TURBULENCE, STABILITY, AND IMAGING OF GAS-LIQUID FLOWS

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Abstract

by

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The shape, location, and stability of a gas-liquid interface are studied. Linear stability analysis was used to determine flow pattern transitions. Two-equation turbulence models are adapted to fit the basestate behavior of stratified and wavy flows. The new model accounts for the interfacial roughening effects of waves and the corresponding vertical shift in the maximum velocity. A conversion from channel to pipe flow is also prescribed. Numerically, the profile is determined via an under-relaxation iteration scheme and utilizes efficient banded matrix solvers in implementing a direct method approach to the boundary value problem. These results are used in a stability analysis that incorporates the Boussinesq approximation into the Orr-Sommerfeld equation. A Chebyshev-Tau method is employed to solve the resulting eigenvalue problem. The growth rate curves provide a quantitative analysis for experimental results performed in finite pipes and also give a true neutral stability line, which is not possible to determine in the laboratory. Fundamentally, the linear stability growth rates of various waves are the necessary information needed to understand phase transitions and form the basis for non-linear studies of turbulent-turbulent flow.
An experimental investigation of dispersed multiphase flow was undertaken to determine a parameter space for gravity independent flow. The force of gravity compared to inertia, capillary pressure, and viscosity are determined by the values of the Froude, Capillary, and Stokes numbers, respectively. The bubble lengths, curvature, and film thickness were analyzed to quantify the influence of gravity on the position and shape of the interface.
For Amy and Kaylee
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CHAPTER 1:

INTRODUCTION

1.1. Overview

The performance of multiphase systems is strongly dependent on the shape and position of the interface. In microchannel flows the pressure drop is determined primarily as a function of capillary forces rather than viscous forces. In large channels the presence of liquid droplets in the gas phase necessitates gas-liquid separation processes in petroleum production. Bubbly, wavy, stratified, annular, slug, churn, and wispy are classifications of two phase flow. Each classification – or flow pattern – behaves uniquely, and thus pressure drop, mass transport, and heat transfer depend on the interfacial dynamics.

Flow patterns are classified by whether the phases are continuous or dispersed and whether the conduit wall is in contact with one or both phases. Appearing only in a specific range of flow rates, a flow pattern is traditionally mapped as a function of superficial gas and liquid velocities. This common exercise in multiphase fluid dynamics is purely empirical and cannot be generalized for all fluid properties and flow rates. So while flow pattern maps are invaluable for air-water systems at 1 atm, they fail to provide any meaningful information when designing subsea natural gas and oil pipelines operating at 20 atm, or heat exchanger in zero-gravity. It is a goal of this research to lift
these restrictions of dimensionality and develop criteria for flow pattern transitions that are independent of fluid properties.

In this research, flow patterns are studied via both experimental and computational means. The behavior of the interface is captured in small capillaries with the use of high resolution photography and high speed video. In the turbulent regime, interfacial stability of stratified flow is modeled.

The experiments in small diameter tubes were performed to measure the effects of gravity on flow patterns. When gravity becomes negligible in the flow, the position of the interface will be independent orientation. A multiphase heat exchanger operating under such conditions would perform similarly in orbit as on Earth’s surface. Testing a gravity independent heat exchanger on the ground requires far fewer resources than testing it on an aircraft flying in parabolic arcs or on the international space station.

The turbulence modeling and stability analysis of stratified flow provides a fundamental approach to studying the onset of atomization in oil-gas pipeline flow. The tools provided by this work will enable the petroleum industry to design more efficient phase separation units and to engineer pipelines that require less maintenance.

The structure of the thesis is as follows: the turbulence and stability modeling will be presented in chapters 2-4, followed by the experimental results of laminar, 1mm scale, two-phase flow in chapter 5.

1.2. Motivation

Even though interfacial stability has been studied extensively, a comprehensive approach to turbulence and stability eludes the literature. Experiments in channels have
been performed by Abrams and Hanratty (1985), spectral analysis of experiments in channels were carried out by Bruno and McCready (1988). Analytical solutions of interfacial stability using a laminar boundary layer mimicked as a Poiseuille-Couette flow was done by Blennerhassett (1980). Inviscid theories such as those presented by Jeffreys (1935) and in stratified pipe flow by Taitel & Dukler (1976) are other approximations of stability analysis of the interface. A solution that accounts for turbulence, a wavy-roughened interface, and viscous effects is desired in order to properly determine stability at an interface.

More importantly, challenges in advanced drilling and separations in the petroleum industry make the issue of interfacial stability – as it is related to pressure drop and liquid entrainment – of paramount importance. The modeling tools of the industry are limited to empirical correlations and cause excessive maintenance, downtime, and safety concerns. The mathematics of the current models involves mass and shear stress averaging equations (Holmas et al. 2008), which have their place in design, but it is possible to go beyond these methods. Since the current techniques consider dynamics only in the flow direction, the velocity and shear stress are calculated for the transverse flow direction. These quantities are critical in predicting the stability of the interface and thus the initial conditions for liquid droplets to enter the gas phase. The important contributions of this work are in developing a two-phase, two-equation turbulence model and successfully combining the turbulence with stability theory to produce growth rates and wave speeds at large Reynolds numbers (greater than $10^5$).

In addition to interfacial stability, the capillary experiments are to aid the National Aeronautical and Space Agency (NASA) in developing a heat transfer device and fluid
flow systems for long duration space flights. This consisted of investigating flow regimes that ensured inertia, viscosity, and surface tension dominated gravity. The effects of gravity were measured on various length scales and reported in dimensionless parameters so that the results were independent of geometry and fluid properties.

1.3. Structure of Thesis

Chapter 2 describes how standard turbulence models were adapted for wavy flow. This required implementing a scheme for interfacial roughening caused by the waves, and enhanced momentum transport from the gas to the liquid phase. The numerical scheme is presented in detail, and a conversion between channel and pipe geometries is proposed. Asymptotic limits, empirical correlations, profile data, and pressure drop data are in good agreement with the calculations.

The stability analysis is covered in Chapter 3. The profiles produced in chapter 2 are incorporated into the Orr-Sommerfeld equation, which is necessary so that for the expansion of the Navier-Stokes equations to be consistent with the turbulent basestate. The set of 4th order differential equations that make up the stability analysis are solved both analytically by a long wave asymptotic analysis and numerically by the Chebyshev-Tau spectral method. A detailed account of the numerical methods is given, which extends the validity of the spectral method to Reynolds numbers into the millions.

Chapter 4 combines the efforts of the previous two chapters and summarizes the results. The turbulence and stability method cannot be developed without consideration for the other. Numerical consistency, speed, and convergence are improved with methods described here. Data for the onset of atomization and the initiation of long
waves are compared to stability calculations. A narrative of the physics and stability of two phase flow is given using results from the program; this provides insight into wave formation.

Chapter 5 discusses the experiments performed in 0.94mm to 5.5mm capillaries. Glycerin and water are pumped using a pressurized water tank. The liquid is mixed with air and sent through an optically invisible tube where the bubbles can be captured without refractive distortion. The orientation of the test section is varied so the effects of gravity are captured. High resolution digital photographs and high speed video were used to capture the two phase flow patterns. These were analyzed with functions from Matlab’s Image Analysis Toolbox. The degree of gravity dependence was quantified by measuring the bubble length, curvature, and film thickness.
2.1. Introduction

The discrepancy of length scales in turbulent flow makes direct simulations a very time intensive process, and prohibitively so for many industry related fluid flow problems. Heuristics and empirical correlations are usually sufficient for heat exchanger and pipe design. However, as the need arises for remote drilling and expensive subsea pipelines, more precise predictions of fluid behavior are needed. Turbulent velocity profiles are approximated by von Karman’s law of the wall (1930), the 1/7 power law profile (Nikuradse 1933), or Pai’s power law relationship (1953). These profiles do not provide the accuracy needed for a stability analysis and only apply to single phase flows at high Reynolds numbers. The models adopted here are based on Prandtl’s mixing length theory (1927), which relates the energy dissipation in turbulence to the corresponding small scale motion.

Prior to discussing the turbulence models, it is important to review the fundamental difficulty in solving the equations of momentum. Using the Reynolds decomposition $u_i = U_i + u'_i$, the Navier Stokes equations are
\[
\frac{dU_i}{dt} + U_j \frac{dU_i}{dx_j} = -\frac{1}{\rho} \frac{dP}{dx_i} + \frac{d}{dx_j} \left( 2 \nu S_{ij} - R_{ij} \right) \quad (2.1.a)
\]
\[
\frac{dU_i}{dx_i} = 0, \quad (2.1.b)
\]

where \(S_{ij}\) is the rate of strain tensor, \(\frac{1}{2} \left( \frac{dU_i}{dx_j} + \frac{dU_j}{dx_i} \right)\), and \(R_{ij}\) is the Reynolds stress tensor containing the perturbation velocities. The Reynolds stresses introduce enough degrees of freedom that the equation cannot be directly solved. This is where closure models are used to make the turbulence terms solvable.

These multidirectional fluctuations that produce the Reynolds stresses are very small and three dimensional. For quick efficient calculations, an ensemble averaging of these eddies is used rather than direct numerical simulation; this is known as the Boussinesq approximation. Analogous to molecular or viscous dissipation, the Boussinesq approximation models turbulent energy dissipation as a coefficient of the velocity gradient:

\[
\bar{\tau}_T = \mu_T \frac{dU}{dy}, \quad (2.2)
\]

where \(\mu_T\) is the position dependent turbulent viscosity, and \(\bar{\tau}_T\) is the time averaged turbulent shear stress. This approach is also consistent with von Karman’s boundary layer scaling and Prandtl’s mixing length theory. The challenge at this point is to develop a closure model that is applicable for flow conditions found in oil-gas pipelines. This range includes Gas Reynolds numbers up to \(10^7\), liquid Reynolds numbers to 50,000, liquid viscosities of 1 poise, and pressures up to 20 atm.
2.2. Closure Models

In general there is a tradeoff between empirical constants that fit certain data very well and fundamental models that fit a broader spectrum of data but which are more costly. For example, the zero-equation model, which computes $\mu_T$ from an algebraic scaling argument, requires only one fitting parameter, but this form is best applied only to jets and requires an experimentally determined coefficient for each fluid. Here we seek a closure model that is not only applicable to a wide range of fluids, but also applicable to the flow pattern of interest: wavy flow.

2.2.1. Two-Equation Models

The Prandtl mixing length and eddy viscosity are related to two terms found in the Reynolds Averaged Navier-Stokes (RANS) equations. These are the turbulent kinetic energy ($k$) and turbulent dissipation ($\varepsilon$). With two differential equations governing $k$ and $\varepsilon$, plus the momentum equation, a constitutive equation for eddy viscosity, and continuity, the system is solvable. As will be shown later, it is also flexible and general enough to be adapted to wavy flows in a wide range of operating conditions.

The general form of the k-epsilon, 1-D, unidirectional, momentum equations is

$$\frac{d}{dy}\left((\mu_j + \mu_T)\frac{dU}{dy}\right) = \frac{dP}{dx} + g (\rho_j - \rho_g) \sin(\theta)$$  \hspace{1cm} (2.3.a.)

$$\frac{d}{dy}\left((\mu_j + \frac{\mu_T}{\sigma_k})\frac{dk}{dy}\right) = \rho_j \varepsilon - \mu_T \left(\frac{dU}{dy}\right)^2$$  \hspace{1cm} (2.3.b.)

$$\frac{d}{dy}\left((\mu_j + \frac{\mu_T}{\sigma_\varepsilon})\frac{d\varepsilon}{dy}\right) = c_{e2} f_2 \frac{\varepsilon^2}{k} - c_{e1} f_1 \frac{\varepsilon}{k} \mu_T \left(\frac{dU}{dy}\right)^2 - \rho_j E,$$  \hspace{1cm} (2.3.c.)
\[ \mu_T = \rho j c_\mu f_\mu \frac{k^2}{\bar{\varepsilon}} \]  

(2.3.d.)

where \( c_\mu \) are constants, \( f_\mu \) are smoothing functions, \( E \) changes depending on the form of the model, \( k \) is the turbulent kinetic energy, and \( \bar{\varepsilon} \) is the isotropic component of the turbulent dissipation, \( \varepsilon = \bar{\varepsilon} + D \), where \( D \) is a term that is changes depending on the model.

Different forms of the two equation model exist. The ones scrutinized and borrowed from in this work are those developed by Chien (1982), Dutoya and Michard (1981), Hoffman (1975), Lam and Bremhorst (1981), Launder and Sharma (1974), and Reynolds (1976). The models were tested to match the near wall asymptotic limit, the log-law region, and profile data where available. Also, the proposed roughening method, which replaces the homogeneous kinetic energy condition with a finite value of \( k \) at the interface, is compared to pressure drop and profile data. It is important to note that all the models were developed for single phase flows, many are limited to high Reynolds numbers, and not all are adaptable to a two-phase system. Selection of the best model was based on its ability to match turbulence profile data at moderate and high Reynolds numbers, match stratified flow data, and accurately model a wavy interface by manipulating the boundary conditions.

The final form used for wavy flow is most similar to the Launder and Sharma model but uses the smoothing function \( f_\mu \) from Chien. The model parameters are

\[ D = 2\mu_j \left( \frac{d\sqrt{k}}{dy} \right)^2 \]
\[ E = 2\frac{\mu_T}{\rho_j} \left( \frac{d^2 U}{dy^2} \right)^2 \]  

(2.4.a.)

\[ f_\mu = 1 - \exp(-0.0115y^+), \quad f_1 = 1, \quad f_2 = 1 - 0.3 \exp(-R_T^2) \]  

(2.4.b.)
The boundary conditions also differ by model, here Lam and Bremhorst were followed, which prescribe homogeneous Dirichlet boundary conditions for \( k \) and homogeneous von Neumann conditions for \( \bar{\varepsilon} \).

For scaling the equations, the interfacial velocity is chosen as the characteristic velocity and the liquid depth as the characteristic length. Inertial scaling is used for the pressure and the eddy viscosity is scaled by the liquid dynamic viscosity, thus the dimensional quantities are defined as

\[
U^d = U_0 U, \quad k^d = k U_0^2, \quad \varepsilon^d = \varepsilon U_0^3 / d_1, \quad y^d = y d_1, \quad \mu_Y^d = \mu_T \mu_L, \\
p^d = \rho_L U_0^2 p, \quad x^d = x d_1,
\]

where the superscript \( d \) denotes dimensional quantities. The differential equations are now

\[
\frac{d}{dy} \left( \Gamma \frac{dU}{dy} \right) = R \frac{dP}{dx} + g \left( 1 - \tau \right) \sin(\theta) \tag{2.6.a}
\]

\[
\frac{d}{dy} \left( \Gamma_k \frac{dk}{dy} \right) = R \frac{\tau_j}{m_j} \varepsilon + 2 \left( \frac{d\sqrt{k}}{dy} \right)^2 - \frac{\mu_T}{m_j} \left( \frac{dU}{dy} \right)^2 \tag{2.6.b}
\]

\[
\frac{d}{dy} \left( \Gamma_\varepsilon \frac{d\varepsilon}{dy} \right) = R \frac{\tau_j}{m_j} c_\varepsilon f_2 \frac{\varepsilon^2}{k} - c_\varepsilon f_1 \frac{\varepsilon}{k m_j} \left( \frac{dU}{dy} \right)^2 \tag{2.6.c}
\]

\[
- 2 \frac{1}{R \tau_j} \frac{1}{\mu_T} \left( \frac{d^2 U}{dy^2} \right)^2,
\]

\[
\mu_T = \tau_j R c_\mu f_\mu \frac{k^2}{\varepsilon} \tag{2.6.d}
\]

where \( \Gamma \) is the total viscosity, \( \tau_j \) is the density ratio of phase \( j \) with respect to the liquid density \( (r_2 = \rho_G / \rho_L, r_1 = 1) \), \( m_j \) is the viscosity ratio, \( R \) is the Reynolds number based
on the interfacial velocity and liquid depth. The three forms of total viscosity vary slightly $\Gamma = 1 + \mu_T/m_j$, $\Gamma_k = 1 + \mu_T/m_j\sigma_k$, and $\Gamma_\varepsilon = 1 + \mu_T/m_j\sigma_\varepsilon$.

2.3. Verifying the Model

The purpose of a turbulence model is to provide information about the transition from the free-stream or bulk velocity to the linear boundary layer and ultimately to the wall. This is why these are sometimes called relaminarization models. Therefore it is the profile near the wall ($y^+ < 200$) that is most important to consider in assessing a model’s accuracy.

2.3.1. Velocity Profile

From the no-slip condition and no flux through the boundary, the velocities near the wall have the form

\begin{align*}
  u &= a_1 y + b_1 y^2 + \\
  v &= + b_2 y^2 + \cdots \\
  w &= a_3 y + b_3 y^2 + \cdots.
\end{align*}

(2.7.

Using the definitions

\begin{align*}
  k &= \frac{1}{2} (u^2 + v^2 + w^2) \\
  \varepsilon &= \mu \frac{d u_j}{d x_i} \frac{d u_i}{d x_j} + D
\end{align*}

(2.8.

where $D$ is model dependent, the equations near the wall become

\begin{align*}
  k^+ &\sim A^+ y^{+2} + B^+ y^{+3} + \cdots \\
  \varepsilon^+ &\sim 2A^+ + 4B^+ y^+ + \cdots,
\end{align*}

(2.9.)
where $A^+$ and $B^+$ are determined empirically. It is convenient and empirically acceptable to choose $B^+ = 0$, this is also consistent with the boundary conditions for $\varepsilon$. The value of $A^+$ is 0.035 as determined from experiments.

These asymptotic limits are used in the program where, by any number of means, creates negative values in intermediate iterations. This can be caused by poor interpolation of the basestate or a combination of stiff matrices and machine precision. These negative values destabilize the calculation. However, this can be corrected by looking for values of $y^+ < 0$. This occurs when the friction velocity has become negative and thus the values very close to the wall are negative. Since this falls in the linear velocity region, values of $k$ and $\varepsilon$ are redefined according to the asymptotic wall limit.

In addition to these very-near-wall limits, there is the law of the wall. Outside linear wall region, at roughly $40 < y^+ < 200$, the velocity profile has the form

$$U^+ = \frac{1}{\kappa} \log y^+ + C,$$

where $\kappa$ is the the von Karman constant, roughly 0.41, and $C$ is approximately 5.5. This equation supposes a self-similar behavior of turbulent flows, which is characteristic of semi-infinite domains. The flows of interest here are more confined. Thus the following log-law is proposed for use in stratified flow of moderate Reynolds number

$$U^+ = \frac{1}{\kappa} \log y^+ + \frac{q}{y^+} + C^+,$$

where $q$ is approximately -40, and $C^+$ is 7.5. Again using similitude, the form of the total viscosity is
Figure 2.1: The modified log-law is more accurate for both gas (left) and liquid (right). The plots shown here are the velocity and total viscosity terms near the two walls for a water-air flow $Re_G=500,000$, $Re_L=13,000$, and $h=0.3$

\[
\Gamma^+ = \frac{k y'^+^2}{y'^+ - k q'}
\]  \hfill (2.12.)
Figure 2.1 compares the k-epsilon model with the log-law and modified log-law equations. A near wall comparison is omitted from the figure because it is identical to the model for $y^+ < 5$. The model and modified log-law equation match remarkably well, particularly for the gas phase. This has been confirmed for other holdups and flow rates. Velocity profiles at Re=13,000 are not turbulent enough for the logarithmic profile to be applicable. Still, the modified log-law equation is an improvement over the standard equation. In addition to providing a better fit, equations 2.11 and 2.12 are valid beyond the log-law region down to $y^+ = 10$.

2.3.2. $k$-$\varepsilon$ Terms

Data for eddy viscosity, turbulent kinetic energy, and turbulent dissipation are not readily available. The data from Coles (1978) are used here. These data are only available for the dissipation and the Reynolds shear stress, $\overline{u\nu}$, but since kinetic energy is related by a constitutive equation, matching the available data is sufficient to verify the model. The relationship between the Reynolds shear stress and the eddy viscosity is $-\rho \overline{u\nu} = \mu_T \frac{dU}{dy}$. Dissipation in the log-law region is roughly equal to the production of kinetic energy, $\varepsilon \approx P_k = -\overline{u\nu} \frac{dU}{dy}$ (Patel et al. 1985). Using equations 2.11 and 2.12, the dissipation terms are

\begin{align}
-\overline{u\nu}^+ &= m_j \left( \frac{1}{ky^+} - \frac{1}{y^+} \right) \left( \frac{ky^{+2}}{y^+ - kq} - 1 \right) \\
\varepsilon^+ &= \frac{1}{k(y^+ - kq)} - \frac{2q}{y^+(y^+ - kq)} + O(y^+^{-3})
\end{align}

(2.13.a, 2.13.b)
Comparisons of the k-epsilon model, the log-law relation, the modified log-law relation, and the data are given in figure 2.2.

Figure 2.2: The calculated near wall profiles of (a) turbulent dissipation and (b) Reynolds stress compared to data and the log-law profiles

2.4. Modeling Turbulence at a Wavy Interface

The interface is not smooth, and thus it is inappropriate to use homogeneous wall conditions for $k$ and $\varepsilon$ at the interface. However, when the flow is such that a smooth interface is stable, the interface does behave like a wall with respect to $k$ and $\varepsilon$.

Boundary conditions for wavy flows have been developed by Lorencez et al. (1997), Biberg (2007), and Akai et al. (1981). It was found that they did not work for the conditions of interest, or as in the case of Akai, relied on wave data rarely available and certainly not generalizable.
2.4.1. Boundary Conditions for Roughness

The asymmetry seen in wavy flow as documented by Abrams and Hanratty (1986), and Thorsness et al. (1978) is similar to that seen in asymmetric roughening of single phase flow, where one wall is roughened and the other smooth. The roughened surface initiates the cascade of eddies responsible for turbulent dissipation. For smooth walls these eddies do not appear until some distance away from the wall. Based on these observations, we hypothesize that roughening is akin to truncating the viscosity dominated part of the flow.

Adjusting the model and boundary conditions so that the interface bypasses the near-wall region does not satisfactorily approximate wavy flow. Doing such simply stretches the profile, making the maximum gas velocity closer to the interface; the opposite, however, is observed. This is caused by the interface posing as a significant resistor of fluid flow. Just like with any resistance, the current will flow through the path of least resistance, which is manifested in a shift of the maximum velocity away from the interface and an increase in the pressure drop.

Using the physical reasoning laid out above, the model was modified first by changing the boundary condition of the kinetic energy to some finite value. This value is then allowed to change according to the profile determined in each subsequent iteration. Before giving the details of how this was done, it is worth mentioning the coordinates used. Turbulence is best studied in “wall units” defined as $\nu/\nu^*$, where $\nu^*$ is the friction velocity and $\nu$ is the kinematic viscosity. The ordinate $y^+ = y^d \nu^*/\nu$ gives the distance from the nearest boundary in wall units. By using $y^+$ the following discussion is applicable to all wavy flows.
The first step is to determine at what value of $y^+$ the profile should be truncated. In other words, we want to determine what turbulent behavior is most descriptive of the interface. Along with finding an effective $y^+$ at the interface, the boundary conditions need to be updated to reflect the roughened surface. Clearly, the shear stress and velocity remain continuous, however, it would be appropriate to introduce a $\Delta U$ across the wavy interface, which some authors (Lorencez, 1997) have done to account for the difference in velocity between the trough and peak of the wave. It was found that this has little effect on the profile so the matching conditions are retained. Also, the homogeneous von Neumann condition for $\varepsilon$ remains unchanged. The turbulent kinetic energy is the key parameter as it is directly related to the eddy viscosity, which is precisely the quantity that models resistance to flow.

The value of $k$ at the interface is determined by its value at the opposite end of the phase: near the gas wall. For the reasons listed above, the conditions at the wall and interface must be uncoupled, but at the same time they are related by pressure drop and velocity. The solution is to determine $k$ at a particular value of $y^+$ from the wall, which is then scaled by the local near-wall maximum of $k$. Multiplying this value now by the local near-interface maximum determines the boundary condition of $k$. A schematic of choosing the boundary condition is shown in figure 2.3. The result is a profile of kinetic energy that is truncated, and similar in shape to the near-wall profile, but with a magnitude that varies freely in order to satisfy the differential equations.
Figure 2.3: Roughness is determined by the value of $k$ at the interface, and this interfacial condition is set proportional to the ratio of local maximum of $k$ to the value of $k$ at a specified $y^+$ value near the wall.

This method of determining the boundary condition is only possible because of the iterative process of the problem, and as with any iterative solution an initial condition is required. The system is not likely to converge given an initial guess for $k$ that is homogeneous, or if it does it converges slowly. The homogeneous initial guess is adjusted by truncating the kinetic energy profile at the corresponding $y^+$ value near the interface, this produces both a non-homogeneous value for the initial guess and also gives the desired qualitative shape. The trimmed profile is then stretched across the gas phase and then interpolated to match the grid points. The program loads an initial guess from a .mat file. It is better to trim a profile with homogeneous boundary conditions – a smooth interface profile – than it is to load a profile that is already non-homogeneous in
Using a stored smooth profile as an initial guess gives the user more flexibility in choosing the value of $k$, improves computational stability, and is applicable to both smooth and wavy flows.

Figure 2.4: Berthelsen's method of calculating roughness at the interface produced results very similar to the calculations made here. This validates the proposed method.
Figure 2.5: Turbulence in the liquid phase deviates a bit more from Berthelsen’s solution, but the degree of turbulence makes the discrepancy unimportant.

Data for kinetic energy and dissipation near a wavy interface are unavailable. A 2-D CFD analysis of wavy pipe-flow by Berthelsen and Ytrehus (2005) is the best approximation to compare turbulence calculations. In their work, the level set and immersed interface methods are used with a 2-equation turbulence model. The empirical Charnock (1955) correlation for roughness is used. A comparison of their results with our calculations is given in figure 2.4. There is good agreement between the methods and
the one presented here is based on fundamental physics. Cast in terms of friction variables \((y^+ \text{ and } k^+)\), the interfacial conditions are universally valid for all flows and fluid properties.

2.4.2. Determining the Roughness Parameter

Turbulence models in other works have been adjusted to include roughening. Correlations for roughness seem to do a good job of matching the data. What these correlations lack are a quantitative tool to determine the bounds of roughening. Here, if short waves are calculated to be stable then clearly no roughening is necessary, and if the long waves are unstable then the effects of entrained liquid need to be included. In other works, roughness is correlated to holdup and superficial velocities. These properties of the flow are unrelated to the fundamental physics that prevent or produce the growth of short waves, which are shear stress, strain, pressure fluctuations, and surface tension. Using the linear stability analysis of a smooth interface, these forces are calculated and used to determine the degree of roughening.

As mentioned, the roughness can be adjusted by changing the value of kinetic energy at the interface. By using turbulence scaling \((y^+, u^+, k^+, \varepsilon^+)\), the goal is to find a boundary condition for a wavy interface that can be applied to any stratified flow. And since kinetic energy is the key quantity to manipulate, we seek to correlate \(k_i^+\) with the growth rate of short waves as determined by a linear stability analysis of a smooth interface.

Determining the roughening parameter was done by comparing the calculations to the 1” channel experiments of Bruno (1988). Having these experiments done by the same research group ensures a high degree of confidence in the data. While, profiles are
not available for these data, the pressure drops and flow rates are used and matched by varying the roughness parameter in the model. The values of $k_i^+$ necessary to match the data are plotted in figure 2.6. Within reasonable scatter, the roughness parameter in the data is unchanged by the flow rate, and so a value of $k_i^+ = 4$ is used in the model. The only stipulation of this condition is that the growth rate must be greater than zero. It is expected that some transition from $k^+ = 0$ at zero growth rate to $k^+ = 4$ should be used but was not determined in this work.

![Figure 2.6: Plotted are the values of the key roughening quantity ($k^+$) that are used to match pressure drop.](image)

Pipe data were also used to check this roughening parameter. Using the geometry conversion that will be described in §2.6, the same process was repeated with the experiments by Mantilla (2008). Although, there are fewer data and more scatter, the results are roughly the same.

While the value $k_i^+ = 4$ is a good approximation, the interfacial kinetic energy ranged between 0 and 6 when all data were compared to the model. This variability
could just as easily result from imprecision in the experiments. Other methods for
determining \( k_i \) are possible such as searching for an equivalent \( y^+ \) as described above or
improving upon the geometry conversion (§2.6) might also yield slightly different
relationships for roughness. It is worth noting that generic wavy flow data are difficult to
obtain due to persistent hydraulic gradients.

![Figure 2.7: Values of \( k_i^+ \) that match pressure drop data in pipe flow at three
liquid mass flow rates (\( w_L = [\text{kg/s}] \))](image)

The interfacial conditions used in this work were done without much
consideration of what was presented in literature, but upon review of modifications made
by others, similar conclusions can be found. Lorencez used

\[
\varepsilon_i = 5 \frac{k_i^{3/2}}{d_L}
\]

(2.14.a.

\[
k_i = \left( \frac{\rho_G}{\rho_L} \right) v_{iG}^* \left[ \frac{2 \Delta h}{T} \right]^2
\]

(2.14.b.

where \( \Delta h \) and \( T \) are the root mean square wave amplitude and wave period, respectively.
The first term in \( k_i \) is the same as \( k_i^+ = 1 \) if the density ratio is eliminated. Upon testing
these conditions it was found that the first $k_i$ term has a negligible effect on the flow. It is likely that the density ratio here is an error. The second term involving wave amplitudes and frequencies was proposed by Akai (1981) but clearly these values are rarely known and are not able to be included in a fundamental model of wavy flow. This term also has very little effect on the flow. The condition for $\varepsilon$ is derived from a scaling argument which is based on the Prandtl mixing theory, the coefficient of 5 is a fitted parameter. There is no need for this condition in our model because the von Neumann boundary condition and the form of the non-isotropic dissipation term automatically satisfy the Prandtl scaling.

The Charnock scaling used by Berthelsen has a slightly different form which includes gravity.

\[
\begin{align*}
k_{iG} &= \frac{\tau_i}{\rho_g \sqrt{\kappa \mu}} \left( \frac{1238}{90} \left( \frac{\rho_g}{\rho_w} \right) \left( \frac{v_{iG}^3}{g \nu} \right) \right)^2 \quad \text{for } h > 0.1 \quad (2.15.a) \\
\text{or } k_{iG}^+ &= 630 h^2 \frac{v_{iG}^6}{v^2 g^2} = 630 h^2 (Re^* Fr^*)^2, \quad (2.15.b)
\end{align*}
\]

where $Re^*$ and $Fr^*$ are the Reynolds and Froude numbers based on the friction velocity. It is not clear why gravity is appropriate in modeling the roughness. The roughened surface is caused by short waves, which are very weakly affected by gravity (see §4.2.1). A possible form of $k_{iG}^+$ that would be consistent with the stability of short waves would be

\[
k_{iG}^+ = a Re^{*b} We^{*c},
\]

where $We$ is the Weber number and the constants $a$, $b$, and $c$ are to be determined from data.
2.4.3. Profile Calculations on a Roughened Surface

Velocity profiles over wavy interfaces have been measured by Akai (1981), Lorencez *et al.* (1997), and Cohen and Hanratty (1968). Figure 2.7 compares the calculations to data.

![Graphs comparing calculations to data](image)

Figure 2.8: Profile data from Akai (top) and Cohen and Hanratty (bottom) are compared with calculations. The shift in maximum gas velocity, and thus roughness, is appropriately accounted for in the model.
2.5. Finite Differencing

The direct method is adopted here for solving the boundary value problem (BVP). Harris and Street (1994) use the shooting method, which is a fast algorithm to solve the turbulence equations. However, this advantage requires a good initial guess. Since the program being designed here is to handle a broad range of flow conditions, the shooting method is not a good choice. The direct method allows the entire domain to be solved in a single matrix inversion. Numerical methods will be discussed in chapter 4 that can be used to make the direct method competitive with, and better than, the shooting method.

2.5.1. Non-Uniform Grid and Discretized Equations

The high velocity gradients in turbulent flow require several nodes in that region, but use of many grid points unnecessarily slows the program. It becomes clear that efficient calculations of turbulence require a non-uniform grid. Therefore, the spacing of nodes should be sparse in the center of the phases and dense in areas of high stress.

One such grid structure that provides this spacing is the roots of a high order Chebyshev polynomial. These nodes are easy to calculated and are defined at

\[ x_i = \cos\left(\frac{2i - 1}{2N} \pi\right), \quad i = 1, \ldots, N \]  

(2.17.)

for the order \( N \) Chebyshev polynomial. These values are bounded in the interval (-1,1).

A set of NL points are used in the liquid phase and NG points in the gas phase. By a linear transformation, the points are then shifted to correspond to the dimensionless liquid range \([0,1]\) and the gas range \([1,1 + n_2]\) where \( n_2 \) is the ratio of the phase depths \( d_2/d_1 \). This determines the location of nodes in the channel and can be written as

\[ y_L = (x_i + 1)/2, \quad y_G = n_2 (x_i + 1)/2 + 1. \]  

(2.18.)
One correction has to be made to this formulation. The intervals do not include the endpoints and thus do not include the walls and interface. Simply changing the endpoints to the location of the nodes either makes the initial step size too large leading to poor resolution or too small leading to a poorly conditioned matrix. Instead, the NL and NG points are redistributed from \((-1,1)\) to \([-1,1]\) using linear interpolation.

In this formulation, there are two nodes that correspond to the interface. This is done because each phase is solved independently of the other, but they are related by the boundary conditions at the interface. The method of treating the flow as a single domain with jump discontinuities was used in Blennerhassett (1981) for boundary layer flow or is known as the immersed boundary method in computational fluid dynamics.

2.5.2. The Discretized Equations

The discretizing and solution procedure follows practices that ensure stability and accuracy, most of which can be found in Patankar (1980). These include a control volume approach to grid generation, and negative-slope linearization of source terms. How these methods are implemented into an under-relaxation method will be briefly covered next.

The important task in developing a discretized equation is to ensure the conservation of energy and mass from one finite element to the next, or what is called eliminating "numerical diffusion". In the case of fluid dynamics, particular attention is paid to conserving momentum. To start this process we propose the staggered grid shown in figure 2.9 where the mid-points between nodes indicate the boundaries of the control volume. The boundary need not lie at the midpoint but that is the choice made for
this work, and the discussion will proceed with this choice. What is important is that the amount of momentum stored at point \( i \) does not change when it gets passed to \( i+1 \) through the boundary at \( i^+ \).

![Diagram showing control volume depiction of grid structure](image)

Figure 2.9: control volume depiction of grid structure

The velocity and the derivatives of velocity are defined at the nodes, whereas the total viscosity is defined at the boundaries. This staggered approach ensures that no saw tooth type functions become solutions, which are numerically allowable but physically infeasible (Patankar p.115). The flux of momentum is represented by the shear stress, and so it is the fluxes on either side of the boundary that must equal one another. The shear stress at the left boundary is defined as

\[
\tau_{i^-} = \Gamma^- \left( \frac{u_i - u_{i-1}}{\Delta y_{i^-}} \right). \tag{2.19}
\]

The shear stress to the left of the boundary is \( \tau_{i-1}^+ = \Gamma^- (u_{i-1} - u_{i-2})/\Delta y_{i-1}^+ \). The flux across the boundary is equal only if \( \Gamma^- \) is defined correctly. To do this analysis, consider the resistance of transport across the composite space from \( i-1 \) to \( i \). Figure 2.10 shows an element with a composite fluid consisting of two different viscosities. The points are relabeled "Left" and "Right" for clarity. As with electrical circuits, the potential is equal to the flux times the resistance so the potential to momentum flux (the velocity
difference) is equal to the resistance (the length divided by viscosity) times the flux (shear stress). This gives

\[ u_R - u_L = \tau_m \left( \frac{\Delta h_L}{\Gamma_L} + \frac{\Delta h_R}{\Gamma_R} \right) \]  

(2.20)

when the resistances are in series. The correct form of the midpoint value of the total viscosity is

\[ \Gamma_m = \left( \frac{1 - f_m}{\Gamma_L} + \frac{f_m}{\Gamma_R} \right)^{-1}. \]  

(2.21)

The value of \( f_m \) is the ratio \( \Delta h_L / (\Delta h_R + \Delta h_L) \), and is 0.5 if the boundary of the control volume is chosen to lie in the center. In the case of a centered control volume boundary, the value of \( \Gamma_m \) becomes the harmonic mean of the total viscosities defined at the nodes.

Figure 2.10: An element of fluid in the numerical scheme is considered as a composite with two different viscosities. It is analogous to an electrical circuit with two resistors in series. This is used to formulate a value of the viscosity at the staggered point between 'L' and 'R'.

In addition to numerical conservation, the system must be numerically stable. Boundary value problems can be solved by either a shooting method or direct method. The direct method is used for the \( k \) and \( \varepsilon \) equations while an algebraic set of equations are used to solve the momentum equations. The momentum equation solution procedure will be explained in more detail in §2.5.4.
Following Patankar, the general form of a discretized, second-order, central
difference equations is

\[ a^* \, u_i = b \, u_{i+1} + c \, u_{i-1} + S_C - S_P \, u_i \]  (2.22)

where all the terms \( a^* \), \( b \), \( c \), \( S_C \), and \( S_P \) are positive and the source term, \( S \), has been split
into a constant term, \( S_C \), and a coefficient, \( S_P \), of the dependent variable, \( u \). The value of
\( S_P \) must be chosen carefully. It usually comes from a linearization of the source term or
from the stability requirements that \( S \) must be positive. To simplify the equation slightly
let \( a = a^* + S_P \); this is how \( a \) will be defined for the remainder of the discussion.

The equation for turbulent kinetic energy in phase \( j \) is

\[ a_j \, k_i = b_j \, k_{i+1} + c_j \, k_{i-1} + \frac{\mu_{\tau,j}}{m_j} \left( \frac{d\bar{U}}{dy} \right)^2 \]  (2.23)

where the symbol \( \sim \) refers to the known value from the previous iteration. The
coefficients are defined

\[ a_j = \left( \frac{\Gamma_{k,i}}{h_{i+1}} + \frac{\Gamma_{k,i}}{h_i} \right) + \frac{r_j \, \tilde{\epsilon}_i}{m_j \, k_i} + \frac{2 \, \left( \frac{d\bar{k}_i}{dy} \right)^2}{k_i} \]  (2.24.a)

\[ b_j = \frac{\Gamma_{k,i}}{h_{i+1} \bar{h}_i} \]  (2.24.b)

\[ c_j = \frac{\Gamma_{k,i}}{h_i \bar{h}_i} \]  (2.24.c)

where \( h_i = y_i - y_{i-1}, h_{i+1} = y_{i+1} - y_i \), and \( \bar{h}_i = (h_{i+1} + h_i)/2 \). As defined above,
\( \Gamma_{k,i}^+ \) and \( \Gamma_{k,i}^- \) are the harmonic means that insure equal flux between control volumes.
The last terms in 2.23 and 2.24.a (involving the derivatives of velocity and the
square root of kinetic energy) are calculated using a second-order accurate, central differencing scheme.

The boundary conditions that complete the matrix for the direct method are

\[ k_1 = k_N = 0 \] (2.25.a)

\[ k_{NL} \left( \frac{\rho_L}{\rho_G} \right) = k_{NL+1} = \frac{k^+ v_r^2}{U_0^2} \] (2.25.b)

where \( i \) refers to the interface. The points \( NL \) and \( NL + 1 \) correspond to the liquid and gas side of the interface, respectively.

The equation for turbulent dissipation is similar but with more terms in the forcing function:

\[ a_j \epsilon_i = b_j \epsilon_{i+1} + c_j \epsilon_{i-1} + c_1 f_1 \frac{\mu_{T,i}}{m_j} \frac{\tilde{e}_i}{k_i} \left( \frac{d \tilde{U}}{dy} \right)^2 + 2 \frac{\mu_{T,i}}{r_j R} \left( \frac{d^2 \tilde{U}}{dy^2} \right)^2. \] (2.26)

The first and second derivatives are again calculated with a second-order accurate central differencing scheme. Since the grid is not uniform, the numerical accuracy is slightly less than second order. However, because the step size is appropriately small in critical areas, the overall accuracy is greater than would be afforded by a uniform grid. The coefficients \( a_j, b_j, \) and \( c_j \) are

\[ a_j = \left( \frac{\Gamma_{k,i}^+}{h_{i+1}} + \frac{\Gamma_{k,i}^-}{h_i} \right) \frac{1}{h_{i+1}} + R \frac{r_j}{m_j} c_2 f_2 \frac{\tilde{e}_i}{k_i} \] (2.27.a)

\[ b_j = \frac{\Gamma_{\epsilon,i}^+}{h_{i+1}} \] (2.27.b)

\[ c_j = \frac{\Gamma_{\epsilon,i}^-}{h_i} \] (2.27.c)

The discretized von Neumann boundary conditions for \( \epsilon \) are
This boundary condition is the second-order accurate, one-sided, finite difference formulation of the first derivative.

Caution should be used with the discretized differential equations above. They are not consistent – in the numerical sense of the word – because as $k$ goes to zero the solution blows up. This problem only appears when the step size becomes very small. A fix for this is given in §4.4.1. The main issue here is stiffness of the matrix.

2.5.3. Matrix Inversion

The matrices formed from the equations above are sparse, banded, and can be decoupled. The forms of the matrices are illustrated in figure 2.11 from which it is easy to see that the interface splits the matrix into two independent sub matrices. Since Gaussian elimination requires $N^2$ computations where $N$ is the size of the matrix, splitting the matrix into two smaller matrices means the problem scales as $NL^2 + NG^2$ rather than $(NL + NG)^2$, a savings of $2 (NL \times NG)$ calculations.
Figure 2.11: Sparse $k$ and epsilon matrices. The gas phase is the lower right block, and the liquid phase is the upper left blocks. The boundary conditions for epsilon require three points whereas the Dirichlet conditions for $k$ only require 1.

The banded and sparse nature of the matrices can be exploited to substantially speed up the computational time of inverting these matrices. To take advantage of this, the matrices are built using Matlab’s `sparse.m` function and left divide operator. This leads to an extremely significant increase in speed. For instance, consider a problem with 700 nodes. Using a sparse matrix to compute $K \backslash b_k = k$ ten thousand times takes only 1.7 seconds. If the same procedure is repeated with the non-sparse matrix $K = \text{full}(K)$, the computation takes 102 seconds! In figure 2.12 it is shown that indeed the improvement in time reduces a problem that scales as $N^{2.5}$ to one that scales with $N$. The line representing inefficient Gaussian elimination is not straight partly because the matrix is still decoupled into gas and liquid phases and the number of nodes in each phase was not increased at the same rate when making this plot.
2.5.4. Under Relaxation

While the system is stable in the sense that all of the terms have the correct sign, the system is not likely to approach a physically realistic solution if the initial guess is far from the final solution. The problem is alleviated by incorporating a fraction of the previous solution of \( k \) and \( \epsilon \) as part of the current value, which is written

\[
\bar{k}_i = \frac{k_i}{\alpha} + \frac{1 - \alpha}{\alpha} k_{i-1},
\]  

(2.29)

where \( k_i \) is the unknown value of the previous iteration. The kinetic energy for the next iteration \( \bar{k}_i \) is a weighted average of the current and previous iteration. This particular form of \( \bar{k} \) is consistent with maintaining positive coefficients in the numerical scheme since \( \alpha < 1 \). The under-relaxed equation is

\[
\frac{a}{\alpha} k_i = b k_{i+1} + c k_{i-1} + S + a \frac{1 - \alpha}{\alpha} k_i.
\]  

(2.30)
The equations are solved iteratively until the previous solution is similar to enough to the current solution. The solution of the basestate is known when \( h^+ = \frac{v_L d_1}{\nu_L} \) has converged within the tolerance set by the user. Because the solution may change very little from one iteration to the next, the test for convergence only occurs at every 10th iteration. The condition for convergence is \( \frac{h^+_n - h^+_n}{h^+_n} < \text{tolerance} \). This is used so that tolerances can be set far from floating point precision and also prevents the program from converging pre-maturely.

The term “relaxation” refers to making the solution dependent on the previous iteration. The numerical method “Successive over relaxation” (SOR) actually uses an \( \alpha \) larger than 1, which speeds up convergence. This method is possible here too, but the user must take special care to ensure that the source term does not become negative. If SOR is used correctly, it optimizes the iterative methods by reducing the size of the source term but still adhering to the rule of positive coefficients.

2.5.5. Integrated Momentum Equation

A solution method exists to solve the momentum equation that does not require a matrix inversion nor does it require an initial guess like the shooting method. Integrating the momentum equation uses the eddy viscosity from the previous iteration to update the velocity profile. There is no loss of accuracy nor is the rate of convergence diminished. The differential equation, which produces an \( N \times N \) matrix by the direct method, is simplified to a system of four algebraic equations. This is possible by exploiting the properties of the iterative method. The algebraic equations were solved symbolically and the solution imported into Matlab. Effectively, the matrix inversion is replaced by numerical integration of the eddy viscosity yielding a significant improvement to speed.
Formulation of this efficient method starts by writing the momentum equations in terms of shear stresses so that

\[ S + F = \frac{d}{dy}(-\tau_L) \quad \text{and} \quad S = \frac{d}{dy}(-\tau_G), \]  
(2.31)

where \( F \) is the Froude number, and \( S = R \, dP/dy \). Integrating from the walls to some point \( y \), the unknown shear stresses at the wall, \( \tau_{wL} \) and \( \tau_{wG} \), are used as the constants of integration yielding

\[ \tau_L = \tau_{wL} - Fy - Sy \quad \text{and} \quad \tau_G = \tau_{wG} + S(1 + n_2 - y). \]  
(2.32)

Now, imposing the first interfacial condition that the shear stresses must match at the interface gives

\[ \frac{\tau_{wL} - \tau_{wG} - F}{1 + n_2} = S. \]  
(2.33)

This equation is the same as an overall macrobalance of momentum in a channel. At this point, use of the iterative method is critical to make this method possible. The total viscosity is known from the previous iteration so the differential equations can be integrated again. The shear stress is the product of the negative total viscosity from the previous iteration and the velocity gradient. Making this substitution and integrating yields

\[ U_L(y) - U_L(0) = \int_0^y \frac{\tau_{wL} - y^* (F + S)}{\tilde{\tau}_L(y^*)} \, dy^*, \]  
(2.34)

and a similar result is obtained for the gas phase, but without the Froude number. The no-slip condition can be applied so that \( U_L(0) \) and \( U_G(1 + n_2) \) are equal to zero. From velocity matching at the interface, and by scaling the equations to the unknown interfacial velocity \( U_o \), two more algebraic conditions are generated
Note that this equation does not suppose anything about the interfacial velocity. The system has not been overdefined because the interfacial Reynolds number, $R$, is an unknown and will need to be solved for in this system of equations. Finally, if the gas Reynolds number and holdup are specified then an integral condition can be written down

$$Re_G = R \frac{r_2}{m_2} \int_{1}^{1+n_2} y^* \frac{\tau_{wG} + S (1 + n_2 - y^*)}{m_2 \tilde{\Gamma}_L(y^*)} dy^* dy^+. \quad (2.36)$$

This completes the set of equations needed to solve the pressure drop, interfacial velocity, and the two wall shear stresses. The equations are now easily solved if it is recognized that the unknowns are constants and can be pulled from the integral. The integrals can be numerically integrated because the total viscosities are given by the previous iteration.

Rarely is the holdup known a priori, and therefore it is advantageous to develop a program that can calculate a holdup given the flow rates. However, unlike the previous set of equations, the location of the interface is not fixed and thus many of the integrals contain the unknown $n_2$. The holdup cannot be solved directly so we presume a value and use the equations above to calculate the interfacial velocity. This produces a family of solutions for the values of $n_2 = (0, \infty)$. To determine a unique solution, the liquid flow rate and corresponding integral condition is used. Replacing the integral condition (2.36) above with
produces a second family of solutions. The intersection of the solutions is the unique solution. In practice, the interfacial velocity is calculated using two subsets of the conditions above. A Newton-Raphson method is used until the interfacial velocities match.

\[
Re_L = R \int_0^1 \int_0^{y_r} \frac{\tau_{WL} - S y^+}{\tilde{\Gamma}_G(y^+)} dy^+ dy^* \tag{2.37.}
\]

2.6. Channel and Pipe Geometry Conversion

In the region near the center of a pipe, symmetry implies that the flow will behave similarly to channel flow. The influence of walls will push the maximum gas velocity closer to the interface, but at moderate holdups in large pipes, a 1-D turbulence model is assumed to provide the pertinent information. However, using centerline conditions and assuming the output can be equated to pipe flow misses critical considerations about the energy dissipated in the two different geometries.

A seemingly simple conversion between channel and pipe geometries does not exist. Yu and Sparrow (1967) formulated a solution for single phase laminar flow in ducts of arbitrary shape. Here we try to develop such a geometrical equivalency for turbulence in pipes and channels.

2.6.1. Stratified Laminar 2-D Pipe-Flow

In order to obtain insight into stratified flows, an analytical solution of flow in pipes was used following Brauner et al. (1996). By using conformal mapping it was hypothesized that a geometry factor might be extracted from the transforming equation.
The mapping consisted of converting the pipe geometry into bipolar coordinates, which can be written

\[ x = \frac{r \sin \phi_0 \sinh(\xi)}{\cosh(\xi) - \cos(\phi)} \quad y = \frac{r \sin \phi_0 \sin(\phi)}{\cosh(\xi) - \cos(\phi)}, \]

(2.38)

where the radius of the pipe is \( r \), and \( \phi_0 \) is the angle made by two straight lines extending from the contact points to any point on the gas wall. The equations in 2.38 map the coordinate \( z = x + iy \) into \( w = \phi + i\xi \). In the transformed coordinates the system looks like a channel of infinite aspect ratio. Unfortunately, this rectilinear mapping is merely a mathematical convenience to simplify the boundary conditions. The height of this channel in transformed space is always equal to \( \pi \) and thus unrelated to any physical channel geometry. Therefore, no direct conversion of channels and pipes is gained directly from the bipolar coordinate transformation. A schematic of the coordinate system and a sample of the two-phase laminar solution are given in figure 2.13.

With the 2-D analytical solution, the two geometries (pipe and channel) can be computed exactly and compared quickly. The parameters to match are the velocities, the shear stresses at the wall and interface, and the pressure drop. An initial approach to comparing geometries was to find a characteristic slice or vertical chord in the 2-D geometry that could be reproduced by a 1-D channel calculation. A series of channel calculations were performed that mimicked the geometry and flow conditions at different vertical slices. The results are given in figure 2.14, which shows that the equivalent chord is approximately halfway between centerline and contact line. This result, however, is dependent on holdup and cannot be generalized.
Figure 2.13: a) lines of constant $\xi$ and $\phi$ in bipolar coordinates and b) lines of constant velocity in stratified laminar-laminar flow
A heuristic that works well is to use 60% of the centerline line gas depth in the pipe and 60% of the average liquid depth. This yields good matches for pipe data and are found in figure 2.15. This prescribed geometry conversion is not unfounded. The walls in a pipe will affect the flow more so than in a channel. Immediately, it is recognized that an equivalent channel height will be less than the corresponding pipe diameter.

Furthermore, the liquid will be affected by the walls more so than the gas. From these considerations it is assume that the channel height should be proportional to the centerline gas depth and average liquid depth. To make the comparison quantitative, overall momentum balances are made on the two geometries. The macro balances relate the shear stress to the pressure drop. Pressure drop in channel flow is

\[ \frac{dp}{dx} = \frac{\tau_{WL} - \tau_{WG}}{d}. \]  \hspace{1cm} (2.39)

and in pipes the momentum balance is

\[ \frac{dp}{dx} = \frac{\tau_{WL}S_L - \tau_{WG}S_G}{A_c}. \]  \hspace{1cm} (2.40)

where \( A_c \) is the cross sectional area of the pipe, \( S_L = 2r \cos(1 - 2h) \) is the arc length of the liquid wetted perimeter, and \( S_G = 2\pi r - S_L \) is the gas wetted perimeter. The choice of 60% was made by matching the pressure drops. If the centerline depth of liquid had been used then the constant of proportionality would have varied greatly with holdup. By using the average liquid and maximum gas depths the difference in pressure drops only varies from 1% to 8% for holdups between 0.03 and 0.5.

The average liquid depth is found by using Matlab’s adaptive quadrature integration function, \texttt{quad.m} and is defined by
\[
d_{L,avg} = \frac{\int_r^R r(1-2h) \sqrt{r^2 - y^2} \, dy}{2 \sqrt{(1 - h_{pipe}) h_{pipe} r^2}}
\]  

(2.41.)

where \( r \) is the radius of the pipe and \( h_{pipe} = d_{L,max}/d_{G,max} \). Now, the phase heights are defined as \( d_1 = 3/5 \, d_{L,avg} \) and \( d_2 = 3/5 \, d_{G,max} \). For the 1-D calculation the holdup is redefined as \( h = d_1/(d_1 + d_2) \).

Figure 2.14: From the analytical 2-D solution, vertical slices in the pipe were taken and the shear stresses and velocities were compared to those found in the channel. The difference between the two was then plotted, which for this particular holdup showed that the two geometries were most similar half-way between the centerline and three-phase contact line.
Figure 2.15: Data from Espedal (1998) taken in pipes at small inclines. The calculated pressure drop utilized the overall force balance to convert from the 1-D channel geometry to the 2-D pipe. The holdup used a heuristic of assuming $d_1$ and $d_2$ are equal to half the maximum or centerline depths.
The Reynolds numbers also must reflect this difference in geometry and are written

\[ Re_G = \frac{U_{SG} A_c \rho_G d_2}{\mu_G} \quad \text{and} \quad Re_L = \frac{U_L d_1 \rho_L}{\mu_L} \],

where \( U_{SG} \) is the superficial gas velocity, and thus this definition is easily adapted for mass and volumetric flow rate inputs. Since the liquid Reynolds number is an output of the 1-D calculation there is no need for a special definition. Obviously, for code where flow rates are inputs, the liquid Reynolds number is calculated similarly to the gas Reynolds number.

Applying the geometry conversion, the 1-D calculations can now be compared to turbulent pipe flow. Holdups and pressure drops in pipe flow from Espedal (1998) are plotted with the calculations for smooth and rough interfaces in figure 2.15. The geometry conversion produces values that are in good agreement with experiments. It is concluded that 1-D modeling is sufficient for both channel and pipe geometries.

2.6.2. Hydraulic Diameters

Friction factors are a standard design convention when dealing with turbulent flows. Essentially a macro momentum balance, the friction factor relates the pressure drop to energy sinks such as walls, pipe contractions, expansions, fittings, and elbows. It would seem possible to develop a similar macrobalance in stratified flow to relate pipe and channel geometries.

The use of a friction factor requires that the appropriate characteristic length scale be identified. This length scale, known as the hydraulic diameter, relates to the macro balance by dividing the cross sectional area on which the body force (e.g. pressure) acts,
with the wall perimeter on which friction equalizes the body force. The value may differ by a factor of 4 depending on whether a Moody or Fanning Friction factor is used. For a pipe, the hydraulic diameter is

\[ d_{hyd} = 4 \frac{\pi r^2}{2\pi r} = 2r \tag{2.43} \]

and for a square duct

\[ d_{hyd} = 4 \frac{d^2}{4d} = d. \tag{2.44} \]

These definitions can now be extrapolated to stratified flow.

To start, the momentum balances for stratified flow in each phase are written down. For a pipe the, equations are

\[ -A_{cl} \frac{dP}{dx} = \tau_{wL} S_L + \tau_{i} S_i \tag{2.45.a} \]

\[ \tau_{i} S_i - A_{cl} \frac{dP}{dx} = \tau_{wL} S_L \tag{2.45.b} \]

and in a channel of large aspect ratio the balances are

\[ -d_G \frac{dP}{dx} = \tau_{wG} - \tau_{i}. \tag{2.46.a} \]

\[ \tau_{i} - d_L \frac{dP}{dx} = -\tau_{wL}. \tag{2.46.b} \]

In these equations the sources of momentum are on the left hand side and balanced by the sinks on the right. Before continuing with the stratified flow, the hydraulic diameter for a channel of infinite width is

\[ \lim_{W \to \infty} 4 \frac{Wd}{2 (d + W)} = 2d. \tag{2.47} \]

From this equation it is easy to show that for phase \( j \) in stratified flow that the hydraulic diameter is simply \( 2d_j \).
Now, we come to the crux of the matter, which is to determine what to do about interfacial stress in a pipe. In the liquid, it acts as a source of momentum, and in the gas it acts as a sink. This problem is averted in the channel since the boundaries (walls and interface) are all the same length and cancel out of the hydraulic diameter equation. Since the average gas velocity is much higher than the interfacial velocity it would not be too crude of an approximation to define the wetted perimeter as $S_l + S_{gw}$. However, the velocity at $S_l$ “slips” or behaves like Couette flow so a better approximation to the hydraulic diameter would be

$$d_{G,hyd} = 4 \frac{A_{cG}}{S_G + q S_l}$$

where $q$ is less than but close to 1.

The liquid is more difficult. The interface does not extract energy from the phase but rather supplies it. The hydraulic diameters in the liquid therefore has the form

$$d_{L,hyd} = 4 \frac{A_{cL}}{S_L - p S_l}$$

where $p$ is bounded between 0 and $S_L/S_l$.

The equivalent channel diameter based on the proposed hydraulic diameters above is

$$\frac{d_{L,hyd} + d_{G,hyd}}{2} = d_{channel}$$

and the new channel holdup is $h = d_{L,hyd}/(d_{L,hyd} + d_{G,hyd})$. This approach was developed before adjusting the pressure drop equation for geometry, and it was not revisited. It may provide a more formal method for determining the equivalent gas and liquid depths than assuming the heuristic proposed above. The terms $p$ and $q$ would still need to be determined empirically, but the values would reflect the geometry (e.g.
account for the fact that $S_i$ is smaller at lower holdups) and also the fluid properties because the values would depend on the average to interfacial velocity difference.
3.1. Introduction

The presence of surface waves can have many adverse effects. In coating technology they distort an otherwise smooth surface, in aerodynamics the presence of slow waves affect the skin friction and causes excessive drag, and in pipelines they increase the pressure drop, form slugs, and propel liquid into the gas phase. On the other hand, waves enhance transport processes. Solids beds are broken up in laminar flows if the surface is susceptible to wave formation (Kuru et al. 1995). Also, mixing afforded by waves is an important factor in improving dissolved oxygen levels in stagnant bodies of water (Toukala et al. 2010).

Surface waves result from an unstable interface or from mechanical forcing. In the case of instabilities, speed and growth rate of a particular wave depends on the forces acting on it. How the wave speed and growth rate depend on wavelength is called the dispersion relation (Lighthill, 1980). The goal here is to determine the dispersion relation of turbulent-turbulent wavy and smooth stratified flow.

Linear stability theory has long helped to understand the formation of waves in thin films. The stability of a film rolling down an incline was solved by Benjamin (1957). Turbulent gas flows and very thin liquid layers produce slow waves (Craik, 1966). Blennerhassett applied a laminar boundary layer approximation in an attempt to
model turbulent gas and large liquid layers. None of these works ever included fully
turbulent-turbulent basestates, or basestates that accounted for a wavy interface. Further
studies in laminar-laminar flow were performed by Yih (1967) and expanded by
Yiantsios & Higgins (1988). The stability of single phase laminar flow was carried out
by Orszag (1971); this work in particular is borrowed from heavily to study the stability
of wavy flow.

What is missing from the literature is a proper treatment of wavy flow. As shown
in Chapter 2, a basestate can be developed that account for roughening by using a
modified k-epsilon model. The information from the basestate is now incorporated into
the stability equations

3.2. Modified Orr-Sommerfeld Equation

If there is an instability it will arise from a perturbation of the basestate. Recall
that the RANS equations are derived from the Reynolds decompositions $u(x, y, t) =
U(y) + \gamma \psi(x, y, t)$ and $v(x, y, t) = V(y) + \gamma \nu(x, y, t)$, where $\gamma$ is much smaller than
one. The perturbation velocity is assumed to have the form of a traveling wave
$\psi(x, y, t) = u(y) \exp(\imath \alpha(x - c t))$. Using a stream function, the perturbation velocities
can be written as

\begin{align}
\psi(x, y, t) &= \phi'(y) \exp(\imath \alpha(x - c t)) \quad (3.1.a) \\
\nu(x, y, t) &= \imath \alpha \phi(y) \exp(\imath \alpha(x - c t)). \quad (3.1.b)
\end{align}

The Boussinesq approximation remains in the momentum equations, but the turbulent
viscosity is not split into a basestate plus perturbation viscosity, the reasons for this will
be clarified in §3.7. The pressure is also represented as a sum of the basestate and normal
wave perturbation: $p = P + \gamma \rho(y) \exp(i \alpha(x - c t))$. Upon substitution, the order $\gamma^1$ terms for the horizontal and vertical momentum equations are

$$\frac{d}{dy}(\Gamma(y)\phi''(y)) - \alpha^2 \Gamma \phi'(y)$$

$$= \alpha \frac{R}{m_k} \mathbf{x} \cdot \mathbf{r}(\phi(y) - c \phi'(y) - \phi(y)U(y)$$

$$+ \phi'(y)U(y)) \quad (3.2.a.)$$

and

$$\alpha(\alpha^2 \Gamma(y)\phi(y) - \Gamma'(y)\phi'(y) - \Gamma(y)\phi''(y))$$

$$= \frac{R}{m_k} \mathbf{x} \cdot \mathbf{r} \left( \alpha^2 c \phi(y) - \phi'(y) - \alpha^2 \phi(y)U(y) \phi(y), \right) \quad (3.2.b.)$$

respectively. Differentiating the horizontal momentum equation with respect to $y$, multiplying the vertical momentum equation by $\alpha$, and adding the two, the pressure perturbation can be eliminated giving

$$\frac{\partial^2}{\partial y^2}(\Gamma(y)\phi''(y)) - 2\alpha^2 \frac{\partial}{\partial y}(\Gamma(y)\phi'(y)) + \alpha^4 \Gamma(y)\phi(y)$$

$$= -\frac{iR\alpha \mathbf{x} \cdot \mathbf{r}}{m_k} \left( (U(y) - c)(\alpha^2 \phi(y) - \phi''(y)) + U''(y)\phi(y) \right), \quad (3.3.)$$

which is a modified Orr-Sommerfeld equation. Unlike the standard form, it includes a basestate eddy viscosity as well as the basestate velocity profile.

3.2.1. Boundary Conditions

As with all perturbation problems, the boundary conditions must also be expanded. The no-slip and zero flux at the walls remain homogeneous:
\[ \phi = 0 \]
\[ \phi' = 0. \]  

Since stability analysis is used to understand how the flow responds to perturbations, it cannot be assumed that the interface is flat. Therefore, the boundary conditions do not occur at precisely \( y = 1 \), but on an undulating interface of the form \( h(x, t) = yh \exp(i\alpha (x - c t)) \), where \( h \) is a complex scalar. The displaced velocities are written as a Taylor series expansion with respect to \( h \) around \( y = 1 \) giving
\[ u(1 + h) = u(1) + h \frac{du(1)}{dy} \]
\[ v(1 + h) = v(1) + h \frac{dv(1)}{dy}. \]

Substituting the normal wave expansion and setting the interfacial velocities equal gives
\[ \phi_L = \phi_G \]  
\[ \phi'_L(y) + U'_L(y) h = \phi'_G(y) + U'_G(y) h. \]

Matching the shear stress condition requires both horizontal and vertical velocity shear stresses \( \tau = \Gamma (du/dy + dv/dx) \). Expanding and equating this term across the interface gives
\[ \Gamma_L \phi''_L + h\tau'_L + \alpha^2 \phi_L = m_2 \Gamma_G \phi''_G + h\tau'_G + \alpha^2 m_2 \Gamma_G \phi_G. \]

This is simplified by recognizing that the gradient of the shear stress is equal in both phases, and thus cancels from the equation (Yih 1967).

Finally, the normal stress must be equal across the interface. In addition to strain \( (\Gamma \, dv/dy) \) there is a perturbation pressure that must be included. Also in the normal stress condition are the interfacial restoring forces: capillary pressure and hydrostatic pressure. Recall that in the expansion of the Navier-Stokes equations, pressure was
eliminated to form the Orr-Sommerfeld equation. Instead of getting rid of pressure, it is solved for and yields the perturbation pressure at the interface, which for phase $j$ is

$$\rho_j = i \frac{m_j}{r_j R} \left( \alpha^2 \Gamma_j \phi - \Gamma_j \phi_j'' + \Gamma_j \phi_j^{(3)} \right) + \alpha \left( U_j' \phi_j - U_j \phi_j' + c \phi_j \right). \quad (3.9)$$

Adding all the forces together gives the normal stress condition

$$2 \left( \Gamma_L \frac{d v_L}{d y} - m_2 \frac{d v_G}{d y} \right) - R \left( \rho_L - \rho_G \right)$$

$$= -T R \frac{d h^2(x,t)}{d x^2} - R F \left( 1 - r_2 \right) h,$$

where $T$ is the Weber number. Substituting the normal wave expansion, the pressure, and collecting $\gamma^1$ terms the normal stress condition becomes

$$-T \alpha^3 h - \alpha F \left( 1 - r_2 \right) + \frac{3i\alpha^2}{R} \left( \Gamma_L \phi_1' - m_2 \Gamma_G \phi_2' \right)$$

$$+ \alpha \left( \phi_1 U_1' - U_1 \phi_1' - r_2 \phi_2 U_2' + r_2 \phi_2 \phi_2 \right)$$

$$- \frac{i}{R} \left( \frac{d}{d y} \left( \Gamma_L \phi_L'' \right) - m_2 \frac{d}{d y} \left( \Gamma_G \phi_G'' \right) \right) = -c \alpha \left( \phi_L' - r_2 \phi_G' \right). \quad (3.11)$$

There are 8 boundary conditions, 2 fourth order equations, and an unspecified interfacial perturbation variable $h$. The system is underdetermined; the kinematic condition provides the necessary information for solvability:

$$h - \phi_1 = c \ h. \quad (3.12)$$

Before proceeding to the solution method, it should be noted that one of the boundary conditions can be dropped if $h$ is eliminated. While much of the literature contains procedures that solve the kinematic condition, this leads to an eigenvalue in the denominator and thus the matrix formulation that is about to be presented does not work. Therefore, the value of $h$ determined by the continuity of tangential velocity (3.7) is
substituted into all the boundary conditions containing $h$. It is possible to keep $h$ as a variable in the vector of unknowns. In this case this vector would include the value of the series coefficients followed by the perturbation height.

In the final program, both choices (to solve for $h$ or leave it in the vector of unknowns) are available and the one that is used is determined by the value of $h_{\text{from BC}4}$ where a value of 1 replaces $h$ as determined by the velocity matching condition, and a value of 0 leaves it in the vector of unknowns.

3.3. Long Wave Stability

A spectral solution of the stability equation will be explained in detail in §3.4. In the current section, the focus is on developing a solution that can be used to verify the solutions obtained by the spectral method. The long-wave expansion of the Orr-Sommerfeld equation is useful for this, and as will be shown, it avoids the numerical pitfalls of the spectral method.

The long wave limit is of particular interest since it is hypothesized that unstable long waves provide the mechanism for atomization. The asymptotic limit yields an analytical solution for laminar flow and requires very little computing power. For turbulent flow, there still exists a closed form solution but requires numerical integration of the eddy viscosity. The solution also does not rely on derivatives of the turbulence profiles, which is the source of many numerical problems in the spectral method. The solution method follows Blennerhassett (1981). Unlike that work, the velocity profile here is not simplified to Couette-Poiseuille flow and also the Orr-Sommerfeld equation with eddy viscosity is solved. The analysis also differs from Uphold (1997) in that it
does not assume a homogeneous eddy viscosity at the interface. Additionally, the reader should be aware of mistakes in Uphold's solution of the leading order problem.

3.3.1. The Leading Order Solution

To get the long wave limit, the Modified Orr-Sommerfeld equation is expanded using

$$\phi_1 = \phi_{10} + \phi_{11}\alpha + O(\alpha^2)$$
$$\phi_2 = \phi_{20} + \phi_{21}\alpha + O(\alpha^2)$$
$$c = c_0 + c_1\alpha + O(\alpha^2),$$

where $\alpha \ll 1$. This yields the leading order differential equations

$$\frac{d^2}{dy^2} \left( \Gamma(y) \phi_{10}''(y) \right) = 0$$
$$\frac{d^2}{dy^2} \left( \Gamma(y) \phi_{20}''(y) \right) = 0.$$

The expanded boundary conditions at the wall are

$$\phi_{10} = \phi_{20} = 0$$
$$\phi_{10}' = \phi_{20}' = 0,$$

and

$$\phi_{10}' + \phi_{10} \frac{U_1'}{c_0 - 1} = \phi_{20}' + \phi_{10} \frac{U_2'}{c_0 - 1}$$
$$\Gamma_1 \left( \frac{U_1''}{c_0 - 1} + \phi_{10}'' \right) = m_2 \Gamma_2 \left( \frac{U_2''}{c_0 - 1} + \phi_{20}'' \right)$$
$$\frac{d}{dy} (\Gamma_1 \phi_{10}'') = m_2 \frac{d}{dy} (\Gamma_2 \phi_{20}''),$$

are the no-slip, no-flux, and shear stress matching conditions at the interface. Notice that the kinematic equation has been used to eliminate $h$.

The solution is obtained by integrating four times, which gives
\[ \int_0^y \int_0^{y'} \left( \frac{c_3 y^+}{\Gamma_1} + \frac{c_4}{\Gamma_1} \right) dy^+ dy' + c_1 y + c_2 = \phi_{10} \]
\[ \int_y^{1+n_2} \int_y^{1+n_2} \left( \frac{c_7 y^+}{\Gamma_2} + \frac{c_8}{\Gamma_2} \right) dy^+ dy' + c_5 y + c_6 = \phi_{20}, \] (3.17.

where \( c_1-8 \) are the constants of integration. The wall conditions simplify the solution determining that \( c_1, c_2, c_5, c_6 = 0 \). The eddy viscosity is known from the basestate and can be integrated numerically. To reduce the integration error, the double integrals are integrated by parts yielding

\[ y \int_0^y \left( \frac{c_3 y^+}{\Gamma_1} + \frac{c_4}{\Gamma_1} \right) dy^+ - \int_0^y y' \left( \frac{c_3 y^+}{\Gamma_1} + \frac{c_4}{\Gamma_1} \right) dy' = \phi_{10} \]
\[ y \int_y^{1+n_2} \left( \frac{c_7 y^+}{\Gamma_2} + \frac{c_8}{\Gamma_2} \right) dy^+ - \int_y^{1+n_2} y' \left( \frac{c_7 y^+}{\Gamma_2} + \frac{c_8}{\Gamma_2} \right) dy' = \phi_{20}. \] (3.18.

This is the same solution obtained by integrating the 4\(^{th}\) order equation twice and then solving the resulting 2\(^{nd}\) order equations by the method of variation of parameters. The remaining constants are determined by substituting the equations for \( \phi_{10} \) and \( \phi_{20} \) into the interfacial conditions. This leads to a generalized eigenvalue problem of 4 equations and 5 unknowns \( c_{0-4} \). The solution gives the wave speed and stream functions to leading order.

### 3.3.2. The First Order Solution

The growth rate of waves is still unknown. The first order problem is used to determine it. The expansion is repeated and the order \( \alpha^1 \) terms are collected. This gives the non-homogeneous equation

\[ \frac{d^2}{dy^2} (m_j \Gamma_j \phi_j''') = i \mathbf{R} ( (U_j - c_0) \phi_j'' - \phi_j' U_j'' ). \] (3.19.)
It is non-homogeneous because the leading order terms $\phi_{10}$ are known. The wall conditions are the same as the leading order problem,

$$\phi_{11} = \phi_{21} = 0$$

$$\phi'_{11} = \phi'_{21} = 0,$$

and the interfacial conditions are

$$\phi_{11} = \phi_{21}$$

$$\phi'_{11} + \frac{U'_1}{c_0 - 1} (\phi_{11} - \frac{c_1 \phi_{10}}{c_0 - 1}) = \phi'_{21} + \frac{U'_2}{c_0 - 1} (\phi_{21} - \frac{c_1 \phi_{20}}{c_0 - 1})$$

$$\Gamma_1 \phi_{11}'' + \frac{U''_1 \Gamma_1}{c_0 - 1} (\phi_{11} - \frac{c_1 \phi_{10}}{c_0 - 1}) = m_2 \Gamma_2 \phi_{21}'' + \frac{m_2 \Gamma_2 U''_2}{c_0 - 1} (\phi_{21} - \frac{c_1 \phi_{20}}{c_0 - 1})$$

$$\frac{d}{dy} (\Gamma_1 \phi_{11}'') - m_2 \frac{d}{dy} (\Gamma_2 \phi_{21}'') - i R F (1 - r_2) \frac{\phi_{10}}{c_0 - 1}$$

$$= i R (+ r_2 (\phi_{20} U'_2 + (c_0 - 1) \phi'_{20})$$

$$- (\phi_{10} U'_{1} + (c_0 - 1) \phi'_{10})).$$

The solution to the first order problem is not as straight forward. The differential equation is no longer homogeneous. By the Fredholm alternative, a unique solution exists only if the corresponding homogeneous problem has only the trivial solution. It is easy to see from the leading order solution that this is the case, that indeed the homogeneous problem ($\phi = \phi' = 0$ on all boundaries) yields $\phi = 0$ everywhere. Also from the homogeneous equation we obtain the set of fundamental solutions which are found in equation 3.18.

Before describing the solution process, it is necessary to recall characteristics of solvability, fundamental solutions, and particular solutions. The fundamental solutions are linearly independent. Also, the particular solution must be linearly independent from the fundamental solutions and therefore must be orthogonal in vector space.
Orthogonally is represented by the inner product $\int_{a}^{b} \phi_{i} \phi_{p} \, dy = 0$, where $\phi_{i}$ is any fundamental solution and $\phi_{p}$ is the particular solution. If we now take the matrix property $\langle Ax, y \rangle = \langle x, A^{*}y \rangle$ where $A^{*}$ is the classical adjoint of $A$ and the angle brackets denote inner product, then the adjoint function is defined by $\int_{a}^{b} \psi L[\phi] \, dy = \int_{a}^{b} \phi L^{*}[\psi] \, dy$, where $\psi(y)$ is the adjoint function of $\phi(y)$, $L[\ ]$ is the linear operator and $L^{*}[\ ]$ is its adjoint. The adjoint function has the same span as $\phi$ and so it too is orthogonal to the forcing function and particular solution.

The task is now to determine the adjoint problem in order to solve for the eigenvalue $c_{1}$ and eigenfunction $\phi_{j1}$. To find the adjoint problem, the left hand side of 3.19 is multiplied by the adjoint function producing $[\psi \cdot d^{2}/dy^{2}(\Gamma \phi'')]$. This term is successively integrated by parts until $\phi L^{*}[\psi]$ is formed. The resulting equation is

$$\int_{a}^{b} \{\psi L[\phi] - \phi L^{*}[\psi]\} \, dy = \pi(\psi, \phi) \bigg|_{a}^{b}. \quad (3.22)$$

The terms evaluated at the boundaries (the right hand side of equation 3.22) form the bilinear concomitant $\pi(\psi, \phi)$ which is algebraic.

Boundary conditions for the adjoint problem come from setting $\pi(\psi, \phi) = 0$, which is the necessary condition for $\psi = \text{adj}(\phi)$ and thus for solvability. The bilinear concomitant for either phase at the interface is

$$\pi_{j} = \phi_{j1} \left( -\Gamma_{j}' \psi_{j}'' - \Gamma_{j} \psi_{j}^{(3)} \right) + \phi_{j1}' \Gamma_{j} \psi_{j}' + \phi_{j1}'' \left( \Gamma_{j}' \psi_{j} - \Gamma_{j} \psi_{j}' \right) + \phi_{j1}^{(3)} \Gamma_{j} \psi_{j}. \quad (3.23)$$

Since the domain (liquid wall to gas wall) is piecewise continuous, Green's formula (3.22) can be used for both phases. Realizing that the terms of $\pi_{j}$ at the walls are homogeneous, only the interfacial terms are retained giving
\[ \int_0^1 \{ \psi_L L[\phi_L] - \phi_L L^*[\psi_L] \} \, dy + \int_1^{1+\pi_2} \{ \psi_G L[\phi_G] - \phi_G L^*[\psi_G] \} \, dy = \pi_L \big|_1 - m_2 \pi_G \big|_1. \] (3.24)

Using the boundary conditions for \( \phi_{j1} \) any derivative of \( \phi_{21} \) can be replaced by an equivalent expression in terms of \( \phi_{11} \). Making this substitution produces

\[
\phi_{11}^{(3)} \Gamma_1 (\psi_1 - \psi_2) + \phi_{11}' \Gamma_1 (\psi_2' - \psi_1') + \phi_{11}' (\Gamma_1 \psi_1'' - m_2 \Gamma_2 \psi_2'') \\
+ \phi_{11} \left( \Gamma_1 \frac{U_1'' \psi_1 - U_1' \psi_1'}{c_0 - 1} + \frac{d}{dy} (\Gamma_1 \psi_1') - m_2 \Gamma_2 \frac{U_2'' \psi_2 - U_2' \psi_2'}{c_0 - 1} \right) \\
- \frac{d}{dy} (\Gamma_2 \psi_2''). \] (3.25)

The equation needs to equal zero, and since \( \phi_{11} \) and its derivatives are not zero, the terms multiplying it must be. These four quantities give the four boundary conditions of the adjoint problem.

The adjoint problem \( L^*[\psi] = \frac{d^2}{dy^2} \left( \Gamma_j \psi_j'' \right) = 0 \) is the same as the leading order problem. Notice that the adjoint differential equation is homogeneous, since the forcing function is not applicable to a discussion of adjoint operators. Also, the wall conditions are homogeneous, but the interfacial conditions written above are different leading to different eigenvectors. The eigenvalue of the adjoint problem, however, is the same as the leading order problem.

Solving he generalized eigenvalue equation for the adjoint problem takes advantage of having the same eigenvalue as the leading order problem. The matrix problem \( A - c_0 B \) has rank 3 and thus the eigenvector can be obtained by row reduction without having to solve the eigenvalue problem. This is advantageous because the other eigenvalues are unphysical and writing an algorithm to pick the correct one is tricky. In
practice the opposite procedure is used, the more difficult eigenvector is the leading order problem so the adjoint problem is solved first, and the value of $c_0$ is used to get the eigenvectors of the leading order problem. This is also possible because $\phi_{j0}$ does not appear in the adjoint problem.

Now the first order problem is ready to be solved. The bilinear concomitant was used to ensure adjointness and to determine $\psi$. For the particular solution, Greens’s formula is written down again, but now substituting the forcing functions of $L[\phi]$ and substituting $L^*[\psi] = 0$ produces

$$
\int_0^1 \psi_1 iR \left( (U_1 - c_0)\phi_{10}'' - \phi_{10}U_1'' \right) dy \\
+ \int_1^{1+n_2} \psi_2 iRr_2 \left( (U_2 - c_0)\phi_{20}'' - \phi_{20}U_2'' \right) dy
$$

(3.26.

$$
= \pi_1 \left| \psi_1 \right|_1 - m_2 \pi_G \left| \psi_2 \right|_1.
$$

While the bilinear concomitant is zero, the values of $\phi_{j1}$ and $c_1$ are still unknown. The stream function is irrelevant when the quantities of interest are the growth rate and wave speed. Equation 3.26 can be simplified by using the interfacial conditions so that all terms containing $\phi_{21}^{(n)}$ are written in terms of $\phi_{11}$ and its derivatives, this is the same tactic used to derive 3.25. Further simplification is obtained by repeating the process for the adjoint function so that $\psi_1^{(n)}$ for $n = 0, 1, 2, 3$ replace $\psi_2^{(n)}$. Finally, the zero order boundary conditions are substituted ($\phi_{20}^{(n)} \rightarrow f(\phi_{10}^{(n)})$ ) for the final simplification. This yields
\[ \int_0^1 \psi_1 iR(U_1 - c_0)\phi_{10}'' - \phi_{10}U_1'') \, dy \]
\[ + \int_1^{1+n_2} \psi_2 iRr_2((U_j - c_0)\phi_{20}'' - \phi_{20}U_2'') \, dy \]
\[ = \psi_1 \left[ -iR(1-r^2)(\phi_{10}U_1' + (c_0 - 1)\phi_{10}' - \frac{F\phi_{10}}{c_0 - 1}) \right] \]
\[ + \psi_1' c_1 \left[ \frac{\phi_{10}}{(c_0 - 1)^2}(m_2\Gamma_2 U_2'' - \Gamma_1 U_1'') \right] \]
\[ + \psi_1'' c_1 \left[ \frac{\Gamma_1\phi_{10}}{(c_0 - 1)^2}(U_1' - U_2') \right]. \]

All non-integrated terms are evaluated at the interface. In this equation the only unknown is \( c_1 \).

**3.3.3. Summary of Long Wave Stability**

As will be shown in the following section, the spectral solution has many numerical difficulties arising from resolving the derivatives of the basestate profiles with orthogonal basis functions. The asymptotic limit of the stability equation circumvents this problem giving a solution that uses integrals of the eddy viscosity, and only in the boundary conditions do derivatives appear. However, the long wave solution is applicable only for very small wavenumbers and therefore its only practical application is in determining the neutral stability line.

The fundamental solutions for both \( \phi \) and \( \psi \) are the same and are given in 3.18; the particular solution for the first order problem is determined from 3.27. The coefficients of integration in each case are the eigenvectors, and the eigenvalue is the complex wave speed.
Inspecting the value of $c = c_0 + c_1 \alpha$ reveals that $c_0$ is real and is the speed of the long downstream traveling waves. Whereas $c_1 \alpha$ is imaginary and the value $c_1 \alpha^2$ corresponds to the growth rate of the downstream wave. Because the growth rate is only approximated by one term, it is not a very precise expansion and the magnitude of $c_1$ is not a good indication of growth rate. What is relevant is the sign of $c_1$ which determines where long waves are stable $c_1 < 0$ or unstable $c_1 > 0$.

Figure 3.1: The neutral stability line is the same whether it is calculated by the long wave asymptotic limit or the spectral method. The difference is insignificant, if the contours from the spectral method were plotted the Long Wave neutral stability line would lie between $0/s$ and $0.01/s$.

The wavelengths of interest in this work are on the order of half a meter. For thin films this does not correspond to the long wave limit and so the procedure describe above is not applicable to all flow conditions. We therefore limit the applicability of the long wave solution as a numerical check on the spectral method. Figure 3.1 shows that the two methods produce the same neutral stability line.
3.4. Chebyshev-Tau Method

The solution method implemented is the same as the one described by Orszag (1971). A brief description of the Chebyshev-Tau method will be given followed by considerations taken to handle turbulent flow.

First, the stream function is assumed to have the series solution

\[ \phi(y) = \sum_{i} T_i(y) a_i, \]  

(3.28)

where \( T_i(y) \) are Chebyshev polynomials. This orthogonal basis was chosen because the functions vary rapidly near the boundaries and thus quickly resolve turbulent profiles.

Substituting this definition into the modified Orr-Sommerfeld equation produces a general eigenvalue problem. However, as suggested by Orszag, it is advantageous to use the recursion properties of the Chebyshev coefficients. This recursion relationship says that there are linear combinations of \( a_i \) that would produce the derivatives of the function without changing the basis functions, in other words if \( \phi(x_i) = T_{ij} a_j \), then \( \phi'(x_i) = T_{ij} A_{jk} a_k = T_{ij} a_j^1 \). Furthermore, if \( \phi(y) \) were multiplied by another function represented by \( f(x_i) = T_{ij} b_j \) there exists a not so trivial matrix relationship that provides \( f(x_i) \phi(x_i) = T_{ij} B_{jk} a_k \), where \( B_{jk} \) contains the chebyshev coefficients \( b_j \). Finally, all combinations of derivatives can be written as matrix multiplications such as \( f''(x_i) \phi'(x_i) = T_{ij} B_{jk} A_{kl}^1 a_k \). What should be apparent is that \( T_{ij} \) and \( a_i \) remain on the outside of the matrix multiplication regardless of the form of the linear term. Thus the eigenvalue problem reduces to

\[ T_{ij} A_{jk} a_k = c T_{ij} B_{jk} a_k, \]  

(3.29)

and the Chebyshev polynomials themselves cancel out from the equation.
The solution is approximated numerically by a finite series. The size of the matrix problem for two phases is \(2N+2\) plus the order of the problem. Since there are two \(4^{th}\) order Orr-Sommerfeld equations (1 for each phase) the matrix problem is \(2N+10 \times 2N+10\), leaving 8 degrees of freedom. The additional rows needed to make the system solvable are given by the boundary conditions. Alternatively, Lanczos (1956) proposes to fill the 8 degrees of freedom with 8 more approximations to the differential equation. In this case, the left hand side no longer equals the right hand side so 8 new unknowns are introduced, and it is this problem that is solved exactly. The two possible ways of writing the last 8 rows of the matrix problem are

\[
L_k \{ \phi_n \} = \tau_k \quad n = 0, 1, \ldots, N + N_b \quad k = 1, \ldots, N_b
\]

\[
B_k \{ \phi_n |_{\partial \Omega} \} = 0 \quad n = 0, 1, \ldots, N + N_b \quad k = 1, \ldots, N_b
\] \hspace{1cm} (3.30)

where \(L_n\) is the linear operator, \(B_n\) is the boundary condition operator, \(\partial \Omega\) refers to the boundary of the domain, and \(N_b\) is equal to the order of the equations or the number of boundary conditions (8 in this case). This approach is known as the tau method and solves the modified problem exactly. The values of \(\tau\) give an indication of the error. In practice the boundary conditions take the place of the last \(N_b\) rows and the general eigenvalue problem is solved.

3.4.1. Eliminating Infinite Eigenvalues

A challenge of solving fourth order eigenvalue problems and also using the tau method is the existence of spurious or non-physical modes. In the discussion above, the
problem is homogenous and has homogeneous and interface matching boundary conditions. The eigenvalue problem has the form

\[
\begin{pmatrix}
A_{jk}

\end{pmatrix}_{kj}^LHS a_k = c \begin{pmatrix}
B_{jk}^RHS

\end{pmatrix}_{kj} a_k,
\]

and in the case homogeneous BCs \(B_{jk}^RHS = 0\). These zero rows create infinite eigenvalues. This problem is allievated by factoring the bottom rows into

\[
B_1 a_1 + B_2 a_2 = 0
\]

where the length of \(a_1\) is equal to \(N+1\), and the length of \(a_2\) is \(N_b\). Thus \(B_1\) is \((N + 1) \times (N + 1)\), and \(B_2\) is \(N_b \times N_b\). Since both matrices are square, the equation is solved and substituted into the original problem giving

\[
\begin{pmatrix}
A_{jk1} + A_{jk2}(-B_2^{-1}B_1)

\end{pmatrix} a_1 = \begin{pmatrix}
B_{jk1} + B_{jk2}(-B_2^{-1}B_1)

\end{pmatrix} a_1
\]

or \(A_{jk}^* a^* = B_{jk}^* a^*\),

where now \(B_{jk}^*\) has full rank.

The method described above needs to be modified slightly for wavy flow. The kinematic condition and the normal stress condition both include the eigenvalue so the right hand side of these conditions are not zero rows. These two conditions are simply put at the top of the boundary condition matrices and included as part of \(A_{jk}\) and \(B_{jk}\).

This means that the size of \(B_2\) will only be \(N_b - 2 \times N_b - 2\).

### 3.4.2. Factorization and Eliminating Spurious Eigenvalues

Boundary value problems differ from initial value problems in that they are not typically reduced to a series of first order differential equations. The 4\(^{th}\) derivatives of the Orr-Sommerfeld equation can be computed directly by any finite differencing...
scheme. To do so, however, is not recommended as the solution will contain spurious eigenvalues and highly oscillatory stream functions. In order to eliminate these problems, the equation is factored into a set of two, second-order ODEs as described by Gardner (1989). A description of this factorization and its application to two phase flow is given below.

The form of the factorization will be determined by the operator multiplying the eigenvalue. In the case of the Orr-Sommerfeld equation the system of equations is

\[ v = \phi''(y) - \alpha^2 \phi(y) \]  
\[ -v''(y)\Gamma(y) - 2v'(y)\Gamma'(y) + \Gamma(y)v(y)\alpha^2 + \phi''(y)\Gamma''(y) \]  
\[ + aiR r_k / m_k (v(y)U(y) - \phi(y)U''(y)) = ir_k / m_k \omega v(y). \]  

Each equation now has two tau values, but since the factorization is a definition these values are exactly zero, which is easily confirmed after calculating the eigenvalues and eigenvectors. The factorization and the tau method can be written using the following equations involving linear operators:

\[ v = \sum T_n b_n \]  
\[ L_v \{ v \} + L_u \{ u \} = \omega \ v \]  
\[ Q \{ u \} = v. \]

The application of the tau-method becomes more apparent in matrix form

\[ L \begin{pmatrix} b_n \\ y_n \end{pmatrix} + Ma_n = \omega \begin{pmatrix} b_n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ r_n \end{pmatrix} \]

where \([b_n \ y_n]^T\) is the same size as \(a_n\).

Applying this to stratified flow the matrices double in size, and the boundary conditions become sparse block matrices which can be written
The matrices are split up in order to remove tau and make room for the boundary conditions. After the number of rows is reduced, the matrix is made square again by appending the boundary conditions. Figure 3.2 shows how the matrix is split into sub-matrices, which correspond to equations 3.39.a-c. The empty spaces represent zeros in the matrix.

Writing the matrices as equations yields

\[
\begin{pmatrix}
L_L & 0 \\
0 & L_G
\end{pmatrix}
\begin{pmatrix}
b_{Ln} \\
y_{Ln}
\end{pmatrix}
+ \begin{pmatrix}
M_L & 0 \\
0 & M_G
\end{pmatrix}
\begin{pmatrix}
a_{Ln} \\
y_{Ln}
\end{pmatrix}
= \omega
\begin{pmatrix}
b_{Ln} \\
y_{Ln}
\end{pmatrix}
+ \begin{pmatrix}
0 \\
\tau_{Ln}
\end{pmatrix}
\]

(3.38.a)

\[
\begin{pmatrix}
Q_L & 0 \\
0 & Q_G
\end{pmatrix}
\begin{pmatrix}
a_{Ln} \\
b_{Ln}
\end{pmatrix}
= \begin{pmatrix}
b_{Ln} \\
b_{Ln}
\end{pmatrix}
\]

(3.38.b)

The procedure is to solve 3.39.b for \( y \) and substitute it into 3.39.a and remove 3.39.c which makes the matrix \( N+1 \times N+1+N_b \). The boundary conditions can then be inserted to make it square and solvable. After \( \omega \) and \( a_1 \) are solved the rest of the eigenvector can be determined using 3.39.a and if so desired equation 3.39.a can be used to determine \( \tau \).

In the matrix equation, the Chebyshev polynomials have canceled out making the problem a spectral solution dealing only with the coefficients and recursion relationships of the basis functions. This cancelation however, cannot be applied in the boundary conditions. The boundary conditions are homogeneous at the walls and therefore cannot cancel with zero, and at the interface the liquid values are evaluated at the right side of the Chebyshev interval \( y = 1 \) whereas the gas phase is evaluated at \( y = -1 \). Therefore
the polynomials are found in the boundary conditions and are $T_n(-1) = (-1)^n$ and $T_n(1) = 1^n$.

![Figure 3.2: Banding of matrices necessary for using the tau method to solve a factored set of differential equations](image)

A detail that has been left out of this discussion is the handling of $h$. This is only a problem if it has not already been substituted by rearranging the tangential velocity condition at the interface. For completeness, this will be addressed. Because $h$ appears only in the boundary condition matrix, the above discussion is unchanged. However, the inclusion of $h$ into the unknowns means that both the kinematic and velocity matching condition must be included so solving the equations produces

$$y_j = L_j^{-1} (\omega b_{j2} - L_{j2} b_j - M_{j2} a_j)$$  \hspace{1cm} (3.40.a)

$$T_j^{LHS} \equiv L_{j1} b_j - L_{j4} L_j^{-1} (L_{j2} b_j + M_{j2} a_j) + M_{k1} a_j$$  \hspace{1cm} (3.40.b)

$$T_j^{RHS} \equiv b_{j1} - L_j^{-1} b_{j2}$$  \hspace{1cm} (3.40.c)

$$T_j^{LHS} = \omega \ T_j^{RHS}$$  \hspace{1cm} (3.40.d)
where $T_j$ is still only $N + 1 + N_b \times N + 1$ and there are $N + N_b + 2$ variables. A column of zeros are added to the $T_j$ matrices and the $N_b + 1$ boundary conditions are appended to the bottom to make the matrix problem solvable.

Finally, as with eliminating the infinite eigenvalues, the normal shear matching condition presents a problem when it comes to handling the spurious eigenvalues. The problem occurs because the equation is third order and factorization of the boundary condition is needed. Substituting 3.40.a into 3.11 yields

$$-(1 - r_2)F \alpha h - T \alpha^3 h + \frac{2i \alpha^2}{R} \left( \Gamma_j(y) \phi_1'(y) - m_2 \Gamma_2(y) \phi_2'(y) \right)$$

$$+ a \left( U_1(y) \phi_1(y) - r_2 U_2 \phi_2(y) - U_1(y) \phi_1'(y) + r_2 U_2 \phi_2'(y) \right)$$

$$= \frac{i}{R} \left( \Gamma_1(y) \nu'(y) - m_2 \Gamma_2(y) \nu_2'(y) + \Gamma_1'(y) \phi_1''(y) - m_2 \Gamma_2'(y) \phi_2''(y) \right)$$

$$= \omega \left( \phi_1'(y) - r_2 \phi_2'(y) \right).$$

In order to implement a factorized version of the boundary condition it is first necessary to recognized that $y_j$ from 3.40.a does not involve the boundary conditions and neither does $Q_j$ from 3.39.b. The normal stress condition (3.41) can be rewritten in matrix form $L \nu + M \phi = \omega Q \phi$ in the same way as the differential equations. The matrices are then factored following 3.39. However, there is only room in the matrix problem for this condition to take up one row. The boundary condition is reduced to a row vector by multiplying the matrices pertaining to the liquid by $T_{1:ML}(1)$ and $T_{1:MG}(-1)$ for the gas, where $T_{1:N}(x)$ is value of the first $N$ Chebyshev polynomials at point $x$.

A similar procedure is needed for the intraphase boundary conditions of the four layer method described in §3.6. While the conditions are simplified by eliminating all terms involving $h$, the matrices are more complicated and sparse.
3.4.3. Chebyshev Decomposition of Basestates

As discussed above, the basestate profiles must be represented by a Chebyshev expansion in order for the basis functions to cancel from the eigenvalue problem. For a laminar profile the velocity basestate is parabolic and the eddy viscosity is zero. Therefore, only the first three Chebyshev polynomials are needed to resolve the basestates. As turbulence increases, the number of required Chebyshev polynomials sharply increases. Additionally, the derivatives of the basestates, which appear in the Orr-Sommerfeld equation, also become increasingly difficult to resolve. There are two ways in which this problem is addressed. The first is to use finite differencing rather than the recursion relationships to calculate the derivatives of the basestate profiles. The second option, which is done in conjunction with the first, is to introduce intraphase boundaries, which splits the basestates into parts that converge with many fewer polynomials.

The basestate coefficients are determined using the orthogonality property of Chebyshev polynomials which is

\[
\int_{-1}^{1} \frac{T_n(y)T_m(y)}{\sqrt{1-x^2}} \, dx = \begin{cases} 
0 & \text{for } n \neq m \\
\frac{\pi}{2} & \text{for } n = m = 0 \\
\pi & \text{for } n = m \neq 0 
\end{cases}
\] (3.42)

When dealing with computations rather than analytics, the discrete form of this condition is faster and more accurate when it comes to determining the coefficients of the expansion. If the basestate profiles are defined at the zeros or roots of the polynomials then it is possible to write

\[
\sum_{k=1}^{N} f(y_k)T_m(y_k) = \sum_{k} f_nT_n(y_k)T_m(y_k) \equiv f_n \times \begin{cases} 
0 & \text{for } n \neq m \\
N & \text{for } n = m = 0 \\
N/2 & \text{for } n = m \neq 0 
\end{cases}
\] (3.43)

from which the coefficients of \( f(y) \) can be determined.
The mathematics is simple enough, but implementing it for turbulent stratified flow is more challenging than the equations suggest. Examples of poor expansions and ways to remedy these problems will be presented next.

3.4.4. Combining Finite Differencing and Recursion

Following the equations above and the recursion relationships found in the literature (Orszag 1971, Uphold 1997), the velocity profile and eddy viscosity can be reproduced as a sum of Chebyshev polynomials, which gives the results shown in figure 3.3.

The best way to determine if the numerical approximation has sufficiently converged is to look at the values of the coefficients. Normalized coefficients are given in table 3.1 to illustrate how the higher derivatives of the basestates become increasingly difficult to resolve. For $U$ and $\Gamma$ the coefficients of the last three modes are several orders of magnitude smaller than unity. However, looking at the coefficients of the second derivatives, the $39^{th}$ value in the expansion of $\Gamma''$ is only 5 times smaller than 1. This means that these highly oscillatory functions are still relevant when trying to resolve the basestate functions and their derivatives. Cutting the series short will produce a poor approximation.
Figure 3.3: A comparison of the finite difference solution of the basestate and the Chebyshev expansion

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode</th>
<th>U</th>
<th>U'</th>
<th>U''</th>
<th>Γ</th>
<th>Γ'</th>
<th>Γ''</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-0.005</td>
<td>-0.579</td>
<td>1</td>
<td>-0.003</td>
<td>0.385</td>
</tr>
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<tr>
<td>3</td>
<td>-0.466</td>
<td>-0.008</td>
<td>-1</td>
<td>-0.958</td>
<td>-0.012</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>...</td>
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<td></td>
</tr>
<tr>
<td>38</td>
<td>4.6E-06</td>
<td>-0.0104</td>
<td>-0.004</td>
<td>-9.7E-07</td>
<td>0.002</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>-0.0002</td>
<td>-0.0002</td>
<td>-0.039</td>
<td>-0.0001</td>
<td>-2.3E-05</td>
<td>0.196</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>4.71E-07</td>
<td>-0.0083</td>
<td>-0.003</td>
<td>-3E-06</td>
<td>0.006</td>
<td>0.004</td>
<td></td>
</tr>
</tbody>
</table>

Increasing the flow rates and thus the turbulence accentuates this problem. Using hundreds of modes is not only time consuming but also makes the matrix problem stiff.

TABLE 3.1: CHEBYSHEV COEFFICIENTS OF BASESTATE PROFILES AND THEIR DERIVATIVES

The limits on the number of modes is shown in figure 3.4. At very high Reynolds number the range of acceptable modes shrinks as more nodes are needed to resolve the highly turbulent profiles, but matrix conditioning or stiffness prevents too many nodes from being used. A method is proposed in §3.6 to overcome this barrier, which
introduces two fictitious boundaries in the gas phase that splits the gas into three phases thus making each subphase simpler to represent by a series of Chebyshev polynomials.

Figure 3.4: The wave speeds (top plots) and growth rates (bottom plots) for cases where a) too few modes were used ML=10, MG=1, b) a proper number of modes were used ML=MG=50, and c) too many modes were used ML=150, MG=300

3.4.5. Conditioning and Interpolating the Basestate

Basestates formed from the k-epsilon turbulence models are not smooth functions everywhere in the phase. That is, they are not infinitely differentiable at all points. Therefore, deriving terms to be used in the stability analysis is done with careful differencing and interpolation as to not introduce unrealistic profiles with singularities that lead to spurious stability results.

The second derivatives of velocity and eddy viscosity are the most problematic. Since the relaminarization models are concerned with behavior near the wall, the second
derivative is not continuous in the middle of the phase. Taking a second order accurate, central difference, second derivative of the eddy viscosity in the liquid gives the dashed line in figure 3.5. This arises from the first derivative having a jump discontinuity.

Clearly, the resulting impulse is a model artifact and trying to resolve it with Chebyshev polynomials will require an extremely large number of modes. Furthermore, like all finite approximations of an impulse this problem gets worse with more nodes since the amplitude increases as the step size decreases. The solution then is to take the center point along with the two neighboring points of the eddy viscosity profile and delete them. Then, using a Piecewise Cubic Hermite Interpolating Polynomial (PCHIP), the points are replaced with realistic values. The finite difference derivatives are calculated using the uninterpolated profile and they are smoothed by deleting the three center points and interpolating. The smoothed second derivative is shown in the figure as a solid line and the Chebyshev polynomial representation is given by the dots.

In general, numerical differentiation is an ominous task, the higher the derivative, the more likely oscillations and singularities will occur. As shown in the integration approach to the momentum equation, alternatives to direct differentiation are always welcome and often produce smoother, more precise, and convergent results. In the Orr-Sommerfeld equation we are faced with second derivatives of the basestate functions. To reduce the effects of finite differencing, the momentum equation is rearranged to determine the second derivative as a function of first derivatives, which is

\[ U'' = \left( R \frac{r_j \frac{dp}{dx} - U'T'}{\Gamma} \right) \]  

(3.44.
while this definition of $U''$ is incorporated in the code, the results are fairly independent of the form of $U''$. This would be expected for a consistent program (where the numerical approximation approaches the analytical solution). Calculating the velocity’s second

Figure 3.5: The turbulence model does not produce smooth eddy viscosity or velocity profiles at the center of the phase (dashed). This point must be eliminated and replaced with an interpolation (solid). The resulting Chebyshev representation (dots) is much easier to obtain and more realistic.
derivative two ways served as a check on the numerics, which revealed that more nodes were needed in the basestate program than were originally assumed. This introduced very important information about the dependence of the Chebyshev approximation on the basestate resolution; details on convergence of the basestate will be discussed more in §4.4.1.

3.5. A 3-Layer Gas Phase Model

Again, the problems of turbulence modeling and stability analysis become exceedingly difficult at high Reynolds numbers. Even with the non-uniform distribution of points, at high flow rates an even higher concentration of grid points are needed at the boundary and fewer in the bulk. While excessive number of points can be piled onto the basestate, Chebyshev modes are no longer able to resolve the profiles without producing an incredibly stiff eigenvalue problem. To circumvent the latter issue we impose faux boundaries in the gas phase so that the profiles in the 3-sublayers are easier to resolve than the entire domain of the gas phase. The location of this boundary is defined in terms of $y^+$ which makes this multilayer method independent of the Reynolds number. A value of $y^+ = 60$ was found to optimize the split; it is here where the fewest number of modes are needed to represent the sub and bulk layers. For low flow rates an upper limit to the subphase thickness is set such that $y^+ = \min(60, 1/3 \, d_2 \, w_G^* / v_G)$. With this condition the subphase cannot be larger than a third of the gas phase. The maximum $y^+$ value kicks in at laminar flow where very few modes are needed and so the optimized value of $y^+ = 60$ is irrelevant.
Figure 3.6: To improve the solutions of the spectral method and Chebyshev representation of the basestate profiles, two additional layers are introduced to make the gas phase easier to resolve.

Boundary conditions for the intraphases are similar to the interfacial conditions. Continuity of velocity and stress is required, but since there is no change in fluid properties across the boundary there are no hydrostatic or capillary restoring forces. Thus there is no need for a perturbation height. The conditions are simpler and are easily
derived by setting $h$ equal to zero in the equations given in §3.2.1 and dropping the kinematic condition.

Figure 3.7: Improved convergence with a 3 layer gas phase compared to resolving the entire gas profile with one set of Chebyshev polynomials. The asymmetry comes from the interfacial conditions for roughness.

The resulting matrix is sparse containing 4 coupled blocks along the diagonal, which is shown in figure 3.8. However, this matrix problem ($A_L a = c A_R a$) is underdetermined, the boundary conditions, Gardner factorization, and the tau-method are yet to be applied. After implementing all these changes give the matrices in figure 3.9. These matrices require some explanation. The 4 blocks along the diagonal are clearly the differential equations. The remaining lines are the boundary conditions. The first three lines are the normal shear stress matching conditions at the three interfaces (gas-liquid,
gas-gas near interface, and gas-gas near wall). These need to come first because they involve the eigenvalue and so the right hand matrix is not zero. This is important if the infinite eigenvalues are to be eliminated by factoring the matrix further. The 4th line of the boundary condition submatrix is the kinematic condition. The 5th and 6th lines are short because they are the homogeneous wall conditions. Lines 7 and 8 are the no-flux and tangential shear stress matching condition, remember that the velocity matching condition has already been used to eliminate \( h \). The following 6 lines are the no-flux, tangential shear stress, and velocity matching conditions at the two gas-gas interfaces. The last two lines are the homogeneous gas-wall conditions.

Figure 3.8: Eigenvalue problem matrices consisting of terms only from the differential equations. The right hand side does not have any convolutions with basestate terms which is why the sub matrices are upper-triangular.

Again, to eliminate spurious modes, the matrix and boundary conditions are factored in the same way as outlined in §3.3.1 and §3.3.2. An example of the improved convergence of the 3-layer gas phase is shown in figure 3.7. The primary difficulty and
thus the motivation for the 3-layer approach is in resolving the second derivatives of the velocity and eddy viscosity. As turbulence increases, the rate of change near the wall increases, and this change requires high-order Chebyshev polynomials; the problem quickly becomes unsolvable. A 3-layer gas phase is necessary starting around a Reynolds number of $3 \times 10^4$. This method can then be used up to at least $\text{Re}_G = 3 \times 10^7$ as long as the long wave growth rates are near zero; a linear stability analysis is not applicable much beyond neutral stability. Since the gas phase is split into three regions based on a $y^+$ value, the 3-layer gas phase – with some minor exceptions – produces reasonable stability results for all incompressible pipe flows. The current version of the program does not split the liquid phase like the gas phase is. This will cause issues for large liquid flow rates; the same 3 layer approach would need to be applied to the liquid phase.

\[0\]
3.6. Justification for Eddy Viscosity Simplification

A very significant simplification is made by assuming that the eddy viscosity perturbation is zero. Consider the full perturbation equation where \( u = U + \gamma \frac{d\phi(x, y, t)}{dy}, k = k_b + k', \text{ and } \varepsilon = \varepsilon_b + \varepsilon' \), and \( \mu_T = \mu_{T,b} + \mu' \), where subscript \( b \) indicates the basestate. The basestates are already solved for so moving onto the perturbation equation we use lexicographic ordering to define all dependent variables together as \( a = [\phi, k', \varepsilon']^T \) where each function is represented by a vector of Chebyshev polynomial coefficients. A generalized eigenvalue problem \( Aa = cBa \) still results, and the elements of the matrices are

\[
A = \begin{pmatrix}
A_{\phi\phi} & 0 & 0 \\
A_{k\phi} & A_{kk} & A_{k\varepsilon} \\
A_{\varepsilon\phi} & A_{\varepsilon k} & A_{\varepsilon\varepsilon}
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
B_{\phi\phi} & 0 & 0 \\
0 & B_{kk} & 0 \\
0 & 0 & B_{\varepsilon\varepsilon}
\end{pmatrix}
\] (3.45)

where \( A_{ij} \) is a term containing \( j \) (either \( \phi, k', \) or \( \varepsilon' \)) in the differential equation for \( i \).

Because neither \( k' \) nor \( \varepsilon' \) appear in the equation for \( \phi \), the eigenvalue problem, which requires the determinant of \( A \) and \( B \) is reduced to

\[
\det(A - cB) = AB_{\phi\phi}(AB_{kk}AB_{\varepsilon\varepsilon} - A_{k\varepsilon}A_{\varepsilon k}) = 0,
\]

where we’ve used the shorthand \( AB_{\phi\phi} = A_{\phi\phi} - cB_{\phi\phi} \) and \( AB_{kk} = A_{kk} - cB_{kk} \). Clearly, only \( AB_{\phi\phi} \) needs to equal zero to get the eigenvalue meaning that the problem simplifies to

\[
A_{\phi\phi}\phi = cB_{\phi\phi}\phi,
\] (3.46)

The perturbation velocity can be decoupled from the other quantities, but this means that \( k' \) and \( \varepsilon' \) will remain unknowned, and the choice to exclude them is subject to an \textit{a posteriori} justification. For completeness, if the full eigenvalue problem were to be
solved then perturbation eddy viscosity could be obtained from $\hat{\mu} = c_\mu f_\mu r_j R^{k_j \over \varepsilon_j} \left( 2 \hat{k}_j - \frac{k_j}{\varepsilon_j} \right)$.

In neglecting the perturbation eddy viscosity, we have assumed that the simplified form must still adequately model the flow without it. To prove that this is the case, the calculated shear stress phase shift is compared to measurements. If this phase shift can be modeled then this is a good indication that the simplified model will have sufficiently resolved turbulence in wavy flow.

Experiments measuring the shear stress variation along a wavy wall were carried out by Thorsness et al. (1979) and Abrams and Hanratty (1985). Water was pumped into a 2” channel with a series of 0.012” solid waves. The shear stress is calculated from measuring the current at electrodes on the surface of the waves, this information is then used to report the phase shift between the shear stress and the wave heights.

To model the liquid-solid wave experiments, the liquid-gas models must reflect the observed dynamics. The relaxation of stress in the liquid at the interface which causes the "S" shaped velocity profile is not indicative of a liquid-solid boundary. This is better modeled by laminar flow. Second, the liquid layer needs to be deep enough to form waves similar to the 0.012” waves in the channel, thus there is a minimum linear growth rate that would mimic the solid waves. For these reasons, calculations are performed in a 3” channel with a holdup of 1/3, and viscosity and density ratios of 1000.

From the spectral solution and equation 3.28, the eigenvectors are used to form the stream functions. Also, the relative wave height is known now that the eigenvalue $c$ is known. Notice that since the height is scaled by the perturbation variable, $\gamma$, the
amplitude has no physical meaning, only the phase shift is relevant. The equation for shear stress at the interface is found in equation 3.8.

\[ \tau_i = m_2 \Gamma_\sigma \left( \phi_\sigma' + h \tau_\sigma' + \alpha^2 \phi_\sigma \right). \] (3.47)

The phases are determined by taking the argument of \( h \) and \( \tau_i \) at any value of \( x \) and \( t \).

The phase shift is the difference of the arguments. This process is repeated for all wavenumbers and plotted in figure 3.10. Not only do the calculations match the data, but at small \( \alpha^+ \) the model does a much better job of predicting the known behavior than the models proposed by either Thorsness or Abrams. The slight kink at large wavenumbers comes from interactions of stable modes (very large negative growth rates) where the program has difficulty deciphering between interfacial and internal modes, but this stable region is of no relevance. Thus we conclude that the perturbation equations sufficiently model turbulence without the perturbation of eddy viscosity. Additionally, if the perturbation eddy viscosity were to be included, the derivatives of this modeling convenience may not carry any physical meaning in the stability analysis. So by neglecting small fluctuation in the eddy viscosity there is no penalty in modeling the flow and in all likelihood it enables the k-epsilon model’s applicability to stability theory.
Figure 3.10: The simplification to not include the perturbation eddy viscosity is supported by the fact that the calculation of the interfacial phase shift of the shear stress matches data. Calculation conditions: upper-phase is water, lower-phase is 1000 times denser and more viscous, and a laminar lower layer is forced. $\text{Re}_c=18000$, $h=0.5$, $d=3''$, $\sigma=0.1$, $\alpha^+$ is determined in the log-law region.
CHAPTER 4:  
SUMMARY OF RESULTS

4.1. Numerical Considerations

There has been only a brief discussion of numerical convergence and computational speed. However, these were given substantial attention because a program that converges only for a narrow parameter and takes several minutes to solve makes it virtually useless for the industrial application it is intended. Many of the numerical tricks used to get a robust and speedy code will be addressed in this section.

To preface the need for improved numerics, the major challenge is in calculating high Reynolds number flows. As the Reynolds numbers increase, more and more nodes are required to accurately model the viscous and transition sublayers. If a brute force Gaussian elimination method is used, the computational cost scales as $N^2$ so that even the 1-D approach becomes impractical.

The situation facing the stability analysis is even more dire because the high Reynolds number flows are simply incompatible with a blind application of the Chebyshev-Tau method. As with the basestate, more Chebyshev polynomials are required at higher Reynolds numbers. However, pushing more modes greatly increases the stiffness of the matrices, making the program useless above $Re \approx 50,000$. The following methods were used to improve computational time and to produce realistic stability results.
4.1.1. Convolution and Grid Spacing

As the Reynolds number increases the boundary layers thin and the bulk region grows. An increase in Chebyshev nodes does not distribute the new nodes to efficiently account for this physical phenomenon. The result is a much more computationally expensive calculation but with little improvement in accuracy.

One solution is to iterate the Chebyshev spacing. The program is given the number of desired nodes $N$; a uniform spacing $(1, 2, 3, \ldots, N)$ is used to produce a non-uniform grid based on the roots of the order $N$ Chebyshev polynomial. By treating this non-uniform grid as uniform, the Chebyshev spacing can be repeated. Instead of trying to explain this process further, the code is more instructive. Let $T_N^*(x_i)$ be the location of the $i^{th}$ root of the order $N$ Chebyshev polynomial rescaled from $(-1,1)$ to $(0,1)$, then using Matlab's `interp1` function the spacing is redefined by

$$z^{j+1} = \text{interp1}(0:N-1, z^j, T_N^*(x_i) \ast (N-1)),$$

where $z^j$ is the location of nodes for the $j^{th}$ iteration, and $z^0 = T_N^*(x_i)$.

The grids produced by this algorithm are illustrated in figure 4.1. A uniform grid corresponds to a linear line. The usefulness of this approach is limited by its inflexibility. Spacing is not continuously variable; it is set by the integer value of the iteration. Shown in the figure are three possible spacings. We still seek a spacing that is continuously variable, but based on the Chebyshev polynomials.
Figure 4.1: Non-uniform spacing created by iterating the Chebyshev-node algorithm.

The solution is to convolve the node locations with a ‘vector-map’. The vector-map is defined in a way that will cluster more points near the boundaries. For clustering points towards the lower boundaries (liquid wall and gas side of the interface), the map has the form \( [a, 1, 1]/(a + 2) \) where increasing \( a \) above 1 puts more points in the vicinity of the wall and fewer in the bulk. For the upper half of each phase this vector is flipped. The built-in Matlab function \( \text{conv} \) was used, but mathematically the convolution of say \( u \) with \( v \) is defined by

\[
w(k) = \sum_i u(j)v(k - j + 1).
\] (4.2)

The size of \( w \) is not the same as \( u \) so in application the ends of the vectors are cut off and replaced by the exact values of the boundary locations.

The problem with the standard grid at high Reynolds numbers is that the viscous boundary layer and log-law regions of the flow are not properly resolved and so the derivatives of the profiles used in the stability give erroneous results. As seen in figure
4.2, when the density of points near the wall is too low, the upstream mode (dashed) is calculated as unstable in the long wave region even though the downstream mode (solid line) is supposedly stable. Intuition quickly refutes this possibility since long waves traveling solely upstream is not physical reality, nor is it consistent with the damping and forcing effects at the interface. Increasing the boundary resolution by performing a convolution of the node vector gives the plot on the right, which properly resolves the wave speed and growth rate curves.

![Image of growth rate curves and wave speed curves](image.png)

Figure 4.2: The growth rate curves (bottom) and wave speed curves (top) using standard Chebyshev spacing (left) and increased boundary points using the convolution process ($a=5$). $ReG=3.2 \times 10^5$, $ReL=4000$, $NL=80$, $NG=350$, $ML=50$, $MG=50$, $MG1=MG2=20$, $P=5$ atm, $d=20''$, $h=0.1$

There is currently no optimized method of determining the convolution parameter. Obtaining clearly delineated and physically consistent growth rate curves relies on a trial and error approach. Increasing the number of nodes has the same effect.
but also increases the computational time. Efficient and accurate calculations depend on both the number of nodes and the redistribution of these via convolution. The basestate and stability calculations do converge, and that will be discussed in §4.5.

4.1.2. Under to Over Relaxation

The differential equations are discretized in such a way as to ensure numerical stability of a linearized system. However, if the initial guess is far from the final solution the intermediate profiles may produce negative terms that will cause the solution to blow up. To model a 1-D profile, the time dependent or 2-D form of the differential equations can be used so that as time or position progresses the profile eventually comes to steady state in time. A Newton-Cotes or S.I.M.P.L.E. numerical method would be used to step forward, where information from the previous point in time or space is used at the new point. Analogously, there is a need to constrain the solution in the iterative method implemented here by using information from the previous iteration. This is done by introducing a relaxation parameter, which was discussed in §2.5.4.

The relaxation parameter \( \alpha \) slows convergence by making the forcing term of the differential equation large. As discussed in Patankar, the correct linearization of the source term is crucial in speed and stability. It has already been shown that keeping \( \alpha \) less than 1 will improve stability but decrease speed. While the initial guess may be far from the solution, each iteration will bring the profiles closer to satisfying the flow conditions. Therefore, after a few dozen iterations the relaxation parameter is less required for stability but continues to slow down convergence. This is fixed by slowly increasing the relaxation parameter to 1. A simple conditional statement at the end of an
iteration increases $\alpha$ by 0.01 or 0.001 depending on how far the initial guess is from the solution.

Convergence can be accelerated further by extrapolating the solution. This is the basis of Successive Over Relaxation (SOR). In this case, the over relaxation factor optimizes convergence if it is equal to

$$\alpha = \frac{2}{1 + \sqrt{1 - \rho_j^2}}$$

where $\rho_j$ is the spectral radius of the Jacobi iteration (Paolucci, 1996). This value of $\rho_j$ requires some computational resources and also would involve more development time to implement into the code. Since convergence is very rapid due to the efficient banded solvers and the fact that the equations are only 1-D, it is not necessary to expend the resources necessary optimize the relaxation factor.

The combination of increasing the relaxation parameter to 1 and utilizing efficient matrix solvers vastly improved the speed of the basestate calculation.

4.1.1. Tau Values and Convergence

Fox (1962) details the use of tau values in estimating error, here we are content with discussing trends and order of magnitude approximations. It is possible to extract an estimate of error from the tau values by realizing that they are approximately the next coefficient of the solution’s Chebyshev expansion. Since the coefficients decrease in magnitude, a near-zero tau value indicates convergence of the series to the actual solution.

The number of tau values, depends on the order of the problem. This can be reduced if the problem is factored, which is why the 4th order problem in two phases
yields only four tau values. There are, in fact, 4 other tau values, but because they come
from the equation $v = \phi'' - \alpha^2 \phi$, which is a definition, they are exactly zero.

The tau values are derived from the equations in 3.4.2. Only after a solution is
found can the tau values be computed. Since they are unimportant in determining the
eigenvalues and eigenvectors, the tau values are in practice rarely computed. Here they
serve to verify that indeed the difficulties of turbulent-turbulent stability analysis have
been overcome and a converged solution using the Chebyshev-Tau method is possible.

The top plot in figure 4.3 shows tau as a function of wavenumber. The optimum
number of modes is the thick line. Increasing the number of modes beyond this point
creates stiff matrices that yield little to no gain in precision. Prior to the optimum the
precision rapidly increases. In fact, the semi-log scale of the second plot shows that the
convergence is exponential; this is a major benefit of using a spectral technique. The
convergence slows due to stiffness. Computing tau values provides important
information that is used to optimize the code and to better understand the limits of
numerical precision.

In §3.6 the difficulties of resolving the basestate profiles are overcome by
splitting the gas phase into three layers. All of the calculations here are for a single gas
layer. It is expected that the tau values would be smaller and the matrices less susceptible
to stiffness. Since convergence was confirmed using the single gas layer method, the
effort of producing tau values for the three layer case was not worth the utility they
provide.
Figure 4.3: tau values for all wavenumbers (top), which shows that long waves converge more rapidly than sjprt waves. The second plot shows the decrease of error with increasing modes.
An interesting result from computing tau values is the “effective order” of the problem. When computing tau it was seen that there was a rapid increase at large wavenumber. This was troublesome until it was realized that some terms of the differential equation become negligible depending on the size of the wavenumber. At small wavenumbers the high order terms (low order in $\alpha$) dominate. When $\alpha$ becomes large then these high order terms are nearly zero and very large tau values are computed, which indicates that the problem went from 4\textsuperscript{th} order to effectively 2\textsuperscript{nd} order. The corresponding tau values can be ignored because they correspond to terms that are unimportant in the solution. It would be interesting to see how reprogramming the spectral solution to neglect the high order terms at large wavenumber would improve speed and convergence.

4.2. Basestate Data Comparison

The validity of the basestate profiles was established by comparing overall system values such as flow rates and pressure drops, as well as profile data when available. The basestates were strictly 1-D meaning that a geometrical conversion has to be applied for the calculations to be applicable to pipe flow.

Channel data are available from Bruno (1988). Water and water-glycerin were used with air in a 6.95 m long, 2.54 cm by 30.48 cm channel. Smooth, wavy, and rolling wave interfaces were produced. Comparisons of the flow rate and friction velocities are given in table 4.1.
### Table 4.1:

**CHANNEL DATA COMPARED TO COMPUTATIONS**

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<th>Re(_L)</th>
<th>Re(_L) Data</th>
<th>Error</th>
<th>v(^*) [m/s]</th>
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Data for inclined flows were compared in figure 2.9. This was for an incline of only 0.104 degrees. To do an intuitive check of flows in inclines and declines a few cases are presented here. The limits of the program require the interfacial velocity remain positive. Figure 4.4 presents profiles of laminar and turbulent liquid phases. In the turbulent flow the shear stress pulls fluid near the interface, but the momentum transport through the fluid is weak, therefore gravity – a body force – has a more pronounced effect on the liquid than in the case of laminar flow.
4.2.1. Interfacial Restoring Forces

The growth of a wave depends on forces acting on length scales similar to the wavelength. Whereas gravity acts on large waves, which is illustrated by the growth of ocean waves, surface tension acts on small waves where the curvature is high. This is apparent in the stability equations where gravity is multiplied by $\alpha^1$ and surface tension is an order $\alpha^3$ term. These are two principle restoring forces in the flows studied. In thin films, shear stress produced from long waves dampens their growth. In deep liquid phases long waves can grow without producing much stress or strain; this is where the inclusion of gravity is most vital. As the wavelength increases to infinity there is no difference between points $x$ and $x + \Delta x$ so the growth rate goes to zero. On the opposite
end of the spectrum, at very large wavenumbers, surface tension stabilizes small waves (i.e. high surface curvature) and the growth rate is negative.

Figure 4.5: Gravity changes with incline, but here the acceleration due to gravity was simply changed to unambiguously show the effects of increasing this restoring force. Surface tension is only applicable at high wavenumbers thus this has no effect on the growth or speed of long waves. Flow conditions are water-air in a 1” channel at $Re_G=6,700$ and $Re_L=850$.

The speed of a wave and its growth are related by the amount of energy in the system. In other words, a wave can expend energy by growing or by traversing. Growth is also dampened by dissipation, which is increased with increasing wave speeds. Restoring forces transfer the energy from growth to speed, this is seen in figure 4.5 below
where increasing the restoring force (either gravity or surface tension) decreases the growth rate and increases the speed.

Figure 4.6: When the mass of liquid is increased either by raising the holdup or density, it requires more energy to both grow and propagate waves.

To understand the behavior of waves it is instructive to reason through the changes that occur to the growth rate and wave speeds when variations of the flow are made. In general, increasing the holdup decreases the stress required to bring the interface to a velocity that is susceptible to instability, thus increasing holdup while keeping the gas flow rate constant increases the wave speed and growth rate of the waves.
The opposite effect occurs when increasing the liquid density. The momentum from the gas transferred to a denser fluid will result in a slower interface and gravity will help it to resist growth. The effects of both liquid density and hold up are shown in figure 4.6.

4.3. Onset of Atomization

Many possible mechanisms have been proposed to explain the onset of atomization. The mechanism studied here is the splashing caused by rolling waves. These roll waves form when long waves are unstable. The turbulence model and stability analysis developed thus far are combined to determine when long waves begin to grow.

Recall that the long wave asymptotic solution is numerically simple and requires very little computing power. However, this approach only allowed for the leading order term of the growth rate to be determined so to use the magnitude of the growth rate at a wavenumber other than 0 would lead to a gross error. This method is only good for neutral stability, which theoretically is the point of interest, the onset of atomization, in practice however neutral stability has little relevance.

Experiments indicate a substantial difference between neutral stability and realizable waves in short pipes. What is more disheartening are the discrepancies amongst authors when it comes to determining the long wave transition. This can be simply and correctly explained by the finite length of channels, and the finite growth rate of waves. To bolster this claim one only has to look at how the methods of roll wave transition measurements lead to very different onset lines. Andreussi et al. (1985) performed experiments with water and air in 20 ft channels. The existence of roll waves was determined visually. This produced a transition line, which is shown in figure 4.7.
Figure 4.7: Growth rate contours compared to the long wave transitions of Bruno and Andreussi et al.
Roll waves can also be detected by a spectral analysis of wave tracings which was the method adopted by Bruno and McCready (1988). This more careful analysis showed that long waves were growing but were too small to detect visually. This produced another long wave transition line again shown in figure 4.7.

The difference between the two long wave transition lines in figure 4.7 is primarily due to the way in which long waves were detected. Wave heights are limited by the finite length of the pipe and finite growth rate of the pipes. Extrapolating from the differences in the experimental transition lines, another large jump is expected from Bruno’s transition line to a true neutral stability line where an infinite length pipe is needed to see the corresponding long waves. Indeed, this is what has been found. Neutral stability is information not available from lab scale experiments but is important for several mile long pipelines. Two important results come from the stability analysis 1) it gives a lower bound on the expected transition to atomization, and 2) it gives experiments a quantitative context, that is, knowing how long it takes a wave of a particular growth rate to form is essential for extrapolating to long pipes.

Comparisons of growth rates and data for onset are given in figures 4.8 and 4.9. These onset data were determined by a visual detection method and recorded by Mantilla (2008). A video camera was located at the top of the pipes and onset was marked at the point where liquid was first observed hitting the lens. Waves that grow large enough to break and propel liquid to the top of the pipe in only 10m need to have substantial growth rates. The figures show that for wavenumbers of 2/m, the onset data lie between 0.5/s and 1.0/s.
Figure 4.8: Water and air growth rates for a 2” pipe at 2 atm (top) and a 6” pipe at 1 atm (bottom). A wavenumber of 2/m was used. Onset data are from Mantilla (2008).
Figure 4.9: Growth rate contours for a 5% by volume mixture of butanol in water at 2 atm (top), for 47% glycerin in water at 2 atm (bottom). Onset data are from Mantilla (2008).
The difference between the plots in figure 4.9 is more proof of the limitation of experiments to determine onset. The glycerin-water mixture is denser and more viscous than the butanol-water. While the measured onset shifts down and to the left, the calculations predict a more substantial shift. This is not surprising considering that the butanol-water also has a lower surface tension, which will produce smaller droplets. Smaller and lighter butanol-water droplets are more easily transported to the top of the pipe.

4.4. Notes on Convergence

Computationalists have the arduous task of defending their theory and their numerical techniques. Where data are available this is fairly straightforward, but such comparisons are of little interest to goals of a computational approach. Computations are often used to extrapolate to regions where data are unavailable. Here we have shown that the Chebyshev-Tau spectral method gives the same onset line as the long wave solution. It has also been shown that there is good agreement with velocity profiles, pressure drops, and onset lines – if one accounts for the limitation of finite lengths and growth rates. However, all these flows were at relatively sedate conditions. For very large pipes and for Reynolds numbers of $10^6$ or larger, the numerics are far more difficult. Troubleshooting these conditions has led to an in-depth understanding of the models and what to consider before trusting results. Presented here are the problematic areas of the flow field and of the numerics.
4.4.1. Finite Difference

The Lax-Richtmyer equivalence theorem states that for a consistent scheme, stability is the necessary and sufficient condition for the numerical solution to converge to the analytical solution. Thus, it is troublesome when the scheme is found not to be stable for small step sizes. The finite difference equations in §2.5.2 are not consistent for very small step sizes because near the wall the kinetic energy goes to zero and this term is found in the denominator of the diagonal of the matrix, which leads to poor conditioning.

The two terms that cause consistency and stability issues are found in equations 2.24.a and 2.27.a. The stable forms are

\[-R \frac{r_j}{m_j} \tilde{e}_i - 2 \left( \frac{d \sqrt{k_i}}{dy} \right)^2\]  \hspace{1cm} (4.4.a)

\[-R \frac{r_j}{m_j} c_{2f} \frac{\tilde{e}_j^2}{k_j}\]  \hspace{1cm} (4.4.b)

Note that \( k \) in the denominator of 2.26 is not a concern because it is multiplied by \( \mu_r \) which contains \( k^2 \) in the numerator. The corrected terms were the coefficients of \( k_i \) and \( e_i \) (the diagonal terms), which were multiplied by \( \tilde{k} \) and \( \tilde{e} \) (the previous iteration values), and then moved to the source term, which is where the negative signs come from.

However, to improve the speed of convergence this is only done for values of \( k \) that are below \( 10^{-4} \) times the floating point precision, \( 2^{-52} \). The value \( k \) still appears in the denominator in one of these terms but this is okay because \( e^2 \) goes to zero faster so that it remains stable for very small step sizes. It is also necessary to force \( e \) to scale according to the near wall conditions. Where \( k \) falls below the floating point precision the value of \( e \) is redefined to ensure that \( e^2 < k \).
Figure 4.10: With stable and consistent finite difference equations the solution converges. The second derivative of velocity is a particularly slow to converge, here it is shown for $Re_L=2,600$ and $Re_G=127,000$ for water-air in a 10" pipe.
Figure 4.10 shows a successfully converged velocity profile. The second
derivative near the interface is plotted because this region is slow to converge and the
term is important in the stability analysis. For the particular case of 4.10, convergence is
acceptable at about 400 nodes in the gas phase. The smallest step size, $1.7 \times 10^{-10}$ m, is
sufficiently close to zero to claim that the finite difference equations and solution method
achieve numerical consistency.

In §3.5.2 a second way of determining $U''$ was given by rearranging the Navier-
Stokes equation. In the case of the converged profiles the two numerical methods of
finding the second derivative are equivalent. Using the Navier-Stokes formulation means
that the pressure drop and thus the shear stresses – a key parameter in the stability of the
interface – used in the spectral solution are consistent with those determined by the
basestate model.

4.4.2 Functional Decomposition

In §3.5 it was shown that the gas phase needed to be split into three sub-phases.
Without this split an extraordinarily high number of Chebyshev polynomials were needed
to produce smooth profile, which created extremely stiff matrices. This difficulty arises
again in the second derivative of the basestate profiles. In figure 4.11 the finite
difference derivative of the basestate is compared to Chebyshev representations of
varying order. For the gas sublayer near the interface, approximately 35 Chebyshev
modes are needed to produce a close approximation to the finite difference solution.

No comparison of the continuous gas-phase approach is given because at these
flow rates the number of required modes is impractical.
4.4.3. Dispersion Relation

The considerations made above to ensure convergence are necessary, but it is all for naught if the growth rate and wave speed curves do not converge. Even this is not sufficient to insure accurate solutions. The code incorporated dozens of internal consistency checks. This includes factoring the differential equation multiple ways, using a 3-layer gas phase versus a continuous phase, factoring the normal and tangential stress boundary conditions, calculating the derivatives of the basestate by Chebyshev recursion formulas and finite difference schemes, using the Navier-Stokes equation to
simplify the second velocity derivative, eliminating infinite eigenvalues or leaving them, and others that all were used to check that two valid approaches to the problem produce the same result. From that point, each variation of the code is evaluated based on whether it improved or diminished the program’s range of validity. Below is the result of this work. It uses the optimized program and only depends on the number of modes to converge to the correct solution for $Re_G=127,000$ and $Re_L=2,600$. The wave has a finite speed and the relative error quickly vanishes. More modes do not improve the shape of this curve. The wave speed is not a critical value when predicting atomization; it does serve as a check to make sure that the corresponding growth rate is indeed the interfacial downstream traveling mode. In this sense it is a very important plot to have.

The growth rate asymptotically approaches zero as the wavenumber goes to zero. As such, a poorly formulated stability code will not precisely determine long wave neutral stability. Furthermore, when too many modes are added it is the long wave region where the consequences of stiffness are most apparent.

The plots of relative error show that the growth rates converge slowly especially in the long wave region. Whereas the wave speeds quickly converge (error falls below 1% with relatively few modes), the growth rates are more sensitive to the number of modes. Two more important results come from this. The first being that a course representation of the basestate profiles produces no useful results because the difference between a converged solution and a course result is greater than 100% in both growth rate and speed. The second important result is that if stiffness can be held at bay – and that is indeed the primary numerical task when using the Chebyshev-Tau spectral method
– then a converged result can be produced. This would seem obvious but without all the numerical considerations explained above this is impossible.

Figure 4.12: The solution converges to the curves shown on the left. This required 50 liquid modes, 50 modes in the gas core, and 30 in each gas sublayer (MG=). On the right are measures of relative errors for cases where fewer modes were used. Convergence is most critical at low wavenumbers where the growth rates are small.
4.4.4. Wave Speeds for Non-Linear Analysis

Slugging is notoriously difficult to predict and is an extremely important multiphase phenomenon to account for in design. It is supposed that slugs form from non-linear growth caused by the coalescence of smaller amplitude waves. This sort of analysis is outside the scope of this work. However, an accurate wave speed dispersion relation is needed to study non-linear growth. Figure 4.13 shows how the wave speed changes depending on the assumptions of the flow. The wave speeds are for a 10cP glycerin-water mixture in air at 1 atm. Air is flowing at a Reynolds number of 5200 in a 1” channel. The liquid has a Reynolds number of 8 and is 0.18” deep.

Waves produced by inviscid flows are determined solely by surface tension and gravity. As given by Lighthill (p.226), the speed of these waves is

\[ c^2 = \frac{(g + \sigma \alpha^2/\rho)}{\alpha} \tanh(\alpha d_L). \tag{2.14} \]

The other limit is laminar flow, where the viscous terms dampen the speed of gravity waves. For properly modeled turbulence the resulting wave speeds lie between the inviscid and laminar limits.

Non-linear analysis is quite complex, but it is meaningless if it based on poor approximations of wave speeds produced by an accurate linear stability analysis. The application of k-epsilon turbulence models and the Chebyshev-Tau method to wavy flows has great potential to improve non-linear studies.
Figure 4.13: Wave speeds of inviscid, laminar, and turbulent flow over a wavy interface. $Re_G=5200$, $Re_L=8$, $\mu_L=10cP$, $d_L=0.18”$
CHAPTER 5:

EXPERIMENTS IN DISPERSED TWO PHASE FLOW IN CAPILLARIES

5.1. Introduction

Determining the best use of energy and room inside a space craft is one of the obstacles of long duration space missions. The feasibility of such missions is dependent on compact, light, and power efficient components. A two phase cooling system can absorb 2,240kJ for every liter of water vaporized where heating a liter 30°C can only remove 125kJ. Furthermore, boiling increases micro and macro mixing so local heat transfer coefficients increase as $\Delta T^2$; whereas, heat transfer coefficients in a single phase system are constant with respect to the temperature difference (Geankoplis, 1993). The overall heat transfer coefficients are therefore much larger for a two phase heat exchanger, 1200 W/m$^2$K compared to 400 W/m$^2$K. The space and weight savings for a multiphase system makes it an obvious choice for cooling onboard nuclear reactors.

Despite the advantages, two-phase heat exchangers are not used by NASA in space missions because of the uncertainty in performance under varying acceleration and orientation. Performance is uncertain because the pressure drop and heat transfer properties in a two phase system are dependent on the position, size, and shape of the interface, which is affected by gravity. In a single phase system there is no interface, and

\[ \text{http://www.engineeringtoolbox.com/overall-heat-transfer-coefficients-d_284.html} \] 2005
gravity does not significantly affect the heat transfer properties. Therefore, the performance of a single phase heat exchanger is well known regardless of acceleration or orientation.

Multiphase flow data in microgravity are too limited, and the fundamental understanding is too weak to design a reliable heat exchanger for long duration flights. Experiments in space or in parabolic flights are cost prohibitive. As an alternative approach, if the effects of gravity were negligible, a cooling system could be designed and tested on Earth. The performance would then be independent of gravity and could be measured with ground experiments. Clearly, a system will be gravity independent when the other forces in the system – viscosity, surface tension, and inertia – are infinitely larger than gravity. Such a system would require substantial pumping power to push viscous fluid through small tubes at high velocities. To resolve this impracticality, experiments are carried out that vary the relative magnitude of gravity with respect to viscosity, surface tension, and inertia until a gravity independent flow regime is found. These force ratios are known as the Stokes, Bond, and inverse Froude number, respectively.

The goal of this research is to determine the maximum magnitude of the Stokes, Bond, and inverse Froude numbers such that two phase flow remains gravity independent. The influence of gravity was quantified by measuring three characteristics of the flow: the length of bubbles, the curvature of bubbles, and the film thickness between bubbles and the tube. The changes in the characteristics were measured as the Stokes, Bond, and Froude numbers were varied.
The next section explains how two phase flows are qualitatively defined by flow patterns and quantitatively defined by dimensionless parameters. In order to capture global and local effects of gravity, the bubble length, curvature, and film thickness were measured. These measurements describe phenomena at three different length scales. The theory governing the phenomena is explained in §5.3. Since momentum transport was investigated and heat transfer was beyond the scope of the research, the experiments discussed in §5.4 use a gas and liquid mixture to study the fluid dynamics at various orientations. The results are presented in §5.5. Time did not permit a thorough analysis of all data, however, many of the results of the experiments could be useful to other investigators so a cataloguing of observations and data are given in §5.6. A summary of the results are given in §5.7.

5.2. Background

5.2.1. Dimensionless Parameters

Results given in terms of dimensionless parameters are useful because they apply to all systems with similar physics and describe how competing phenomena interact. Dimensionless parameters are used to determine when a fluid flow becomes turbulent, how thick a boundary layer is, or whether a reaction is limited by kinetics or mass transport. Despite these advantages of dimensionless parameters, it is common practice in two phase flow to map flow patterns with respect to superficial velocities. As a result, flow pattern maps cannot be generalized to arbitrary fluids or tube sizes. This is confirmed by Bousman et. al. (1996) who reports that the bubbly to slug flow transition
shifts when the fluid properties or tube size are changed. The inability to collapse data is why a flow map using superficial velocities is not a comprehensive tool for predicting flow regimes. Results in this research however will be presented as functions of dimensionless parameters. What follows is an explanation of the relevant dimensionless parameters, a brief review of published work in gravity independent two phase flow, and a list of flow characteristics used to quantify the effects of gravity.

A two phase flow is defined by the liquid and gas densities \( \rho_L \) and \( \rho_G \), dynamic viscosity \( \mu_L \) and \( \mu_G \), surface tension \( \sigma \), tube diameter \( D \), gravity \( g \), the angle of the flow from horizontal \( \theta \), the three phase contact angle \( \phi \), and the characteristic velocities of the liquid and gas \( U_L \) and \( U_G \). From these 11 properties, the Buckingham Pi Theorem says that 8 dimensionless parameters will define the flow since there are three reference dimensions, mass, length, and time. The eight dimensionless parameters are the Bond number, the liquid and gas Weber numbers, \( \Delta \rho / \rho_L \), \( \mu_G / \mu_L \), \( \theta \), \( \phi \), and either the liquid or gas Reynold’s number (Ghiaasiaan, 2001). In a gas-liquid system, \( \Delta \rho / \rho_L \) is assumed to equal one, the tube is assumed to be completely wetted, and \( \mu_G / \mu_L \) is approximately zero.

Only three orientations are considered limiting the domain of \( \theta \) to three discrete points. The 11 parameters are reduced to 5: the Bond number, Weber number, the gas and liquid Reynold’s numbers, and orientation. This is not a unique list. The Froude, Stokes, and Capillary numbers can be formed by a combination of the other terms.

The dimensionless parameters containing gravity are the Froude, Stokes, and Bond numbers. The definitions and naïve target ranges are

\[
Fr = \frac{U^2}{gD} \gg 1, \quad St = \frac{\rho g D^2}{\mu U} \ll 1, \quad \text{and} \quad Bo = \frac{\rho g D^2}{\sigma} \ll 1. \tag{5.1}
\]
The object of this research is to see how close to unity these numbers can be while maintaining a flow that is independent of gravity. In addition to these numbers, the Reynolds, Weber, and Capillary numbers can be used to understand the interaction between forces. They are defined as

\[ Re = Fr St, \quad We = Fr Bo, \quad \text{and} \quad Ca = Bo/St. \]  

The Reynolds number quantifies the importance of the inertial terms in the Navier-Stokes equations. The Weber number predicts flow pattern transitions in the absence of gravity and at low viscosity (Rezkallah, 1996). The Capillary number determines the film thickness and pressure drop in surface tension dominated flows (Bretherton, 1961).

An obvious way to ensure gravity independence is to increase inertia until it dominates gravity, which is the limit of the Froude number going to infinity. However, this would not be practical. Other criteria for gravity independence have been suggested. For instance, Suo and Griffith (1964) state that the Bond number should be less than 0.3 for gravity independence, while later work claims that the Bond number only needs to be less than about 20 (Brauner and Moalem-Maron, 1992). The reason for the discrepancy is that surface tension is assumed to be the dominating force while inertia and viscosity are neglected. This discrepancy needs to be resolved. Criteria for gravity independence need to be determined by considering all of the forces affecting the flow.

While global qualitative characteristics such as flow patterns are invaluable descriptions of the flow, smaller length scale phenomena provide insight as to why certain patterns form. In axisymmetric flow, the location of the bubbles is predicted by the Capillary number. The ratio of viscous to capillary forces is called the Capillary number, and is defined as
\[ Ca = \frac{a \mu \gamma}{\sigma} \quad \text{or} \quad Ca = \frac{U \mu}{\sigma}, \quad (5.3. \]

where \( a \) is the bubble diameter, \( \mu \) is the dynamic viscosity, \( \gamma \) is the shear rate, and \( \sigma \) is the surface tension. The first definition is more illustrative for the following example.

When bubbles are small or when the viscosity is low and the surface tension is high, bubbles will act as rigid spheres and be dispersed randomly in a tube. High surface tension and low viscosity is why dispersed bubbly flow is common in water air systems. In higher viscous flows, the shear stress deforms bubbles and produce asymmetric velocity disturbances. This disturbance draws the bubble into the center (Chaffey, 1965). This theory is consistent with my observations of glycerin-air mixtures where the bubbles are centered and often coalesce into slugs. This example illustrates the importance of investigating phenomena at multiple length scales and in the context of dimensionless parameters.

This work uses three length scales based on the bubble length, bubble curvature, and liquid film thickness. By investigating phenomena at these three length scales, the effect of gravity at global and local levels can be identified and proper criteria for gravity independence can be determined. The criteria will be presented as functions of dimensionless parameters, which can be applied to any two phase flow.

5.2.2. Flow Patterns

As mentioned above, the shape, size, and position of the interface directly affects the pressure and velocity of the flow. Flow patterns classify the shape of the interface. In small tubes, flow pattern maps have been produced by Barajas and Panton (1993),
Figure 5.1: Flow patterns. The images on the right have more inertia than the ones on the left.
Mishima and Hibiki (1996), Bousman et. al. (1996), Triplett et. al. (1999), and Narrow et. al. (2000). All of these works used water-air system.

The definitions of flow patterns are subjective. There are three distinct classifications of flow patterns and several secondary classifications. The primary patterns can be distinguished by the existence of a three phase contact line (e.g. stratified or wavy flows), the existence of a dispersed phase (e.g. bubbly, slug, slug-annular, or churn flows), and the case of two continuous phases with no contact line (e.g. annular, wispy-annular, or rivulet flows). General flow patterns considered in this work are given in figure 5.1.

5.3. Theory

5.3.1. Bubble Length

The formation of bubbles occurs only when there is enough liquid to bridge the diameter of the tube. The gas and liquid flow rates are, one way of predicting how long gas bubbles will be. However, experiments showed that the bubble lengths were dependent on orientation, which indicates that gravity affects the way bubbles form, split, or coalesce. The flow rate ratio therefore is not able to account for and classify all the dynamics occurring in the flow. Interfacial instabilities were the primary mechanism investigated to explain this. General stability theory is presented next, followed by short explanations of the Rayleigh-Plateau and Kelvin-Helmholtz instabilities.
5.3.2. Formulation of Linear Stability Analysis

An interface may become unstable to infinitesimal disturbances. When interfacial disturbances are small, the equations governing the shape of the interface can be linearized. The shape of the interface is a linear combination of traveling waves. Each wave or mode corresponds to a perturbation with a different wavelength and frequency. Some of these modes are unstable and grow in time. A traveling wave has the form

$$e^{i(kx-\lambda t)}$$

(5.4)

where $\text{Im}[\lambda]$ is the growth rate, $k$ is the wave number, and $\text{Re}[\lambda/k]$ is the wave speed. If the growth rate is positive then the corresponding wavenumber determined by the dispersion relationship is unstable. The dispersion relationship is the equation that results from substituting equation 5.4 into the linearized differential equation and then canceling the exponential terms.

If interfacial instabilities govern bubble break-up then some characteristic length would arise. To bracket what this length would be, capillary instabilities and inertial instabilities are considered in determining the mechanism driving bubble formation.

Capillary Instability

The Rayleigh instability is caused by unbalanced capillary forces in a jet of fluid. It was first studied by Joseph Plateau (1873) and then revisited by Lord Rayleigh (1879). The theory describes how a jet breaks up in an unbounded fluid. This can be applied to a gas jet in a tube as long as the liquid film can provide enough fluid to bridge the tube diameter without producing substantial viscous forces.
Figure 5.2: a) axial curvature pushes fluid into low pressure regions b) the azimuthal curvature has the opposite effect pulling liquid in and breaking the cylinder

The pressure outside of a perturbed cylinder is a function of the axial and azimuthal curvatures of the cylinder. As shown in Figure 5.2, the axial curvature dampens the perturbations by pushing the liquid out of the thinning regions of the cylinder. The azimuthal curvature, however, pulls liquid into the thinning regions, which causes the cylinder to break up.

A long bubble in a small tube surrounded by liquid is similar to the break-up of a jet. A volume balance across an arbitrary control volume in the liquid film can be written as

$$\frac{\partial h}{\partial t} = v - u \frac{\partial h}{\partial x} - w \frac{\partial h}{\partial z} \quad \text{and assuming 1-D} \quad \frac{\partial h}{\partial t} = u \frac{\partial h}{\partial x}$$  \hspace{1cm} (5.5.)
where $h$ is the perturbation height of the interface. The 1-D case is sufficient because disturbances to the flow are assumed to be axisymmetric. This equation is the flat earth limit approximation to the kinematic condition, which is a continuity statement and does not include a force balance. The kinematic condition rewritten in terms of the flow rate is

$$
\lim_{\Delta x \to 0} \frac{\partial h}{\partial t} \Delta x = -(q(x + \Delta x) - q(x)) \rightarrow \frac{\partial h}{\partial t} = \frac{dq}{dx} \tag{5.6}
$$

where $q$ is flow rate defined by the integral of the velocity across the film

$$
q(x) = \int_0^h u \, dy. \tag{5.7}
$$

Now, considering the force balance, the capillary pressure is balanced by viscous dissipation. The capillary pressure is found by multiplying the surface tension by the sum of the azimuthal and axial curvatures. The Navier-Stokes equation in Cartesian coordinates is

$$
\mu \frac{\partial^2 u}{\partial y^2} = \nabla P \quad \text{where} \quad P = \sigma \left( \frac{1}{R} \left( 1 + \frac{h}{R} \right) + \frac{\partial^2 h}{\partial x^2} \right) + \rho gx. \tag{5.8}
$$

The above pressure term is linearized for small perturbations in the film thickness (Probstein, 2003). The Bond number appears when the equation is rendered dimensionless, and if it is small enough then the hydrostatic pressure term drops out.

The axial velocity component, $u$, is a function of $y$ only, so the ODE can be solved directly by integrating twice. The ordinary differential equation of the axial velocity is

$$
\mu \frac{d^2 u}{dy^2} = \sigma \frac{\partial}{\partial x} \left( \frac{1}{R} \left( 1 + \frac{h}{R} \right) + \frac{\partial^2 h}{\partial x^2} \right) \quad \text{with BCs} \quad \left. \frac{du}{dy} \right|_{y=h} = 0 \quad \text{and} \quad u \bigg|_{y=0} = 0 \tag{5.9}
$$
Integrating twice, and applying boundary conditions gives
\[ u(y) = \frac{\sigma}{\mu} \left( \frac{y^2}{2} - yh \right) \left( h_{xxx} + \frac{h_x}{R^2} \right) \]  \quad (5.10)

The equation for \( u \) can now be put into the continuity condition involving the flow rate (Chang, 2002). Integrating the velocity to get the flow rate and substituting it into the PDE results in
\[ \frac{\partial h}{\partial t} = -\frac{\sigma}{3\mu} \frac{d}{dx} \left( h^3 \left( h_{xxx} + \frac{h_x}{R^2} \right) \right). \]  \quad (5.11)

This equation is linearized by an Ansatz expansion, \( h = h_0 + \epsilon h' + O(\epsilon^2) \). The order \( \epsilon \) terms are collected, the normal mode equation \( h = \exp[i(kx - \lambda t)] \) is substituted into this equation, and the exponentials are canceled. The dispersion relation is plotted, and the maximum growth rate and corresponding wave number are found.

![Growth rate chart of Rayleigh instability](image)

Figure 5.3: Growth rate chart of Rayleigh instability. The wavenumber corresponding to the maximum growth rate will create bubbles that are \( 2\pi k^{-1} \) long.

The maximum growth rate in figure 5.3 occurs at a wavenumber of \( 2^{-1/2} \), and experiments have shown that the wavenumber is empirically related to the most unstable
growth rate by $\lambda = 2 \pi / k_{max}$, which means that a jet will break up into drops every $\sqrt{8\pi R}$.

At high Bond number and in asymmetric flows, the effects of gravity on the instability would need to be considered. A bubble in horizontal flow is pushed against the tube wall by buoyancy, this reduces the ability of the film above the bubble to bridge the tube. Capillary instability is thus only considered applicable at low Bond numbers or in axisymmetric flows.

Inertial Instability

When inertia is the dominating force, velocity can be treated as a potential gradient. A simpler form of this statement is the Bernoulli equation where pressure decreases when velocity increases. An infinitesimal disturbance at the interface will produce a slight change in velocity and a corresponding pressure change. As the disturbance begins to grow, the local pressure continues to decrease and the liquid begins to rise. This phenomenon is known as the Kelvin-Helmholtz instability.

The theory follows from a straightforward substitution of the traveling wave into Bernoulli’s equation

$$
\left( \frac{\partial \phi'_{1}}{\partial t} + U_{1} \frac{\partial \phi'_{1}}{\partial x} \right) - \left( \frac{\partial \phi'_{2}}{\partial t} + U_{2} \frac{\partial \phi'_{2}}{\partial x} \right) = \sigma \frac{\partial^{2} y}{\partial x^{2}},
$$

(5.12)

where $\phi$ is the potential function and the subscripts 1 and 2 refer to the liquid and gas phases, respectively (Panton, 1984). The potential functions here are the second terms in the expansion $\phi = \phi_{0} + \epsilon \phi'$. 

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The stability results show that the growth rate increases as the wave number increases until the waves are small enough that surface tension becomes the restoring force, preventing the wave from growing.

![Graph showing growth rate as a function of wavenumber for the Kelvin-Helmholtz instability.](image)

**Figure 5.4:** Growth rate as a function of wavenumber for the Kelvin-Helmholtz instability.

Kelvin-Helmholtz instability, unlike capillary instability, is applicable at high Bond numbers. Even though it is unlikely to be applicable in small tubes, the theory persists in the literature (Taitel, 1976; Milne, 1960) and is considered here for completeness.

**Viscous Potential flow**

Viscosity must be negligible in both of the instabilities mentioned. A correction can be made to this by using viscous potential flow. While potential flow cannot satisfy no-slip conditions at the interface, normal viscous terms can be added to the boundary conditions (Joseph & Liao, 1994).
Figure 5.5: a) Capillary instability growth rate and b) Kelvin Helmholtz instability. Dashed lines correspond to viscous potential flow. There is a large shift in the maximum growth.

The boundary conditions are a balance of the capillary pressure jump across the interface, the static pressure, and the normal viscous stresses. The following equation is the normal stress condition for a planar geometry:

\[-p_G + 2 \mu_G \frac{\partial u_G}{\partial Z} + \rho_G gh - \left(-p_L + 2 \mu_L \frac{\partial u_L}{\partial Z} + \rho_L gh \right) = -\sigma \frac{\partial^2 h}{\partial x^2} \]  

Adding in the azimuthal curvature and scaling the variables, the boundary condition becomes

\[p_G - p_L + \frac{2}{\sqrt{Re}} \frac{\partial u_L}{\partial r} - \frac{2 \mu_G / \mu_L \partial u_G}{\sqrt{Re}} \frac{\partial^2 h}{\partial Z^2} + \frac{h}{R^2} = \]  

(5.14)
Viscosity adds resistance to the growth of the instability. As a result, including viscous terms shifts the maximum growth rate to a smaller wavenumber, which corresponds to a larger wavelength. It also diminishes the amplitude of the growth rate. Figure 5.5 uses viscous potential flow to calculate the growth rate. The plots were reproduced in Mathematica following the work of T. Funada and D.D. Joseph (2001, 2002).

5.3.3. Curvature

The forces of any flow are balanced. In horizontal two-phase flow at low velocities and with large bubbles, the hydrostatic force around the bubble is balanced mainly by surface tension. The result is an asymmetric bubble profile, which provides a means of quantifying the observed effect of gravity. The curvature of a bubble is proportional to the pressure jump across the interface. This is also known as the Young-Laplace equation,

$$\Delta P_l = \sigma \kappa,$$  \hspace{1cm} (5.15)

where $\kappa$ is the curvature, and $\sigma$ is the surface tension and $\Delta P_l$ is the pressure jump across the interface. Since the gas has the same pressure everywhere in the bubble, the curvature measures how the pressure changes in the liquid around the perimeter of the bubble.

In small tubes, capillary pressure can dominate gravity. Rearranging the Bond number gives the Laplace constant which is the tube size where surface tension is equal to gravity (Triplett et. al. 1998). The Laplace constant is defined as
\[ D = \sqrt{\frac{\sigma}{g(\rho_1 - \rho_g)}} \]

and is about 2.7mm for water-air and 2.3mm for glycerin-air. This gives a first order approximation to the size tubes needed if gravity independence is to be achieved.

5.3.4. Film Thickness

Taylor Flow

In 1961 G.I. Taylor and F.P. Bretherton published, respectively, experimental and theoretical results of the film thickness left by a bubble (Taylor, 1961; Bretherton, 1961). At vanishingly small Bond and Reynolds numbers, the film thickness was found to be a function of the Capillary number only. Their work extended earlier work by Fairbother and Stubbs (1935) to larger capillary numbers. The methods of matched asymptotics and scaling analysis were used to determine the Bretherton scaling, which is

\[
\frac{h}{r} = \begin{cases} 
Ca^{2/3} & \text{for } Ca \in (0,0.07) \\
1.34 \frac{Ca^{2/3}}{1 + 2.5 \, 1.34 \, Ca^{2/3}} & \text{for } Ca \in (0,2)
\end{cases}
\]

These equations relate the relative magnitude of the film thickness to the capillary number. They are derived from matching the capillary pressure at the end of the bubble with the pressure in the transition region of the bubble, which is where the bubble transitions from spherical to a cylindrical shape. The second equation accounts for a large film thickness, that is, when \(h/r\) cannot be assumed to be zero.
Inertia Influence

The concepts behind Taylor Flow have been extended by Aussillous and Quèrrè (2000) to include inertia. The following analysis neglects the order one coefficients found in matched asymptotic analysis; Aussillous and Quèrrè do not determine the coefficients either. The Navier-Stokes equation in this capillary driven flow around a bubble front is

\[ \mu \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \sigma \frac{1}{r-h} \right) + \rho \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) \]  (5.18)

If this equation is scaled using \( u^d = U u, x^d = \lambda x, r^d = R r, y = h y \), and then by eliminating the order one variable terms, equation 5.13 becomes

\[ Ca \frac{R-h}{\lambda} = 1 + We \]  (5.19)

This equation is only an order of magnitude approximation to the governing equation, which is the balance of inertial, viscous, and surface tension forces in the transition region of the bubble.

Another equation is needed because both \( \lambda \) and \( h \) are unknown. A second equation is found by matching the capillary pressure in the transition region with the inertia and capillary forces near the bubble tip. Using the Young-Laplace equation for the capillary pressure, the differential equation is

\[- \frac{\sigma}{r-h} - \sigma \frac{\partial^2 h}{\partial x^2} = - \frac{2\sigma}{r-h} + \rho U^2, \]  (5.20)

and after scaling, the equation becomes

\[ \frac{h(r-h)}{\lambda^2} = 1 - We. \]  (5.21)

Solving for \( h/r \) gives
\[ \frac{h}{r} \propto \frac{Ca^{2/3}}{1 + Ca^{2/3} - We} \quad Ca \in (0,2) \] (5.22)

This accurately predicts a critical Weber number when the film thickness deviates from Taylor Flow. The range of validity for this equation is simply to ensure that film thickness is less than the radius \( h < r \) (Aussillous, 2000).

Gravity Influence

Following Aussillous, gravity can be added back into the lubrication limit. The Navier-Stokes equation is

\[ \frac{-\sigma}{r-h} - \sigma \frac{\partial^2 h}{\partial x^2} = -\frac{2\sigma}{r-h} \pm \rho g(r-h) \] (5.23)

Here, the \( \pm \) accounts for the flow direction. Scaling this equation results in a Bond number instead of a Weber number:

\[ \frac{h}{r} \propto \frac{Ca^{2/3}}{1 + Ca^{2/3} \pm Bo^*} \] (5.24)

where \( Bo^* \) is a modified Bond number defined below. The length scales associated with the modified Bond number are not obvious physical parameters. The Bond number has a squared length scale in the numerator, but in this case the length scale is the diameter multiplied by the axial length scale. The modified Bond number is

\[ Bo^* = \frac{\rho g (R-h) \lambda}{\sigma} \quad \text{where} \quad \lambda^3 \frac{Bo^*}{R} + \lambda^2 - Rh = 0 \] (5.25)

The cubic of the axial length scale can be solved analytically and substituted into the Bond number definition. However, the film thickness must be solved iteratively since equation 5.24 cannot be solved algebraically.
5.4. Experiments

Experiments were designed to simulate the momentum transport properties of a two phase heat exchanger. The effects of orientation on the bubble length, interfacial curvature, and film thickness were observed. Experiments were performed with different tube sizes, flow rates, fluid properties, and at three orientations to vary the forces of inertia, viscosity, surface tension, and gravity. High speed video and high resolution photography of the flow patterns were taken and analyzed with Matlab’s image analysis toolbox.

5.4.1. Apparatus

A diagram of the experimental apparatus is given in figure 5.6. The bladder of a water tank is pressurized to supply the liquid. The constant pressure in the bladder eliminates mechanical vibrations that would come from feeding the liquid with a pump. Dry air is supplied from a pressurized cylinder. The liquid and gas are mixed in a ‘T’ fitting and enter an optically invisible tube. The tube’s refractive index matches the surrounding fluid so that the flow inside of the tube is not distorted by the curvature of the tube. The flow pattern is captured with high speed video (up to 4000 frames per second) or high resolution photos (up to 8 megapixels). An 8 bulb halogen lamp was used as a lighting source.

The method used to add one phase to the other can have a major effect on the observed flow pattern (Oya, 1971). Gas can be added symmetrically by pushing it through a needle suspended in the middle of the flow. Different gauge needles will produce different sized bubbles (Kreutzer 2005). However, in heat exchangers, which is
Figure 5.6: Schematic of the experimental apparatus
after all the application of this work, the nucleation sites are on the sides of tubes. As the bubbles grow at the nucleation site in the absence of gravity they are eventually brought into the flow by drag (Keshock 1964) or in more viscosity dominated flows, by shear stress. This analogous phase method mixing is thus adding gas from a tee fitting. This is the method used in the research.

![Flow Calibration](image)

Figure 5.7: Before the phases mix, differential pressure drop measurements provide a means of determining the flow rate.

Rotometers were used to measure the flow of gas. The viscous liquid flow rate was measured by correlating the single phase volumetric flow with differential pressure measurement, which for laminar flow follows an expected linear correlation (Figure 5.7). Additionally, all liquid was collected and the weight was tracked as a function of time. Absolute pressure transducers were used to measure the pressure drop across the test section.

High speed video was taken with a Kodak EctaPro. The upper frame rate limit is 4000fps. A Kiron 105mm f/2.8 macro lens was attached to the video camera. The high resolution photos were taken with a Canon XT digital SLR with a Canon MP-E 5x macro
lens. For each of these devices the pixel to distance ratio was calibrated. The field of view is modeled as a square pyramid with the apex of the pyramid being the focal point. Using simple trigonometry, the relationship between the camera’s distance to the flow and the pixel’s physical size is

$$\frac{\text{pix}}{\text{cm}} = \frac{\text{pixels in frame}}{2 \tan(\theta) \text{ distance}}$$

(5.26)

where the distance refers to the difference between the object and the focal point – an offset is determined to account for the difference between the theoretical focal point and a physical point on the camera – and $\theta$ is the angle found at the apex of the viewing pyramid; it is the only fitted parameter. The result of the calibrations for both the video and photos using the Kiron lens are shown in figure 5.8.

The resolution for the MP-E was high enough that the edge of the inner diameter could be used as a reference and so the pixel/cm ratio was determined on a per trial basis. The typical resolution with this lens was 7500 pixels/cm.

---

**Figure 5.8**: Using basic geometry, the pixel/cm ratio can be determined with one parameter.
5.4.2. Flow Parameters

A maximum of 120psi is used to feed both fluids. This is the limit of the regulators and the water tank, which has a maximum operating pressure of 125psi. The tubes in the test section for the first year of experiments were approximately 17” long and usually required a pressure drop of about 40-80 psi for glycerin at moderate velocities. The current design uses 4” long glass tubes and requires a much smaller pressure drop.

The primary liquid was glycerin. Pure glycerin matches the refractive index of Borosilicate 7740 glass, has a surface tension of 63 dynes/cm, and has a viscosity of approximately 10 poise at room temperature. Decahydronaphthalene is another fluid with a lower viscosity that matches the refractive index of glass. This has been avoided because of toxicity concerns. Matching a refractive index is critical for measuring a film thickness. However, since refraction is not a concern when measuring bubble lengths, water could also be used in that case.

The tube diameters used in the experiments were in the range of 0.94mm to 5.5mm. The diameter of the tube is most important for controlling the Bond number since the Bond number is unaffected by the fluid velocity. For water and air, the Bond number is 5.8 in a 5.5mm diameter tube and 0.17 in a 0.94mm glass tube. The ranges of dimensionless parameters that have been investigated are in table 5.1.

<table>
<thead>
<tr>
<th></th>
<th>Froude</th>
<th>Stokes</th>
<th>Bond</th>
<th>Reynolds</th>
<th>Weber</th>
<th>Capillary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>0.004</td>
<td>0.8</td>
<td>0.18</td>
<td>0.02</td>
<td>0.003</td>
<td>0.03</td>
</tr>
<tr>
<td>Maximum</td>
<td>230</td>
<td>280</td>
<td>5.9</td>
<td>3900</td>
<td>85</td>
<td>33</td>
</tr>
</tbody>
</table>

TABLE 5.1:
RANGES OF DIMENSIONLESS PARAMETERS
5.4.3. Image Analysis with Matlab

Video is streamed to the computer through an analog composite cable from the high speed camera or VCR. The video is captured with the video editing software Studio 9 by Pinnacle. The movies are then loaded into Matlab as a structure type variable using the function aviread.m (This is an obsolete Matlab function, it was replaced by mmreader.m).

The easiest way to analyze the video is to convert the frames to binary images. Each pixel containing a part of a bubble is set equal to one and everything else is zero. The analysis of video is greatly simplified if the frames are binary images. Properties such as an object’s size, shape, and location can be measured automatically with Matlab scripts. These properties can be linked from one frame to the next so dynamic properties can be calculated such as velocity, acceleration, or the length of a bubble spanning several frames. An outline of the image manipulation is briefly given.

Figure 5.9 summarizes the main process for preparing video for quantitative analysis. A movie is imported as a structure variable. The variable stores a three dimensional matrix of RGB intensity values, and color-map information for every frame. Most of the transformations will only work on a 2-D matrix so the picture is cropped and converted to a gray scaled image.

The gray scale intensities are determined by 8-bit integers, meaning they take on values between 0 (black) and 255 (white). Most images will not use the entire range of intensities so the imadjust.m function is used to increase the contrast of the image. With increased contrast it becomes easier to select a gray level threshold. This threshold is an intensity used to determine the cutoff point between a bubble and the background.
Image Transformations

a) image cropped and reduced to a grayscale image

b) Adjusted image used to increase contrast between bubble and fluid

c) Complementary image so that bubbles are now large values and the background is close to one.

d) By an erosion of the image, the light gradient of the background can be extracted and then subtracted from the original

e) Complementary image with the background subtracted.

f) A gray-level threshold is used to convert the gray scaled image into a black and white image

g) Using a dilation followed by an erosion of the image, the bubbles are closed

h) Bubbles are split up that are too close for the program to distinguish

Figure 5.1: The photos are turned into a binary image using the Matlab’s image analysis toolbox. The bubbles in the final image are counted and stored in a structured variable.
As can be seen in figures 5.9.a and b the bubbles are darker than the background meaning that the background has larger values in those pixels than the bubbles do. For edge detection algorithms and the methods used here, light pixels will be considered the important parts of the image. Taking the complement or negative makes the image easier manipulate. This makes the bubble edges bright and the background dark.

A light gradient exists in the image‘s background where in subfigures ‗a‘ and ‗b‘ the left side of the picture has a brighter background than the right. The background might cause the lighter part of the background to be seen as important information. By using a morphological structuring element that is larger than the radius of a bubble the image can be eroded to give the general trend of the background‘s brightness gradient. Subfigure ‗d‘ is the complimentary image of the background.

After the background is subtracted, the resulting image‘s background is almost completely black (subfigure ‗e‘). At this point the gray level threshold is chosen which distinguishes a bubble from the background. This only needs to be performed once for all frames of the video. The threshold is then used to convert the gray image to a binary image (subfigure ‗f‘).

Bubbles in a binary image are easy to count and measure. Still, subfigure ‗f‘ can be simplified further. Even more importantly, if the perimeter of a bubble is not intact it will make filling the entire bubble impossible or will register as two separate bubbles. Subfigure ‗f‘ shows that the perimeter of the bubbles is not completely intact. A morphological structure element is again used to dilate the image and then eroded. This
smooths the perimeter and fills the interior (subfigure \textit{g}). Now, each white region is counted as an object and labeled chronologically from left to right.

The computer cannot distinguish between touching adjacent bubbles. Based on adding the columns of the binary image, an algorithm was written to determine where a perceived single bubble should be split into two or more. Subfigure \textit{h} shows the result of that algorithm.

Other transformations are performed to these images to limit invalid results. Bubbles with a pixel area under a given limit are deleted and bubbles that are not completely filled in by the dilation are filled automatically. Every time one of these transformations is complete the number of bubbles is reevaluated in each frame. The process is iterated to make sure that cut bubbles are not below the pixel area limit and so erroneous cuts in the bubbles are corrected.

The images are stored in a three dimensional matrix with the first two dimensions containing a black and white labeled image with the third dimension containing the frame number. A tracking algorithm was written based on the principle that a bubble moves in one direction and anytime a numbered object moves backwards it indicates that an object has left the frame and the tracking should be adjusted accordingly.

Three matrices of the same size are used to track and number the bubbles. One matrix tracks the leading edge of each object in each frame. A second one tracks the back edge, and with that information the third one keeps a running tally on what number is assigned to each bubble in each frame. This way a video capturing 500 bubbles will label each bubble 1 through 500 even though any single frame may only have 3 bubbles.
By tracking each bubble independently this method also allows for fluctuations in the flow.

The properties of each bubble are compiled into a structure variable. This variable has 12 fields. The static bubble properties that are determined are the chronological number, height, length, volume, velocity, and vertical midpoint. Dynamic properties that change every frame include the first and last pixels of the bubble, the frames for which it appears in, the frames for which the entire bubble can be seen, and the centroid.

5.5. Results

As mentioned, phenomena at small, medium, and large length scales were investigated so that gravity’s influence could be identified and quantified. The three characteristics measured that correspond to the three length scales were length, curvature, and film thickness. The theory corresponding to these characteristics are respectively instability theory and flow ratios, the Young-Laplace equation, and Taylor flow and Bretherton’s equation. The following three subsections present the results for each of the flow characteristics studied.
5.5.1. Bubble Length Distributions

Weber Number > 1

For large Weber numbers, water and air were the working fluids. The Weber number is the ratio of inertia to surface tension. The viscous forces were presumed to be negligible because the capillary number was approximately $10^{-2}$ and the Reynolds number was about $10^3$. At a Froude number around 10, the flow was not gravity independent because the lengths varied with the flow orientation. The results are shown in Figure 5.10. Note that the bubble lengths are reported in pixels. Not having standard units of measure is inconsequential in determining gravity independence as long as the pixel lengths can be compared across different orientations.

Inertia in figure 5.10 is one hundred times greater than gravity. At this point, gravity might be assumed to be negligible. However, figures 5.10 and 5.11 show similar gravity dependent distributions. The upward flow has primarily long bubbles, the horizontal flow creates a bimodal distribution of lengths, and the downward flow has a wide range of bubble lengths. Decreasing inertia to increase viscosity and surface tension may yield a more gravity independent flow.
Figure 5.10: Length distributions for vertical upward flow, $90^\circ$; horizontal flow, $0^\circ$; and vertical downward flow, $-90^\circ$ at $Fr=100$, $Bo=0.5$, $Re=1000$, $We=10$, $Ca=0.003$, $St=200$. The video snapshots above correspond to upward (top), horizontal (middle), and downward flows (bottom).
Figure 5.11: Length distributions for three orientations at \( Fr = 100, \ Bo = 0.5, \ Re = 2000, \ We = 50, \ Ca = 0.01 \). The video snapshots above correspond to upward (top), horizontal (middle), and downward flows (bottom).

**Weber Number < 1**

Glycerin was used to increase viscosity and smaller tubes were used to increase the capillary force. The experiments at small Weber numbers and order one capillary numbers produced histograms with more pronounced peaks than at higher Weber numbers. Furthermore, these experiments included ten times more data than the previous experiments as can be seen on the ordinates of the histograms.
Rayleigh instability predicts a length of 0.4cm which does not agree with the results. Also, Kelvin-Helmholtz instability is not appropriate at low Weber numbers because inertia is small. Furthermore, the Reynolds, Weber, and Froude numbers indicate that inertia is the smallest force in the system.

Figure 5.12: Fr=0.02, St=0.74, Bo=0.18 Bubble lengths in vertical flow correspond to free stream, inertia stabilized Rayleigh instability. Re=0.02, We=0.003, Ca=0.24. The video snapshots above correspond to upward (top), horizontal (middle), and downward flows (bottom)
An interesting result of figure 5.12 is that the two vertical flows have similar modes. High resolution photographs provided some insight to this problem and will be discussed in §5.6.1. Also, the upward flow has longer bubbles which indicate that there is a stabilizing effect of back flow. The photographs indicate that this is caused by dynamics occurring at the back of the bubbles.

Figure 5.13: Fr=1, St=0.2, Bo=0.18 Increased inertia increased the bubble length distribution. Increased inertia also increased the velocity distribution. Re=0.1, We=1, Ca=1. The video snapshots above correspond to upward (top), horizontal (middle), and downward flows (bottom)
These experiments were repeated at higher velocities with a maximum Weber number of 0.2. The Reynolds number changed from $10^{-2}$ to $10^{-1}$ and the Froude number increased from $10^{-2}$ to $10^{0}$. The histograms in figure 5.13 use logarithmic and linear scales on the abscissa in order to best display the results. While there is a shift in the mean bubble length from one orientation to the next, there is not a clearly defined mode for any of the graphs. The largest peak in the downward vertical flow is not a good mode either because it is lost in a logarithmic scale of the abscissa.

Clearly, even though these experiments were performed at much lower Froude numbers they are much more independent of gravity. While the length distributions are not exactly identical, there are no similarities between figures 5.12 and 5.13. The one difference between the orientations in figure 5.13 is that the downward flow generally has longer bubbles.

Velocity Distribution

Further investigation of the small Weber number results revealed that as the flow rates increased so did the distribution of the velocity. This is shown in figure 5.14.

The experiments do not suggest that the Rayleigh-Plateau or viscous potential flows can be used to explain the difference in bubble lengths for different orientations. What has been found is that increased inertia leads to a larger distribution in bubble velocities, which approaches gravity independence. In the low Weber number experiments, the Froude number is the dimensionless parameter farthest from the ideal
value (Fr>>1). Increasing inertia, while maintaining a small Bond and Stokes number, may produce bubble length and velocity distributions that are independent of orientation.

The problem with correlating inertia to a velocity distribution is that the velocity is geometry dependent. By changing the tubing and configuration of valves, the resulting flows can be completely different.

Figure 5.14: a) length distribution as low velocities b) velocity distribution at low velocities. c) length distribution at high velocities. d) velocity distribution at high velocities. This figure shows that a broadening of the length distribution may be more linked to the distribution of the velocities rather than the magnitude of the velocity.
One reason velocity tends to fluctuate is due to the viscous dissipation differences between glycerin and air. When there are long air bubbles in the small diameter glass tube, the resistance to flow decreases and the velocity increases. A way to address oscillations in glycerin versus air dominated flow is to use tubing with uniform diameter. Previous experiments have had a tube diameter step change before and after the test section. The velocity will fluctuate the most when the tubing with the smallest diameter is also the shortest part of the flow path.

Related to the pulsation of the flow is the issue of compressibility. At gas to liquid volumetric flow ratios around 1 and higher, the bubbles will tend to coalesce, this lowers the pressure drop and these bubbles travel faster in a feedback mechanism which eventually transitions to flow from discrete slugs into annular flow. The annular flow allows the pressurized air to release all at once. As the air flow rate slows, the liquid film again fills the entire tube, and bubbles are formed again. However, now the pressure drop increases meaning that the air mixing with the liquid will need to return to the higher pressure. Therefore, the flow of air slows down until the air in the feed line is compressed again. The cycle then repeats itself. The result is highly unsteady flow.

The variance of velocity is an issue for all viscous, large bubble flows. The size of the velocity variation is proportional to the length of bubbles exiting the system. Each time a pressurized air bubble reaches the end of the tubing it quickly releases the air, there is an immediate decrease in the pressure of the system.

In a closed system, the velocity fluctuations would be damped more than they are here. The hope was to find length distributions that became similar as gravity became
less important. The result for these open loop experiments was that gravity independence was best approximated when the distributions became random.

Below are plots for six conditions. The velocity and length distribution are given along with a plot of velocities in time, or in the order that the bubbles were seen. They show the inherent pulsation found in open, highly viscous, two-phase flows loops. The first three are at low velocities and at the three orientations (downward, horizontal, and upward flows). The second three are at higher velocities.
Figure 5.15: The randomness of the bubble length correlated to the randomness of the velocity which were caused by unsteady viscous dissipation (pp. 151-156)
Orientation: 0°

Average Velocity
approx. 1.5 cm/s

Re=1.8x10^-2 Ca=0.24
We=2.9x10^-3

Bo=0.18 Fr=2.4x10^-2
St=0.74
Orientation: 90°

Average Velocity
approx. 1.5 cm/s

Re = 1.8 x 10^-2, Ca = 0.24
We = 2.9 x 10^-3

Bo = 0.18, Fr = 2.4 x 10^-2
St = 0.74
Orientation: -90°

Average Velocity approx. 7 cm/s

Re = 8.3 x 10^{-2}  Ca = 1.1
We = 9.1 x 10^{-2}

Bo = 0.18  Fr = 0.51
St = 0.17
Orientation: 0°

Average Velocity
approx. 7 cm/s

\( Re = 8.3 \times 10^{-2} \) \( Ca = 1.1 \)

\( We = 9.1 \times 10^{-2} \)

\( Bo = 0.18 \) \( Fr = 0.51 \)

\( St = 0.17 \)
Orientation: 90°

Average Velocity
approx. 7 cm/s

Re=0.13 Ca=1.7
We=0.21
Bo=0.18 Fr=1.2
St=0.10

Bubble Velocity Fluctuations

Chronological Bubble Number
Pumping Considerations

A water tank eliminated mechanical vibrations, but not being a positive displacement pump caused more fluctuations in the flow rate. Ideally, the source pressure would be much higher than the system pressure, and the system would be closed so that some back pressure remained at the end of the loop. Since high viscosity fluids were used in tubes less than the Laplace constant, the full potential of the tank was used and not throttled at all. Figure 5.16 shows how the pressure, and thus the flow rate changed with time.

![Flow Rate vs. Differential Pressure Graph]

Figure 5.16: The flow rate from the tank is unsteady, this is caused from start-up times in the system and since the source pressure was not throttled the hydrostatic pressure and expansion of the tank bladder changed the pressure that provided liquid to the system.

The differential pressure here is the difference between the pressure taps on either side of the visualization box. The low side pressure (not shown) indicates that for the
first 100 seconds the pressure it registers increases steadily to a maximum. This is presumably the time it takes for bubbles to populate the exit tube. The steady decrease on the high side also indicates an accumulation of bubbles in the system. After the startup time which is roughly 350s for the flow rates in the figure, the pressure and flow rates continue to decrease. This is caused from a loss in hydrostatic pressure as the tank empties and also from the expansion of the rubber bladder. The inflation of a rubber bladder follows Hooke’s law since it is designed to expand elastically, as opposed to plastic deformation. A linear decrease in pressure drop is expected for both the energy lost in hydrostatic pressure and elastic stretching of the tank bladder, and this is what figure 5.16 shows.

For high velocity situations, not much can be done about the unsteady flow. Videos are taken in short periods of time so that the flow rate can be assumed constant. Low flow rates are used for experiments in the following sections so that the source pressure is much higher than the system pressure. For some experiments the tank system was taken out completely and replaced with syringe pumps. The compressibility issues still existed, but the positive displacement operation of the syringe pump would keep a steady flow of glycerin. However, there were practical limitations to the syringe pump as using two 40ml syringes to pump 10 poise glycerin through several feet of 1/16” tubing put a strain on the pump’s mechanical abilities. These flows at lower velocities and with the syringe pump are unlikely to be gravity independent but are useful in studying the dynamics of large bubbles in small capillaries.
Summary of Results

The high Weber number results are dependent on gravity up to a Froude number of 100. At lower Weber numbers, however, the distributions of bubble lengths became randomized as inertia is increased. This may be due in part to the broad velocity distribution, but the broad bubble length distribution did decrease the distinction between orientations. Because the larger Bond and Stokes number flows did not lose gravity dependence despite the much higher inertia, it appears that these parameters are vital in determining gravity independence.

5.5.2. Curvature

Capillary pressure is the pressure jump across an interface. In horizontal flows the curvature will account for the buoyancy of the lighter fluid. The hydrostatic pressure plus the capillary force at the bottom of the bubble is equal to the capillary force at the top of the bubble. For smaller tubes, the bubble is forced into a narrower path, which increases the curvature and decreases the hydrostatic effect. Increasing inertia and viscosity will also add terms to the force balance (see figure 5.17). The limitation of this approach and the current setup is that while the axial curvature can be measured directly, the azimuthal curvature must be estimated.

In the context of dimensionless parameters, the effects of buoyancy are more prominent at high Bond and Stokes numbers and low Froude numbers. In this parameter range, buoyant forces are stronger than capillary forces and bubbles migrate to the top of
the tube (see figure 5.18b). This results in an asymmetric flow and an asymmetric bubble profile.

\[ \sigma \kappa_1 + \rho u^2 + \mu u_y \]

\[ \sigma \kappa_2 + \rho u^2 + \mu u_y + \rho g D_b \]

**Figure 5.17:** Force balance across a bubble in horizontal flow. The subscripts on the curvature \( \kappa \) refer to the two principle radii of curvature.

Bubble profiles were captured with video and photos. The high speed videos only have a resolution of 352x240 pixels. After cropping an image, the heights of the bubbles are about 10 to 20 pixels. The pixilation cannot resolve the smooth profile of a bubble or be used to calculate the curvature. The digital SLR camera was used instead. The camera has a maximum resolution of 3456x2304 and a minimum exposure of 1/4000th of a second.

Even at these resolutions, calculating the curvature is not a trivial task. The curvature in Cartesian coordinates is defined as

\[ \kappa = \nabla \times N = \frac{y''(x)}{(1 + y'(x)^2)^{3/2}} \]  

(5.27)

where \( N \) is the normal vector of the interface, \( \kappa \) is the curvature, and \( y \) is the height, which is a function of the position, \( x \). Pixels are still discrete approximations to a smooth interface and calculating the second derivatives directly with a finite difference technique produces extremely oscillatory and thus useless curvature profiles. A few options exist to smooth this. Matlab has a built in smoothing interpolation function \texttt{csaps.m}. 

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However, this only smooths the profile, but the second derivative still varies rapidly. Instead of taking differentials of the profile, the profile is considered to be a sum of orthogonal functions. Function decomposition is well known so no background needs to be given, and here we simply employ the tools developed in chapter 3. The profile varies more near the edges than the center and so Chebyshev polynomials provide an efficient basis for these curvature studies. To emphasize, the advantage of using a few polynomials to represent an otherwise jagged profile is that the sum of smooth functions are necessarily a smooth approximation to noisy data.

### TABLE 5.2:
CHEBYSHEV COEFFICIENTS OF CURVATURE PROFILES

<table>
<thead>
<tr>
<th>i</th>
<th>Symmetric</th>
<th>Asymmetric</th>
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<td>4.9</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>3</td>
<td>0.050</td>
<td>0.21</td>
</tr>
<tr>
<td>4</td>
<td>-0.23</td>
<td>0.49</td>
</tr>
</tbody>
</table>

The Chebyshev coefficients provide a first order approximation to symmetry. As with Fourier decompositions, Chebyshev modes alternate between even and odd function. The even numbered modes (0, 2, 4, ...) correspond to the even Chebyshev polynomials (1, 2x^2-1, 8x^4-8x^2+1,...) and the odd numbered modes correspond to the odd polynomials...
(x, 4x^3-3x, 16x^5-20x^3+5x,...). Therefore, if the odd coefficients are relatively large then the profile is asymmetric, the curvature does not even need to be calculated. In table 5.2 the first 5 coefficients are given for a symmetric and asymmetric profile. The curvature is still the quantity of interest. Therefore, the first 2 coefficients (corresponding to the constant and linear terms) are irrelevant since curvature depends on the second derivatives. The odd coefficients in the symmetric profile are not vanishingly small, but smaller than the asymmetric profile and so this table demonstrates that the symmetry can be ascertained from the coefficients, and it also provides a lower bound of odd coefficients.

With the first 5 modes, a smooth curvature can be calculated using equation 5.22. The derivatives are found using the recursion relationships discussed in chapter 3. The symmetry profiles are given in figure 5.18. The first plot is of axisymmetric flow and yet the curvature plot shows an asymmetry. The difference in the local maximums is less than 10%; this is the degree of precision in the experiments and image analysis. With a low number of polynomials the bubble outline is not perfectly represented, as evident by the negative curvature towards the middle of the bubble. This is part of the truncation error. Better approximations can be obtained by focusing on the middle, rather than trying to resolve the entire profile. The edges are included here to show that the curvature does indeed approach zero as expected, and the quantitative comparison of the local maximum is all that is required to determine symmetry.
Figure 5.18: Plots of curvatures in terms of pixels with the corresponding bubble image to the right. The magnitude of the curvature peaks of the symmetric bubble are less than 10% different. (pp. 188-189)
With a little work, direct differentiation can be used. Points along the profile were chosen based on a minimum change in $y$ or a maximum change in $x$, i.e.

$$y_{i+1} \geq \Delta y_{\text{min}} + y_i$$

as long as

$$x_{i+1} \leq \Delta x_{\text{max}} + x_i.$$  

The smoothing spline function was used before taking a second order accurate finite difference derivative to calculate the curvature. The result of this for an asymmetric bubble is show in figure 5.19. This is one of the better profiles, and because the profile was so strongly dependent on lighting and the image analysis tools, determining symmetry with function decompositions is a far more robust method.
5.5.3. Film Thickness

The flat liquid film between a long bubble and the tube wall is affected by gravity. Analytical equations for the film thickness already include viscosity, surface tension, and inertia. The purpose of measuring the film thickness in this work is to validate the addition of gravity to the theory.

The film thickness is a local phenomenon, but previous research has used global mass balances to determine the film thickness. For those experiments, a liquid slug is blown through a dry pipe, and the film thickness is calculated by assuming that the difference between the amount of liquid entering and exiting is evenly distributed in the film coating the tube (Fairbrother, 1935; Taylor, 1961; Aussillous, 2001). This macroscopic balance has the advantage of providing high precision measurements but is
susceptible to false assumptions in wetting, axisymmetry, and uniform coating. In this way, the accuracy of the measurements suffers. The experiments performed here measure the film thickness directly by using high resolution digital images.

Figure 5.20: Glycerin in a 0.94mm tube at Low capillary number. The film thickness increases with viscosity and velocity; it decreases with surface tension.

A high resolution camera was used to capture the bubbles and the film thickness. At 8 megapixels of resolution and 5x focusing, one pixel corresponds to 1.3μm. The image below is of two gas slugs in an upward vertical flow.

At higher velocities, the capillary number is much larger than the Bond number. As the theory above suggests, this limit would collapse to Taylor flow. The film thickness will deviate from Taylor flow when the Bond number reaches the same magnitude as the Capillary number. This effect cannot be studied by a mass balance of a liquid slug in a dry tube because gravity makes the film uneven. The film ratios, $h/r$, measured in these experiments are plotted with the Taylor flow theory. As the ratio of
the Bond number to the capillary number increases, so should the deviations from Taylor flow.

Figure 5.21: As the Bond number approaches the Capillary number (i.e. the gravitational force becomes of the same order as the capillary force) the film thickness tends to deviate from Taylor Flow.

Precision deteriorates as the capillary number decreases because the film thickness decreases. Figure 5.22 is included to demonstrate what happens to the film thickness and symmetry of the flow when surface tension is not a dominant force. The flow geometry of the top image in figure 5.22 is symmetrical (downward flow) but the
bubble is not. The inertia pushes the bubble from one side to the other. When surface tension does not dominate, the measurement of film thickness becomes difficult. The film thickness might not be the best way to characterize the flow pattern in low capillary number flows.

Figure 5.22: Waves on the surface indicate significant inertial influence. Top: Bo=0.6 this is a picture of a bubble moving downward (left to right). Bottom: Bo=5.5 the film thickness and bubble profile are asymmetric due to gravity.

5.6. Miscellaneous Observations

A multitude of experiments went without a thorough analysis, but I would be remiss if observation from the photos and a cataloguing of video results were excluded from the discussion.
5.6.1. Inertia Induced Annular Instability and Break-up

Very small bubbles appear in annular and churn flow. These are far smaller than the tube diameter, and usually persist because of the very high curvature and hence do not easily coalesce. Clearly, the Rayleigh-Plateau instability of large bubbles cannot account for these, nor can any simple break-up that occurs at the tube diameter length scale. The high resolution photographs provide insight into what is occurring. It is hypothesized that when inertia dominates surface tension, the back of the bubble will be breached by the liquid. This forms an annular gas ring, which is unstable and breaks up into many small bubbles.

From the Bretherton problem and direct measurements of curvature, it is shown that the bubble’s curvature is at a minimum in the center of the tube, and thus the capillary pressure jump is smaller. This is caused by the relatively stagnant liquid at this point compared to the rapidly changing velocity near the bubble’s edges. This center point is the most susceptible to rupture. An increase in inertia causes the liquid to penetrate the bubble. This penetration has been captured and is shown in figure 5.25. The difference in orientation makes the phenomena even more clear, where in the vertical downward flow droplets can actually be seen breaking through and falling through the gas bubble. In upwards flow, the slow penetration is prevented by gravity and this rupture phenomenon is only seen at higher flow rates.

The penetration of liquid has profound implications in assessing gravity independent flow. Downward flow creates liquid droplets that fall through the bubble and gravity acting on upward flows adds resistance and prevents these liquid drops from
growing. This would explain why in figure 5.12 that in highly inertial flows the upward and downward flows have similar bubble length distributions but the horizontal flow is dramatically different. In horizontal flow, the inertia pushes the liquid into the bubble, but gravity causes the bubble to fall to the interface. The Froude number in this case should not be based on the bulk velocity, but of the difference in velocity of the two fluids.

Returning to photographic evidence, the next step of this break-up mechanism is the break-up of the annular gas ring that surrounds the liquid. For low flow rates the liquid drops are small and thus the thickness of the gas annulus is large. This length scale \((r_{outer} - r_{inner})\) creates a relatively small capillary pressure and thus the break-up is slow, or the liquid drops are allowed to relax back into the liquid bulk without splitting the bubble at all. As the flow rate increases, the back of the bubble flattens and thus the liquid penetrates a larger area, and does so more rapidly. This creates a very thin annular space which breaks up very quickly and produces very small bubbles. This process is shown in figure 5.27. These bubbles, with very high capillary pressure persist in the flow. They are stationary and spherical when they lie in the stagnant liquid layer around large bubbles, but outside the large bubbles they stretch and move to the center of the tube. This shear induced migration was captured in video and snapshots are shown in figure 5.23.
Figure 5.23: A small bubble remains spherical and stationary in the stagnant film. It becomes a spheroid where a velocity gradient exist producing shear induced migration.

From the large density and viscosity ratios it is readily reasoned that the pressure is constant everywhere inside a bubble. Kreutzer et al. (2005) calculated the wall pressure around the bubble and showed that there was no pressure drop in the flat film and that any energy losses occurred at the ends of the bubble. Therefore, when the bubble is punctured, the liquid is met with very little resistance. This accelerates the flow and especially the center line velocity. The result is that adjacent bubbles will come closer together and coalesce. This creates a positive feedback mechanism in which the bubbles become longer and the flow becomes faster. Annular flow eventually forms, and in an open system all the compressed air quickly escapes until the liquid again bridges the capillary. This describes the oscillating velocities observed back in §5.5.1.
Figure 5.24: At low flow rates $We = O(1)$ the Capillary number describes the flow. The images show bubbles from $Ca=3 \times 10^{-3}$ (top) to $Ca=4.4$ (bottom).
Figure 5.25: At higher velocities $We = O(10)$ the liquid breaks through the interface and starts to fill the bubbles. The size of the liquid drops increase from top to bottom in the images above.
Figure 5.26: In vertical flows the drops break free from the interface and fall through the gas phase.
Figure 5.27: At even higher velocities $We = O(10^2)$ the liquid blasts through the interface and the annular gas ring breaks up into several small drops according to capillary instability.
Figure 5.28: The droplets persist in the flow as they are too small to coalesce. They are pushed to the edges in the presence of large bubbles and then brought to the center of the flow by shear induced migration.

Figure 5.29: The shear induced migration, and bubble ruptures create wispy annular flow.
5.6.2. Video Catalogue

Length and velocity histograms were made for a water-air system in a 2mm ID capillary. Below are plots documenting the findings of the experiments.

### TABLE 5.3:

<table>
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<th>Angle (degrees)</th>
<th>Speed (cm/s)</th>
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<th>Stokes Number</th>
<th>Reynolds Number</th>
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Figure 5.30: Histograms of bubble length and velocity for horizontal flow at Bo=0.54 (pp. 178-180)
Figure 5.31: Histograms of bubble length and velocity for downward flow at Bo=0.54 (pp. 181-183)
Fr=125, St=31.4, Re=3130

Fr=3903, St=1.01, Re=17502
Figure 5.32: Histograms of bubble length and velocity for upward vertical flow at Bo=0.54 (pp. 184-185)
5.7. Conclusions

Length measurements suggest that Bond numbers of order $10^{-1}$ are not necessarily gravity independent. This is not contradictory to previous research because increasing the velocity diminishes the influence of gravity. At Bond numbers of order one and larger, however, the dependence of gravity could not be eliminated in the operating ranges considered. Approximate requirements for gravity independence are that the Bond number should be less than 0.2, the Froude number be larger than 1, and the Stokes number be less than 0.2. The viscosity was not systematically varied, so the Stokes number requirement is not confirmed.

The symmetry in curvature can easily be determined by fitting the profile with Chebyshev polynomials. Calculating the magnitude of the curvature with any degree of accuracy is outside the realistic goals of the current measuring and analysis techniques.

The Stokes number was further investigated in the film thickness measurements. It was thought that as the Bond number reached the same order as the Capillary number that gravitational effects would appear. Figure 5.21 plotted the film thickness as a function of the ratio of Bond to Capillary numbers, which is the Stokes number. The influence of gravity can be seen by the deviation from Taylor flow. It would be safe to assume from this figure and from the bubble length measurements that a Stokes should not be much larger than 0.1.

All three methods used to analyze the flow pattern use a different length scale. The film thickness, $h$; the tube radius for curvature, $r$; and the bubble length, $L$, are
different length scales that correspond to different phenomenon. This multi length scale approach is a comprehensive approach to capture the physics of the flow.


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