SPECTRUM OF HYPERPLANE ARRANGEMENTS IN FOUR VARIABLES

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Hodge spectrum is one of the most important invariants of hypersurface singularities and a hyperplane arrangement contains the simplest higher dimensional singular set. It is known that the Hodge spectra of hyperplane arrangements are combinatorial. Calculating the Hodge spectrum is a difficult task and combinatorial formulas exist for only a few cases. In this thesis the main result is the formula for reduced hyperplane arrangements in four variables.
To Miae, my wife.
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CHAPTER 1

INTRODUCTION

1.1 Motivation and History

Singularities exist everywhere in our life, from the sharpened point of a pin to black holes. Also, we cannot avoid them in many areas of mathematics. Despite its importance, little was known about singularities before Hironaka (1964). Consider the singular points of two varieties, \( V(x^2 + y^3) \in \mathbb{C}^2 \) and \( V(x^2 + y^7) \in \mathbb{C}^2 \). How different are they? Invariants give an answer for this question.

Whenever we define a new concept, we use other concepts which are well known. In geometry, the behavior of smooth objects is relatively well known; thus, we use invariants of smooth objects to define invariants of singularities. Two important smooth objects in the study of singularity are resolution and Milnor fiber. Many people started with isolated singularities and developed several invariants from this point of view. Ever since, we have been trying to generalize these invariants to higher dimensional singularities; geometric genus, log canonical threshold, multiplier ideal, Milnor fiber, zeta functions, \( b \)-function, Hodge structure, monodromy, and Hodge spectrum are such invariants (see [3]). Of course, these invariants have wide applications in many areas of mathematics. For instance, log canonical threshold is a main tool for singular learning theory in statistics (see [21]).

One aspect of singularities is that they can be considered as collapsed points of smooth objects. Indeed, Hironaka proved that every algebraic singularity over \( \mathbb{C} \) can be considered as a collapsed point of subvariety in a smooth space (see [12]).
The map from a smooth variety to the singular variety is called a *resolution of the singularity*. Resolution method is very useful. We could define many singularity invariants from this method. However, there is a problem with this method when we define an invariant from a resolution: resolution is not unique and we cannot choose the canonical one among resolutions in general. Thus, invariants defined from a resolution need to be checked for independency.

Another way of studying a singular point $x$ is to study a small neighborhood of the point $x$ which is smooth. In the case of hypersurface singularities, a variety $Z$ containing a singular point $x$ is locally defined by a single function $f$ on a smooth ambient space $X$. We can think of the function germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ at the point $x$ with $n = \text{dim} \, X$ and its fiber $f^{-1}(\delta)$ in a small neighborhood of $0 \in \mathbb{C}^n$. Milnor proved that if $\delta$ and the neighborhood are small enough, then all these fibers are smooth and diffeomorphic to each other for any $\delta \neq 0$ (see [14]). This fiber is called the *Milnor fiber* $M_{f,0}$ and gives information of the singular point $x$. It is natural to think of the invariants of the Milnor fiber as invariants of the singularity $x$. One of the most important invariants is this cohomology group $H^*(M_{f,0}, \mathbb{C})$. The cohomology group carries the canonical Hodge structure (see [17]-12.1.2). Moreover, there is a so-called monodromy action $T$ on this group. Thus, the eigenspace of $p$-th Hodge graded piece of $j$-th cohomology for given eigenvalue $\lambda$, $Gr^p_T H^j(M_{f,0}, \mathbb{C})_\lambda$, becomes a finer invariant than a cohomology group $H^j(M_{f,0}, \mathbb{C})$.

The *spectrum* $Sp(f)$ of the germ of a hypersurface singularity $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is a fractional Laurent polynomial

$$Sp(f) = \sum_{\alpha \in \mathbb{Q}} n_{f, \alpha} t^\alpha$$

with $n_{f, \alpha} \in \mathbb{Z}$ which is defined from the dimensions of $Gr^p_T H^j(M_{f,0}, \mathbb{C})_\lambda$ (see Section 2.1.2). This invariant gives much information about the discrete invariants of singu-
larities so that we can recover classical invariants, the Milnor number and the geometric genus of isolated hypersurface singularities. In the case of isolated hypersurface singularities, the spectrum satisfies a symmetry \( n_{f,\alpha} = n_{f,n-\alpha} \) and semi-continuity under deformation. The semi-continuity tells us the spectrum is constant under a deformation of isolated hypersurface singularities with constant Milnor number and also tells us semi-continuity of other invariants such as genus (see [13]). Moreover, it is easier to calculate than \( \text{Gr}_F^p H^j(M_{f,0}, \mathbb{C})_\lambda \).

For isolated singularities, Saito and Steenbrink gave a method for the calculation of spectra in terms of a Newton diagram (see [13]-II 8.5). Thom and Sebastiani gave a formula for the join of isolated singularities (i.e. \( f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_k) + f_2(x_{k+1}, \ldots, x_n) \) for some \( 0 < k < n \) (see [13]-II 8.7). Saito proved that the Thom-Sebastiani formula works in the case of non-isolated singularities (see [13]-II (8.10.6)). It is also valid in the case of a zero function in \( k \) variables (i.e. \( f_1(x_1, \ldots, x_k) = 0 \)) so that we can calculate the spectrum of a higher dimensional singularity when the function \( f(x_1, \ldots, x_n) \) can be written as a function \( f_2(x_{k+1}, \ldots, x_n) \) of less variables which has an isolated singularity at the origin in \( \mathbb{C}^{n-k} \).

A hyperplane arrangement is a union of a finite number of hyperplanes in \( \mathbb{C}^n \). A hyperplane arrangement contains the simplest higher dimensional singular set. We may assume the defining function \( f \) of a hyperplane arrangement is central (i.e. \( f \) has only linear forms as factors) since the Milnor fiber is defined locally. We may want to find a combinatorial method for the calculation of each invariant. For instance, the cohomology of the complement is a combinatorial invariant known as Orlik-Solomon algebra (see [16]). Unfortunately, some of invariants are not combinatorial. The \( b \)-function of hyperplane arrangements (see [3]) and the fundamental group of their complement are such cases (see [18]). The combinatorial invariance of the Betti numbers of \( H^*(M_{f,0}, \mathbb{C}) \) or, even stronger, of the dimensions of \( \text{Gr}_F^p H^j(M_{f,0}, \mathbb{C})_\lambda \) is not known. It is a main open problem in the theory of hyperplane arrangements.
since the dimensions of $Gr^F_{r} H^j(M_{f,0}, \mathbb{C})_{\lambda}$ can be regarded as an invariant between fundamental group of the complement and the cohomology of the complement in the order of refinement (see [3]).

Nero and Saito have shown that the spectra of hyperplane arrangements are combinatorial invariant (see [5]). However, combinatorial formulas have been calculated for only a few cases. Calculation of the spectra of general hyperplane arrangements is not easy since we cannot use the Thom-Sebastiani formula. The spectra of generic hyperplane arrangements are known for $\alpha \in \mathbb{Z}$ (see [20]-5.6). If the hyperplane arrangement germ is reduced with $n = 2$, it is easy to calculate. In [5], the reduced hyperplane arrangement case with $n = 3$ is calculated and the case of $n = 4$ is partially calculated.

In this thesis, the main result is a full combinatorial formula for the $n = 4$ case. To our knowledge, this is the first non-trivial case of a complete Hodge spectrum computation for a large class of hypersurface singularities with the dimension of singular locus bigger than 1 in which we cannot use the Thom-Sebastiani formula. Moreover, this induces nice combinatorial formulas for special cases.

1.2 Statements

**Theorem 1.2.1** Assume $f$ is a reduced central hyperplane arrangement with $d$ irreducible components in $\mathbb{C}^4$. Let $m_V$ be the number of hyperplanes which pass through the edge $V$. Let $S$ be the set of dense edges excepting the hyperplanes in the arrangement. Then we have the following formulas for $i \in \{1, \cdots, d\}$,

$$n_{f, \frac{i}{d}} = \eta_{0,i}(\lceil im_V/d \rceil - 1)_{V \in S}$$

and

$$n_{f, 1+\frac{i}{d}} = \eta_{1,i}(\lceil im_V/d \rceil - 1, \lfloor (d - i)m_V/d \rfloor)_{V \in S}.$$
Similarly for $i \in \{0, \cdots, d-1\}$,

$$n_{f,4-\frac{i}{d}} = \eta_{0,i}(\langle \lfloor im_V/d \rfloor \rangle_{V \in S}) \text{ and}$$

$$n_{f,3-\frac{i}{d}} = \eta_{1,i}(\langle \lfloor im_V/d \rfloor, \lceil (d - i) m_V / d \rceil - 1 \rangle_{V \in S}).$$

Otherwise $n_{f,\alpha} = 0$. Here the functions $\eta_{0,i}$ and $\eta_{1,i}$ for each $i$ are defined as

$$\eta_{0,i}(\langle u_V \rangle_{V \in S}) = \binom{i - 1}{3} - \sum_{W \in S^{(3)}} \binom{u_W}{3}$$

$$- \sum_{V \in S^{(2)}} \binom{(i-3)u_V}{2} - 2\binom{u_V}{3}$$

$$- \sum_{V \in S^{(2)}} \sum_{W \subset V} \binom{2u_V}{3} - (u_W - 2)\binom{u_V}{2}) + \delta_{0,i}$$


and

$$\eta_{1,i}(\langle u_V, v_V \rangle_{V \in S}) = (d-i-1)\binom{i - 1}{2} - \sum_{W \in S^{(3)}} v_W \binom{u_W}{2}$$

$$- \sum_{V \in S^{(2)}} u_V v_V (i - 2) + (d - i - 1 - 2v_V)\binom{u_V}{2})$$

$$+ \sum_{V \in S^{(2)}} \sum_{W \subset V} u_V v_V (u_W - u_V) + v_W \binom{u_V}{2}) ,$$

where the set $S^{(k)}$ is the set of the codimension $k$ edges in $S$. The notations $\lfloor \cdot \rfloor$, and $\lceil \cdot \rceil$ mean the round down and the round up respectively. Also, $\delta_{0,i} = 1$ if $0 = i$ and 0 otherwise.

In this thesis, we will follow the convention, $\binom{t}{k} = t(t - 1) \cdots (t - k + 1)/k!$ for $k \in \mathbb{N}$ and any $t$. For the definitions of edge and dense edge see Section 2.3. In this theorem, $S$ can be substituted by any set of edges containing all the dense edges with
codimension $\geq 2$ (see Section 3.1.6). In [5], the formula for $n_{f,i}$ for $i \in \{1, \cdots, d\}$ had been proved in the setting that $S$ is the set of edges in the non normal crossing singular locus of $f$ (see Section 2.3).

If $f$ is not essential (i.e. $f$ is a function of fewer variables for a possibly different choice of coordinates), we can apply the Thom-Sebastiani formula (see [13]-II (8.10.6)) and recover the formulas for $n = 3$ and $n = 2$.

**Corollary 1.2.2** Assume $f$ is a reduced central hyperplane arrangement with $d$ irreducible components in $\mathbb{C}^3$. Let $m_V$ be the number of hyperplanes which pass through the edge $V$. Let $S$ be the set of codimension 2 dense edges of the arrangement. Then we have the following formulas for $i \in \{1, \cdots, d\}$:

$$n_{f,i} = \binom{i-1}{2} - \sum_{V \in S} \binom{\lceil im_V/d \rceil - 1}{2},$$

$$n_{f,1+i} = (i-1)(d-i-1) - \sum_{V \in S} ([im_V/d] - 1)(m_V - [im_V/d]),$$

$$n_{f,2+i} = \binom{d-i-1}{2} - \sum_{V \in S} (m_V - [im_V/d] - \delta_{i,d}).$$

Otherwise $n_{f,\alpha} = 0$.

This had been proved in [5].

**Corollary 1.2.3** Assume $f$ is a reduced central hyperplane arrangement with $d$ irreducible components in $\mathbb{C}^2$. Then we have the following formulas for $i \in \{1, \cdots, d\}$

$$n_{f,i} = i - 1 \text{ and } n_{f,1+i} = d - i - 1 + \delta_{i,d}.$$ 

Otherwise $n_{f,\alpha} = 0$.

For the generic hyperplane arrangements with $n = 4$ case, we do not have codimension 2 and 3 dense edges. Thus, we get the following result.
Corollary 1.2.4 Assume \( f \) is a generic hyperplane arrangement with \( d \) irreducible components in \( \mathbb{C}^4 \). Then we have the following formulas for \( i \in \{1, \cdots, d\} \)

\[
n_{f,i,i} = \binom{i-1}{3} \text{ and } n_{f,1+i,i} = (d - i - 1)\binom{i-1}{2}.
\]

For \( i \in \{0, \cdots, d-1\} \), we have

\[
n_{f,4-i,i} = \binom{i-1}{3} + \delta_{0,i} \text{ and } n_{f,3-i,2} = (d - i - 1)\binom{i-1}{2}.
\]

Otherwise \( n_{f,\alpha} = 0 \). Here \( \delta_{0,i} = 1 \) if \( 0 = i \) and 0 otherwise.

Consider decomposable cases (i.e. after possibly a different change of coordinates, \( f = f_1f_2 \) for non-constant \( f_1 \) and \( f_2 \), two polynomials in two disjoint sets of variables). According to [8]-Theorem 1.2, we know \( n_{f,\alpha} = 0 \) except for \( \alpha \in \mathbb{Z} \) when the degrees of \( f_1 \) and \( f_2 \) are relative primes. Here we give formulas for \( \alpha \in \mathbb{Z} \) in 4 variables.

Corollary 1.2.5 Assume \( f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2, x_3)f_2(x_4) \) for non-constant \( f_1 \) and \( f_2 \). Also assume \( f \) is a reduced central hyperplane arrangement with \( d \) irreducible components in \( \mathbb{C}^4 \). Then we have

\[
n_{f,1} = \binom{d-2}{2} - \sum_{V \in \mathcal{S}^{(2)}} \binom{m_V - 1}{2},
\]

\[
n_{f,2} = -\binom{d-1}{2} + \sum_{V \in \mathcal{S}^{(2)}} \binom{m_V - 1}{2}, \text{ and}
\]

\[
n_{f,3} = d - 1.
\]

Otherwise \( n_{f,\alpha} = 0 \).

Corollary 1.2.6 Assume \( f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2)f_2(x_3, x_4) \) for non-constant \( f_1 \) and \( f_2 \). Also assume \( f \) is a reduced central hyperplane arrangement with \( d \) irreducible components in \( \mathbb{C}^4 \). Let \( s_1 \) and \( s_2 \) be the degree of \( f_1 \) and \( f_2 \) respectively. If

\[
\]
\[ \gcd(s_1, s_2) = 1 \] then we have the following formulas for \( i \in \{1, \cdots, d\} \):

\[ n_{f,1} = (s_1 - 1)(s_2 - 1), \quad n_{f,2} = 1 - s_1s_2, \quad \text{and} \quad n_{f,3} = s_1 + s_2 - 1. \]

Otherwise \( n_{f,\alpha} = 0 \).

1.3 Outline of the thesis

The aim of this thesis is to prove Theorem 1.2.1 and the Corollaries above. Background will be explained in Chapter 2. This chapter contains the definition of Hodge spectrum from the Hodge structure, intersection theory that we will use for the proof of main theorem, and the chern class of a resolution of a hyperplane arrangement. In Chapter 3, we prove our main theorem and corollaries by hand. In Chapter 4, we will double-check our calculation by using a computer program.
CHAPTER 2
PRELIMINARIES

2.1 Hodge structure

Before introducing Hodge spectrum we need to define Hodge structure on the cohomology $H^*(M,\mathbb{Z})$ of a manifold $M$. See [17] for more on Hodge structure.

2.1.1 Pure Hodge Structure

There are two equivalent definitions of pure Hodge structure.

Definition 2.1.1 A pure Hodge structure of weight $k \in \mathbb{Z}$ consists of a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank and a direct sum decomposition of its complexification $H_C := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ with $H^{p,q} = \overline{H^{q,p}}$.

Example 2.1.2 The Hodge decomposition of the $k$-th cohomology group of a compact Kähler manifold gives a pure Hodge structure of weight $k$.

Alternatively we can replace the direct sum decomposition of $H_C$ by a filtration.

Definition 2.1.3 A pure Hodge structure of weight $k \in \mathbb{Z}$ consists of a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank and a finite decreasing filtration $F$ of $H_C$ satisfying

$$ F^p H_C \cap \overline{F^q H_C} = 0 \text{ with } p + q = k + 1. $$

The filtration $F$ is called a Hodge filtration.
The relation between two definitions, 2.1.1 and 2.1.3, is given as

\[ H^{p,q} = F^p H_C \cap \overline{F^q H_C} \text{ and } F^p H_C = \bigoplus_{i \geq p} H^{p,q}. \]

**Example 2.1.4** A Hodge filtration of the \( k \)-th cohomology group of a compact Kähler manifold or a projective variety is induced by the stupid filtration on the De Rham complex. Moreover, it corresponds to the classical Hodge decomposition in the cases of compact Kähler manifolds.

The *canonical Hodge structure* on the cohomology of a variety is the Hodge structure on the cohomology group induced by the stupid filtration on the De Rham complex.

### 2.1.2 Milnor fiber and Monodromy

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be the germ of a non-zero holomorphic function \( f(x) = t \), where \( x = (x_1, \cdots, x_n) \). Consider the restriction \( f' \) of \( f \) onto \( B_{\epsilon} \cap f^{-1}(\Delta') \), where \( B_{\epsilon} = \{x \in \mathbb{C}^n | \|x\| < \epsilon\} \) and \( \Delta' = \{t \in \mathbb{C} | 0 < |t| < \delta\} \). Milnor has shown that if \( \epsilon \) and \( \delta \ll \epsilon \) are sufficiently small, then \( f' \) is a smooth locally trivial fibration; the diffeomorphism type of which only depends on the germ of \( f \) at 0 (see [14]). Then the Milnor fiber \( M_{f,0} \) is defined as any of its fibers \( M_t \) at \( t \in \Delta' \). Let \( S^1 \subset \Delta \) be a circle and \( \theta \) be the angle on \( S^1 \). Any lifting to \( f'^{-1} S^1 \) of associated vector field \( \partial/\partial \theta \) on \( S^1 \) integrates to a flow and induces the *geometric monodromy* of the fibration

\[ h : M_{f,0} \to M_{f,0}, \]

which is well defined up to isotopy (see [17]-C.2.2).
2.1.3 Hodge Spectrum

The geometric monodromy $h$ induces the monodromy action $T$ on the cohomology groups $H^*(M_{f,0}, \mathbb{C})$. It can be explained by means of the Gauss-Manin connection on the vector bundle

$$H = \bigcup_{t \in \Delta'} H^*(M_t, \mathbb{C}) \to \Delta'.$$

By the monodromy theorem, the eigenvalues $\lambda$ of the monodromy action $T$ on $H^*(M_{f,0}, \mathbb{C})$ are roots of unity (see [13]-I.9.2). Thus, if $T = T_s T_u$ is a Jordan decomposition, where $T_s$ and $T_u$ are semi-simple and unipotent parts of the monodromy, then there exists $m \in \mathbb{Z}$ such that $T_s^m = 1$. The cohomology groups $H^*(M_{f,0}, \mathbb{C})$ carry canonical Hodge structures such that the semi-simple part $T_s$ of the monodromy acts as an automorphism of finite order of these Hodge structures (see [17]-12.1.2 and [13]-II (3.4.4)).

**Definition 2.1.5** We define the spectrum multiplicity of $f$ at $\alpha \in \mathbb{Q}$ to be

$$n_{f,\alpha} = \sum_{j \in \mathbb{Z}} (-1)^{j-n+1} \dim Gr^{p}_{F} \tilde{H}^j(M_{f,0}, \mathbb{C})_{\lambda}$$

with $p = \lfloor n - \alpha \rfloor$, $\lambda = \exp(-2\pi i \alpha)$,

where $\tilde{H}^j(M_{f,0}, \mathbb{C})_{\lambda}$ is the $\lambda$-eigenspace of the reduced cohomology under $T_s$ and $F$ is the Hodge filtration.

It is known that $n_{f,\alpha} = 0$ for $\alpha \notin (0, n)$ (see [4]).

**Definition 2.1.6** The Hodge spectrum of the germ $f$ is the fractional Laurent polynomial

$$Sp(f) := \sum_{\alpha \in \mathbb{Q}} n_{f,\alpha} t^\alpha.$$
2.1.4 Spectrum of homogeneous polynomials

Assume that $f$ is homogeneous with degree $d$. Then we can consider the divisor $Z \subseteq \mathbb{P}^{n-1} =: Y$ defined by $f$. Let $\rho : \tilde{Y} \to Y$ be an embedded resolution of $Z$ inducing an isomorphism over $Y \setminus Z$. We have a divisor $\tilde{Z} := \rho^*Z$ with normal crossing on $\tilde{Y}$. Let $\tilde{H}$ be the total transform of general hyperplane $H$ of $Y$. Then the eigenvalues of the monodromy are $d$-th roots of unity (see [1]-4) and the Hodge filtration on the cohomology coincides with the Hodge filtration obtained from the theory of mixed Hodge modules (see [19]-3.11). Thus, we have the following formula for spectrum multiplicity (see [5]-1.5). Note that the formula holds only on $\alpha \in (0,n)$ by [4].

**Proposition 2.1.7** For $\alpha = n - p - \frac{i}{d} \in (0,n)$ with $p \in \mathbb{Z}$ and $i \in [0,d-1] \cap \mathbb{Z}$

$$n_{f,\alpha} = (-1)^{p-n+1} \chi \left( \tilde{Y}, \Omega^p_Y(\log \tilde{Z}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{\tilde{Y}} \left( -i\tilde{H} + \sum_{j \in J} \left\lfloor \frac{m_V}{d} \right\rfloor E^V \right) \right), \quad (2.1)$$

where $\lfloor \cdot \rfloor$ is round down.

2.2 Intersection Theory


2.2.1 Chern Class

Chern classes are invariants of a complex vector bundle $E$ over a variety $X$ (see [10]-3.2).

**Definition 2.2.1** The $k$-th Chern class $c_k(E)$ of $E$ is an element of $H^{2k}(X, \mathbb{Z})$ and
the total Chern class $c(E)$ is the sum of $k$-th Chern classes

$$c(E) = \sum_k c_k(E),$$

which satisfies the following axioms:

Axiom 1. $c_0(E) = 1$ for all $E$.

Axiom 2. (Functoriality) If $g : Y \to X$ is continuous, then $c_k(g^* E) = g^*(c_k(E))$.

Axiom 3. (Whitney sum) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of complex vector bundles, then $c(E) = c(E') c(E'')$, where the product between chern classes is the cup product on $H^*(X, \mathbb{Z})$.

Axiom 4. (Normalization) If $E$ is a line bundle on a variety $X$, and $D$ a Cartier divisor on $X$ with $\mathcal{O}(D) \cong E$, then $c_1(E) \cap [X] = D$, where $\cap$ is the cap product inducing Poincaré dual.

For convenience we define the Chern polynomial as

$$c_t(E) = \sum_k c_k(E) t^k.$$

If the vector bundle $E$ has rank $r$, $c_k(E) = 0$ except when $k = 0, \cdots, r$ from the definition. Thus, the Chern polynomial $c_t(E)$ can be written as

$$c_t(E) = \prod_{i=1}^r (1 + x_i t),$$

where $x_i$ are formal symbols. We will state Chern class formulas for vector bundle operations. The formal symbols make the formulas easy to state (see [10] Remark 3.2.3).

**Proposition 2.2.2** Let $E$ and $F$ be the vector bundle with rank $r$ and $s$. Write

$$c_t(E) = \prod_{i=1}^r (1 + x_i t) \text{ and } c_t(F) = \prod_{j=1}^s (1 + y_j t).$$
Then we have

\[ c_t(E \otimes F) = \prod_{i,j} (1 + (x_i + y_j)t), \]

\[ c_t(\wedge^p E) = \prod_{1 \leq i_1 < \cdots < i_p \leq r} (1 + (x_{i_1} + \cdots + x_{i_p})t), \quad \text{and} \]

\[ c_t(E^\vee) = c_{-t}(E), \]

where \( E^\vee \) is the dual bundle of \( E \).

Notice that each coefficient of \( t^k \) is a polynomial of Chern classes.

2.2.2 The Hirzebruch-Riemann-Roch Theorem

The Hirzebruch-Riemann-Roch formula calculates the Euler characteristic in terms of Chern character (see [10] Example 3.2.3) and Todd class (see [10] Example 3.2.4).

**Definition 2.2.3** Let \( E \) be the vector bundle over \( X \) with rank \( r \) and write

\[ c_t(E) = \prod_{i=1}^{r} (1 + x_i t). \]

Then we define the Chern character of \( E \)

\[ \operatorname{ch}(E) = \sum_{i=1}^{r} e^{x_i} = \sum_{i=1}^{r} \sum_{j=0}^{\infty} \frac{1}{j!} x_i^j, \]

and the Todd class of \( E \)

\[ \operatorname{td}(E) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} = \prod_{i=1}^{r} \left( 1 + \frac{1}{2} x_i + \frac{1}{12} x_i^2 - \frac{1}{720} x_i^3 + \cdots \right). \]
Notice that $ch(E) \in H^*(X, \mathbb{Q})$ and $td(E) \in H^*(X, \mathbb{Q})$. Moreover, they can be written as follows,

$$ch(E) = r + c_1(E) + \frac{1}{2} (c_1(E)^2 - 2c_2(E)) + \frac{1}{6} (c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \cdots$$

(2.2)

and

$$td(E) = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1(E)^2 + c_2(E)) + \frac{1}{24} (c_1(E)c_2(E)) + \cdots.$$  

(2.3)

From now we will denote $c(X) := c(TX)$, $ch(X) := ch(TX)$, and $td(X) := td(TX)$ for the tangent bundle $TX$ of a variety $X$. We now state the Hirzebruch-Riemann-Roch theorem (see [10] Corollary 15.2.1).

**Theorem 2.2.4** For a vector bundle $E$ of finite rank over a nonsingular projective variety $X$ of dimension $n$,

$$\chi(E) = (ch(E) \cdot td(X))_n,$$

where $(\cdot)_n$ is the component in $H^{2n}(X, \mathbb{Q})$.

We can apply this theorem to (2.1) to calculate $n_{f,\alpha}$.

**Corollary 2.2.5** Let $\mathcal{E}_{i,p} := \Omega^p_Y(\log Z) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \left( -i\bar{H} + \sum_{j \in J} \lfloor \text{im} V / d \rfloor E_V \right)$. Then

$$n_{f,\alpha} = (-1)^{p-n+1} (ch(\mathcal{E}_{i,p}) \cdot td(Y))_{n-1}.$$  

(2.4)

2.2.3 Calculation of Chern character

The Chern character has better properties than the Chern class (see [10] Example 3.2.3).
Proposition 2.2.6 Let \( E \) and \( F \) be the vector bundles with finite ranks. Then,

\[
ch(E \oplus F) = ch(E) + ch(F) \quad \text{and} \quad ch(E \otimes F) = ch(E)ch(F).
\]

This property decomposes \( ch(\mathcal{E}_{i,p}) \).

Corollary 2.2.7 Let \( A := \Omega^1_Y(\log \tilde{Z}) \) and \( U_i := \mathcal{O}_Y \left( -i\tilde{H} + \sum_{j \in J} |im_{i'}d|E_{i'} \right) \). Then,

\[
ch(\mathcal{E}_{i,p}) = ch(\wedge^p A) \cdot ch(U_i).
\]

Thus, we have

\[
n_{f,\alpha} = (-1)^{p-n+1}(ch(\wedge^p A) \cdot ch(U_i) \cdot td(\tilde{Y}))_{n-1}. \quad (2.5)
\]

Chen classes calculate Chern character and Todd class (see (2.2) and (2.3)). Hence, we need to calculate \( c(\wedge^p A) \) to get \( ch(\wedge^p A) \). It can be calculated from Proposition 2.2.2. We give an example that we will use for our case.

Example 2.2.8 Let \( A \) be a vector bundle with the rank 3. Then:

\[
c(\wedge^0 A) = 1,
\]

\[
c(\wedge^1 A) = 1 + c_1(A) + c_2(A) + c_3(A),
\]

\[
c(\wedge^2 A) = 1 + 2c_1(A) + (c_1(A)^2 + c_2(A)) + (c_1(A)c_2(A) - c_3(A)), \quad \text{and}
\]

\[
c(\wedge^3 A) = 1 + c_1(A).
\]

Applying (2.2) we get:

\[
ch(\wedge^0 A) = 1, \quad (2.6)
\]

\[
ch(\wedge^1 A) = 3 + c_1(A) + \frac{1}{2} \left( c_1(A)^2 - 2c_2(A) \right) + \frac{1}{6} \left( c_1(A)^3 - 3c_1(A)c_2(A) + 3c_3(A) \right),
\]

\[
ch(\wedge^2 A) = 3 + 2c_1(A) + (c_1(A)^2 - c_2(A)) + \frac{1}{6} \left( 2c_1(A)^3 - 3c_1(A)c_2(A) - 3c_3(A) \right), \quad \text{and}
\]

\[
ch(\wedge^3 A) = 1 + c_1(A) + \frac{1}{2} c_1(A)^2 + \frac{1}{6} c_1(A)^3.
\]
For the calculation of $c(A) = c(\Omega^1_Y(\log \tilde{Z}))$, the following short exact sequence is useful

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log \tilde{Z}) \to \bigoplus_{j \in J} \mathcal{O}_{\tilde{Y}}(E_V) \to 0.$$

This induces

$$c(\Omega^1_Y(\log \tilde{Z})) = c(\Omega^1_Y) \prod_{j \in J} c(\mathcal{O}_{\tilde{Y}}(E_V)).$$

(2.7)

from the Axiom 3 in the definition of Chern class.

In order to finish this section, we should construct $\tilde{Y}$ explicitly to calculate $n_{f,\alpha}$ in the homogeneous cases.

2.3 Spectrum of Hyperplane arrangements

Let $D$ be a hyperplane arrangement defined by $f : \mathbb{C}^n \to \mathbb{C}$ with $D_l (l \in \Lambda)$ the irreducible components of $D$. We say that $D$ is central if all the $D_l$ pass through the origin. $D$ is essential if $\cap_l D_l = \{0\}$. Assume that $D$ is central and essential. Hence, $f$ is homogeneous so that we can apply Corollary 2.2.5. We define the intersection lattice $S(D)$ as

$$S(D) = \{\cap_{l \in I} D_l \}_{I \subset \Lambda, I \neq \emptyset}.$$

Each element in this set is called an edge. For $V \in S(D)$, define $\gamma(V) := \text{codim}_{\mathbb{C}^n} V$. An edge is dense if the subarrangement of hyperplanes containing it is indecomposable (see the definition of decomposable in the introduction). Let $S(D)^{dense}$ be the set of dense edges. Set

$$S(D)^{dense \geq 2} = \{V \in S(D)^{dense} | \gamma(V) \geq 2\}.$$

Let $D^{nnc} \subset D$ denote the complement of the subset consisting of normal crossing singularities. Set

$$S(D)^{nnc} = \{V \in S(D) | V \subset D^{nnc}\}.$$
For a set $S$ of edges, let
\[ S^{(k)} = \{ V \in S | \gamma(V) = k \}. \]

Notice that $S(D)^{nnc}$ includes $S(D)^{dense \geq 2}$ and they coincide at codimension 2.

2.3.1 Construction of $\tilde{Y}$

We construct $\tilde{Y}$ using successive blow-ups (see [5]-2 and [6]-2). Let $Y_0 = Y = \mathbb{P}^{n-1}$. For a vector space $V \subset X = \mathbb{C}^n$, its corresponding subspace of $Y$ will be denoted by $\mathbb{P}(V)$. Let $S$ be any of $S(D), S(D)^{dense}$, and $S(D)^{nnc}$. There is a sequence of blow-ups $\rho_i : Y_{i+1} \to Y_i$ for $0 \leq i < n-2$ whose center is the disjoint union of the proper transforms of $\mathbb{P}(V)$ for $V \in S$ with $\dim \mathbb{P}(V) = i$. Set $\tilde{Y} = Y_{n-2}$ with $\rho : \tilde{Y} \to Y$ the composition of the $\rho_i$. This is the canonical log resolution of $(\mathbb{P}^{n-1}, Z)$ from [7]-4 where $Z$ is the divisor defined by $f$.

Note that $S = S(D)^{dense}$ gives the minimal log resolution (see [6]-2) but is not stable under intersection (i.e. $V \cap V' \in S$ if $V, V' \in S$).

2.3.2 Cohomology of $\tilde{Y}$

Let $S := S(D)^{nnc}$. By [5]-5.3 (see also [7]-5) the cohomology ring of $\tilde{Y}$ is described as
\[ \mathbb{Q}[e_V]_{V \in S}/I_S \xrightarrow{\cdot} H^*(\tilde{Y}, \mathbb{Q}) \] (2.8)

sending $e_V$ to $[E_V]$ for $V \neq 0$ and $e_0$ to $-[E_0]$, where $e_V$ are independent variables for $V \in S$ and $E_0$ is the total transform of a general hyperplane which was denoted by $\tilde{H}$. Moreover, the ideal $I_S$ is generated by

\[ R_{V,W} = \begin{cases} e_V e_W & \text{if } V, W \text{ are incomparable}, \\ e_V \tilde{e}_W^{\gamma(W) - \gamma(V)} & \text{if } W \subsetneq V, \end{cases} \] (2.9)

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where $\tilde{e}_W := \sum_{W' \subset W} e_{W'}$. Here $V, W, W' \in S \cup \{C^n\}$ and $e_{C^n} = 1$.

The stability of $S(D)^\text{mc}$ under intersections was used in [3] to get (2.9) from [7] by observing that the nested condition from [7] becomes always linearly ordered by the inclusion relation. From now we use $e_V$ and $e_0$ for $[E_V]$ and $-[E_0]$ in $H^\bullet(\tilde{Y}, \mathbb{Q})$.

2.3.3 Calculation of $c(\tilde{Y})$

$\tilde{Y}$ was constructed by the successive blow-ups. By [2] (see also [10]-Example 15.4.2), we have a formula for the Chern class of $\tilde{Y}$, $c(\tilde{Y}) = \prod_{V \in S} F_V$, where

$$F_V = \begin{cases}
(1 + e_V - \tilde{e}_V)^{-\gamma(V)} (1 + e_V) (1 - \tilde{e}_V)^{\gamma(V)} & \text{if } V \neq 0, \\
(1 - e_0)^n & \text{if } V = 0.
\end{cases}$$

(2.10)

2.3.4 Duality on $\tilde{Y}$

Let $U$ be a divisor on $\tilde{Y}$. $u := [U]$ can be written as $u = u_0 e_0 + \sum_{V \in S} u_V e_V \in H^2(\tilde{Y})$ where $u_0, u_V \in \mathbb{Z}$. Set $\mathcal{F}_p(U) := \Omega^p_Y(\log \tilde{Z}) \otimes \mathcal{O}_Y(U)$ and consider a function $\mu_p : H^2(\tilde{Y}) \rightarrow \mathbb{Z}$ defined by $\mu_p(u) := (-1)^{p-n+1} \chi(\tilde{Y}, \mathcal{F}_p(U))$ for each $p \in \mathbb{Z}$. By Proposition 2.1.7 $n_{f,n-p-\frac{i}{n}} = \mu_p(ie_0 + \sum_{V \in S} [im_V/d]e_V)$. Using Serre duality, we get the following property.

**Proposition 2.3.1** Assume $f$ is a reduced hyperplane arrangement of degree $d$. Then we have the following for $i \in \{0, \cdots, d-1\}$ and $p \in \{0, \cdots, n-1\}$

$$n_{f,p+1-\frac{i}{d}} = \mu_p \left( (d-i)e_0 + \sum_{V \in S} (m_V - 1 - [im_V/d])e_V \right).$$

**Proof.** By Serre duality we have $H^q(\tilde{Y}, \mathcal{F}_p(U)) \cong H^{n-1-q}(\tilde{Y}, \mathcal{F}_{n-1-p}(-\tilde{Z}_{\text{red}} - U))^\vee$ using $\Omega^p_Y(\log \tilde{Z}) = \Omega^{n-1-p}(\log \tilde{Z})^\vee \otimes \omega_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{Y}}(\tilde{Z}_{\text{red}})$ (see [3]-6.8 (b)). Let $z \in H^2(\tilde{Y})$.
be the corresponding element to $\tilde{Z}_{\text{red}}$. From this duality

$$\mu_p(u) = \mu_{n-1-p}(-z - u). \tag{2.11}$$

Let $d_l (l \in \Lambda) \in H^2(\tilde{Y})$ be the corresponding element to the strict transform of an irreducible component $D_l$ in the hyperplane arrangement $D = \bigcup_{l \in \Lambda} D_l$. We have relations $d_l = -(e_0 + \sum_{V \subset D_l} e_V)$ and $z = \sum_{V \in S} e_V + \sum_{l \in \Lambda} d_l$. Thus,

$$z = \sum_{V \in S} e_V + \sum_{l \in \Lambda} d_l$$
$$= \sum_{V \in S} e_V - \sum_{l \in \Lambda} \left( e_0 + \sum_{V \subset D_l} e_V \right)$$
$$= \sum_{V \in S} e_V - \left( \sum_{V \in S} m_V e_V + de_0 \right)$$
$$= \sum_{V \in S} (1 - m_V) e_V - de_0.$$

Plugging into (2.11) we have

$$\mu_p(u_0 e_0 + \sum_{V \in S} u_V e_V) = \mu_{n-1-p}((d - u_0) e_0 + \sum_{V \in S} (m_V - 1 - u_V) e_V).$$

This equality means

$$n_{f,p+1-\frac{i}{d}} = n_{f,n-(n-1-p)-\frac{i}{d}} = \mu_p \left( (d - i) e_0 + \sum_{V \in S} (m_V - 1 - \lfloor im_V/d \rfloor) e_V \right).$$

\qed
3.1 Proof of Theorem 1.2.1

Recall that $e_V$ and $e_0$ are the generators of $H^\bullet(\tilde{Y}, \mathbb{Q})$. Let $a_V$, $b_W$ and $c$ denote the $e_V$ for $V \in S^{(2)}$, $e_W$ for $W \in S^{(3)}$ and $e_0$ respectively (see 2.3.2).

3.1.1 Calculation of $H^\bullet(\tilde{Y})$

According to Section 2.3.2 we have generators $a_V$, $b_W$ and $c$ and the following vanish in $H^\bullet(\tilde{Y}, \mathbb{Q})$ by (2.9):

$$ a_Va_{V'} (V \neq V'), \ b_Wb_{W'} (W \neq W'), \ a_Vb_W (W \not\subset V), $$

$$ a_V(b_W + c) (W \subset V), \ a_Vc^2, b_Wc, $$

$$ (a_V + \sum_{W \subset V} b_W + c)^2, \ (b_W + c)^3, \text{ and } c^4. $$

Hence, we have the following relations from the above:

$$ a_Va_{V'} = b_Wb_{W'} = b_Wc = a_Vb_W = 0 \ (V \neq V', W \neq W', W \not\subset V), \quad (3.1) $$

$$ a_Vb_W^2 = a_Vc^2 = b_W^2c = b_Wc^2 = a_Vb_Wc = 0, $$

$$ a_V^3 = 2 \left( 1 - \sum_{W \subset V} 1 \right) c^3, \ a_V^2c = b_W^3 = -c^3, c^4 = 0, $$

$$ a_Vb_W = -a_Vc \ (W \subset V), \text{ and } a_V^2b_W = c^3 \ (W \subset V). $$
From relations (3.1) we get:

\[
\left( \sum_{V \in S^{(2)}} N_V a_V^s \right) \left( \sum_{V \in S^{(2)}} N'_V a_V^{s'} \right) = \sum_{V \in S^{(2)}} N_V N'_V a_V^{s+s'},
\]

\[
\left( \sum_{W \in S^{(3)}} N_W b_W^s \right) \left( \sum_{W \in S^{(3)}} N'_W b_W^{s'} \right) = \sum_{W \in S^{(3)}} N_W N'_W b_W^{s+s'},
\]

and

\[
\left( \sum_{V \in S^{(2)}} N_V a_V^s \right) \left( \sum_{W \in S^{(3)}} N'_W b_W^{s'} \right) = \sum_{V \in S^{(2)}} \sum_{W \subset V} N_V N'_W a_V^s b_W^{s'},
\]

for any coefficients \( N_V, N'_V, N_W, N'_W \) and any positive integers \( s \) and \( s' \). These equalities are very useful for the calculations.

### 3.1.2 Calculation of Chern classes

According to Section 2.3.3, we have

\[
c(\tilde{Y}) = (1 - c)^4 \prod_{W \in S^{(3)}} \left( 1 + b_W \left( \frac{1 - c - b_W}{1 - c} \right)^3 \right) 
\cdot \prod_{V \in S^{(2)}} \left( 1 + a_V \left( \frac{1 - c - \sum_{W \subset V} b_W - a_V}{1 - c - \sum_{W \subset V} b_W} \right)^2 \right).
\]

Since \( c(\tilde{Y}) = c((\Omega^1_Y)^\vee) \), it is enough to change the signs of all the generators in the formula above for the calculation of \( c(\Omega^1_Y) \)

\[
c(\Omega^1_Y) = (1 + c)^4 \prod_{W \in S^{(3)}} \left( 1 - b_W \left( \frac{1 + c + b_W}{1 + c} \right)^3 \right) 
\cdot \prod_{V \in S^{(2)}} \left( 1 - a_V \left( \frac{1 + c + \sum_{W \subset V} b_W + a_V}{1 + c + \sum_{W \subset V} b_W} \right)^2 \right).
\]
From equation (2.7), we have

\[
c(\Omega^1_Y(\log \tilde{Z})) = c(\Omega^1_Y) \prod_{W \in S^{(3)}} \frac{1}{1 - b_W} \prod_{V \in S^{(2)}} \frac{1}{1 - a_V} \prod_{l \in \Lambda} \frac{1}{1 - d_l},
\]

where \(d_l (l \in \Lambda)\) correspond to the strict transform of an irreducible component \(D_l\) of hyperplane arrangement \(D = \cup_{l \in \Lambda} D_l\). Recall \(d_l = -(c + \sum_{W \subset D_l} b_W + \sum_{V \subset D_l} a_V)\).

From this relation,

\[
c(\Omega^1_Y(\log \tilde{Z})) = (1 + c)^4 \prod_{W \in S^{(3)}} \left(\frac{1 + c + b_W}{1 + c}\right)^3 \prod_{V \in S^{(2)}} \left(\frac{1 + c + \sum_{W \subset V} b_W + a_V}{1 + c + \sum_{W \subset V} b_W}\right)^2 \prod_{l \in \Lambda} \frac{1}{1 + c + \sum_{W \subset D_l} b_W + \sum_{V \subset D_l} a_V}.
\]

Notice that the Chern classes above \(c(\tilde{Y}), c(\Omega^1_Y)\) and \(c(\Omega^1_Y(\log \tilde{Z}))\) are calculated from the following factors:

\[
\prod_{V \in S^{(2)}} (1 - a_V), \prod_{W \in S^{(3)}} (1 - b_W), (1 + c)^4, \prod_{W \in S^{(4)}} \left(\frac{1 + c + b_W}{1 + c}\right)^3, \prod_{V \in S^{(2)}} \left(\frac{1 + c + \sum_{W \subset V} b_W + a_V}{1 + c + \sum_{W \subset V} b_W}\right)^2, \text{ and } \prod_{l \in \Lambda} \frac{1}{1 - d_l}.
\]

We calculate each factor using the relations (3.1):

\[
\prod_{V \in S^{(2)}} (1 - a_V) = 1 + \sum_{V \in S^{(2)}} (-a_V),
\]

\[
\prod_{W \in S^{(3)}} (1 - b_W) = 1 + \sum_{W \in S^{(3)}} (-b_W),
\]

\[
(1 + c)^4 = 1 + 4c + 6c^2 + 4c^3,
\]
\[
\prod_{W \in S^{(3)}} \left( \frac{1 + c + b_W}{1 + c} \right)^3 = 1 + 3 \sum_{W \in S^{(3)}} b_W + 3 \sum_{W \in S^{(3)}} b_W^2 + \sum_{W \in S^{(3)}} b_W^3,
\]

\[
\prod_{V \in S^{(2)}} \left( \frac{1 + c + \sum_{W \subset V} b_W + a_V}{1 + c + \sum_{W \subset V} b_W} \right)^2 = 1 + 2 \sum_{V \in S^{(2)}} a_V
\]

\[
+ \sum_{V \in S^{(2)}} \left( a_V^2 - 2(1 - \sum_{W \subset V} 1)a_Vc \right) + \sum_{V \in S^{(2)}} a_V^3,
\]

and

\[
\prod_{l \in \Lambda} \frac{1}{1 - d_l} = \prod_{l \in \Lambda} \frac{1}{1 + c + \sum_{W \subset D_l} b_W + \sum_{V \subset D_l} a_V}
\]

\[
= 1 - \left( \sum_{V \in S^{(2)}} m_V a_V + \sum_{W \in S^{(3)}} m_W b_W + d c \right)
\]

\[
+ \left( \sum_{V \in S^{(2)}} \left( \binom{m_V + 1}{2} \right) a_V^2 + \left( d + 1 - \sum_{W \subset V} (m_W + 1) \right) m_V a_V c \right)
\]

\[
+ \sum_{W \in S^{(3)}} \left( \binom{m_W + 1}{2} b_W^2 + \binom{d + 1}{2} c^2 \right)
\]

\[
- \left( \sum_{V \in S^{(2)}} \left( 2 \binom{m_V + 1}{3} - d \binom{m_V + 1}{2} \right) - \sum_{W \in S^{(3)}} \left( \binom{m_W + 2}{3} + \binom{d + 2}{3} \right) - \sum_{V \in S^{(2)}} \sum_{W \subset V} \left( 2 \binom{m_V + 1}{3} - m_W \binom{m_V + 1}{2} \right) \right) c^3,
\]

where \(m_V = \sum_{V \subset D_l} 1\) and \(m_W = \sum_{W \subset D_l} 1\). We need some work for \(\prod_{l \in \Lambda} \frac{1}{1 - d_l}\) (see Section 3.1.3).
From these factors we give formulas for the Chern classes:

\[ c(\Omega Y) = 1 + \left( \sum_{V \in S^{(2)}} a_V + \sum_{W \in S^{(3)}} 2b_W + 4c \right) + \left( \sum_{V \in S^{(2)}} (-a_V^2 + 2a_VC) + 6c^2 \right) \]

\[ + \left( \sum_{V \in S^{(2)}} 2 + \sum_{W \in S^{(3)}} 2 + 4 \right) c^3, \]

\[ c(\tilde{Y}) = 1 - \left( \sum_{V \in S^{(2)}} a_V + \sum_{W \in S^{(3)}} 2b_W + 4c \right) + \left( \sum_{V \in S^{(2)}} (-a_V^2 + 2a_VC) + 6c^2 \right) \]

\[ - \left( \sum_{V \in S^{(2)}} 2 + \sum_{W \in S^{(3)}} 2 + 4 \right) c^3, \]

and

\[ c(\Omega Y (\log \tilde{Z})) = 1 - \left( \sum_{V \in S^{(2)}} (m_V - 2)a_V + \sum_{W \in S^{(3)}} (m_W - 3)b_W + (d - 4)c \right) - \]

\[ + \left( \sum_{V \in S^{(2)}} \left( \binom{m_V - 1}{2} a_V^2 + (m_V - 2) \left( d - 3 - \sum_{W \subset V} (m_W - 2) \right) a_V c \right) \right) \]

\[ + \sum_{W \in S^{(3)}} \left( \binom{m_W - 2}{2} b_W^2 + \left( \frac{d - 3}{2} \right) c^2 \right) \]

\[ - \left( \sum_{V \in S^{(2)}} \left( 2 \binom{m_V - 1}{3} - (d - 4) \binom{m_V - 1}{2} \right) - \sum_{W \in S^{(3)}} \binom{m_W - 1}{3} \right) \]

\[ + \left( \frac{d - 2}{3} - \sum_{V \in S^{(2)}} \sum_{W \subset V} \left( 2 \binom{m_V - 1}{3} - (m_W - 3) \binom{m_V - 1}{2} \right) \right) c^3. \]
3.1.3 Calculation of $\prod_{l \in \Lambda} \frac{1}{1-d_l}$

We calculate $\prod_{l \in \Lambda} (1 - d_l)$ and then invert it. After fixing a total order on $\Lambda,$

$$\prod_{l \in \Lambda} (1 - d_l) = 1 + \sum_l (-d_l) + \sum_{l < l'} d_l d_{l'} + \sum_{l < l' < l''} (-d_l d_{l'} d_{l''}).$$

Since we have the relation $d_l = -\left( c + \sum_{W \subset D_l} b_W + \sum_{V \subset D_l} a_V \right),$ we get the following results (see Section 3.1.3.1 for $\sum_{l < l'} d_l d_{l'}$ and Section 3.1.3.2 for $\sum_{l < l' < l''} (-d_l d_{l'} d_{l''})$):

$$\sum_l (-d_l) = \sum_{V \in S^{(2)}} m_V a_V + \sum_{W \in S^{(3)}} m_W b_W + dc,$$

$$\sum_{l < l'} d_l d_{l'} = \sum_{V \in S^{(2)}} \left( \binom{m_V}{2} d_V^2 + \left( \binom{d}{2} - \sum_{W \subset V} \binom{m_W - 1}{2} \right) m_V a_V c \right)$$

$$\sum_{l < l' < l''} (-d_l d_{l'} d_{l''}) = \sum_{V \in S^{(2)}} \left( \binom{m_V}{3} - (d - 2) \binom{m_V}{2} \right) - \sum_{W \in S^{(3)}} \binom{m_W}{3} + \binom{d}{3}$$

$$- \sum_{V \in S^{(2)}} \left( \sum_{W \subset V} \binom{m_W}{3} - (m_W - 2) \binom{m_V}{2} \right) c^3. \quad (3.4)$$

In terms of $\sum_l (-d_l), \sum_{l < l'} d_l d_{l'}$ and $\sum_{l < l' < l''} (-d_l d_{l'} d_{l''}),$

$$\prod_{l \in \Lambda} \frac{1}{1-d_l} = 1 - \left( \sum_l (-d_l) \right) + \left( \sum_l (-d_l) \right)^2 - \sum_{l < l'} d_l d_{l'}$$

$$- \left( \sum_l (-d_l) \right)^3 - 2 \left( \sum_l (-d_l) \right) \left( \sum_{l < l'} d_l d_{l'} \right) + \sum_{l < l' < l''} (-d_l d_{l'} d_{l''}).$$

Using the equalities from (3.2), we get the previous result (3.3).
3.1.3.1 Calculation of $\sum_{l<l'} dd'$

\[
\sum_{l<l'} dd' = \sum_{l<l'} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 + \sum_{W \subset D_l, D_{l'}} b_W^2 + c^2 + \sum_{V \subset D_l} a_V c + \sum_{V \subset D_{l'}} a_V c \right) \\
+ \sum_{V \subset D_l} a_V b_W + \sum_{W \subset D_l} a_V b_W + \sum_{W \subset D_{l'}} b_W c + \sum_{W \subset D_{l'}} b_W c \right) .
\]

Consider a term $a_V^2$ in

\[
\sum_{l<l'} \sum_{V \subset D_l} a_V^2.
\]

The coefficient of $a_V^2$ is the number of ways of choosing two hyperplanes $D_l$ and $D_{l'}$ out of the hyperplanes containing $V$. Similarly for the fixed $b_W^2$ and $c^2$, their coefficients are the number of ways of choosing two hyperplanes $D_l$ and $D_{l'}$ out of the hyperplanes containing $W$ and 0 respectively. Hence, we get

\[
\sum_{l<l'} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 + \sum_{W \subset D_l, D_{l'}} b_W^2 + c^2 \right) = \sum_{V \in S^{(2)}} \binom{m_V}{2} a_V^2 + \sum_{W \in S^{(3)}} \binom{m_W}{2} b_W^2 + \binom{d}{2} c^2.
\]

Consider a term $a_V c$ in

\[
\sum_{l<l'} \left( \sum_{V \subset D_l} a_V c + \sum_{V \subset D_{l'}} a_V c \right) .
\]

We have two possibilities. Both two hyperplanes $D_l$ and $D_{l'}$ contain $V$ or only one hyperplane $D_l$ or $D_{l'}$ contains $V$. For the first possibility both terms

\[
\sum_{V \subset D_l} a_V c \quad \text{and} \quad \sum_{V \subset D_{l'}} a_V c
\]
survive. We get \( \binom{m_V}{2} \) such cases. For the other possibility only one term survives and we have \( m_V(d - m_V) \) such cases. Thus, we have

\[
\sum_{l < l'} \left( \sum_{V \subset D_l} a_V c + \sum_{V \subset D_{l'}} a_V c \right) = \sum_{V \in \mathcal{S}^{(2)}} \left( 2 \binom{m_V}{2} + m_V(d - m_V) \right) a_V c
\]

\[
= \sum_{V \in \mathcal{S}^{(2)}} (d - 1)m_V a_V c.
\]

Consider a term \( a_V b_W \) in

\[
\sum_{l < l'} \left( \sum_{V \subset D_l \atop W \subset D_{l'}} a_V b_W + \sum_{V \subset D_{l'} \atop W \subset D_l} a_V b_W \right).
\]

We have two possibilities. Both two hyperplanes \( D_l \) and \( D_{l'} \) contain \( V \) so that both contain \( W \) or only one hyperplane \( D_l \) or \( D_{l'} \) contains \( V \) but the other contains only \( W \). For the first possibility both terms

\[
\sum_{V \subset D_l \atop W \subset D_{l'}} a_V b_W \text{ and } \sum_{V \subset D_{l'} \atop W \subset D_l} a_V b_W
\]

survive and we get \( \binom{m_V}{2} \) such cases. For the other possibility only one term survives and we have \( m_V(m_W - m_V) \) such cases. Hence, we have

\[
\sum_{l < l'} \left( \sum_{V \subset D_l \atop W \subset D_{l'}} a_V b_W + \sum_{V \subset D_{l'} \atop W \subset D_l} a_V b_W \right) = \sum_{V \in \mathcal{S}^{(2)}} \sum_{W \subset V} \left( 2 \binom{m_V}{2} + m_V(m_W - m_V) \right) a_V b_W
\]

\[
= \sum_{V \in \mathcal{S}^{(2)}} \sum_{W \subset V} (m_W - 1)m_V a_V b_W.
\]

The other terms \( b_W c \) for any \( W \) vanish by our relation [3.1]. Therefore, we get
the equation

\[
\sum_{l<l'} d_l d_{l'} = \sum_{V \in S^{(a)}} \binom{m_V}{2} a_V^2 + \left( (d-1) - \sum_{W \subset V} (m_W - 1) \right) m_V a_V c + \sum_{W \in S^{(a)}} \binom{m_W}{2} b_W^2 + \binom{d}{2} c^2.
\]

### 3.1.3.2 Calculation of \( \sum_{l<l'<l''} (-d_l d_{l'} d_{l''}) \)

After cutting out vanishing terms, we get the following equality

\[
\sum_{l<l'<l''} (-d_l d_{l'} d_{l''}) = \sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}, D_{l''}} a_V^3 + \sum_{W \subset D_l, D_{l'}, D_{l''}} b_W^3 + c^3 \right)
\]

\[
+ \sum_{V \subset D_l, D_{l'}} a_V^2 c + \sum_{V \subset D_l, D_{l''}} a_V^2 c + \sum_{V \subset D_{l'}, D_{l''}} a_V^2 c + \sum_{V \subset D_l, D_{l'}, D_{l''}} a_V^2 b_W + \sum_{W \subset D_{l'}, D_{l''}} a_V^2 b_W + \sum_{W \subset D_l, D_{l''}} a_V^2 b_W \right).
\]

Consider a term \( a_V^3 \) in

\[
\sum_{l<l'<l''} \sum_{V \subset D_l, D_{l'}, D_{l''}} a_V^3.
\]

The coefficient of \( a_V^2 \) is the number of ways of choosing three hyperplanes \( D_l, D_{l'} \) and \( D_{l''} \) out of the hyperplanes containing \( V \). Similarly for the fixed \( b_W^3 \) and \( c^3 \), their coefficients are the number of ways of choosing two hyperplanes \( D_l, D_{l'} \) and \( D_{l''} \) out of the hyperplanes containing \( W \) and 0 respectively. Thus, we get
\[
\sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}, D_{l''}} a_V^3 + \sum_{W \subset D_l, D_{l'}, D_{l''}} b_W^3 + c^3 \right) \\
= \sum_{V \in S^{(2)}} \binom{m_V}{3} a_V^3 + \sum_{W \in S^{(3)}} \binom{m_W}{3} b_W^3 + \left( \frac{d}{3} \right) c^3.
\]

Consider a term \(a_V^2 c\) in

\[
\sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 c + \sum_{V \subset D_l, D_{l''}} a_V^2 c + \sum_{V \subset D_{l'}, D_{l''}} a_V^2 c \right).
\]

We have two possibilities. All three hyperplanes \(D_l, D_{l'}, D_{l''}\) contain \(V\) or only two hyperplanes contain \(V\). For the first possibility all three terms

\[
\sum_{V \subset D_l, D_{l'}} a_V^2 c, \sum_{V \subset D_l, D_{l''}} a_V^2 c \text{ and } \sum_{V \subset D_{l'}, D_{l''}} a_V^2 c
\]

survive. We get \(\binom{m_V}{3}\) such cases. For the other possibility only one term survives and we have \(\binom{m_V}{2}(d - m_V)\) such cases. Thus, we have

\[
\sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 c + \sum_{V \subset D_l, D_{l''}} a_V^2 c + \sum_{V \subset D_{l'}, D_{l''}} a_V^2 c \right) \\
= \sum_{V \in S^{(2)}} \left( 3 \binom{m_V}{3} + \binom{m_V}{2} (d - m_V) \right) a_V^2 c \\
= \sum_{V \in S^{(2)}} (d - 2) \binom{m_V}{2} a_V^2 c.
\]
Consider a term $a_V^2 b_W$ in

$$
\sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 b_W + \sum_{V \subset D_l, D_{l''}} a_V^2 b_W + \sum_{V \subset D_{l'}, D_{l''}} a_V^2 b_W \right).
$$

We have two possibilities. All three hyperplanes $D_l$, $D_{l'}$, and $D_{l''}$ contain $V$ so that they contain $W$ or only two hyperplanes contain $V$ but the other contains only $W$. For the first possibility all three terms

$$
\sum_{V \subset D_l, D_{l'}} a_V^2 b_W, \sum_{V \subset D_l, D_{l''}} a_V^2 b_W, \text{ and } \sum_{V \subset D_{l'}, D_{l''}} a_V^2 b_W
$$

survive and we get $\binom{m_V}{3}$ such cases. For the other possibility only one term survives and we have $\binom{m_V}{2}(m_W - m_V)$ such cases. Thus, we have

$$
\sum_{l<l'<l''} \left( \sum_{V \subset D_l, D_{l'}} a_V^2 b_W + \sum_{V \subset D_l, D_{l''}} a_V^2 b_W + \sum_{V \subset D_{l'}, D_{l''}} a_V^2 b_W \right)
$$

$$
= \sum_{V \in S^{(2)} \subset V} \sum_{W \subset V} \left( \binom{m_V}{3} + \binom{m_V}{2}(m_W - m_V) \right) a_V^2 b_W
$$

$$
= \sum_{V \in S^{(2)} \subset V} (m_W - 2) \binom{m_V}{2} a_V b_W.
$$

Using our relation (3.1), we get (3.4).
3.1.4 Calculation of Chern character for line bundle

Let $U$ be a divisor on $\tilde{Y}$. Then its class $u := [U]$ can be written as $u = \sum_{V \in S^{(2)}} u_V a_V + \sum_{W \in S^{(3)}} u_W b_W + u_0 c$. Since $c(O_{\tilde{Y}}(U)) = 1 + u$ we have

$$\text{ch}(O_{\tilde{Y}}(U)) = 1 + \left( \sum_{V \in S^{(2)}} u_V a_V + \sum_{W \in S^{(3)}} u_W b_W + u_0 c \right) + \frac{1}{2} \left( \sum_{V \in S^{(2)}} u_V^2 a_V^2 + 2u_V \left( u_0 - \sum_{W \subset V} u_W \right) a_V c \right) + \sum_{W \in S^{(3)}} u_W^2 b_W^2 + u_0^2 c^2 \right)$$

$$+ \frac{1}{6} \left( \sum_{V \in S^{(2)}} u_V^2 \left( 2u_V - 3u_0 + \sum_{W \subset V} (3u_W - 2u_V) \right) - \sum_{W \in S^{(3)}} u_W^3 + u_0^3 \right) c^3$$

by formula (2.2). Here we used (3.2) to simplify the calculation.

3.1.5 Calculation of $n_{f,\alpha}$

Let $\mu_p(u) = (-1)^{p-n+1}(\text{ch}(\Omega_{\tilde{Y}}^p(\log \tilde{Z})) \cdot \text{ch}(U) \cdot \text{td}(\tilde{Y}))_{n-1}$ as before (see Section 2.3.4). We have calculated $\text{ch}(O_{\tilde{Y}}(U))$ in Section 3.1.4. We will calculate $\text{td}(\tilde{Y})$, $\text{ch}(\Omega_{\tilde{Y}}^p(\log \tilde{Z}))$, and $\mu_p(u)$ for each $p \in \{0, 1, 2, 3\}$. Using (3.2) is crucial for the simplicity of calculation.

From Section 3.1.2 and formula (2.3) we calculate

$$\text{td}(\tilde{Y}) = 1 + \left( - \sum_{V \in S^{(2)}} \frac{a_V}{2} - \sum_{W \in S^{(3)}} b_W - 2c \right)$$

$$+ \sum_{V \in S^{(2)}} \left( \frac{5 - 2 \sum_{W \subset V} 1}{6} a_V c \right) + \sum_{W \in S^{(3)}} b_W^2 \frac{1}{3} + \frac{11c^2}{6} + (-c^3).$$

We can also calculate $\text{ch}(\Omega_{\tilde{Y}}^p(\log \tilde{Z}))$ for $p = 0, 1, 2, 3$ from Section 3.1.2 and formulas (2.6):
\[ \text{ch}(\Omega^0_Y(\log \tilde{Z})) = \text{ch}(\mathcal{O}_Y) = 1, \]

\[ \text{ch}(\Omega^1_Y(\log \tilde{Z})) = 3 - \left( \sum_{V \in S^{(2)}} (m_V - 2)a_V + \sum_{W \in S^{(3)}} (m_W - 3)b_W + (d - 4)c \right) \]

\[ - \frac{1}{2} \left( \sum_{V \in S^{(2)}} (m_V - 2) \left( a_V^2 + 2a_Vc \right) - 2 \sum_{W \subset V} a_Vc \right) + \sum_{W \in S^{(3)}} (m_W - 3)b_W^2 + (d - 4)c^2 \]

\[ + \frac{1}{6} \left( \sum_{V \in S^{(2)}} (m_V - 2) \left( 1 - \sum_{W \subset V} 1 \right) + \sum_{W \in S^{(3)}} (m_W - 3) - (d - 4) \right) c^3, \]

\[ \text{ch}(\Omega^2_Y(\log \tilde{Z})) = 3 - 2 \left( \sum_{V \in S^{(2)}} (m_V - 2)a_V + \sum_{W \in S^{(3)}} (m_W - 3)b_W + (d - 4)c \right) \]

\[ + \left( \sum_{V \in S^{(2)}} \left( \frac{m_V - 2}{2} \right) a_V^2 + (m_V - 2) \left( d - 5 - \sum_{W \subset V} (m_W - 4) \right) a_Vc \right) \]

\[ + \sum_{W \in S^{(3)}} \left( \frac{m_W - 3}{2} \right) b_W^2 + \left( d - 4 \right) \left( \frac{1}{2} \right) c^2 \]

\[ - \frac{1}{6} \left( \sum_{V \in S^{(2)}} (m_V - 2) \left( 3d - 11 - \sum_{W \subset V} (3m_W - 8) \right) \right) \]

\[ + \sum_{W \in S^{(3)}} (3m_W - 8)(m_W - 3) - (3d - 11)(d - 4) \right) c^3, \text{ and} \]

\[ \text{ch}(\Omega^3_Y(\log \tilde{Z})) = 1 - \left( \sum_{V \in S^{(2)}} (m_V - 2)a_V + \sum_{W \in S^{(3)}} (m_W - 3)b_W + (d - 4)c \right) \]

\[ + \frac{1}{2} \left( \sum_{V \in S^{(2)}} (m_V - 2) \left( m_V - 2 \right) a_V^2 + 2 \left( d - 4 - \sum_{W \subset V} (m_W - 3) \right) a_Vc \right) \]

\[ + \sum_{W \in S^{(3)}} (m_W - 3)^2 b_W^2 + (d - 4)^2 c^2 \]

\[ + \frac{1}{6} \left( \sum_{V \in S^{(2)}} (m_V - 2)^2 \left( 3d - 2m_V - 8 \right) + \sum_{W \subset V} (2m_V - 3m_W + 5) \right) \]

\[ + \sum_{W \in S^{(3)}} (m_W - 3)^3 - (d - 4)^3 \right) c^4. \]
To calculate $\mu_p(u)$, we multiply $ch(\mathcal{O}_Y(U))$ in Section 3.1.4, $td(\tilde{Y})$, and $ch(\Omega^p_Y(\log \tilde{Z}))$ above. The result is the following, and again, here we used (3.2) to simplify calculation:

$$
\mu_0(u) = \left(\frac{u_0 - 1}{3}\right) - \sum_{W \in S^{(3)}} \left(\frac{u_W}{3}\right) - \sum_{W \in S^{(2)}} \left(\frac{(u_0 - 3)\left(u_V\right)}{2}\right) - 2\left(\frac{u_V}{3}\right)
$$

$$
= \sum_{V \in S^{(2)}} \sum_{W \subseteq V} \left(2\left(\frac{u_V}{3}\right) - (u_W - 2)\left(\frac{u_V}{2}\right)\right),
$$

$$
\mu_1(u) = (d - u_0 - 1)\left(\frac{u_0 - 1}{2}\right) - \sum_{W \in S^{(3)}} (m_W - u_W - 1)\left(\frac{u_W}{2}\right)
$$

$$
= \sum_{V \in S^{(2)}} \left(u_V(m_V - u_V - 1)(u_0 - 2) + (d - u_0 - 1 - 2(m_V - u_V - 1))\left(\frac{u_V}{2}\right)\right)
$$

$$
+ \sum_{V \in S^{(2)}} \sum_{W \subseteq V} \left(u_V(m_V - u_V - 1)u_V - u_V + (m_V - u_V - 1)\left(\frac{u_V}{2}\right)\right),
$$

$$
\mu_2(u) = (u_0 - 1)\left(\frac{d - u_0 - 1}{2}\right) - \sum_{W \in S^{(3)}} u_W\left(\frac{m_W - u_W - 1}{2}\right)
$$

$$
= \sum_{V \in S^{(2)}} \left((m_V - u_V - 1)u_V(d - u_0 - 2) + (u_0 - 1 - 2u_V)(m_V - u_V - 1)\right)
$$

$$
+ \sum_{V \in S^{(2)}} \sum_{W \subseteq V} \left((m_V - u_V - 1)u_V((m_W - u_W - 1) - (m_V - u_V - 1))
$$

$$+ u_W\left(\frac{m_V - u_V - 1}{2}\right)\right), \text{ and}
$$

$$
\mu_3(u) = \left(\frac{d - u_0 - 1}{3}\right) - \sum_{W \in S^{(3)}} \left(\frac{m_W - u_W - 1}{3}\right)
$$

$$
= \sum_{V \in S^{(2)}} \left((d - u_0 - 3)(m_V - u_V - 1) - 2\left(\frac{m_V - u_V - 1}{3}\right)\right)
$$

$$
- \sum_{V \in S^{(2)}} \sum_{W \subseteq V} \left(2\left(\frac{m_V - u_V - 1}{3}\right) - (m_W - u_W - 3)\left(\frac{m_V - u_V - 1}{2}\right)\right).\]

By formula (2.5), $n_{f,\alpha} = \mu_p\left((ic + \sum_{W \in S^{(3)}} [im_W/d]b_W + \sum_{V \in S^{(2)}} [im_V/d]a_V)\right)$ with $\alpha = 4 - p - i \in (0,4)$. From the relation $m_V - [im_V/d] = \lceil (d - i)m_V/d \rceil$ and substituting $d - i$ for $i$ for $\alpha \in (0, 2]$, we get the formula in Theorem 1.2.1 for

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$S = S(D)^{une}$.

**Remark 3.1.1** Here we calculated all the $\mu_p$ without using Proposition 2.3.1. We can and did use Proposition 2.3.1 to double-check the formulas for $\mu_p$. In other words, we can get the formulas for $n_{f,1-\frac{i}{d}}$ and $n_{f,3-\frac{i}{d}}$ from the formulas for $n_{f,1-\frac{i}{d}}$ and $n_{f,2-\frac{i}{d}}$ respectively and vice versa. Also, all computations were double-checked by computer. The implementation of the symbolic computation is possible because of the relations in (3.2).

### 3.1.6 Non-dense edges

In this section, we will prove that $S$ in Theorem 1.2.1 can be substituted by any set of edges containing all the dense edges with codimension $\geq 2$ by showing vanishing of all the terms which depend on edges in $S\setminus S(D)^{dense\geq 2}$. It implies Theorem 1.2.1.

Fix a set $S$ containing $S(D)^{dense\geq 2}$.

First of all, the edges in $S^{(1)}$ are not used in our formula. The terms $\eta_{0,i,V}$ in $\eta_{0,i}(\langle u_V \rangle_{V \in S})$ and the terms $\eta_{1,i,V}$ in $\eta_{1,i}(\langle u_V, v_V \rangle_{V \in S})$ depending on $V \in S^{(2)}$ are

\[
\eta_{0,i,V} = -\left( (i-3) \left( \frac{u_V}{2} \right) - 2 \left( \frac{u_V}{3} \right) \right) - \sum_{W \subset V, W \in S^{(3)}} \left( 2 \left( \frac{u_{W}}{3} \right) - (u_{W} - 2) \left( \frac{u_{V}}{2} \right) \right) \quad \text{and}
\]

\[
\eta_{1,i,V} = -\left( u_V v_V (i-2) + (d - i - 1 - 2v_V) \left( \frac{u_V}{2} \right) \right) + \sum_{W \subset V, W \in S^{(3)}} \left( u_V v_V (u_W - u_V) + v_W \left( \frac{u_V}{2} \right) \right).
\]

If an edge $V \in S^{(2)}$ is not dense, then $m_V = 2$. Thus, $\eta_{0,i,V} = \eta_{1,i,V} = 0$ because $u_V + v_V = 1$ and $u_V$ is 0 or 1 for any formula of $n_{f,\alpha}$ in Theorem 1.2.1.
The terms $\eta_{0,i,W}$ in $\eta_{0,i}(\langle u_V \rangle_{V \in S})$ and the terms $\eta_{1,i,W}$ in $\eta_{1,i}(\langle u_V, v_V \rangle_{V \in S})$ depending on $W \in S^{(3)}$ are

$$
\eta_{0,i,W} = -\left( \begin{array}{c} u_W \\ 3 \end{array} \right) - \sum_{\substack{V \supset W \\ V \in S^{(2)}}} \left( 2 \left( \begin{array}{c} u_V \\ 3 \end{array} \right) - (u_W - 2) \left( \begin{array}{c} u_V \\ 2 \end{array} \right) \right) \quad \text{and}
$$

$$
\eta_{1,i,W} = -v_W \left( \begin{array}{c} u_W \\ 2 \end{array} \right) + \sum_{\substack{V \supset W \\ V \in S^{(2)}}} \left( u_V v_V (u_W - u_V) + v_W \left( \begin{array}{c} u_V \\ 2 \end{array} \right) \right).
$$

If $V \in S^{(2)}$ in $\eta_{0,i,W}$ and $\eta_{1,i,W}$ (i.e. $V \supset W$) is not dense, the terms depending on $V$ vanish since $m_V = 2$. Thus, we may assume that $V \in S^{(2)}$ in $\eta_{0,i,W}$ and $\eta_{1,i,W}$ are dense edges with codimension 2.

When an edge $W \in S^{(3)}$ is not dense, we have two possibilities: either $W \notin S(D)^{nnc}$ or $W \in S(D)^{nnc} \setminus S(D)^{dense \geq 2}$. If $W \notin S(D)^{nnc}$, then $m_W = 3$ and we do not have dense edge $V \in S^{(2)}$ in $\eta_{0,i,W}$ and $\eta_{1,i,W}$. Also, $m_W = 3$ implies that $u_W + v_W = 2$ and $u_W$ is 0,1 or 2 for any formula of $n_{f,\alpha}$ in Theorem 1.2.1. Hence, $\eta_{0,i,W} = \eta_{1,i,W} = 0$. In the case of $W \in S(D)^{nnc} \setminus S(D)^{dense \geq 2}$, we have exactly one codimension 2 dense edge $V_W \in S^{(2)}$ such that $W \subset V_W$ since the subarrangement of hyperplanes containing $W$ is decomposable. Moreover, $m_W = m_{V_W} + 1$. Thus,

$$
\eta_{0,i,W} = -\left( \begin{array}{c} u_W \\ 3 \end{array} \right) - \left( 2 \left( \begin{array}{c} u_{V_W} \\ 3 \end{array} \right) - (u_W - 2) \left( \begin{array}{c} u_{V_W} \\ 2 \end{array} \right) \right) \quad \text{and}
$$

$$
\eta_{1,i,W} = -v_W \left( \begin{array}{c} u_W \\ 2 \end{array} \right) + \left( u_{V_W} v_{V_W} (u_W - u_{V_W}) + v_W \left( \begin{array}{c} u_{V_W} \\ 2 \end{array} \right) \right).
$$

For any formula of $n_{f,\alpha}$ in Theorem 1.2.1 we have $u_W + v_W = m_W - 1$ and $u_{V_W} + v_{V_W} = m_{V_W} - 1$. From $m_W = m_{V_W} + 1$ we have only two possibilities for each formula of $n_{f,\alpha}$: either $u_W = u_{V_W}$ or $u_W = u_{V_W} + 1$. For the first case, $v_W = m_W - 1 - u_W =$
\[ m_{VW} - u_{VW} = v_{VW} + 1. \] For the second case, \( v_W = m_W - 1 - u_W = m_{VW} - u_{VW} - 1 = v_{VW} \).

Both cases make \( \eta_{0,i,W} = 0 \) and \( \eta_{1,i,W} = 0 \).

Hence, only the terms depending on dense edges can survive. This proves Theorem 1.2.1.

\[ \square \]

3.2 Proof of Corollaries

We need the Thom-Sebastiani formula for the Corollaries 1.2.2 and 1.2.3. Hyperplane arrangements have non-isolated singularities but the formula still holds in our case (see [13]-II (8.10.6)). Here we state a special case.

**Lemma 3.2.1** Assume that \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) can be written as \( f(x_1, \cdots, x_n) = g(x_1, \cdots, x_m) \) for \( g : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0) \). Then \( Sp(f) = (-t)^{n-m} Sp(g) \).

3.2.1 Proof of Corollary 1.2.2

Consider \( f(x_1, x_2, x_3) \) with degree \( d \) in \( \mathbb{C}^3 \). Let \( f(x_1, x_2, x_3) = g(x_1, x_2, x_3, x_4) \). We will calculate \( Sp(g) \). First, we assume that \( S^{(3)} \neq \emptyset \). The hyperplane arrangement \( g \) has only one codimension 3 dense edge \( W = \{x_1 = x_2 = x_3 = 0\} \). Moreover the multiplicity \( m_W \) is \( d \). Thus, \( \lceil im_W/d \rceil - 1 = i - 1 \), \( \lfloor (d-i)m_W/d \rfloor = d-i \), \( \lfloor im_W/d \rfloor = i \), and \( \lceil (d-i)m_W/d \rceil - 1 = d - i - 1 \). Using Theorem 1.2.1 we get for \( i \in \{1, \cdots, d\} \):

\[
\begin{align*}
    n_{g, \frac{i}{2}} &= 0, \\
    n_{g, 1 + \frac{i}{2}} &= -\binom{i-1}{2} + \sum_{V \in S^{(2)}} \left( \lceil im_V/d \rceil - 1 \right), \\
    n_{g, 2 + \frac{i}{2}} &= -(i - 1)(d - i - 1) + \sum_{V \in S^{(2)}} (\lceil im_V/d \rceil - 1)(m_V - \lceil im_V/d \rceil), \text{ and} \\
    n_{g, 3 + \frac{i}{2}} &= -\binom{d - i - 1}{2} + \sum_{V \in S^{(2)}} \left( m_V - \lceil im_V/d \rceil \right) + \delta_{i,d}.
\end{align*}
\]
Otherwise $n_{f,\alpha} = 0$.

When $\mathcal{S}^{(3)} = \emptyset$, $\mathcal{S}^{(2)}$ has only one element, $V$, or no elements. In the case that $\mathcal{S}^{(2)} = \{V\}$, $m_V = d$ or $d - 1$. The calculation from Theorem 1.2.1 for both cases coincides with the calculation from the formula (3.5). If $\mathcal{S}^{(2)} = \emptyset$, the hyperplane arrangement is a generic case with $1 \leq d \leq 3$ which also satisfy the formula (3.5).

The set $\mathcal{S}^{(2)}$ has a one-to-one correspondence to the set $\mathcal{S}$ of codimension 2 dense edges of $f$. This proves Corollary 1.2.2 by Lemma 3.2.1.

3.2.2 Proof of Corollary 1.2.3

Consider $f(x_1, x_2)$ with degree $d$ in $\mathbb{C}^2$. Let $f(x_1, x_2) = h(x_1, x_2, x_3, x_4)$. We will calculate $Sp(h)$. The hyperplane arrangement $h$ has no codimension 3 dense edge. First, we assume that $\mathcal{S}^{(2)} \neq \emptyset$. The hyperplane arrangement $h$ has only one codimension 2 dense edge $V = \{x_1 = x_2 = 0\}$. Moreover the multiplicity $m_V$ is $d$. Thus, $\lceil im_V/d \rceil - 1 = i - 1$, $\lfloor (d - i)m_V/d \rfloor = d - i$, $\lceil im_V/d \rceil = i$, and $\lfloor (d - i)m_V/d \rfloor - 1 = d - i - 1$. Using Theorem 1.2.1 we get for $i \in \{1, \cdots, d\}$:

\[
\begin{align*}
n_{h,\frac{i}{d}} &= 0, \\
n_{h,1+\frac{i}{d}} &= 0, \\
n_{h,2+\frac{i}{d}} &= i - 1, \text{ and} \\
n_{h,3+\frac{i}{d}} &= d - i - 1 + \delta_{i,d}.
\end{align*}
\]

Otherwise $n_{f,\alpha} = 0$.

If $\mathcal{S}^{(2)} = \emptyset$, then $d = 1$ or 2. The calculation from Theorem 1.2.1 coincide with the calculation from the formula above. This proves Corollary 1.2.3 by Lemma 3.2.1.
3.2.3 Proof of Corollary 1.2.5

Notice that $f_2(x_4) = x_4$ and $S^{(2)}$ is the set of codimension 2 dense edges of subarrangement $f_1(x_1, x_2, x_3)$ in $\mathbb{C}^4$. First, we assume $S^{(3)} \neq \emptyset$. We have only one codimension 3 dense edge $W_\infty = \{x_1 = x_2 = x_3 = 0\}$ with the multiplicity $m_{W_\infty} = d - 1$. Applying these to Theorem 1.2.1 we get,

$$
\eta_{0,i}(\langle u_V \rangle_{V \in S}) = \left(\frac{i - 1}{3}\right) - \left(\frac{u_{W_\infty}}{3}\right) + \sum_{V \in S^{(2)}} (u_{W_\infty} - i + 1)\left(\frac{u_V}{2}\right) + \delta_{0,i} \text{ and }
$$

$$
\eta_{1,i}(\langle u_V, v_V \rangle_{V \in S}) = (d - i - 1)\left(\frac{i - 1}{2}\right) - v_{W_\infty}\left(\frac{u_{W_\infty}}{2}\right)
$$

$$
+ \sum_{V \in S^{(2)}} \left( u_V v_V (u_{W_\infty} - u_V - i + 2) + (v_{W_\infty} + 2v_V - d + i + 1)\left(\frac{u_V}{2}\right) \right).
$$

We calculate $n_{f,\alpha}$ for $\alpha \in (0, 2]$. If $i \in \{1, \ldots , d-1\}$, then $u_{W_\infty} = \lceil i(d-1)/d \rceil - 1 = i - 1$ and $v_{W_\infty} = \lfloor (d-i)(d-1)/d \rfloor = d - i - 1$. We get

$$
n_{f,i/d} = 0 \text{ and } n_{f,1+i/d} = 0.
$$

If $i = d$, then $u_V = m_V - 1$, $v_V = 0$, $u_{W_\infty} = d - 2$ and $v_{W_\infty} = 0$. We get

$$
n_{f,1} = \left(\frac{d - 2}{2}\right) - \sum_{V \in S^{(2)}} \left(\frac{m_V - 1}{2}\right) \text{ and }
$$

$$
n_{f,2} = -\left(\frac{d - 1}{2}\right) + \sum_{V \in S^{(2)}} \left(\frac{m_V - 1}{2}\right).
$$

We calculate $n_{f,\alpha}$ for $\alpha \in (2, 4]$. If $i \in \{1, \ldots , d-1\}$, then $u_{W_\infty} = \lfloor im_{W_\infty}/d \rfloor = i - 1$ and $v_{W_\infty} = \lfloor (d-i)m_{W_\infty}/d \rfloor - 1 = d - i - 1$. We get

$$
n_{f,3-i/d} = 0 \text{ and } n_{f,4-i/d} = 0.$$

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If $i = 0$, then $u_V = \lfloor im_V/d \rfloor = 0$, $v_V = \lceil (d - i)m_V/d \rceil - 1 = m_V - 1$, $u_{W_\infty} = \lfloor im_{W_\infty}/d \rfloor = 0$ and $v_{W_\infty} = \lceil (d - i)m_{W_\infty}/d \rceil - 1 = d - 2$. We get

$$n_{f,3} = d - 1 \text{ and } n_{f,4} = 0.$$

When $S^{(3)} = \emptyset$, $S^{(2)}$ has only one element $V$ or no elements. In the case that $S^{(2)} = \{V\}$, $m_V = d - 1$ or $d - 2$. The calculation from Theorem 1.2.1 for both cases coincides with the calculation from Corollary 1.2.5. If $S^{(2)} = \emptyset$, the hyperplane arrangement is a generic case with $2 \leq d \leq 4$ since $f_1$ and $f_2$ are not constant. The calculation from Theorem 1.2.1 for both cases coincides with the calculation from Corollary 1.2.5 for any of $2 \leq d \leq 4$. This proves Corollary 1.2.5.

\[\square\]

3.2.4 Proof of Corollary 1.2.6

In the case that $1 \leq s_1 \leq 2$ and $1 \leq s_2 \leq 2$, we can apply Corollary 1.2.4 and get the result of Corollary 1.2.6. We may assume $s_1 > 2$. If $s_2 = 1$, we use the formula (3.5) with $S^{(2)} = \{V = \{x_1 = x_2 = 0\}\}$ and $s_1 = m_V = d - 1$. This proves Corollary 1.2.6 for this case. If $s_2 = 2$, we can apply Corollary 1.2.5 since $f_2$ can be written as $x_3x_4$ after a suitable change of coordinates. In this case $S^{(2)} = \{V = \{x_1 = x_2 = 0\}\}$ and $s_1 = m_V = d - 2$. The result satisfies Corollary 1.2.6. Thus, we assume that $s_1 > 2$ and $s_2 > 2$. We have only two dense edges in $S^{(2)}$

$$V_1 = \{x_1 = x_2 = 0\} \text{ and } V_2 = \{x_3 = x_4 = 0\}.$$
Their multiplicity $m_{V_1}$ and $m_{V_2}$ are $s_1$ and $s_2$ respectively. Notice that $\mathcal{S}^{(3)} = \emptyset$. Applying these to Theorem 1.2.1 we get,

$$
\eta_{0,i}(\langle u_V \rangle_{V \in \mathcal{S}}) = \left( \frac{i - 1}{3} \right) - \sum_{j=1,2} \left( (i - 3) \left( \frac{u_{V_j}}{2} \right) - 2 \left( \frac{u_{V_j}}{3} \right) \right) + \delta_{0,i} \text{ and} \\
\eta_{1,i}(\langle u_V, v_V \rangle_{V \in \mathcal{S}}) = (d - i - 1) \left( \frac{i - 1}{2} \right) - \sum_{j=1,2} \left( u_{V_j} v_{V_j}(i - 2) + (d - i - 1 - 2v_{V_j}) \left( \frac{u_{V_j}}{2} \right) \right).
$$

First, we calculate $n_{f,\frac{i}{d}}$ and $n_{f,1+\frac{i}{d}}$ with $m_{V_1} = s_1$, $m_{V_2} = s_1$, and $s_1 + s_2 = d$. We get

$$
n_{f,\frac{i}{d}} = -\frac{1}{6} \left( \left\lfloor is_1/d \right\rfloor + \left\lfloor is_2/d \right\rfloor - i - 1 \right) \left( (i - 1)(i - 6) + (i + 1) \left( \left\lfloor is_1/d \right\rfloor + \left\lfloor is_2/d \right\rfloor \right) \right) - 2 \left( \left\lfloor is_1/d \right\rfloor^2 - \left\lfloor is_1/d \right\rfloor \left\lfloor is_2/d \right\rfloor + \left\lfloor is_2/d \right\rfloor^2 \right) + \delta_{0,i} \text{ and} \\
n_{f,1+\frac{i}{d}} = \frac{1}{2} \left( \left\lfloor is_1/d \right\rfloor + \left\lfloor is_2/d \right\rfloor - i - 1 \right) \left( \left( i^2 - 3i - 2 \right) + (i + 1) \left( \left\lfloor is_1/d \right\rfloor + \left\lfloor is_2/d \right\rfloor \right) \right) - (i - 2)(s_1 + s_2) + (\left\lfloor is_1/d \right\rfloor - \left\lfloor is_2/d \right\rfloor)(s_1 - s_2) - 2 \left( \left\lfloor is_1/d \right\rfloor^2 - \left\lfloor is_1/d \right\rfloor \left\lfloor is_2/d \right\rfloor + \left\lfloor is_2/d \right\rfloor^2 \right).
$$

If $gcd(s_1, s_2) = 1$ and $i \in \{1, \cdots, d - 1\}$, then the common factor $\left\lfloor is_1/d \right\rfloor + \left\lfloor is_2/d \right\rfloor - i - 1$ of $n_{f,\frac{i}{d}}$ and $n_{f,1+\frac{i}{d}}$ vanishes because $\left\lfloor is_2/d \right\rfloor = \left\lfloor i(d - s_1)/d \right\rfloor = i - \left\lfloor is_1/d \right\rfloor$ and $gcd(s_1, d) = 1$. Hence, we have

$$
n_{f,\frac{i}{d}} = 0 \text{ and } n_{f,1+\frac{i}{d}} = 0.
$$

If $gcd(s_1, s_2) = 1$ and $i = d$, then we get

$$
n_{f,1} = (s_1 - 1)(s_2 - 1) \text{ and } n_{f,2} = 1 - s_1 s_2.
$$
Similarly, we calculate $n_{f,3-\frac{i}{d}}$ and $n_{f,4-\frac{i}{d}}$. We get

$$n_{f,4-\frac{i}{d}} = -\frac{1}{6} ([is_1/d] + [is_2/d] - i + 1) ((i - 2)(i - 3) + (i - 1) ([is_1/d] + [is_2/d])
\quad - 2 ([is_1/d]^2 - [is_1/d][is_2/d] + [is_2/d]^2)) + \delta_{0,i} \text{ and}
$$

$$n_{f,3-\frac{i}{d}} = \frac{1}{2} ([is_1/d] + [is_2/d] - i + 1) ((i - 2)(i + 1 - (s_1 + s_2))
\quad + (s_1 - s_2) ([is_1/d] - [is_2/d]) + (i - 1) ([is_1/d] + [is_2/d])
\quad - 2 ([is_1/d]^2 - [is_1/d][is_2/d] + [is_2/d]^2))\).$$

If $gcd(s_1, s_2) = 1$ and $i \in \{1, \cdots, d-1\}$, then the common factor $[is_1/d] + [is_2/d] - i + 1 = [is_1/d] - [is_1/d] + 1$ of $n_{f,3-\frac{i}{d}}$ and $n_{f,4-\frac{i}{d}}$ vanishes. Therefore, we have

$$n_{f,3-\frac{i}{d}} = 0 \text{ and } n_{f,4-\frac{i}{d}} = 0.$$

If $gcd(s_1, s_2) = 1$ and $i = 0$, then we get

$$n_{f,3} = s_1 + s_2 - 1 \text{ and } n_{f,4} = 0.$$

This proves Corollary 1.2.6.
CHAPTER 4
CALCULATION

We can use a symbolic system of computer program to check our proof. We use Mathematica.

4.1 Symbols

Each \( a_V, b_W, \) and \( c \) has degree 1. We use a variable \( t \) to trace their degree for each term. Also, \( a[t], b[t], c[t], d[t], e[t], \) and \( f[t] \) denote the following:

\[
\prod_{V \in S(2)} \left( \frac{1 + c + \sum_{W \subseteq V} b_W + a_V}{1 + c + \sum_{W \subseteq V} b_W} \right)^2, \quad \prod_{W \in S(3)} \left( \frac{1 + c + b_W}{1 + c} \right)^3,
\]

\[
(1 + c)^4, \quad \prod_{t \in \Lambda} \frac{1}{1 - d_t}, \quad \prod_{V \in S(2)} (1 - a_V), \quad \text{and} \quad \prod_{W \in S(3)} (1 - b_W)
\]
in 3.1.2 respectively. These can be written in terms of \( \sum_{V \in S(2)} \) and \( \sum_{W \in S(3)} \). We will use symbols \( S_V \) and \( S_W \) for them. Multiplication * with these symbols means summation with index \( V \) and \( W \). For instance \( m_V * a_V * S_V = a_V * S_V * m_V = S_V * m_V * a_V \)
reads \( \sum_{V \in S(2)} m_V a_V \). Addition and distribution hold for summation but commutativity between summations does not hold in general. However in our case we can assume that the commutativity holds and each term is simplified by means of (3.2). Thus any polynomial \( P(S_V, m_V, a_V, u_V, S_W, m_W, b_W, u_W, c) \) in our calculation can be uniquely written as \( S_V P_{V,2}(m_V, a_V, u_V, c) S_W P_{W,2}(m_W, a_W, u_W, c) + S_V P_{V,1}(m_V, a_V, u_V, c) + S_W P_{W,1}(m_W, a_W, u_W, c) + P_0(c) \) for some polynomial functions \( P_{V,2}, P_{W,2}, P_{V,1}, P_{W,1} \) and \( P_0 \).
Due to (3.1) and (3.2) we can simplify formula by defining a function \( \text{simple}[x] \).

\[
(*) \text{ Defining simplifying function } *)
\]

\[
\text{Simple}[x] := \begin{align*}
\text{Collect}[
\text{PolynomialMod}[x, \{t^4, b \cdot c, a \cdot b^2, a \cdot c^2, S_w - S_v, S_v^2 - S_w\}] /: \{a \cdot b^2 \to -a \cdot c, \\
a \cdot c^3 \to -2 \cdot (S_w - 1) \cdot c^3, b \cdot c^3 \to -c^3, a \cdot c^2 \to -c^3, a \cdot b^2 \to c^3\}, \{t, S_v, S_w\}, \text{FullSimplify}\}
\end{align*}
\]

4.2 Double-checking of the Proof of Theorem 1.2.1

First of all we calculate \( d[t] \) by using the program.

\[
(*) \text{ Define (Power series to polynomial) function } *)
\]

\[
\text{Poly}[x_] := \text{Normal}[\text{Series}[x, \{t, 0, 3\}]]
\]

\[
(*) \text{ Calculation of d-factor } *)
\]

\[
d_1 := S_v + S_w + a \cdot c \to (S_v + S_w - 1) \cdot a \cdot c + (S_v + S_w - 1) \cdot a \cdot c \to c^3
\]

\[
d_2 := S_v \times (\text{Binomial}[m_w, 2] + a \cdot c^2 + (m_w + (d + S_v + S_w + 1) \cdot a \cdot c)) + S_v \times (\text{Binomial}[m_w, 2] + b \cdot c^2) + \text{Binomial}[d, 2] \times c^2
\]

\[
d_3 := S_v \times (\text{Binomial}[m_v, 3] \times a \cdot c^3) + S_v \times (\text{Binomial}[m_w, 3] \times b \cdot c^3) + \text{Binomial}[d, 3] \times c^3 + S_v \times (\text{Binomial}[m_v, 2] \times (d - 2) \times a \cdot c^2) + S_v \times S_w \times (\text{Binomial}[m_v, 2] \times (d_w - 2) \times a \cdot c^2 \times b \cdot c)
\]

\[
d[t_] := \text{Poly}[1 / (1 + d_1 \times t + d_2 \times t^2 + d_3 \times t^3)]
\]

We define \( a[t], b[t], c[t], c[t], \) and \( f[t] \) from 3.1.2.

\[
(*) \text{ The other Input Factors } *)
\]

\[
a[t_] := 1 + (S_v + 2 \cdot a \cdot c) \times t + (S_v + (a \cdot c^2 + 2 \cdot (S_w - 1) \cdot a \cdot c)) \times t^2 + (S_v + a \cdot c^3) \times t^3
\]

\[
b[t_] := 1 + (3 \cdot S_v + b \cdot c) \times t + (3 \cdot S_v + b \cdot c^2) \times t^2 + (S_v + b \cdot c^3) \times t^3
\]

\[
c[t_] := (1 + c + t)^4
\]

\[
e[t_] := 1 - (S_v + a \cdot c) \times t
\]

\[
f[t_] := 1 - (S_v + b \cdot c) \times t
\]

We define \( CY_1[t], CYT[t], \) and \( Td[t] \) to be \( c(\Omega^1_v), c(\tilde{Y}), \) and \( td(\tilde{Y}) \).

\[
(*) \text{ Calculation of Chern class and Todd Class } *)
\]

"Chern class of cotangent bundle on resolution"

\[
CY_1[t_] := \text{Collect}[
\text{PolynomialMod}[a[t] \times b[t] \times c[t] \times e[t] \times f[t], \\{(t^4, b \cdot c, a \cdot (b \cdot c^2), a \cdot (c^2))\}, \{t, S_v, S_w, c, a \cdot c\}, \text{Simplify}]
\]

Simple[CY_1[t]]

"Chern class of tangent bundle on resolution"

\[
CYT[t_] := \text{Collect}[\text{CY}_1[t], \{t, S_v, S_w, c, a \cdot c\}, \text{Simplify}]
\]

Simple[CYT[t]]

"Todd class of tangent bundle on resolution"

\[
\text{CT}_1 := \text{Coefficient}[\text{CYT}[t], t, 1]
\]

\[
\text{CT}_2 := \text{Coefficient}[\text{CYT}[t], t, 2]
\]

\[
\text{CT}_3 := \text{Coefficient}[\text{CYT}[t], t, 3]
\]

\[
Td[t_] := \text{Collect}[
\text{PolynomialMod}[1 + (1 / 12 \cdot \text{CT}_1 \times t + (1 / 12 \cdot \text{CT}_1 \times t^2 + (1 / 24 \cdot (\text{CT}_1 + \text{CT}_2) \times t^3, \\{(t^4, b \cdot c, a \cdot (b \cdot c^2), a \cdot (c^2))\}, \{t, S_v, S_w, c, a \cdot c\}, \text{Simplify}]
\]

\]

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To calculate Chern character we first define a function $Ch[R, C_1, C_2, C_3]$ which gives Chern character from the rank $R$, first Chern class $C_1$, second Chern class $C_2$, and the third Chern class $C_3$. Using this function we calculate the Chern character $ch(\mathcal{O}_Y(U))$ of line bundle on $\tilde{Y}$ for a divisor $U$.

We can also calculate $ch(\Omega_p^p(\log \tilde{Z}))$ and $ch(\Omega_p^p(\log \tilde{Z}) \otimes \mathcal{O}_Y(U))$ for $p = 0, 1, 2, 3$ from Section 3.1.2 and formulas (2.6).
Finally we calculate $n_{f,a}$.

This double-checks the computation on the proof of Theorem 1.2.1.
BIBLIOGRAPHY


