EXAMPLES OF RIEMANNIAN FUNCTORIAL QUANTUM FIELD THEORY

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Abstract

by

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Following Atiyah, Segal, Kontsevich and others, a $d$-dimensional Riemannian Functional Quantum Field Theory $E$ assigns to a closed $d - 1$ dimensional oriented Riemannian manifold a Hilbert space $E(Y)$ and to a bordism $\Sigma$ from $Y_1$ to $Y_2$ (which is a compact oriented Riemannian manifold with $\partial \Sigma = Y_2 \sqcup Y_1$) a Hilbert-Schmidt operator $E(\Sigma) : E(Y_1) \to E(Y_2)$ so that gluing bordisms corresponds to composing the associated operators. If we forget the Riemannian structure on the $Y$’s and the bordisms, then there are many examples which are known has Topological Quantum Field Theories. In 2007, Douglas Pickrell ([18]) constructed a family of examples of 2-dimensional theory. In this dissertation, we construct examples of $d$-dimensional theory when $d$ is even.
To my grand parents
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SYMBOLS

\( \mathcal{B}(X) \) Borel \( \sigma \)-algebra on \( X \), Page 14

\( \mathcal{B}(H) \) The set of bounded operators on \( H \), Page 67

\( C \) Cayley Transform, Page 41

\( D'(\Sigma) \) Space of distributions on \( \Sigma \), Page 17

\( \text{Exp} \) Exponential map, Page 22

\( \mathcal{E} \) Another “Exponential map”, Page 26

\( \text{Sym}^*(H) \) Bosonic Fock space of \( H \).

\( \mathcal{S}(H) \) Page 46

\( \mathcal{Z}(H) \) Page 41

\( \mathcal{Z}_2(H) \) Page 46

\( d\text{-RBord} \) \( d \)-dimensional Riemannian bordism category, Page 2
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CHAPTER 1

INTRODUCTION

There are a number of mathematical approaches to rigourously define Quantum Field Theory. In the framework pioneered by Atiyah, Kontsevich, Segal and many others, a $d$-dimensional Riemannian QFT $E$ is a rule that assigns to a $d$-dimensional closed oriented Riemannian manifold $\Sigma$ a number $E(\Sigma)$ depending only on the isomorphism class of $\Sigma$. More generally, $E$ assigns to a $d$-dimensional compact oriented Riemannian manifold $\Sigma$ with $\partial \Sigma = Y_0 \sqcup Y_1$ Hilbert spaces $E(Y_0)$ and $E(Y_1)$ together with a bounded operator $E(\Sigma) : E(Y_0) \to E(Y_1)$. The main requirement is that $E$ is local. Here by local we mean that if $\Sigma$ is obtained by gluing $\Sigma_1$ and $\Sigma_2$ along the common boundary $Y$, then the contributions from $\Sigma_1$ and $\Sigma_2$ are sufficient to compute $E(\Sigma)$.

Motivation for the definition comes from the heuristic notion of path integral quantization of a classical $\sigma$-model. A $d$-dimensional classical $\sigma$-model with target a Riemannian manifold $M$ consists of a $d$-dimensional compact oriented Riemannian manifold $\Sigma$, the space of “fields” $\mathfrak{F}(\Sigma)$ which depends on $M$, and a local action functional $S : \mathfrak{F}(\Sigma) \to \mathbb{R}$. For example, the space of “fields” is $C^\infty(\Sigma, M)$ and the action functional is the energy functional. The path integral quantization heuristically means making sense of the integral

$$E(\Sigma) = \int_{\mathfrak{F}(\Sigma)} e^{-S(\phi)} D\phi$$

where $D\phi$ is a “volume measure” on $\mathfrak{F}(\Sigma)$. 

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If $\partial \Sigma = Y$ then

$$E(\Sigma)(\psi) = \int_{\{\phi \in \mathcal{H}(\Sigma) : \phi|_{\mathcal{H}(Y)} = \psi\}} e^{-S(\phi)} \, D\phi$$

defines a function on $\mathcal{H}(Y)$. Hence $E(\Sigma)$ is a vector in $E(Y)$, the space of “functions” on $\mathcal{H}(Y)$. $E(Y)$ is called the quantum Hilbert space of states. More generally, if $\partial \Sigma = Y_0 \sqcup Y_1$ then for $\psi_0 \in \mathcal{H}(Y_0)$ and $\psi_1 \in \mathcal{H}(Y_1)$,

$$E(\Sigma)(\psi_0, \psi_1) = \int_{\{\phi \in \mathcal{H}(\Sigma), \phi|_{\mathcal{H}(Y_0)} = \psi_0, \phi|_{\mathcal{H}(Y_1)} = \psi_1\}} e^{-S(\phi)} \, D\phi$$

can be thought as a kernel of the linear operator $E(\Sigma) : E(Y_0) \to E(Y_1)$, given by

$$E(\Sigma)(\alpha)(\psi_1) = \int_{\mathcal{H}(Y_0)} E(X)(\psi_0, \psi_1) \alpha(\psi_0) \, D\psi_0$$

In particular, if $\Sigma = \Sigma_1 \cup_Y \Sigma_2$ and $\Sigma$ is closed then the number $E(\Sigma)$ ought to be equal to $E(\Sigma_2)(E(\Sigma_1))$. Furthermore, heuristic arguments suggest that if one could give a rigorous meaning to these path integrals, they would satisfy:

$$E(\sqcup_i Y_i) = \otimes_i E(Y_i)$$

$$E(\overline{Y}) = E(Y)^*$$

$$E(\Sigma_2 \circ \Sigma_1) = E(\Sigma_2) \circ E(\Sigma_1)$$

$$E(\sqcup_i \Sigma_i) = \otimes_i E(\Sigma_i).$$

Let $d$-$\text{RBord}$ be the category whose objects are oriented closed $d - 1$ dimensional Riemannian manifolds. A nondegenerate morphism $\Sigma : Y_0 \to Y_1$ in $d$-$\text{RBord}$ is a $d$-dimensional compact oriented Riemannian manifold such that $\partial \Sigma = \overline{Y_0} \sqcup Y_1$. Here $\overline{Y_0}$ means that the intrinsic orientation on $Y_0$ is opposite of the induced orientation coming from $\Sigma$. We require that a nondegenerate morphism $\Sigma$ in $d$-$\text{RBord}$ has product metric.
near the boundary so that we have a well defined composition law by glueing. We also consider degenerate morphisms in \( d\text{-RBord} \). A degenerate morphism from \( Y_0 \) to \( Y_1 \) is an isometry \( Y_0 \to Y_1 \). Composition of a nondegenerate morphism is done in the obvious way. The identity isometry \( Y \to Y \) provides the identity morphism. We note that \( d\text{-RBord} \) is a symmetric monoidal category where the symmetric monoidal structure is given by disjoint union.

Let \( \text{Hilb} \) be the category whose objects are Hilbert spaces and morphisms are continuous linear maps. We note that \( \text{Hilb} \) is a symmetric monoidal category with the symmetric monoidal structure is given by Hilbert space tensor product.

The discussion above suggests the following definition:

**Definition 1.0.1.** (Atiyah-Segal) A \( d \)-dimensional Riemannian Functorial QFT is a symmetric monoidal functor \( E: d\text{-RBord} \to \text{Hilb} \).

Naively one thing to try to construct such examples would be to give rigorous meaning to the path integrals (1), (2), (3) and (4). In general it is very difficult. But, using ideas from Constructive Quantum Field Theory \([6]\), it is possible to give rigorous meaning to these path integrals in some linear \( \sigma \)-models i.e. in the case when \( M = \mathbb{R}^n \) by enlarging the space of fields. The measures which are required to define the path integrals live on the space of distributions. This is the point of view Douglas Pickrell follows in \([18]\) to construct examples of 2-dimensional Functorial QFTs called \( P(\phi)_2 \)-interaction theories.

Let us briefly discuss the construction from \([18]\). To a 1-dimensional closed oriented Riemannian manifold \( Y \), Pickrell’s construction assigns the Hilbert space

\[
E(Y) = L^2(D'(Y), \mu_Y)
\]

where \( \mu_Y \) is the Gaussian measure on \( D'(Y) \) with the corresponding symmetric bilinear form given by \( (\Delta_Y + m^2)^{-\frac{1}{2}} \) (we refer to chapter \([2]\) for the definition of Gaussian
measure). For a 2-dimensional oriented compact Riemannian manifold with boundary $Y$, the construction of the vector $E(\Sigma) \in E(Y)$ is as follows. First double $\Sigma$ along $Y$ to get $\hat{\Sigma}$. Then construct a “map” $\tilde{\gamma} : D'(\hat{\Sigma}) \to D'(Y)$. Now pushforward the $P(\phi)_2$ measure on $D'(\hat{\Sigma})$ to a measure on $D'(Y)$. It turns out that pushforward of the $P(\phi)_2$ measure and $\mu_Y$ are mutually absolutely continuous. Now, define the vector $E(\Sigma)$ to be a “suitable multiple” of the squareroot of the Radon-Nikodym derivative. The existence of $\tilde{\gamma}$ is a problem but there are ways to fix it. It is also desirable to get rid of doubling of $\Sigma$ in order to construct the vector $E(\Sigma)$.

In this thesis, we construct examples of $d$-dimensional Functorial QFT when $d$ is even. Our construction has two stages. We first show that

**Theorem 1.0.2.** There is a projective representation of $d$-RBord in Hilb.

Then we can turn the projective representation given by previous theorem to get a Functorial QFT when $d$ is even. More precisely, we prove that

**Theorem 1.0.3.** There exists $d$-dimensional Functorial QFT when $d$ is even.

Our construction is inspired by the construction in [18] but we formulate the Quantum Hilbert space and the squareroot of Radon-Nikodym derivative in algebraic language in the Free Theory case. One advantage is that we don’t have to deal with the existence of $\tilde{\gamma}$. Another advantage is that we do not need the doubling.
CHAPTER 2

GAUSSIAN MEASURES

Gaussian measures are ubiquitous in Mathematical Physics. One particular instance Gaussian measures appear which is of our interest is the Constructive Quantum Field Theory ([6]). In this chapter, we recall Gaussian measure and various facts surrounding it. We mention Gaussian measures of interest in the thesis which are associated to Riemannian manifolds. Our main references are [1] and [3]. In particular, materials in 2.1 and 2.1.1 are taken from chapter 2 in [1] and lemma 2.1.11 is taken from [3].

2.1 Gaussian measure preliminaries

A Gaussian measure \( \mu \) on \( \mathbb{R} \) is a Borel probability measure such that it is either the Dirac measure \( \delta_a \) or it has density

\[
x \rightarrow \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-a)^2}{2\sigma}}
\]

with respect to the Lebesgue measure on \( \mathbb{R} \). In the later case the measure \( \mu \) is called a nondegenerate. When \( a = 0 \) and \( \sigma = 1 \), then \( \mu \) is called standard and \( \mu \) is centered if \( a = 0 \).

We want to define Gaussian measures on infinite dimensional vector spaces. We begin by discussing Gaussian measures on finite dimensional vector spaces in such a way that it can be generalized for infinite dimensional ones.
Definition 2.1.1. A Borel probability measure $\mu$ on $\mathbb{R}^n$ is called a Gaussian measure if the measure $f_*\mu$ is a Gaussian measure on $\mathbb{R}$ for all $f \in (\mathbb{R}^n)^\vee$.

This definition does not say much about structure of the measure $\mu$. The following proposition 2.1.3 provides an explicit description of the measure $\mu$. Let $\langle , \rangle$ denote the standard inner product on $\mathbb{R}^n$. We first need the following definition.

Definition 2.1.2. Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$. The Fourier transform of $\mu$ is a function $\tilde{\mu} : \mathbb{R}^n \to \mathbb{C}$ given by

$$\tilde{\mu}(y) = \int_{\mathbb{R}^n} e^{iy \cdot x} \, d\mu.$$ 

Proposition 2.1.3. A Borel measure $\mu$ on $\mathbb{R}^n$ is Gaussian if and only if its Fourier transform has the form

$$\tilde{\mu}(y) = e^{i\langle y, a \rangle - \frac{1}{2} \langle Ky, y \rangle}$$

where $a \in \mathbb{R}^n$ and $K$ is a nonnegative matrix. Moreover, $\mu$ has density with respect to the Lebesgue measure if and only if $K$ is positive. In this case, the density of $\mu$ is given by the function

$$x \mapsto \frac{1}{\sqrt{\det(K)} \cdot (2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \langle K^{-1}(x-a), (x-a) \rangle}$$

Proof of proposition 2.1.3 is a well known fact; for example Proposition 1.2.2 in [1].

Let $X$ be a locally convex space and $X^\vee$ be its topological dual. A cylindrical set in X has the form

$$C = \{x \in X : (f_1(x), \ldots, f_n(x)) \in C_0\}$$

where $C_0$ is a Borel subset on $\mathbb{R}^n$ and $f_i \in X^\vee$. We use the notation $\mathcal{B}(X)$ to denote the Borel $\sigma$-algebra of $X$ and $\mathcal{F}(X)$ to denote the $\sigma$-algebra generated by the cylinder
sets. From definition it follows that \( \mathcal{F}(X) \) is contained in \( \mathcal{B}(X) \) but may not coincide. It is a fact that \( \mathcal{B}(X) \) and \( \mathcal{F}(X) \) coincide for Nuclear spaces.

**Assumption 2.1.4.** In this thesis, we assume that \( \mathcal{B}(X) \) and \( \mathcal{F}(X) \) coincide and this is an assumption on \( X \).

**Definition 2.1.5.** A Borel probability measure \( \mu \) on \( X \) is called Gaussian if the measure \( f_*\mu \) is Gaussian on \( \mathbb{R} \) for all \( f \in \mathcal{X}^\vee \). The measure \( \mu \) is called centered Gaussian if all the measures \( f_*\mu \), \( f \in \mathcal{X}^\vee \) are centered Gaussian. Moreover if \( f_*\mu \) is nondegenerate Gaussian on \( \mathbb{R} \) for all \( f \in \mathcal{X}^\vee \), then \( \mu \) is called nondegenerate Gaussian.

**Definition 2.1.6.** Let \( \mu \) be a Borel probability measure on a locally convex space \( X \). The Fourier transform of the measure \( \mu \) is the function \( \tilde{\mu} : \mathcal{X}^\vee \to \mathbb{C} \) given by

\[
\tilde{\mu}(f) = \int_X e^{if(x)} d\mu.
\]

Let \( \mu \) be a centered Gaussian measure on a locally convex space \( X \). Then the bilinear form on \( \mathcal{X}^\vee \) defined by

\[
B(f, g) = \int_X f g d\mu
\]

is continuous, nonnegative and symmetric. In fact, \( B \) completely determines \( \mu \) in the following sense.

**Theorem 2.1.7.** A measure \( \mu \) on a locally convex space \( X \) is centered Gaussian if and only if its Fourier transform has the form

\[
\tilde{\mu}(f) = e^{-\frac{1}{2}B(f, g)}
\]

where \( B \) is a continuous nonnegative symmetric bilinear form on \( \mathcal{X}^\vee \).
We refer to Theorem 2.2.4 of [1] for the proof of this theorem.

Definition 2.1.8. A continuous $\mathbb{C}$-valued function $C$ on a locally convex space $X$ is called a characteristic function if

$$\sum_{i,j=1}^{n} \beta_i \bar{\beta}_j C(v_i - v_j) \geq 0$$

for all $\beta_1 \ldots \beta_n \in \mathbb{C}$ and $v_1 \ldots v_n \in X$ and $C(0) = 1$.

Let $X$ be a Nuclear space and $\mu$ be a Gaussian measure on $X^\vee$ then $\tilde{\mu}$ is a characteristic function on $X$. In fact, Bochner-Milnos theorem (Theorem A.6.1 in [6]) states that there is a bijection between the set of probability measures on $X^\vee$ and the characteristic functions on $X$.

Definition 2.1.9. A finite Borel measure $\mu$ on topological space $X$ is called Radon if for every $B \in \mathcal{B}(X)$ and every $\epsilon > 0$ there exists a compact subset $K_\epsilon$ of $X$ such that $\mu(B \setminus K_\epsilon) < \epsilon$.

A large class of spaces have the property that all Borel measures are Radon. This class contains Nuclear spaces; For details we refer to Chapter 3 in [1].

Assumption 2.1.10. From now on we assume that all the Gaussian measures are Radon measures.

2.1.1 Cameron-Martin Spaces

A lot of information of a nondegenerate Gaussian measure $\mu$ on a locally convex space is contained in a subspace of $X$ called the Cameron-Martin space of the pair $(X, \mu)$. Here, we define and collect a few properties of Cameron-Martin space.

To motivate we assume that $X$ is a finite dimensional real vector space and

$$B : X^\vee \times X^\vee \rightarrow \mathbb{R}$$
be a non negative symmetric bilinear form on $X^\vee$. Let

$$\text{Null}(B) = \{ f \in X^\vee : B(f, f) = 0 \}$$

and

$$H = \{ v \in X : f(v) = 0 \text{ for all } f \in X^\vee \};$$

We can think of $H$ as an “annihilator” of $\text{Null}(B)$.

**Lemma 2.1.11.** There exists a unique inner product on $<,>_H$ on $H$ such that for any orthonormal basis $\{h_1, \ldots, h_k\}$ of $H$ where $\dim(H) = k$,

$$B(f, g) = \sum_{j=1}^{k} f(h_j)g(h_j)$$

Moreover, let

$$\|h\|_H = \sup_{f \in X^\vee} \frac{|f(h)|}{\sqrt{B(f, f)}},$$

then

$$H = \{ h \in X : \|h\|_H < \infty \}$$

and

$$\|h\|_H^2 = <h, h>_H.$$ 

**Proof.** We note that there is an isomorphism of vector spaces

$$X^\vee/\text{Null}(B) \rightarrow H^\vee$$

given by the map

$$f + \text{Null}(B) \mapsto f|_H$$

and $B$ induces an inner product $\tilde{B}$ on $X^\vee/\text{Null}(B)$. Hence there is the inner product
on $H^\vee$ induced by $\bar{B}$. Using this inner product on $H^\vee$ we construct an inner product $\langle,\rangle_H$ on $H$. Now, suppose that $\{h_1, \ldots, h_k\}$ is any orthonormal basis and $f, g \in X^\vee$. Then,

$$B(f, g) = \langle f|_H, g|_H \rangle_H = \sum_{j=1}^{k} f(h_j)g(h_j).$$

This proves the first part of the lemma.

Suppose that $v \notin H$. Then there is $f \in \text{Null}(B)$ such that $f(v) \neq 0$ and this means that

$$\|v\|_H = \infty.$$

Also, we note that $\|v\|_H < \infty$ implies that $v \in H$. This shows that

$$H = \{h \in X : \|h\|_H < \infty\}.$$

To see

$$\|h\|_H^2 = \langle h, h \rangle_H,$$

let us write

$$h = \sum_{j=1}^{k} \langle h, h_j \rangle_H h_j$$

and $f \in X^\vee$. Now,

$$|f(h)|^2 = \left| \sum_{j=1}^{k} \langle h, h_j \rangle_H f(h_j) \right|^2 \leq \langle h, h \rangle_H \cdot B(f, f)$$
with equality if $f$ is such that

$$f(h_j) = \langle h, h_j \rangle_H.$$

This shows that

$$\|h\|^2_H = \langle h, h \rangle_H.$$

\[\square\]

**Note 2.1.12.** If the form $B$ is nondegenerate then $H = X$.

**Note 2.1.13.** Suppose that $(X, \mu)$ is a nondegenerate Gaussian measure space and $\dim(X) < \infty$. As a consequence of Fernique’s Theorem \[\text{[3, Theorem 3.4]}, one can define a map $J_\mu : L^2(X, \mu) \to X$ given by

$$J_\mu f = \int_X x f(x) \, d\mu(x).$$

It turns out that

$$J_\mu(L^2(X, \mu)) = H$$

and if

$$K = \{f \in L^2(X, \mu) : f \in X^\vee\},$$

then $J_\mu : K \to H$ is an isomorphism of Hilbert spaces.

In the finite dimensional case, it is difficult to distinguish $H$ from $X$ when the Gaussian measure $\mu$ is nondegenerate. As a preparation for the infinite dimensional case, we mention a different way to think about $H$. Let $T_h : X \to X$ denote the translation by $-h$. Then

$$(T_h)_* \mu \sim \mu$$
if and only if \( h \in H \). In this case, \[
\frac{d(T_h)_* \mu}{d\mu} = e^{J^*_\mu(h) - \|h\|_H^2}
\] where \( J^*_\mu \) is the adjoint of \( J_\mu \).

Let \( X \) be a locally convex space, \( \mu \) be a nondegenerate Gaussian measure on \( X \) and \( B_\mu \) be the corresponding symmetric bilinear form on \( X^\vee \). Define
\[
\|h\|_{H(\mu)} = \sup_{f \in X^\vee} \frac{f(h)}{\sqrt{B_\mu(f,f)}}.
\]

The space
\[
H(\mu) = \{h \in X : \|h\|_{H(\mu)} < \infty\}
\]
is called the Cameron-Martin space of \((X, \mu)\).

**Example 2.1.1.**

1. Let \( X \) be a finite dimensional real vector space and \( \mu \) be a nondegenerate Gaussian measure on \( X \) then \( H \) constructed above is the Cameron-Martin space of \((X, \mu)\).

2. If \((i, H, W, \mu)\) be an abstract Wiener space, then \( H = H(\mu) \) (we refer to the appendix for the definition of abstract Wiener space).

We mention a characterization theorem for the Cameron-Martin space which is Theorem 3.2.3 in [1].

**Theorem 2.1.14.** Let \( X \) be a locally convex space and \( \mu \) be a nondegenerate Gaussian measure on \( X \). Then
\[
H(\mu) = \{h \in X : (T_h)_* \mu \sim \mu\}.
\]

Cameron-Martin space captures a lot of properties of the Gaussian measures. We mention few of them without proofs; For details we refer to 3.7 in [1].
1. Let $H(\mu)$ and $H(\nu)$ be Cameron-Martin spaces of $(X, \mu)$ and $(Y, \nu)$ respectively and $A : H(\mu) \to H(\nu)$ be a continuous linear operator. Then $A$ extends to a $\mu$-measurable linear operator $\hat{A} : X \to Y$. Moreover, $A_*\mu$ is a Gaussian measure on $Y$ with Cameron-Martin space $A(H(\mu))$.

2. Let $A : X \to Y$ be a $\mu$-measurable operator where $\mu$ is a Gaussian measure on $X$. Then $A : H(\mu) \to A(H(\mu))$ is a continuous linear map with norm one.

We can use Cameron-Martin space to study when two Gaussian measures are equivalent or singular. Let $\mu$ and $\mu'$ be two Gaussian measures on $X$. Then $\mu \sim \nu$ if and only if $H(\mu)$ coincides with $H(\nu)$ as vector space and the two inner product are close to each other in the sense that there exists a symmetric Hilbert-Schmidt operator $S$ on $H(\nu)$ such that

$$<h, h'>_{H(\mu)} = <h, h'>_{H(\nu)} + <Sh, h'>_{H(\nu)}$$

for all $h, h' \in H(\nu)$. For a more general statement we first need the following definition.

**Definition 2.1.15.** Let $A : H_1 \to H_2$ be a continuous surjective linear operator between two Hilbert spaces. We can define a new inner product on $H_2$ by identifying $H_2$ with the orthogonal complement of kernel of $A$. We call this inner product the pushforward inner product on $H_2$. We denote the pushforward inner product by $A_* <,>$. 

Let $A : (X, \mu) \to (Y, \nu)$ be a “$\mu$-measurable” linear operator. Suppose that $A_*\mu \sim \nu$. Then $A|_{H(\mu)} : H(\mu) \to H(\nu)$ is continuous surjective linear operator such that the pushforward inner product on $H(\nu)$ is close to the inner product of $H(\nu)$ in the sense mentioned above. Conversely, if $A : H(\mu) \to H(\nu)$ is a continuous surjective linear operator such that the pushforward inner product on $H(\nu)$ is close
to the inner product of $H(\nu)$, then there is a measurable linear extension $\hat{A}$ of $A$ such that $\hat{A}_* \mu \sim \nu$.

The assignment of the Cameron-Martin space $H(\mu)$ to $(X, \mu)$ is Functorial in the following sense. We define a category $\mathcal{AWS}$ whose objects are pairs $(X, \mu)$ where $X$ is locally convex space and $\mu$ is a Radon Gaussian measure on $X$. A morphism $A : (X, \mu) \to (Y, \nu)$ is a $\mu$-measurable linear operator such that $A_* \mu \sim \nu$. Let $\mathbf{Hilb}$ be the category of Hilbert spaces with morphisms continuous linear operators. Then from the discussions above, we see that we have a functor $\mathcal{CMS} : \mathcal{AWS} \to \mathbf{Hilb}$.

**Definition 2.1.16.** We say two inner products $<,>$ and $<,>'$ on a real separable Hilbert space $H$ are close to each other if there is a symmetric Hilbert-Schmidt operator $S$ such that

$$<h, h'> = <h, h'> + <Sh, h'>$$

for all $h, h' \in H$. When $<,>$ and $<,>'$ are close to each other we write $<,> \sim <,>'$.

**Lemma 2.1.17.** Let $A : H_1 \to H_2$ be a surjective continuous linear operator between two separable real Hilbert spaces. Let $<,>_1$ and $<,>_2$ be the inner products in $H_1$ and $H_2$ respectively. Then $A_* <,>_1 \sim <,>_2$ if and only if $AA^* - I$ is a Hilbert-Schmidt operator.

**Proof.** Let $A = UC$ be the polar decomposition of $A$ where $C = \sqrt{AA^*}$ and

$$U : H_1 \to H_2$$

such that $U^*$ is an isometry. We note that $C$ is injective as $A$ is surjective. Moreover, $A$ is surjective also implies that $C$ is surjective. Hence $C$ has continuous inverse. This implies that $(AA^*)^{-1}$ is a continuous operator. Now we note that the pushforward inner product is given by $(AA^*)^{-1}$. 

14
Hence

\[ A_1 < ,> _1 \sim < ,> _2 \text{ if and only if } (AA^*)^{-1} - I \text{ is Hilbert-Schmidt.} \]

But

\[ (AA^*)^{-1} - I \]

is Hilbert-Schmidt if and only if

\[ AA^* - I \]

is Hilbert-Schmidt. This completes the proof of the lemma. \( \square \)

**Lemma 2.1.18.** Let \( A : H_1 \to H_2 \) and \( B : H_2 \to H_2 \) be surjective linear maps such that \( A_1 < ,> _1 \sim < ,> _2 \) and \( B_1 < ,> _2 \sim < ,> _3 \). Then \( B_1 \circ A_1 < ,> _1 \sim < ,> _3 \).

**Proof.** By lemma 2.1.17 there exists a symmetric Hilbert Schmidt operator

\[ S_2 : H_2 \to H_2 \]

such that

\[ AA^* = I + S_2 \]

Similarly, there exists a symmetric Hilbert-Schmidt operator

\[ S_3 : H_3 \to H_3 \]

such that

\[ BB^* = I + S_3. \]

Hence

\[ (BA)(BA)^* = BAA^*B^* = B(I + S_2)B^* = BB^* + BS_2B^* = I + S_3 + BS_2B^*. \]
Now the lemma follows from the fact that $BS_2B^*$ is Hilbert-Schmidt.

As a consequence of the lemma 2.1.18 we can consider the category $\text{Hilb}^1$ whose objects are Hilbert spaces and morphisms $A : H_1 \to H_2$ are continuous surjective linear operator such that the pushforward inner product on $H_2$ is close to the inner product on $H_2$ in the sense described above. Now, we can see that $\mathcal{CM}\mathcal{S}$ defines a functor $\text{AWS} \to \text{Hilb}^1$.

2.2 Finite Dimensional Gaussian Integrals

We will compute few finite dimensional Gaussian Integrals which will be used in the thesis. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a positive operator.

**Lemma 2.2.1.** \[ \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} <Ax,x>} e^{<v,x>} \, dx = \frac{1}{\text{det}(A)^{\frac{n}{2}}} e^{\frac{1}{2} <A^{-1}v,v>} . \]

**Proof.** The strategy here is to complete the square. To do this, we first observe that the function $x \mapsto -\frac{1}{2} <Ax,x> + <v,x>$ attains minimum value at

\[ x = A^{-1}v. \]

Now let

\[ y = x - A^{-1}v, \]

then

\[ -\frac{1}{2} <Ax,x> + <v,x> = -\frac{1}{2} <Ay,y> + \frac{1}{2} <A^{-1}v,v>. \]
Now,

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle Ax,x \rangle} e^{\langle v,x \rangle} \, dx \\
= e^{\frac{1}{2} \langle A^{-1}v,v \rangle} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle Ay,y \rangle} \, dy \\
= \frac{1}{\det(A)^{1/2}} e^{\frac{1}{2} \langle A^{-1}v,v \rangle}.
\]

Let \( A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^m \oplus \mathbb{R}^n \) and \( D : \mathbb{R}^n \to \mathbb{R}^n \) be a positive operators.

**Corollary 2.2.2.**

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} , \begin{bmatrix} y \\ x \end{bmatrix} \rangle} \, dx \\
= \frac{1}{\det(D + D_1)^{1/2}} e^{-\frac{1}{2} \langle (A_1 - B_1(D + D_1)^{-1}B_1^t)y,y \rangle}.
\]

**Proof.** We note that

\[
\begin{align*}
\langle \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} , \begin{bmatrix} y \\ x \end{bmatrix} \rangle &+ \langle Dx,x \rangle \\
= \langle A_1y,y \rangle + 2 \langle B_1^t y,x \rangle + \langle (D + D_1)x,x \rangle.
\end{align*}
\]

This implies that

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} , \begin{bmatrix} y \\ x \end{bmatrix} \rangle} \, dx \\
= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \langle A_1y,y \rangle} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle (D + D_1)x,x \rangle} e^{\langle 2B_1^t y,x \rangle} \, dx \\
= \frac{1}{\det(D + D_1)^{1/2}} e^{-\frac{1}{2} \langle A_1y,y \rangle} e^{\frac{1}{2} \langle B_1(D + D_1)^{-1}B_1^t y,y \rangle} \\
= \frac{1}{\det(D + D_1)^{1/2}} e^{-\frac{1}{2} \langle (A_1 - B_1(D + D_1)^{-1}B_1^t)y,y \rangle}.
\]
More generally if \( A_1 : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^m \oplus \mathbb{R}^n \) and \( A_2 : \mathbb{R}^p \oplus \mathbb{R}^m \to \mathbb{R}^p \oplus \mathbb{R}^m \) are positive operators given by

\[
A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix}
\]

and

\[
A_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^t & D_2 \end{bmatrix},
\]

then

\[
\frac{1}{(2\pi)^{\frac{m+n}{2}}} \int_{\mathbb{R}^m} e^{-\frac{1}{2} \left\langle \begin{bmatrix} A_2 & B_2 \\ B_2^t & D_2 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}, \begin{bmatrix} z \\ y \end{bmatrix} \right\rangle} \cdot e^{-\frac{1}{2} \left\langle \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix} \right\rangle} \, dx
\]

\[
= \frac{1}{\det(D_1 + D_2)^{\frac{1}{2}}} e^{-\frac{1}{2} \left\langle A_3 \begin{bmatrix} z \\ x \end{bmatrix}, \begin{bmatrix} z \\ x \end{bmatrix} \right\rangle}
\]

where

\[
A_3 = \begin{bmatrix} A_2 - B_2(A_1 + D_2)^{-1}B_1^t & -B_2(A_1 + D_2)^{-1}B_1 \\ B_1^t(A_1 + D_2)^{-1}B_2^t & D_1 - B_1^tB_2(A_1 + D_2)^{-1}B_1 \end{bmatrix}.
\]

### 2.3 Gaussian Measures associated to Riemannian Manifolds

Let \( \Sigma \) be an oriented closed Riemannian manifold, \( \Delta_\Sigma \) the nonnegative Laplacian on \( \Sigma \) and \( m \) be a positive real number. Consider the positive operator \((\Delta_\Sigma + m^2)^{-s}\) where \( s \) is a positive real number and define an inner product on \( C^\infty(\Sigma) \) by

\[
<f, g> = \int_\Sigma f(\Delta_\Sigma + m^2)^{-s} g \, dvol\Sigma.
\]

Then there is associated nondegenerate Gaussian measure \( \mu_\Sigma \) on \( D'(\Sigma) \). In this case the Cameron-Martin space is the Sobolev space \( W^s(\Sigma) \) which is the completion of
\( C^\infty(\Sigma) \) with respect to the inner product

\[
<f, g> = \int_\Sigma f(\Delta_\Sigma + m^2)^s \, g \, d\text{vol}_\Sigma.
\]

**Remark 2.3.1.** We can start with the Hilbert space \( W^s(\Sigma) \) and consider an abstract Wiener space \((X, \mu)\) whose Cameron-Martin space is \( W^s(\Sigma) \). It is not necessary that \( X = D'(\Sigma) \) and \( \mu_\Sigma = \mu \) however we will see that \( L^2(X, \mu) \) only depends on \( W^1(\Sigma) \).

Let \( \Sigma \) be an oriented compact Riemannian manifold possibly with boundary. We will follow previous remark to construct Gaussian measures on \( D'(\Sigma) \).

For \( f, g \in C^\infty(\Sigma) \) define

\[
<f, g> = \int_\Sigma df \wedge *_{\Sigma} dg + m^2 *_{\Sigma} fg
\]

where \( *_{\Sigma} \) is the Hodge star operator associated to the Riemannian metric on \( \Sigma \). Define the Sobolev space \( W^1(\Sigma) \) to be the completion of \( C^\infty(\Sigma) \) with respect to this inner product. When \( \partial \Sigma = \phi \) this inner product defines the \( W^1(\Sigma) \) as above when \( s = 1 \). Then there exists a nondegenerate Gaussian measure \( \mu_\Sigma \) on \( D'(\Sigma) \) whose Cameron-Martin space is \( W^1(\Sigma, m) \). We want to stress that choosing \( \mu_\Sigma \) is a choice however as in remark 2.3.1 \( L^2(D'(\Sigma), \mu_\Sigma) \) depends only on \( W^1(\Sigma) \).
CHAPTER 3

BOSONIC FOCK SPACES

The main goal of this chapter is to lay the foundation for our construction by developing the necessary language and tools.

We begin with collecting some definitions. For the definition of Bosonic Fock space, subsection 3.1.1, proposition 3.2.20 and proposition 3.2.21 we follow Chapter IV in [9]. All the materials in subsection 3.1.1 comes from Chapter 7 in [8]. The definition of $E$ in 3.1.1.2 is inspired by section 5 of [21], and we took proposition 3.1.16 from there. We use materials from section 2 in [7] for the subsection 3.1.1.1 and subsection 3.2.3.

3.1 Bosonic Fock Spaces

Bosonic Fock space is a very important construction. It is used to describe the quantum state space for an unknown number of Bosonic particles. Roughly, a Bosonic Fock space is a “graded” Hilbert space

$$H = \bigoplus_{n=0}^{\infty} H_n$$

where $H_n$ are Hilbert spaces and $H_0 = \mathbb{R}$ or $\mathbb{C}$. The Hilbert spaces $H_n$ are the “state” spaces for $n$ Bosonic particles. We know from Quantum Mechanics that such a Hilbert space can be represented by $L^2$ space of a Gaussian measure. In this section, we define Bosonic Fock Spaces and relate these spaces with $L^2$ spaces of Gaussian measures.
Let $\mu$ the standard Gaussian measure on $\mathbb{R}$. Let us consider the following toy problem.

**Question:** Can we decompose $L^2(\mathbb{R}, d\mu)$ into orthogonal subspaces in a “canonical” way?

One solution is given by *Hermite polynomials*. Let us recall that the degree $n$ Hermite polynomial is an element $h_n$ of $L^2(\mathbb{R}, d\mu)$ defined by

$$h_n : \mathbb{R} \to \mathbb{R}, \quad h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}).$$

A direct computation shows that

$$<h_n, h_m>_{L^2(\mathbb{R}, d\mu)} = n! \delta_{nm}.$$ 

In fact $\{h_n\}$ form an orthogonal basis of $L^2(\mathbb{R}, d\mu)$. Let $H_n$ be the subspace spanned by $h_n$. Then see that

$$L^2(\mathbb{R}, d\mu) = \bigoplus_{n=0}^{\infty} H_n.$$ 

The upshot here is that the hermite polynomials provide a canonical orthogonal decomposition of $L^2(\mathbb{R}, d\mu)$.

We recall that a Gaussian measure $\mu$ on a finite dimensional real vector space $V$ induces an inner product on $V^\vee$. Our goal is to use $V^\vee$ to construct a family of finite dimensional Hilbert spaces $H'_n$ such that if

$$H' = \bigoplus_{n=0}^{\infty} H'_n,$$ 

then $H'$ will be isomorphic to $L^2(V, \mu)$. We want this construction to be “canonical”.

Given two Hilbert spaces $H_1$ and $H_2$ we can define Hilbert space tensor product $H_1 \hat{\otimes} H_2$ which is the completion of the algebraic tensor product $H_1 \otimes H_2$ with respect
to the inner product

\[ <h_1 \otimes h_2, h'_1 \otimes h'_2> = <h_1, h'_1>_{H_1} \cdot <h_2, h'_2>_{H_2}. \]

From now on we just write \( H_1 \otimes H_2 \) for the Hilbert space tensor product. The Hilbert space tensor product of several Hilbert spaces are defined in the similar way. In particular, we can define the \( n \)th power \( H^{\otimes n} \) of a Hilbert space \( H \).

We recall that \( S_n \) acts on \( H^{\otimes n} \) by

\[ \sigma \cdot (h_1 \otimes \cdots \otimes h_n) = h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}. \]

We define \( \text{Sym}^n H \) to be the closed subspace of \( H^{\otimes n} \) that is invariant under the action of \( S_n \). The map \( P_n : H^{\otimes n} \to \text{Sym}^n H \) given by

\[ P_n(h_1 \otimes \cdots \otimes h_n) = \frac{1}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)} \]

and extended linearly is a projection map.

If we define

\[ h_1 \otimes_s \cdots \otimes_sh_n = \sqrt{n!}P_n(h_1 \otimes \cdots \otimes h_n), \]

then

\[ <h_1 \otimes_sh_n, h'_1 \otimes_sh'_n> = \sum_{\sigma \in S_n} \prod_{i=1}^n <h_i, h'_{\sigma(i)}>_{H} \]

and the closure of the subspace spanned by

\[ \{h_1 \otimes_sh_n | h_i \in H \} \]

is \( \text{Sym}^n H \).
Consider the bilinear map

\[ \otimes_s : \text{Sym}^n H \times \text{Sym}^m H \to \text{Sym}^{n+m} H \]

defined by

\[
(h_1 \otimes_s \cdots \otimes_s h_n) \otimes_s (h_{m+1} \otimes_s \cdots \otimes_s h_{m+n})
= \frac{\sqrt{(m+n)!}}{\sqrt{n!} \cdot \sqrt{m!}} P_{n+m}((h_1 \otimes_s \cdots \otimes_s h_n) \otimes (h_{m+1} \otimes_s \cdots \otimes_s h_{m+n})) \quad (3.1)
\]

and extended linearly. We note that

\[
\frac{\sqrt{(m+n)!}}{\sqrt{n!} \cdot \sqrt{m!}} P_{n+m}((h_1 \otimes_s \cdots \otimes_s h_n) \otimes (h_{m+1} \otimes_s \cdots \otimes_s h_{m+n}))
= h_1 \otimes_s \cdots \otimes_s h_n \otimes_s h_{n+1} \otimes_s \cdots \otimes_s h_{m+n}. \quad (3.2)
\]

The map \( \otimes_s \) is continuous; [9, Page 330].

**Notation 3.1.1.** For \( h_1 \ldots h_n \in H \) we denote \( h_1 \otimes_s \cdots \otimes_s h_n \in \text{Sym}^n H \) by

\[ h_1 \cdot h_2 \cdots h_n. \]

According to our convention

\[ h^n = \sqrt{n!}h \otimes \cdots \otimes h. \]

**Example 3.1.1.** 1. Let \( \{e_i\}_{i=1}^\infty \) be an orthonormal basis of \( H \). Then,

\[ \|e_i^n\|^2 = n!. \]
In general,

$$\|e_1^{k_1} \ldots e_r^{k_r}\| = \sqrt{k_1! \cdot k_2! \ldots k_r!}.$$ 

2. Consider $\mathbb{R}^\vee$ with the standard inner product and $x \in \mathbb{R}^\vee$ be dual to $1 \in \mathbb{R}$. Then,

$$\|x^n\|^2 = \|h_n\|^2_{L^2(\mathbb{R}, d\mu)}$$

where $h_n$ is the degree $n$ Hermite polynomial.

Let $H$ be a separable real Hilbert space. Let $\rho_1, \rho_2 : H^n \to \mathbb{R}$ be continuous multilinear forms where $H^n = H \times \cdots \times H$. Then,

$$\langle \rho_1, \rho_2 \rangle_{\text{Mult}_n(H, \mathbb{R})} = \sum_{h_1, \ldots, h_n \in S} \rho_1(h_1, \ldots, h_n) \rho_2(h_1, \ldots, h_n)$$

where $S$ is an orthonormal basis of $H$, is independent of $S$; \cite[Page 50]{3}. Let $\text{Mult}_n(H, \mathbb{R})$ denote the set of continuous multilinear forms $\rho : H^n \to \mathbb{R}$ such that

$$\langle \rho, \rho \rangle_{\text{Mult}_n(H, \mathbb{R})} < \infty.$$ 

Then $\text{Mult}_n(H, \mathbb{R})$ is a Hilbert space and it is canonically isomorphic to $(H^\vee)^{\otimes n}$.

Let $\text{Sym}_n(H, \mathbb{R})$ denote the set of symmetric multilinear forms $\rho \in \text{Mult}_n(H, \mathbb{R})$. Then $\text{Sym}_n(H, \mathbb{R})$ is a closed subspace of $\text{Mult}_n(H, \mathbb{R})$.

Now, let $\rho_1, \ldots, \rho_n \in H^\vee$. This defines an element $\rho_1 \otimes \cdots \otimes \rho_n$ defines an element of $\text{Mult}_n(H, \mathbb{R})$ given by

$$\rho_1 \otimes \cdots \otimes \rho_n(h_1, \ldots, h_n) = \prod_{i=1}^n \rho_i(h_i).$$

In fact, this map gives the canonical isomorphism of Hilbert spaces between $(H^\vee)^{\otimes n}$
and Mult\(_n(H, \mathbb{R})\) mentioned above. Moreover,

\[
\|\rho_1 \otimes_s \cdots \otimes_s \rho_n\|_{\text{Sym}_n(H, \mathbb{R})}^2 = \sum_{\sigma \in S_n} \prod_{i=1}^n <\rho_i, \rho_{\sigma(i)}>_{H^\vee}.
\]

This implies that we can canonically identify Sym\(_n(H^\vee)\) with Sym\(_n(H, \mathbb{R})\).

**Definition 3.1.2.** The Bosonic Fock space of a real separable Hilbert space \(H\) is the Hilbert space direct sum

\[
\bigoplus_{n=0}^{\infty} \text{Sym}^n H = \{(\alpha_n)_{n=0}^{\infty} : \alpha_n \in \text{Sym}^n H \text{ with } \sum_{n=0}^{\infty} \|\alpha_n\|^2 < \infty\}.
\]

We use \(\text{Sym}^*(H)\) to denote the Bosonic Fock space of \(H\). One defines the Bosonic Fock space of a complex Hilbert space in a similar fashion.

**Remark 3.1.3.** \(\text{Sym}^*(H)\) is not an algebra. Even though there is a continuous map

\[
\otimes_s : \text{Sym}^n H \times \text{Sym}^m H \rightarrow \text{Sym}^{n+m} H
\]

it does not extend to a continuous map

\[
\text{Sym}^*(H) \times \text{Sym}^*(H) \rightarrow \text{Sym}^*(H).
\]

3.1.1 Exponential Properties of \(\text{Sym}^*\)

Define a map \(\text{Exp} : H \rightarrow \text{Sym}^*(H)\) given by

\[
\text{Exp}(h) = \sum_{n=0}^{\infty} \frac{h^n}{n!}.
\]

From definition, we see that

\[
<\text{Exp}(h), \text{Exp}(h')> = e^{<h, h'>}.
\]
This shows that Exp is continuous. Moreover, Exp is injective.

**Lemma 3.1.4.** The set \{Exp(h) : h ∈ H\} is linearly independent and the closure of the subspace generated by this is equal to \(\text{Sym}^*(H)\).

**Proof.** Let \(K\) be the closure of the subspace generated by \{Exp(h) : h ∈ H\}. We claim that \(h^n ∈ K\) for each \(h ∈ H\). Then, the polarization identity implies that \(\text{Sym}^n H ⊂ K\). We note that map \(γ : \mathbb{R} → K\) given by

\[γ(t) = \text{Exp}(th)\]

is a smooth map. In particular, \(\frac{d^n}{dt^n}γ(t)|_{t=0} ∈ K\). But

\[\frac{d^n}{dt^n}γ(t)|_{t=0} = h^n\]

which implies the claim. This shows that \(K = \text{Sym}^*(H)\).

To show linear independence we assume

\[\sum_{i=1}^{n} c_i \text{Exp}(h_i) = 0\]

for some scalars \(c_1, \ldots, c_n\). Define the function \(f : H → \mathbb{R}\) by

\[f(x) = \sum_{i=1}^{n} c_i e^{〈h_i, x〉} .\]

Our hypothesis implies that \(f\) is indentically zero. Now taking the \(p\)th derivative of \(f\) at zero and evaluating at \((b, \ldots, b)\) we get

\[\sum_{i=1}^{n} c_i < h_i, b > = 0\]

for all \(b ∈ H\). This shows that \(c_i = 0\) for all \(i = 1 \ldots n\).
In the next proposition we show that the assignment $H \to \text{Sym}^*(H)$ behaves like the usual exponential map. This justifies the notation Exp.

**Proposition 3.1.5.** Let $H_1$ and $H_2$ be two Hilbert spaces. Then there is a unique isomorphism of Hilbert spaces

$$E : \text{Sym}^*(H_1 \oplus H_2) \to \text{Sym}^*(H_1) \otimes \text{Sym}^*(H_2)$$

such that

$$E(\text{Exp}(h_1, h_2)) = \text{Exp}(h_1) \otimes \text{Exp}(h_2).$$

**Proof.** We note that the uniqueness follows from lemma 3.1.4. As a consequence, we don’t have a choice but to define the map

$$E : \text{Sym}^*(H_1 \oplus H_2) \to \text{Sym}^*(H_1) \otimes \text{Sym}^*(H_2)$$

by setting

$$E(\text{Exp}(h_1, h_2)) = \text{Exp}(h_1) \otimes \text{Exp}(h_2)$$

and extending linearly. We check that

$$< \text{Exp}(h_1, h_2), \text{Exp}(h'_1, h'_2)> = e^{<h_1, h'_1>} e^{<h_2, h'_2>} = e^{<h_1, h'_1>} \cdot e^{<h_2, h'_2>} = <\text{Exp}(h_1), \text{Exp}(h'_1)> \cdot <\text{Exp}(h_2), \text{Exp}(h'_2)>$$

This fact together with lemma 3.1.4 implies that $E$ is indeed an isomorphism of Hilbert spaces. □
3.1.1.1 Bosonic Fock Space Functor

Let $H_1$ and $H_2$ be separable Hilbert spaces and $A : H_1 \to H_2$ be a continuous linear map. Then $A$ induces a bounded linear map

$$\text{Sym}^n(A) : \text{Sym}^n(H_1) \to \text{Sym}^n(H_2)$$

given by

$$\text{Sym}^n(A)(u_1 \cdot u_2 \ldots u_n) = Au_1 \cdot Au_2 \ldots Au_n.$$

By definition $\|\text{Sym}^n(A)\| = \|A\|^n$. Moreover, when $\|A\| \leq 1$, it induces a unique bounded linear map ([9, Page 45])

$$\text{Sym}^*(A) : \text{Sym}^*(H_1) \to \text{Sym}^*(H_2)$$

such that

$$\text{Sym}^*(A)|_{\text{Sym}^n(H_1)} = \text{Sym}^n(A).$$

Let $A : H_1 \to H_2$ and $B : H_2 \to H_3$ then

$$\text{Sym}^n(B \circ A) = \text{Sym}^n(B) \circ \text{Sym}^n(A).$$

In other words the construction $\text{Sym}^n$ is functorial. This shows that

$$\text{Sym}^*(B \circ A) = \text{Sym}^*(B) \circ \text{Sym}^*(A)$$

whenever $\text{Sym}^*(B)$ and $\text{Sym}^*(A)$ are defined. We can summarize this as the construction $\text{Sym}^*$ defines an exponential functor from the category whose objects are separable Hilbert spaces and whose morphisms are $A : H_1 \to H_2$ are bounded linear operators such that $\|A\| \leq 1$ to itself.
3.1.1.2 Another Exponential Map

Let $\mu$ be the standard Gaussian measure on $\mathbb{R}$. We recall that the function

$$ x \mapsto ax^2 $$

is in $L^2(\mathbb{R}, \mu)$ for all $a$. However, the function

$$ x \mapsto e^{\frac{1}{2}ax^2} $$

is in $L^2(\mathbb{R}, \mu)$ if and only if $a < \frac{1}{2}$. There is a similar phenomenon in $\text{Sym}^2(H^\vee)$. It says that it is not possible to “exponentiate” each and every element in $\text{Sym}^2(H^\vee)$.

Let $a \in \mathbb{R}$ and $x \in \mathbb{R}^\vee$ be dual to $1 \in \mathbb{R}$. Define

$$ \text{Exp}(ax^2) = \sum_{n=0}^{\infty} \frac{(ax^2)^n}{n!}. $$

**Lemma 3.1.6.** $\|\text{Exp}(ax^2)\|^2 = \sum_{n=0}^{\infty} \frac{a^{2n}(2n)!}{(n!)^2}. $

**Proof.** We note that

$$ \|x^n\|^2 = n!, $$

using this we get

$$ \|\text{Exp}(ax^2)\|^2 = \sum_{n=0}^{\infty} \frac{a^{2n}\|x^{2n}\|^2}{(n!)} = \sum_{n=0}^{\infty} \frac{a^{2n}(2n)!}{(n!)^2}. $$

We recall that

$$ \sum_{n=0}^{\infty} \frac{a^{2n}(2n)!}{(n!)^2} = (1 - 4a^2)^{-\frac{1}{2}}, $$

whenever $|a| < \frac{1}{2}$. 

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Corollary 3.1.7. \( \exp(\frac{1}{2}ax^2) \in \text{Sym}^*(\mathbb{R}^\vee) \) if and only if \(|a| < 1\). If \( \exp(\frac{1}{2}ax^2) \in \text{Sym}^*(\mathbb{R}^\vee) \), then
\[
\| \exp(\frac{1}{2}ax^2) \|^2 = (1 - a^2)^{-\frac{1}{2}}.
\]

In particular,
\[
\langle \exp(\frac{1}{2}ax^2), \exp(\frac{1}{2}bx^2) \rangle = (1 - ab)^{-\frac{1}{2}}
\]
whenever \(|a| < 1\) and \(|b| < 1\).

Proof. We recall that
\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n}(2n)!}{(n!)}
\]
is convergent if and only if \(|\alpha| < \frac{1}{2}\). Hence,
\[
\exp(\alpha x^2) \in \text{Sym}^*(\mathbb{R}^\vee)
\]
if and only if \(|\alpha| < \frac{1}{2}\). Now \(\alpha = \frac{1}{2}a\) implies the corollary. \(\square\)

Let \(H\) be a finite dimensional real Hilbert space and \(A : H \to H\) be a symmetric operator. Define a map \(S(A) : H \times H \to \mathbb{R}\) by
\[
S(A)(u \otimes v) = \langle Au, v \rangle.
\]
Since \(A\) is symmetric, we see that \(S(A) \in \text{Sym}^2(H^\vee)\). Moreover, if \(O : H \to H\) is an orthogonal operator then \(S(OAO^{-1}) = S(A)\).

Note 3.1.8. From now on we write \(A\) for \(S(A)\).

Definition 3.1.9. Let \(H\) be a finite dimensional real Hilbert space and let \(A : H \to H\)
symmetric. We define 

\[ \mathcal{E}(A) = \exp\left(\frac{1}{2} A\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} A)^n}{n!}. \]

**Notation 3.1.10.** Let \( H \) is a separable real Hilbert space. We use \( \mathcal{Z}(H) \) to denote the set of all symmetric operators \( A \) on \( H \) such that \( \|A\| < 1 \). Here the norm means the operator norm.

**Lemma 3.1.11.** Assume that \( H \) is a finite dimensional real Hilbert space. Then, \( \mathcal{E}(A) \in \text{Sym}^*(H^\vee) \) if and only if \( A \in \mathcal{Z}(H) \). When \( A, B \in \mathcal{Z}(H) \), then

\[ <\mathcal{E}(A), \mathcal{E}(B)> = \det(1 - AB)^{-\frac{1}{2}}. \]

**Proof.** \( A : H \to H \) symmetric. Let \( \{e_i\} \) be an orthonormal basis of \( H^\vee \). Diagonaling \( A \) if necessary we can always assume that

\[ A = \sum_{i=1}^{n} \lambda_i e_i^2 \]

where \( n \) is the dimension of \( H \). This allows us to reduce to the case when \( n = 1 \). Now by lemma 4.2.2 we conclude that \( \mathcal{E}(A) \in \text{Sym}^*(H^\vee) \) if and only if \( A \in \mathcal{Z}(H) \).

Let \( B \in \mathcal{Z}(H) \). Without loss of generality (we can always modify \( B \) by conjugating with an orthogonal operator) we can assume that

\[ B = \sum_{i=1}^{n} \gamma_i e_i^2. \]
Therefore,

\[ <\mathcal{E}(A),\mathcal{E}(B)> = <\text{Exp}(\frac{1}{2}\sum_{i=1}^{n}\lambda_i e_i^2),\text{Exp}(\frac{1}{2}\sum_{i=1}^{n}\gamma_i e_i^2)> \]

\[ = \prod_{i=1}^{n} <\text{Exp}(\frac{1}{2}\lambda_i e_i^2),\text{Exp}(\frac{1}{2}\gamma_i e_i^2)> \]

\[ = \prod_{i=1}^{n} (1 - \lambda_i \cdot \gamma_i)^{-\frac{1}{2}} \text{ [Using corollary 3.1.7]} \]

\[ = \det(1 - AB)^{-\frac{1}{2}} \]

Hence, we have a well defined map \( \mathcal{E} : \mathcal{Z}(H) \rightarrow \text{Sym}^*(H^\vee) \). Moreover,

**Corollary 3.1.12.** \( \mathcal{E} \) is continuous.

**Proof.** Using previous lemma, we compute that

\[ \|\mathcal{E}(A) - \mathcal{E}(B)\|^2 = \det(1 - A^2)^{-\frac{1}{2}} + \det(1 - B^2)^{-\frac{1}{2}} - 2(\det(1 - AB)^{-\frac{1}{2}}). \]

When \( A \) is close to \( B \) the quantity on the right hand side approached to zero. This proves the corollary.

We will use this corollary to define the exponential map \( \mathcal{E} \) in infinite dimension.

Let \( H \) be a separable real Hilbert space. We recall the space of Hilbert Schmidt operators forms a Hilbert space with respect to the Hilbert Schmidt norm. We refer to the appendix for this fact.

**Definition 3.1.13.** We define \( \mathcal{Z}_2(H) \) to be the set of symmetric Hilbert Schmidt operators of operator norm less than one.
We note that \( \mathcal{Z}_2(H) \) is an open subset of the Hilbert space of Hilbert Schmidt operators on \( H \). We will use the following lemma to define

\[
\mathcal{E} : \mathcal{Z}_2(H) \to \text{Sym}^*(H^\vee).
\]

We will need the following lemma whose proof is obvious.

**Lemma 3.1.14.** The map \( \mathcal{Z}_2(H) \times \mathcal{Z}_2(H) \to \mathbb{R} \) defined by

\[
(A, B) \to (\det(I - A^2))^{-\frac{1}{2}} + (\det(I - B^2))^{-\frac{1}{2}} - 2(\det(I - AB))^{-\frac{1}{2}}
\]

is a continuous map.

Let \( \{A_n\} \) be a sequence of finite rank operators in \( \mathcal{Z}_2(H) \) converging to \( A \). Then previous lemma implies that \( \{\mathcal{E}(A_n)\} \) is Cauchy in \( \text{Sym}^*(H^\vee) \). Hence the sequence \( \{\mathcal{E}(A_n)\} \) converges in \( \text{Sym}^*(H^\vee) \). Now, we define

\[
\mathcal{E}(A) = \lim_{n \to \infty} \mathcal{E}(A_n).
\]

Then from finite dimensional case it follows that,

\[
< \mathcal{E}(A), \mathcal{E}(B) >_{\text{Sym}^*(H^\vee)} = (\det(I - AB))^{-\frac{1}{2}}.
\]

(3.3)

for all \( A, B \in \mathcal{Z}_2(H) \). We also get

\[
\|\mathcal{E}(A) - \mathcal{E}(B)\|^2 = (\det(I - A^2))^{-\frac{1}{2}} + (\det(I - B^2))^{-\frac{1}{2}} - 2(\det(I - AB))^{-\frac{1}{2}}.
\]

This shows that

**Proposition 3.1.15.** \( \mathcal{E} : \mathcal{Z}_2(H) \to \text{Sym}^*(H^\vee) \) is continuous.
We also see that the map

$$A \mapsto \det(I - A^2)^{\frac{1}{2}} E(A)$$

is an embedding of $\mathcal{Z}_2(H)$ into the unit ball in $\text{Sym}^*(H^\vee)$.

Let $\mathcal{F}$ denote the completion free vector space generated by the set

$$\{ E(A) : A \in \mathcal{Z}_2(H) \}$$

with respect to the inner given by (3.3). By construction there is an isometry $\mathcal{F} \to \text{Sym}^*(H^\vee)$. In fact,

**Proposition 3.1.16.** $\mathcal{F} = \text{Sym}^{\text{even}}(H^\vee)$.

**Proof.** Let $h \in H$ such that $\|h\| = 1$ and $f = \langle h, . . \rangle$. Define a linear operator on $A_f$ on $H$ which identity on the one dimensional subspace of $H$ spanned by $\{h\}$ and zero on the orthogonal complement of $h$. Then the map $t \to E(tA_f)$ is smooth for small values of $t$. Now taking derivative we see that $f^{\otimes 2n}$ is in $\mathcal{F}$ for all $n$. We can think of $f^{\otimes 2n}$ as Taylor coefficient of $t \to E(tA_f)$ as a function of $t$. Now using polarization identity we see that $\mathcal{F}$ contains $\text{Sym}^m(H^\vee)$ whenever $m$ is even and hence $\mathcal{F} = \text{Sym}^{\text{even}}(H^\vee)$. \hfill $\square$

### 3.2 Relation between Bosonic Fock Space and Gaussian Measures

#### 3.2.1 Relation between Bosonic Fock Space and Gaussian Measures Finite Dimensional Case

Let $\mu$ be standard Gaussian measure on $\mathbb{R}$. We recall that it defines an inner product on $\mathbb{R}^\vee$.

**Lemma 3.2.1.** There is unique isomorphism $Q : \text{Sym}^*(\mathbb{R}^\vee) \to L^2(\mathbb{R}, d\mu)$ such that $Q(x^n) = h_n$ where $x \in \mathbb{R}^\vee$ is dual to $1 \in \mathbb{R}$ and $h_n$ is the degree $n$ Hermite polynomial.
Proof. Define \( Q : \text{Sym}^*(\mathbb{R}^\vee) \to L^2(\mathbb{R}, d\mu) \) which on \( \text{Sym}^n(\mathbb{R}^\vee) \) is given by

\[
Q(x^n) = h_n.
\]

We recall that \( \|x^n\|^2 = n! \)

and

\[
\|h_n\|^2 = n!.
\]

Thus,

\[
\|Q(x^n)\|^2 = \|x^n\|^2.
\]

This shows that \( Q \) is an isomorphism of Hilbert spaces. \( \square \)

Fact 3.2.2. \( e^{ax-\frac{1}{2}a^2} = \sum_{n=0}^{\infty} \frac{1}{n!} a^n h_n. \)

Proof. We note that

\[
e^{ax-\frac{1}{2}a^2} = e^{\frac{1}{2}x^2-\frac{1}{2}(x-a)^2}
= e^{\frac{1}{2}x^2} e^{-\frac{1}{2}(x-a)^2}
\]

Now we write

\[
e^{-\frac{1}{2}(x-a)^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{da^n} e^{-\frac{1}{2}(x-a)^2} \bigg|_{a=0}
\]

which is the Maclaurin series expansion in the variable \( a \). We note that

\[
\frac{d^n}{da^n} e^{-\frac{1}{2}(x-a)^2} \bigg|_{a=0} = (-1)^n \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}.
\]
This gives us

\[ e^{ax - \frac{1}{2}a^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} a^n h_n \]

Hence,

\[ Q(\text{Exp}(ax)) = e^{ax - \frac{1}{2}a^2} \]

where \( e^{ax - \frac{1}{2}a^2} \) is considered as function and it follows that

\[ Q(\text{Exp} h) = e^{h - \frac{1}{2}||h||^2} \]

for all \( h \in \mathbb{R}^\vee \). Moreover,

\[ Q(\text{Exp} h) = e^{h - \frac{1}{2}||h||^2} \]

implies that \( Q \) is unique.

Let us denote by \( L^2(\mathbb{R}, \mu, \mathbb{C}) \) the complex valued \( L^2 \) functions. Then \( Q \) induces an unitary map \( Q : \text{Sym}^*(\mathbb{C}^\vee) \to L^2(\mathbb{R}, \mu, \mathbb{C}) \).

Let \( H \) be a finite dimensional real vector space and \( \mu_H \) a centered Gaussian measure on \( H \). Lemma 3.2.1 can be generalized in a routine way to construct the unique isomorphism

\[ Q : \text{Sym}^*(H^\vee) \to L^2(H, \mu_H) \]

with the property that

\[ Q(\text{Exp} h) = e^{h - \frac{1}{2}||h||^2}. \]

As in the one dimensional case , \( Q \) induces an unitary map

\[ Q : \text{Sym}^*(H_\mathbb{C}^\vee) \to L^2(H, \mu_H, \mathbb{C}) \]
where $H_C$ is the complexification of $H$ and $L^2(H, \mu_H, \mathbb{C})$ is the $L^2$ space of complex valued functions. This map $Q$ can be thought as a “backward heat” operator which maps polynomials to $L^2$ functions. Next, we will attempt to justify this assertion.

Let us identify $H$ with $\mathbb{R}^n$ such that $\mu_H$ corresponds to the standard Gaussian measure $\mu$ on $\mathbb{R}^n$. Let $\mu_C$ be the Gaussian measure on $\mathbb{C}^n$ with the density $\pi^{-n} e^{-\|z\|^2}$ with respect to the standard Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Let $\mathcal{H}(\mathbb{C}^n)$ denote the space of holomorphic functions on $\mathbb{C}^n$ and $\mathcal{H}L^2(\mathbb{C}^n, \mu_C)$ denote the space of $L^2$-holomorphic functions on $\mathbb{C}^n$.

**Notation 3.2.3.** For a multiindex

$$\alpha = (k_1, \ldots, k_n)$$

we write

$$z^\alpha = \prod_{i=1}^n z_i^{k_i}$$

and

$$\alpha! = \prod_{i=1}^n k_i!.$$

It is a well known fact ([7, page 79]) that the set $\{z^\alpha\}$ where $\alpha$ runs over all the multiindices forms an orthogonal basis for $\mathcal{H}L^2(\mathbb{C}^n, \mu_C)$ and

$$\|z^\alpha\|_{\mathcal{H}L^2(\mathbb{C}^n, \mu_C)} = \sqrt{\alpha!}.$$

In particular, if $f \in \mathcal{H}L^2(\mathbb{C}^n, \mu_C)$ and

$$f(z) = \sum_{\alpha} c_{\alpha} z^\alpha,$$

then

$$\|f\|_{\mathcal{H}L^2(\mathbb{C}^n, \mu_C)}^2 = \sum_{\alpha} |c_{\alpha}|^2 \alpha!.$$

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Define $S : L^2(\mathbb{R}^n, d\mu, \mathbb{C}) \to \mathcal{H}L^2(\mathbb{C}^n, \mu_\mathbb{C})$ by

$$
Sf(z) = e^{-\frac{1}{2} \langle z, z \rangle} \int_{\mathbb{R}^n} f(u)e^{\langle z, u \rangle} d\mu(u).
$$

It is well known that the map $S$ is a unitary map; [7, Theorem 2.3]. Here $\langle, \rangle$ is the complex bilinear extension of the standard inner product on $\mathbb{R}^n$. The map $S$ takes Hermite polynomials to the $z^\alpha$s. Note that

$$
Sf(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(z - u)e^{-\frac{1}{2} \langle u, u \rangle} du. \tag{3.4}
$$

We recall that the density function

$$
\rho(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \langle x, x \rangle}
$$

of $\mu$ with respect to the Lebesgue measure can be thought as the kernel function of the forward heat operator $e^{-\frac{1}{2} \Delta}$ where

$$
\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}
$$

the Laplacian on $\mathbb{R}^n$. This says that $Sf$ can be interpreted as the forward heat operator applied to $f$ at time $t = \frac{1}{2}$. There is a tautological unitary map $T$ called the Taylor map

$$
T : \mathcal{H}L^2(\mathbb{C}^n, \mu_\mathbb{C}) \to \text{Sym}^*((\mathbb{C}^n)^\vee).
$$

It turns out that $T \circ S$ maps the Hermite polynomials to $z^\alpha$s. This shows that

$$
T \circ S = Q^{-1}.
$$

Since $T$ is tautological, we can treat $T$ as identity operator. Then we can write
$Q = S^{-1}$. Now $S$ is a forward heat operator implies that $Q$ is a backward Heat operator.

Now we construct the unitary map

$$T : \mathcal{H}L^2(\mathbb{C}^n, \mu_{\mathbb{C}}) \to \text{Sym}^*((\mathbb{C}^n)^\vee).$$

Let $\{z_1, z_2, \ldots, z_n\}$ be the basic of $(\mathbb{C}^n)^\vee$ dual to the standard basis on $\mathbb{C}^n$. Now, we define the Taylor map

$$T : \mathcal{H}L^2(\mathbb{C}^n, \mu_{\mathbb{C}}) \to \text{Sym}^*((\mathbb{C}^n)^\vee)$$

by requiring

$$T(z^\alpha) = z_1^{k_1} \cdots z_n^{k_n}$$

and then extending linearly.

From now on we identify $T \circ S$ with $S$. Hence we have a unitary map

$$S : L^2(\mathbb{R}^n, d\mu, \mathbb{C}) \to \text{Sym}^*((\mathbb{C}^n)^\vee).$$

Moreover, the maps $Q$ and $S$ commute with complex conjugation. Hence, we get the following commutative diagram.

\[
\begin{array}{ccc}
\text{Sym}^*((\mathbb{C}^n)^\vee) & \xrightarrow{S} & L^2(\mathbb{C}^n, d\mu, \mathbb{C}) \\
\Downarrow & & \Downarrow \\
\text{Sym}^*((\mathbb{R}^n)^\vee) & \xrightarrow{S} & L^2(\mathbb{R}^n, d\mu) \\
\end{array}
\]

**Proposition 3.2.4.** The inverse of $Q : \text{Sym}^*(\mathcal{H}^\vee) \to L^2(\mathcal{H}, d\mu_{\mathcal{H}})$ is $S$. 

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Proof. From the discussion above we notice that for any polynomial \( P \) on \( H \), we have

\[
S(Q(P)) = P.
\]

Let \( K \) be a subspace of \( H \) where \( H \) is a finite dimensional real Hilbert space and \( \mu_K \) be the standard Gaussian measure on \( F \). If \( f \in L^2(H, \mu, \mathbb{C}) \) is such that

\[
f(x) = f(\pi_K(x))
\]

where

\[
\pi_K : H \to K
\]

is the orthogonal projection, then

\[
\int_H f \, d\mu_H = \int_K f \, d\mu_K.
\]

This means that we have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Sym}^*(H^\times) & \xrightarrow{S} & L^2(H, d\mu, \mathbb{C}) \\
P_F & & \downarrow P_F \\
\text{Sym}^*(K^\times) & \xleftarrow{S} & L^2(K, d\mu_K, \mathbb{C})
\end{array}
\]

Hence we get a commutative diagram as above whenever there is an isometry \( K \to H \).

3.2.1.1 Radon-Nikodym Derivative(Finite Dimensional Case):

**Definition 3.2.5.** Let \( H \) be a finite real Hilbert space. An operator \( A : H \to H \) is called positive if \( \langle Ah, h \rangle \) is positive for all nonzero \( h \in H \). It is an elementary fact
from linear algebra that a positive operator is symmetric i.e.

\[ <Au,v> = <u,Av> \]

for all \( u,v \in H \).

**Notation 3.2.6.** For a finite dimensional real Hilbert space \( H \) the notation \( S(H) \) means the set of all inner products on \( H \). We note that there is a canonical bijection between the set \( S(H) \) and the set of positive operators on \( H \). We usually do not distinguish between an inner product and the corresponding positive operator.

Consider the map \( C : (0, \infty) \to (-1, 1) \) given by \( x \mapsto (1-x)(1+x)^{-1} \). It is easy to see that \( C \) is a homeomorphism. Now we use \( C \) to define the **Cayley transform** of an operator.

**Definition 3.2.7.** Let \( H \) be a real Hilbert space and let \( A : H \to H \) be a bounded positive operator. The Cayley transform of \( A \) is defined by \( C(A) = (I-A)(I+A)^{-1} \). We note that \( C(A) \in \mathcal{Z}(H) \) whenever \( A \) is a positive operator on \( H \) where \( \mathcal{Z}(H) \) is the set of symmetric operator on \( H \) with norm less than 1.

**Lemma 3.2.8.** Assume that \( H \) is finite dimensional. Then, \( C : \mathcal{S}(H) \to \mathcal{Z}(H) \) is a homeomorphism.

**Remark 3.2.9.** There are many version of Cayley transforms in the literature.

Let \( A \) be a positive linear operator on a \( n \)-dimensional real Hilbert space \( H \) and let \( \nu \) be the Gaussian measure on \( H \) with the density

\[ (2\pi)^{-\frac{n}{2}} \sqrt{\det(A)} e^{-\frac{1}{2} <Ax,x>}. \]

**Proposition 3.2.10.** \( S \left( \frac{d\nu}{d\mu} \right)^{\frac{1}{2}} = \frac{\mathcal{E}(C(A))}{\|\mathcal{E}(C(A))\|} \).
Proof. Without loss of generality we assume that $H = \mathbb{R}^n$ with the standard inner product and $\mu$ the standard Gaussian measure on $\mathbb{R}^n$. We recall that

$$\frac{d\nu}{d\mu} = (\det(A))^{\frac{1}{2}} e^{-\frac{1}{2}<(A-I)x,x>}.$$ 

Now,

$$S \left( \frac{d\nu}{d\mu} \right)^{\frac{1}{2}} (z) = \frac{\det(A)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}<z,z>} \int_{\mathbb{R}^n} e^{-\frac{1}{4}<(A-I)u,u>} e^{<z,u>} e^{-\frac{1}{2}<u,u>} \, du$$

$$= \frac{\det(A)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}<z,z>} \int_{\mathbb{R}^n} e^{-\frac{1}{4}<(A+I)u,u>} e^{<z,u>} \, du$$

We recall that for a positive operator $B : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}<Bu,u>} e^{<z,u>} \, du = \frac{(2\pi)^{\frac{n}{2}}}{\det(B)^{\frac{1}{2}}} e^{\frac{1}{2}<B^{-1}z,z>}.$$ 

Using this we get

$$\frac{\det(A)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}<z,z>} \int_{\mathbb{R}^n} e^{-\frac{1}{4}<(A+I)u,u>} e^{<z,u>} \, du$$

$$= \frac{\det(A)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}} \det(A+I)^{\frac{1}{2}}} e^{-\frac{1}{2}<z,z>} e^{<(A+I)^{-1}z,z>}$$

$$= \det(2\sqrt{A}(A+I)^{-1})^{\frac{1}{2}} e^{\frac{1}{2}<(I-A)(I+A)^{-1}z,z>}$$

This means that

$$S \left( \frac{d\nu}{d\mu} \right)^{\frac{1}{2}} = \det(2\sqrt{A}(A+I)^{-1})^{\frac{1}{2}} \mathcal{E}(C(A)).$$
Now using lemma 3.1.11 we recognize that

\[ \| \mathcal{E}(C(A)) \| = \frac{1}{\det(2\sqrt{A}(A + I)^{-1})^{\frac{1}{2}}}. \]

3.2.1.2 The semicategories \( T_{1, \text{fin}}^{1, \text{fin}} \) and \( T_{2, \text{fin}}^{2, \text{fin}} \)

We can use square roots of the Radon-Nikodym derivatives to define semicategories \( T_{pol}^{1, \text{fin}} \) and \( T_{pol}^{2, \text{fin}} \). The main idea is that we can think of the Radon-Nikodym derivative as “path integral”. We will use infinite dimensional version of these categories to give a functorial interpretation of Free Classical Field Theories.

Let \( H_1, H_2, \) and \( H_3 \) be a finite dimensional real Hilbert Spaces with the standard Gaussian measures \( \mu_1 \) and \( \mu_2 \) and \( \mu_3 \) respectively. Any

\[ \alpha \in L^2(H_1, \mu_1) \otimes L^2(H_2, \mu_2) \]

induces a Hilbert-Schmidt operator

\[ K_\alpha : L^2(H_1, \mu_2) \to L^2(H_2, \mu_2). \]

By abuse of notation we denote the kernel of \( K_\alpha \) by \( K_\alpha \). Let

\[ \beta \in L^2(H_2, \mu_2) \otimes L^2(H_3, \mu_3) \]

and \( K_\beta \) be the corresponding Hilbert-Schmidt operator from \( L^2(H_2, \mu_2) \) to \( L^2(H_3, \mu_3) \). We recall that \( K_\beta \circ K_\alpha \) is the operator with kernel

\[ \int_{H_2} K_\alpha(h_1, h_2) K_\beta(h_2, h_3) d\mu_{H_2}. \quad (3.6) \]
Let $H_1$ and $H_2$ be finite dimensional real Hilbert spaces and $A \in \mathcal{S}(H_1 \oplus H_2)$. Let $\nu_A$ be the Gaussian measure on $H_1 \oplus H_2$ corresponding to $A$. Then,

$$\left(\frac{d\nu_A}{d(\mu_1 \otimes \mu_2)}\right)^{\frac{1}{2}}$$

defines a Hilbert Schmidt operator

$$K_A : L^2(H_1, d\mu_{H_1}) \to L^2(H_2, d\mu_{H_2}).$$

The map $K$ from $\mathcal{S}(H_1 \oplus H_2)$ to the space of Hilbert Schmidt operators

$$L^2(H_1, d\mu_{H_1}) \to L^2(H_2, d\mu_{H_2})$$

is a continuous.

**Lemma 3.2.11.** Let $H_1$, $H_2$ and $H_3$ be finite dimensional real Hilbert spaces,

$$A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^T & D_1 \end{bmatrix} \in \mathcal{S}(H_1 \oplus H_2) \text{ and } A_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^T & D_2 \end{bmatrix} \in \mathcal{S}(H_2 \oplus H_3).$$

Then

$$K_{A_2} \circ K_{A_1} = cK_{A_3}$$

where

$$A_3 = \begin{bmatrix} A_2 - B_2(A_1 + D_2)^{-1}B_1^T & -B_2(A_1 + D_2)^{-1}B_1 \\ B_1^T(A_1 + D_2)^{-1}B_2^T & D_1 - B_1^T(A_1 + D_2)^{-1}B_1 \end{bmatrix} \in \mathcal{S}(H_1 \oplus H_3)$$

and

$$c = \frac{\det(A_1)^{\frac{1}{4}} \det(A_2)^{\frac{1}{4}}}{\det(A_2 + D_2)^{\frac{1}{2}}}$$

**Proof.** We know that the kernel of the operator $K_{A_2} \circ K_{A_1}$ is given by the integral

$$\int_{H_2} K_{A_2}(z, y) K_{A_1}(y, x) \, d\mu_{H_2}(y).$$
Using results of 2.2, we compute that

\[ \int_{H_2} K_{A_2}(z, y) K_{A_1}(y, x) \, d\mu_{H_2}(y) \]

\[ = \frac{\det(A_1)^{\frac{1}{2}} \det(A_2)^{\frac{1}{2}}}{\det\left(\frac{A_2 + D_1}{2}\right)^{\frac{1}{2}}} K_{A_3}(z, x) \]

where \( A_3 \) is given as above. We note that \( K_{A_3}(z, x) \in L^2(H_1, \, d\mu_{H_1}) \otimes L^2(H_3, \, d\mu_{H_3}) \) automatically implies that \( A_3 \) is a positive operator.

\[ \square \]

**Corollary 3.2.12.** The set \( \{ cK_A | A \in S(H \oplus H), c \in (0, \infty) \} \) forms a semigroup.

We want to point out that corollary 3.2.12 is a special case of the oscillator semigroup associated the Siegel upper half plane \( \Sigma_{2n} \) which is the set of symmetric complex \( 2n \times 2n \) matrices such that \( A \) such that \( \text{Re}(A) \) is positive definite. The construction of the oscillator semigroup goes as follows. One can define a map

\[ \gamma : \Sigma_{2n} \to L^2(\mathbb{R}^{2n}) \text{ by } \gamma(A) = e^{-\frac{1}{2} \langle Ax, x \rangle}. \]

Then \( \gamma \) is well defined because \( \text{Re}(A) \) is positive definite. Now, think of \( \gamma(A) \) as a Hilbert Schmidt operator

\[ K_{\gamma(A)} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \]

As in 3.2.11 it turns out that \( K_{\gamma(A_2)} \circ K_{\gamma(A_1)} = c K_{\gamma(A_3)} \) where \( c \) is a nonzero complex number and \( A_3 \in \Sigma_{2n} \). This shows that the set

\[ \{ cK_{\gamma(A)} | A \in \Sigma_{2n}, c \in \mathbb{C} \setminus \{0\} \} \]

is a semigroup which is known as oscillator semigroup; For details we refer to Chapter 4 in [4]. Now, we see that the semigroup in the corollary 3.2.12 can be obtained from the oscillator semigroup by taking the real matrices.
In fact, one can use the same calculation which is used to show semigroup law to show that there is a well defined map

\[ \Sigma_{m+n} \times \Sigma_{n+p} \to \Sigma_{m+p} \]

as in \[3.2.11\].

We can use this idea to construct a semicategory \( \mathcal{G}M \). An object in \( \mathcal{G}M \) is a finite dimensional real Hilbert space. Morphisms \( H_1 \to H_2 \) are nondegenerate centered Gaussian measures on \( H_1 \oplus H_2 \). Let \( \nu_1 : H_1 \to H_2 \) and \( \nu_2 : H_2 \to H_3 \) be morphisms in \( \mathcal{G}M \). Let \( A_1 \in \mathcal{S}(H_2 \oplus H_1) \) and \( A_2 \in \mathcal{S}(H_3 \oplus H_2) \) corresponding to \( \nu_1 \) and \( \nu_2 \) respectively. Let \( A_3 \) be as in \[3.2.11\]. Now by lemma \[3.2.11\], \( A_3 \in \mathcal{S}(H_3 \oplus H_1) \). Now, we define \( \nu_2 \circ \nu_1 \) be the Gaussian measure on \( H_2 \oplus H_3 \) that corresponds to \( A_3 \). Since the composition law for the morphisms comes from the composing corresponding operators, it is associative. Hence, we proved that

**Lemma 3.2.13.** \( \mathcal{G}M \) is a semicategory.

We used the canonical bijection between \( \mathcal{S}(H) \) and the set of nondegenerate Gaussian measures on the finite dimensional Hilbert space \( H \) to define the composition of morphisms in the category \( \mathcal{G}M \). This motivates construction of a semicategory \( T_{pol}^{1, fin} \) which can be thought as a linear algebraic avatar of \( \mathcal{G}M \).

The semicategory \( T_{pol}^{1, fin} \). The objects in \( T_{pol}^{1, fin} \) are finite dimensional real Hilbert spaces and the set of morphisms \( H_1 \to H_2 \) is given by \( \mathcal{S}(H_2 \oplus H_1) \). For \( A_1 \in \mathcal{S}(H_2 \oplus H_1) \) and \( A_2 \in \mathcal{S}(H_3 \oplus H_2) \) we define

\[ A_2 \circ A_1 = A_3 \]

where \( A_3 \) is as in lemma \[3.2.11\]. The fact that \( \mathcal{G}M \) is a semicategory automatically implies that \( T_{pol}^{1, fin} \) is a semicategory as well. Moreover, there is an obvious isomorphism
between these categories.

To $H \in T_{pol}^{1, fin}$ assign $L^2(H, d\mu_H)$ and to a morphism $A$ assign $K_A$. Lemma 3.2.11 implies that this assignment defines a “Projective Functor” (See Chapter 5 for details) from $T_{pol}^{1}$ to the category of Hilbert spaces whose objects are Hilbert spaces and morphisms are continuous linear operators.

**Remark 3.2.14.** Fix an object $H \in T_{pol}^{1, fin}$. Then the endomorphism of $H$ parametrizes the semigroup in the corollary 3.2.12 “upto a constant”.

There is another semigroup isomorphic to the semigroup corresponding to the Siegel Upper half plane $\Sigma_{2n}$. The Siegel disc $D_{2n}$ is the set of symmetric $2n \times 2n$ matrices $A$ such that

$$\|A\| < 1$$

where $\|A\|$ is the operator norm of $A$.

**Note 3.2.15.** The Cayley transform $C$ defines a homoemorphism between the Siegel half plane and Siegel half disk.

There is a continuous map

$$\Gamma : D_{2n} \to \mathcal{H}L^2(\mathbb{C}^{2n}, \mu_\mathbb{C})$$

given by

$$\Gamma(A)(z) = e^{\frac{1}{2} <Az, z>}.$$ 

One can then interpret $\Gamma(A)$ as a Hilbert Schmidt operator

$$K_{\Gamma(A)} : \mathcal{H}L^2(\mathbb{C}^n, \mu_\mathbb{C}) \to \mathcal{H}L^2(\mathbb{C}^n, \mu_\mathbb{C}).$$
In [4] chapter 5 it is shown that the set

$$\{cK_{\Gamma(B)}|A \in D^{2n}, c \in \mathbb{C} \setminus \{0\}\}$$

is a semigroup and this is canonically isomorphic to the semigroup corresponding to $\Sigma_{2n}$. The construction of the isomorphism is done using the “Cayley” transform. The philosophy here is that the composition law in the semigroup associated to $\Sigma_{2n}$ comes from how the associated operators act on $L^2(\mathbb{R}^n)$ which can be thought as the “Schrodinger representation”. One the other hand, the composition law in the semigroup corresponding to $D_{2n}$ comes from how they act on the Fock space and this can be thought as the “Segal-Bargman representation” or the “Fock representation”.

In fact, for any $A \in \Sigma_{2n}$

$$\begin{array}{ccc}
L^2(\mathbb{R}^n) & \xrightarrow{K_A} & L^2(\mathbb{R}^n) \\
S & & S \\
\mathcal{H}L^2(\mathbb{C}^n, \mu_{\mathbb{C}}) & \xrightarrow{K_{\Gamma(C(A))}} & \mathcal{H}L^2(\mathbb{C}^n, \mu_{\mathbb{C}})
\end{array}$$

is commutative. We refer to chapter 5 section 4 of [4] for this fact.

This is the idea behind the construction of the semicategory $T^{2,\text{fin}}_{\text{pol}}$ which can be thought as another linear algebraic avatar of $\mathcal{G} \mathcal{M}$. We will see that it’s relation to $T^{1,\text{fin}}_{\text{pol}}$ is analogous to the the relation of semigroup associated to disk model to the semigroup associated to the upper half plane.

The semicategory $T^{2,\text{fin}}_{\text{pol}}$: We recall that the unitary map

$$S : L^2(H, \mu) \rightarrow \text{Sym}^*(H^\vee)$$

maps $\left(\frac{d\nu_A}{d\mu}\right)^{\frac{1}{2}}$ to $\frac{\xi(C(A))}{\|\xi(C(A))\|}$. More generally, if $H_1$ and $H_2$ are finite dimensional real Hilbert spaces then the operator $K_A$ for $A \in \mathcal{S}(H_1 \oplus H_2)$ corresponds to $\frac{\xi(C(A))}{\|\xi(C(A))\|}$ and
is commutative. In fact, the vector $\frac{E(C(A))}{\|E(C(A))\|}$ is an element of $\Sym^*(H_1) \otimes \Sym^*(H_2)$ and hence defines a Hilbert-Schmidt operator

$$\Sym^*(H_1) \rightarrow \Sym^*(H_2).$$

Let $A_1 : H_1 \rightarrow H_2$ and $A_2 : H_2 \rightarrow H_2$ be morphisms in $T_{poly}^{1,fin}$. A direct application of lemma 3.2.11 gives

**Proposition 3.2.16.** $E(C(A_2)) \circ E(C(A_1)) = c(A_2, A_1)E(C(A_2 \circ A_1))$ where

$$c(A_2, A_1) = \frac{\text{det}(A_1)^{\frac{1}{2}} \cdot \text{det}(A_2)^{\frac{1}{2}} \cdot \|E(C(A_1))\| \cdot \|E(C(A_2))\|}{\text{det}(\frac{A_2 + D_1}{2})^{\frac{1}{2}} \text{det}(A_2 \circ A_1)^{\frac{1}{2}} \|E(C(A_2 \circ A_1))\|}$$

and the norms are taken in the Fock spaces.

We have all the ingredients needed to define $T_{pol}^{2,fin}$. Objects in $T_{pol}^{2,fin}$ are finite dimensional real Hilbert spaces. The set of morphisms $H_1 \rightarrow H_2$ is $\mathcal{Z}(H_2 \oplus H_1)$.

We use composition of morphisms in $T_{pol}^{1,fin}$ and the Cayley Transform defines a continuous map

$$\circ : \mathcal{Z}(H_2 \oplus H_1) \times \mathcal{Z}(H_3 \oplus H_2) \rightarrow \mathcal{Z}(H_3 \oplus H_1).$$

Let $A_1 : H_1 \rightarrow H_2$ and $A_2 : H_2 \rightarrow H_3$ be morphisms in $T_{pol}^{2,fin}$. We define,

$$A_2 \circ A_1 = C(C(A_2) \circ C(A_1)).$$
where \( C(A_2) \circ C(A_1) \) is the composition in \( T^1_{pol} \). It is now obvious that \( T^2_{pol} \) is a semicategory. Moreover, the assignment

\[
H \mapsto H, \quad A \in \mathcal{S}(H_2 \oplus H_1) \mapsto C(A) \in \mathcal{Z}(H_2 \oplus H_1)
\]

defines a functor

\[
C : T^1_{pol} \to T^2_{pol}.
\]

Moreover,

**Proposition 3.2.17.** Then Functor \( C : T^1_{pol} \to T^2_{pol} \) is an isomorphism.

In fact, it is possible to write down the map \( \circ : \mathcal{Z}(H_3 \oplus H_2) \times \mathcal{Z}(H_2 \oplus H_1) \to \mathcal{Z}(H_3 \oplus H_1) \) explicitly.

**Lemma 3.2.18.** Let \( A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^* & D_1 \end{bmatrix} \in \mathcal{Z}_2(H_2 \oplus H_1) \) and \( A_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^* & D_2 \end{bmatrix} \in \mathcal{Z}_2(H_3 \oplus H_2) \).

Then,

\[
A_2 \circ A_1 = \begin{bmatrix} A_2 + B_2(I-A_1D_2)^{-1}A_1B_2^* & B_2(I-A_1D_2)^{-1}B_1 \\ B_2^*(I-D_2A_1)^{-1}B_2^* & D_1 + B_1^*(I-D_2A_1)^{-1}D_2B_1 \end{bmatrix}.
\]

Proof of the lemma can be found in [4] Proposition 5.60.

We outline may be a more conceptual way to define the composition of morphisms in \( T^2_{pol} \). Let \( H \) be a finite dimensional real Hilbert space. Let \( V = H \oplus H \). For notational simplicity we write

\[
V = V^+ \oplus V^-.
\]

Let us define a “category” \( C \) whose objects are pairs \( V \)s as above. A morphism \( P : V \to W \) is a linear relation from \( V \) to \( W \). We recall that a linear relation from \( V \) to \( W \) is a subspace \( P \) of \( V \oplus W \). It turns out that \( C \) is a “category” where the composition of morphisms is composition of linear relations. We refer to chapter 4 for the details.

To a morphism \( A : H_1 \to H_2 \) be a morphism in \( T^2_{pol} \), the graph of \( A \) defines a morphism \( V \to W \) in \( C \) where \( V = H_1 \oplus H_1 \) and \( W = H_2 \oplus H_2 \). Here we are thinking
Now, using Theorem 4.3.3 of [14], we realize that the formula in lemma 3.2.18 for the composition comes from composing the corresponding morphisms in $\mathcal{C}$. Moreover if $\mathcal{A}_1 : H_1 \rightarrow H_2$ and $\mathcal{A}_2 : H_2 \rightarrow H_3$ are morphisms in $T_{pol}^{2,fin}$ and $\mathcal{A}_2 \circ \mathcal{A}_1$ is defined as in lemma 3.2.18 then $\mathcal{A}_2 \circ \mathcal{A}_1 : H_1 \rightarrow H_3$ is a morphism in $T_{pol}^{2,fin}$ by Theorem 1.1 in [16].

Combining all these we have now shown that $T_{pol}^{2,fin}$ is a category without using $T_{pol}^{1,fin}$. Another way to define the category $T_{pol}^{2,fin}$ would be make use of the category $T_{pol}^{2,fin}$ and the Cayley transform.

We want to point out that the assignment $H \rightarrow \text{Sym}^*(H^\vee)$ and $\mathcal{A} \rightarrow \mathcal{E}(\mathcal{A})$ defines a “Projective Functor” (Chapter 5 for details) from $T_{pol}^{2,poly}$ to the category of Hilbert spaces whose objects are Hilbert spaces and morphisms are continuous linear operators.

3.2.2 Relation between the Bosonic Fock Space and Gaussian Measures for the Infinite Vector spaces

Let $X$ be a locally convex topological vector space. Let $\mu$ a nondegenerate Gaussian measure on $X$.

**Definition 3.2.19.** A Hilbert space $H$ is called a Gaussian Hilbert space of the pair $(X, \mu)$ if there is an isometry $H \rightarrow L^2(X, \mu)$ such that each $f \in H$ is a Gaussian Random variable i.e. $f_\ast \mu$ is a Gaussian measure on $\mathbb{R}$.

**Example 3.2.1.** We recall that we denote the Cameron-Martin space of the pair $(X, \mu)$ by $H(\mu)$. An example of a Gaussian Hilbert space is $H(\mu)^\vee$ as there is an
When $f \in H(\mu)^\vee$ we denote the corresponding element in $L^2(X, \mu)$ by $\tilde{f}$.

The main goal here is to prove the following proposition.

**Proposition 3.2.20.** There exists a unique isomorphism of Hilbert spaces $Q : \text{Sym}^* (H(\mu)^\vee) \to L^2(X, d\mu)$ with the property that

$$Q(\text{Exp } h) = e^{\left(\tilde{h} - \frac{1}{2}\|h\|^2\right)}$$

for every $h \in H(\mu)^\vee$. Moreover, if $e_1 \ldots e_n$ are mutually orthonormal then

$$Q(e_1^{k_1} \ldots e_n^{k_n}) = \prod_{i=1}^{n} h_{k_i}(\tilde{e}_i)$$

where $h_{n_i}$ are Hermite polynomials.

Let $H$ be a Gaussian Hilbert space on $(X, \mu)$ and define

$$P_n(H) = \{ p(h_1, h_2, \ldots, h_n) : h_i \in H, p \text{ is a polynomial of degree } \leq n \}.$$ 

For $n \geq 0$, $\bar{P}_n(H)$ be the closure of $P_n(H) \subset L^2(X, \mu)$ of the linear space $P_n(H)$ and let

$$H^{n:} = \bar{P}_n(H) \cap (\bar{P}_{n-1}(H))^\perp.$$ 

We let

$$H^{0:} = \bar{P}_0(H),$$

the space of scalars. From definition we see that

$$P_n(H) = \bigoplus_{k \leq n} H^k.$$
It turns out that the subspaces $H^k$: are mutually orthogonal. In fact, if $H$ generates the $\sigma$-algebra on $X$, then we recall the following well known fact (see page 19 in [9]).

**Proposition 3.2.21.** $L^2(X, \mu) = \bigoplus_{n=0}^{\infty} H^{\infty}$. 

Let $h_1, \ldots, h_n \in H$ and 

$$: h_1 \cdots h_n :$$

denote the orthogonal projection of 

$$h_1 \cdots h_n$$

onto $H^{\infty}$. If $\{e_i\}$ be an orthonormal basis of $H$, then 

$$: \prod_i e_i^{k_i} := \prod_i h_{k_i}(e_i)$$

where $h_{k_i}$ are Hermite polynomials. Moreover, 

$$: e^{h} := e^{h - \frac{1}{2} \|h\|^2}.$$ 

**proof of proposition 3.2.20**: For notational convenience let us write $H = H(\mu)$. 

We note that $Q : \text{Sym}^n(H) \to H^{\infty}$ defined by 

$$Q(h_1^{k_1} \cdots h_r^{k_r}) = : h_1^{\bar{k}_1} \cdots h_r^{\bar{k}_r} :$$

and extended linearly is an isomorphism of Hilbert spaces. Now, we extend $Q$ in the obvious way to get the desired isomorphism between the Hilbert spaces $\text{Sym}^*(H(\mu))$ and $L^2(X, d\mu)$. 

3.2.2.1 Radon-Nikodym Derivative in the Infinite Dimensional Case

Here, we will prove the infinite dimensional version of proposition 3.2.10.
Notation 3.2.22. Let $H$ be a separable real Hilbert space. The set $S(H)$ denotes the set of continuous positive linear operators on $H$ such that $A - I$ is Hilbert-Schmidt. We note that each $A \in S(H)$ has a bounded inverse.

There is an obvious map $S(H)$ to the Hilbert space of Hilbert Schmidt operators on $H$ given by $A \rightarrow A - I$. We give $S(H)$ the weakest topology such that the map $A \rightarrow A - I$ is a continuous map. In other words $A_n$ converges to $A$ in $S(H)$ if and only if $A_n - I$ converges to $A - I$ with respect to the Hilbert Schmidt norm.

Let us consider the pair $(X, \mu)$ where $X$ is a locally convex space and $\mu$ is non-degenerate Gaussian measure on $X$ and let $H(\mu)$ be the Cameron-Martin space. Let $\nu_A$ be the Gaussian measure on $X$ whose Cameron Martin inner product is given by the operator $A \in S(H(\mu))$. To each $A \in S(H(\mu))$ we assign

$$\sqrt{\frac{d\nu_A}{d\mu}} \in L^2(X, \mu).$$

Proposition 3.2.23. The map $S(H(\mu)) \rightarrow L^2(X, \mu)$ described above is continuous.

Proof. It is sufficient to show that if $\{K_n\}$ is a sequence of finite rank operators such that $I + K_n$ converges to $A$ then $\sqrt{\frac{d\nu_{(I+K_n)}}{d\mu}}$ converges to $\sqrt{\frac{d\nu_A}{d\mu}}$ in $L^2(X, \mu)$. But this statement follows from proof of the Theorem 6.4.5 [1].

Let us consider the pair $(X, \mu)$ as above and let $\nu$ be another Gaussian measure on $X$ such that $\mu \sim \nu$. Then we recall from Chapter 2 that there exists a $A \in S(H(\mu))$ such that the inner product on $H(\nu)$ is given by the operator $A$. Now we prove the following theorem\footnote{The author does not claim the originality of the statement. But the author was unable to find the statement in the literature.}

Theorem 3.2.24. $Q\left(\frac{\mathbb{E}(C(A))}{\|\mathbb{E}(C(A))\|}\right) = \sqrt{\frac{d\nu}{d\mu}}.$
Proof. Let $K_n$ be a sequence of finite rank operators such that $A_n = I + K_n$ converging to $A$. Then by continuity of $E$ we see that

$$\frac{E(C(A_n))}{\|E(C(A_n))\|} \text{ converges to } \frac{E(C(A))}{\|E(C(A))\|}.$$ 

On the other hand if $\nu_n$ is the Gaussian measure on $X$ with the property that the inner product on $H(\mu)$ is given by $I + K_n$, then $\sqrt{\frac{d\nu_n}{d\mu}}$ converges to $\sqrt{\frac{d\nu}{d\mu}}$ in $L^2(X, \mu)$; see for example Theorem 6.4.5 [1]. Hence it is enough to show that

$$Q\left(\frac{E(C(A_n))}{\|E(C(A_n))\|}\right) = \sqrt{\frac{d\nu_n}{d\mu}}.$$ 

This assertion follows from the finite dimensional case because $\mu$ and $\nu_n$ differ only in a finite dimensional subspace. The finite dimensional case is already proved in proposition [3.2.10].

3.2.3 Another model for the Bosonic Fock Space $\text{Sym}^*(H^\vee)$

In this section we assume that $H$ is a separable real Hilbert space.

3.2.3.1 Cylinder Set Measure

A subset $C$ of $H$ is called a cylinder set based on a finite dimensional subsapce $K$ of $H$ is it is of the form

$$C = \pi_K^{-1}(B)$$

where $\pi_K$ is the orthogonal projection of $H$ onto $K$. Let $S_K$ denote the set of all cylinder sets based at $K$. Then $S_K$ is a $\sigma$-algebra. Let $\mathcal{R}$ be the union of $S_K$ where $K$ ranges over the finite dimensional subspaces of $H$, then $\mathcal{R}$ is an algbra but not a $\sigma$-algebra.

Definition 3.2.25. A cylinder set measure on $H$ is a set function $\mu : \mathcal{R} \to [0, 1]$
such that $\mu|_{S_K}$ is countably additive for all finite dimensional subspace $K$ of $H$. It is equivalent to say that $(\pi_K)_*(\mu)$ is a probability measure on $K$. A function $f$ on $H$ is called a cylinder function based on a finite dimensional subspace $K$ if it is $S_K$ measurable and it is of the form

$$f = \phi \circ \pi_K$$

for some borel measurable function $\phi$ on $K$.

For a cylinder function based on $K$ one can define $\int_H |f| \, d\mu$. Moreover if the last integral is finite then the integral $\int_H f \, d\mu$ is well defined. For $K_1 \subset K$ and $f$ is based at $K_1$ then it $\int_H f \, d\mu$ is not altered by viewing $f$ as based on $K$. In fact if $f$ is a cylinder function based on $F$ then

$$\int_H f \, d\mu = \int_K \phi \, d(\pi_K)_*(\mu).$$

This shows that there is a unitary map

$$\pi_K^*: L^2(F, d(\pi_K)_*(\mu), \mathbb{C}) \to L^2(H, S_K, \mu, \mathbb{C})$$

where $L^2(H, S_K, \mu, \mathbb{C})$ is the space of square integrable cylinder functions based on $K$. If $G$ is subspace of $K$ then we have the following commutative diagram:

$$L^2(G, d(\pi_G)_*(\mu), \mathbb{C}) \to L^2(H, S_G, \mu, \mathbb{C})$$

$$\begin{array}{ccc}
L^2(K, d(\pi_K)_*(\mu), \mathbb{C}) & \to & L^2(H, S_K, \mu, \mathbb{C})
\end{array}$$

Now let $\mu$ be the “standard cylinder set Gaussian measure” on $H$. By this we
mean that \((\pi_K)_*(\mu)\) is the standard Gaussian measure on \(K\) for all finite dimensional subspace \(K\) of \(H\). We define

\[
L^2_0(H, \mu, \mathbb{C}) = \cup K L^2(H, S_K, \mu, \mathbb{C}).
\]

For \(f \in L^2_0(H, \mu, \mathbb{C})\) let

\[
\|f\|_{L^2(H, \mu, \mathbb{C})}^2 = \int_H |f|^2 \, d\mu.
\]

and \(L^2(H, d\mu, \mathbb{C})\) to be the completion of \(L^2_0(H, \mu, \mathbb{C})\) with respect to this norm. Then as in the finite dimensional case, there is a unitary map

\[T \circ S : L^2(H, d\mu, \mathbb{C}) \to \text{Sym}^*(H\vee^C)\]

with the property that

\[
\begin{array}{ccc}
L^2(F, d(\pi_F)_*(\mu), \mathbb{C}) & \xrightarrow{T \circ S} & \text{Sym}^*(F\vee^C) \\
\circ & & \circ \\
L^2(H, d\mu, \mathbb{C}) & \xrightarrow{T \circ S} & \text{Sym}^*(H\vee^C)
\end{array}
\]

Moreover each of these maps commute with complex conjugation to give:

\[
\begin{array}{ccc}
L^2(F, d(\pi_F)_*(\mu)) & \xrightarrow{T \circ S} & \text{Sym}^*(F\vee) \\
\circ & & \circ \\
L^2(H, d\mu) & \xrightarrow{T \circ S} & \text{Sym}^*(H\vee)
\end{array}
\]

**Remark 3.2.26.** This model for the Bosonic Fock is convenient to do calculations. If we take a abstract Weiner space \((B, \mu)\) whose Cameron-Martin space is \(H\) then there is canonical isomorphism \(L^2(H, \mu_H) \to L^2(B, d\mu)\).
3.2.4 The semicategories \( T_{pol}^{1} \) and \( T_{pol}^{2} \)

In this section, we will construct the infinite dimensional version of \( T_{pol}^{1,fin} \) and \( T_{pol}^{2,fin} \). The difference from the finite dimensional case is that we will first construct \( T_{pol}^{2} \) and use it to define \( T_{pol}^{1} \).

**Notation 3.2.27.** We use \( \mathcal{HS}(H) \) to denote the Hilbert space of Hilbert-Schmidt operators on the separable Hilbert space \( H \).

We also recall that \( \mathbb{Z}_2(H) \) is the set of symmetric Hilbert-Schmidt operators on \( H \) which have operator norm less than one and \( \mathcal{S}(H) \) is the set of bounded positive operators \( A \) on \( H \) such that \( A - I \in \mathcal{HS}(H) \).

**Fact 3.2.28.** The map \( \mathcal{HS}(H) \times \mathcal{B}(H) \to \mathcal{HS}(H) \) defined by

\[
(A, B) \mapsto A \circ B
\]

is a continuous map which follows from the simple observation that

\[
\|A \circ B\|_{\mathcal{HS}(H)} \leq \|A\|_{\mathcal{HS}(H)} \|B\|.
\]

This implies that the map \( \mathcal{HS}(H) \times \mathcal{HS}(H) \to \mathcal{HS}(H) \) given by

\[
(A, B) \mapsto A \circ B
\]

is a continuous map. To see this one uses \( \|B\| \leq \|B\|_{\mathcal{HS}(H)} \).

Let \( H_1, H_2 \) and \( H_2 \) be separable real Hilbert spaces. Also, let \( A_1 \in \mathbb{Z}_2(H_2 \oplus H_1) \) and \( A_2 \in \mathbb{Z}_2(H_3 \oplus H_1) \). Assume that

\[
A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^* & D_1 \end{bmatrix}
\]
and

\[
\mathcal{A}_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^* & D_2 \end{bmatrix}.
\]

Define

\[
\mathcal{A}_2 \circ \mathcal{A}_1 = \begin{bmatrix} A_2 + B_2 (I - A_1 D_2)^{-1} A_1 B_2^* & B_2 (I - A_1 D_2)^{-1} B_1 \\ B_1^* (I - D_2 A_1)^{-1} B_2 & D_1 + B_1^* (I - D_2 A_1)^{-1} D_2 B_1 \end{bmatrix}
\]  (3.7)

First of all we observe that \( \mathcal{A}_2 \circ \mathcal{A}_1 \) is a symmetric Hilbert-Schmidt operator on \( H_3 \oplus H_1 \). In fact, we have a continuous map

\[
\circ : \mathcal{Z}_2(H_3 \oplus H_1) \times \mathcal{Z}_2(H_2 \oplus H_1) \rightarrow \mathcal{HS}(H_3 \oplus H_1).
\]

We observe that all the operators in the equation (7) are bounded operators and Hilbert-Schmidt operators. Now continuity of composition follows from the fact 3.2.28.

In Theorem 1 of [16] it is shown that \( \|\mathcal{A}_2 \circ \mathcal{A}_1\| < 1 \). Thus we have a continuous map

\[
\circ : \mathcal{Z}_2(H_3 \oplus H_1) \times \mathcal{Z}_2(H_2 \oplus H_1) \rightarrow \mathcal{Z}_2(H_3 \oplus H_1).
\]

Now, we are in a position to define the category \( T^2_{pol} \). An object in this category is a separable real Hilbert space and a morphism \( H_1 \rightarrow H_2 \) is an element of \( \mathcal{Z}_2(H_2 \oplus H_1) \). Composition of morphisms is given by (7) It is still not clear that the composition is associative. As in the second construction of \( T^2_{pol,fin} \) one can view a morphism in \( T^2_{pol} \) as a linear relation which is a graph. Then we observe that the composition of morphisms in \( T^2_{pol} \) is essentially the composition of linear relations. Using the associativity of composition of linear relations we conclude that the composition is associative and hence

**Proposition 3.2.29.** \( T^2_{pol} \) is a category.

**Note 3.2.30.** For a more conceptual explanation of the composition of morphisms in \( T^2_{pol} \) we refer to the definition of the category \( \mathcal{SP} \) in Chapter 4 which provides a
new proof of associativity.

**Note 3.2.31.** To an object \( H \) in \( T^2_{pol} \), assign the Hilbert space \( \text{Sym}^*(H^\vee) \) and to a morphism \( A : \{0\} \to H \) assign the vector \( \mathcal{E}(A) \in \text{Sym}^*(H^\vee) \). This assignment defines a “projective” functor from \( T^2_{pol} \) to \( \text{Hilb} \). The composition of morphisms in \( T^2_{pol} \) is guided by the composition of corresponding morphisms in \( \text{Hilb} \).

**Lemma 3.2.32.** \( H \) be a separable real Hilbert space. The Cayley transform \( C : \mathcal{S}(H) \to \mathcal{Z}_2(H) \) defined by \( C(A) = (I - A)(I + A)^{-1} \) is a homeomorphism.

Now we define the category \( T^1_{pol} \). Objects in \( T^1_{pol} \) are separable real Hilbert spaces and morphisms from \( H_1 \) to \( H_2 \) are elements of \( \mathcal{S}(H_2 \oplus H_1) \). For \( A_1 \in \mathcal{S}(H_2 \oplus H_1) \) and \( A_2 \in \mathcal{S}(H_3 \oplus H_1) \) we define

\[
A_2 \circ A_1 = C(C(A_2) \circ C(A_1)).
\]

By definition the Cayley transform defines a functor \( T^1_{pol} \) to \( T^2_{pol} \) which assigns to an object \( H \in T^1_{pol} \) the object \( H \in T^2_{pol} \) and to a morphism

\[
A : H_1 \to H_2
\]
the morphism

\[
C(A) : H_1 \to H_2.
\]

Now, by definition of \( C \) we see that \( C \) defines an isomorphism between \( T^1_{pol} \) and \( T^2_{pol} \).

We will present a different construction of \( T^1_{pol} \) in chapter 4. For now we like to describe perhaps a more conceptual hint for the existence of \( T^1_{pol} \). As a consequence of proposition **3.2.23** we note that there is a continuous map

\[
K : \mathcal{S}(H_1 \oplus H_2) \to L^2(H_1, \mu_{H_1}) \otimes L^2(H_2, \mu_{H_2})
\]
where $\mu_H$ and $L^2(H, \mu_H)$ are as in section 3.2.3.

Let $\mathcal{A}_1 \in S(H_1 \oplus H_2)$ such that $\mathcal{A}_1 - I$ is finite rank operator and $\mathcal{A}_2 \in S(H_2 \oplus H_3)$ such that $\mathcal{A}_2 - I$ is finite rank operator. Then we saw in lemma 3.2.1 that the kernel of the operator $K_{\mathcal{A}_2} \circ K_{\mathcal{A}_1}$ is given by $K_{\mathcal{A}_3}$ up to a nonzero constant where $\mathcal{A}_3$ is given as

$$\mathcal{A}_3 = \begin{bmatrix} A_2 - B_2(A_1 + D_2)^{-1} B_2' & -B_2(A_1 + D_2)^{-1} B_1 \\ B_1'(A_1 + D_2)^{-1} B_2' & D_1 - B_1'(A_1 + D_2)^{-1} B_1 \end{bmatrix}. \tag{3.8}$$

whenever

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & B_1 \\ B_1' & D_1 \end{bmatrix} \in S(H_1 \oplus H_2)$$

and

$$\mathcal{A}_2 = \begin{bmatrix} A_2 & B_2 \\ B_2' & D_2 \end{bmatrix} \in S(H_2 \oplus H_3).$$

Moreover, $\mathcal{A}_3 - I$ is a finite rank operator. This suggests that we can simply follow the path suggested by the finite dimensional case and define the category $T^1_{pol}$. The point we wanted to emphasize is that the hint for composition of morphisms comes convoluting the kernels of associated operators.
In this chapter, we recall few symplectic categories and relate them with the categories $T_1^{pol}$ and $T_2^{pol}$. All the materials for section 4.1 and section 4.2 are taken from chapter 2 in [15], and section 4.3 is taken from chapter 6 in [14].

4.1 Preliminaries

Let $V = \mathbb{R}^n \oplus \mathbb{R}^n$ and $\Omega$ be the standard symplectic structure on $V$. For notational convenience write $V = V^+ \oplus V^-$. Then $\Omega$ is defined as $\Omega((v_1^+, v_1^-), (v_2^+, v_2^-)) = <v_1^+, v_2^-> - <v_2^+, v_1^->$. Let $V_C$ be the complexification of $V$ and $\Omega_C$ be the complex bilinear extension of $\Omega$ to $V_C$. We recall that $V_C$ acts on $L^2(\mathbb{R}^n)$ as follows:

$$(\hat{a}(v)f)(x) = \sum_{j=1}^{n} \left[ v_j^- x_j + iv_j^+ \frac{\partial}{\partial x_j} \right] f(x)$$

where $v = v^+ \oplus v^-$. 

**Definition 4.1.1.** A Lagrangian subspace of $L$ of $V_C$ is a subspace such that $L$ is a maximal isotropic with respect to $\Omega_C$. We note that the form $i \Omega(v, \bar{w})$ is a Hermitian form on $L$. We call $L$ is positive if the form $i \Omega(v, \bar{w})$ is an Hermitian inner product on $L$.

Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a symmetric operator such that $Re(A)$ is positive definite i.e. $A \in \Sigma_n$. Then $Graph(iA)$ is a positive Lagrangian subspace of $V_C$.

On the other hand, the constant multiples of the functions $e^{-\frac{1}{2} \langle Ax, x \rangle}$ are in $L^2(\mathbb{R}^n)$ whenever $A \in \Sigma_n$. Scalar multiples of such functions are called Gaussian functions.
on $\mathbb{R}^n$ or Gaussian vectors in $L^2(\mathbb{R}^n)$. We will see that there is a bijection between the set of positive Lagrangian subspaces of $V_C$ and Gaussian functions on $\mathbb{R}^n$ up to constant multiples. More precisely,

**Proposition 4.1.2.** Let $\Gamma(A) = e^{-\frac{1}{2}Ax,x}$ be the Gaussian vector associated to $A \in \Sigma_n$. Define

$$L_A = \{v \in V_C : \hat{a}(v)\Gamma(A) = 0\}.$$

Then, $L_A$ is a positive Lagrangian subspace of $V_C$. Moreover $A \neq A'$ implies that $L_A \neq L_{A'}$. Conversely if $L$ is a positive Lagrangian subspace of $V_C$, then there is a Gaussian vector $\Gamma(A)$ such that $L = L_A$.

**Proof.** By computation $v \in L_A$ if and only if $v^- = iAv^+$. This shows that $L_A$ is a positive Lagrangian subspace. Conversely assume that $L$ is a positive Lagrangian subspace. Now $L$ is a positive subspace implies $L \cap V^+ = \{0\}$ and this together with $dim(L) = n$ implies $L$ is a graph of an operator $iA : \mathbb{C}^n \to \mathbb{C}^n$. Now $L$ is Lagangian implies $A$ is symmetric and the positivity implies that $Re(A)$ is positive definite.

Now, we see that $L = L_A$. \qed

Let $W = W^+ \oplus W^-$, $W^+ = \mathbb{R}^m$ and $W^- = \mathbb{R}^m$ and $\chi$ be a Gaussian vector in $L^2(\mathbb{R}^m \oplus \mathbb{R}^n)$. Then $\chi$ defines a Hilbert Schmidt operator

$$B(\chi) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$$

whose integral kernel is $\chi$.

Let

$$P(\chi) = \{(w^+, w^-, v^+, v^-) : \hat{a}(w) \circ B(\chi) = B(\chi)\hat{a}(w)\}.$$

Assume that

$$\chi(y, x) = exp\left(-\frac{1}{2}\left\langle \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix} \right\rangle\right).$$
Lemma 4.1.3. \( (w^+, w^-, v^+, v^-) \in P(\chi) \) if and only if

\[
\begin{bmatrix}
  w^+ \\
  v^+
\end{bmatrix} = i \begin{bmatrix}
  A & B \\
  -B^\top & -D
\end{bmatrix}
\begin{bmatrix}
  w^- \\
  v^-
\end{bmatrix}.
\]

Proof. Let \( f \in C^\infty(\mathbb{R}^n) \) with compact support. We know that

\[
B(\chi)f(y) = \int_{\mathbb{R}^n} \chi(y, x) f(x) \, dx.
\]

Now, we compute that

\[
\hat{a}(w)B(\chi)f(y) = \int_{\mathbb{R}^n} \left[ \sum_{i=1}^m w_i^+ y_i + iw_i^- \frac{\partial}{\partial y_i} \right] K(y, x) f(x) \, dx.
\]

Similarly,

\[
B(\chi)(\hat{a}(v)f)(y) = \int_{\mathbb{R}^n} \left[ \sum_{i=1}^n v_i^+ x_i + iv_i^- \frac{\partial}{\partial x_i} \right] f(x) \, dx.
\]

Using integration by parts, we get

\[
B(\chi)(\hat{a}(v)f)(y) = -\int_{\mathbb{R}^n} \left[ \sum_{i=1}^n v_i^+ x_i + iv_i^- \frac{\partial}{\partial x_i} \right] K(y, x) f(x) \, dx.
\]

This shows that \( (w^+, w^-, v^+, v^-) \in P(\chi) \) if and only if

\[
\begin{bmatrix}
  \sum_{i=1}^m w_i^+ y_i + iw_i^- \frac{\partial}{\partial y_i}
\end{bmatrix} K(y, x) = -\begin{bmatrix}
  \sum_{i=1}^n v_i^+ x_i + iv_i^- \frac{\partial}{\partial x_i}
\end{bmatrix} K(y, x) \quad (4.1)
\]

By computation,

\[
\begin{bmatrix}
  \sum_{i=1}^m w_i^+ y_i + iw_i^- \frac{\partial}{\partial y_i}
\end{bmatrix} K(y, x) = \{ <w^+, y> -i <w^-, Ay + Bx> \} K(y, x)
\]
and
\[
\left[ \sum_{i=1}^{n} v_i^+ x_i + iv_i^- \frac{\partial}{\partial x_i} \right] K(y, x) = \{ < v^+, y > - i < v^-, B^t y + Bx > \} K(y, x).
\]

This gives that
\[
< w^+, y > - i < w^-, Ay + Bx >= - < v^+, y > + i < v^-, B^t y + Bx >
\]
for all \( x \) and \( y \). Now from this we deduce that
\[
< w^+ - iAw^- - iBv^-, y >= 0 \quad \text{and} \quad < v^+ + iB^t w^- + Dv^-, x >= 0
\]
for all \( x \) and \( y \) and this implies that
\[
\begin{bmatrix} w^+ \\ v^+ \end{bmatrix} = i \begin{bmatrix} -B & B \\ -D & -D \end{bmatrix} \begin{bmatrix} w^- \\ v^- \end{bmatrix}.
\]

\hfill \Box

**Note 4.1.4.** \( P(\chi) \) is a positive Lagrangian subspace of \( W_\mathbb{C} \oplus V_\mathbb{C} \) with respect to the symplectic form \( \Omega_W \oplus (-\Omega_V) \) where \( \Omega_W \) and \( \Omega_V \) are the standard symplectic forms on \( W_\mathbb{C} \) and \( V_\mathbb{C} \) respectively.

Let \( \Omega_{W \oplus V} \) be the standard symplectic form on \( \mathbb{C}^{m+n} \oplus \mathbb{C}^{m+n} \).

Let \( L \) be a subspace of \( W_\mathbb{C} \oplus V_\mathbb{C} \). Let
\[
\tilde{L} = \{ (w^+, v^+, w^-, v^-) | (w^+, -v^+, w^-, v^-) \in L \}.
\]

Then \( L \) is a Lagrangian subspace of \( W_\mathbb{C} \oplus V_\mathbb{C} \) with respect to \( \Omega_W \oplus (-\Omega_V) \) if and only if \( \tilde{L} \) is a Lagrangian subspace of \( (W \oplus V)_\mathbb{C} \) with respect to \( \Omega_{W \oplus V} \). Moreover, \( L \) is a positive Lagrangian subspace of \( W_\mathbb{C} \oplus V_\mathbb{C} \) if and only if \( \tilde{L} \) is a positive Lagrangian subspace of \( (W \oplus V)_\mathbb{C} \).
Let \( \chi \) be the Gaussian vector in \( L^2(\mathbb{R}^{m+n}) \) as before i.e.

\[
\chi(y, x) = \exp \left( -\frac{1}{2} \langle \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} ; \begin{bmatrix} y \\ x \end{bmatrix} \rangle \right).
\]

Let \( L(\chi) \) be the positive Langangian subspace of \((W \oplus V)_c\) that annihilates \( \chi \). Then, from the previous paragraph we have \( L(\chi) = \overline{P(\chi)} \).

Let \( \chi \) be a Gaussian vector in \( L^2(\mathbb{R}^{m+n}) \) and

\[
B(\chi) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)
\]

be the integral operator with the kernel \( \chi \). Similarly, let \( \chi' \) be a Gaussian vector in \( L^2(\mathbb{R}^{p+m}) \) and

\[
B(\chi) : L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^p)
\]

be the integral operator with the kernel \( \chi' \). Then the kernel of the composition \( B(\chi') \circ B(\chi') \) is again a Gaussian vector in \( L^2(\mathbb{R}^{p+m}) \). Let us write \( \chi' * \chi \) for the kernel of \( B(\chi') \circ B(\chi') \). It turns out that \( P(\chi' * \chi) \) is “composition” of \( P(\chi') \) and \( P(\chi) \). In the next subsection, we will make this precise.

4.2 Symplectic Category Finite Dimensional Case

**Definition 4.2.1.** Let \( V \) and \( W \) be vector spaces. A linear relation \( P : V \rightarrow W \) is a subspace of \( W \oplus V \).

Let \( P : V \rightarrow W \) be a linear relation and \( Q : W \rightarrow U \) be another linear relation. The composition \( Q \circ P \) of linear relations is defined as

\[
Q \circ P = \{(u, v) | \exists w \in W \text{ with } (u, w) \in U \oplus W, (w, v) \in W \oplus V \}.
\]

An object of the Symplectic category \( \text{Sp} \) is \((V_c, V_c^+, V_c^-, \Omega_V)\). Here, \( V = V^+ \oplus V^- \).
and $V^+ = V^- = H$ where $H$ is a finite dimensional real Hilbert space. $\Omega_V$ is defined as in the previous section. We write $V_C$ for $(V_C, V^+_C, V^-_C, \Omega_V)$. A morphism $P : V_C \to W_C$ is a linear relation which is positive Lagrangian with respect to the symplectic form $\Omega_W \oplus (-\Omega_V)$. The composition of morphisms in $\mathbf{Sp}$ is the composition of linear relations. We need the following lemma to guarantee that $\mathbf{Sp}$ is a semicategory.

**Lemma 4.2.2.** Let $P : V_C \to W_C$, $Q : W_C \to U_C$ be morphisms in $\mathbf{Sp}$. Then $Q \circ P : V_C \to U_C$ is a morphism in $\mathbf{Sp}$.

**Proof.** It is well known that the composition of Lagrangian linear relations is again a Lagrangian linear relation. Hence $Q \circ P$ is a Lagrangian linear relation. To complete the proof we show that $Q \circ P$ is positive Lagrangian with respect to $\Omega_U \oplus (-\Omega_V)$. Let $(u, v) \in Q \circ P$. Then there is $w \in W$ such that $(u, w) \in P$ and $(w, v) \in P$. Now

$$i \left[ \Omega_U \oplus (-\Omega_V)((u, v), (\bar{u}, \bar{v})) \right] = i \left[ \Omega_U(u, \bar{u}) - \Omega_W(w, \bar{w}) + \Omega_W(w, \bar{w}) - \Omega_V(v, \bar{v}) \right]$$

which is positive as $(u, w) \in P$ and $(w, v) \in P$. \qed

4.2.1 A Variant of $\mathbf{Sp}$

Here we will construct a variant of $\mathbf{Sp}$ which is isomorphic to $\mathbf{Sp}$.

**Definition 4.2.3.** We recall that if $A \in D_n$ where $D_n$ is the set of symmetric linear operators $\mathbb{C}^n \to \mathbb{C}^n$ of norm less than one, then

$$\mathcal{E}(A) \in \text{Sym}^*(\mathbb{C}^n)^\vee.$$  

A vector in $\text{Sym}^*(\mathbb{C}^n)^\vee$ is called Gaussian if it is of the form $c \mathcal{E}(A)$ for some $A \in D_n$ and for some $c \in \mathbb{C}$.

Let $V = \mathbb{R}^n \oplus \mathbb{R}^n$ and $\Omega_V$ be the standard symplectic structure on $V_C$ which is
the complex bilinear extension of the standard symplectic form $\Omega_V$ on $V$. We first note that there is a canonical bijection between the following sets.

- $\{\text{Gaussian vectors in } L^2(\mathbb{R}^n)\}/\sim$ where $f \sim g$ if and only if $f = c \cdot g$ for some non zero constant $c$.

- $\Sigma_n$

- $\{L : L \text{ is a positive Lagrangian subspace of } V_{\mathbb{C}}\}$

- $\{\text{Gaussian vectors in } \text{Sym}^*(\mathbb{C}^n)\}/\sim$ where $f \sim g$ if and only if $f = c \cdot g$ for some non zero constant $c$.

- $\{A : \mathbb{C}^n \to \mathbb{C}^n : A \text{ linear, symmetric and } \|A\| < 1\}$ i.e $D_n$.

Define a Hermitian form $M_V$ on $V_{\mathbb{C}}$ by

$$M_V((v_1^+, v_1^-), (v_2^+, v_2^-)) = \langle v_1^+, v_2^+ \rangle - \langle v_1^-, v_2^- \rangle.$$

**Definition 4.2.4.** A subspace $L$ of $V_{\mathbb{C}}$ is called $M_V$ negative if $M_V((v_1^+, v_1^-), (v_1^+, v_1^-)) < 0$ for all nonzero $(v_1^+, v_1^-) \in L$.

**Note 4.2.5.** There is bijection between the following sets.

- $\{A : \mathbb{C}^n \to \mathbb{C}^n : A \text{ linear, symmetric and } \|A\| < 1\}$ i.e $D_n$.

- $\{L : L \text{ is a } M_V \text{ negative Lagrangian subspace of } V_{\mathbb{C}}\}$

Let us recall from Chapter 3 that there is a bijection $\Sigma_n \to D_n$ given by the Cayley transform. Let $L$ be a positive Lagrangian subspace of $V_{\mathbb{C}}$. Let $C(L)$ be the graph of $C(A)$ where $A \in \Sigma_n$. Then, $C(L)$ is $M_V$ negative Lagrangian subspace of $V_{\mathbb{C}}$. We can summarize this as follows.
Lemma 4.2.6. The assignment $L \rightarrow C(L)$ is a canonical bijection from the set of positive Lagrangian subspaces of $V_{\mathbb{C}}$ to the set of $M_V$ negative Lagrangian subspaces of $V_{\mathbb{C}}$.

Note 4.2.7. We can replace $\mathbb{R}^n$ by any finite dimensional real Hilbert space.

This motivates the definition of a category $\widehat{\text{Sp}}$. An object of the category $\widehat{\text{Sp}}$ is an object of $\text{Sp}$. A morphism $P : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ is a $\Omega_W \oplus (-\Omega_V)$ Lagrangian linear relation which is $M_W \oplus (-M_V)$ negative. We can follow the proof of lemma 4.2.2 to show that the composition of morphisms in $\widehat{\text{Sp}}$ is well defined. There is a a functor $C : \text{Sp} \rightarrow \widehat{\text{Sp}}$ which is constructed as follows. At the level of objects $C$ does not do anything. To a morphism morphism $L$ in $\text{Sp}$ it assigns $C(L)$. It is obvious that

Proposition 4.2.8. $C : \text{Sp} \rightarrow \widehat{\text{Sp}}$ is an isomorphism.

4.3 Symplectic Category Infinite Dimensional Case

4.3.1 Category $\overline{\text{Sp}}$

The category $\widehat{\text{Sp}}$ can be generalized to the infinite dimensional case.

An object in the category $\overline{\text{Sp}}$ is $H \oplus H$ where $H$ is a seperable complex Hilbert space. For the notational convenience we write $V^-$ for $0 \oplus H$, $V^+$ for $H \oplus 0$ and $V = V^+ \oplus V^-$. We note that $V$ has the following structures.

- the Hermitian indefinite form

$$M_V((f, u), (g, v)) = <f, g> - <u, v>,$$

- the skew-symmetric bilinear form

$$\Omega_V((f, u), (g, v)) = f(v) - g(u).$$
A morphism $P : V \to W$ is a linear relation which is a graph of an operator

$$S(P) = \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} : W^- \oplus V^+ \to W^+ \oplus V^-$$

such that

1. $S(P) = S(P)^t$,
2. $\|S(P)\| < 1$,
3. $\|A\| < 1, \|D\| < 1$,
4. $A$ and $D$ are Hilbert-Schmidt operators.

Condition (1) means that $P$ is maximal isotropic subspace of $V \oplus W$ with respect to the skew symmetric bilinear form $\Omega W \oplus (-\Omega V)$. Similarly condition (2) means that the sesquilinear form $M W \oplus (-M V)$ is negative definite on $P$. Finally, condition (3) means that there exists $\epsilon > 0$ such that

$$M(v, v) \geq \epsilon ||v||^2$$

for all $v \in V$ such that $(v, 0) \in P$ and

$$M(v, v) \leq -\epsilon ||w||$$

for all $w \in W$ such that $(0, w) \in P$; for details see page 169 [14].

Composition of morphisms is given by the composition of linear relations. Let $A_1 : V \to W$ be a morphism such that

$$S(A_1) = \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix}$$

and let $A_2 : W \to U$ be a morphism such that

$$S(A_2) = \begin{bmatrix} A_2 & B_2 \\ B_2^t & D_2 \end{bmatrix}.$$
Then $A_2 \circ A_1$ is graph of the operator

$$S(A_2 \circ A_1) = \begin{bmatrix}
A_2 + B_2(I-A_1 D_2)^{-1} A_1 B_2' & B_2(I-A_1 D_2)^{-1} B_1 \\
B_2'(I-D_2 A_1)^{-1} B_2 & D_1 + B_1'(I-D_2 A_1)^{-1} D_2 B_1
\end{bmatrix}. \quad (4.2)$$

Now we give a proof for the associativity for composition of morphisms in $T_{pol}^2$. To an object $H$ in $T_{pol}^2$ we assign the object $H_C \oplus H_C$ in $Sp$ and to a morphism in $A$ in we assign the graph of $A$. Now we observe that the composition of morphisms in $T_{pol}^2$ is given by the operator associated with the composition of morphisms in $\overline{Sp}$. Since $\overline{Sp}$ is a category we conclude that the composition of morphisms in $T_{pol}^2$ is associative.

The discussion above also shows that there is an obvious functor $T_{pol}^2$ to $\overline{Sp}$.

4.3.2 The category $S$ and $T_{pol}^1$

Let $H$ be a seperable real Hilbert space and $V = H \oplus H$ and $\Omega_V$ be the standard symplectic form on $V$.

Let $L$ an isotropic subspace of $V_C$ such that

- $V_C = L \oplus L$
- $L$ is positive

then $L$ is a Lagrangian subspace of $V_C$ which means that $L$ is a maximal isotropic space of $V_C$.

**Example 4.3.1.** Let $A : H \rightarrow H$ be a bounded invertible positive operator. Then the graph of $iA$ is a Lagrangian subspace of $V_C$. In particular if $A$ is Fredholm positive operator on $H$ then the graph of $iA$ is a Lagrangian subspace of $V_C$.

Let us consider a category $S$ whose objects are pairs $(V_C, \Omega_V)$ as above and morphisms $(V_C, \Omega_V) \rightarrow (U_C, \Omega_U)$ are isotropic subspaces of $(U_C \oplus \overline{V_C}, \Omega_U \oplus -\Omega_V)$. The composition of morphisms is defined as a composition of linear relations. Since composition of isotropic relations is isotropic $S$ is a category.
Let $A_1 \in S(H_2 \oplus H_1)$ and $A_2 \in S(H_3 \oplus H_2)$ be morphisms in $T^1_{pol}$. Let

$$A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix}$$

and let

$$A_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^t & D_2 \end{bmatrix}.$$ 

We define

$$\tilde{A}_1 = \begin{bmatrix} A_1 & B_1 \\ -B_1^t & -D_1 \end{bmatrix}$$

and similarly we define $\tilde{A}_2$. The subspace $L_1$ which is the graph of $i\tilde{A}_1$ is a positive Lagrangian subspace of

$$(H_2 \oplus H_2)_C \oplus ((H_1 \oplus H_1)_C).$$

Similarly, $L_2$ which is the graph of $i\tilde{A}_2$ is a positive Lagrangian subspace of

$$(H_3 \oplus H_3)_C \oplus ((H_2 \oplus H_2)_C).$$

By computation, $L_2 \circ L_1$ is a graph of $i\tilde{A}_2 \circ \tilde{A}_1$ where

$$A_2 \circ A_1 = \begin{bmatrix} A_2 - B_2(A_1 + D_2)^{-1}B_1^t & -B_2(A_1 + D_2)^{-1}B_1 \\ B_1^t(A_1 + D_2)^{-1}B_2^t & D_1 - B_1^t(A_1 + D_2)^{-1}B_1 \end{bmatrix}.$$ 

Since composition of positive linear relations is again a positive linear relation, it follows that $A_2 \circ A_1$ is a positive operator. We also see that $A_2 \circ A_1 - I$ is Hilbert-Schmidt from the definition. This says that $L_2 \circ L_1$ is a positive Lagrangian subspace. In particular, it shows that $A_2 \circ A_1 \in S(H_3 \oplus H_1)$ and hence the composition of morphisms is well defined in $T^1_{pol}$. This also shows that the composition is associative as well and hence $T^1_{pol}$ is a category.

We want to note that the construction of $T^1_{pol}$ here is independent of $T^2_{pol}$. Now one can reverse the process to define $T^2_{pol}$ using the Cayley transform and $T^1_{pol}$. In any
case the Cayley transform defines isomorphism between $T_{pol}^1$ and $T_{pol}^2$.

4.4 Relation between $T_{pol}^1$ and $\overline{Sp}$

We begin with relating the $T_{pol}^{1, fin}$ and $Sp$. To an object $H$ in $T_{pol}^{1, fin}$ let us assign the symplectic vector space $H_C \oplus H_C$ and to a morphism $A : H_1 \to H_2$ in $T_{pol}^{1, fin}$ assign the graph of $iA_C$. This assignment defines a functor $T_{pol}^1 \to Sp$.

Similarly, there is an obvious functor $T_{pol}^{2, fin} \to \widehat{Sp}$. Moreover the following diagram is commutative.

\[
\begin{array}{c}
T_{pol}^{1, fin} \\
\downarrow C \\
T_{pol}^{2, fin} \\
\downarrow C
\end{array}
\begin{array}{c}
\rightarrow Sp \\
\rightarrow \widehat{Sp}
\end{array}
\]

Composing the obvious functor $T_{pol}^2 \to Sp$ with the functor $C : T_{pol}^1 \to T_{pol}^2$ we get

**Proposition 4.4.1.** A functor $T_{pol}^1 \to \overline{Sp}$.

Proposition 4.4.1 can be interpreted as relating “Gaussian measure picture” to the “symplectic picture”.

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CHAPTER 5

EXAMPLES OF FUNCTORIAL QUANTUM FIELD THEORIES

This chapter contains our main results. We construct examples of $d$-dimensional Functorial Quantum Field Theories when $d$ is even. We follow the path below to construct examples:

$$d\text{-RBord} \rightarrow T^1_{pol} \rightarrow T^2_{pol} \rightarrow \text{Hilb}.$$ 

5.1 Dirichlet-to-Neumann Map

Let $\Sigma$ be a $d$-dimensional compact oriented Riemannian manifold, $\Delta_{\Sigma}$ be the nonnegative Laplacian on $\Sigma$ and $m > 0$.

**Note 5.1.1.** We use the notation $C^\infty(\Sigma)$ to denote the space of real valued smooth functions on $\Sigma$.

Define an inner product on $C^\infty(\Sigma)$ by

$$<f, g>_{W^1(\Sigma)} = \int_\Sigma df \wedge *_\Sigma dg + *_\Sigma m^2 fg$$

where $*_\Sigma$ is the Hodge star operator given by the Riemannian metric on $\Sigma$. We define the Sobolev space $W^1(\Sigma)$ to be the completion of $C^\infty(\Sigma)$ with respect to this metric. We recall that

$$-\Delta_{\Sigma} f = *_{\Sigma} d *_{\Sigma} df$$

for all $f \in C^\infty(\Sigma)$(Page 155, [17]).
Assume that $\partial \Sigma \neq \emptyset$. Let $\vec{n}$ denote the outward unit normal vector to $\partial \Sigma$ and \( i : \partial \Sigma \to \Sigma \) be the inclusion. Let us recall the Green's formula (page 155 \[17\])

\[
\int_{\Sigma} f \wedge \ast_{\Sigma} (\Delta_{\Sigma} g) = \int_{\Sigma} df \wedge \ast_{\Sigma} dg - \int_{\partial \Sigma} i^{\ast} \left( f \wedge \ast_{\Sigma} \frac{\partial g}{\partial \vec{n}} \right)
\]

(5.1)

Let $g \in C^\infty(\Sigma)$ such that

\[(\Delta_{\Sigma} + m^2)g = 0.\]

Then,

\[f \wedge \ast_{\Sigma} (\Delta_{\Sigma} g) = -\ast_{\Sigma} (m^2fg).\]

Hence,

\[
\int_{\Sigma} df \wedge \ast_{\Sigma} dg + \ast_{\Sigma} m^2fg = \int_{\Sigma} df \wedge \ast_{\Sigma} dg - \int_{\Sigma} f \wedge \ast_{\Sigma} (\Delta_{\Sigma} g)
\]

\[= \int_{\partial \Sigma} i^{\ast} \left( f \wedge \ast_{\Sigma} \frac{\partial g}{\partial \vec{n}} \right) \quad \text{[Using Green's Formula]} \]

\[= \int_{\partial \Sigma} i^{\ast}(f) \cdot i^{\ast} \left( \frac{\partial g}{\partial \vec{n}} \right) \, d\text{vol}(\partial \Sigma) \]

We have used the fact

\[i^{\ast}(f \wedge \ast_{\Sigma} dg) = i^{\ast}(f) \cdot i^{\ast} \left( \frac{\partial g}{\partial \vec{n}} \right) \, d\text{vol}(\partial \Sigma)\]

(Page 154, \[17\]) in the previous calculation.

Let $Y$ be a closed oriented Riemannian manifold and $m > 0$. Using spectral calculus we can define $(\Delta_{Y} + m^2)^s$ for any real number $s$. Define a bilinear form on $C^\infty(Y)$ by

\[< f, g >_{W^s(Y)} = \int_{Y} f(\Delta_{Y} + m^2)^s g \, d\text{vol}(Y).\]

This defines an inner product on $C^\infty(Y)$. We define the Sobolev space $W^s(Y)$ as the completion of $C^\infty(Y)$ with respect to this inner product. Hence $W^s(Y)$ is a real
Hilbert space.

Let $\Sigma$ be a compact oriented Riemannian manifold with $\partial \Sigma = Y$. Then we have the restriction map $\gamma : C^\infty (\Sigma) \to C^\infty (Y)$.

**Fact 5.1.2.**

- The restriction map $\gamma$ extends to a surjective continuous linear operator $\gamma : W^1(\Sigma) \to W^{\frac{1}{2}}(Y)$ called the trace map.

- The Dirichlet Problem

$$(\Delta_\Sigma + m^2)u = 0 \text{ with } \gamma(u) = f$$

has a unique solution $u_f$ in $W^1(\Sigma)$ for all $f \in W^{\frac{1}{2}}(Y)$ (Page 193, [20]).

**Definition 5.1.3.** For $f \in W^{\frac{1}{2}}(Y)$ the unique solution $u_f$ to the Dirichlet problem is called the Helmholtz extension of $f$.

We note that the Helmholtz extension gives a splitting of the short exact sequence

$$0 \to W^1_0(\Sigma) \to W^1(\Sigma) \to W^{\frac{1}{2}}(Y) \to 0 \quad (5.2)$$

where $W^1_0(\Sigma)$ is the kernel of the trace map $\gamma$.

Let $\Sigma$ and $Y$ be as above. Let $u_f$ be the Helmholtz extension of $f \in W^{\frac{1}{2}}(Y)$. If $u_f$ is “nice” i.e. it is continuously differentiable, then we define

$$D_\Sigma f = \frac{\partial u_f}{\partial n}|_Y.$$ 

In fact when $f$ is smooth then it is a fact from Elliptic PDE that $u_f$ is smooth and hence it is nice. The operator $D_\Sigma$ is called the Dirichlet-to-Neumann operator associated to the Helmholtz operator.

It is well known that $D_\Sigma$ is a positive elliptic pseudo differential operator of order one as an operator on $Y$ and it has the same principal as the operator $(\Delta_Y + m^2)^{\frac{1}{2}}$ (for
example [III]). In particular, this means that $D_\Sigma(\Delta_Y + m^2)^{-\frac{1}{2}}$ is a pseudo differential operator of order zero and extends to a unique bounded operator on $W^{\frac{1}{2}}(Y)$.

**Lemma 5.1.4.** The pushforward inner product on $W^{\frac{1}{2}}(Y)$ by the trace map $\gamma$ is given by

$$(\Delta_Y + m^2)^{-\frac{1}{2}}D_\Sigma$$

on $W^{\frac{1}{2}}(Y)$. (For the definition of push forward inner product we refer to chapter 2)

**Proof.** Let $f, g \in C^\infty(Y)$ and $u_f$ and $u_g$ be their respective Helmholtz extensions which means that $u_f$ and $u_g$ are in the orthogonal complement of $\gamma$. Now we want to claim that

$$\int_\Sigma du_f \wedge *_\Sigma du_g + *_\Sigma m^2 u_f u_g = < f, (\Delta_Y + m^2)^{-\frac{1}{2}}D_\Sigma g >_{W^{\frac{1}{2}}(Y)}$$

We recall that

$$\int_\Sigma du_f \wedge *_\Sigma du_g + *_\Sigma m^2 u_f u_g = \int_Y f D_\Sigma g \, dvol(Y) = < f, D_\Sigma g >_{L^2(Y)}.$$

But

$$< f, D_\Sigma g >_{L^2(Y)}$$

$$= < f, (\Delta_Y + m^2)^{\frac{1}{2}}((\Delta_Y + m^2)^{-\frac{1}{2}}D_\Sigma g) >_{L^2(Y)}$$

$$= < f, (\Delta_Y + m^2)^{-\frac{1}{2}}D_\Sigma g >_{W^{\frac{1}{2}}(Y)}.$$

This proves our claim and now the proof of the lemma follows because of the fact that $C^\infty(Y)$ is dense in $W^{\frac{1}{2}}(Y)$.

Let $\Sigma_1 : Y_1 \to Y_2$ be a morphism in $d$-**RBord** which means that $\Sigma$ is a $d$ dimen-
sional oriented compact Riemannian manifold with

$$\partial \Sigma = Y_2 \sqcup Y_1$$

where $Y_1$ highlights our assumption that the intrinsic orientation on $Y_1$ is opposite of the induced orientation on $Y_1$. We recall that $D_{\Sigma_1} : C^\infty(\partial \Sigma) \to C^\infty(\partial \Sigma)$ is a positive operator and hence symmetric. Hence, we can write

$$D_{\Sigma_1} = \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} : C^\infty(Y_2) \oplus C^\infty(Y_1) \to C^\infty(Y_2) \oplus C^\infty(Y_1)$$

where

$$A : C^\infty(Y_2) \to C^\infty(Y_2), B : C^\infty(Y_1) \to C^\infty(Y_2),$$

$$D : C^\infty(Y_1) \to C^\infty(Y_1) \text{ and } B^t : C^\infty(Y_1) \to C^\infty(Y_2).$$

Let us describe $A$, $B$, $B^t$ and $D$. Given $\phi_2 \in C^\infty(Y_2)$, calculate the Helmholtz solution on $\Sigma_1$ with boundary value $\phi_2$ on $Y_2$ and vanishing boundary value on $Y_2$; then $A(\phi_2)$ is the outward normal derivative of the Helmholtz solution along $Y_2$. Given $\phi_1 \in C^\infty(Y_1)$, calculate the Helmholtz solution on $\Sigma_1$ with boundary value $\phi_1$ on $Y_1$ and vanishing boundary value on $Y_1$; then $D(\phi_1)$ is the inward normal derivative of the Helmholtz solution along $Y_1$. Similarly $B$ and $B^t$ are defined. The notation $B^t$ reminds us the fact that $D_{\Sigma_1}$ is a symmetric operator.

Let $\Sigma_2 : Y_2 \to Y_3$ be a morphism in $d$-RBord and write

$$D_{\Sigma_2} = \begin{bmatrix} K & L \\ L' & M \end{bmatrix} : C^\infty(Y_3) \oplus C^\infty(Y_2) \to C^\infty(Y_3) \oplus C^\infty(Y_2)$$

We need the following simple key lemma.
Lemma 5.1.5.

\[ D_{\Sigma_2 \circ \Sigma_1} = \begin{bmatrix} K-L(A+M)^{-1}L^t & -L(A+M)^{-1}B \\ -B'(A+M)^{-1}L' & D-B'(A+M)^{-1}B \end{bmatrix}. \]

Proof. Let us write

\[ D_{\Sigma_2 \circ \Sigma_2} = \begin{bmatrix} \alpha & \beta \\ \beta^t & \delta \end{bmatrix} \]

where \( \alpha : C^\infty(Y_3) \to C^\infty(Y_3) \), \( \beta : C^\infty(Y_1) \to C^\infty(Y_3) \) and \( \delta : C^\infty(Y_3) \to C^\infty(Y_1) \). Let \( \phi \in C^\infty(Y_3) \). Let \( \Phi \) be the Helmholtz solution on \( \Sigma_2 \circ \Sigma_1 \) such that

\[ \Phi|_{Y_3} = \phi \text{ and } \Phi|_{Y_1} = 0. \]

We recall that \( \Phi \) is smooth. Let \( \psi = \Phi|_{Y_2} \). First of all we claim that

\[ K\phi = \alpha \phi - L\psi. \quad (5.3) \]

Let \( \Psi_2 \in C^\infty(\Sigma_2) \) be the Helmholtz solution on \( \Sigma_2 \) such that

\[ \Psi_2|_{Y_3} = 0 \text{ and } \Psi_2|_{Y_2} = \psi. \]

Then, we observe that \( \Phi - \Psi_2 \) is the Helmholtz solution on \( \Sigma_2 \) such that

\[ \Phi - \Psi_2|_{Y_3} = \phi \text{ and } \Phi - \Psi_2|_{Y_2} = 0. \]

Now \( K\phi \) is the outward normal derivative of \( \Phi - \Psi_2 \) along \( Y_3 \). Now, claim (5.3) follows from the definitions of \( K \), \( \alpha \) and \( L \). Rewriting claim (2) we get

\[ \alpha \phi = K\phi + L\psi. \]

Next, we will show that

\[ L^t\phi = -(A+M)\psi. \quad (5.4) \]
We know that $\Phi|_{\Sigma_1}$ is the Helmholtz solution on $\Sigma_1$ such that

$$\Phi|_{Y_2} = \psi \text{ and } \Phi|_{Y_1} = 0.$$ 

This means that $A\psi$ is the outward normal derivative of $\Phi$ along $Y_2$ with respect to $\Sigma_1$. Let $\Psi \in C^\infty(\Sigma_2)$ be the Helmholtz solution on $\Sigma_2$ such that

$$\Psi|_{Y_2} = \psi \text{ and } \Psi|_{Y_3} = 0.$$ 

Then $M\psi$ is the inward normal derivative of $\Psi$ along $Y_2$ with respect to $\Sigma_2$. Moreover we see that $\Phi - \Psi$ is the Helmholtz solution on $\Sigma_2$ such that it restricts to $\phi$ on $Y_3$ and zero on $Y_2$. Hence $L^t\phi$ is the inward normal derivative of $\Phi - \Psi$ along $Y_2$ with respect to $\Sigma_2$. But notice that the inward normal derivative of $\Phi$ along $Y_2$ with respect to $\Sigma_2$ is the negative of the outward normal derivative of $\Phi$ along $Y_2$ with respect to $\Sigma_1$ because of our orientation assumptions on $Y_2$. Putting these things together we get claim (5.4). From (5.4)

$$(A + M)^{-1}L^t\phi = -\psi.$$ 

Now using

$$\alpha\phi = K\phi + L\psi,$$

we conclude that

$$\alpha = K - L(A + M)^{-1}L^t.$$ 

Similarly,

$$\beta = -L(A + M)^{-1}B \text{ and } \delta = D - B^t(A + M)^{-1}B.$$
Let us recall that from chapter 3 that for positive operators

\[ A_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^t & D_1 \end{bmatrix} \]

and

\[ A_2 = \begin{bmatrix} A_2 & B_2 \\ B_2^t & D_2 \end{bmatrix}, \]

we defined

\[ A_2 \circ A_1 = \begin{bmatrix} A_2 - B_2(A_1 + D_2)^{-1}B_1^t & -B_2(A_1 + D_2)^{-1}B_1 \\ -B_1^t(A_1 + D_2)^{-1}B_2 & D_2 - B_1^t(A_1 + D_2)^{-1}B_1 \end{bmatrix}. \]

As a corollary of lemma 5.1.5 we get

**Corollary 5.1.6.**

\[ (\Delta_{Y_3 \sqcup Y_2} + m^2)^{-\frac{1}{2}} D_{\Sigma_2} \circ (\Delta_{Y_2 \sqcup Y_1} + m^2)^{-\frac{1}{2}} D_{\Sigma_1} = (\Delta_{Y_3 \sqcup Y_1} + m^2)^{-\frac{1}{2}} D_{\Sigma_2 \circ \Sigma_1}. \]

Let us try to understand the meaning of corollary 5.1.6. We will see that \((\Delta_{Y_2 \sqcup Y_1} + m^2)^{-\frac{1}{2}} D_{\Sigma_1}\) is a morphism from \(W^{\frac{1}{2}}(Y_1)\) to \(W^{\frac{1}{2}}(Y_2)\) in \(T_{pol}^1\). Similarly \((\Delta_{Y_3 \sqcup Y_2} + m^2)^{-\frac{1}{2}} D_{\Sigma_2}\) is a morphism from \(W^{\frac{1}{2}}(Y_2)\) to \(W^{\frac{1}{2}}(Y_2)\). The operation \(\circ\) in corollary 5.1.6 is the composition of morphisms in \(T_{pol}^1\).

5.1.1 The Functor \(\mathcal{CL}_1\)

**Notation 5.1.7.** We denote \((\Delta_{Y_1 \sqcup Y_2} + m^2)^{-\frac{1}{2}} D_{\Sigma}\) by \(\alpha_{\Sigma}\) where \(\Sigma : Y_1 \to Y_2\) is a morphism in \(d-RBord\).

**Note 5.1.8.** Theorem 2.1 of [13] implies that \((\Delta_{Y_1 \sqcup Y_2} + m^2)^{\frac{1}{2}} - D_{\Sigma}\) is a smoothing operator when the metric is a product metric near the boundary. In particular this implies that \(\alpha_{\Sigma} - I\) is a trace class operator.

To an object \(Y\) in \(d-RBord\), we assign the Hilbert space \(W^{\frac{1}{2}}(Y)\) and to a mor-
we assign the operator $\alpha_\Sigma \in S(W^\frac{1}{2}(Y_1) \oplus W^\frac{1}{2}(Y_2))$. We denote this assignment by $\mathcal{CL}_1$. Now, corollary 3.1.6 immediately implies that

**Proposition 5.1.9.** $\mathcal{CL}_1$ is a functor from $d$-$\text{RBord}$ to $T^1_{pol}$.

### 5.2 Construction of Theories

We recall that a Functorial Riemannian QFT is a symmetric monoidal functor $E : d$-$\text{RBord} \to \text{Hilb}$ and that one source of examples is path integral quantization of a Classical Field Theory. The basic idea is to enlarge the space of fields so that it indeed supports a measure. More precisely the desirable measures live on the space of distribution on manifolds. Roughly, path integral quantization assigns to an object $Y$ in $d$-$\text{RBord}$ the Hilbert space $L^2(D'(Y), \mu_Y)$ where $\mu_Y$ is a measure on $D'(Y)$.

To assign a vector to a morphism $\Sigma : \phi \to Y$ one may proceed as follows. First consider a measure $\mu_\Sigma$ on $D'(\Sigma)$ and then “restrict” $\mu_\Sigma$ to a Gaussian measure on $D'(Y)$. Then the desired vector is given by a suitable multiple of the square root of the Radon-Nikodym derivative. The problem is that one has to deal with restricting the distributions. We avoid this difficulty in our construction. Our construction can be thought of a mix of “path integral quantization” and the following beautiful idea.

Formally a classical field theory defines a functor from $d$-$\text{RBord}$ to a “category” $\overline{Sp}$ of symplectic manifolds. Then there is the “quantization” functor $\overline{Sp} \to \text{Hilb}$ which is a “projective” functor. Composing these two functors we get a “projective” version of Functorial Quantum Field Theory. In the case of Free linear $\sigma$-model we can turn this formal picture into a mathematical statement. Then we “deprojectivize” the “projective” version by inserting suitable correction factors to get a Functorial Quantum Field Theory.
5.2.1 Projective representation of a Category in \textbf{Hilb}

Here we define what we mean by “projective” Functor.

\textbf{Definition 5.2.1.} A projective representation $T$ of a category $\mathcal{C}$ in $\textbf{Hilb}$ assigns to an object $C$ in $\mathcal{C}$ a Hilbert Space $T(C)$, and to a morphism $P : C \to D$ a continuous linear operator $T(P)$ such that for any pair of morphisms $P : C \to D$ and $Q : D \to E$ we have

$$T(Q \circ P) = \lambda(Q, P) T(Q) \circ T(P)$$

where $\lambda(Q, P)$ is a nonzero complex number.

\textbf{Note 5.2.2.} $\lambda$ satisfies “cocycle” condition in the following sense. If $P$, $Q$ and $R$ are morphisms in $\mathcal{C}$ then $\lambda(QP, R)\lambda(Q, P) = \lambda(Q, PR)\lambda(P, R)$ whenever $Q \circ P \circ R$ is defined.

\textbf{Example 5.2.1.} To an object $H \in T_{\text{pol}}^{1, \text{fin}}$ assign the Hilbert space $L^2(H, \mu_H, \mathbb{C})$ and to a morphism $A : H_1 \to H_2$ assign the operator $K_A$ as in 3.2.1.2. Now as a consequence of lemma 2.5 we see that this assignment is a projective representation of $T_{\text{pol}}^{1, \text{fin}}$ in $\textbf{Hilb}$.

\textbf{Example 5.2.2.} To an object $H \in T_{\text{pol}}^{2, \text{fin}}$ assign the Hilbert space $\text{Sym}^*(H_C^\vee)$ and to a morphism $A : H_1 \to H_2$ assign the operator corresponding to the vector $\frac{\xi(A)}{\|\xi(A)\|}$ as in 3.2.1.2. This assignment is a projective representation of $T_{\text{pol}}^{2, \text{fin}}$ in $\textbf{Hilb}$.

\textbf{Note 5.2.3.} The unitary map “forward heat operator” $S : L^2(H, \mu_H, \mathbb{C}) \to \text{Sym}^*(H_C^\vee)$ defines a natural “isomorphism” between these projective representations.

Now, we construct a projective representation of $d\text{-RBord}$ in $\textbf{Hilb}$. First we need

\textbf{Proposition 5.2.4.} Let $H_1$, $H_2$ and $H_3$ be real seperable Hilbert spaces, $A_1 \in \mathcal{S}(H_2 \oplus H_1)$ and $A_2 \in \mathcal{S}(H_3 \oplus H_2)$ such that $\det(A_1)$, $\det(A_2)$ and $\det(A_2 \circ A_1)$ exist.
Then,

\[ E(C(A_2)) \circ E(C(A_1)) = c(A_2, A_1)E(C(A_2 \circ A_1)) \]  

(5.5)

where

\[ c(A_2, A_1) = \frac{\det(A_1) \cdot \det(A_2)}{\det(A_2 + D_{\Sigma})} \cdot \frac{||E(C(A_1))|| \cdot ||E(C(A_2))||}{||E(C(A_2 \circ A_1))||} \]

and the norms are taken in the Fock spaces. (For matrix decomposition of \( A_2 \) and \( A_1 \) we refer to proposition 3.2.16.)

Proof. First of all we note that \( \det(A_1) \) and \( \det(A_2) \) implies that \( \det(A_2 + D_{\Sigma}) \) exists. We recall that \( E, C \) and the composition map in \( T_{pol}^{1} \) are continuous. Hence, it is sufficient to show that the equation (5.5) holds for \( A_1 \) and \( A_2 \) such that \( A_1 - I \) and \( A_1 - I \) are finite dimensional operators. This case is the content of proposition 3.2.16.

To an object \( Y \) in \( d\text{-RBord} \), we assign \( E(Y) = \text{Sym}^* W^{\frac{1}{2}}(Y)^\vee \). We define \( E(\overline{Y}) = E(Y)^\vee \). By construction, we get

\[ E(\phi) = \mathbb{R} \]

\[ E(\sqcup_i Y_i) = \otimes_i E(Y_i) \]

Let \( \Sigma : \phi \rightarrow Y \) be a morphism in \( d\text{-RBord} \) i.e. a \( d \)-dimensional compact oriented Riemannian manifold with boundary \( Y \). We recall that

\[ \alpha_{\Sigma} = (\Delta_Y + m^2)^{-\frac{1}{2}} D_\Sigma \]

is continuous positive operator on \( W^{\frac{1}{2}}(Y, m) \). Moreover, \( \alpha_{\Sigma} - I \) is trace class operator because of our assumption on metric being a product near the boundary. This means that \( \alpha_{\Sigma} \in \mathcal{S}(W^{\frac{1}{2}}(Y)) \) and \( C(\alpha_{\Sigma}) \in \mathcal{Z}_2(W^{\frac{1}{2}}(Y)) \). Here, \( C \) is the Cayley Transform.
We also recall from 3.1.1.2 the continuous map $\mathcal{E} : \mathbb{Z}_2(W^{1/2}(Y)) \to \text{Sym}^*(W^{1/2}(Y)^\vee)$ given by $\mathcal{E}(A) = \text{EXP}(\frac{1}{2}(A))$.

**Theorem 5.2.5.** The assignment $Y \to E(Y)$ where $Y$ is an object in $d$-RBord and $\Sigma \to E(C(\alpha_\Sigma))$ where $\Sigma : \phi \to Y$ is a morphism in $d$-RBord defines a projective representation of $d$-RBord in Hilb.

*Proof.* Let $\Sigma_1 : Y_1 \to Y_2$ and $\Sigma_2 : Y_2 \to Y_3$ be morphisms in $d$-RBord. Corollary 5.1.6 gives us $\alpha_{\Sigma_2} \circ \alpha_{\Sigma_1} = \alpha_{\Sigma_2 \circ \Sigma_1}$. Now using proposition 5.2.4 to $\alpha_{\Sigma_2}$, $\alpha_{\Sigma_1}$ and $\alpha_{\Sigma_2 \circ \Sigma_1}$ we get

$$E(C(\alpha_{\Sigma_2}) \circ E(C(\alpha_{\Sigma_1}) = c(\alpha_{\Sigma_2}, \alpha_{\Sigma_1})E(C(\alpha_{\Sigma_2 \circ \Sigma_1}))$$

where

$$c(\alpha_{\Sigma_2}, \alpha_{\Sigma_1}) = \frac{\det(\alpha_{\Sigma_1})^{1/4} \cdot \det(\alpha_{\Sigma_1})^{1/4}}{\det((\Delta_{\Sigma_2} + m^2)^{-1/2}(A+M)^{1/2}) \cdot \det(\mathcal{E}(C(\alpha_{\Sigma_2 \circ \Sigma_1}))^{1/4})} \frac{||\mathcal{E}(C(\alpha_{\Sigma_2})|| \cdot ||\mathcal{E}(C(\alpha_{\Sigma_1})||}{||\mathcal{E}(C(\alpha_{\Sigma_2 \circ \Sigma_1}))||}$$

(5.6)

which is nonzero. \qed

5.2.2 Free Scalar Theory

We now “deprojectivize” the projective representation constructed above to construct Free Scalar Theory.

To an object $Y$ in $d$-RBord, we assign $E(Y) = \text{Sym}^*(W^{1/2}(Y,m)^\vee$. We define $E(\overline{Y}) = E(Y)^\vee$. By construction, we get

$$E(\phi) = \mathbb{R}$$

$$E(\sqcup_i Y_i) = \otimes_i E(Y_i)$$
To a morphism $\Sigma : \phi \to Y$ we assign

$$E(\Sigma) = \frac{1}{\det_\zeta(\Delta_{\Sigma, D} + m^2)^{\frac{1}{2}} \cdot \det_\zeta(2D_{\Sigma})^{\frac{1}{2}}} \cdot \frac{\mathcal{E}(C(\alpha_\Sigma))}{\|\mathcal{E}(C(\alpha_\Sigma))\|}$$

More generally if

$$\Sigma : Y_1 \to Y_2$$

is a morphism in $d$-$\text{RBord}$, we think of

$$\Sigma : \phi \to \overline{Y}_1 \sqcup Y_2$$

and then define $E(\Sigma)$ as before. This means that $E(\Sigma) \in E(Y_1) \otimes E(Y_2)$ and consequently we get a Hilbert-Schmidt operator

$$E(\Sigma) : E(Y_1) \to E(Y_2).$$

Also, by construction, we have

$$E(\sqcup_i \Sigma_i) = \otimes_i E(\Sigma_i).$$

**Lemma 5.2.6.** Let $\Sigma : \phi \to Y$ then

$$\det(\alpha_\Sigma) = \frac{\det_\zeta(2D_\Sigma)}{\det_\zeta(2(\Delta_Y + m^2)^{\frac{1}{2}})}.$$

**Proof.** We note that

$$2(\Delta_Y + m^2)^{\frac{1}{2}} \alpha_\Sigma = 2D_\Sigma.$$

Since $\det(\alpha_\Sigma)$ exists, a fact from the appendix implies that

$$\det_\zeta(2(\Delta_Y + m^2)^{\frac{1}{2}} \alpha_\Sigma) = \det_\zeta(2(\Delta_Y + m^2)^{\frac{1}{2}}) \det(\alpha_\Sigma).$$
Now we see \( \det(\alpha_\Sigma) = \frac{\det_\zeta(2D_\Sigma)}{\det_\zeta(2(\Delta_Y + m^2)^{\frac{1}{2}})} \).

In Theorem 5.2.7 we show that \( E(\Sigma_2) \circ E(\Sigma_1) = E(\Sigma_2 \circ \Sigma_1) \) when \( d \) is even. This shows that \( E \) constructed above is indeed a symmetric monoidal functor from \( d\text{-RBord} \) to \( \text{Hilb} \) whenever \( d \) is even.

**Theorem 5.2.7.** Let \( \Sigma_1 : Y_1 \to Y_2 \) and \( \Sigma_2 : Y_2 \to Y_3 \) be two morphisms in \( d\text{-RBord} \).

(i) There exists a non zero constant \( C_{\Sigma_2, \Sigma_1} \) such that

\[
E(\Sigma_2) \circ E(\Sigma_1) = C_{\Sigma_2, \Sigma_1} E(\Sigma_2 \circ \Sigma_1).
\]

(ii) When \( d \) is even, \( C_{\Sigma_2, \Sigma_1} = 1 \).

**Proof.** For \( \Sigma_1 : Y_1 \to Y_2 \), we write

\[
D_{\Sigma_1} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}
\]

where

\[
D_{\Sigma_1} : C^\infty(Y_2) \oplus C^\infty(Y_1) \to C^\infty(Y_2) \oplus C^\infty(Y_1).
\]

Similarly for \( \Sigma_2 : Y_2 \to Y_3 \) we write

\[
D_{\Sigma_2} = \begin{bmatrix} K & L \\ L^* & M \end{bmatrix}
\]

where

\[
D_{\Sigma_2} : C^\infty(Y_3) \oplus C^\infty(Y_2) \to C^\infty(Y_3) \oplus C^\infty(Y_2).
\]
We note that

\[
E(\Sigma_2) \circ E(\Sigma_1)
\]

\[
= \frac{1}{\det_\zeta(\Delta_{\Sigma_1,D} + m^2)^\frac{1}{2}} \cdot \frac{1}{\det_\zeta(\Delta_{\Sigma_2,D} + m^2)^\frac{1}{2}} \cdot \frac{1}{\|E(C(\alpha_{\Sigma_1}))\|} \cdot \frac{1}{\|E(C(\alpha_{\Sigma_2}))\|} \cdot \frac{\det_\zeta(\Delta Y_1, \Sigma_1, \Sigma_2)}{\det_\zeta(\Delta Y_2, \Sigma_1, \Sigma_2)} \cdot \frac{\det_\zeta(2\Delta Y_1, \Sigma_1)}{\det_\zeta(2\Delta Y_2, \Sigma_1)} \cdot \frac{\det_\zeta(2\Delta Y_3, \Sigma_2)}{\det_\zeta(2\Delta Y_2, \Sigma_2)} \cdot \frac{\det_\zeta(2\Delta Y_3, \Sigma_2)}{\det_\zeta(2\Delta Y_1, \Sigma_1)}
\]

By proof of theorem 5.2.5 we have

\[
\frac{1}{\|E(C(\alpha_{\Sigma_1}))\|} \cdot \frac{1}{\|E(C(\alpha_{\Sigma_2}))\|} \cdot \frac{\det_\zeta(\Delta Y_2, \Sigma_1, \Sigma_2)}{\det_\zeta(\Delta Y_1, \Sigma_1, \Sigma_2)} \cdot \frac{\det_\zeta(2\Delta Y_1, \Sigma_1)}{\det_\zeta(2\Delta Y_2, \Sigma_1)} \cdot \frac{\det_\zeta(2\Delta Y_3, \Sigma_2)}{\det_\zeta(2\Delta Y_2, \Sigma_2)} \cdot \frac{\det_\zeta(2\Delta Y_3, \Sigma_2)}{\det_\zeta(2\Delta Y_1, \Sigma_1)} = \frac{\det(\alpha_{\Sigma_1})}{\det(\alpha_{\Sigma_2})} \cdot \frac{\det_\zeta(2\Delta Y_1, \Sigma_1)}{\det_\zeta(2\Delta Y_2, \Sigma_1)} \cdot \frac{\det_\zeta(2\Delta Y_3, \Sigma_2)}{\det_\zeta(2\Delta Y_2, \Sigma_2)}
\]

where \(D_{\Sigma_1, \Sigma_2} = A + M\).

We recall from previous lemma that

\[
\det(\alpha_{\Sigma_1}) = \frac{\det_\zeta(2\Delta Y_1, \Sigma_1)}{\det_\zeta(2\Delta Y_2, \Sigma_2) \cdot \det_\zeta(2\Delta Y_3, \Sigma_2) \cdot \det_\zeta(2\Delta Y_1, \Sigma_1)}
\]

\[
\det(\alpha_{\Sigma_2}) = \frac{\det_\zeta(2\Delta Y_2, \Sigma_2)}{\det_\zeta(2\Delta Y_3, \Sigma_2) \cdot \det_\zeta(2\Delta Y_1, \Sigma_1) \cdot \det_\zeta(2\Delta Y_2, \Sigma_2)}
\]

\[
\det(\alpha_{\Sigma_2 \circ \Sigma_1}) = \frac{\det_\zeta(2\Delta Y_3, \Sigma_2 \circ \Sigma_1)}{\det_\zeta(2\Delta Y_1, \Sigma_1) \cdot \det_\zeta(2\Delta Y_2, \Sigma_2) \cdot \det_\zeta(2\Delta Y_3, \Sigma_2)}
\]

and using argument of the lemma above

\[
\det(\frac{1}{2}(\Delta Y_2 + m^2)^{-\frac{1}{2}} D_{\Sigma_1, \Sigma_2}) = \frac{\det_\zeta(D_{\Sigma_1, \Sigma_2})}{\det_\zeta(2\Delta Y_1, \Sigma_1)}
\]
Using these relations we see that

\[
\frac{\det(\alpha \Sigma_1)^{\frac{1}{4}} \cdot \det(\alpha \Sigma_2)^{\frac{1}{4}}}{\det(\frac{1}{2}(\Delta Y_2 + m^2)^{-\frac{1}{2}} D_{\Sigma_1, \Sigma_2})^{\frac{1}{2}} \cdot \det(\alpha_{\Sigma_2 \circ \Sigma_1})^{\frac{1}{4}}} = \frac{\det(2D_{\Sigma_1})^{\frac{1}{4}} \cdot \det(2D_{\Sigma_2})^{\frac{1}{4}}}{\det(2D_{\Sigma_1, \Sigma_2})^{\frac{1}{2}} \cdot \det(2D_{\Sigma_3})^{\frac{1}{4}}}
\]

This gives us

\[
\frac{1}{\|E(C(\alpha \Sigma_1))\|} \cdot \frac{1}{\|E(C(\alpha \Sigma_1))\|} \cdot \frac{\det(\zeta(\Delta \Sigma_1, D + m^2)) \cdot \det(\zeta(\Delta \Sigma_2, D + m^2))}{\det(\zeta(\Delta \Sigma_1, D + m^2)) \cdot \det(\zeta(\Delta \Sigma_2, D + m^2)) \cdot \det(\zeta(2D_{\Sigma_1}))}
\]

Hence,

\[
E(\Sigma_2) \circ E(\Sigma_1) = \frac{1}{\det(\zeta(\Delta_{\Sigma_1, D} + m^2)) \cdot \det(\zeta(\Delta_{\Sigma_2, D} + m^2)) \cdot \det(\zeta(2D_{\Sigma_1}))}
\]

Using a result of [2], we have that

\[
\det(\zeta(\Delta_{\Sigma_2 \circ \Sigma_1, D} + m^2)) = K_{\Sigma_1, \Sigma_2} \det(\zeta(\Delta_{\Sigma_1, D} + m^2)) \det(\zeta(\Delta_{\Sigma_1, D} + m^2)) \det(\zeta(D_{\Sigma_1, \Sigma_2}))
\]

for some nonzero constant \(K_{\Sigma_1, \Sigma_2}\). Hence,

\[
E(\Sigma_2) \circ E(\Sigma_1) = \frac{1}{\sqrt{K_{\Sigma_1, \Sigma_2}}} \cdot E(\Sigma_2 \circ \Sigma_1).
\]

When \(d\) is even, then \(K_{\Sigma_1, \Sigma_2} = 1\). The \(d = 2\) case is proved in [2] and the general case \(d\) even is proved in [12].

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Corollary 5.2.8. If $\Sigma = \Sigma_2 \circ \Sigma_1$ and $\Sigma$ is closed, then

$$E(\Sigma) = (\det_\zeta(\Delta_\Sigma + m^2))^{-\frac{1}{2}}$$

As a consequence of the corollary we see that the partition function of the Theory is not a Topological invariant. In particular, this theory is not Topological.

5.2.3 $\mathbb{R}^n$-valued Free Theory

The construction above has an immediate generalization to linear $\sigma$-model. For convenience we take the target to be $\mathbb{R}^n$.

To an object $Y$ in $d$-$\textbf{R Bord}$, we assign $E_n(Y) = \text{Sym}^\dagger(W^{1,2}(Y, m) \oplus \cdots \oplus W^{1,2}(Y, m))^\vee \cong E(Y) \otimes \cdots \otimes E(Y)$. We define $E_n(Y) = E_n(Y)^\vee$. By construction, we get

$$E_n(\phi) = \mathbb{R}$$

$$E_n(\sqcup_i Y_i) = \otimes_i E(Y_i)$$

To a morphism $\Sigma : \phi \to Y$ we assign $E_n(\Sigma) = E(\Sigma) \otimes \cdots \otimes E(\Sigma)$. Now using the arguments above we see that $E_n$ is a Functorial Riemannian Quantum Field Theory.

5.2.4 Category $\overline{Sp}$, Weil Representation and Free Theory

Here we give a more conceptual construction of projective version of the Free Theory. For that we first construct a functor $\mathcal{CL} : d$-$\textbf{R Bord} \to \overline{Sp}$. Here $\overline{Sp}$ is infinite dimensional of $\hat{Sp}$. We call this functor $\mathcal{CL}$ the Classical Field Theory. Then there is a functor called Weil representation ‘functor” $\overline{Sp} \to \textbf{Hilb}$. Composing these two “functors” we the projective version of the Free theory constructed in theorem ??.

Let us recall that an object in the category $\overline{Sp}$ is $H \oplus H$ where $H$ is a seperable
complex Hilbert space. For the notational convenience we write $V^{-}$ for $0 \oplus H$, $V^{+}$ for $H \oplus 0$ and $V = V^{+} \oplus V^{-}$. We note that $V$ has the following structures.

- the Hermitian indefinite form

\[ M_{V}((f, u), (g, v)) = \langle f, g \rangle - \langle u, v \rangle, \]

- the skew-symmetric bilinear form

\[ \Omega_{V}((f, u), (g, v)) = f(v) - g(u). \]

A morphism $P : V \to W$ is a graph of an operator $S(P) = \begin{bmatrix} A & B \\ B^{t} & D \end{bmatrix} : W^{-} \oplus V^{+} \to W^{+} \oplus V^{-}$ such that

1. $S(P) = S(P)^{t}$,
2. $\|S(P)\| < 1$,
3. $\|A\| < 1$, $\|D\| < 1$,
4. $A$ and $D$ are Hilbert-Schmidt operators.

Condition (1) means that $P$ is maximal isotropic subspace of $V \oplus W$ with respect to the skew symmetric bilinear form $\Omega_{W} \oplus (-\Omega_{V})$. Similarly condition (2) means that the sesquilinear form $M_{W} \oplus (-M_{V})$ is negative definite on $P$. Finally, condition (3) means that there exists $\epsilon > 0$ such that

\[ M(v, v) \geq \epsilon \|v\|^{2} \]

for all $v \in V$ such that $(v, 0) \in P$ and

\[ M(v, v) \leq -\epsilon \|w\| \]
for all \( w \in W \) such that \((0, w) \in P\); for details we refer to page 169 [14].

In [14], Neretin constructs a projective representation \( W \) of \( \overline{Sp} \) on \( \text{Hilb} \) as follows. Let \( H \) be a complex Hilbert space. Then it is possible to identify \( \text{Sym}^* (H^\vee) \) with “\( L^2(C^\infty, \nu) \)” where \( \nu \) is the infinite product of the Gaussian measure \( \frac{1}{\pi} e^{-|z|^2} \) on \( \mathbb{C} \); see [14] page 159.

To an object \( V = V^+ \oplus V^- \) assign the \( W(V) = \text{Sym}^*((V^+)^\vee) \) where \( (V^+)^\vee \) is the canonical dual of \( V^+ \). To a morphism \( P : V \to W \) as above assign the “integral operator” \( W(P) \) whose “kernel” is given by \( \exp \left( \frac{1}{2} \langle \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} [\hat{z}, [\hat{z}]] \right) \).

In [14] Theorem 6.2.3, it is shown that if \( P : V \to W \) and \( Q : W \to U \) are morphisms in \( \overline{Sp} \) given by respectively \( \begin{bmatrix} A & B \\ B^t & D \end{bmatrix} \) and \( \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} \), then

\[
W(Q) \circ W(P) = \det(1 - CK)^{-\frac{1}{2}} W(Q \circ P).
\]

We recall from Chapter 4 that there is a canonical the functor \( T^1_{\text{pol}} \to \overline{Sp} \). Now composing this functor with the functor \( \mathcal{C}L_1 : d\text{-}RBord \to T^1_{\text{pol}} \), we get

**Proposition 5.2.9.** There is a canonical functor \( \mathcal{C}L : d\text{-}RBord \to \overline{Sp} \).

**Definition 5.2.10.** We call the functor \( \mathcal{C}L \) the Classical Field Theory.

The discussion above can be summarized as

**Theorem 5.2.11.** \( W \circ \mathcal{C}L \) is a projective representation of \( d\text{-}RBord \) in \( \text{Hilb} \).

We can think of \( W \circ \mathcal{C}L \) as the complex version of the projective representation we constructed in theorem 5.2.5.

**Remark 5.2.12.** We may use the corrections similar to the corrections in theorem 5.2.7 to get a Functorial Quantum Field Theory.
5.2.5 More Examples

Let $Y$ be a $d-1$ dimensional closed oriented Riemannian manifold and $m$ and $k$ be positive real numbers. We define $kW^{\frac{1}{2}}(Y)$ to be the completion of $C^\infty(Y)$ with respect to the inner product

$$<f,g>_{kW^{\frac{1}{2}}(Y)} = \int_Y kf(\Delta_Y + m^2)^{\frac{1}{2}} g \, dvol(Y).$$

Note that when $k = 1$ gives the usual $W^{\frac{1}{2}}(Y)$. Let $\Sigma : \phi \rightarrow Y$ be a morphism in $d$-RBord. Then the operator $\alpha_\Sigma$ is a positive symmetric continuous operator on $kW^{\frac{1}{2}}(Y)$. In fact $\alpha_\Sigma \in S(kW^{\frac{1}{2}}(Y))$ and hence the Cayley transform $C(\alpha_\Sigma) \in Z_2(kW^{\frac{1}{2}}(Y))$.

Now to an object $Y$ in $d$-RBord we assign the Hilbert space $E(Y) = \text{Sym}^*(kW^{\frac{1}{2}}(Y)^\vee)$ and to a morphism $\Sigma : \phi \rightarrow Y$ assign the vector $E(C(\alpha) \in E(Y))$. As in theorem 5.2.5 this assignment gives a projective representation of $d$-RBord in Hilb. If we define

$$E(\Sigma) = \frac{1}{\det_\zeta(\Delta_{\Sigma_1,D} + m^2)^{\frac{1}{2}} \cdot \det_\zeta(2D_{\Sigma_1})^{\frac{1}{4}}} \cdot \frac{E(C(\alpha_\Sigma))}{\|E(C(\alpha_\Sigma))\|^2},$$

then by the same calculations in theorem 5.2.7 we see that $E$ is indeed a Functorial Quantum Field Theory when $d$ is even. Therefore, theorem 5.2.7 is the case when $k = 1$. Similarly, there are obvious $\mathbb{R}^n$ valued versions.
6.1 Basic facts about operators on a Hilbert space

We always consider separable complex Hilbert space unless it is specified. The main reference here is [19].

Notation 6.1.1. let $H$ be a Hilbert Space. We write $\mathcal{B}(H)$ for the set of bounded operators $A : H \to H$.

6.1.1 Adjoints

Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$. The adjoint of $A$ is an operator $A^*$ on $H$ such that

$$< u, Av > = < A^* u, v >$$

for all $u, v \in H$. Proof of the following proposition can be found in [19, page 186].

Proposition 6.1.2. (a) The map $A \mapsto A^*$ is conjugate linear and $\|A\| = \|A^*\|$.

(b) $(AB)^* = B^* A^*$.

(c) $(A^*)^* = A$.

(d) If $A$ has bounded inverse then $(A^{-1})^* = (A^*)^{-1}$.

Definition 6.1.3. A bounded operator $A : H \to H$ is called self adjoint if $A^* = A$. 
6.1.2 Trace class and Hilbert Schmidt operators

**Definition 6.1.4.** Let $H$ be a Hilbert space Let $A : H \to H$ be bounded operator. We define $|A| = \sqrt{A^*A}$ ([19, Page 196]).

**Definition 6.1.5.** Let $H_1$ and $H_2$ be two Hilbert spaces and $A : H_1 \to H_2$ be a bounded operator. We say $A$ is compact if for every bounded sequence $\{u_n\}_{n=1}^{\infty}$ in $H_1$ the sequence $\{Au_n\}_{n=1}^{\infty}$ in $H_2$ has a convergent subsequence ([19, Page 199]).

**Fact 6.1.6.** Let $A \in \mathfrak{B}(H)$ be a positive operator i.e. $< Au, u >$ is positive for all non-zero $u \in H$. Then the sum

\[
\sum_{n=1}^{\infty} < Ae_n, e_n >
\]

is independent of the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $H$ ([19, Page 207]). We define

\[
tr(A) = \sum_{n=1}^{\infty} < Ae_n, e_n >
\]

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H$.

**Definition 6.1.7.** An operator $A \in \mathfrak{B}(H)$ is called trace class if $tr(|A|)$ is finite. The set of trace class operators $A \in \mathfrak{B}(H)$ is denoted by $\mathcal{I}_1(H)$.

For $A \in \mathcal{I}_1(H)$ define $\|A\|_1 = tr(A)$. The following fact is taken from chapter VI ([19, Page 209]).

**Fact 6.1.8.** (a) $(\mathcal{I}_1(H), \|\cdot\|_1)$ is a Banach space.

(b) The finite rank operators are dense in $\mathcal{I}_1(H)$.

(c) $B \in \mathfrak{B}(H)$ and $A \in \mathcal{I}_1(H)$ implies that $AB$ and $BA$ are in $\mathcal{I}_1(H)$.
Definition 6.1.9. For $A \in \mathcal{J}_1(H)$ we define

$$tr(A) = \sum_{n=1}^{\infty} < Ae_n, e_n>$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H$ [19, Page 211].

Definition 6.1.10. An operator $A \in \mathfrak{B}(H)$ is Hilbert Schmidt if $tr(A^* A)$ is finite. The set of Hilbert Schmidt operators on $H$ will be denoted by $\mathcal{I}_2(H)$.

Now we state few facts about Hilbert Schmidt operators which are taken from chapter VI [19].

Fact 6.1.11. (a) For $A, B \in \mathcal{I}_2(H)$, for any orthonormal basis $\{e_n\}_{n=1}^{\infty}$,

$$\sum_{n=1}^{\infty} < Ae_n, Be_n>$$

is absolutely convergent, and the sum is independent of the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. The sum defines an inner $\langle, \rangle_2$ product in $\mathcal{I}_2(H)$.

(b) $(\mathcal{I}_2(H), \langle, \rangle_2)$ is Hilbert Space.

(c) The finite rank operators are dense in $\mathcal{I}_2(H)$.

(d) $B \in \mathfrak{B}(H)$ and $A \in \mathcal{I}_2(H)$ implies that $AB$ and $BA$ are in $\mathcal{I}_2(H)$.

(e) $A \in \mathcal{J}_1(H)$ if and only if $A = BC$ and $B, C \in \mathcal{I}_2(H)$.

Definition 6.1.12. For $A \in \mathcal{I}_2(H)$, $\|A\|_2 = \sqrt{\langle A, A \rangle_2}$ is called the Hilbert Schmidt norm of $A$.

6.2 $\zeta$-regularized determinants

Our main reference is Leonid Friedlander’s thesis [5].
Let $\Sigma$ be a compact Riemannian manifold and $A$ be a classical elliptic pseudo-differential operator of order $r > 0$ and assume that the spectrum of $A$ is discrete. Then the $\zeta$-function of $A$ is defined as

$$\zeta_A(s) = \sum \lambda_j^s \tag{6.1}$$

where $\lambda_j$s are eigenvalues of $A$. For $\text{Re}(s) > \frac{d}{r}$ where $d$ is the dimension of $\Sigma$, the series (1) converges absolutely. Moreover, $\zeta_A(s)$ has a meromorphic extension on the complex planes with poles at

$$\frac{d - k}{r}, k = 1, 2, 3 \ldots$$

The $\zeta$-regularized determinant of $A$ is defined by the formula

$$\log(\det \zeta A) = -\zeta'_A(0).$$

We use the following proposition which is Proposition 4.2 in [5].

**Proposition 6.2.1.** Let $A$ be a classical elliptic pseudo differential operator of order $r > 0$ on a compact Riemannian manifold $\Sigma$ and $T$ be a trace class operator on $\Sigma$. Then

$$\det \zeta(A(I + T)) = \det \zeta(A) \cdot \det(I + T).$$

6.3 Polarization Identity

Let $H$ be a vector space and $\rho \in \text{Sym}_n(H, \mathbb{C})$ which means that $\rho : H \times \cdots \times H : \mathbb{C}$ is continuous, linear in each argument and symmetric. Define $F_\rho(v) = \rho(v, \ldots, v)$. 

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The relation
\[
\rho(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\epsilon_i \in \{0, 1\}} (-1)^{n-\epsilon_1-\cdots-\epsilon_n} F_\rho(\epsilon_1 v_1 + \cdots + \epsilon_n v_n)
\]

Hence, using the polarization identity one can completely determine \( \rho \in \text{Sym}_n(\mathbb{H}, \mathbb{C}) \) if its values along the long diagonal is known. We refer to chapter 5 in [1] for details.

6.4 Measure Theory related definitions

Push forward of a measure: Let \((X, \mathcal{B}_X)\) and \((Y, \mathcal{B}_Y)\) be two measurable spaces and \(\alpha : X \to Y\) be a measurable map and \(\mu\) be a measure on \(X\). Then the assignment \(C \mapsto \mu(\alpha^{-1}(C))\) defines a measure on \(Y\). This measure is called the push forward of \(\mu\) and it is denoted by \(\alpha_* \mu\).

Let \(f \in L^1(Y, \alpha_* \mu)\). Then the “pull back” \(\alpha^* f \in L^1(X, \mu)\) and it is easy to check that
\[
\int_Y f \, d(\alpha_* \mu) = \int_X \alpha^* f \, d\mu.
\]

Hence \(f \mapsto \alpha^* f\) defines an isometry of \(L^p(Y, \alpha_* \mu)\) into \(L^p(X, \mu)\) for all \(0 < p < \infty\).

Absolutely continuous measures:

**Definition 6.4.1.** Let \(\mu\) and \(\nu\) be two measures on a measurable space \((X, \mathcal{B}_X)\). We recall that \(\mu\) is absolutely continuous with respect to \(\nu\) if
\[
\nu(A) = 0
\]
implies
\[
\mu(A) = 0.
\]

We say \(\mu \sim \nu\) if \(\mu\) and \(\nu\) are mutually absolutely continuous with respect to each
Abstract Wiener space: Let $H$ be a separable real Hilbert space and $\mu$ be the canonical cylinder set measure on $H$. Then it is not possible to “promote” $\mu$ to a Gaussian measure defined on the Borel subsets of $H$ unless $H$ is finite dimensional ([10, Corollary 2.1]). Assume that $H$ is infinite dimensional. However, it is possible to “enlarge” $H$ to a Banach space so that there is a nondegenerate Gaussian measure on $B$ which “extends” $\mu$.

**Notation 6.4.2.** We use $\mathfrak{P}(H)$ to denote the set of all projections of $H$ onto finite dimensional subspaces.

**Definition 6.4.3.** Let $H$ be a separable real Hilbert space and $\mu$ be the canonical cylinder set measure on $H$. A seminorm $q$ on $H$ is measurable if each positive $\epsilon$ there is $P_\epsilon \in \mathfrak{P}(H)$ such that

$$\mu(\{x \in H : q(Px) > \epsilon\}) < \epsilon$$

for each $P \in \mathfrak{P}(H)$ which is orthogonal to $P_\epsilon$.

**Definition 6.4.4.** Let $H$ be a separable real Hilbert space, $\mu$ be the canonical cylinder set measure on $H$ and $q$ be a measurable seminorm on $H$. Let $B$ be the Banach space which is the completion of $H$ with respect to $q$. There $\mu$ can be extended to a Gaussian measure on $B$ such that the corresponding nondegenerate symmetric bilinear form on $B^\vee$ is the restriction of the inner product on $H$ ([7, Theorem 3.9.5]). The triple $(i, H, B)$ where $i : H \to B$ is the inclusion is called an abstract Wiener space.

**Note 6.4.5.** Abstract Wiener space associated to a Hilbert space is not unique.


