgetful map to the base smooth manifold) in this way give us a first example of what is called a fibered category.

**Definition A.2.** A functor \( p : C \to B \) is a fibered category if for every morphism \( f : B' \to B \) in \( B \) and object \( C \in C \) with \( p(C) = B \) there is a Cartesian morphism \( \hat{f} : C' \to C \) covering \( f \), that is, satisfying \( p(\hat{f}) = f \).

**Remark A.3.** Note that the definition of fibered category contains an unusual aspect, categorically. We require that \( p(C_0) \) is equal to \( B \), rather than simply isomorphic to it. At various points in the definition of fibered categories we will use equalities rather than isomorphisms. Often when working with categories it is preferable to allow isomorphisms other than the identity. In the case of the definition of fibered categories it is useful to restrict to equalities.

Having defined fibered categories, it is natural to consider what the appropriate notion of a functor between such categories is. We fix the base category rather than allowing it to vary, i.e. a fibered functor covers the identity on the base.

**Definition A.4.** A functor \( F : C \to D \) of categories fibered over \( B \) is a fibered functor
if the following two conditions hold.

- \( p_C(c) = p_D(F(c)) \) where the maps \( p_i \) are the relevant functors to \( \mathcal{B} \).
- \( F \) takes Cartesian morphisms to Cartesian morphisms

Again, note the presence in the first condition of the usually-unacceptable equality, rather than more general isomorphism.

A.2 Topologies and Stacks

Once again we closely follow the presentation of [32].

Given a fibered category \( p : \mathcal{C} \to \mathcal{B} \), if the base category \( \mathcal{B} \) has some notion of an object being built up out of other objects, we might wish to consider the extent to which the collection of objects in \( \mathcal{C} \) over some fixed object \( B \) in the base is determined by the collections of objects over the constituent parts of \( B \).

We first formalize the idea of an object being built up out of other objects. An example to keep in mind is that of an ordinary open cover. An open cover \( \{ U_i \} \) of a smooth manifold \( M \) can be seen as the collection of morphisms in \( \text{Man} \) given by the inclusions \( U_i \to M \).

**Definition A.5.** A Grothendieck topology on a category \( \mathcal{B} \) is a collection of families of morphisms \( \{ U_i \to M \} \) called coverings which satisfies three conditions.

- Every isomorphism is a covering (the family of morphisms consisting of a single element, the isomorphism).
- A covering can be pulled back along an arbitrary morphism \( f : N \to M \) in \( \mathcal{B} \) to obtain a covering \( \{ f^*U_i \to N \} \) of \( N \).
- Iterating coverings leads to coverings, in that if \( \{ U_i \to M \} \) is a covering of \( M \), and \( \{ V_{ij} \to U_i \} \) is a covering of each \( U_i \), then composition gives another covering \( \{ V_{ij} \to U_i \to M \} \) of \( M \).

For a fibered category whose base has a Grothendieck topology, we now introduce a category that captures the extent to which covering families and pullbacks along them determine the fibered category.
Recall that in the situation of sheaves on a topological space, we require that elements on two open sets that restrict to the same element on an intersection patch together to form an element on the union. We express this condition more categorically by saying that the restriction maps exhibit $F(U_1 \cup U_2)$ as a limit of the diagram $F(U_1) \coprod F(U_2) \to F(U_1) \cap F(U_2)$.

Let $\mathcal{U} := \{U_i \to M\}$ be a covering family. Denote by $Y_\mathcal{U}$ the subfunctor of the presheaf represented by $M$ consisting of maps factoring through some element of the covering. The category $Y_\mathcal{U}$ is fibered over $\mathcal{B}$ in the following way. An object $S \to U_i \to M$ is sent to $S \in \mathcal{B}$. Precomposition with morphisms in $\mathcal{B}$ leads to Cartesian morphisms in $Y_\mathcal{U}$.

Suppose we are given a fibered functor $f : Y_\mathcal{U} \to \mathcal{C}$ of categories over $\mathcal{B}$. Such a functor includes the following data. For any element $U_i \to M$ in the covering family, we have an object $f(U_i)$ of $\mathcal{C}$ over the object $U_i \in \mathcal{B}$ given by evaluating the functor on the identity morphism $U_i \to U_i$. Let $U_{ij}$ be any fibered product of $U_i$ and $U_j$ over $U$. The projections $U_{ij} \to U_i$ and $U_{ij} \to U_j$ are Cartesian, and hence must be mapped to Cartesian arrows in $\mathcal{C}$. By the universal property characterizing Cartesian arrows, there is a unique isomorphism between the domains of these two arrows. In other words, we have an isomorphism $\phi_{ij} : p_1^*f(U_i) \to p_2^*f(U_j)$. The commutativity of cubes involving triples of indices implies a relation among the isomorphisms $\phi_{ij}$ when they are pulled-back to fibered products $U_{ijk}$. In this way (although we’ve omitted discussion of morphisms) we can relate the category of fibered functors $Y_\mathcal{U} \to \mathcal{C}$ with a category referred to as a category of descent data associated to the cover $\mathcal{U}$. This category is further discussed in 4.1 of [32].

Given any map to $M$ which factors through some element of the cover, we can forget the factorization, and thereby obtain a $Y_\mathcal{U} \to Y_M$, where the latter is the image of $M$ under the Yoneda embedding. By precomposition, we get a natural map $\text{hom}_\mathcal{B}(Y_M, \mathcal{C}) \to \text{hom}_\mathcal{B}(Y_\mathcal{U}, \mathcal{C})$. By the 2-Yoneda lemma (given as 3.6.2 of [32]) we
identify the domain with $\mathcal{C}(M)$.

With this functor in hand we may make the following definition.

**Definition A.6.** A fibered category over a site is a stack if for every cover $\mathcal{U} = \{U_i \to M\}$ the natural map $\mathcal{C}(M) \to \text{hom}_B(Y_\mathcal{U}, \mathcal{C})$ is an equivalence of categories.

We again consider smooth vector bundles in order to make the definition more tangible. Surjective submersions lead to a Grothendieck topology on the category of smooth manifolds. The category of smooth vector bundles, with its forgetful map, is a fibered category over this site, since pullbacks of vector bundles always exist and satisfy the necessary universal property. What does it mean to say that this fibered category is in fact a stack? The commutativity of the sides of the cube (descent data) amounts to saying that the cocycle condition is satisfied by the maps that patch together the vector bundles on overlaps.

A.2.1 Producing a weak functor from a fibered category

In 3.1.2 of [32] one may find a comprehensive discussion of the relationship between fibered categories $\mathcal{C} \to \mathcal{B}$ and (weak) functors from $\mathcal{B}^{op}$ to the (2-)category of categories. We summarize that treatment here, and then focus on the relationship as it applies for the work of this thesis.

The idea is the following. A cleavage is defined as a collection of cartesian morphisms in $\mathcal{C}$, subject to the condition that the collection contains, for every morphism $f$ in the base $\mathcal{B}$, a unique cartesian morphism in $\mathcal{C}$ covering $f$ for each object in $\mathcal{C}$ over the codomain of $f$. In other words, among all cartesian morphisms satisfying the universal property of Definition A.1 we pick exactly one.

A fibered category together with a cleavage can be used to produce a weak functor $F : \mathcal{B}^{op} \to \text{Cat}$. The category $F(b)$ associated to an object $b \in \mathcal{B}$ is defined to be the collection of objects in $\mathcal{C}$ over $\mathcal{B}$, together with morphisms in $\mathcal{C}$ which cover the
identity of \(b\). Given a morphism \(b' \to b\) in \(\mathcal{B}\), the induced functor \(F(b) \to F(b')\) is defined on objects by using the cleavage to pull back objects. This is done uniquely, by the property defining the cleavage. On morphisms, we use the universal property of cartesian morphisms, together with the fact that morphisms in \(F(b)\) cover the identity on \(b\), to induce unique morphisms between the pulled back objects.

Since the functors \(F(b) \to F(b')\) we produce in this way will not, in general, compose associatively, we do not get a strict functor.

For the purpose of this dissertation the preceding discussion simplifies drastically. The use of the ambient Hilbert space \(H\) leads to strictly associative pullbacks (and a distinguished cleavage of the fibered category of functors \(1|1-\mathcal{EBord} \to TV(H)\)), so that we get an ordinary strict functor \(\mathcal{B} \to \text{Cat}\), where for us the base category is supermanifolds. We then take the composition \(\mathcal{B} \to \text{Cat} \to \text{Set}\), with the second map forgetting morphisms, in order to obtain the generalized manifolds (i.e. functors \(\text{Man}^{op} \to \text{Set}\) or \(\text{SMan}^{op} \to \text{Set}\)) used define our simplicial sets.

There will of course only be a functor \(\text{Cat} \to \text{Set}\) if the domain is small categories; this is so in our case, again because of the use of the ambient Hilbert space.

A.3 Internal categories for future work

As mentioned in the text, it suffices for the purposes of this dissertation to treat \(1|1-\mathcal{EBord}\) as a category (fibered, with flip). In the future it will not be possible to reduce things to a purely 1-categorical setting by taking diffeomorphism classes of bordisms. The reason for this is as follows. In order to broaden the Euclidean field theory picture to \(K\)-theory of non-zero degree, and more generally to twisted \(K\)-theory, we modify the algebraic target with a delooping of vector spaces and consider field theories as natural transformations between certain simple (i.e. factoring through topological bordisms) functors called twist functors. This approach is described in section 5 of [28].
The algebraic target must become some sort of 2-category. One way to produce something 2-categorical is by working with categories internal to an ambient strict 2-category. We describe such an approach here.

A.3.1 Strict 2-Categories

The definition of an ordinary category includes the data of a collection of objects (a set, or even a proper class, of them) and a set of morphisms between each pair of objects. An enriched category has morphisms being more than just a set, as we now explain.

**Definition A.7.** A category enriched in a category \( \mathcal{D} \) is a category \( \mathcal{C} \) together with an identification \( \text{Hom}_\mathcal{C}(A, B) \in \mathcal{D} \). Moreover, composition is a morphism in \( \mathcal{D} \).

A simple example of such an enriched category is the category of topological spaces. It is enriched in topological spaces when we give the set of morphisms the compact-open topology. The category of finite-dimensional vector spaces is another example of a category enriched over itself, since the collection of linear maps between two such spaces is naturally a finite dimensional vector space.

The category of categories is perfectly valid as an ordinary category. When treating it as such, though, we omit natural transformations, which are of central importance. Fixing two categories \( \mathcal{A} \) and \( \mathcal{B} \), the collection of morphisms between them is naturally itself a category. The objects of this category are functors \( \mathcal{A} \to \mathcal{B} \), and morphisms are natural transformations. Thus the category of categories is enriched in categories. We now make the following definition.

**Definition A.8.** A strict 2-category is a category enriched in categories.

So the collection of categories forms a strict 2-category, as described above.

**Remark A.9.** There are a variety of forms of 2-categories, which are in general more flexible than the one we adopt here. Our 2-categories are called 'strict' because
various statements hold exactly, rather than up to a 2-morphism. For example, when three 1-morphisms are composed with two different bracketings they are equal, rather than just isomorphic.

A.3.2 Categories internal to a strict 2-category

This dissertation uses categories whose morphisms are bordisms. The failure of the disjoint union of sets to be strictly associative means that the bordism category (whose composition involves a disjoint union) naturally lives in a bigger world which contains higher morphisms expressing the associativity of composition. One way of meeting the need for such higher morphisms is by working with categories internal to a 2-category.

The following sketch is given precise form in [28] as well as [22].

Definition A.10. A category $\mathcal{C}$ internal to a strict 2-category $\mathcal{D}$ consists of the data

- An object $\mathcal{C}_0 \in \mathcal{D}$ of objects.
- An object $\mathcal{C}_1 \in \mathcal{D}$ of morphisms.
- A morphism of $\mathcal{D}$ of the form $\mathcal{C}_0 \to \mathcal{C}_1$ which assigns to each object its identity morphism
- Morphisms $s, t : \mathcal{C}_1 \to \mathcal{C}_0$ which are the source and target maps
- A morphism $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$ which determines composition

These data are required to satisfy, in a certain sense, the conditions that an ordinary category must, for example that pre- and post-composition with the relevant identity morphism leaves a general morphism unchanged, and that composition is associative. Rather than having strict equalities, though, one has instead the extra data of 2-morphisms of the ambient 2-category that implement the relations.

Since one is interested in obtaining field theories as smooth functors, it is necessary to work with categories internal to a 2-category of categories fibered over a site of
smooth manifolds or supermanifolds. The definition above goes without a problem, since for any base $\mathcal{B}$ the category of categories fibered over $\mathcal{B}$ is a strict 2-category, with fibered functors and natural transformations as the 1- and 2-morphisms.

Since $\mathcal{B}$ is the terminal object of the category of categories fibered over $\mathcal{B}$, one can define monoidal categories over $\mathcal{B}$, etc.

A.4 Presheaf determined by fibered functors

For the following discussion is is useful to consider the proof of Proposition 1 of I.6 in [19], which shows that the category of presheaves of sets on a small category is cartesian closed.

The Euclidean field theories of this dissertation are a presheaf of sets on the category of supermanifolds (or smooth manifolds, or the ordinal category $\Delta$, by restriction). In this section we describe more fully the passage from maps between fibered categories to (pre)sheaves of groupoids or of sets.

Suppose that one wishes to endow the category $\text{Cat}_{\mathcal{B}}$ of categories fibered over a fixed base $\mathcal{B}$ with an internal hom which makes $\text{Cat}_{\mathcal{B}}$ a cartesian closed category. That means that the internal hom is right adjoint to cartesian product. With this constraint, there is only one possible way to define $\text{Hom}(X, Y)$. Here we pass freely between fibered categories and weak functors from $\mathcal{B}$ to the 2-category of categories.

The $S$-points of $\text{Hom}(X, Y)$ must correspond to maps of fibered categories $X \times S \rightarrow Y$. Here $S \in \mathcal{B}$ may be considered a fibered category by the forgetful functor $\mathcal{B} \downarrow S \rightarrow \mathcal{B}$, i.e. by the Yoneda embedding.

In order to justify the approach taken in this work we must relate $\text{Hom}(X \times S, Y)$ and $\text{Hom}(X, Y_S)$. The fibered category $Y_S$ is defined as the category over $\mathcal{B}$ with objects over $T \in \mathcal{B}$ consisting of objects over $S \times T$ in $Y$, and morphisms covering the identity on $S$.

A fibered functor $f : X \rightarrow Y_S$ gives, for each $x \in X$ over some $b \in \mathcal{B}$, an element
$f(x)$ over $S \times b$. We now see how to use such data to produce a fibered functor $X \times S \to Y$.

Given an $x \in X$ over $b$, and a morphism $b \to S$, we must produce an element of $Y$ over $b$. The functor $f$ gives us an element $f(x)$ over $S \times b$. Pulling this element back along $b \to b \times b \to S \times b$, the composition of the diagonal and the map $(g \times id)$, we obtain an element of $Y$ over $b$.

Such a construction leads to an equivalence of categories of the following form.

$$\text{Hom}(X \times S, Y) \simeq \text{Hom}(X, Y_S)$$

By means of this equivalence we view a family of field theories parameterized by $S$ as we do, namely as a fibered functor from the bordism category to the category $TV(H)_S$.

A.5 Some simplicial remarks

An excellent introduction to simplicial sets is [9]. For much of the following we refer to the clear and comprehensive [10].

A.5.1 Very brief review of simplicial sets

The category $\Delta$ is defined as the category having as objects finite sets $[k] := \{0, 1, \ldots, k\}$ and as morphisms order preserving maps (where we use the standard order on the natural numbers). Certain morphisms in fact generate all the morphisms in the category. One kind of generating morphism is the map $d^i : [k] \to [k + 1]$ whose image omits the $i$th element of $[k + 1]$ and is otherwise an order preserving bijection between ordered sets with $k + 1$ elements. These maps are referred to as coface maps. The other kind of generating morphism is the maps $s^i : [k] \to [k - 1]$ which sends both $i$ and $i + 1$ in $[k]$ to $i$ in $[k - 1]$ and which is otherwise an order preserving bijection.
between ordered sets containing $k-1$ elements. These are referred to as codegeneracy maps. Note that the domain (or codomain) is not included in the notation for the maps $d^i$ and $s^i$, so that there are many such maps.

A simplicial set is a functor $\Delta^{op} \to \text{Set}$, where the superscript $op$ denotes the opposite category, and $\text{Set}$ denotes the category of sets. Note that elements of $\Delta$ define simplicial sets by the Yoneda embedding. We denote the functor represented by $[k]$ as $\Delta^k$.

Given a simplicial set $X$, we refer to the set $X_k := X([k])$ as the set of $k$-simplices of $X$. Note that since $X$ is a functor, precomposition with the coface and codegeneracy maps leads to maps $d_i : X_{k+1} \to X_k$ and $s_i : X_{k-1} \to X_k$ referred to as face and degeneracy maps.

A map between simplicial sets $X$ and $Y$ is a natural transformation of functors $\Delta^{op} \to \text{Set}$, hence a collection of set maps $X_k \to Y_k$ which are compatible with the face and degeneracy maps. We use $s\text{Set}$ to denote the category of simplicial sets.

One familiar simplicial set is the singular set of a topological space $A$, which is defined to have $k$-simplices given by continuous maps $|\Delta^k| \to A$, where $|\Delta^k|$ denotes the standard $k$-simplex (a topological space) and the face and degeneracy maps arise via certain natural affine linear maps among the standard simplices.

There is a functor from $s\text{Set}$ to the category of topological spaces (in fact, CW-complexes) called geometric realization. A powerful, but less elementary, way of describing this functor is as a coend. More concretely, the geometric realization of a simplicial set $X$ is a collection of standard $k$-simplices, one for each element in the set $X_k$, which are glued together using the face and degeneracy maps. A more precise description is available in Definition 4.1 of [9]. We use $|X|$ to denote the geometric realization of a simplicial set $X$.  

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A.5.2 Collection of definitions and theorems for simplicial homotopy

All the material of this subsection may be found in Chapter I of [10].

Definition A.11. The simplicial set $\Lambda^k_j$ is generated by all faces $d_i$ of $\Delta^k$ except for the face $d_j$. In other words, it is the simplicial set having $k$ non-degenerate $(k-1)$-simplices $d_i(\Delta^k)$ for $i \neq j$, subject to the relations given by the simplicial identities relating compositions of face and degeneracy maps. It is referred to as the $j$th horn of $\Delta^k$.

Definition A.12. A map $f : X \to Y$ of simplicial sets is a fibration if there is a lift for every diagram of the following form.

\[
\begin{array}{ccc}
\Lambda^k_j & \longrightarrow & X \\
\downarrow i & & \downarrow f \\
\Delta^k & \longrightarrow & Y \\
\end{array}
\]

The vertical map $i$ is the inclusion of the horn.

Definition A.13. The simplicial set $X$ is Kan if the unique map $X \to \Delta^0$ is a fibration.

Given a vertex (0-simplex) $v$ in a Kan simplicial set $X$, we may define simplicial homotopy groups. These are the groups given by simplicial homotopy classes of maps $(\Delta^k, \partial \Delta^k) \to (X, v)$. We also define the set of path components $\pi_0$ as the set of vertices modulo simplicial homotopy.

Definition A.14. A weak equivalence of Kan simplicial sets is a map $f : X \to Y$ which induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomorphisms $\pi_k(X,v) \cong \pi_k(Y,f(v))$ for all vertices $v$ in $X$ and natural numbers $k$.

It is in fact true that the previous definition is compatible with the one that follows, which generalizes the notion of weak equivalence of Kan complexes to general simplicial sets.
Definition A.15. A weak equivalence of (general) simplicial sets is a map \( X \rightarrow Y \) inducing a weak equivalence of geometric realizations \( |X| \rightarrow |Y| \).

Lemma A.16. A map \( f : X \rightarrow Y \) of simplicial sets is a fibration and weak equivalence if and only if all lifting problems of the following form may be solved.

\[
\begin{array}{ccc}
\partial \Delta^k & \overset{j}{\rightarrow} & X \\
\downarrow \quad & \quad & \downarrow f \\
\Delta^k & \longrightarrow & Y \\
\end{array}
\]

This is I.11.2 of [10].

A.5.3 Embedding the ordinal category in the category of smooth manifolds

Here we provide details of the embedding \( \Delta \hookrightarrow \text{Man} \) and establish the homogeneous coordinates that we use in some constructions.

Denote by \( A^k \) the hyperplane in \( \mathbb{R}^{k+1} \) consisting of points \( (x_0, x_1, \ldots, x_k) \) satisfying \( \sum x_i = 1 \). Let \( d^i : A^k \rightarrow A^{k+1} \) denote the map defined by inserting a coordinate 0 between \( x_{i-1} \) and \( x_i \). Denote by \( s^i \) the map \( A^k \rightarrow A^{k-1} \) the map which replaces \( x_i \) with \( x_i + x_{i+1} \) and shifts the other coordinates down. In other words, \( s^i \) is given by the following formula.

\[
s^i : (x_0, x_1, \ldots, x_k) \mapsto (x_0, x_1, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_k)
\]

With these smooth maps in hand, we now define the embedding \( \Delta \hookrightarrow \text{Man} \) as the functor which maps \( [k] \) to \( A^k \) and which sends morphisms \( d^i, s^i \) to the smooth maps with the same names. According to this embedding we refer to the smooth manifolds \( A^k \) as the extended \( k \)-simplices.
APPENDIX B

TOPOLOGICAL VECTOR SPACES

We have consulted [31, 15, 5] for material regarding topological vector spaces. More specific references to these works are given below.

B.1 Elementary Facts

See I.2 and I.7 of [31] for topological vector spaces and locally convex topologies.

A topological vector space is a vector space endowed with a topology compatible with the linear structure. Finite dimensional vector spaces have a unique Hausdorff topology making them topological vector spaces. Infinite dimensional vector spaces do not have uniquely determined vector space topologies. Topologies which are locally convex, and which we now define, turn out to be useful.

**Definition B.1.** A topological vector space $V$ is locally convex if the origin has a neighborhood basis of convex sets.

The topology of a locally convex topological vector space can be defined by a collection of seminorms (compare with the more restricted cases of Hilbert and Banach spaces, whose topology is determined by a single norm). An example of a locally convex space which is not a Hilbert of Banach space is the space of smooth functions on a compact smooth manifold $S$ endowed with the family of seminorms which are the sup norms of the first $k$ derivatives, as $k$ ranges over the natural numbers.
Definition B.2. The projective topology on the algebraic tensor product of (locally convex topological) vector spaces $V$ and $W$ is $V \otimes_{\text{alg}} W$ endowed with the finest locally convex topology such that the canonical bilinear map $V \times W \to V \otimes_{\text{alg}} W$ is continuous.

Definition B.3. The completed projective tensor product $V \otimes W$ of locally convex topological vector spaces $V$ and $W$ is the completion of the algebraic tensor product $V \otimes_{\text{alg}} W$ with respect to the projective topology.

The following is Theorem 45.1 of [31] and makes more explicit the elements in the completed projective tensor product. A Fréchet space is locally convex, complete, and metrizable.

Theorem B.4. For Fréchet spaces $E$ and $F$, every element of the completed projective tensor product $E \otimes F$ can be expressed as the sum of an absolutely convergent series $\sum \lambda_n x_n \otimes y_n$ where $\lambda_n$ is a sequence of complex numbers satisfying $\sum |\lambda_n| < 1$ and $x_n$ and $y_n$ are sequences converging to zero in their respective vector spaces.

We now state a useful theorem regarding smooth functions taking values in a locally convex space. This may be found in Chapter 51 of [31] as an application of Theorem 44.1 of the same work.

Theorem B.5. For a smooth manifold $S$ and locally convex complete Hausdorff space $E$, the natural map $C^\infty(S) \otimes E \cong C^\infty(S; E)$ is an isomorphism.

B.2 Open Mapping Theorem

The following theorem may be found as Theorem 17.1 in [31].

Theorem B.6. A continuous linear bijection between metrizable complete topological vector spaces is an open map.
B.3 Extracting subbundles

In this section, we use $\sigma(T)$ to denote the spectrum of a linear map $T$.

Holomorphic functional calculus gives a way of taking an operator $T$ and a holomorphic function $f$ and producing a new operator $f(T)$ with the property that the spectrum of $f(T)$ is $f$ applied to the elements of $\sigma(T)$. In this section we deal with the details of such a construction in order to demonstrate that given a smooth family of operators, we can produce a smooth family of projections onto some group of eigenspaces. This section relies closely on the excellent treatment of [15] (section I.5.3).

B.3.1 Eigenprojections for a single operator

Suppose $T$ is an endomorphism of a finite dimensional complex vector space. For each $\zeta \in \mathbb{C}$ in the resolvent set of $T$ (meaning $\zeta$ is not in the spectrum) there is the operator $R(\zeta) := (T - \zeta)^{-1}$. The function $R(\zeta)$ is a holomorphic function on the complement of $\sigma(T)$.

Given an eigenvalue $\lambda$, we can choose a positively-oriented curve $\Gamma$ about $\lambda$ which includes no other eigenvalues. Then the operator

$$P = -\frac{1}{2\pi i} \int_{\Gamma} R(\zeta) d\zeta \quad \text{(B.1)}$$

is defined (sometimes called the Riesz projection), and is in fact a projection onto the (generalized) eigenspace associated to the eigenvalue $\lambda$. If the reader is unfamiliar with this, it is helpful to note that in the case that the vector space is $\mathbb{C}$ this is Cauchy’s integral theorem.
B.3.2 Projections from smooth families

Given a smooth family of endomorphisms, we wish to conclude that a separation in the spectrum which persists across the parameter space can be used to introduce a smooth family of projections and thereby express the original (trivial) bundle as a sum of subbundles consisting of neighboring eigenspaces.

**Lemma B.7.** Projection onto a group of eigenspaces yields a smooth vector bundle.

*Proof.* Observe that the resolvent, which is the integrand of [B.1](#), depends smoothly on the parameter space if the family of operators does. Hence we can differentiate the projections. □

B.4 Semigroups of Endomorphisms of Finite Dimensional Vector Spaces

The following is Theorem 2.8 of [6](#).

**Theorem B.8.** A continuous semigroup \( T : \mathbb{R}_{\geq 0} \to \text{End}(V) \) of endomorphisms of a finite dimensional vector space which satisfies \( T(0) = \text{id}_V \) is given by exponentiating an endomorphism.

We provide here the idea of the proof. One notes that \( T(t) \) is invertible for small \( t \). Integrating \( T(t) \) on this small interval gives an operator that enables the expression of \( T(t) \) for general \( t \) as a certain integral, so that \( T(t) \) is differentiable, and then has infinitesimal generator \( T'(0) \).

**Lemma B.9.** A finite field theory determines and is determined by an infinitesimal generator, which is an odd endomorphism of \( V \).

*Proof.* The family of (even) endomorphisms \( A(t) \) form a semigroup. One of the relations in the bordism category implies that \( A(t) \) approaches the identity on \( V \) as \( t \) goes to 0. Then by [B.8](#) there is an element \( M \) which determines the semigroup
as $e^{-tM}$. The relation $B(s + t) = B(s)A(t)$ may be evaluated with $s = 0$ (there is no ambiguity in the topology on the space of odd endomorphisms) in order to see that $B(t) = B(0)A(t)$. Then, using the other relation $A'(s + t) = -B(s)B(t)$, we see that $-Me^{-(s+t)M} = B(0)^2e^{-(s+t)M}$, so that $B(0)^2 = M$. We conclude that the super-semigroup is entirely determined by $B(0)$. On the other hand any odd endomorphism, by exponentiation, will give a super semigroup of endomorphisms, thanks to the standard properties of the exponential function. \[\square\]
APPENDIX C

EUCLIDEAN SUPERMANIFOLDS

An excellent introduction to supermanifolds is [3]. One may find a more detailed exposition in [24].

C.1 Supermanifold Basics

The local model for a supermanifold is Euclidean space equipped with a sheaf which is smooth functions tensored with an exterior algebra on a finite number of generators.

Definition C.1. A supermanifold of dimension $p|q$ is $X_{\text{red}}$, a second countable Hausdorff space, and a sheaf $C^\infty(X_{\text{red}})$ of super (i.e. $\mathbb{Z}/2$-graded) commutative algebras which is locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes \Lambda \mathbb{R}^q$.

We will use $X$ to refer to the supermanifold as a whole (the space plus the sheaf). When we wish to refer to the space alone, we will use $X_{\text{red}}$.

Morphisms of supermanifolds are morphisms of ringed spaces. If the morphism is $X \to Y$, then we have a continuous map $X_{\text{red}} \to Y_{\text{red}}$ and a morphism of sheaves of rings over $Y$, $C^\infty(Y) \to f_*C^\infty(X)$, where $f_*$ denotes direct image. As a simple example; given a smooth map of ordinary manifolds from $M$ to $N$, and a smooth function on $N$, then by precomposition we get a smooth function on $M$.

When we quotient out the nilpotent elements in the sheaf of graded commutative algebras, we see that $X_{\text{red}}$ has the structure of a smooth manifold of dimension $p$. This quotient map on sheaves of rings corresponds to a morphism of supermanifolds $X_{\text{red}} \to X$. Note that there is not a canonical map $X \to X_{\text{red}}$. 
C.1.1 S-point formalism

When we talk about maps between supermanifolds, we at times use the notion of S-points. This means that we consider a morphism $X \to Y$ by observing what it does to a morphism $S \to X$ when post-composing. The following lemma will be useful.

**Lemma C.2.** There is a bijective correspondence of sets

$$SMan(S, \mathbb{R}^{p|q}) \cong (C^\infty(S)^{ev})^{\times p} \times (C^\infty(S)^{odd})^{\times q}$$

One proves the lemma by observing that the ring of functions $C^\infty(\mathbb{R}^{p|q})$ is generated by $p$ even and $q$ odd coordinate functions. Their images completely determine the map of algebras. On the other hand, given a collection of odd and even functions, one extends to a map of super-algebras by using the Taylor expansion. See also the discussion of this fact in Proposition 2.4 of [3].

We expand here briefly on the Taylor expansion in composing maps of supermanifolds. Suppose $S \to \mathbb{R}^{p|q}$ is a map of supermanifolds denoted, as in C.2, by $(s_1, s_2, \ldots, s_p, \psi_1, \psi_2, \ldots, \phi_q)$ with $s_i$ even elements of $C^\infty(S)$ and $\psi_j$ odd elements. Suppose that we have, in addition, a function $f : \mathbb{R}^{p|q} \to \mathbb{R}$. We now consider the composition $S \to \mathbb{R}^{p|q} \to \mathbb{R}$. Let $x$ be the standard coordinate function on $\mathbb{R}$, and let $x_1, x_2, \ldots, x_p$ be the even coordinate functions on $\mathbb{R}^{p|q}$ and $\theta_1, \theta_2, \ldots, \theta_q$ the odd coordinate functions. The composition $S \to \mathbb{R}$ is determined by the image of $x$ under the map of super algebras $C^\infty(\mathbb{R}) \to C^\infty(S)$. We wish to describe this element in terms of $f$ and $(s_i, \psi_j)$.

Note that $f$, an even element of $C^\infty(\mathbb{R}^{p|q})$, may be written as $f = f_0 + f_\alpha \theta_\alpha$, where $f_0$ as well as each $f_\alpha$ is an ordinary smooth function on $\mathbb{R}^p$ and the $\alpha$ are multi-indices consisting of an even number of distinct indices in $\{1, 2, \ldots, q\}$, with $\theta_\alpha$ the product of $\theta_i$ determined by $\alpha$. We already know that the elements $\theta_\alpha$ map
to the corresponding products $\psi_\alpha$ of odd elements of $C^\infty(S)$. So we have reduced the problem to composing an ordinary smooth function such as $f_0$ with the $S$-point $(s_1, s_2, \ldots, s_p, \psi_1, \psi_2, \ldots, \phi_q)$. The idea of how one proceeds is the following. Consider the Taylor expansion of $f_0$ at the origin. We write this as

$$f_0 + x_i \frac{\partial f_0}{\partial x_i} + \frac{1}{2} x_i x_j \frac{\partial^2 f_0}{\partial x_i \partial x_j} + \ldots$$

in which we suppress the evaluation of the function and its derivatives to keep the notation more compact. The image of any Taylor polynomial already has an explicit description in terms of the elements $(s_1, s_2, \ldots, s_p, \psi_1, \psi_2, \ldots, \psi_q)$, since we know that the element $s_i$ is the image of $x_i$ under the induced map of super algebras, and the map of algebras is unital so that the coefficients are determined as well.

Without claiming to having proven anything, we state that the expression below may be considered in light of the preceding discussion. Write the functions $s_i$ as $s_i^0 + \nu_i$, where $s_i^0$ is the reduced part and $\nu_i$ is the nilpotent part. Then our maps compose in the following way.

$$f_0(s_1, s_2, \ldots, s_p) = f_0(s_1^0, s_2^0, \ldots, s_p^0) + \nu_i \frac{\partial}{\partial x_i} f_0(s_1^0, s_2^0, \ldots, s_p^0) + \ldots$$

The next terms will involve mixed partial derivatives and products $\nu_i \nu_j$. Note that there will only be finitely many terms in the sum, since the functions $\nu_i$ are all nilpotent. We can similarly define the compositions $f_\alpha(s_1, s_2, \ldots, s_p)$. As was observed earlier, when we produce the single even element of $C^\infty(S)$, these elements will be multiplied by products $\psi_\alpha$.

The formula for composition of functions on supermanifolds, together with more references, is available at Proposition 2.4 of [3].
C.1.2 Batchelor’s Theorem

Given a real vector bundle $E$ on a smooth manifold $M$, one may construct a supermanifold in the following way. The reduced manifold is $M$, and the sheaf of super-algebras consists of sections of the exterior algebra bundle $\Lambda^*E$ graded by parity. Up to isomorphism, this produces all vector bundles. This result is a theorem of Batchelor [2].

**Theorem C.3.** Every supermanifold is isomorphic to one of the form $(M, \Lambda^*E)$.

We emphasize that the category of supermanifolds has more morphisms than the category of vector bundles, so that the isomorphism is not canonical.

C.1.3 Super Lie groups

The group structure on the set of maps from a supermanifold $S$ to $\mathbb{R}^{1|1}$ is central to our discussion of families supermanifolds with geometry. We now make the following definition.

**Definition C.4.** A super Lie group is a group object in the category of supermanifolds.

Thus we have $G \times G \to G$ giving the operation, and $pt \to G$ giving the identity element, and these morphisms satisfy diagrams expressing associativity and unitality.

We now give $\mathbb{R}^{1|1}$ the structure of a super Lie group. Here, the group operation is defined by means of the $S$-points described above, where we identify maps $S \to \mathbb{R}^{1|1}$ with pairs $(s, \theta)$ consisting of an even function $s$ and an odd function $\theta$ on $S$. The operation is defined as follows.

$$(s, \theta), (t, \eta) \mapsto (s + t + \theta \eta, \theta + \eta)$$
This gives an associative operation extending ordinary addition on \( \mathbb{R} \leftrightarrow \mathbb{R}^{1|1} \). The inverse of \((s, \theta)\) is \((-s, -\theta)\), where we rely on the fact that \(\theta^2 = 0\) since it is an odd function.

C.1.3.1 Lie Algebra of \(\mathbb{R}^{1|1}\)

The following lemma is not used in this thesis. Nonetheless it provides a useful principle by which to understand the super-semigroups of operators obtained from a family of field theories. The following statement and its proof are presented in 2.4 of [4].

**Lemma C.5.** The Lie algebra of \(\mathbb{R}^{1|1}\) is the free super Lie algebra on a single odd generator.

**Proof.** The Lie algebra of a 1|1-dimensional Lie group is of dimension 1|1. The derivations \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial \theta}\) are indeed left invariant and, respectively, even and odd. So they form a basis of the Lie algebra. They can both be obtained from a single element, though, namely \(D := \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x}\). Observe that \(D^2 = -\frac{\partial}{\partial x}\). So \(\frac{\partial}{\partial x}\) is in the Lie algebra generated by \(D\) and is even (as the square of an odd element). Similarly, \(\frac{\partial}{\partial \theta}\) is contained in the algebra generated by \(D\) as \(D + \theta D^2\), and is odd, as the sum of two odd elements. \(\square\)

While the universal family of endomorphisms does not exactly provide a map of super Lie groups \(\mathbb{R}^{1|1} \rightarrow \text{End}(V)\), in which case we could directly consider the induced map on Lie algebras, it is nonetheless helpful to think of the infinitesimal generator of the semigroup of endomorphisms as the image under such a map of the odd generator of the Lie algebra of \(\mathbb{R}^{1|1}\).
C.2 Euclidean Isometries of $\mathbb{R}^{1|1}$

In order to define the isometries of $\mathbb{R}^{1|1}$ we could look simply at some subgroup of the set of invertible elements in $SMan(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$. This is not the right approach, though. Rather, we should think about how things work in families. This is clearer if we think in terms of generalized supermanifolds, i.e. functors $SMan^{op} \to Set$ from the opposite category of supermanifolds to the category of sets. Let $Aut(\mathbb{R}^{1|1})$ be the functor which assigns to each supermanifold $S$ the set of invertible elements in $SMan(S \times \mathbb{R}^{1|1}, \mathbb{R}^{1|1})$. Invertible here means that there exists another such element so that the composition of maps over $S$ gives the identity on $S \times \mathbb{R}^{1|1}$.

We define a subgroup of $Aut(\mathbb{R}^{1|1})$. This will be the Euclidean group of isometries of $\mathbb{R}^{1|1}$.

Note that given an $S$-point of $\mathbb{R}^{1|1}$, we can obtain an invertible element in $Aut(\mathbb{R}^{1|1})$ by translation as follows.

$$\mu \circ (f \circ p_1 \times id_{\mathbb{R}^{1|1}} \circ p_2) : S \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$$

Then the $S$-point $(s, \theta)$ is invertible with inverse $(-s, -\theta)$.

There are also reflections. The assignment $(s, \theta) \mapsto (s, -\theta)$ gives an action of $\mathbb{Z}/2$ on $\mathbb{R}^{1|1}$. This action in fact gives an automorphism of $\mathbb{R}^{1|1}$ as a super Lie group, so we have the ingredients for a semi-direct product. The subgroups $\mathbb{Z}/2$ and $\mathbb{R}^{1|1}$ are in fact a semi-direct product in the group $Aut(\mathbb{R}^{1|1})$, and we define the Euclidean isometries of $\mathbb{R}^{1|1}$ to be exactly this semi-direct product $\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$. 
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