PASSIVITY ANALYSIS AND PASSIVATION IN THE DESIGN OF CYBER-PHYSICAL SYSTEMS

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Abstract

by

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This dissertation focuses on the analysis and control of cyber-physical systems (CPS) using dissipativity and passivity theory. Cyber-physical systems, as a new generation of systems with integrated computational and physical capabilities, present significant challenges in control design and analysis, due to non-traditional modeling, uncertain environment and highly coupled discrete-event and continuous-time dynamics. On the other hand, it is well known that passive and dissipative systems have modeling, compositionality advantages and stability-guaranteed performance, which are desirable requirements in CPS design. However, it is not straightforward to apply dissipativity and passivity theory to CPS directly in general.

The main contribution of this dissertation is to provide systematic and computational methods of passivity analysis and passivation for continuous, networked and hybrid dynamical systems, which provide modeling foundations for CPS. These methods are originally developed for classical nonlinear systems. They include passivity analysis and passivation for interconnected systems using passivity indices and a transformation-based passivation scheme for individual systems. Later, it is shown that the proposed methods can address the issues in the design of CPS, by considering hybrid systems and networked control systems (NCS), respectively. For hybrid systems, the transformation-based passivation scheme provides valuable re-
sults on preserving passivity of switched systems under quantization. For networked control systems, the problems of passivity analysis and passivation using passivity indices for interconnected event-triggered feedback systems are investigated. The co-design of passivity levels and event-triggering conditions demonstrates how the trade off between required passivity levels and communication resource utilization can be achieved in NCS.

Overall, this dissertation provides new approaches to passivity analysis and passivation of CPS with the focus being on hybrid systems and networked control systems. Numerical simulations and relevant examples are also provided to demonstrate the practical applications of these methods.
DEDICATION

To my dearest wife Jin
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CHAPTER 1

INTRODUCTION

The notion of dissipativity, and its special case of passivity, is a characterization of systems input and output behavior of systems based on a generalized notion of energy. The definition of a dissipative system is based on a storage function (energy stored in the system) and a supply function (externally supplied energy). The basic idea behind dissipativity is that the increase of the stored energy is bounded by the supplied energy. Since passivity and dissipative theory is based on the abstraction of the connections between input-output behavior and energy functions, it can be applied to the analysis of chemical, mechanical, electromechanical and electrical systems where the definition of energy has both concrete mathematical representation and clear physical meaning. Due to the fact that Lyapunov functions can serve as the candidate energy functions in dissipative and passive systems, dissipativity and passivity theory acts as powerful tool for analyzing a large class of system behavior by utilizing Lyapunov function techniques. Passivity theory, as a special case of dissipativity, is widely used in the analysis and synthesis of nonlinear systems because of its compositional property and implications on the Lyapunov stability. The compositional property mainly implies that the systems consisting of parallel and negative feedback interconnections of two passive systems are still passive. The advantage of using this property is that one can always guarantee passivity of interconnected passive systems and thus the whole system is stable. Moreover, passivity also leads to asymptotic stability and $\mathcal{L}_2$ stability under additional assumptions. When the system is not passive, it can be passivated using feed-forward or feedback passivation.
schemes. In general, passivity and dissipativity theory has been extensively studied and are well developed in the control theory of linear and nonlinear dynamics.

On the other hand, the notion of cyber-physical systems (CPS) has been only recently introduced and has attracted significant attention recently from both computer and control communities. The term “cyber-physical systems (CPS)” refers to a new generation of systems with integrated computational and physical capabilities that can interact with humans through many new modalities. By the term “cyber”, we usually refer to computation, communication, and control systems that are discrete, logical, and switched while the term “physical” refers to natural and human-made systems governed by the laws of physics and operating in continuous time. Applications of CPS include high confidence medical devices, advanced automotive systems, distributed robotics, smart electric grids and so on. Although computer science and control science communities have independently developed enormously successful theories and technologies in their own areas, a wide range of issues arises when it comes to CPS integration. From a control perspective, the effort to address the challenges in CPS resides mainly in two areas, namely hybrid systems and networked control systems. The research of both areas has longer history than the emergence of CPS and dates back to the time when digital controllers and communication networks were introduced in control systems.

Hybrid systems are heterogeneous dynamical systems which involve both continuous models, describing the physical and mechanical part, and discrete event models, describing the software and logical behavior. As such, hybrid systems theory combines ideas originating in the computer science and software engineering disciplines on one hand, and systems theory and control engineering on the other. There is a variety of approaches to modeling of hybrid systems and they are distinguished by their different origins, the complexity of the continuous and discrete dynamics and the emphasis on analysis and synthesis abilities. Among the models of hybrid systems...
are switched systems that can be viewed as extensions of continuous systems towards hybrid systems. A variety of control problems, such as stability analysis, stabilization and optimal control problems, have been addressed in switched systems. It is noted that the notion of passivity for switched systems is also introduced.

In many practical control systems, sometimes it is difficult and inefficient to locate the physical plant, controller, sensor and actuator in the same location due to practical constraints or requirements. If these components are distributed in different locations, it is necessary that they should be connected over network media so that signals can be transmitted from one place to another, which gives rise to the so-called networked control systems (NCS). Introduction of communication networks has led to great advantages, such as lower cost, reduced weight and power requirements, simple installation and maintenance. However, various networked-induced factors, such as time delay, packet dropouts and quantization, set up new challenges and obstacles for controller design, which make a NCS distinct from traditional feedback control systems. It is shown that issues brought by the communication networks can be dealt with by passivity. Several passivity-based control schemes are proposed in the literature by considering different network-induced factors.

This dissertation focuses on the design of CPS using passivity and dissipativity theory. Passivity analysis and passivation for interconnected systems using passivity indices is first developed, as extensions to conventional passivity and dissipativity theory. Passivity analysis for individual systems under approximation is considered and then a transformation-based passivation scheme is introduced. Later these approaches, originally proposed for general nonlinear systems, are applied to CPS design. Passivity The transformation-based passivation scheme is applied for stabilization of quantized switched systems. The passivity analysis and passivation problems for event-triggered systems using passivity indices are discussed.

This dissertation is organized as follows. Chapter 2 contains background material
on passivity and dissipativity. This chapter summarizes dissipativity and passivity
properties of interconnected systems and their implications to stability. The results
on passivation techniques are also included.

Chapter 3 begins with the notion of cyber-physical systems (CPS) and the design
challenges due to CPS’s non-traditional modeling, uncertain environment and highly
coupled discrete-event and continuous-time dynamics. More details are focused on
hybrid systems and networked control systems. A variety of hybrid models are dis-
cussed and the existing results of passivity and dissipativity for switched and hybrid
systems are introduced. For networked control systems, different techniques on pas-
viation and passivity analysis under network-induced factors, such as time delay and
discretization, are investigated.

Chapter 4 moves to the fundamental study of passivity analysis and passivation
using passivity indices. We focus on passivity and passivation of the feedback in-
terconnection of two input feed-forward output-feedback (IF-OF) passive systems.
The conditions are given to determine passivity indices in feedback interconnected
systems. The results can be viewed as the extension of the well-known composi-
tional property of passivity. We also consider the passivation problem which can be
used to render a non-passive plant passive using a feedback interconnected passive
controller. The passivity indices of the passivated system are also determined. The
results derived do not require linearity of the systems as it is commonly assumed in
the literature.

Chapter 5 studies the passivity and passivation of quantized systems. The theory
of passivity analysis for systems under approximation shows that passivity of quan-
tized systems cannot be guaranteed in general, which motivates the introduction of
a transformation-based passivation scheme for non-switched and switched systems.
The chapter begins with some background on passive quantizers and passive switched
systems. Then passivity analysis is presented for quantized systems. To preserve
passivity property in general, the scheme centering on the use of an input-output coordinate transformation to passivate a non-switched system is introduced. It is shown that the same scheme can be applied to switched systems and thus the stability of interconnected passive switched systems can be guaranteed from the results. The example demonstrated how these methods can be applied to a quantized switched system.

Chapter 6 considers the problems of passivity analysis and passivation using passivity indices for interconnected event-triggered feedback systems. Two event-triggered control schemes are investigated, i.e. sampler at plant output and sampler at controller output, based on the location of the event-triggered samplers. For both control schemes, we first derive the conditions to characterize the level of passivity for the interconnected system using passivity indices. The event-triggering condition proposed guarantees that these indices can be achieved. Then the passivation problem is considered and passivation conditions are provided. The passivation conditions depend on the passivity indices of the plant and controller and also the event-triggering condition, which reveals the trade off between passivity levels and communication resource utilization.

Chapter 7 presents a robust stabilizing output feedback nonlinear model predictive control (NMPC) scheme by using passivity and dissipativity. Model discrepancy between the nominal model and the real system is considered and characterized by comparing the outputs for the same excitation function. With this kind of characterization, we are able to compare the supply rate between the nominal model and the real system based on their passivity indices. Then, by introducing specific stabilizing constraint based on the passivity indices of the nominal model into the MPC, it is shown that our proposed NMPC scheme can stabilize the real system to be controlled.

Concluding remarks where future directions and open questions are also discussed are given in Chapter 8.
CHAPTER 2

BACKGROUND ON DISSIPATIVITY AND PASSIVITY THEORY

2.1 Introduction

The notions of dissipativity, and its special case of passivity, are input and output characterizations of systems behavior of systems based on generalized notions of energy. The ideas of passivity first emerged from the phenomenon of dissipation of energy across passive components in the circuit theory field \[5, 46, 47\]. Then dissipativity was introduced and formalized by \[119\], and it is a generalized notion of passivity. Since passive and dissipative theories are based on the abstraction of the connections between input-output behavior and energy functions, it can be applied to the analysis of chemical, mechanical, electromechanical and electrical systems where the definition of energy has both clear physical meaning and concrete mathematical representation.

Due to the fact that Lyapunov functions can serve as the candidate energy functions in dissipative and passive systems, dissipativity and passivity theory act as powerful tools for analyzing a large class of systems behavior by utilizing Lyapunov function techniques. There is no doubt that Lyapunov functions techniques have been the main tools to deal with stability or stabilization problems in control systems. Moreover, the concepts of dissipativity and passivity generalize the concept of Lyapunov functions, which are connected to the stability of the system, to systems with inputs and outputs. Besides stability, it is possible to address other requirements, such as robustness and performance issues in control systems, within the
framework of dissipativity and passivity. Over the past decades, dissipativity and passivity have received constantly high attention by the systems and control community both in theory and practice \[18, 48, 57, 59, 98, 114\]. Recent summaries of dissipativity and passivity theory can be found in \[25, 91\].

In this chapter, the preliminaries and notation are first introduced. Then the main results in dissipativity and passivity theory are briefly reviewed. The topic of passivation is presented at the end where the different passivation approaches are discussed.

### 2.2 Preliminaries and Notation

Before introducing the dissipativity and passivity theory, we will first introduce the notation to be used in this chapter and briefly discuss basic concepts in systems theory.

#### 2.2.1 Notation

The notation used in this chapter is standard. A function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) is called positive semi-definite if \( V(x) \geq 0 \) for all \( x \) and positive definite if \( V(0) = 0, \ V(x) > 0 \) for all nonzero \( x \). \( V \) is called proper (or radially unbounded) if \( V(x) \rightarrow \infty \) whenever \( \|x\| \rightarrow \infty \). \( \|x\| \) denotes the Euclidean norm of a vector \( x \in \mathbb{R}^n \). A continuously differentiable, positive definite, radially unbounded function \( V \) is called a Lyapunov function candidate.

A function \( \alpha : [0, \infty) \rightarrow [0, \infty) \) is said to be of class \( \mathcal{K} \) if it is continuous, strictly monotonically increasing, and \( \alpha(0) = 0 \). \( \mathcal{K}_\infty \) is the subset in the class of \( \mathcal{K} \) functions that are unbounded. A function \( \alpha : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) is said to be of class \( \mathcal{KL} \) if it is of class \( \mathcal{K} \) in the first argument and if it is converging to zero whenever the second argument goes to infinity.

Furthermore, \( P \geq 0 \ (P > 0) \) indicates that a matrix \( P \) is symmetric and positive.
semi-definite (positive definite). Finally, \( y(t) \equiv 0 \), where \( y : \mathbb{R}^+ \to \mathbb{R}^p \), is used as a short form for \( y(t) = 0, \forall t \geq 0 \). \( \mathbb{R}^+ \) denotes the non-negative real line.

2.2.2 Systems Theory Preliminaries

We will first introduce the models of nonlinear and linear systems (both continuous-time and discrete-time) on which the conventional dissipativity (and passivity) theory is based.

**Definition 2.1.** A *nonlinear continuous-time system* which is driven by an input \( u(t) \) and has an output \( y(t) \) can be characterized by the following nonlinear differential equations

\[
\dot{x} = f(x, u) \tag{2.1}
\]
\[
y = h(x, u) \tag{2.2}
\]

where \( x : \mathbb{R}^+ \to \mathbb{R}^n, u : \mathbb{R}^+ \to \mathbb{R}^m \) and \( y : \mathbb{R}^+ \to \mathbb{R}^p \) are the state, input and output of the system.

To guarantee that there exists a unique, continuous trajectory for \( x \) that satisfies the nonlinear continuous-time ODE equations defined in (2.1) and (2.2), we assume that the vector function \( f(x, 0) \) is locally Lipschitz w.r.t \( x \), i.e., \( \| f(x_2, 0) - f(x_1, 0) \| \leq L \| x_2 - x_1 \| \) for all \( x_1 \) and \( x_2 \) in some neighborhood of the initial condition \( x_0 \), where \( L \) is a positive constant. Due to the time-invariant nature of the system, it can be assumed that the initial time \( t_0 = 0 \). Without loss of generality, we also assume that \( f(0, 0) = 0 \) and \( h(0, 0) = 0 \). If the system is time-varying, we can add time as an additional argument in \( f(x, u) \) and \( h(x, u) \), that is, \( f(t, x, u) \) and \( h(t, x, u) \).

**Definition 2.2.** A *nonlinear discrete-time system* which is driven by an input \( u(k) \) and has an output \( y(k) \) can be characterized by the following nonlinear difference
equations

\[ x(k+1) = f(x(k), u(k)) \]  
\[ y(k) = h(x(k), u(k)) \]

where \( x : \mathbb{N} \to \mathbb{R}^n \), \( u : \mathbb{N} \to \mathbb{R}^m \) and \( y : \mathbb{N} \to \mathbb{R}^p \) are the state, input and output of the system. \( \mathbb{N} \) denotes the set of non-negative integers.

Depending on the specific forms of \( f(x, u) \) and \( h(x, u) \), the notions of linear systems and input affine systems can be defined accordingly.

**Definition 2.3.** A linear continuous-time system which is driven by an input \( u(t) \) and has an output \( y(t) \) can be characterized by the following linear differential equations

\[ \dot{x} = Ax + Bu \]  
\[ y = Cx + Du \]

where \( x : \mathbb{R}^+ \to \mathbb{R}^n \), \( u : \mathbb{R}^+ \to \mathbb{R}^m \) and \( y : \mathbb{R}^+ \to \mathbb{R}^p \) are the state, input and output of the system. \( A, B, C \) and \( D \) are constant matrices with proper dimensions.

**Definition 2.4.** An input affine continuous-time system which is driven by an input \( u(t) \) and has an output \( y(t) \) can be characterized by the following differential equations

\[ \dot{x} = f(x) + g(x)u \]  
\[ y = h(x) + j(x)u \]

where \( x : \mathbb{R}^+ \to \mathbb{R}^n \), \( u : \mathbb{R}^+ \to \mathbb{R}^m \) and \( y : \mathbb{R}^+ \to \mathbb{R}^p \) are the state, input and output of the system.

The definitions of discrete-time linear systems and input affine systems can be obtained analogously to that of discrete-time nonlinear systems and thus the formal
definitions are omitted here.

The two notions of stability introduced here are the Lyapunov stability and $\mathcal{L}_2$ stability. In general, an equilibrium point is Lyapunov stable if all solutions starting nearby points stay nearby. Asymptotic stability is a stronger stability notion in the sense that all solutions starting at nearby points not only stay nearby but also tend to the equilibrium point as time approaches infinity.

**Definition 2.5.** \[59\] A system (2.1)-(2.2) with equilibrium $x = 0$ is **Lyapunov stable** if there exists a Lyapunov candidate function $V(x)$ such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, 0) \leq 0$$

Additionally, this system is **asymptotically stable** if

$$\dot{V} = \frac{\partial V}{\partial x} f(x, 0) < 0.$$  

An alternative notion of stability, $\mathcal{L}_2$ stability is from an input/output perspective. It characterizes how the output is bounded by the input in terms of $\mathcal{L}_2$ norm.

**Definition 2.6.** \[59\] The $\mathcal{L}_2$ norm of a signal is defined as

$$||x(t)||_2 = \sqrt{\int_0^\infty x(t)^T x(t) dt}.$$  

**Definition 2.7.** \[101\] A system (2.1) and (2.2) is **$\mathcal{L}_2$ stable** if there exist constants $\gamma$ and $\beta$ such that the $\mathcal{L}_2$ norm of the output is bounded by the following expression for all $u(t)$,

$$||y(t)||_2 \leq \gamma ||u(t)||_2 + \beta$$

The infimum of all $\gamma$ such that there exists a finite $\beta$ to satisfy the above inequality is the $\mathcal{L}_2$ gain of the system.
Note that Definition 2.6 defines the standard $L_2$ space. It is also possible to define the extended $L_2$ space by introducing truncated signals. For a discrete-time signal $u(k)$, the truncation of $u(k)$ is defined as

$$u_N = \begin{cases} u(k), & 0 \leq k \leq N \\ 0, & k > N \end{cases}.$$ 

The extended space is defined by $L_{2e} = \{ u \mid u_N \in L_2, \forall N \geq 0 \}$. The truncation of a continuous signal $u(t)$ (up to time $T$ ($0 \leq T < \infty$)) can be defined similarly, denoted by $u_T(t)$. The $L_2$-induced norm of the signal $u_T(t)$ is denoted by $\| u_T(t) \|_{L_2}$, where $\| u_T(t) \|_{L_2}^2 \triangleq \int_0^T u_T(t)u(t)dt$.

### 2.3 Dissipativity Theory

#### 2.3.1 Definition

The definition of a dissipative system is based on a storage function (energy stored in the system) and a supply function (externally supplied energy). The basic idea behind dissipativity is that the increase of the stored energy is bounded by the supplied energy.

**Definition 2.8.** [25] A system (2.1)-(2.2) is said to be dissipative with respect to the supply rate $w : \mathbb{R}^n \times \mathbb{R}^{m \times p} \rightarrow \mathbb{R}$, if there exists a positive semi-definite storage function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the (integral) dissipation inequality

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} w(x(t), u(t), y(t)) dt \quad (2.9)$$

is satisfied for all $t_0, t_1$ with $t_0 \leq t_1$ and all solutions $x = x(t), u = u(t), y = y(t)$, $t \in [t_0, t_1]$.

If the storage function is smooth, then the integral dissipation inequality (2.9)
can be rewritten as
\[ \dot{V} (x(t)) \leq w(x(t), u(t), y(t)), \forall t \]  (2.10)

Similarly, dissipativity can also be defined for the discrete-time (2.3)-(2.4)
\[ \Delta V x(k) = V (x(k+1)) - V (x(k)) \leq w(x(k), u(k), y(k)), \forall k \]  (2.11)

It can be seen that dissipativity involves three components: The first component is the positive semi-definite storage function $V$ that can be interpreted as a generalized energy function. Note that the positive (semi)definiteness of the storage function $V$ is not always necessary or desirable, but it is often needed in the context of stability. The second component is the supply rate $w$ that can be interpreted as a generalized power supply and the third one is the dissipation inequality that relates the storage function and the supply rate.

Although Definition 2.8 is based on a time-invariant system, it should be pointed out that dissipativity can also be extended to time-varying systems by (possibly) redefining $V(x)$ as $V(t, x)$.

### 2.3.2 Dissipativity Properties

One of the most important dissipativity results is its close connection to Lyapunov stability. It can be seen that by choosing $u \equiv 0$ and $w \equiv 0$ and assuming $V$ is positive (semi)definite, the dissipation inequality reduces to $\dot{V} (x(t)) \leq 0$, which is the well known Lyapunov stability condition.

Other properties of the system related to dissipativity depend on the specific forms of supply rate $w$. Table 2.1 summarizes certain systems properties [25].

It is important to note that passivity requires the dimensionality of input $u$ and output $y$ to be the same, which restricts its application. As a special case of dissi-
## TABLE 2.1
SYSTEM PROPERTIES WITH THEIR CORRESPONDING SUPPLY RATES

<table>
<thead>
<tr>
<th>System Property</th>
<th>Supply Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptotic Stability</td>
<td>$-\alpha (|x|)$, where $\alpha (\cdot) &gt; 0$</td>
</tr>
<tr>
<td>Passivity</td>
<td>$u^T y$</td>
</tr>
<tr>
<td>$L_2$-Gain</td>
<td>$\gamma^2 |u|^2 - |y|^2$</td>
</tr>
<tr>
<td>Input-to-state Stability</td>
<td>$-\alpha (|x|) + \sigma (|u|)$, where $\alpha \in K_\infty$, $\sigma \in K$</td>
</tr>
</tbody>
</table>

pativity and as a generalized notion of passivity, *QSR-dissipativity* was proposed in [51] and developed in [52, 53, 88, 89], whose supply rate is defined as

$$w(u, y) = y^T Q y + 2y^T S u + u^T R u \quad (2.12)$$

where $Q$, $S$ and $R$ are matrix with appropriate dimensions.

In addition to the Lyapunov stability result, further stability results can be derived due to the explicit form of the supply rate for QSR-dissipativity.

**Theorem 2.1.** [51] If an input affine system (2.7)-(2.8) is QSR-dissipative with $Q < 0$, then it is $L_2$-stable.

**Theorem 2.2.** [51] If an input affine system (2.7)-(2.8) is detectable and QSR-dissipative with $Q < 0$, then it is asymptotically stable.

In addition to characterizing the stability of an individual system, QSR-dissipativity can also be applied to the stability analysis of interconnected systems. Consider the negative feedback interconnection of two systems, as in Fig. 2.1.

**Theorem 2.3.** [52] Consider the feedback of two dissipative systems ($i = 1, 2$) (Fig. 2.1)
Figure 2.1. The negative feedback interconnection of two systems

\[ w_i(u_i, y_i) = y_i^T Q_i y_i + 2 y_i^T S_i u_i + u_i^T R_i u_i. \]  

(2.13)

This interconnection is stable (asymptotically stable) if the following matrix

\[
\hat{Q} = \begin{bmatrix}
Q_1 + \alpha R_2 & -S_1 + \alpha S_2^T \\
-S_1^T + \alpha S_2 & R_1 + \alpha Q_2
\end{bmatrix}
\]

(2.14)

is positive semi-definite (positive definite) for some \( \alpha > 0 \).

The result shows that it is possible to stabilize a QSR-dissipative system through negative feedback interconnection of a QSR-dissipative controller. It is an appealing advantage in dissipativity and passivity theory in the sense that it simplifies the design of a stabilizing controller when both the plant and controller are dissipative or passive systems.
2.4 Passivity Theory

2.4.1 Definition

As a special case of dissipativity, passivity theory is a widely used tool for the analysis and synthesis of nonlinear systems. Passivity for state space systems was first studied at the same time as dissipativity by [119, 120]. Summaries of passivity and its applications can be found in the literature [18, 48, 59, 98, 101, 114]. The aim of this section is first to define passivity at first and then briefly discuss how different system properties and performances can be considered from one single point of view when employing dissipation inequalities. The system properties we are interested in are the compositional properties of passivity and its implications to the Lyapunov stability and $L_2$ stability.

**Definition 2.9.** [18] A system (2.1)-(2.2) with $m = p$ is passive if there exists a positive semi-definite storage function $V(x)$ such that the following inequality holds for all $t_1, t_2 \in [0, \infty)$ such that $t_1 \leq t_2$,

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} u^T y dt. \quad (2.15)$$

If the storage function is smooth, then the integral dissipation inequality (2.15) can be rewritten as

$$\dot{V}(x(t)) \leq u^T y. \quad (2.16)$$

Similarly, passivity can also be defined for the discrete-time nonlinear systems (2.3)-(2.4) as

$$V(x(k+1)) - V(x(k)) \leq u(k)^T y(k). \quad (2.17)$$

It can be seen that there is no essential difference in passivity definition between continuous-time and discrete-time systems, other than replacing the differentiation ($\dot{V}(x(t))$) with the difference $V(x(k+1)) - V(x(k))$. Therefore, the properties of
passivity and its implications for continuous-time systems are naturally transferable to discrete-time systems without additional assumptions in general. Thus we will concentrate on the results for continuous-time systems and mention their counterparts in discrete-time systems only when it is necessary. Summaries of passivity and dissipativity results for discrete-time systems can be found in [48].

For linear and input affine systems, the passivity condition [2.16] can be characterized by the explicit algebraic conditions related to the system dynamics.

**Lemma 2.1.** [68] The input affine systems (2.7)-(2.8) with \( y = h(x) \) and \( m = p \) is said to be passive if and only if there exists a positive semi-definite storage function \( V \) such that the following dissipation inequality is satisfied:

\[
\frac{\partial V}{\partial x} f(x) \leq 0 \tag{2.18}
\]
\[
\frac{\partial V}{\partial x} g(x) = h^T(x). \tag{2.19}
\]

**Lemma 2.2.** [68] The linear systems (2.5)-(2.6) with \( m = p \) is said to be passive if and only if there exists a positive definite storage function \( V = x^T P x \) \((P > 0)\) such that the following LMI condition is satisfied:

\[
\begin{bmatrix}
A^T P + PA & P B - C^T \\
B^T P - C & -D - D^T \\
\end{bmatrix} \leq 0. \tag{2.20}
\]

The two lemmas above are also known as “Kalman-Yakubovich-Popov Lemma” for both input affine and linear systems. One can easily check if a system is passive or not (with respect to a certain storage function) using the condition (2.18)-(2.19) or (2.20). For linear systems, positive realness is also used for passive systems [64].
2.4.2 Passivity Properties

One useful property of passive systems in systems theory is the fact that the parallel interconnection and the negative feedback interconnection of two passive systems is again a passive system. Therefore in a very fundamental sense, passivity is a *compositional property*.

Consider the parallel interconnection (Fig. 2.2) and negative feedback interconnection (Fig. 2.1) of two passive systems. The following theorems show that passivity is preserved under parallel and negative feedback interconnections.

**Theorem 2.4.** [18] The parallel interconnection of two passive systems (Fig. 2.2) is passive, with respect to the input $u$ and the output $y$.

**Theorem 2.5.** [18] The negative feedback interconnection of two passive systems (Fig. 2.1) is passive, with respect to the input $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ and the output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Theorems 2.4 and 2.5 can be easily derived by defining the storage function $V$ as the sum of the two individual storage functions $V_1, V_2$, i.e. $V(x) = V_1(x) + V_2(x)$. The recent results in [131] show that the cascaded passive systems can also remain passive.
under certain additional assumptions. It is noted that the compositional property is often used in large-scale network design of nonlinear interconnected systems and related topics [16, 90]. The advantage of using this property is that one can always guarantee passivity of the interconnected passive systems and thus the whole system is stable.

As pointed out previously, dissipativity with a positive definite storage function leads to Lyapunov stability. Since passivity is a special class of dissipativity, the same conclusion applies to passivity also, as shown in Theorem 2.6.

**Theorem 2.6.** [59] A passive system with a positive definite storage function $V(x)$ is stable with Lyapunov function $V(x)$ for zero input $u(t) = 0$.

To show stronger stability results, such as asymptotic stability and $L_2$ stability, we need to define a subclass of passive systems.

**Definition 2.10.** [59] The system (2.1)-(2.2) is said to be strictly passive if $\dot{V} \leq u^T y - \psi(x)$ for some positive definite function $\psi$, i.e. $\psi(x) > 0 \forall x \neq 0$.

**Definition 2.11.** [59] The system (2.1)-(2.2) is said to be input strictly passive if $\dot{V} \leq u^T y - u^T \varphi(u)$ and $u^T \varphi(u) > 0$, $\forall u \neq 0$.

**Definition 2.12.** [59] The system (2.1)-(2.2) is said to be output strictly passive if $\dot{V} \leq u^T y - y^T \rho(y)$ and $y^T \rho(y) > 0$, $\forall y \neq 0$.

It is noted that the strictly passive systems are passive systems strictly satisfying the dissipation inequality $\dot{V} < u^T y$.

**Lemma 2.3.** [59] Consider the system (2.1)-(2.2). The origin of $\dot{x} = f(x, 0)$ is asymptotically stable if the system is either strictly passive or output strictly passive and zero-state observable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.
Lemma 2.4. [59] If the system (2.1)-(2.1) is output strictly passive with \( \dot{V} \leq u^T y - \delta y^T y \) for some \( \delta > 0 \), then it is finite-gain \( \mathcal{L}_2 \) stable and its \( \mathcal{L}_2 \) gain is less than or equal to \( 1/\delta \).

Theorem 2.6 and Lemmas 2.3, 2.4 summarize the stability analysis of an individual passive system using passivity. As for the interconnected systems, one can use the same techniques to prove the stability of the two systems under feedback interconnection. Though it has been shown that the feedback interconnection of two passive systems remain passive and thus is stable, the result is conservative since in most cases stability is the primary concern while passivity imposes additional restrictions on the system. The results on asymptotic stability and \( \mathcal{L}_2 \) stability are as follows.

Theorem 2.7. [59] Consider the feedback connection of two systems of the form (2.1)-(2.2). The origin of the closed-loop system (when \( u = 0 \)) is asymptotically stable if

- both feedback components are strictly passive,
- both feedback components are output strictly passive and zero-state observable,

or

- one component is strictly passive and the other one is output strictly passive and zero-state observable.

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable.

It is worth pointing out that the notions of passivity and dissipativity (and their related definitions) can be defined without defining the storage function \( V \) (as in Definition 2.9). It can be shown that these notions (with or without the storage function \( V \)) are equivalent [103].

Definition 2.13. [54, 68, 113, 120] Consider a system (2.1)-(2.2) with input \( u \) and output \( y \) where \( u(t), y(t) \in \mathbb{R}^m \). It is said to be
passive, if there exists a constant $\beta \leq 0$ such that

$$\langle u, y \rangle_T \geq \beta.$$  (2.21)  

input strictly passive (ISP), if there exist constants $\nu > 0$ and $\beta \leq 0$ such that

$$\langle u, y \rangle_T \geq \beta + \nu \langle u, u \rangle_T.$$  (2.22)  

output strictly passive (OSP), if there exist constants $\rho > 0$ and $\beta \leq 0$ such that

$$\langle u, y \rangle_T \geq \beta + \rho \langle y, y \rangle_T.$$  (2.23)  

very strictly passive (VSP), if there exist constants $\rho > 0$, $\nu > 0$ and $\beta \leq 0$ such that

$$\langle u, y \rangle_T \geq \beta + \rho \langle y, y \rangle_T + \nu \langle u, u \rangle_T.$$  (2.24)  

finite-gain $L_2$ stable, if there exist constants $\kappa > 0$ and $\beta \leq 0$ such that

$$\langle y, y \rangle_T \leq -\beta + \kappa^2 \langle u, u \rangle_T.$$  (2.25)  

$(Q, S, R)$-dissipative, if there exist $Q = Q^T$, $R = R^T$ and $S$ and a constant $\beta \leq 0$, such that

$$r(u, y) \triangleq \langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T \geq \beta.$$  (2.26)  

where $\langle u, y \rangle_T \triangleq \int_0^T u^T(t)y(t)dt$.

In all cases, we require that the inequality holds $\forall u(t)$, $\forall T \geq 0$ and the corresponding $y(t)$.  

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2.5 Passivity Indices and Passivity Levels

Passivity indices are used to characterize how passive a system is.

**Definition 2.14.** [59, 98] A system is input feed-forward output feedback passive (IF-OFP) if it is dissipative with respect to the supply rate

\[ \omega(u, y) = u^T y - \nu u^T u - \rho y^T y, \quad \forall t \geq 0, \tag{2.27} \]

for some \( \rho, \nu \in \mathbb{R} \).

Definition 2.14 is often used in passivity analysis, passivation and passivity-based control [95, 126, 133, 143]. We can denote an IF-OFP system by IF-OFP(\( \nu, \rho \))\textsuperscript{m}. When \( \rho = \nu = 0 \) an IF-OFP system is simply a passive system.

Based on Definition 2.14, one can further have the definitions of input feed-forward (strictly) passive, output feedback (strictly) passive and very strictly passive.

1. When \( \rho = 0 \) and \( \nu \neq 0 \), the system is said to be input feed-forward passive (IFP), denoted as IFP(\( \nu \)). When in addition \( \nu > 0 \), the system is input feed-forward strictly passive (ISP).

2. When \( \rho \neq 0 \) and \( \nu = 0 \), the system is said to be output feedback passive (OFP), denoted as OFP(\( \rho \)). When in addition \( \rho > 0 \), the system is output feedback strictly passive (OSP).

3. When \( \rho > 0 \) and \( \nu > 0 \), the system is said to be very strictly passive (VSP).

Note that positive \( \rho \) or \( \nu \) means that the system has an excess of passivity, such as ISP, OSP and VSP. If either \( \rho \) or \( \nu \) is negative, the system has a shortage of passivity and thus is non-passive. When one of indices is zero and the other is non-zero (i.e. IFO and OFP), \( \rho \) or \( \nu \) is called “passivity index”, defined as the largest value such that (2.27) holds for \( \forall u \) and \( \forall t \geq 0 \) (See [18]).

\footnote{Although the definition is called input feed-forward output feedback “passive”, the system is not necessarily passive since we do not require \( \rho \) and \( \nu \) to be non-negative.}
The physical significance of output feedback passivity (OFP) index \( \rho \) is that it is the largest gain that can be placed in positive feedback with a system such that the interconnected system is passive. The input feedforward passivity (IFP) index \( \nu \) is the largest gain that can be put in a negative parallel interconnection with a system such that the interconnected system is passive. The formal definitions of passivity indices for linear systems are provided in Definition 2.15 and Definition 2.16.

Definition 2.15. [18] The input feed-forward passivity index for a stable linear system \( G(s) \) is defined as

\[
\nu = \frac{1}{2} \min_{\omega \in \mathbb{R}} \lambda(G(j\omega) + G^*(j\omega))
\]

where \( \lambda \) denotes the minimum eigenvalue.

Definition 2.16. [18] The output feedback passivity index for a minimum phase linear system \( G(s) \) is defined as

\[
\rho = \frac{1}{2} \min_{\omega \in \mathbb{R}} \lambda(G^{-1}(j\omega) + [G^{-1}(j\omega)]^*)
\]

For linear single-input-single-output systems, it can be shown that the indices can be obtained from (2.28) and (2.29)

\[
\nu = \min_{\omega \in \mathbb{R}} \text{Re}[G(j\omega)], \quad (2.28)
\]

\[
\rho = \min_{\omega \in \mathbb{R}} \text{Re}[G^{-1}(j\omega)]. \quad (2.29)
\]

It means that the indices can be determined from Nyquist plots [18, 124–126]. [64] further showed that the indices can also be obtained by solving the LMIs.

Lemma 2.5. A linear system (2.5)-(2.6) is \((Q,S,R)\)-dissipative if there exists a
\[ P = P^T > 0 \text{ such that} \]
\[
\Pi \triangleq \begin{bmatrix}
A^T P + PA - \hat{Q} & PB - \hat{S} \\
(PB - \hat{S})^T & -\hat{R}
\end{bmatrix} \leq 0,
\]
(2.30)

where \( \hat{Q}, \hat{S}, \hat{R} \) are given by \( \hat{Q} = C^T QC, \hat{S} = C^T S + C^T QD, \hat{R} = D^T QD + (D^T S + S^T D) + R \).

This lemma can be used to test whether a linear system is passive by setting \( S = 1/2 I, Q = 0, R = 0 \), or ISP by setting \( S = 1/2 I, Q = 0, R = -\nu I \), or OSP by setting \( S = 1/2 I, Q = -\rho I, R = 0 \) or VSP by setting \( S = 1/2 I, Q = -\rho I, R = -\nu I \), see e.g. [62].

Recently, an experimental method of determining indices from input-output data was reported in [123]. A numerical optimization method was used to find the indices.

It can be seen that the definitions of passivity indices require \( \rho \) and \( \nu \) to be the largest values to make (2.27) hold, which imposes hard restrictions on calculation. In general, it is not easy to calculate passivity indices for nonlinear systems. Moreover, for VSP, it may not make sense to define the largest \( \rho \) and the largest \( \nu \) simultaneously because the indices are not unique and dependent on each other. On the other hand, the search of valid \( \rho \) and \( \nu \) satisfying (2.27) is less demanding. The methods proposed in [64, 123] can be used to find (or verify) valid values of \( \rho \) and \( \nu \). Although the found values are (possibly) not strict passive indices, they have demonstrated the value in passivity-based system analysis and design. Therefore, the notions of passivity levels are proposed.

**Definition 2.17.** Consider a system with input \( u \) and output \( y \) where \( u(t), y(t) \in \mathbb{R}^m \),

- any \( \tilde{\nu} \leq \nu \) is an *IFP level* if the system has \( \nu \) as the IFP index;
- any \( \tilde{\rho} \leq \rho \) is an *OFP level* if the system has \( \rho \) as the OFP index;
- any \( (\tilde{\rho}, \tilde{\nu}) \) satisfying \( 0 < \tilde{\rho} \leq \rho \) and \( 0 < \tilde{\nu} \leq \nu \) are *VSP levels* if the system is VSP for \( (\rho, \nu) \).
By introducing passivity levels, we do not need to distinguish between passivity indices and levels as long as there exist $\rho$ and $\nu$ such that (2.27) holds. The two notions will be used interchangeably unless clarification is needed from the context.

The valid domain of $\rho$ and $\nu$ has been proposed in [73, 134].

**Lemma 2.6.** [134] The domain of $\rho$ and $\nu$ in IF-OFP system is $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{ \rho, \nu \in \mathbb{R} | \rho \nu < \frac{1}{4} \}$ and $\Omega_2 = \{ \rho, \nu \in \mathbb{R} | \rho \nu = \frac{1}{4}; \rho > 0 \}$.

**Proof.** If $\rho, \nu \in \bar{\Omega} = \Omega_3 \cup \Omega_4$ with $\Omega_3 = \{ \rho, \nu \in \mathbb{R} | \rho \nu \geq \frac{1}{4}; \rho < 0 \}$ and $\Omega_4 = \{ \rho, \nu \in \mathbb{R} | \rho \nu > \frac{1}{4}; \rho > 0 \}$, degenerate cases occur. In case $\rho, \nu \in \Omega_3$, multiplying (2.27) with $\rho < 0$ and taking the square complement it follows

$$
\rho \left[ V(x_t) - V(x_0) \right] + \int_0^t \left[ \rho^2 \| y(\tau) \|^2_2 - \rho u^T(\tau)y(\tau) \right. \\
+ \left. \frac{1}{4} \| u(\tau) \|^2_2 + (\rho \nu - \frac{1}{4}) \| u(\tau) \|^2_2 \right] d\tau \geq 0. 
$$

(2.31)

Let $\beta_1 = -\rho \min \{ V(x_0) \}$, then $\rho \left[ V(x_t) - V(x_0) \right] \leq -\rho V(x_0) \leq \beta_1$, in view of (2.31), we have

$$
\beta_1 + \int_0^t \left[ \| \rho y(\tau) - \frac{1}{2} u(\tau) \|^2_2 + (\rho \nu - \frac{1}{4}) \| u(\tau) \|^2_2 \right] d\tau \geq 0, 
$$

(2.32)

which is satisfied for any pair of $(u, y)$, since $\rho \nu - \frac{1}{4} \geq 0$ imposing no restriction to the system’s input-output behavior. In case $\rho, \nu \in \Omega_4$, multiplying (2.27) with $\rho > 0$ and taking the square complement it follows

$$
V(x_t) - V(x_0) \leq -\frac{1}{\rho} \int_0^t \left[ \| \rho y(\tau) - \frac{1}{2} u(\tau) \|^2_2 \\
+ (\rho \nu - \frac{1}{4}) \| u(\tau) \|^2_2 \right] d\tau \leq 0, 
$$

(2.33)

which indicates that $V(x_t) \leq V(x_0), \forall t \geq 0$. Thus

$$
0 = \max \{ V(x_t) - V(x_0) \} \\
\leq -\frac{1}{\rho} \int_0^t \left[ \| \rho y(\tau) - \frac{1}{2} u(\tau) \|^2_2 + (\rho \nu - \frac{1}{4}) \| u(\tau) \|^2_2 \right] d\tau, 
$$

(2.34)
which can be only satisfied for $u(t) = 0$ since $\rho \nu - \frac{1}{4} > 0$. The proof is completed. \qed

With the help of passivity indices (levels), $L_2$ stability conditions for the interconnected system can be derived.

**Theorem 2.8.** [59] Consider the feedback interconnection of Fig. 2.1 and suppose each feedback component satisfies the inequality

$$\dot{V}_i \leq u_i^T y_i - \nu_i u_i^T u_i - \rho_i y_i^T y_i,$$

for some storage function $V_i(x_i)$ where $i = 1, 2$. Then, the closed-loop system from

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \text{ to } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is finite gain $L_2$ stable if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$.

### 2.6 Passivation

Making a system passive is called passivation. Since passive systems are stable and easy to control, passivation is often a useful step in control design. For example, we may passivate a system and then stabilize the passivated system using a passive controller, as stated in Theorem 2.9.

**Theorem 2.9.** [59] If an input affine system is zero-state observable and passive with a radially bounded positive definite storage function, the origin $x = 0$ can be globally stabilized by $u = -\phi(y)$, where $\phi$ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T \phi(y) > 0$ for all $y \neq 0$.

There is great freedom in the choice of $\phi$. We can select it to meet any constraint on the magnitude of $\phi$. For example, if $u$ is constrained so that $|u_i| \leq k_i$ for $1 \leq i \leq p$, we can choose $\phi_i(y) = k_i \text{sat}(y_i)$ or $\phi_i(y) = (2k_i/\pi) \tan^{-1}(y_i)$.

Traditionally, there are two approaches to passivate a system: input feed-forward passivation and output feedback passivation. The input feed-forward passivation applies to a stable system with a Lyapunov function. Consider an input affine system
which has a stable equilibrium at $x = 0$ with a Lyapunov function $V(x)$. Then we assume that feed-forward system $G_2$ (as in Fig. 2.2) has the form $vI$, where $v$ is a constant. Since $G_1$ is stable, it is easy to see that

$$\dot{V} = \frac{\partial V}{\partial x} f_1(x) + \frac{\partial V}{\partial x} g_1(x) u \leq \frac{\partial V}{\partial x} g_1(x) u. \tag{2.35}$$

Since $G_2 = vI$, we have $y = y_1 + y_2 = h_1(x) + vu$. Then $y^T u = h_1^T(x) u + vu^T u$.

As long as there exist a $v$ such that

$$vu^T u > \left[ \frac{\partial V}{\partial x} g_1(x) - h_1^T(x) \right] u, \tag{2.36}$$

we can guarantee that $\dot{V} \leq y^T u$.

The result can also be generalized to dynamic feed-forward systems. Any stable input affine systems (of which a Lyapunov function can be found) can be passivated with a feed-forward dynamic systems, as shown in Theorem 2.10.

**Theorem 2.10.** [18] A non-passive system $G_1$ with a stable equilibrium point $x = 0$ and a Lyapunov function $V$ can be passivated by a feed-forward passivater $G_2$ (as in Fig. 2.2), which can be constructed as

$$\begin{align*}
\dot{x} &= f_1(x) + g_1(x) u \\
y_2 &= \left[ \frac{\partial V}{\partial x} g_1(x) \right]^T - h_1(x).
\end{align*}$$

For linear systems, the feed-forward system can be easily obtained using the linear version of the KYP lemma. Detailed discussion can be found in [18]. It should be noted that it is not possible to passivate an unstable system with feed-forward control because feed-forward control does not affect the free dynamics of the process. Such systems can only be passivated via feedback.

For feedback passivation, most work is concerned with passivation by state feed-
back. A thorough development of this topic can be found in [24]. An input affine system is said to be feedback passive (or feedback equivalent to a passive system) if there exists a feedback transformation [24]:

\[ u = \alpha(x) + \beta(x) v \]

with invertible \( \beta(x) \) such that the system

\[
\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\
y = h(x),
\]

is passive. The condition for feedback passivity is given in the following theorem.

**Theorem 2.11.** ([59]) An input affine system is locally feedback equivalent to a passive system with a positive definite storage function if and only if \( \text{rank}\left\{ \frac{\partial h}{\partial x} g(x) \big|_{x=0} \right\} = p \) and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function.

Theorem 2.11 shows that the relative degree and the zero dynamics cannot be altered by feedback. In this case, passivation is only possible via feed-forward.

For the direct output feedback case, there are some additional conditions which has been shown in [18].

### 2.7 Summary

In this chapter, we presented the main results in dissipativity and passivity theory. The properties of passivity and dissipativity were provided. To characterize how passive the system is, the notions of passivity indices and levels are introduced and their relations were discussed. In general, one can assume a system to be IF-OFP when either passivity indices or levels are used. The two passivation schemes were
mentioned at the end. It is noted that the results require special system structures and do not provide a straightforward way for passivation. To address these problems, different passivation schemes were proposed in Chapter 4 and 5.
CHAPTER 3

OVERVIEW OF CYBER-PHYSICAL SYSTEMS

3.1 Introduction

The term Cyber-Physical Systems (CPS) refers to a new generation of systems with integrated computational and physical capabilities that can interact with humans through many new modalities. According to [3], CPS are “physical, biological, and engineered systems whose operations are integrated, monitored, and/or controlled by a computational core. Components are networked at every scale. Computing is deeply embedded into every physical component, possibly even into materials. The computational core is an embedded system, usually demands real-time response, and is most often distributed. The behavior of a cyber-physical system is a fully-integrated hybridization of computational (logical), physical, and human action”. By the term Cyber, we usually refer to computation, communication, and control that are discrete, logical, and switched while the term Physical refers to natural and human-made systems governed by the laws of physics and operating in continuous time. Applications of CPS include high confidence medical devices, advanced automotive systems, distributed robotics, smart electric grids and so on.

Over the years, computer science and control science communities have independently developed enormously successful theories and technologies in their own areas. Systems and control researchers have pioneered the development of powerful system science and engineering methods and tools, such as time and frequency domain methods, state space analysis, system identification, filtering, prediction, optimiza-
tion, robust control, and stochastic control. At the same time, computer science researchers have made major breakthroughs in new programming languages, real-time computing techniques, visualization methods, compiler designs, embedded systems architectures and systems software, and innovative approaches to ensure computer system reliability, cyber security, and fault tolerance. Computer science researchers have also developed a variety of powerful modeling formalisms and verification tools. However, when it comes to CPS integration by combining the results from the two communities, the desired “separation of concerns” property does not hold, which leads to a wide range of issues due to lack of integrated science, technology and engineering in CPS. The shortcomings and challenges of current science and technology are identified in [1–3]. A few important issues are summarized below.

Composition is a technical foundation of all engineering disciplines. It helps us manage complexity, decrease time-to-market, and reduce costs. In academia, it means that system-level properties can be computed from local properties of components and component properties are not changing as a result of interactions with other components. Lack of compositionality means that systems don’t behave well outside of a small operational envelope. Cyber-physical systems are inherently heterogeneous not only in terms of their components but also in terms of essential design requirements. In addition to functional properties, CPS are subject to a wide range of physical requirements, such as dynamics, power, physical size and other systems-level requirements, such as safety, security and fault tolerance. This heterogeneity does not go well with current design methods and practices for several reasons.

System integration is the common problem in large-scale system design. It is expected that we can specify components, design them independently, and then easily plug them together to create complex systems. Unfortunately, history is full of examples that show we cannot reliably integrate complex components into complex systems. Our current technology cannot provide predictability for partially composi-
tional properties, which is a common situation in all large scale system development. We treat system integration as a management problem instead of a well-formulated science and engineering problem. A scientific and systematic basis for system integration is needed.

Security and privacy are necessary for both economic security and quality of life. Cyber-physical systems open up new threats: physical systems can now be attacked through cyberspace and cyberspace can be attacked through physical devices.

Certification is a key problem for safety critical systems. Safety critical systems must be certified as being safe by governmental agencies. Certification is currently done through exhaustive testing of the cyber-physical system. This practice will not scale well as increase the size of the system. In particular, if we have a certified a large-scale CPS and then add onto that system, will we need to replicate the certification process? This can be extremely expensive and will inhibit the development of plug-and-play safety-critical systems.

Although the notion of CPS is newly introduced and has attracted significant attentions recently from both computer and control communities, the attempt to deal with integration of CPS dates back to the origin of digital control theory when computers started to be used as controllers. In the design of computer-controlled systems, digital control theory focuses on designing a discrete-time digital controller for a continuous-time physical plant so that certain control specifications can be met. It is well known that a physical plant can be first discretized into its equivalent discrete-abstraction and then various digital controller synthesis methods are able to be applied. On the other hand, one can also first obtain a continuous-time controller through classical controller design processes and then discretize it into a digital controller which is suitable for computer implementation. Both methods have been successful in theory and practice and their success relies on certain assumptions on samplers and hold devices introduced in digital control. It is assumed that sampling
is periodic and synchronized with the hold operation. Moreover, we assume there is no delay in both plant and controller process. The output of the controller is assumed to be accurately computed. If there are sufficient computational and communication resources, one can ensure that the above requirements are always met.

However, cyber processes may have access only to finite resources and the use of these resources has a price in terms of energy or system complexity. Moreover, recent technologies in communication and computer science profoundly have changed the traditional infrastructure of control systems. Because of communication networks, it is necessary to consider various network-induced factors such as delay, bandwidth, quantization and scheduling. Due to intelligent components embedded in the system, cyber processes have rich and complicated behaviors that closely interact with plant processes, which makes the embedded system hard to model, analyze and control. Therefore, the challenges mentioned above in CPS requires the interdisciplinary research among control, communication and computer science. From control perspective, the recent effort to tackle the challenges are mainly in two areas, namely hybrid systems and networked control systems.

In this chapter, we will briefly introduce these two important research areas. The focus is on passivity-based methods for hybrid and networked control systems. The main results in this area are presented for completeness.

3.2 Hybrid Systems

3.2.1 Introduction

In the literature of hybrid dynamical systems, there exist several approaches to modeling. They can be characterized and described along several dimensions. We group modeling frameworks according to their roots, to the emphasis on or the complexity of the continuous and discrete dynamics and on whether they emphasize
analysis and synthesis results or analysis only or simulation only. On one end of
the spectrum there are approaches to hybrid systems that represent extensions of
system theoretic ideas for systems (with continuous-valued variables and continuous
time) that are described by ordinary differential equations to include discrete time
and variables that exhibit jumps or extend results to switching systems. Typically
these approaches are able to deal with complex continuous dynamics and empha-
size stability results. On the other end of the spectrum there are approaches to
hybrid systems that are embedded in computer science models and methods that
represent extensions of verification methodologies from discrete systems to hybrid
systems. Typically these approaches are able to deal with complex discrete dynamics
described by finite automata and emphasize analysis results (verification) and sim-
ulation methodologies. There are additional methodologies spanning the rest of the
spectrum that combine concepts from continuous control systems described by linear
and nonlinear differential/difference equations, and from supervisory control of dis-
crete event systems that are described by finite automata and Petri nets to derive,
with varying success, analysis and synthesis results. Optimization and mathematical
programming methodologies are also used in hybrid systems study.

Hybrid systems are heterogeneous dynamical systems which involve both contin-
uous models describing the physical and mechanical part and discrete event models
describing the software and logical behavior [13]. Continuous models are time-driven
continuous variable dynamics, which is usually described as differential or differ-
ence equations; discrete event models are event-driven discrete logic dynamics, often
described by finite-state machines or Petri nets. As such, hybrid systems theory com-
bines ideas originating in the computer science and software engineering disciplines
on one hand, and systems theory and control engineering on the other. This mixed
color character explains the terminology “hybrid systems”, which was used in this context
for the first time by Witsenhausen [21].
Hybrid systems have been identified in a wide variety of applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, among many other fields. As a simple example, the temperature regulation system in a typical room can be considered as a hybrid system. The furnace or air conditioner, along with the heat flow characteristics of the room, form the continuous variable dynamics, whereas the on-off thermostat can be modeled as a discrete event system. Also, hybrid systems have a central role in understanding the behavior of systems interacting with communication networks, such as automated highway and air-traffic management \[71, 111\]. In these systems, the discrete logic dynamics arises due to the asynchronous and even-driven nature of the data transmission in communication networks. Moreover, hybrid systems can be used to describe the physical processes which exhibit both fast and slow changing behaviors, such as non-smooth mechanics \[23\]. By modeling them as hybrid systems, more accurate models can be obtained, which will contribute to better performance benefits.

More detailed historical review of hybrid systems can be found in \[9, 10, 14\], and more recent reviews are given in \[11, 45, 142\]. The published books on hybrid systems include \[70, 74, 100, 102, 107, 112, 118\].

3.2.2 Passivity and Dissipativity Analysis in Hybrid Systems

In the literature of hybrid systems, there exist a variety of approaches to modeling of hybrid systems. However, there does not exist a unified framework which can be used to address all the issues in hybrid systems. Although the common features of all the modeling paradigms are integration of discrete event dynamics and continuous time dynamics and the interaction between them, they are distinguished by their different origins, the complexity of the continuous and discrete dynamics and the emphasis on analysis and synthesis abilities. For example, the models called switched systems are natural extensions of ordinary differential equations in system
theory by including switchings or jumps as discrete event variables. They are suitable to address stability analysis and controller synthesis problems. On the other hand, there are models which originated from computer science models whose emphasize is on verification and simulation problems. In addition, there are other models combining concepts and techniques from continuous control systems and from supervisory control of discrete event systems. It is important to choose the right model for a given analysis or control problem in hybrid systems.

In the following, three frameworks for hybrid systems are introduced and discussed, i.e. switched systems, piecewise affine systems and hybrid automata. We mainly focus on the results on passivity/dissipativity characterization for these models. Difficulties of extending the classical passivity/dissipativity to hybrid systems lie in switchings and other event-driven dynamics. As energy-based notions, passivity/dissipativity for discrete event dynamics need to be properly modeled and energy generated from discrete transitions should be considered additionally.

Switched systems:

Switched systems represent a type of models that are close to “non-hybrid” systems and can be viewed as an extension of the theory of continuous systems towards hybrid system. A switched system consists of a finite number of continuous-variable subsystems and a logical rule that orchestrates switching between these subsystems. Although its model is relatively simple and straightforward, this system class exhibits several typical behaviors of hybrid dynamical systems.

The mathematical representation of switched systems is based on the state-space model.

**Definition 3.1.** A switched system is described by a collection of indexed differential
(or difference) equations

\[ \dot{x}(t) = f_{q(t)}(x(t), u(t)), \quad x(0) = x_0 \]  
\[ y(t) = g_{q(t)}(x(t), u(t)). \]  

(3.1)  
(3.2)

where the input \( x \in \mathbb{R}^n \) is the continuous state vector and \( q(t) \in \{1, 2, \cdots, M\} \triangleq Q \).

The discrete event dynamics are modeled by a switching law, which is usually described as a piecewise constant map, \( \sigma : \mathbb{R}^+ \rightarrow Q \). The switching law introduced contributes to the hybrid behaviors of switched systems and distinguish them to the conventional nonlinear systems.

The extension of defining dissipative and passivity on switched systems are considered in \[77, 83, 138, 140, 141\]. The definitions provide a general characterization of dissipativity for switched systems by utilizing multiple storage functions, which are similar to multiple Lyapunov functions, to reduce conservativeness of assuming the existence of a common storage function. It is shown that all these definitions of passivity for switched systems reduce to the classical notion of passivity for the case of no switching. Furthermore, they can be used to show the Lyapunov stability of passive switched systems, as the main expected result of traditional passivity. Among the definitions, \[141\] is the most general and comprehensive one, which hows the stability of the system and passivity is preserved under interconnection. Compared to other definitions, \[136\] is restrictive by assuming each subsystem is passive “on average” while inactive and \[140\] does not cover whether this definition of passivity is preserved under parallel or negative feedback interconnection. Recently, \[26, 44\] showed the conditions of passivity for linear switched systems when the switching is subject to dwell time constraint.

**Piecewise affine systems:**

Piecewise affine systems (PWA) are a special class of switched systems where the
vector field \( f \) and output function \( g \) are affine for all modes. PWA systems have been studied extensively as they form the “simplest” extension of linear systems, which still can characterize many nonlinear and non-smooth dynamical systems by approximating nonlinear dynamics via multiple linearizations at different operating points \([104, 105]\).

**Definition 3.2.** A piecewise affine system is described by the equations

\[
\dot{x}(t) = A_q x(t) + B_q u(t) + e_q, \\
y(t) = C_q x(t) + D_q u(t) + f_q,
\]

for \( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \Omega_q \), where \( \Omega_q \subseteq \mathbb{R}^n \) are convex polyhedral (i.e. given by a finite number of linear inequalities) in the input/state space with non-overlapping interiors.

\( \Omega_q \) defines a partition of input/state space. Once the partition characterized by \( \Omega_q \) is given, the current operation mode \( q \) only depends on the current state \( x(t) \) and input \( u(t) \). Therefore, piecewise affine systems can be viewed as autonomously switched systems with the switching signal as a function of the state \( x(t) \) and input \( u(t) \).

[20] studied the passivity analysis of discrete-time piecewise affine and piecewise polynomial systems. It proposed sufficient passivity analysis and synthesis criteria based on the computation of piecewise quadratic or piecewise polynomial storage functions using linear matrix inequality techniques and sum of squares decomposition methods.

**Hybrid automata:**

A hybrid automaton is a transition system that is extended with continuous dynamics, which can be seen as a cross product of finite state machine and differential or
difference equations. The finite state machine components describe the discrete event dynamics while differential or difference equations model the evolution of continuous variables in time. Typically, a hybrid automaton consists of discrete states, continuous state space, vector fields, initial sets, invariants, transitions, guards, and resets. Various definitions exist in the literature which differ only in details. We adopt the definition from \cite{70}, which include all the basic elements.

**Definition 3.3.** \cite{70} A hybrid automaton $H$ is a collection

$$H = \{Q, X, f, Init, Inv, E, G, R\}$$

where

- $Q$ is a finite set of discrete states or control locations;
- $X \in \mathbb{R}^n$ is the continuous state space;
- $f : Q \times X \to X$ is an analytic vector field $f(q, \cdot)$ associated with each location $q \in Q$;
- $Init \subseteq Q \times X$ is the set of initial states;
- $Inv : Q \to 2^X$ describe the invariants of the locations;
- $E \subseteq Q \times Q$ is the transition relation;
- $G : E \to 2^X$ is the guard conditions;
- $R : E \times X \to 2^X$ is the reset map.

Hybrid automata have been originally proposed in \cite{4}. Many extensions of hybrid automata exist, such as hybrid I/O automata \cite{72}, to model embedded systems. There has been little work proposed on passivity or dissipativity for hybrid automata. Recently, \cite{79} proposed a new dentition of dissipativity for hybrid systems. The hybrid model used was the hybrid input-output automata model. The work followed closely previous work on dissipativity for switched systems. Stability results on bounded stability and Lyapunov stability were demonstrated.
3.3 Networked Control Systems

3.3.1 Introduction

In many practical control systems, because of certain practical constraints or requirements it is difficult and inefficient in many cases to have the physical plant, controller, sensor and actuator in the same location. If these components are distributed in different locations, it is necessary that they should be connected over a network so that signals can be transmitted from one place to another, which gives rise to networked control systems (NCS). Due to the introduction of communication network media, the great advantages can be expected, such as low cost, reduced weight and power requirements, simple installation and maintenance. Therefore, NCSs have become more and more popular in many practical applications and thus they have attracted a lot of attention in the theoretical research of NCSs. In the control community, modeling, analysis and synthesis of networked-based feedback systems with limited communication capacity has emerged as a topic of significant interest. Various results have been reported [22, 50, 76, 78, 92, 110, 116, 129, 135, 139].

The basic infrastructure of a networked-based feedback system is illustrated in Figure 3.1.
Fig. 3.1 where $G_p$ is the plant and $G_c$ is the controller. The two systems are feedback interconnected through the communication network. From a control perspective, what makes a NCS distinct from traditional feedback control system is the various networked-induced factors in the communication path, such as time delay, packet dropouts and quantization. Among those factors, time delay is probably the most significant one, which is caused by the limited bit rate of the communication channels, by possible waiting time when signals transmit through a busy channel, or by processing of the signal or propagation. It is well known that time delays will bring negative effects on the stability and performance of NCSs. Different from the state-delayed models, time delays in NCSs typically take the form of input delays or output delays. The second interesting problem in an NCS is the packet dropout phenomenon, which is usually caused by unavoidable errors and losses due to unreliability of the network. Though many NCSs employ automatic repeat request mechanisms, packet dropout phenomenon still can happen. The situation may appear that one packet sampled at the sensor node reaches the destination later than its successors. If this is the case, it is natural to use the most updated packet by dropping out the old ones, giving rise to packet dropout phenomenon. Another important issue in NCSs is the quantization effect. In NCSs, it is standard that the measurement of output and input command signals are usually quantized before feeding into the network. Since quantized signals are represented by number of quantization levels, they only have finite precision and therefore the standard assumption that data transmission can be performed with infinite precision is not valid. Most of the work concentrates on understanding and mitigating the effects of quantization for feedback stability and stabilization.

The notion of passivity plays an important role in the analysis and design of linear and nonlinear systems. As elaborated in the previous chapter, passive systems have many desired properties, such as stability and robustness to noise and uncertainty. Although traditional passivity theory has been applied successfully to
various classical control systems design, this property is vulnerable to time delay, discretization, quantization or other factors introduced by digital controllers or communication channels in modern control systems. Passivation and passivity analysis problems need to be investigated under these network-induced factors. As for time delays, [29, 31] pointed out that the feedback interconnection of two passive systems may not result in a passive system when time delays appear in the feedback path. In order to preserve passivity under time-varying delays, it has been shown that the wave variable transformation and time-varying gains can be applied [29, 33, 130]. Other than time delays, discretization is another factor under which passivity may not be preserved. Since it is required that continuous-time signals need to be discretized into discrete-time signals before transmitting over network, a continuous-time system is first discretized into a sampled-data system. However, it is pointed out in [34, 36, 85, 95, 109] that passivity is not preserved under discretization, which means the discretized system may not be passive even if the original continuous-time system is passive. Exactly how much passivity is lost under standard discretization has been quantified in [95]. The passivity degradation under the standard discretization can be characterized in terms of passivity indices and sampling time. In [85, 109], a novel average passivity for discrete-time systems was proposed in order to preserve the passivity property losslessly under any sampling time.

3.3.2 Passivity Analysis and Passivation in Networked Control Systems

Time delays:

As mentioned in the previous chapter, passivity is a compositional property: when two passive systems are combined in negative feedback the resulting interconnected system is still passive. This simple but powerful paradigm has led to a wide variety of constructive control design methods. However, such property may not hold when systems are interconnected over a network with delays. Therefore, it is important to
extend the standard passivity and dissipativity results for feedback interconnections that may have delays in communication. Some results in this direction have been reported in [29, 31].

Consider the systems interconnection when delays are inserted in the communication path, as shown in Fig. 3.2. When delays are constant, it was demonstrated in [31] that under appropriate assumptions, a negative feedback interconnection of output strictly passive systems is passive. In [29], the results were further extended to include time-varying delays. It showed that a feedback interconnection of output strictly passive systems, with non-increasing time delays, is passive independent of the time-varying delays. The main result is stated as in Theorem 3.1.

**Theorem 3.1.** [29] Consider two output strictly passive systems $G_1$ and $G_2$ in Fig. 3.2 with

$$
\dot{V}_i \leq u_i^T y_i - \delta_i y_i^T y_i,
$$

where $V_i$ are valid energy storage functions, $\delta_i > 0$ and $i = 1, 2$. If the time-varying delays $T_i(t)$ are non-increasing (i.e. $\dot{T}_i(t) \leq 0$), the feedback interconnection is passive.
Figure 3.3. A feedback interconnection of passive systems with time-varying delays and gains

if $\delta_1 = \delta_2 = 1$ or strictly output passive if $\delta_1, \delta_2 > 1$, from the input $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Remark 3.1. Theorem 3.1 shows that passivity in feedback interconnection can be preserved for certain passive systems and certain class of time delays. It is noted that the rate of change of the delay plays is a key factor that affects passivity.

In Theorem 3.1, passivity of the feedback interconnection is shown only for non-increasing time delays. However, in a practical scenario, the time delay may be increasing or decreasing, which can be determined a priori. Therefore, it is important to ensure passivity for a larger class of time delays. In fact, [29, 69] show that passivity can be recovered provided time-varying gains, dependent on the maximum rate of change of delay, are used in the communication path. The proposed time-varying gains are added in the communication network as shown in Figure 3.3.
Theorem 3.2. [29] Consider two output strictly passive systems $G_1$ and $G_2$ in Fig. 3.3 with

$$\dot{V}_i \leq u_i^T y_i - \delta_i y_i^T y_i,$$

where $V_i$ are valid energy storage functions, $\delta_i > 0$ and $i = 1, 2$. If the time-varying gains $d_i^2(t) \leq 1 - \dot{T}_i^{\text{max}}$, $i = 1, 2$, where $\dot{T}_i^{\text{max}}$ is the maximum rate of change of the delay $T_i(t)$ and $\dot{T}_i^{\text{max}} < 1$, the feedback interconnection is passive if $\delta_1 = \delta_2 = 1$ or strictly output passive if $\delta_1, \delta_2 > 1$, from the input $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Remark 3.2. The condition $\dot{T}_i^{\text{max}} < 1$ implies that the time delays cannot grow faster than time itself, which is a statement about the causality of the system. Therefore, it is a condition that all practical systems and delays will satisfy. Moreover, it is shown that the time-varying gains $d_i(t)$ only depends on the upper bound of the rate of change of delay, which is often assumed in time-delayed systems.

For Theorem 3.1 and 3.2, one has to assume that the individual systems are output strictly passive systems with passivity indices not less than 1, which restricts its applications in practice. To tackle this issue, a transformation scheme named "wave variable transformation" [93] or, equivalently "scattering transformation" [6] was utilized to ensure passivity of the feedback interconnection but only assumed that the individual systems to be passive. It turns out using such transformation scheme passivity of the feedback interconnection can be guaranteed for constant time delays [6, 29, 93, 94]. In the time-varying delay case, as before, time-varying gains can be used together with wave variable transformation to guarantee passivity of the feedback interconnection [29, 33, 130]. Further results on considering date dropouts can be found in [60, 61]. Recent results show that the wave variable transformation technique has been successfully applied to networked control systems [63] and control of haptic teleoperation [55].

Consider the system interconnection with constant delays and wave variable trans-
Figure 3.4. A feedback interconnection with delays and the wave variable transformation (WVT) inserted in the communication path, as shown in Fig. 3.4. $G_1, G_2$ are passive systems and $T_1, T_2$ are constant time delays. The block “WVT” denotes the wave variable transformation. The wave variable transformation is defined as in [40], which is a linear transformation. In Fig. 3.4, $v_i$ and $z_i$ where $i = 1, 2$, are wave variables and can be obtained as

$$\begin{bmatrix} v_1 \\ z_1 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} I & bI \\ -I & bI \end{bmatrix} \begin{bmatrix} y_{1s} \\ y_1 \end{bmatrix}$$

(3.3)

$$\begin{bmatrix} v_2 \\ z_2 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} I & bI \\ -I & bI \end{bmatrix} \begin{bmatrix} y_{2s} \\ y_2 \end{bmatrix}.$$  

(3.4)

With inputs and outputs of the two wave variable transformation blocks as defined
in the figure, the transformation can be implemented as

\[
\begin{bmatrix}
  v_1 \\
y_{1s}
\end{bmatrix}
= \begin{bmatrix}
  -I & \sqrt{2b}I \\
  -\sqrt{2b}I & bI
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
y_1
\end{bmatrix}
\]

(3.5)

\[
\begin{bmatrix}
  z_2 \\
y_{2s}
\end{bmatrix}
= \begin{bmatrix}
  I & -\sqrt{\frac{2}{b}}I \\
  \sqrt{\frac{2}{b}}I & -\frac{1}{b}I
\end{bmatrix}
\begin{bmatrix}
  v_2 \\
y_2
\end{bmatrix}
\]

(3.6)

**Theorem 3.3.** Consider the wave variable transformation in (3.5)-(3.6) and two passive systems \( G_1 \) and \( G_2 \) in Fig. 3.4 with

\[
\dot{V}_i \leq u_i^T y_i
\]

where \( V_i \) are valid energy storage functions for \( i = 1, 2 \). The feedback interconnection is passive from the input \( \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \) to output \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \), independent of the constant time delay.

**Remark 3.3.** With the wave variable transformation, the energy in the network can be characterized by the energy difference of a wave going out over the network and a wave coming in from the network. It can be shown that the energy in the network is always non-negative when \( T_1 \) and \( T_2 \) are non-increasing time-varying delays.

For the general time-varying delays, passivity of the feedback interconnection can be ensured by introducing time-varying gains, as shown in Theorem 3.2. Theorem 3.4 implies that the time-varying gains can be used together with wave variable transformation to guarantee passivity of the feedback interconnection under arbitrary time-varying delays with the change rate less than 1. Fig. 3.5 shows the scheme of joint utilization of the wave variable transformations and time-varying gains.

**Theorem 3.4.** Consider the wave variable transformation in (3.5)-(3.6) and two
passive systems $G_1$ and $G_2$ in Fig. 3.5 with
\[
\dot{V}_i \leq u_i^T y_i
\]
where $V_i$ are valid energy storage functions for $i = 1, 2$. If the time-varying gains $d_i(t)$ are selected such that $d_i^2(t) \leq 1 - \dot{T}_{i\max}$, where $i = 1, 2$ and $\dot{T}_{i\max}$ is the maximum rate of change of the delay $T_i(t)$ satisfying $\dot{T}_{i\max} < 1$, the feedback interconnection is passive from the input \[
\begin{bmatrix}
  e_1 \\
  e_2
\end{bmatrix}
\] to output \[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
\], independent of the time-varying delays.

Remark 3.4. In this framework, the time-varying delays are compensated by time-varying gains in the wave variable transformation. Each received wave variable is scaled before the transformation is applied. Similar to the result in constant delay
case, it can be shown that the energy in the network is always non-negative regardless of time-varying delays $T_1$ and $T_2$.

Discretization:

As pointed out in the previous section, it is noted that individual systems in the feedback interconnection are assumed to be passive at first. However, the assumption may not always be guaranteed in practice. It is important to find approaches to passivate a system before it is connected to the network. In conventional passivity theory, feed-forward and output feedback are two schemes to passivate a nonlinear system under certain conditions [18, 59]. After introducing digital controllers or communication channels in modern control systems, it is pointed out in [34, 36, 85, 95, 109] that passivity is not preserved under discretization, which means the discretized system may not be passive even if the original continuous-time system is passive. Therefore, it is necessary to find out how to discretize the system so that passivity can be preserved.

Consider a passive system being discretized with a hold and a sampler, as shown in Fig. 3.6. The continuous-time system $H$ is an operator mapping a continuous
signal \( u(t) \) to a continuous signal \( y(t) \), which satisfies the passivity inequality

\[
\int_0^T u(t)^T y(t) \, dt \geq \epsilon \int_0^T u(t)^T u(t) \, dt + \delta \int_0^T y(t)^T y(t) \, dt
\]

for any \( T \geq 0 \) and any piecewise continuous input \( u(t) \), where \( \epsilon \) and \( \delta \) are non-negative real numbers. Recall that \( \epsilon \) and \( \delta \) are also referred as passive indices in the previous chapter. Then the continuous-system \( H \) is discretized into a discrete-time system \( H_d \) with an input \( u_d(k) \) and an output \( y_d(k) \) by a hold and a sampler. With \( h \) being the sampling period, the hold maps a discrete-time input \( u_d(k) \) to a continuous-time input \( u(t) \) while the sampler relates a continuous-time output \( y(t) \) to a discrete-time output \( y_d(k) \). How the discrete-time signals are related to their continuous-time counterparts is dependent on the specific choice of the hold and sampler. Analogous to (3.7), passivity of the discrete-time system \( H_d \) is characterized as

\[
\sum_{k=0}^{K-1} u_d(k)^T y_d(k) \geq \epsilon_d \sum_{k=0}^{K-1} u_d(k)^T u_d(k) + \delta_d \sum_{k=0}^{K-1} y_d(k)^T y_d(k)
\]

for any positive integer \( K \) and any input \( u_d(k) \), where \( \epsilon_d \) and \( \delta_d \) are non-negative real numbers.

The most commonly used discretization is using a zero-order hold and an ideal sampler such that

\[
u(t) = u_d(k), \; kh \leq t < (k + 1)h, \; k = 0, 1, \ldots
\]

and

\[
y_d(k) = y(kh), \; k = 0, 1, \ldots
\]

hold. The question of interest is whether passivity of the discrete-time system \( H_d \) will be preserved if the passive continuous-system \( H \) is discretized under standard
discretization. The result is summarized in the following theorem.

**Theorem 3.5.** Suppose that the continuous-system $H$ is passive and satisfies \ref{eq:passivity}. If there exist a finite $L_2$ gain $\gamma > 0$ such that

$$
\int_0^T \|\dot{y}(t)\|^2 dt \leq \gamma^2 \int_0^T \|u(t)\|^2 dt
$$

\[\text{(3.11)}\]

is satisfied for any $T \geq 0$ and admissible $u(t)$, the discrete-time system $H_d$ under standard discretization satisfies \ref{eq:discretization} with

$$
\epsilon_d = \epsilon - h\gamma - h\gamma |\delta| - h^2\gamma^2 |\delta|
$$

\[\text{(3.12)}\]

$$
\delta_d = \delta - h\gamma |\delta|.
$$

\[\text{(3.13)}\]

**Remark 3.5.** The theorem highlights the importance of input strict passivity for preservation of passivity. From \text{(3.12)}-\text{(3.13)}, it is obvious that passivity or output passivity of $H_d$ may not be preserved however small the sampling period $h$ is if $H$ lacks input strict passivity.

**Remark 3.6.** It is noted that we need to choose $h$ small to make the passivity degradation small. When $\gamma$ in \text{(3.11)} is large, it is understandable that $h$ should be small since such a oscillatory system requires fast enough sampling for its behavior to be captured.

As shown in Theorem 3.5, passivity of discretized systems can not always be preserved under standard discretization. Therefore, \cite{85,109} proposed a new output sampling scheme, named “average discretization”, such that it can always preserve passivity with any sampling period $h$ and zero-order hold. The new discrete-time output is defined by

$$
y_d(k) = \frac{1}{h} \int_{kh}^{(k+1)h} y(t)dt, \ k = 0,1,\ldots
$$

\[\text{(3.14)}\]
Theorem 3.6. [95] Suppose that the continuous-system $H$ is passive and satisfies \[ 3.7 \]. If the sampler is defined in \( 3.14 \), the discrete-time system $H_d$ under standard discretization satisfies \( 3.8 \) with

\[
\begin{align*}
\epsilon_d &= \epsilon \quad (3.15) \\
\delta_d &= \delta. \quad (3.16)
\end{align*}
\]

Remark 3.7. It can be seen that passivity of the system $H$ is losslessly preserved under average discretization. The average discretization is theoretically attractive because it makes the passivity-based method available after the discretization. However, it should be noted that the computation of the average sampler may not be easy since it requires the future output value.

3.4 Summary

In this chapter, we surveyed the main results on passivity-based analysis and design of CPS, focusing on hybrid systems and networked control systems.

For hybrid systems, most of the work was on extending the conventional passivity and dissipativity definitions to different hybrid models. Such extension is possible since passivity and dissipativity are energy-based notions. Although hybrid systems can demonstrate complex and “heterogeneous” dynamical behaviors in terms of their internal states, their external input and output representation can be characterized by the notions of passivity and dissipativity. Therefore passive hybrid systems, though different in dynamics, are “homogeneous” in the passivity-based design. It shows great promise in CPS design since compositional design of large-scale systems is allowed [15, 76].

For networked control systems, it is noted that most of the work assumed systems connected in communication network to be passive or strictly passive. Such require-
ment is restrictive since it is possible that not all systems are passive in practice.
Also, additional network-induced factors, other than time delay and discretization, need to be investigated in passivity analysis. In view of these considerations, Chapter 5 considers passivity analysis and passivation for quantized effects and Chapter 6 considers the same problems for event-triggered feedback systems. The requirement of passivity in this work is relaxed.
4.1 Introduction

Due to the fact that Lyapunov functions can serve as candidate energy functions in dissipative and passive systems, dissipativity and passivity theory act as powerful tools for analyzing a large class of systems behavior by utilizing Lyapunov function techniques \cite{12, 59}. Other than stability, the significant benefit of passivity is that when two passive systems are interconnected in parallel or in feedback, the overall system is still passive. Thus passivity is preserved when large-scale systems are combined from components of passive subsystems. Recent results \cite{8, 15, 65} showed its power in the compositional design of cyber-physical systems.

In order to measure the excess or shortage of passivity, passivity indices \cite{18, 59, 98, 124, 126} were introduced. The indices can be used to render the system passive using feedback and feed-forward gains, and describe the performance of passive systems. Various stability conditions based on passivity indices are derived to assess the stability of interconnected systems \cite{59}. In addition to characterizing stability, passivity indices can also be used in passivity analysis and passivation of interconnected systems. \cite{59} and \cite{115} gave the passivity indices for the closed-loop system when the subsystems are passive. \cite{79, 80} showed the passivity condition for the feedback interconnected linear systems. \cite{122} considered the schemes of altering the passivity indices of a given system using constant feedback and feed-forward interconnection.
matrices. A passivity measure of system interconnections in series using passivity indices is reported in [131].

Although it is well known that the negative feedback interconnection of two passive systems is still passive, the quantitative characterization of passivity for the closed-loop system has not been addressed previously. We propose a measure of passivity indices for the negative feedback interconnection of two input feed-forward output-feedback (IF-OF) passive systems. The two systems need to be neither passive nor linear in general. It is shown that passivity, with respect to the full input and output, may be reinforced under feedback interconnection. Then the problem of partial passivation is considered. We present the conditions under which passivity for a desired input and output pair can be guaranteed. The conditions are identical to the conditions in [79, 80] but the linearity assumption is no longer needed. Moreover, a measure of passivity indices for the passivated system is provided in the end.

The paper is organized as follows. In Section 4.2, the previous work on passivity analysis and passivation using passivity indices is presented. The problems of passivity analysis and passivation using passivity indices for feedback interconnected systems are introduced in Section 4.3. In Section 4.4, we derive the conditions to measure the passivity indices for the negative feedback interconnection of two input feed-forward output-feedback (IF-OF) passive systems. Passivation by feedback interconnection and its measure of passivity indices are discussed in Section 4.5. Two examples are discussed in Section 4.6. The conclusion is provided in Section 4.7. Results in this chapter are have been reported in [145].

4.2 Preliminaries and Background

Passivity indices can be used in the problem of passivity analysis and passivation. For nonlinear systems, [59] and [115] gave the passivity indices for the closed-loop system (as in Fig. 2.1) when $G_1$ and $G_2$ are either output strictly passive or input
strictly passive.

For linear systems, \cite{79, 80} showed when the closed-loop system (Fig. 2.1, assuming \( r_2 = 0 \)) is passive with respect to the input \( r_1 \) and output \( y_1 \) using a slightly different definition of passivity indices. By assuming \( G_1 \) and \( G_2 \) are both linear systems with \( (\nu_1, \rho_2) \) and \( (\nu_2, \rho_2) \) respectively, a sufficient passivity condition on the closed-loop system is given in Theorem \ref{thm:passivity_conditions}.

**Theorem 4.1.** \cite{79} Consider the feedback interconnection (Fig. 2.1) when \( G_1 \) is a linear system with a shortage of OFP, i.e. \( \rho_1 < 0 \) and \( \nu_1 \geq 0 \). \( G_2 \) is also linear and passive with \( (\nu_2, \rho_2) \), i.e. \( \rho_2 \geq 0 \) and \( \nu_2 \geq 0 \). Then this interconnection is passive if \( \rho_1 + \nu_2 \geq 0 \).

Another work related to passivity-based design using passivity indices appeared in \cite{122}. It considered the schemes of altering the passivity indices of a given system using constant feedback and feed-forward interconnection matrices. It assumed that \( G_1 \) is a diagonal transfer function matrix and \( G_2 \) (denoted as \( H_\rho \) in \cite{122}) is a constant output feedback matrix. As in \cite{79, 80}, the passivity is defined with respect to the input \( r_1 \) and output \( y_1 \), assuming \( r_2 = 0 \).

The recent work on the passivity analysis for parallel and series interconnections using passivity indices is reported in \cite{122, 131}.

### 4.3 Problem Formulation

We consider two problems in feedback interconnected systems. The interconnection considered here is the negative feedback interconnection of two input feed-forward output-feedback (IF-OF) passive systems (see Definition \ref{def:IF-OF}), as shown in Fig. 4.1. It is assumed that the passivity indices of the two systems are known, denoted as \( (\nu_p, \rho_p) \) for the system \( G_p \) and \( (\nu_c, \rho_c) \) for the system \( G_c \).

The first problem is to determine the passivity indices of the interconnected sys-
tem if the passivity indices of each individual systems are known. Although it is
well known that the negative feedback interconnection of two passive systems is still
passive, the quantitative characterization of passivity for the closed-loop system has
not been addressed.

The second problem considered is the “partial passivation” problem, which refers
to finding the conditions for which the closed-loop system is passive with respect to
the input $w_1$ and output $y_p$ when $w_2 = 0$. The condition can be used to passivate a
non-passive plant $G_p$ using a passive controller $G_c$. This problem has been considered
in [79] for linear systems. Here we consider nonlinear systems. Moreover, the passivity
indices of the passivated system are also given.

![Figure 4.1. Feedback connection of two IF-OF systems](image)

$G_p$

$\{v_p, \rho_p\}$

$G_c$

$\{v_c, \rho_c\}$

$w_1$ $+$ $-$$u_p$ $+$ $y_p$ $+$

$u_c$ $+$ $w_2$ $+$

Figure 4.1. Feedback connection of two IF-OF systems
4.4 Passivity/Dissipativity Analysis for Feedback Interconnected Systems

We first present the result relating the interconnected system to QSR-dissipative systems.

**Lemma 4.1.** Consider the feedback interconnection of two IF-OF systems with the passivity indices $\nu_p$, $\rho_p$ and $\nu_c$, $\rho_c$ respectively. The interconnected system with the input $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and output $y = \begin{bmatrix} y_p \\ y_c \end{bmatrix}$ is QSR-dissipative (See Fig. 4.1) with

$$\dot{V} \leq y^T Q y + 2w^T S y + w^T R w$$

where

$$Q = \begin{bmatrix} -(\rho_p + \nu_c) I & 0 I \\ 0 I & -(\nu_p + \rho_c) I \end{bmatrix},$$

$$S = \begin{bmatrix} \frac{1}{2} I & \nu_p I \\ -\nu_c I & \frac{1}{2} I \end{bmatrix},$$

and

$$R = \begin{bmatrix} -\nu_p I & 0 I \\ 0 I & -\nu_c I \end{bmatrix}.$$

**Proof.** Since $G_p$ and $G_c$ are IF-OF systems with the passivity indices $\nu_p$, $\rho_p$, $\nu_c$ and $\rho_c$, there exist $V_p$ and $V_c$ such that

$$\dot{V}_p \leq u_p^T y_p - \nu_p u_p^T u_p - \rho_p y_p^T y_p$$

and

$$\dot{V}_c \leq u_c^T y_c - \nu_c u_c^T u_c - \rho_c y_c^T y_c.$$
Then we have

\[
\dot{V} = \dot{V}_p + \dot{V}_c \\
\leq u_p^T y_p - \nu_p u_p^T u_p - \rho_p y_p^T y_p + u_c^T y_c \\
- \nu_c u_c^T u_c - \rho_c y_c^T y_c.
\]

(4.1)

Consider that \( u_p = w_1 - y_c \) and \( u_c = y_p + w_2 \). (4.1) can be rewritten as

\[
\dot{V} \leq w_1^T y_p + w_2^T y_c + 2\nu_p w_1 y_c - 2\nu_c w_2^T y_p \\
- \nu_p w_1^T w_1 - \nu_c w_2^T w_2 - (\nu_p + \rho_c) y_c^T y_c \\
- (\rho_p + \nu_c) y_p^T y_p \\
+ w^T \begin{bmatrix}
-\nu_p I & 0 I \\
0 I & -\nu_c I
\end{bmatrix} w + 2w^T \begin{bmatrix}
\frac{1}{2} I & \nu_p I \\
-\nu_c I & \frac{1}{2} I
\end{bmatrix} w
\]

\[
= y^T Q y + 2w^T S y + w^T R w
\]

(4.2)

Remark 4.1. From Theorem 2.1, the interconnected system is \( \mathcal{L}_2 \) stable if \( Q < 0 \). For this particular system, a sufficient condition for \( Q < 0 \) is \( \nu_p + \rho_c > 0 \) and \( \nu_c + \rho_p > 0 \). Therefore, we can recover the stability condition stated in Theorem 2.8.

Remark 4.2. Although we have the stability conditions for the interconnected system, it is not clear how the passivity indices of the closed-loop system can be characterized.

Theorem 4.2 shows how to determine the passivity indices of the closed-loop system.

**Theorem 4.2.** Consider the feedback interconnected system in Fig. 4.1. Suppose the
passivity indices $\nu_p$, $\rho_p$, $\nu_c$ and $\rho_c$ are known. If we choose $\epsilon$ and $\delta$ such that

$$\left\{ \begin{array}{l}
\epsilon < \min \{ \nu_p, \nu_c \} \\
\delta \leq \min \{ \rho_c - \frac{\epsilon \nu_p}{\nu_p - \epsilon}, \rho_p - \frac{\epsilon \nu_c}{\nu_c - \epsilon} \}
\end{array} \right., \quad (4.3)$$

then the closed-loop system has passivity indices $\epsilon$ and $\delta$ satisfying

$$\dot{V} \leq w^T y - \epsilon w^T w - \delta y^T y \quad (4.4)$$

where $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_p \\ y_c \end{bmatrix}$.

Proof. From (4.1), we have

$$\dot{V} = \dot{V}_p + \dot{V}_c \leq w_1^T y_p + w_2^T y_c + 2\nu_p w_1 y_c - 2\nu_c w_2^T y_p - \nu_p w_1^T w_1 - \nu_c w_2^T w_2 - (\nu_p + \rho_c) y_c^T y_c - (\rho_p + \nu_c) y_p^T y_p$$

$$= w^T y - \begin{bmatrix} w_1^T \\ y_c^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix}$$

$$- \begin{bmatrix} w_2^T \\ y_p^T \end{bmatrix} \begin{bmatrix} \nu_c & \nu_c \\ \nu_c & \rho_p + \nu_c \end{bmatrix} \begin{bmatrix} w_2 \\ y_p \end{bmatrix}. \quad (4.5)$$

Since $\epsilon$ and $\delta$ are chosen such that (4.3) is satisfied, (4.6) holds for the chosen $\epsilon$.
\[ \begin{align*}
\epsilon & \leq \nu_p \\
\epsilon & \leq \nu_c \\
(\nu_p - \epsilon)(\nu_p + \rho_c - \delta) & \geq \nu_p^2 \\
(\nu_c - \epsilon)(\rho_p + \nu_c - \delta) & \geq \nu_c^2 \\
\nu_p + \rho_c - \delta & \geq 0 \\
\rho_p + \nu_c - \delta & \geq 0
\end{align*} \]  

(4.6)

Further implies that the matrices

\[ M = \begin{bmatrix}
\nu_p - \epsilon & -\nu_p \\
-\nu_p & \nu_p + \rho_c - \delta
\end{bmatrix} \]

and

\[ N = \begin{bmatrix}
\nu_c - \epsilon & \nu_c \\
\nu_c & \rho_p + \nu_c - \delta
\end{bmatrix} \]

are positive semi-definite. Therefore, we have

\[ \begin{bmatrix} w_1^T & y_c^T \end{bmatrix} M \begin{bmatrix} w_1 \\
y_c \end{bmatrix} + \begin{bmatrix} w_2^T & y_p^T \end{bmatrix} N \begin{bmatrix} w_2 \\
y_p \end{bmatrix} \geq 0 \]  

(4.7)

for \( \forall w_1, w_2, y_c \) and \( y_p \). After re-arranging the terms in (4.7), one can obtain

\[ -\begin{bmatrix} w_1^T & y_c^T \end{bmatrix} E \begin{bmatrix} w_1 \\
y_c \end{bmatrix} - \begin{bmatrix} w_2^T & y_p^T \end{bmatrix} \Delta \begin{bmatrix} y_p \\
y_c \end{bmatrix} \geq \]

\[ -\begin{bmatrix} w_1^T & y_c^T \end{bmatrix} O \begin{bmatrix} w_1 \\
y_c \end{bmatrix} - \begin{bmatrix} w_2^T & y_p^T \end{bmatrix} P \begin{bmatrix} w_2 \\
y_p \end{bmatrix} . \]  

(4.8)
where

\[ E = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}, \]

\[ \Delta = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \]

\[ O = \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix}, \]

and

\[ P = \begin{bmatrix} \nu_c & \nu_c \\ \nu_c & \rho_p + \nu_c \end{bmatrix}. \]

From (4.8) and (4.5), we can finally show that

\[
\dot{V} \leq w^T y - \begin{bmatrix} w_1^T \\ y_c^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix} \\
- \begin{bmatrix} w_2^T \\ y_p^T \end{bmatrix} \begin{bmatrix} \nu_c & \nu_c \\ \nu_c & \rho_p + \nu_c \end{bmatrix} \begin{bmatrix} w_2 \\ y_p \end{bmatrix} \\
\leq w^T y - \epsilon w^T w - \delta y^T y
\]

(4.9)

**Remark 4.3.** (4.3) can be used to obtain an estimate of the passivity indices for the closed-loop system. The condition implies that the interconnected system may have smaller passivity indices than each subsystems. Note that the passivity considered here is with respect to the input \( w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \) and output \( y = \begin{bmatrix} y_p \\ y_c \end{bmatrix} \).
4.5 Passivation by Feedback Interconnection

Based on Theorem 4.3, passivity with respect to the full input and output (i.e. input $w$ and output $y$), may not be guaranteed to be reinforced under feedback interconnection. However, by selecting different inputs and outputs the corresponding passivity may change accordingly. As in Fig. 4.1, if our goal is to passivate a non-passive plant $G_p$ using a passive controller $G_c$ we only consider whether the closed-loop system is passive with the input $w_1$ and output $y_p$ by assuming $w_2$ is zero. Theorem 4.3 shows that it is possible to guarantee passivity for the desired input and output although passivity for full input and output may not hold.

**Theorem 4.3.** Assume $w_2 = 0$. The closed-loop system is passive with respect to the input $w_1$ and output $y_p$ if the passivity indices satisfy the conditions

$$\nu_p \geq 0 \quad \rho_c \geq 0 \quad \rho_p + \nu_c \geq 0.$$  

Proof. If $w_2 = 0$, (4.1) becomes

$$\dot{V} \leq \dot{w}_1^T y_p + 2\nu_P w_1 y_c - \nu_p w_1^T w_1$$

$$\leq \dot{w}_1^T y_p - \begin{bmatrix} w_1^T & y_1^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix} - (\rho_p + \nu_c) y_p^T y_p$$  

$$\leq \begin{bmatrix} w_1^T & y_1^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix}$$  

$$- (\rho_p + \nu_c) y_p^T y_p.$$  

$$\dot{V} \leq \dot{w}_1^T y_p - \begin{bmatrix} w_1^T & y_1^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix} - (\rho_p + \nu_c) y_p^T y_p$$  

(4.13)
Since we have $\nu_p \geq 0$, $\rho_c \geq 0$ and $\rho_p + \nu_c \geq 0$, it can be shown that

$$\begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \geq 0$$

(4.14)

$$\rho_p + \nu_c \geq 0.$$  

(4.15)

Therefore, we can conclude that

$$\dot{V} \leq w_1^T y_p - \begin{bmatrix} w_1^T \\ y_c^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix}$$

$$- (\rho_p + \nu_c) y_p^T y_p$$

$$\leq w_1^T y_p.$$ 

(4.16)

Remark 4.4. When the plant $G_p$ is non-passive (i.e. $\rho_p < 0$), the closed-loop system can be rendered passive by choosing a passive controller $G_c$ with $\rho_c \geq 0$ and $\nu_c \geq -\rho_p$.

It is noted that the conditions are identical to the conditions in Theorem 4.1 but here we do not need to assume linearity of the systems.

Remark 4.5. (4.16) shows that closed-loop system is OSP with the OFP index $\rho_p + \nu_c$.

We can recover the conditions in [59] where $G_p$ and $G_c$ are assumed to be OSP and ISP, respectively. The conditions (4.10)-(4.12) are similar to the conditions in Theorem 2.8 but are more restrictive since OSP is more conservative than $L_2$ stable.

We can also obtain an estimate of the passivity indices for the passivated closed-loop system, as shown in Theorem 4.4.

**Theorem 4.4.** Suppose that the conditions (4.10)-(4.12) are satisfied and $\nu_p + \rho_c > 0$. 

If we choose $\epsilon$ and $\delta$ such that

$$
\begin{cases}
\epsilon \leq \frac{\nu_p \rho_c}{\nu_p + \rho_c}, \\
\delta \leq \nu_c + \rho_p
\end{cases}
$$

(4.17)

the closed-loop system has the passivity indices $\epsilon$ and $\delta$ satisfying

$$
\dot{V} \leq w_1^T y_p - \epsilon w_1^T w_1 - \delta y_p^T y_p
$$

(4.18)

Proof. If the condition (4.10)-(4.12), (4.17) and $\nu_p + \rho_c > 0$ are satisfied, we have

$$
\begin{bmatrix}
\nu_p - \epsilon & -\nu_p \\
-\nu_p & \nu_c + \rho_c
\end{bmatrix} \geq 0
$$

(4.19)

$$
\nu_c + \rho_p - \delta \geq 0
$$

(4.20)

Then it implies

$$
\begin{bmatrix}
w_1^T & y_c^T
\end{bmatrix}
\begin{bmatrix}
\nu_p - \epsilon & -\nu_p \\
-\nu_p & \nu_c + \rho_c
\end{bmatrix}
\begin{bmatrix}
w_1 \\
y_c
\end{bmatrix} + (\rho_p + \nu_c - \delta) y_p^T y_p \geq 0,
$$

which can be written as

$$
-\begin{bmatrix}
w_1^T & y_c^T
\end{bmatrix}
\begin{bmatrix}
w_1 \\
y_c
\end{bmatrix}
- \epsilon w_1^T w_1 - \delta y_p^T y_p
$$

(4.21)

where

$$
O = \begin{bmatrix}
\nu_p & -\nu_p \\
-\nu_p & \nu_c + \rho_c
\end{bmatrix}
$$
Since it is already known that

\[
\dot{V} \leq w_1^T y_p - \begin{bmatrix} w_1^T & y_c^T \end{bmatrix} \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c \end{bmatrix} \begin{bmatrix} w_1 \\ y_c \end{bmatrix} - (\rho_p + \nu_c) y_p^T y_p
\]

\begin{align}
- (\rho_p + \nu_c) y_p^T y_p
\end{align}

we can conclude that

\[
\dot{V} \leq w_1^T y_p - \epsilon w_1^T w_1 - \delta y_p^T y_p
\]

holds for \(\forall w_1\).

**Remark 4.6.** Because of the conditions (4.10)-(4.12) and \(\nu_p + \rho_c > 0\), the passivity indices \(\epsilon\) and \(\delta\) are upper bounded by positive numbers. For feedback passivation, (4.17) provides a way to obtain the desired passivity indices of the closed-loop system by choosing a passive \(G_c\) with proper indices.

**Remark 4.7.** When \(G_c\) is a constant feedback with gain \(K_c\) where \(K_c\) is a positive definite matrix, we can show that \(G_c\) is IFP(\(\Lambda(K_c)\)). Consider that \(G_p\) is OFP(\(\rho_p\)) with \(\rho_p < 0\). If we choose \(K_c\) such that \(\Lambda(K_c) + \rho_p \geq 0\), the closed-loop system is passive. Moreover, if \(\Lambda(K_c) + \rho_p > 0\) Theorem 4.4 shows that the passivated system is OFP(\(\Lambda(K_c) + \rho_p\)). The result here is consistent with the previous result [18] but can be applied even when \(G_c\) is a dynamical controller.

### 4.6 Examples

In this section, two examples are presented to show how Theorem 4.3 and 4.4 can be applied for partial passivation. In order to be able to conveniently verify the results, we will focus on linear systems. Note that the methods can be applied to nonlinear systems in the same way.

Both examples consider a feedback system as in Fig. 4.1 with \(w_2 = 0\).
**Example 4.1.** The first example assumes a plant

\[ G_p = \frac{s + 0.5}{s - 0.1} \]

and a controller

\[ G_c = \frac{s + 4}{s + 2}. \]

It can be calculated (from the Nyquist plots [18]) that the plant is OFP with \( \rho_p = -0.2 \) and the controller is IFP with \( \nu_c = 1 \). From Theorem 4.4, the closed-loop system is OSP with the estimated output feedback passivity level \( \delta = \rho_p + \nu_c = 0.8 \).

We can further verify that the closed-loop transfer function

\[ \frac{s^2 + 2.5s + 1}{2s^2 + 6.4s + 1.8} \]

is OSP with actual output feedback passivity index \( \delta = 1.8 \).

**Example 4.2.** The second example assumes the plant is a 5th-order linear system with

\[
A = \begin{bmatrix}
-1.8 & 0.1 & 1.2 & 0 & 0 \\
0.1 & -0.5 & 0 & -0.3 & 0 \\
1.2 & 0 & -3 & -3 & 0.5 \\
0 & -0.3 & -2 & -3 & 0.4 \\
0 & 0 & 0.5 & 0.4 & -1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1 \\
4 \\
1 \\
5 \\
1
\end{bmatrix},
\]
Here the controller is a 2nd-order system with

\[
A = \begin{bmatrix}
-2 & -1 \\
-3 & -5
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1 \\
2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 1
\end{bmatrix}, \quad D = 1.
\]

We can determine the passivity levels of \( G_p \) and \( G_c \) to be \((0.18, 0.02)\) and \((0.3, 0.5)\), respectively. From Theorem 4.4, the closed-loop system has passivity indices \((0.1324, 0.32)\). It can be verified by the KYP lemma that \((0.1324, 0.32)\) are the valid passivity indices for the closed-loop system. This example also shows that it is possible to increase the OFP of the plant by choosing a proper controller, as pointed out in Remark 4.6.

**Example 4.3.** The third example assumes that the plant \( G_p \) is a nonlinear system (the model \( H_3 \) used in [131]). \( G_p \) is given by

\[
G_p \begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.5x_1^3 + 0.5x_2 + 2u_p \\
y_p &= x_2 + u_p
\end{cases}
\]

\( G_p \) admits a storage function given by

\[
V(x) = \frac{1}{8}x_1^4 + \frac{1}{2}x_2^2.
\]
\[ \dot{V}(x) = u_p y_p - 1.5u_p^2 + 0.5y_p^2. \]

Therefore the passivity levels for \( G_p \) are \( \rho_p = -0.5 \) and \( \nu_p = 1.5 \). Here we still use the controller \( G_c \) from Example 4.2 whose IFP level is \( \nu_c = 1 \). Again, using the conditions in Theorem 4.4, we know that the closed-loop system is OSP with the estimated OFP level \( \delta = \rho_p + \nu_c = 0.5 \). Fig. 4.2 shows the trajectory of the function \( \int (w_1 y_p - 0.5y_p^2)dt \) is always above 0 along the time of simulation.

Figure 4.2. The trajectory of the function \( \int (w_1 y_p - 0.5y_p^2)dt \) over time \( t \) for \( w_1(t) = \sin(2\pi t) \)
4.7 Summary

In this chapter, we considered passivity analysis and passivation using passivity indices. Measures of passivity indices for two input feed-forward output-feedback (IF-OF) interconnected system were provided. We also presented conditions for partial passivation and discussed passivity indices for the passivated system. In contrast to previous results, we do not need to assume that the systems are linear.
5.1 Introduction

Although traditional passivity theory has been applied successfully in various classical nonlinear systems, this property is vulnerable to discretization, quantization and other factors introduced by digital controllers or communication channels in modern control systems. In digital control system design, a continuous-time system is first discretized into a sampled-data system. However, it is pointed out in [34, 36, 85, 95, 109] that passivity is not preserved under discretization, which means the discretized system may not be passive even if the original continuous-time system is passive. Exactly how much passivity is lost under standard discretization (using zero order hold and ideal sampler) has been quantified in [95]. The passivity degradation under the standard discretization can be characterized in terms of passivity indices and sampling time. In [63, 85, 109], a novel average passivity for discrete-time systems was proposed in order to preserve the passivity property losslessly under any sampling time. Besides preserving passivity in discrete-time, stability and stabilization of discrete-time passive systems were also considered in recent work [27, 86]. The problem of finding the maximum sampling time preserving passivity for linear discrete-time systems was considered in [27]. It was shown that the feedback system is exponentially stable if the time-varying asynchronous sampling times embedded in feedback connection are bounded by the maximum sampling time. Two passivity-
based control strategies for the problem of stabilizing sampled-data systems were presented in [86].

In addition to discretization, the effect of quantization also needs to be considered when digital controllers interact with the environment by means of analog-to-digital converters or digital-to-analog converters that have a finite resolution. Moreover, quantization is necessary when the information between plants and controllers is transmitted through communication networks. In fact, the problem of control using quantized feedback has been an active research area for a long time. Most of the work [35, 37, 42, 58, 84] concentrates on understanding and mitigating the effects of quantization for feedback stability and stabilization. The existing results on passivity and quantization effects mainly focus on certain specific problems, depending on what kind of systems are considered. In signal processing systems [128], passivity analysis and passification of LTI systems with quantization was treated as an uncertainty described by integral quadratic constraints. In networked control systems, conditions were derived [43] under which the closed-loop networked control system is passive in the presence of sensor quantization and network induced delay. The problem of closed-loop stability for input-affine passive systems with quantized output feedback was investigated in [28]. Recent results [132, 133] used passivity to achieve $L_2$ stability in the presence of communication delays and signal quantization for networked control systems.

In this chapter, passivity analysis and passivation for quantized systems are provided. Its application to stability for passive switched systems with passive quantizers is also discussed. Since the quantized system can be viewed as the approximation of the original system, the theory of passivity analysis for systems under approximation can be used to show when the quantized system is passive. Considering that the conditions can not guarantee that passivity is preserved for quantized systems in general, a transformation-based passivation scheme is introduced. The passivity
preservation relies on an input/output transformation on the quantized input and output. The result shows that one can find such transformation so that the same passivity index of the original OSP system, with respect to the transformed input and output, will be recovered. The result is relatively general since we only require the system to be OSP and the quantizers to be passive, which characterize many practical quantizers. Although the passivity preserving condition is initially derived for non-switched systems, it can be extended to passive switched systems where the input/output transformation can switch between different transformations according to the current active subsystem. Therefore, passivity of passive switched systems under quantization can be guaranteed and the stability conditions in [81, 82] can be applied.

The rest of the chapter is as follows. In Section 5.2, background material on passive switched systems is covered. The notion of passive quantizers is introduced. Section 5.3 presents the main results in passivity analysis for systems under approximation. The results are then applied to analyze passivity for quantized systems. The transformation-based passivation scheme is given in Section 5.5. Section 5.6 extends the passivity-preserving conditions for non-switched systems to passive switched systems and then the stability conditions on passive switched systems are obtained. An example is provided in Section 5.7 to demonstrate the methods used. Some conclusions are provided in Section 5.8. Results in this chapter are have been reported in [143] and [124].

5.2 Preliminaries and Background

5.2.1 Finite Gain $L_2$ Stability

**Definition 5.1.** [59] A discrete-time system 2.2 is $L_2$ stable with a finite gain $\gamma$ if
there exist non-negative constants $\gamma$ and $\beta$ such that

$$\|y_N\|_2 \leq \gamma \|u_N\|_2 + \beta$$  \hspace{1cm} (5.1)

for all $u \in \mathcal{L}_{2e}$ and $N \geq 0$.

**Theorem 5.1.** [39] If there exists a positive storage function $V$ and a non-negative constant $\gamma$ such that the following inequality is satisfied

$$\Delta V(x(k)) \leq \gamma^2 u^T(k)u(k) - y^T(k)y(k)$$  \hspace{1cm} (5.2)

for all $x, u$ and $y$, then the system (2.3)-(2.4) is $\mathcal{L}_2$ stable with a finite gain $\gamma$.

Note that Theorem [39] is sufficient condition. However, it can be shown that it is also a necessary condition for linear systems. In this paper, we assume that for a finite gain $\mathcal{L}_2$ stable system one can find such storage function $V$ satisfying (5.2).

Note that similar results can be applied to continuous systems too.

5.2.2 Passive Quantizers

Consider a quantizer $Q(\cdot)$ with an input $v$ and an output $u$, where $v \in \mathbb{R}$ and $u \in \mathcal{U}$. $\mathcal{U} \subset \mathbb{R}$ is a quantized set whose elements are distinct quantized levels.

**Definition 5.2.** [133] A quantizer is called a passive quantizer if its input $v$ and output $u$ satisfy

$$av^2 \leq uv \leq bv^2$$  \hspace{1cm} (5.3)

where $u = Q(v)$ and $0 \leq a \leq b < \infty$.

If the input to the quantizer $v$ is vector, the quantization function acts component-
wise on the input vector such that

$$av^Tv \leq v^Tu \leq bv^Tv,$$

One can further verify that

$$\|u\|_2^2 \leq b^2\|v\|_2^2$$

is satisfied for passive quantizers.

Figure 5.1. A general quantizer bounded by a cone

The notion of a passive quantizer is based on conic systems theory [137]. A passive quantizer is a special case of a memory-less conic system. This can be seen
in Fig. 5.1, where a quantizer satisfying (5.3) has its input and output mapping bounded in a cone characterized by two lines with slope $a$ and $b$. The quantizer is called “passive” since the condition $uv \geq 0$ holds for all inputs $v$. This is the general condition for a memory-less non-linearity to be passive [59]. The notion of passivity for quantizers can capture many quantizers used in practice, such as the uniform mid-tread quantizer (Fig. 5.2), the logarithmic quantizer (Fig. 5.3) and many non-standard quantizers (Fig. 5.1).

Figure 5.2. (a) A uniform quantizer with infinite quantization levels (b) A uniform quantizer with finite quantization levels
Figure 5.3. (a) A logarithmic quantizer with infinite quantization levels (b) A logarithmic quantizer with finite quantization levels

We can find the values of $a$ and $b$ from a quantizer's input and output mapping. For example, we can show that $a = 0, b = 2$ for a uniform mid-tread quantizer with infinite/finite quantization levels; $a = 0, b = 1 + \delta$ for a logarithmic quantizer with finite quantization levels; and $a = 1 - \delta, b = 1 + \delta$ for a logarithmic quantizer with infinite quantization levels, where $1 > \delta > 0$ is a constant quantization gain.

5.2.3 Passivity for Switched Systems

A nonlinear switched system consists of a finite set of subsystems with nonlinear dynamics. The finite number of subsystems can be enumerated, $\{1, 2, ..., P\}$. At any point in time, a single subsystem $i$ is active and the dynamics are nonlinear and time-invariant. The time-varying nature of these systems comes from the switching behavior. The switching signal $\sigma(k)$ is a function that maps the time to the index of
the active subsystem, \( \sigma : \mathbb{Z}^+ \to \{1, \ldots, P\} \). This function is piecewise constant and only changes at switching instants. The model with the switching signal is given by

\[
\begin{align*}
    x(k+1) &= f_{\sigma(k)}(x(k), u(k)) \\
    y(k) &= h_{\sigma(k)}(x(k), u(k)).
\end{align*}
\] (5.6)

The switching instants can be listed in order \( k_1, k_2, \) etc. Alternatively, the notation \( k_{i_p} \) will be used to denote the \( p^{th} \) time that subsystem \( i \) becomes active. For example, the first subsystem \( (i = 1) \) becomes active for the first time \( (p = 1) \) at time \( k_0 \) \( (k_0 = k_{1_1}) \). The second subsystem \( i = 2 \) becomes active at time \( k_1 \) \( (k_1 = k_{2_1}) \) and so forth. By using these two notations in conjunction, it is possible to list completely the times that a system becomes active as well as the times it becomes inactive. Subsystem \( i \) becomes active the \( p^{th} \) time at time \( k_{i_p} \) and then inactive at time \( k_{i_p+1} \). That same subsystem becomes active again at time \( k_{i_p+1} \).

An indicator set will be defined to signify regions where a particular subsystem is active. Consider subsystem \( i \) that is active from \( k_{i_1} \) to \( k_{(i_1+1)} \), \( k_{i_2} \) to \( k_{(i_2+1)} \), etc. The set of times \( I_i \) can be defined to indicate those time intervals where subsystem \( i \) is active,

\[
I_i = \bigcup_{p=1}^{K_i} \{k_{i_p}, \ldots, k_{i(p+1)}\}.
\] (5.7)

This notation will be used to draw a distinction between the active and inactive time intervals of a system.

The notion of passivity for switched systems used in this chapter is based on previous work on decomposable dissipativity for switched systems. This approach has been used in continuous-time \([108, 141]\) and in discrete-time \([67]\). The concept of decomposable dissipativity is based on the fact that systems typically store energy differently when they are active compared to when they are inactive. The solution is to decompose the supply rate into an active portion and an inactive portion. When
a subsystem is inactive, it may have a different supply rate depending on which other subsystem is active. The definition given here is a special case of \[67\]. While that work presented a very general definition, the authors didn’t consider stability of interconnected systems. Traditionally, stability of feedback interconnections is one of the main benefits of dissipativity theory.

In decomposable dissipativity, the multiple energy storage function approach is taken. This allows for each subsystem $i$ to have a unique notion of energy captured by the storage function $V_i(x)$. This notion of energy is positive, i.e. for all $i$, $V_i(x) > 0$ for all $x \neq 0$. The notion of supplied energy for a subsystem $i$ while it is inactive may be unique for each active subsystem $j \neq i$. This results in several inactive energy supply rates for each $i$ and $j$. These rates may be a function of input, output, state, and time and will be denoted as $\omega_i^j(u, y, x, k)$. When each subsystem is inactive, the following inequality holds for each active subsystem $j$ at an appropriate time $t \in I_j (\forall i)$.

Passivity for discrete-time switched systems is given in the following definition. Recall that a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is class $K_\infty$ if $\alpha(0) = 0$, $\alpha$ is non-decreasing, and $\alpha$ is radially unbounded.

**Definition 5.3.** Consider a discrete-time switched system (5.6). This system is passive if there exists a positive storage function $V_i(x)$, for each subsystem $i$, with the property that for some $K_\infty$ functions $\alpha_i$ and $\overline{\alpha}_i$,

$$\alpha_i(\|x\|) \leq V_i(x) \leq \overline{\alpha}_i(\|x\|),$$

such that the following conditions hold for all $i$.

During the active time period $k \in I_i$ of each subsystem $i$, the system is passive ($\rho_i \geq 0$)

$$V_i(x(k+1)) - V_i(x(k)) \leq u^T y - \rho_i y^T y.$$  \hspace{1cm} (5.8)
When each subsystem $i$ is inactive, it is dissipative with respect to a cross supply rate that may be specific to the active subsystem $j$. For $k \in I_j$

$$V_i(x(k+1)) - V_i(x(k)) \leq \omega_i^j(u, y, x, k).$$

(5.9)

The cross supply rates are absolutely summable for all switching sequences $\forall i$ and $\forall j \neq i$,

$$\sum_{k=k_0}^{\infty} |\omega_i^j(u, y, x, k)| < L,$$

(5.10)

where $L$ is an arbitrarily large finite constant.

When $\rho_i > 0$ for all $i$, the switched system is called output strictly passive.

This definition is a natural extension of passivity for non-switched systems. Consider the case when there exists a common storage function for the switched system such that equation (5.8) holds for all $i$. In this case, passivity for switched systems reduces to the traditional notion of passivity for non-switched systems.

### 5.3 Passivity Analysis for Quantized Systems

The theory of passivity analysis for systems under approximation is given. Then the results are used for passivity analysis of quantized systems.

#### 5.3.1 Passivity under Approximation

Consider two system models $\Sigma_1$ and $\Sigma_2$ as shown in Fig. 5.4. One can view $\Sigma_1$ as the system we are interested in and $\Sigma_2$ as an approximation of $\Sigma_1$. A commonly used measure for judging how well $\Sigma_2$ approximates $\Sigma_1$ is to compare the outputs for the same excitation function $u$. We denote the difference in the outputs by $\Delta y$. Note that in general $\Delta y$ will depend on the exact function $u$. The error may be due to modeling, linearization, model reduction, or a host of other reasons.
For a good approximation, a reasonable requirement is that the worst case $\Delta y$ over all control inputs $u$ be “small” in terms of a suitably defined norm. More formally, with every approximate model, we associate two non-negative constants $\gamma > 0$ and $\epsilon \geq 0$ (if they exist) such that

$$\langle \Delta y, \Delta y \rangle_T \leq \gamma^2 \langle u, u \rangle_T + \epsilon, \quad \forall u \text{ and } \forall T \geq 0. \quad (5.11)$$

where $\langle u, y \rangle_T \triangleq \int_0^T u^T(t)y(t)dt$. The values of $\gamma$ and $\epsilon$ obviously reflect how good the approximation is. In fact, (5.11) requires the “error system” with input $u$ and output $\Delta y$ to be $L_2$ stable.

The problem of interest is considered as following. Assume that $\Sigma_2$ has an excess of passivity. What passivity property for $\Sigma_1$ can be inferred from that of $\Sigma_2$? For the case when $\Sigma_2$ does not have an excess of passivity but is $(Q_2, S_2, R_2)$-dissipative, we may pose the same problem in terms of obtaining conditions under which $\Sigma_1$ is $(Q_1, S_1, R_1)$-dissipative as well. The problem is summarized as follows.
Problem 5.1. Suppose that an approximate model $\Sigma_2$

1. has IFP($\nu$); or
2. has OFP($\rho$); or
3. is VSP for ($\rho, \nu$); or
4. is ($Q_2, S_2, R_2$)-dissipative.

What passivity or ($Q, S, R$)-dissipativity properties can be derived if the models $\Sigma_1$ and $\Sigma_2$ satisfy the relation (5.11)?

The aim is to characterize the passivity levels or the supply rate of the system $\Sigma_1$ by analyzing the (hopefully more tractable) approximate model $\Sigma_2$. Note that the approximate model $\Sigma_2$ may be nonlinear or even unstable. When the system $\Sigma_1$ is not passive (e.g. unstable) but is guaranteed to be ($Q_1, S_1, R_1$)-dissipative, then we can use the answer from Problem 1 to design controllers to render the system $\Sigma_1$ to be passive (and hence stable) \[55, 56\]. It can be verified that the results are symmetric in $\Sigma_1$ and $\Sigma_2$. In other words, it does not matter whether we view $\Sigma_1$ as an approximation of $\Sigma_2$ or $\Sigma_2$ as an approximation of $\Sigma_1$. In practice, however, a simpler model is usually used as an approximation of a complex system.

[124–126] provided the results for all the cases in Problem 5.1. Here only the results for the cases when the approximate model is ISP and VSP are presented (as shown in Theorem 5.2 and Theorem 5.3), due to their close connection to the passivity analysis of quantized systems.

Theorem 5.2. Consider $\Sigma_1$ and $\Sigma_2$ in Fig. 5.4. Suppose (5.11) is satisfied for some $\gamma > 0$ and $\epsilon \geq 0$. If $\Sigma_2$ has IFP($\nu$) and $\gamma < \nu$, then $\Sigma_1$ is ISP for $\tilde{\nu} = \nu - \gamma$.

Proof. Consider systems $\Sigma_1$ and $\Sigma_2$ with an arbitrary input $u$, so that the corresponding outputs are $y_1$ and $y_2$, where $y_2 = y_1 + \Delta y$. First we note that for any
\( \gamma > 0, \) \( |u^T \Delta y| \leq \frac{1}{2\gamma} \Delta y^T \Delta y + \frac{\gamma}{2} u^T u, \) thus we have the following relation

\[
|\langle u, \Delta y \rangle_T| \leq \int_0^T |u^T \Delta y| \, dt \leq \frac{1}{2\gamma} \langle \Delta y, \Delta y \rangle_T + \frac{\gamma}{2} \langle u, u \rangle_T.
\] (5.12)

Thus, from assumption (5.11), we obtain

\[
|\langle u, \Delta y \rangle_T| \leq \frac{\gamma}{2} \langle u, u \rangle_T + \frac{1}{2\gamma} (\gamma^2 \langle u, u \rangle_T + \epsilon) = \gamma \langle u, u \rangle_T + \frac{\epsilon}{2\gamma}.
\] (5.13)

Now for the system \( \Sigma_2 \), we have

\[
\langle u, y_2 \rangle_T - \nu \langle u, u \rangle_T = \langle u, y_1 \rangle_T - \nu \langle u, u \rangle_T + \langle u, \Delta y \rangle_T \\
\leq \langle u, y_1 \rangle_T - \nu \langle u, u \rangle_T + |\langle u, \Delta y \rangle_T| \\
\leq \langle u, y_1 \rangle_T - (\nu - \gamma) \langle u, u \rangle_T + \frac{\epsilon}{2\gamma}.
\]

By assumption, \( \Sigma_2 \) is ISP for a given \( \nu > 0 \). Thus, for a given constant \( \beta \leq 0 \),

\[
\langle u, y_2 \rangle_T - \nu \langle u, u \rangle_T \geq \beta.
\]

By defining \( \tilde{\nu} = \nu - \gamma, \tilde{\beta} = \beta - \frac{\epsilon}{2\gamma} \leq 0 \), we obtain for the system \( \Sigma_1 \),

\[
\langle u, y_1 \rangle_T - \tilde{\nu} \langle u, u \rangle_T + \frac{\epsilon}{2\gamma} \geq \tilde{\beta},
\]

or equivalently \( \langle u, y_1 \rangle_T - \tilde{\nu} \langle u, u \rangle_T \geq \tilde{\beta} \) for a constant \( \tilde{\beta} \leq 0 \). Now note that \( u \) and \( T \) are arbitrary. Therefore, if \( \gamma < \nu \), then \( \Sigma_1 \) is ISP for \( \tilde{\nu} > 0 \).}

Note that \( \tilde{\nu} \) may not represent the input feed-forward passivity index (IFP) of \( \Sigma_1 \), since \( \Sigma_1 \) may have IFP larger than \( \tilde{\nu} \). If we are interested merely in determining whether \( \Sigma_1 \) is passive (rather than characterizing the ISP level of \( \Sigma_1 \)), we can allow

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γ to be equal to ν.

**Corollary 5.1.** Consider Σ₁ and Σ₂ in Fig. 5.4. Suppose (5.11) is satisfied for some γ > 0 and ϵ ≥ 0. If Σ₂ has IFP(ν) and γ ≤ ν, then Σ₁ is passive.

**Proof.** Since γ ≤ ν, using (5.11), we have

\[ \langle \Delta y, \Delta y \rangle_T \leq \nu^2 \langle u, u \rangle_T + \epsilon, \quad \forall u \text{ and } \forall T \geq 0. \]  

(5.14)

Setting γ = ν in (5.12), from (5.14), we obtain \(|\langle u, \Delta y \rangle_T| \leq \nu \langle u, u \rangle_T + \frac{\epsilon}{2\nu}\). The following relation then holds for Σ₁ with \(y_1 = y_2 - \Delta y\),

\[
\langle u, y_1 \rangle_T = \langle u, y_2 \rangle_T - \langle u, \Delta y \rangle_T \\
\geq \langle u, y_2 \rangle_T - |\langle u, \Delta y \rangle_T| \\
\geq \langle u, y_2 \rangle_T - \nu \langle u, u \rangle_T - \frac{\epsilon}{2\nu}.
\]

By assumption Σ₂ has IFP(ν), so that \(\langle u, y_2 \rangle_T - \nu \langle u, u \rangle_T \geq \beta\). Thus, defining \(\tilde{\beta} = \beta - \frac{\epsilon}{2\nu} \leq 0\), we obtain \(\langle u, y_1 \rangle_T \geq \tilde{\beta}\). Therefore, Σ₁ is passive.

One possible interpretation of these results is that the IFP(ν) of Σ₂ provides an upper bound on the error γ caused by the approximation of Σ₁ into Σ₂. Further, the difference of these two values, ν − γ, provides a lower bound on the IFP of Σ₁. It is apparent that there exists a trade-off between how good the approximation is (corresponding to the value of γ) and how passive we can guarantee Σ₁ to be (corresponding to the value of ν − γ).

The following result presents the two passivity levels of a system Σ₁ from those of its approximation Σ₂ when Σ₂ is VSP.

**Theorem 5.3.** Consider Σ₁ and Σ₂ in Fig. 5.4. Suppose (5.11) holds for some γ > 0. Suppose Σ₂ is VSP for \((\rho, \nu)\) and γ < \(\min\{\rho, \nu\}\). Then, Σ₁ is VSP for
\[(\rho - \gamma, \nu - \gamma) \text{ if} \]
\[
\gamma^2 - (\rho - \frac{2}{\rho})\gamma + \nu^2 - 2 \geq 0. \tag{5.15}
\]

**Proof.** Since \( \Sigma_2 \) is VSP for \((\rho, \nu)\), then \( \Sigma_2 \) is ISP for \( \nu > 0 \) and OSP for \( \rho > 0 \). Thus,
\[
\langle y_2, y_2 \rangle_T \geq \nu^2 \langle u, u \rangle_T + 2\beta\nu. \tag{5.16}
\]

Together with (5.11), (5.13) and \( |\langle y_2, \Delta y \rangle_T| \leq \sqrt{\langle \Delta y_2, \Delta y_2 \rangle_T} \sqrt{\langle y_2, y_2 \rangle_T} \), if we define \( a \triangleq \rho - \gamma > 0 \) and \( \psi \triangleq 2a \langle y_2, \Delta y \rangle_T - \langle u, \Delta y \rangle_T - a \langle \Delta y, \Delta y \rangle_T \), then we obtain
\[
|\psi| \leq |\langle u, \Delta y \rangle_T| + 2a|\langle y_2, \Delta y \rangle_T| + a|\Delta y, \Delta y \rangle_T
\]
\[
\leq \left( \gamma + 2a \frac{\gamma}{\rho} + a\gamma^2 \right) \langle u, u \rangle_T + \frac{\epsilon}{2\gamma} + 2a\left( \frac{\epsilon}{\gamma\rho} - 2\beta\gamma \right) + a\epsilon.
\]

Thus, the following relation holds by substituting (5.16),
\[
\gamma \langle u, u \rangle_T + \gamma \langle y_2, y_2 \rangle_T + \psi
\]
\[
\geq \gamma (1 + \nu^2) \langle u, u \rangle_T + 2\beta\nu\gamma - |\psi|
\]
\[
\geq \gamma \left( \nu^2 - \frac{2a}{\rho} - a\gamma \right) \langle u, u \rangle_T - \frac{\epsilon}{2\gamma} - 2a\left( \frac{\epsilon}{\rho\gamma} - 2\beta\gamma \right) - a\epsilon + 2\beta\nu\gamma.
\]

If (5.15) is satisfied, we obtain \( \nu^2 - \frac{2a}{\rho} - a\gamma \geq 0 \). Thus, we have
\[
\gamma \langle u, u \rangle_T + \gamma \langle y_2, y_2 \rangle_T + \psi
\]
\[
\geq - \frac{\epsilon}{2\gamma} - 2a\left( \frac{\epsilon}{\gamma\rho} - 2\beta\gamma \right) - a\epsilon + 2\beta\nu\gamma.
\]
For $\Sigma_1$ with input $u$ and output $y_1 = y_2 - \Delta y$, we have

\[
(u, y_1)_T - (\nu - \gamma)(u, u)_T - (\rho - \gamma)(y_1, y_1)_T \\
= (u, y_2)_T - \nu(u, u)_T - \rho(y_2, y_2)_T \\
+ \gamma(u, u)_T + \gamma(y_2, y_2)_T + \psi.
\]

$\Sigma_2$ is assumed to be VSP for $(\rho, \nu)$ and therefore there exists a constant $\beta \leq 0$ such that

\[
(u, y_2)_T - \nu(u, u)_T - \rho(y_2, y_2)_T \geq \beta.
\]

Defining $\tilde{\beta} \triangleq \beta - \frac{\epsilon}{2\gamma} - 2a(\frac{\epsilon}{\gamma\rho} - 2\beta\gamma) - a \epsilon + 2\beta \nu \gamma \leq 0$, we obtain for $\Sigma_1$,

\[
(u, y_1)_T - (\nu - \gamma)(u, u)_T - (\rho - \gamma)(y_1, y_1)_T \geq \tilde{\beta}.
\]

Note $u$ and $T$ are arbitrary. Thus, if $\gamma < \rho$ and $\gamma < \nu$, $\Sigma_1$ is VSP for $(\rho - \gamma, \nu - \gamma)$. □

The result about passivity of $\Sigma_1$ follows.

**Corollary 5.2.** Consider $\Sigma_1$ and $\Sigma_2$ in Fig. 5.4. Suppose (5.11) holds for some $\gamma > 0$. If $\Sigma_2$ is VSP for $(\rho, \nu)$ and $\rho \nu^2 + \nu - \gamma \geq 0$, then, $\Sigma_1$ is passive.

**Proof.** From (5.13) and (5.16), we obtain

\[
\chi \triangleq -|\langle u, \Delta y \rangle_T| + \rho\langle y_2, y_2 \rangle_T + \nu\langle u, u \rangle_T \\
\geq (\rho \nu^2 + \nu - \gamma)(u, u)_T + 2\beta \rho \nu - \frac{\epsilon}{2\gamma}.
\]

Thus, if $\rho \nu^2 + \nu - \gamma \geq 0$, we obtain $\chi \geq 2\beta \rho \nu - \frac{\epsilon}{2\gamma}$. $\Sigma_2$ is VSP for $(\rho, \nu)$, then $\langle u, y_2 \rangle_T - \rho\langle y_2, y_2 \rangle_T - \nu\langle u, u \rangle_T \geq \beta$. For $\Sigma_1$ with input $u$ and output $y_1 = y_2 - \Delta y$,
we have

\[ \langle u, y_1 \rangle_T \geq \langle u, y_2 \rangle_T - \rho \langle y_2, y_2 \rangle_T - \nu \langle u, u \rangle_T + \chi \geq \beta + 2\beta \rho \nu - \frac{\epsilon}{2\gamma}. \]

Thus, define \( \tilde{\beta} = \beta + 2\beta \rho \nu - \frac{\epsilon}{2\gamma} \leq 0 \), we have \( \langle u, y_1 \rangle_T \geq \tilde{\beta} \), i.e. \( \Sigma_1 \) is passive.

Remark 5.1. To ensure passivity of \( \Sigma_1 \) (instead of positive passivity levels), \( \gamma \) may be larger than the passivity levels \( \rho \) and \( \nu \) of \( \Sigma_2 \) because for passivity, the only requirement is that \( \gamma \leq \rho \nu \).  

5.3.2 Passivity Analysis of Quantized Systems

After introducing the results of passivity analysis for systems under approximation, the passivity analysis problem for quantized systems can be solved using the results previously provided.

Consider the system \( \Sigma_1 \) with input \( u \) and output \( y_1 \), as shown in Fig. 5.5. For simplicity, we assume zero initial conditions. \( \Sigma_1 \) is finite-gain stable if there exists a
\( \kappa > 0 \) such that \( \forall T \geq 0 \) and \( \forall u \),
\[
\langle y_1, y_1 \rangle_T \leq \kappa^2 \langle u, u \rangle_T.
\] (5.17)

The system \( \Sigma_2 \) is the quantized system of \( \Sigma_1 \), which forms a series connection of \( \Sigma_1 \) and the quantizers (\( Q_1 \) and \( Q_2 \)). Assume that \( Q_1 \) and \( Q_2 \) are passive quantizers. In Proposition 5.1, the measure of goodness of the approximation (i.e. \( \gamma \) and \( \epsilon \) defined in 5.11) is obtained for \( \Sigma_1 \) and \( \Sigma_2 \).

**Proposition 5.1.** Consider the two systems in Fig. 5.5, where \( \Sigma_1 \) is \( \mathcal{L}_2 \) stable with some \( \kappa > 0 \). The quantizer \( Q_i \) satisfies \( a_i v^T v \leq v^T Q_i(v) \leq b_i v^T v \), where \( 0 \leq a_i \leq b_i < \infty \). Then, (5.11) is satisfied for \( \gamma \triangleq \kappa (1 + b_1 b_2) \) and \( \epsilon = 0 \).

**Proof.** Denote the input to quantizer \( Q_2 \) as \( y \); then we have \( y_2 = Q_2(y) \) and \( a_2 y^T y \leq y^T Q_2(y) \leq b_2 y^T y \). Therefore, we obtain \( Q_2^T(y)Q_2(y) \leq b_2^2 y^T y \).

Because \( \Sigma_1 \) is stable, we have \( \langle y, y \rangle_T \leq \kappa^2 \langle Q_1(u), Q_1(u) \rangle_T \). Also, from \( a_1 u^T u \leq u^T Q_1(u) \leq b_1 u^T u \), we obtain \( Q_1^T(u)Q_1(u) \leq b_1^2 u^T u \). Then, we have
\[
\langle Q_2(y), Q_2(y) \rangle_T \leq \kappa^2 b_2^2 b_1^2 \langle u, u \rangle_T.
\]

From (5.17) and Cauchy-Schwarz inequality, we can derive that
\[
|\langle y_1, Q_2(y) \rangle_T| \leq \kappa^2 b_2 b_1 \langle u, u \rangle_T.
\]

From the above relations, we can derive that
\[
\langle \Delta y, \Delta y \rangle_T \triangleq \langle y_2 - y_1, y_2 - y_1 \rangle_T
= \langle Q_2(y), Q_2(y) \rangle_T + \langle y_1, y_1 \rangle_T - 2 \langle y_1, Q_2(y) \rangle_T
\leq (1 + b_1 b_2)^2 \kappa^2 \langle u, u \rangle_T.
\]

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Therefore, \((5.11)\) holds for \(\gamma \triangleq \kappa (1 + b_1 b_2)\) and \(\epsilon = 0\). This completes the proof. \(\Box\)

Note that a tighter bound for the error \(\gamma\) in \((5.11)\) can be derived when system \(\Sigma_1\) is linear. More details can be found in [124]. With Proposition 5.1 the following result is immediate from Theorem 5.2 and Theorem 5.3.

**Corollary 5.3.** Consider the two systems in Fig. 5.5, where \(\Sigma_1\) is \(\mathcal{L}_2\) stable, i.e. \((5.17)\) holds for some \(\kappa > 0\). The quantizer \(Q_i\) satisfies \(a_i v^T v \leq v^T Q_i(v) \leq b_i v^T v\), where \(0 \leq a_i \leq b_i < \infty\).

- If \(\Sigma_1\) has IFP(\(\nu\)) and \(\kappa (1 + b_1 b_2) < \nu\), then \(\Sigma_2\) has IFP no less than \(\nu - \kappa (1 + b_1 b_2)\).
- If \(\Sigma_1\) is VSP for \((\rho, \nu)\) and \(\rho \nu^2 + \nu - \kappa (1 + b_1 b_2) \geq 0\), then \(\Sigma_2\) is passive.

Corollary 5.3 presents a sufficient condition under which passivity (or an excess of passivity) is preserved after quantization for IFP and VSP systems. It is also possible to consider OFP systems, although additional assumptions are needed [124, 126]. It is noted that passivity can not be guaranteed after quantization in general. However, passivity of \(\Sigma_2\) is desired in many cases, especially when \(\Sigma_2\) as a subsystem to be interconnected with another passive systems through parallel or feedback configurations, see e.g. [18]. Therefore, a transformation-based passivation scheme is proposed in Section 5.4.

### 5.4 Transformation-Based Passivation Scheme

The passivation scheme proposed is shown in Fig. 5.6. Here assume that \(H\) is a discrete-time \(\mathcal{L}_2\) stable system with \(V\) satisfying

\[
\Delta V(x(k)) \leq \gamma^2 u^T(k) u(k) - y^T(k) y(k).
\]

\((5.18)\)

\(Q_I\) and \(Q_O\) are input and output passive quantizers satisfying
\( Q_I : \ a_I v_I^2 \leq u_I v_I \leq b_I v_I^2, \)  \ with \( 0 \leq a_I < b_I < \infty; \)  \ (5.19) \\
\( Q_O : \ a_O v_O^2 \leq u_O v_O \leq b_O v_O^2, \)  \ with \( 0 \leq a_O < b_O < \infty; \)  \ (5.20) \\

where \( u_I \in \) represents the input of the quantizer \( Q_I \) and \( y_I \) is the output of \( Q_I \). The same holds for \( Q_O \). \( \hat{H} \) is the quantized system.

The block \( M \) shown in Fig. 5.6 is an input/output coordinate transformation such that

\[ \hat{y} = m_{11} \tilde{y} + m_{12} \tilde{u} \]  \ (5.21) \\
\[ \hat{u} = m_{21} \tilde{y} + m_{22} \tilde{u} \]  \ (5.22) \\

where \( m_{ij} \in \mathbb{R}, \hat{u}, \hat{y} \in \mathbb{R}^m \) and \( \tilde{u}, \tilde{y} \in \mathbb{R}^m \). \( \hat{H} \) is the rendered systems after the transformation \( M \).
The objective is to find a realizable transformation $M$ so that the rendered system $\tilde{H}$ is passive with respect to the new input $\tilde{u}$ and output $\tilde{y}$. We first show that a finite gain $\mathcal{L}_2$ stable system with passive quantizers remains a finite gain $\mathcal{L}_2$ stable system.

**Lemma 5.1.** Consider the quantized system $\hat{H}$ as shown in Fig. 5.6. $H$ is a finite gain $\mathcal{L}_2$ stable system with a positive storage function $V$ satisfying (5.18). $Q_I$ and $Q_O$ are input and output passive quantizers satisfying (5.19)-(5.20). The quantized system $\hat{H}$ with input $\hat{u}(k)$ and output $\hat{y}(k)$ is $\mathcal{L}_2$ stable with finite gain $\gamma_{b_I} b_O$. 

![Figure 5.6. Transformation-based scheme to passivate a quantized system](image-url)
Proof. Since $Q_I$ and $Q_O$ are passive quantizers, from (5.5) we have

\begin{align}
    u(k)^T u(k) &\leq b_I^2 \hat{u}(k)^T \hat{u}(k) \tag{5.23} \\
    \hat{y}(k)^T \hat{y}(k) &\leq b_O^2 y(k)^T y(k). \tag{5.24}
\end{align}

From (5.18), (5.23) and (5.24), we obtain

\[
    \Delta V \leq \gamma^2 u(k)u(k) - y^T(k)y(k) \leq \gamma^2 b_I^2 \hat{u}(k)^T \hat{u}(k) - \frac{1}{b_O^2} \hat{y}(k)^T \hat{y}(k). \tag{5.25}
\]

By redefining

\[
    \hat{V} = b_O^2 V,
\]

(5.25) leads to

\[
    \Delta \hat{V} \leq \gamma^2 b_I^2 b_O^2 \hat{u}(k)^T \hat{u}(k) - \hat{y}(k)^T \hat{y}(k). \tag{5.26}
\]

From Lemma [5.1] we can conclude that the quantized system is $L_2$ stable with finite gain $\gamma b_I b_O$.

Although the quantized system $\hat{H}$ preserves $L_2$ stability under passive quantizers, it is not necessarily a passive system. Therefore, the input-output transformation $M$ in (5.28)-(5.29) is needed to render the quantized system into a passive system.

**Theorem 5.4.** Consider a quantized system $\hat{H}$ with a positive storage function satisfying (5.26). The rendered system $\tilde{H}$ with input $\tilde{u}$ and output $\tilde{y}$ is passive with a positive storage function $\tilde{V}$ satisfying

\[
    \Delta \tilde{V}(x(k)) \leq \tilde{u}^T(k)\tilde{y}(k) - \nu \tilde{u}^T(k)\tilde{u}(k) - \rho \tilde{y}^T(k)y(k) \tag{5.27}
\]
if a transformation $M$ is chosen such that

\begin{align}
\hat{\gamma}^2 m_{21} m_{22} &> m_{11} m_{12} \\
m_{12}^2 &\geq \hat{\gamma}^2 m_{22}^2 \\
m_{11}^2 &\geq \hat{\gamma}^2 m_{21}^2 \\
m_{11} m_{22} &\neq m_{21} m_{12}
\end{align}

(5.28)  
(5.29)  
(5.30)  
(5.31)

where $\hat{\gamma} = \gamma b_1 b_0$ is the finite $L_2$ gain in (5.26). Moreover, the resulting $v$ and $\rho$ in (5.27) can be obtained as $v = (m_{12}^2 - \hat{\gamma}^2 m_{22}^2)/(2\hat{\gamma}^2 m_{21} m_{22} - 2m_{11} m_{12})$ and $\rho = (m_{11}^2 - \hat{\gamma}^2 m_{21}^2)/(2\hat{\gamma}^2 m_{21} m_{22} - 2m_{11} m_{12})$.

Proof. Given $H_q$ is finite gain $L_2$ stable and has a positive storage function $\hat{V}$, $\hat{V}$ satisfies (5.26).

Consider the transformation defined in (5.21)-(5.22), (5.26) can be written as

\[
\Delta \hat{V}(x(k)) \leq \gamma^2 q (m_{21} \tilde{y} + m_{22} \tilde{u})^T (m_{21} \tilde{y} + m_{22} \tilde{u}) - (m_{11} \tilde{y} + m_{12} \tilde{u})^T (m_{11} \tilde{y} + m_{12} \tilde{u}) \\
= (2\gamma^2 m_{21} m_{22} - 2m_{11} m_{12}) \tilde{u}^T \tilde{y} - (m_{12}^2 - \gamma^2 m_{22}^2) \tilde{u}^T \tilde{u} - (m_{11}^2 - \gamma^2 m_{21}^2) \tilde{y}^T \tilde{y}.
\]

With the parameters of $M$ as chosen in (5.28)-(5.30), by redefining

\[
\hat{V}(x) = \frac{1}{2\gamma^2 m_{21} m_{22} - 2m_{11} m_{12}} \hat{V}(x)
\]

one can verify that

\[
\Delta \hat{V}(x(k)) \leq \tilde{u}^T \tilde{y} - \frac{m_{12}^2 - \gamma^2 m_{22}^2}{2\gamma^2 m_{21} m_{22} - 2m_{11} m_{12}} \tilde{u}^T \tilde{u} - \frac{m_{11}^2 - \gamma^2 m_{21}^2}{2\gamma^2 m_{21} m_{22} - 2m_{11} m_{12}} \tilde{y}^T \tilde{y}
\]

which shows that the rendered system is passive with a positive storage function $\hat{V}$ satisfying (5.27).

Moreover, (5.31) guarantees that the transform $M$ is invertible and thus can be
Remark 5.2. The implementation of the transformation $M$ chosen in Theorem 5.4 is illustrated in Fig. 5.7, where $\Delta = \text{det} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. It can be seen that one can render a system into a passive, input strictly passive, output strictly passive or very strictly passive system, depending on the specific conditions on $v$ and $\rho$. This gives designers freedom to choose the specific type of passivity based on requirements. For example, if the objective is to obtain a passive system with $v = \rho = 0$, the transformation conditions 5.28-5.31 can be simplified and are equivalent to the conditions
as

\begin{align*}
m_{11} &= \gamma m_{21} \\
m_{12} &= -\gamma m_{22} \\
m_{21}m_{22} &> 0.
\end{align*}

Remark 5.3. Although Theorem 5.4 is derived based on finite gain $L_2$ stable, the result remains valid for continuous-time finite gain $L_2$ stable systems and the same transformation can also be applied.

Remark 5.4. For the system with the input $u(k) \in \mathbb{R}^m$ and output $y(k) \in \mathbb{R}^p$ where $m \neq p$, the transformation can be generalized to passivate the system into QSR-dissipative system by replacing scalars $m_{ij}$ with proper matrices.

5.5 Application in Preserving Passivity Under Quantization

In this section, we show how passivity of the system $H$ (in Fig. 5.6) can be preserved under quantization. It is assumed that $H$ is a discrete-time output strictly passive system such that

\[ \Delta V(k) = V(k+1) - V(k) \leq u^T(k)y(k) - \rho y^T(k)y(k), \quad (5.32) \]

where $u, y \in \mathbb{R}^m$, $0 < \rho < \infty$, $V \in \mathbb{R}^+$ is the storage function of $H$. $Q_I$ and $Q_O$ are passive quantizers satisfying (5.19)-(5.20). The result is stated in Theorem 5.5.

**Theorem 5.5.** Consider an OSP system $H$ satisfying (5.32) in Fig. 5.6 with passive quantizers $Q_I$ and $Q_O$ satisfying (5.19)-(5.20). If a transformation $M$ is chosen such that

\begin{align*}
m_{21} &= 0, \quad m_{11}^2 = 2b_O^2 \\
m_{12} &= \frac{-b_I^2}{\rho}, \quad m_{22}^2 = \frac{b_I^2b_O^2}{\rho^2}m_{22}^2,
\end{align*}

(5.33)
then the system $\tilde{H}$ is OSP such that

$$\Delta \tilde{V}(k) = \tilde{V}(k + 1) - \tilde{V}(k) \leq \tilde{u}^T(k)\tilde{y}(k) - \rho \tilde{y}^T(k)\tilde{y}(k).$$

**Proof.** From Theorem (2.4), the system $H$ being output strictly passive implies that it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\frac{1}{\rho}$. For the quantized system $\hat{H}$ with the quantizers $Q_I$ and $Q_O$, Lemma (5.1) shows that it is also finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\hat{\gamma} = \frac{1}{\rho}b_Ib_O$.

Then it can be seen that the transformation $M$ characterized by (5.33) satisfies the conditions (5.28)-(5.31) in Theorem 5.4. Therefore, the rendered system $\tilde{H}$ is passive with a positive storage function $\tilde{V}$ satisfying

$$\Delta \tilde{V}(x(k)) \leq \tilde{u}^T(k)\tilde{y}(k) - \tilde{\nu} \tilde{u}^T(k)\tilde{u}(k) - \tilde{\rho} \tilde{y}^T(k)\tilde{y}(k)$$

(5.34)

where

$$\tilde{\nu} = (m_{12}^2 - \hat{\gamma}^2 m_{22}^2)/(2\hat{\gamma}^2 m_{21} m_{22} - 2m_{11} m_{12})$$

and

$$\tilde{\rho} = (m_{11}^2 - \hat{\gamma}^2 m_{21}^2)/(2\hat{\gamma}^2 m_{21} m_{22} - 2m_{11} m_{12}).$$

Considering the (5.33), we can further show that $\tilde{\nu} = 0$ and $\tilde{\rho} = \rho$, which means that $\tilde{H}$ is OSP such that

$$\Delta \tilde{V}(k) = \tilde{V}(k + 1) - \tilde{V}(k) \leq \tilde{u}^T(k)\tilde{y}(k) - \rho \tilde{y}^T(k)\tilde{y}(k).$$

\[\square\]

**Remark 5.5.** The implementation of the transformation $M$ chosen in Theorem 5.5 is illustrated in Fig. 5.5. The transformation chosen is a special case of the general transformation illustrated in Fig. 5.7 ($\Delta = m_{11} m_{22}$). In fact, the choice of transfor-
mation $M$ is not unique. One can find a different transformation from (5.33), which gives designers freedom to choose from various transformation candidates according to different specifications.

![Figure 5.8. Implementation of $M$ in Theorem 5.5](image)

*Remark 5.6.* Although Theorem 5.5 is derived based on discrete-time OSP systems, the result remains valid for continuous-time OSP systems and the same transformation can be applied to preserve passivity.

*Remark 5.7.* For the case where only one of the quantizers is needed, one can choose $b = 1$ when only input quantizer $Q_I$ is present or $b_I = 1$ when only output quantizer $Q_O$ is present.
Remark 5.8. Since $\tilde{H}$ is an OSP system after rendering by $M$, the negative feedback interconnection of $\tilde{H}$ with another OSP system is also passive and thus the stability condition can be derived from traditional passive systems theory. Therefore, this result can be used to guarantee stability of two OSP systems ($G_p$ and $G_c$ in Fig. 5.9) feedback interconnected through communication networks. The same idea is extended to switched systems in Section 5.6.

![Diagram of negative feedback interconnection of two OSP systems](image-url)
5.6 Stability of Passive Switched Systems with Quantization

5.6.1 Stability of Passive Switched Systems

Passive systems form an important class of dynamical systems. For one, these systems are common in practice. Additionally, passivity can be used to simplify analysis. Passivity is a property that implies stability and the property is preserved when systems are combined in feedback. Combining these two results gives open-loop conditions for closed-loop stability. Additionally, large scale systems can be shown to be stable if each component is passive and the components are sequentially combined in feedback or in parallel. The following results are discrete-time extensions of the work presented in [81]. The first result concerns stability of a single passive switched system.

**Theorem 5.6.** A passive discrete-time switched system is stable for zero input \( u(k) = 0, \forall k \).

The passivity property can be used when considering interconnections of systems. The following result shows stability of the feedback interconnection of two passive systems.

**Theorem 5.7.** The feedback interconnection (Fig. 2.1) of two passive switched systems \( G_1 \) and \( G_2 \) forms a passive switched system.

As in the non-switched case, these results can be used to verify closed loop stability by showing that the two systems in feedback are passive. This result can also be used from a design perspective. When controlling a passive switched system, any passive controller is stabilizing without additional conditions. This allows for a large class of controllers to be applied directly including traditional PI controllers.
5.6.2 Passivation of Quantized Switched Systems

The work presented in Section 5.5 can be extended to switched systems. The structure of the passification scheme remains the same (Fig. 5.6) with the system $H$ being modeled as a switched system according to the dynamics (5.6). Now that the system dynamics are time-varying, the transformation $M$ must also be time-varying

$$M(k) = \begin{bmatrix} m_{11}(k) & m_{12}(k) \\ m_{21}(k) & m_{22}(k) \end{bmatrix}. \quad (5.35)$$

The matrix $M(k)$ will be piecewise constant, belonging to a finite set of constant matrices. There will be at most one constant matrix for each subsystem of the given switched system. Fig. 5.10 shows the passivation scheme for quantized switched systems.
The transformation $M$ can switch as $H$ switches. In order for this to be allowable, the switching signal of $H$ must be known or measurable in real time. From the perspective of this chapter, the system $H$ is a designed controller so it should be possible to measure the switching signal. Additionally, the set of $\rho_i$ that define the OSP switched system should be known. A function $\rho(k)$ can be defined such that

$$\rho(k) = \rho_i \quad \text{for active subsystem } i.$$  \hspace{1cm} (5.36)

This function is piecewise constant and changes as the switching signal changes. This function is used to demonstrate passivity in the following theorem.
Theorem 5.8. Consider an output strictly passive discrete-time switched system \( H (5.6) \). This system is placed in the structure (Fig. 5.6) with passive quantizers defined by the constants \( a_I, b_I, a_O, \) and \( b_O \). This control structure preserves the output strict passivity property of system \( H \) if the transformation \( M(k) \) is chosen according to the following time-varying equations

\[
m_{21}(k) = 0, \quad m_{11}^2(k) = 2b_c^2, \quad m_{12}^2(k) = \frac{-b_c^2}{\rho(k)}, \quad m_{22}^2(k) = \frac{b_c^2 b_p^2}{\rho^2(k)} m_{22}(k), \hspace{1cm} (5.37)
\]

\[
m_{11}m_{12}(k) = \frac{-b_c^2}{\rho(k)}, \quad m_{21}m_{12}(t) = b_c b_p^2 \rho^2(k) m_{22}(k), \hspace{1cm} (5.38)
\]

Proof. Since \( H \) is OSP, for each subsystem \( i \) there exists a \( V_i \) to satisfy the passive inequality with \( \rho_i > 0 \) for \( i \in \{1, ..., P\} \),

\[
V_i(x(k+1)) \leq V_i(x(k)) + u^T(k)y(k) - \rho_i y(k)^T y(k). \hspace{1cm} (5.39)
\]

The quantizers satisfy the following inequalities,

\[
\|u_I\|_2 \leq b_I \|v_I\|_2 \text{ and } \|u_O\|_2 \leq b_O \|v_O\|_2.
\]

Applying Theorem 5.5, the OSP structure of each active subsystem is preserved at each time step by the transformation \( M(k) \). The storage functions \( V_i \) are also preserved with the structure.

Now the inactive behavior can be analyzed. For each inactive subsystem \( i \) and for all active subsystems \( j \neq i \), there exists a cross supply rate \( \omega_i^j \). For each one, a modified supply rate can be introduced such that

\[
\tilde{\omega}_i^j(\tilde{u}, \tilde{y}, x, k) = \omega_i^j(u, y, x, k), \forall i, j. \hspace{1cm} (5.40)
\]
These new cross supply rates imply

\[ V_i(x(k+1)) \leq V_i(x(k)) + \tilde{\omega}_i^j(\tilde{u}, \tilde{y}, x, k) \]  \hspace{1cm} (5.41)

and

\[ \sum_{k=k_0}^{\infty} |\tilde{\omega}_i^j(\tilde{u}, \tilde{y}, x, k)| < L, \]  \hspace{1cm} (5.42)

where \( L \) is an arbitrarily large finite constant given by (5.10). Since these hold for all \( i \) and \( j \), the inactive behavior is dissipative and the supply rates are still absolutely summable. All the conditions for the switched system to be passive are satisfied. The proposed scheme maintains passivity of the switched systems.

Remark 5.9. As mentioned earlier, this choice of transformation \( M(k) \) is not unique. This result can be used to preserve passivity of an individual system. This can be used with previous results to show stability of feedback interconnections. When this system is combined in negative feedback with another passive switched system, the overall interconnection is a passive switched system so is stable using Theorem 5.6 and 5.7. Therefore, the result can be applied to guarantee stability of two OSP switched systems (\( G_p \) and \( G_c \) in Fig. 5.11) feedback interconnected through communication networks. An example is provided in the following section to demonstrate how this result can be used.
Figure 5.11. Negative feedback interconnection of two OSP systems
Example 5.1. The following example illustrates how this method can be applied to a practical system, as in Fig. 5.12. The plant $G_p$ is a LTI system, with the following
dynamics

\[
x(k + 1) = \begin{bmatrix} -0.020 & 0.865 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(k) \quad (5.43)
\]

\[
y(k) = \begin{bmatrix} -0.330 & 0.865 \end{bmatrix} x(k) + 2u(k). \quad (5.44)
\]

The switched system \( G_c \) chosen is a switched system with two subsystems \( G_1 \) and \( G_2 \). The first subsystem \( G_1 \) is modeled by the following dynamics

\[
x(k + 1) = \begin{bmatrix} -0.060 & 0.173 \\ 0.125 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \quad (5.45)
\]

\[
y(k) = \begin{bmatrix} -0.74 & 0.346 \end{bmatrix} x(k) + 2u(k). \quad (5.46)
\]

The second subsystem \( G_2 \) is

\[
x(k + 1) = \begin{bmatrix} -0.179 & 0.169 \\ 0.125 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \quad (5.47)
\]

\[
y(k) = \begin{bmatrix} -0.667 & 0.158 \end{bmatrix} x(k) + 0.94u(k). \quad (5.48)
\]

\( G_c \) can be shown to be a passive switched system using the definition given in this chapter. The storage functions to show passivity \((5.8)\) are

\[
V_1(x) = x^T(k) \begin{bmatrix} 0.761 & -0.016 \\ -0.016 & 0.96 \end{bmatrix} x(k) \quad (5.49)
\]

\[
V_2(x) = x^T(k) \begin{bmatrix} 0.671 & -0.019 \\ -0.019 & 0.989 \end{bmatrix} x(k) \quad (5.50)
\]
with cross supply rates

\[ \omega_1^2(u, y, x, k) = u_T(k)y(k) + \frac{1}{10}(x_1^2 + x_2^2) \quad (5.51) \]

\[ \omega_2^2(u, y, x, k) = u_T(k)y(k) + \frac{2}{5}x_1^2. \quad (5.52) \]

These rates satisfy (5.9-5.10). The system \( G_c \) is OSP with \( \rho_1 = 0.202 \) and \( \rho_2 = 0.295 \).

Quantization effects in networks are considered by introducing input and output quantizers \( Q_I \) and \( Q_O \). The two quantizers are uniform, with quantization interval 0.1. It can be shown that these are passive quantizers with \( a = 0 \) and \( b = 2 \).

The transformation \( M(k) \) can take on values in the set \( \{M_1, M_2\} \) where

\[
M_1 = \begin{bmatrix} 2.83 & -7.00 \\ 0 & 0.354 \end{bmatrix}, \quad \text{(5.53)}
\]

\[
M_2 = \begin{bmatrix} 2.83 & -4.79 \\ 0 & 0.354 \end{bmatrix}, \quad \text{(5.54)}
\]

given by (5.37-5.38). Transformation \( M(k) = M_1 \) when the subsystem \( G_1 \) is active and \( M(k) = M_2 \) when the subsystem \( G_2 \) is active.

The switched controller \( G_c \) with quantization and transformation \( M(k) \) is simulated in feedback with the passive plant \( G_p \). The feedback interconnection of these two systems forms a passive switched system. When simulated, both the state of the plant and the controller converge to a set near the origin for arbitrary switching. The convergence of the plant state and output are as shown in Fig. 5.13 with switching signal Fig. 5.14.
5.8 Summary

In this chapter, we presented the results of passivity analysis and passivation for quantized systems. The passivity conditions were given with the help of passivity analysis for systems under approximation. Since the conditions cannot guarantee passivity property in general, we introduced a method to preserve the output strict passivity property of a system with passive input and output quantization by using an input-output coordinate transformation. Then we showed that the same approach can be applied to switched systems and thus the stability of interconnected passive
Figure 5.14. The switching signal of controller $H_c$ that switches between subsystems 1 and 2 is shown.

switched systems can be guaranteed. The example demonstrated how these methods can be applied to a practical quantized switched system.
6.1 Introduction

As pointed out in Chapter 4, the passivity property is vulnerable to discretization, quantization and other factors introduced by digital controllers or communication channels in modern control systems. Existing results in the literature mainly considered passivity analysis and passivation for a single dynamical system under different network effects [95, 128, 143]. On the other hand, it is also important to study passivity and passivation of interconnected systems, considering the advantages in the analysis and design of large-scale interconnected systems. [117] considered passivity analysis for discrete-time periodically controlled nonlinear systems, where the system switches between open and closed loop periodically. As the extension to the well-known compositional property of passivity, Chapter 4 considered the passivity and passivation problems for feedback interconnection of two input feed-forward output-feedback (IF-OF) passive systems.

In this chapter we consider the passivity and passivation problems for event-triggered feedback interconnected systems. Instead of stability studied in [133], we focus on passivity properties of the interconnected system. Based on the location of event-triggered sampler implemented, we have two event-triggered control schemes to consider: event-triggered sampler at the plant output (Fig. 6.1) and event-triggered sampler at the controller output (Fig. 6.2). For each control scheme, the condition
to characterize the level of passivity for the interconnected system using passivity indices is derived. Event-triggering conditions are proposed to guarantee that these indices can be achieved. For the passivation problem, the condition to render the interconnected system passive is given. The condition depends on the passivity indices of the plant and controller and the event-triggering condition. Moreover, we discuss the trade off between passivity levels and resource utilization by choosing appropriate passive controllers and event-triggering conditions. The results presented in this chapter are extensions of the corresponding results in Chapter 4 by considering, in addition, the effect of event-triggered samplers.

The chapter is organized as follows. The passivity analysis and passivation problems for event-triggered systems are stated in Section 4.3. Based on the location of the event-triggered samplers, two event-triggered control schemes are considered, namely event-triggered sampler at the plant output and event-triggered sampler at the controller output. Section 6.3 considers passivity analysis and passivation problems for the system with event-triggered sampler at the plant output while Section 6.4 considers the same problems for the system with event-triggered sampler at the controller output. Two examples are discussed in Section 6.5. The conclusion is provided in Section 6.6. The work in this chapter was originally published in [144, 146].

6.2 Problem Formulation

We first consider a feedback interconnection of two systems with an event-triggered sampler at the plant output, given in Fig. 6.1. We assume $G_p$ is IF-OFP($\nu_p, \rho_p$) and $G_c$ is IF-OFP($\nu_c, \rho_c$) with known passivity indices. Instead of assuming continuous communication in the feedback loop, an event-triggered feedback scheme is introduced. Event-triggered control has been introduced for the possibility of reducing resources usage (i.e., sampling rate, CPU time, network access frequency), see e.g. [17, 38, 49, 66, 75, 96, 106, 133]. The triggering mechanisms are referring to the
situation in which the control signals are kept constant until the violation of a condition on certain signals triggers the re-computation of the control signals. As in Fig. 6.1, new output information of $G_p$ is sent to the controller $G_c$ only when the output novelty error $e_p = y_p - y_p(t_k)$ in the event-triggered sampler satisfies a triggering condition. $y_p(t_k)$ denotes the last output information sent to the controller $G_c$ at the event time $t_k$. Note that [133] considered the same control scheme but focused on deriving the triggering condition to guarantee stability of the closed-loop system.

In this chapter, we focus on characterizing dissipativity/passivity properties of the closed-loop system. The main questions investigated are summarized as follows.

1. Given the passivity indices of $G_c$ and $G_p$, how can we determine the passivity indices for the closed-loop systems and accordingly, what is the event-triggering condition to guarantee that these indices can be achieved?

2. For a non-passive plant $G_p$ and a passive controller $G_c$, what condition on the passivity indices of both systems should be satisfied to render the closed-loop system passive and accordingly, what is the event-triggering condition to guarantee that the condition can be satisfied?
In addition to feedback connection with an event-triggered sampler of plant output, another similar scheme can be considered as in Fig. 6.2, where the event-triggered sampler is implemented in the output path of the controller $G_c$. The new output information of $G_c$ is sent to the plant $G_p$ only when the output novelty error $e_c = y_c - y_c(t_k)$ in the event-triggered sampler satisfies a triggering condition. $y_c(t_k)$ denotes the last output information sent to the controller $G_c$ at the event time $t_k$. Analogously, same questions listed above also need to be considered and answered.

![Figure 6.2. Feedback connection of two IF-OFP systems with event-triggered sampler at controller output](image)

Figure 6.2. Feedback connection of two IF-OFP systems with event-triggered sampler at controller output

In this chapter, we consider the passivity analysis and passivation problems (two problems proposed in Section 4.3) for event-triggered feedback interconnected systems using passivity indices. Based on the location of event-triggered sampler implemented, we have two event-triggered control schemes to consider: sampler at the
plant output and sampler at the controller output. For both schemes, we first derive
the conditions to characterize the level of passivity for the closed-loop system using
passivity indices. Then the passivation problem is considered and the passivation
conditions are provided.

6.3 Passivity Analysis and Passivation for Event-Trigerred Sampler at Plant Output

We first consider the passivity analysis problem for the feedback system with
an event-triggered sampler of the plant output (Fig. 6.1). Lemma 6.1 relates the
interconnected system to QSR-dissipative systems.

Lemma 6.1. Consider the feedback interconnection of two IF-OF systems with the
passivity indices $\nu_p$, $\rho_p$ and $\nu_c$, $\rho_c$ respectively (Fig. 6.1). If the event time $t_k$ is
explicitly determined by the following triggering condition

$$\| e_p(t) \|_2 > \frac{\beta_p}{\sqrt{\nu_c^2 + m_p \beta_p + |\nu_c|}} \| y_p(t) \|_2$$  \hspace{1cm} (6.1)

where $m_p = \frac{1}{4\alpha_p} + |\nu_c| - \nu_c$, $\alpha_p > 0$ and $\beta_p > 0$, then the interconnected system is QSR-
dissipative (with respect to the input $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ and output $y(t) = \begin{bmatrix} y_p(t) \\ y_c(t) \end{bmatrix}$),
satisfying the inequality

$$\dot{V}(t) \leq y(t)^T Q y(t) + 2w(t)^T S y(t) + w(t)^T R w(t)$$  \hspace{1cm} (6.2)

where

$$Q = \begin{bmatrix} - (\rho_p + \nu_c - \beta_p) I & 0 I \\ 0 I & - (\nu_p + \rho_c - \alpha_p) I \end{bmatrix}.$$
\[ S = \begin{bmatrix} \frac{1}{2}I & \nu_p I \\ -\nu_c I & \frac{1}{2}I \end{bmatrix}, \]

and
\[ R = \begin{bmatrix} -\nu_p I & 0I \\ 0I & -(\nu_c - |\nu_c|)I \end{bmatrix}. \]

**Proof.** Since \( G_p \) and \( G_c \) are IF-OF systems with the passivity indices \( \nu_p, \rho_p, \nu_c \) and \( \rho_c \), there exist \( V_p(t) \) and \( V_c(t) \) such that
\[
\dot{V}_p(t) \leq u_p^T(t)y_p(t) - \nu_p u_p^T(t)u_p(t) - \rho_p y_p^T(t)y_p(t) \\
\dot{V}_c(t) \leq u_c^T(t)y_c(t) - \nu_c u_c^T(t)u_c(t) - \rho_c y_c^T(t)y_c(t).
\]

Consider a storage function for the interconnected system given by \( V(t) = V_p(t) + V_c(t) \), we have
\[
\dot{V}(t) = \dot{V}_p(t) + \dot{V}_c(t) \\
\leq u_p^T(t)y_p(t) - \nu_p u_p^T(t)u_p(t) - \rho_p y_p^T(t)y_p(t) \\
+ u_c^T(t)y_c(t) - \nu_c u_c^T(t)u_c(t) - \rho_c y_c^T(t)y_c(t). \tag{6.3}
\]

Consider that \( u_p(t) = w_1(t) - y_c(t), \ u_c(t) = y_p(t_k) + w_2(t) \) and \( y_p(t_k) = y_p(t) - e_p(t) \).
For any $t \in [t_k, t_{k+1})$, (6.3) can be rewritten as

\[
\dot{V}(t) \leq (w_1^T(t) - y_c(t)) y_p(t) - \rho_p y_p^T(t) y_p(t) - \nu_p (w_1(t) - y_c(t))^T (w_1(t) - y_c(t)) + (w_2(t) + y_p(t_k))^T y_c - \rho_c y_c^T(t) y_c(t) - \nu_c (w_2(t) + y_p(t_k))^T (w_2(t) + y_p(t_k))
\]

\[
= w_1^T(t) y_p(t) + w_2^T(t) y_c(t) + 2\nu_p w_1^T(t) y_c(t)
\]

\[-2\nu_c w_2^T(t) y_p(t) - \nu_p w_1^T(t) w_1(t) - \nu_c w_2^T(t) w_2(t)
\]

\[-(\nu_p + \rho_p) y_c^T(t) y_c(t) - (\rho_p + \nu_c) y_p^T(t) y_p(t)
\]

\[+2\nu_c w_2^T(t) e_p(t) + 2\nu_c y_p^T(t) e_p(t) - \nu_c e_p^T(t) e_p(t) - y_c^T(t) e_p(t).
\]

Since

\[2\nu_c w_2^T(t) e_p(t) \leq |\nu_c| w_2^T(t) w_2(t) + |\nu_c| e_p^T(t) e_p(t),\]

we can obtain that

\[
\dot{V}(t) \leq \begin{bmatrix}
    w_1^T(t) & w_2^T(t)
\end{bmatrix}
\begin{bmatrix}
    1 & 2\nu_p \\
    -2\nu_c & 1
\end{bmatrix}
\begin{bmatrix}
    y_p(t) \\
    y_c(t)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
    w_1^T(t) & w_2^T(t)
\end{bmatrix}
\begin{bmatrix}
    -\nu_p & 0 \\
    0 & -\nu_c + |\nu_c|
\end{bmatrix}
\begin{bmatrix}
    w_1(t) \\
    w_2(t)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
    y_p^T(t) & y_c^T(t)
\end{bmatrix}
\begin{bmatrix}
    -\rho_p + \nu_c & 0 \\
    0 & -(\nu_p + \rho_c)
\end{bmatrix}
\begin{bmatrix}
    y_p(t) \\
    y_c(t)
\end{bmatrix}
\]

\[+2\nu_c y_p^T(t) e_p(t) + (|\nu_c| - \nu_c) e_p^T(t) e_p(t) - y_c^T(t) e_p(t).
\]

With

\[y_c^T(t) e_p(t) = \left\| \sqrt{\alpha_p} y_c(t) + \frac{1}{2\sqrt{\alpha_p}} e_p(t) \right\|_2^2 - \alpha_p y_c^T(t) y_c(t) - \frac{1}{4\alpha_p} e_p^T(t) e_p(t).
\]
where $\alpha > 0$, we can further get

$$
\hat{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t) - \left\| \sqrt{\alpha_p} y_c(t) + \frac{1}{2\sqrt{\alpha_p}} c_p(t) \right\|_2^2 \\
+ m_p \| e_p(t) \|_2^2 + 2\nu_c y_p^T(t)e_p(t) + \frac{\nu_c}{m_p} \| y_p(t) \|_2^2 - \left( \frac{\nu_c^2}{m_p} + \beta_p \right) \| y_p(t) \|_2^2
$$

(6.4)

where

$$Q = \begin{bmatrix}
-(\rho_p + \nu_c - \beta_p)I & 0I \\
0I & -(\nu_p + \rho_c - \alpha_p)I
\end{bmatrix},
$$

$$S = \begin{bmatrix}
\frac{1}{2}I & \nu_p I \\
-\nu_c I & \frac{1}{2}I
\end{bmatrix},
$$

$$R = \begin{bmatrix}
-\nu_p I & 0I \\
0I & -(\nu_c - |\nu_c|)I
\end{bmatrix},
$$

$\beta > 0$ and $m_p = \frac{1}{4\alpha_p} + |\nu_c| - \nu_c$.

Note that

$$2\nu_c y_p^T(t)e_p(t) \leq 2 |\nu_c| \| y_p(t) \|_2 \| e_p(t) \|_2.$$

Then we can show

$$
\hat{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t)
+ \left( \sqrt{m_p} \| e_p(t) \|_2 + \frac{|\nu_c|}{\sqrt{m_p}} \| y_p(t) \|_2 \right)^2
- \left( \frac{\nu_c^2}{m_p} + \beta_p \right) \| y_p(t) \|_2^2
$$

$$
+ \left( \sqrt{m_p} \| e_p(t) \|_2 + \frac{|\nu_c|}{\sqrt{m_p}} \| y_p(t) \|_2 + \sqrt{\frac{\nu_c^2}{m_p} + \beta_p} \| y_p(t) \|_2 \right) \times
$$

$$\left( \sqrt{m_p} \| e_p(t) \|_2 + \frac{|\nu_c|}{\sqrt{m_p}} \| y_p(t) \|_2 - \sqrt{\frac{\nu_c^2}{m_p} + \beta_p} \| y_p(t) \|_2 \right).
$$
From (6.1), one can verify that

\[ \dot{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t) \]

which completes the proof.

**Remark 6.1.** Although Lemma 6.1 does not explicitly characterize passivity indices for the closed-loop system, it determines an event-triggering condition (6.1) which guarantees that the closed-loop system is QSR-dissipative. After preserving QSR-dissipativity of the closed-loop system, same proof techniques used in Chapter 4 can be applied to further explore passivity properties of the system.

**Remark 6.2.** As pointed out in Theorem 2.1, the closed-loop system (Fig. 6.1) is $L_2$ stable if $Q < 0$. It can be seen that the sufficient conditions for $Q < 0$ is $\nu_p + \rho_c > \alpha_p$ and $\nu_c + \rho_p > \beta_p$, which are similar to the conditions derived in [133]. Also note that the triggering condition here is different from the condition in [133].

**Remark 6.3.** It can be seen from (6.1) that larger $\alpha_p$ and $\beta_p$ result in a larger triggering threshold. A large triggering threshold implies lower sampling rate and thus lower resources usage. Later we will show how these parameters affect passivity of the system.

Next, Theorem 6.1 shows how to determine the passivity indices for the feedback system with event-triggered sampler of plant output.

**Theorem 6.1.** Consider the feedback interconnected system in Fig. 6.1. Suppose the passivity indices $\nu_p$, $\rho_p$, $\nu_c$ and $\rho_c$ are known and the triggering condition is determined by (6.1). If we choose $\epsilon$ and $\delta$ such that

\[
\left\{ \begin{array}{l}
\epsilon < \min \{\nu_p, \nu_c - |\nu_c|\} \\
\delta \leq \min \left\{ \rho_c - \alpha_p - \frac{\epsilon \nu_p}{\nu_p - \epsilon}, \rho_p - \beta_p - \frac{(|\nu_c| + \epsilon) \nu_c}{\nu_c - |\nu_c| - \epsilon} \right\}
\end{array} \right.,
\]

(6.5)
then the interconnected system has passivity indices $\epsilon$ and $\delta$ satisfying

$$\dot{V} \leq w^T(t)y(t) - \epsilon w^T(t)w(t) - \delta y^T(t)y(t)$$  \hspace{1cm} (6.6)

where $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ and $y = \begin{bmatrix} y_p(t) \\ y_c(t) \end{bmatrix}$.

Proof. From (6.2), we have

$$\dot{V}(t) \leq w^T(t)y(t) - \begin{bmatrix} w_1^T(t) \\ w_2^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - \begin{bmatrix} w_2^T(t) \\ y_p^T(t) \end{bmatrix} P \begin{bmatrix} w_2(t) \\ y_p(t) \end{bmatrix}.$$  \hspace{1cm} (6.7)

where

$$O = \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c - \alpha_p \end{bmatrix}$$

and

$$P = \begin{bmatrix} \nu_c - |\nu_c| & \nu_c \\ \nu_c & \rho_p + \nu_c - \beta_p \end{bmatrix}.$$  \hspace{1cm} (6.8)

Since $\epsilon$ and $\delta$ are chosen such that (6.5) is satisfied, (6.8) holds for the chosen $\epsilon$ and $\delta$. 

$$\begin{cases} 
\epsilon \leq \nu_p \\
\epsilon \leq \nu_c - |\nu_c| \\
(\nu_p - \epsilon)(\nu_p + \rho_c - \alpha_p - \delta) \geq \nu_p^2 \\
(\nu_c - |\nu_c| - \epsilon)(\rho_p + \nu_c - \beta_p - \delta) \geq \nu_c^2 \\
\nu_p + \rho_c - \alpha_p - \delta \geq 0 \\
\rho_p + \nu_c - \beta_p - \delta \geq 0
\end{cases}.$$  \hspace{1cm} (6.8)
(6.8) further implies that the matrices

\[
M = \begin{bmatrix}
\nu_p - \epsilon & -\nu_p \\
-\nu_p & \nu_p + \rho_c - \alpha_p - \delta
\end{bmatrix}
\]

and

\[
N = \begin{bmatrix}
\nu_c - |\nu_c| - \epsilon & \nu_c \\
\nu_c & \rho_p + \nu_c - \beta_p - \delta
\end{bmatrix}
\]

are positive semi-definite. Therefore, we have

\[
\begin{bmatrix}
w_1^T(t) \\
y_c^T(t)
\end{bmatrix}
M
\begin{bmatrix}
w_1(t) \\
y_c(t)
\end{bmatrix}
+ \begin{bmatrix}
w_2^T(t) \\
y_p^T(t)
\end{bmatrix}
N
\begin{bmatrix}
w_2(t) \\
y_p(t)
\end{bmatrix}
\geq 0
\]

(6.9)

for \( \forall w_1(t), w_2(t), y_c(t) \) and \( y_p(t) \). After re-arranging the terms in (6.9), one can obtain

\[
-w^T E w - y^T \Delta y \geq - \begin{bmatrix}
w_1^T(t) \\
y_c^T(t)
\end{bmatrix} O \begin{bmatrix}
w_1(t) \\
y_c(t)
\end{bmatrix}
- \begin{bmatrix}
w_2^T(t) \\
y_p^T(t)
\end{bmatrix} P \begin{bmatrix}
w_2(t) \\
y_p(t)
\end{bmatrix}
\]

(6.10)

where

\[
E = \begin{bmatrix}
\epsilon & 0 \\
0 & \epsilon
\end{bmatrix},
\]

and

\[
\Delta = \begin{bmatrix}
\delta & 0 \\
0 & \delta
\end{bmatrix}.
\]
From (6.10) and (6.7), we can finally show that
\[
\dot{V}(t) \leq w^T(t)y(t) - \begin{bmatrix} w_1^T(t) & y_p^T(t) \end{bmatrix} \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - \begin{bmatrix} w_2^T(t) \\ y_p(t) \end{bmatrix} \begin{bmatrix} w_2(t) \\ y_p(t) \end{bmatrix} \leq w^T(t)y(t) - \epsilon w^T(t)w(t) - \delta y^T(t)y(t).
\] (6.11)

Remark 6.4. (6.5) can be used to obtain an estimate of the passivity indices for the closed-loop system, with respect to the input \( w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \) and output \( y = \begin{bmatrix} y_p \\ y_c \end{bmatrix} \). The condition is similar to its counterpart in Chapter 4. Additionally, (6.5) quantifies the impact of triggering condition on the passivity indices of the closed-loop system using the parameters \( \alpha_p \) and \( \beta_p \).

Now we introduce the passivation problem for the feedback system with event-triggered sampler of plant output. For this problem, the goal is to passivate a non-passive plant \( G_p \) using a passive controller \( G_c \). Here passivity of the interconnected system is defined on the input \( w_1 \) and output \( y_p \). We also assume that \( w_2 \) is zero.

One may observe from Theorem 6.1 that passivity with respect to the full input and output (i.e. input \( w \) and output \( y \)) may not be guaranteed to be reinforced under feedback interconnection and event-triggering scheme. However, since we have selected different inputs and outputs, the corresponding passivity may change accordingly. Theorem 6.2 shows that it is possible to guarantee passivity for the desired input and output although passivity for full input and output may not hold.

**Theorem 6.2.** Assume \( w_2 = 0 \) and let the triggering condition be determined by (6.1). The interconnected system (Fig. (6.1)) is passive with respect to the input \( w_1 \)
and output $y_p$ if the passivity indices satisfy the conditions

$$\nu_p \geq 0 \quad (6.12)$$

$$\rho_c \geq \alpha_p \quad (6.13)$$

$$\rho_p + \nu_c \geq \beta_p. \quad (6.14)$$

Proof. If $w_2(t) = 0$, (6.7) becomes

$$\dot{V}(t) \leq w_1^T(t) y_p(t) - (\rho_p + \nu_c - \beta_p) y_p^T(t) y_p(t)$$

$$- \begin{bmatrix} w_1^T(t) & y_c^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} \quad (6.15)$$

where

$$O = \begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c - \alpha_p \end{bmatrix}.$$ 

Since we have $\nu_p \geq 0$, $\rho_c \geq \alpha_p$ and $\rho_p + \nu_c \geq \beta_p$, it can be shown that

$$\begin{bmatrix} \nu_p & -\nu_p \\ -\nu_p & \nu_p + \rho_c - \alpha_p \end{bmatrix} \geq 0 \quad (6.16)$$

$$\rho_p + \nu_c \geq \beta_p. \quad (6.17)$$

Therefore, we can conclude that

$$\dot{V}(t) \leq w_1^T(t) y_p(t) - (\rho_p + \nu_c - \beta_p) y_p^T(t) y_p(t)$$

$$- \begin{bmatrix} w_1^T(t) & y_c^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix}$$

$$\leq w_1^T(t) y_p(t). \quad (6.18)$$
Remark 6.5. When the plant $G_p$ is non-passive (i.e. $\rho_p < 0$), the closed-loop system can be rendered passive by choosing a passive controller $G_c$ with $\rho_c \geq \alpha_p$ and $\nu_c \geq -\rho_p + \beta_p$. Compared with the passivation conditions in Chapter 4, the conditions (6.12)-(6.14) imply that one needs a passive controller with higher passivity indices to passivate a non-passive plant for a triggering condition with fixed $\alpha_p$ and $\beta_p$. On the other hand, the conditions also give the upper bounds for $\alpha_p$ and $\beta_p$ to guarantee closed-loop passivity for a given plant and controller with known passivity indices. The results provide certain flexibility for designers by trade off between passivity level of the controller and resource utilization.

Moreover, we can also obtain an estimate of passivity indices for the passivated system, as shown in Corollary 6.1.

**Corollary 6.1.** Assume the triggering condition is determined by (6.1). Suppose that the conditions (6.12)-(6.14) are satisfied and $\nu_p + \rho_c > \alpha_p$. If we choose $\epsilon$ and $\delta$ such that

\[
\begin{cases}
0 \leq \epsilon \leq \frac{\nu_p (\rho_c - \alpha_p)}{\nu_p + \rho_c - \alpha_p}, \\
0 \leq \delta \leq \nu_c + \rho_p - \beta_p
\end{cases}
\]  

(6.19)

then the interconnected system (Fig. 6.1) has passivity indices $\epsilon$ and $\delta$ satisfying

\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - \epsilon w_1^T(t)w_1(t) - \delta y_p^T(t)y_p(t).
\]  

(6.20)

Proof. If the condition (6.12)-(6.14), (6.19) and $\nu_p + \rho_c > \alpha$ are satisfied, we have

\[
L \geq 0
\]  

(6.21)

\[
\nu_c + \rho_p - \beta_p - \delta \geq 0.
\]  

(6.22)
where
\[ L = \begin{bmatrix} \nu_p - \epsilon & -\nu_p \\ -\nu_p & \nu_p + \rho_c - \alpha_p \end{bmatrix}. \]

Then it implies
\[
\begin{bmatrix} w_1^T(t) \\ y_c^T(t) \end{bmatrix} L \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} + (\rho_p + \nu_c - \beta_p - \delta) y_p^T(t)y_p(t) \geq 0 \quad (6.23)
\]

which can be written as
\[
-\begin{bmatrix} w_1^T(t) \\ y_c^T(t) \end{bmatrix} L \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - (\rho_p + \nu_c - \beta_p) y_p^T(t)y_p(t) \leq -\epsilon w_1^T(t)w_1(t) - \delta y_p^T(t)y_p(t). \quad (6.24)
\]

Since it is already known that
\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - \begin{bmatrix} w_1^T(t) \\ y_c^T(t) \end{bmatrix} L \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - (\rho_p + \nu_c - \beta_p) y_p^T(t)y_p(t) \quad (6.25)
\]

we can conclude that
\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - \epsilon w_1^T(t)w_1(t) - \delta y_p^T(t)y_p(t) \quad (6.26)
\]
holds for \( \forall w_1. \)

**Remark 6.6.** Because of the conditions (6.12)-(6.14) and \( \nu_p + \rho_c > \alpha_p, \) the passivity indices \( \epsilon \) and \( \delta \) are upper bounded by positive numbers. (6.19) provides a way to obtain the desired passivity indices of the closed-loop system by choosing a passive
with proper indices and a triggering condition with proper $\alpha_p$ and $\beta_p$. As we point out in Remark \ref{rem:6.5}, the trade off between passivity levels and resource utilization can be considered. For instance, if an OFP index given by $\delta = \nu_c + \rho_p - \beta_p$ is desired, one can either choose a passive controller with high $\nu_c$ and a triggering condition with low $\beta_p$ to conserve more communication resources, or a triggering condition with high $\beta_p$ and a passive controller with low $\nu_c$ to impose less restrictions on the controller design.

6.4 Passivity Analysis and Passivation for Event-Triggered Sampler at Controller Output

For the feedback system with event-triggered sampler at the controller output (Fig. \ref{fig:6.2}), we can follow the same rationale as for the feedback system with event-triggered sampler at the plant output. We first consider the passivity analysis problem and then move to the passivation problem.

**Lemma 6.2.** Consider two IF-OF systems with the passivity indices $\nu_p$, $\rho_p$ and $\nu_c$, $\rho_c$ respectively. If the event time $t_k$ is explicitly determined by the following triggering condition

$$
\|e_c(t)\|_2 > \frac{\beta_c}{\sqrt{\nu_p^2 + m_c \beta_c + |\nu_p|}} \|y_c(t)\|_2
$$

where $m_c = \frac{1}{4\alpha_c} + |\nu_p| - \nu_p$, $\alpha_c > 0$ and $\beta_c > 0$, then the interconnected system with the event-triggered sampler (Fig. \ref{fig:6.2}) is QSR-dissipative (with respect to the input $w(t)$ = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ and output $y(t)$ = \begin{bmatrix} y_p(t) \\ y_c(t) \end{bmatrix}$), which satisfies the inequality

$$
\dot{V}(t) \leq y(t)^TQy(t) + 2w(t)^TSy(t) + w(t)^TRw(t)
$$
where
\[
Q = \begin{bmatrix}
-(\rho_p + \nu_c - \alpha_c) I & 0 I \\
0 I & -(\nu_p + \rho_c - \beta_c) I
\end{bmatrix},
\]
\[
S = \begin{bmatrix}
\frac{1}{2} I & \nu_p I \\
-\nu_c I & \frac{1}{2} I
\end{bmatrix},
\]
and
\[
R = \begin{bmatrix}
-(\nu_p - |\nu_p|) I & 0 I \\
0 I & -\nu_c I
\end{bmatrix}.
\]

Proof. Since \(G_p\) and \(G_c\) are IF-OF systems with the passivity indices \(\nu_p, \rho_p, \nu_c\) and \(\rho_c\), there exist \(V_p(t)\) and \(V_c(t)\) such that
\[
\dot{V}_p(t) \leq u_p^T(t)y_p(t) - \nu_p u_p^T(t)u_p(t) - \rho_p y_p^T(t)y_p(t)
\]
and
\[
\dot{V}_c(t) \leq u_c^T(t)y_c(t) - \nu_c u_c^T(t)u_c(t) - \rho_c y_c^T(t)y_c(t).
\]

Consider a storage function for the interconnected system given by \(V(t) = V_p(t) + V_c(t)\), we have
\[
\dot{V}(t) = \dot{V}_p(t) + \dot{V}_c(t) \leq u_p^T(t)y_p(t) - \nu_p u_p^T(t)u_p(t) - \rho_p y_p^T(t)y_p(t) + u_c^T(t)y_c(t) - \nu_c u_c^T(t)u_c(t) - \rho_c y_c^T(t)y_c(t).
\](6.29)

Consider that \(u_p(t) = w_1(t) - y_c(t_k)\), \(u_c(t) = y_p(t) + w_2(t)\) and \(y_c(t_k) = y_c(t) - e_c(t)\).
For any $t \in [t_k, t_{k+1})$, (6.29) can be rewritten as

$$
\dot{V}(t) \leq (w_1^T(t) - y_c(t_k)) y_p(t) - \rho_p y_p^T(t) y_p(t) - \nu_p (w_1(t) - y_c(t_k))^T (w_1(t) - y_c(t_k)) + (w_2(t) + y_p(t))^T y_c - \rho_c y_c^T(t) y_c(t) - \nu_c (w_2(t) + y_p(t))^T (w_2(t) + y_p(t))
$$

$$
= w_1^T(t)y_p(t) + w_2^T(t)y_c(t) + 2\nu_p w_1^T(t)y_c(t)
$$

$$
-2\nu_c w_2^T(t)y_p(t) - \nu_p w_1^T(t)w_1(t) - \nu_c w_2^T(t)w_2(t)
$$

$$
- (\nu_p + \rho_c) y_c^T(t) y_c(t) - (\rho_p + \nu_c) y_p^T(t) y_p(t) - 2\nu_p w_1^T(t)e_c(t)
$$

$$
+ 2\nu_p y_c^T(t)e_c(t) - \nu_c e_c^T(t) e_c(t) + y_p^T(t)e_c(t).
$$

Since

$$
-2\nu_p w_1^T(t)e_c(t) \leq |\nu_p| w_1^T(t)w_1(t) + |\nu_p| e_c^T(t)e_c(t),
$$

we can obtain that

$$
\dot{V}(t) \leq \begin{bmatrix} \begin{array}{cc} w_1^T(t) & w_2^T(t) \end{array} \end{bmatrix} \begin{bmatrix} 1 & 2\nu_p \\ -2\nu_c & 1 \end{bmatrix} \begin{bmatrix} \begin{array}{c} y_p(t) \\ y_c(t) \end{array} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \begin{array}{cc} w_1^T(t) & w_2^T(t) \end{array} \end{bmatrix} \begin{bmatrix} -\nu_p + |\nu_p| & 0 \\ 0 & -\nu_c \end{bmatrix} \begin{bmatrix} \begin{array}{c} w_1(t) \\ w_2(t) \end{array} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \begin{array}{cc} y_p^T(t) & y_c^T(t) \end{array} \end{bmatrix} \begin{bmatrix} -(\rho_p + \nu_c) & 0 \\ 0 & -(\nu_p + \rho_c) \end{bmatrix} \begin{bmatrix} \begin{array}{c} w_1(t) \\ w_2(t) \end{array} \end{bmatrix}
$$

$$
+ 2\nu_p y_c^T(t)e_c(t) + (|\nu_p| - \nu_p) e_c^T(t) e_c(t) + y_p^T(t)e_c(t).
$$

With

$$
y_p^T(t)e_c(t) \leq \alpha_c \|y_p(t)\|_2^2 + \frac{1}{4\alpha_c} \|e_c(t)\|_2^2
$$
where $\alpha_c > 0$, we can further get

$$
\dot{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t) + 2\nu_p y_c^T(t)e_c(t) \\
+ \left( \frac{1}{4\alpha_c} + |\nu_p| - \nu_p \right) e_c^T(t)e_c(t) - \beta y_c^T(t)y_c
$$

(6.30)

where

$$
Q = \begin{bmatrix}
-(\rho_p + \nu_c - \alpha_c)I & 0I \\
0I & -(\nu_p + \rho_c - \beta_c)I
\end{bmatrix},
$$

$$
S = \begin{bmatrix}
\frac{1}{2}I & \nu_p I \\
-\nu_c I & \frac{1}{2}I
\end{bmatrix},
$$

$$
R = \begin{bmatrix}
-(\nu_p - |\nu_p|)I & 0I \\
0I & -\nu_c I
\end{bmatrix}
$$

and $\beta > 0$.

Note that

$$
2\nu_p y_c^T(t)e_c(t) \leq 2 |\nu_p| \|y_c(t)\|_2 \|e_c(t)\|_2.
$$

Then we can show

$$
\dot{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t) \\
+ \left( \sqrt{\frac{\nu_p}{m_c}} \|e_c(t)\|_2 + \frac{|\nu_p|}{\sqrt{m_c}} \|y_c(t)\|_2 \right)^2 - \left( \frac{\nu_p^2}{m_c} + \beta_c \right) \|y_c(t)\|_2^2
$$

$$
= 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t) + \\
\left( \sqrt{\frac{\nu_p}{m_c}} \|e_c(t)\|_2 + \frac{|\nu_p|}{\sqrt{m_c}} \|y_c(t)\|_2 + \sqrt{\frac{\nu_p^2}{m_c} + \beta_c} \|y_c(t)\|_2 \right)
$$

$$
\times \left( \sqrt{\frac{\nu_p}{m_c}} \|e_c(t)\|_2 + \frac{|\nu_p|}{\sqrt{m_c}} \|y_c(t)\|_2 - \sqrt{\frac{\nu_p^2}{m_c} + \beta_c} \|y_c(t)\|_2 \right)
$$

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From (6.27), one can verify that

$$\dot{V}(t) \leq 2w^T(t)Sy(t) + w^T(t)Rw(t) + y^T(t)Qy(t)$$

which completes the proof.

\[\Box\]

**Theorem 6.3.** Suppose that the passivity indices \(\nu_p, \rho_p, \nu_c\) and \(\rho_c\) are known and the triggering condition is determined by (6.27). If we choose \(\epsilon\) and \(\delta\) such that

\[
\begin{align*}
\epsilon &< \min \{\nu_p - |\nu_p|, \nu_c\} \\
\delta &\leq \min \left\{\rho_p - \alpha_c - \frac{\epsilon \nu_c}{\nu_c - \epsilon}, \rho_c - \beta_c - \frac{(|\nu_p| + \epsilon)\nu_p}{|\nu_p| - \epsilon}\right\},
\end{align*}
\]

(6.31)

the interconnected system with the event-triggered sampler (Fig. 6.2) has the passivity indices \(\epsilon\) and \(\delta\) satisfying

\[
\dot{V} \leq w^T(t)y(t) - \epsilon w^T(t)w(t) - \delta y^T(t)y(t)
\]

(6.32)

where \(w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}\) and \(y = \begin{bmatrix} y_p(t) \\ y_c(t) \end{bmatrix}\).

**Proof.** From (6.28), we have

\[
\dot{V}(t) \leq w^T(t)y(t) - \begin{bmatrix} w_1^T(t) & y_c^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - \begin{bmatrix} w_2^T(t) & y_p^T(t) \end{bmatrix} P \begin{bmatrix} w_2(t) \\ y_p(t) \end{bmatrix},
\]

(6.33)

where

\[
O = \begin{bmatrix} \nu_p - |\nu_p| & -\nu_p \\ -\nu_p & \nu_p + \rho_c - \beta_c \end{bmatrix}
\]

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and
\[ P = \begin{bmatrix}
\nu_c & \nu_c \\
\nu_c & \rho_p + \nu_c - \alpha_c
\end{bmatrix}. \]

Since \( \epsilon \) and \( \delta \) are chosen such that (6.31) is satisfied, (6.34) holds for the chosen \( \epsilon \) and \( \delta \).

\[ \begin{cases}
\epsilon \leq \nu_p - |\nu_p| \\
\epsilon \leq \nu_c \\
(\nu_p - |\nu_p| - \epsilon)(\nu_p + \rho_c - \beta_c - \delta) \geq \nu_p^2 \\
(\nu_c - \epsilon)(\rho_p + \nu_c - \alpha_c - \delta) \geq \nu_c^2 \\
\nu_p + \rho_c - \beta_c - \delta \geq 0 \\
\rho_p + \nu_c - \alpha_c - \delta \geq 0
\end{cases} \tag{6.34} \]

(6.34) further implies that the matrices
\[
M = \begin{bmatrix}
\nu_p - |\nu_p| - \epsilon & -\nu_p \\
-\nu_p & \nu_p + \rho_c - \beta_c - \delta
\end{bmatrix}
\]
and
\[
N = \begin{bmatrix}
\nu_c & \nu_c \\
\nu_c & \rho_p + \nu_c - \alpha_c - \delta
\end{bmatrix}
\]
are positive semi-definite. Therefore, we have
\[
\begin{bmatrix}
w_1^T(t) & y_c^T(t)
\end{bmatrix} M
\begin{bmatrix}
w_1(t) \\
y_c(t)
\end{bmatrix}
+ \begin{bmatrix}
w_2^T(t) & y_p^T(t)
\end{bmatrix} N
\begin{bmatrix}
w_2(t) \\
y_p(t)
\end{bmatrix} \geq 0 \tag{6.35}
\]
for \( \forall w_1(t), w_2(t), y_c(t) \) and \( y_p(t) \). After re-arranging the terms in (6.35), one can
obtain

\[-w^T E w - y^T \Delta y \geq - \begin{bmatrix} w_1^T(t) & y_c^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - \begin{bmatrix} w_2^T(t) & y_p^T(t) \end{bmatrix} P \begin{bmatrix} w_2(t) \\ y_p(t) \end{bmatrix} \]

(6.36)

where

\[ E = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \]

and

\[ \Delta = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}. \]

From (6.36) and (6.33), we can finally show that

\[ \dot{V}(t) \leq w^T(t)y(t) - \begin{bmatrix} w_1^T(t) & y_c^T(t) \end{bmatrix} O \begin{bmatrix} w_1(t) \\ y_c(t) \end{bmatrix} - \begin{bmatrix} w_2^T(t) & y_p^T(t) \end{bmatrix} P \begin{bmatrix} w_2(t) \\ y_p(t) \end{bmatrix} \]

\[ \leq w^T(t)y(t) - \epsilon w^T(t)w(t) - \delta y^T(t)y(t). \]  

(6.37)

Remark 6.7. The results in Lemma 6.2 and Theorem 6.3 are similar to their counterparts for the feedback system with event-triggered sampler of plant output. However, note that the triggering condition (6.27) now depends on \( \alpha_c, \beta_c \) and \( \nu_p \). Moreover, the matrices \( Q, S \) and \( R \) in (6.28) are different from those in (6.2).

For the passivation problem, Theorem 6.4 gives the conditions of rendering the
Theorem 6.4. Assume $w_2 = 0$ and the triggering condition is determined by (6.27). The interconnected system with the event-triggered sampler (Fig. 6.2) is passive with respect to the input $w_1$ and output $y_p$ if the passivity indices satisfy the conditions

$$\nu_p = 0$$ \hspace{1cm} (6.38)

$$\rho_c \geq \beta_c$$ \hspace{1cm} (6.39)

$$\rho_p + \nu_c \geq \alpha_c.$$ \hspace{1cm} (6.40)

Proof. If $w_2(t) = 0$ and $\nu_p = 0$, (6.33) becomes

$$\dot{V}(t) \leq w_1^T(t)y_p(t) - (\rho_c - \beta_c)y_c^T(t)y_c(t) - (\rho_p + \nu_c - \alpha_c)y_p^T(t)y_p(t)$$ \hspace{1cm} (6.41)

Since we have $\rho_c \geq \beta_c$ and $\rho_p + \nu_c \geq \alpha_c$, we can conclude that

$$\dot{V}(t) \leq w_1^T(t)y_p(t) - (\rho_c - \beta_c)y_c^T(t)y_c(t) - (\rho_p + \nu_c - \alpha_c)y_p^T(t)y_p(t) \leq w_1^T(t)y_p(t).$$ \hspace{1cm} (6.42)

Remark 6.8. The condition (6.38) requires the plant $G_p$ to be a OFP system. Because of (6.38), the triggering condition (6.27) can be further simplified as $\|e_c(t)\|_2 = 2\sqrt{\alpha_c\beta_c} \|y_c(t)\|_2$, which shows that the triggering condition is independent of the passivity indices of the plant $G_p$ and controller $G_c$. Therefore, one can first design a desired triggering condition by choosing $\alpha_c$ and $\beta_c$, and then design a passive controller satisfying the conditions (6.38)-(6.40), or vice versa.

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Corollary 6.2. Suppose that the conditions (6.38)-(6.40) are satisfied. If we choose $\epsilon$ and $\delta$ such that
\[
\epsilon = 0 \\
0 \leq \delta \leq \rho_p + \nu_c - \alpha_c,
\]
the interconnected system with event-triggering (Fig. 6.2) has the passivity indices $\epsilon$ and $\delta$ satisfying
\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - \epsilon w_1^T(t)w_1(t) - \delta y_p^T(t)y_p(t). 
\] (6.44)

Proof. If the condition (6.38)-(6.40) are satisfied, we have
\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - (\rho_c - \beta)y_c^T(t)y_c(t) \\
- (\rho_p + \nu_c - \alpha) y_p^T y_p \\
\leq w_1^T(t)y_p(t) - (\rho_p + \nu_c - \alpha) y_p^T y_p 
\] (6.45)

If $\delta$ is chosen such that
\[
0 \leq \delta \leq \rho_p + \nu_c - \alpha,
\]
we can show that
\[
(\rho_p + \nu_c - \alpha - \delta) y_p^T y_p(t) \geq 0,
\]
which implies that
\[
\dot{V}(t) \leq w_1^T(t)y_p(t) - \epsilon w_1^T(t)w_1(t) - \delta y_p^T(t)y_p(t) 
\] (6.46)

where $\epsilon = 0$. \qed
Remark 6.9. The condition (6.43) implies that the closed-loop system is actually an OSP system with an OFP index $\delta \leq \rho_p + \nu_c - \alpha_c$. The ideas of passivity indices design and passivity-resource trade off discussed in Remark 6.6 apply likewise.

6.5 Examples

The examples show how to passivate a nonlinear plant with a linear feedback controller with event-triggered samplers at the plant output (Fig. 6.1) and at the controller output (Fig. 6.2), respectively. For both examples, it is assumed that $w_2 = 0$.

**Example 6.1.** We first consider the case that the event-triggered sampler is implemented at the plant output (as shown in Fig. 6.1). Assume that the plant $G_p$ is a nonlinear system (an adapted model of $G_p$ in Example 4.3), given by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.5x_1^3 + 0.1x_2 + u_p \\
y_p &= x_2.
\end{align*}
\]

$G_p$ admits a storage function given by

\[
V(x) = \frac{1}{8}x_1^4 + \frac{1}{2}x_2^2,
\]

and

\[
\dot{V}(x) = u_py_p + 0.1y_p^2.
\]

Therefore the OFP level for $G_p$ are $\rho_p = -0.1$. 

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The feedback controller $G_c$ is a 2nd-order system with

$$A = \begin{bmatrix} -2 & -1 \\ -3 & -5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 1.$$ We can determine the passivity levels of $G_c$ to be $\nu_c = 0.3$ and $\rho_c = 0.5$.

It can be seen that the triggering condition (6.1) depends on two non-negative scalars $\alpha_p$ and $\beta_p$, in addition to the passivity levels of $G_p$ and $G_c$. In order to guarantee passivity of the closed-loop system, $\alpha_p$ and $\beta_p$ need to satisfy the conditions (6.13)-(6.14), proposed in Theorem 6.2. Therefore, we choose $\alpha_p = 0.3 < \rho_c$ and $\beta_p = \rho_p + \nu_c = 0.2$ so that the obtained triggering condition is

$$\|e_p(t)\|_2 > 0.2497 \|y_p(t)\|_2.$$ 

(6.47)

It is noted the closed system is now a passive system with the passivity levels $\epsilon = 0$ and $\delta = 0$, given by (6.19).

The simulation results are shown in Fig. 6.3-6.5. Fig. 6.3 verifies that the trajectory of the function $\int w_1 y_p dt$ is always above 0 along the time of simulation. Fig. 6.4 shows the event-triggered sampler only samples the plant output at certain time instants determined by the triggering condition. Fig. 6.5 presents the evolutions of $\|e_p\|_2$ and $0.2497 \|y_p(t)\|_2$, illustrating the time instants when the triggering condition is satisfied.

As discussed in Remark 6.5 and 6.6, it is possible to increase the passivity levels of
Figure 6.3. The trajectory of the function $\int w_1 y_p \, dt$ over time $t$ for $w_1(t) = \sin(2\pi t) + 1$, under the triggering condition (6.47).
Figure 6.4. The trajectories of the event-triggered sampler output $y_p(t_k)$ and the plant output $y_p$ over time $t$, under the triggering condition (6.47)
Figure 6.5. The trajectories of the error $\|e_p\|_2$ and $0.2497\|y_p(t)\|_2$ in the triggering condition (6.47).
the closed-loop systems at the cost of a “tighter” bound on the triggering condition. We can choose $\beta_p = 0.05$ so that the triggering condition becomes

$$\|e_p(t)\|_2 > 0.0754 \|y_p(t)\|_2. \quad (6.48)$$

Although the new triggering condition results in more frequent sampling of the plant output, based on Corollary 6.1 the closed system is an OSP system with OFP level $\delta = 0.15$.

Similarly, the simulation results under the triggering condition (6.48) are shown in Fig. 6.6-6.8. Note that Fig. 6.6 verifies that the trajectory of the function $\int w_1y_p - 0.15y_p^2dt$ is always above 0 along the time of simulation. Compared to Fig. 6.4 and 6.5, Fig. 6.7 and 6.8 show that the sampler is triggered more frequently due to the tighter triggering condition.

To summarize, Fig. (6.9) shows how different triggering thresholds lead to different OFP levels of the interconnected systems by varying $\beta_p$, which makes the co-design of communication transmission and passivity possible.

**Example 6.2.** We then consider the case that the event-triggered sampler is implemented at the controller output (as shown in Fig. 6.2). We still use the same plant $G_p$ and controller $G_c$ as defined in Example 6.1.

In this case, the triggering condition (6.27) depends on two non-negative scalars $\alpha_c$ and $\beta_c$, other than the passivity levels of $G_p$ and $G_c$. We can choose $\alpha_c = 0.2$ and $\beta_c = 0.5$ so that the conditions (6.38)-(6.40) in Theorem (6.4) are satisfied. Therefore the obtained triggering condition is

$$\|e_c(t)\|_2 > 0.6325 \|y_c(t)\|_2. \quad (6.49)$$

Moreover, Corollary (6.2) shows that the closed system is a passive system with the passivity levels $\epsilon = 0$ and $\delta = 0$. The simulation results are shown in Fig. 6.10-6.12.
Figure 6.6. The trajectory of the function $\int (w_1 y_p - 0.15 y_p^2) dt$ over time $t$ for $w_1(t) = \sin(2\pi t) + 1$, under the triggering condition (6.48).
Figure 6.7. The trajectories of the event-triggered sampler output $y_p(t_k)$ and the plant output $y_p$ over time $t$, under the triggering condition (6.48).
Figure 6.8. The trajectories of the error $\|e_p\|_2$ and $0.0754 \|y_p(t)\|_2$ in the triggering condition (6.48)
Figure 6.9. The evolution of triggering threshold in (6.1) and OFP level of the interconnected system by varying $\beta_p$. 
Fig. 6.10 shows the evolution of the function \( \int w_1 y_p dt \). Fig. 6.11 shows the evolution of the outputs of the controller and sampler. Fig. 6.12 shows the evolution of the signals in the triggering condition (6.49).

Figure 6.10. The trajectory of the function \( \int w_1 y_p dt \) over time \( t \) for \( w_1(t) = \sin(2\pi t) + 1 \), under the triggering condition (6.49).

Analogously to the previous example, we can also enhance the passivity levels of the closed-loop system by choosing an event-triggering condition which leads to more frequent sampling. Therefore we can change \( \alpha_c \) to 0.05 so that the same OFP index \( (\delta = 0.15) \) is obtained, as in Example 6.1. The resulting triggering condition is

\[
\|e_c(t)\|_2 > 0.3162 \|y_c(t)\|_2 .
\] (6.50)
Figure 6.11. The trajectories of the event-triggered sampler output $y_c(t_k)$ and the controller output $y_c$ over time $t$, under the triggering condition (6.49).
Figure 6.12. The trajectories of the error $\|e_c\|_2$ and $0.6325 \|y_c(t)\|_2$ in the triggering condition (6.49)
The simulation results under the triggering condition (6.50) are shown in Fig. 6.13-6.15. It can be seen that the OFP index has been increased to 0.15 by increasing sampling frequency of the sampler.

![Graph showing the trajectory of the function \( \int_0^t (w_1 y_p - 0.15 y_p^2) \, dt \) over time for \( w_1(t) = \sin(2\pi t) + 1 \), under the triggering condition (6.50).](image)

Similarly, the evolution of triggering threshold in (6.27) and OFP level by varying \( \alpha_c \) is shown in Fig. (6.16).

6.6 Summary

In this chapter, we considered the problems of passivity analysis and passivation using passivity indices for interconnected event-triggered feedback systems. The
Figure 6.14. The trajectories of the event-triggered sampler output \( y_c(t_k) \) and the controller output \( y_p \) over time \( t \), under the triggering condition (6.50).
Figure 6.15. The trajectories of the error $\|e_c\|_2$ and $0.3162\|y_c(t)\|$ in the triggering condition (6.50).
Figure 6.16. The evolution of triggering threshold in (6.27) and OFP level of the interconnected system by varying $\beta_p$ when $\alpha_p = 0.3$.
present work extended our previous work presented in Chapter 4 for feedback interconnected systems assuming continuous communication in the feedback loop. We considered two event-triggered control schemes: an event-triggered sampler at the plant output and an event-triggered sampler at the controller output. Using the passivity indices of the plant and controller, the conditions to determine the passivity indices of the interconnected system were given, under a proposed event-triggering condition. We also showed the passivation conditions in terms of the passivity indices of the plant and controller and the triggering condition. The trade off between passivity and communication resources utilization was also discussed.
7.1 Introduction

Model predictive control (MPC), is an effective control technique to deal with multi-variable constrained control problems and it has been widely used in a variety of industrial applications. The success of MPC can be attributed to its effective computational control algorithm and its ability to impose various constrains when optimizing the plant behavior. Unlike the conventional feedback control, MPC allows one to first compute an open-loop optimal control trajectory by using an explicit model over a specified prediction horizon, then only the first part of the calculated control trajectory is actually implemented and the entire process is repeated for the next prediction intervals. For extensive surveys on MPC, one can refer to [19, 41, 87] and the references therein.

Although MPC has many advantages, several issues, such as feasibility, closed-loop stability, non-linearity and robustness still need to be studied. If either models of the plant or constraints are nonlinear, nonlinear MPC (NMPC) schemes are required to be used. However, it was pointed out in [21] that NMPC does not always guarantee closed-loop stability. Moreover, robustness of MPC is also an issue when model uncertainty or noise are present [19]. Model uncertainty usually exists in MPC because the model used for prediction cannot perfectly match the real dynamics of the plant to be controlled.

On the other hand, passivity theory is a powerful tool in analysis and control of nonlinear systems [39, 51, 57, 59]. Recently, a passivity-based NMPC scheme
was proposed in [99], motivated by the relationship between optimal control and passivity as well as by the relationship between optimal control and NMPC. It is shown that closed-loop stability and feasibility can be guaranteed by introducing specific passivity-based constraints.

Motivated by [99], in this chapter, we propose a robust stabilizing output feedback NMPC scheme by using passivity and dissipativity. Instead of assuming that both the nominal model and the plant are passive systems with the same dynamics as reported in the previous work, we assume that the model for prediction and the actual plant dynamics are dissipative (which are more general than passive systems since they could be non-passive), and they do not have to possess the same dynamics. Model discrepancy between the nominal model and the real system is characterized by comparing the outputs for the same excitation function. With this characterization of model discrepancy, we are able to compare the supply rate between the nominal model and the real system based on their passivity indices [127]. Then, by introducing specific stabilizing constraints into the MPC based on the passivity indices of the nominal model, we can show that the control input calculated using the nominal model can guarantee stability of the plant to be controlled.

The rest of this chapter is organized as follows: in Section 7.2, we give a brief review on the results of passivity-based NMPC; in Section 7.3, model discrepancy between the nominal model and the real system is characterized, and conditions under which the supply rate of the real system is bounded above by the supply rate of the nominal model is provided; our proposed stabilizing output feedback NMPC scheme is presented in Section 7.4; simulations are provided to validate our results in Section 7.5; finally, conclusions are made in Section 7.6. Results in this chapter are have been published in [134].
7.2 Passivity-Based NMPC

Different from many other NMPC schemes, which achieve stability by enforcing a decrease of the control Lyapunov function (CLF) along the solution trajectory, for the passivity-based NMPC scheme stability is achieved by using a nonlinear input-output constraint, which is implemented as an additional condition within the NMPC set-up. The passivity-based NMPC scheme in [99] was given by

$$\min_{u(t)} \int_{t_k}^{t_k+T_p} \left[q(x(\tau)) + u(\tau)^T u(\tau)\right] d\tau$$

subject to

$$\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t)) u(t) \\
y(t) &= h(x(t)) \\
u^T(t)y(t) + y^T(t)y(t) &\leq 0,
\end{align*}$$

(7.1)

where $t_k$ denotes the time instant at which state measurement of the controlled system is available to the MPC, $q(x(t))$ is a positive semi-definite function and $T_p$ denotes the finite time horizon for prediction. The passivity-based constraint $u^T(t)y(t) + y^T(t)y(t) \leq 0$ guarantees stability of the plant to be controlled, where the dynamics of the plant is assumed to be passive and zero state-detectable. Feasibility is also guaranteed due to the already known stabilizing output feedback control law $u = -y$. It is also shown in [99] that as $T_p \to 0$, the passivity-based NMPC recovers the known stabilizing output feedback $u = -y$.

Motivated by the work reported in [99], in this chapter we propose a robust stabilizing output feedback NMPC scheme. The nominal model and the real plant are assumed to be IF-OFP, and they do not have to possess the same dynamics. Compared to the previous work, our results can also be applied to a class of non-passive systems, and model discrepancy between the real system and the nominal model can be accommodated as well.
7.3 Characterization of Model Discrepancy

Consider two systems denoted by $\Sigma$ and $\hat{\Sigma}$ as shown in Fig. 7.1. One can view $\hat{\Sigma}$ as an approximation of $\Sigma$, and $\hat{\Sigma}$ describes some behavior of interests of $\Sigma$. A common used measure for judging how well $\hat{\Sigma}$ approximates $\Sigma$ is to compare the outputs for the same excitation function $u$. We denote the difference in the output by $\Delta y$. The error may be due to the modeling, linearization or model reduction, etc. For a “good” approximation, we require that the “worst” case $\Delta y$ over all control inputs $u$ be small. Thus $\hat{\Sigma}$ is a good approximation of $\Sigma$ if there exists a positive constant $\gamma > 0$ such that

$$\|\Delta y\|_t \leq \gamma \|u\|_t, \ \forall u \text{ and } \forall t \geq 0,$$  \hfill (7.2)

where $\|\cdot\|_t$ denotes the truncated $L_2$-norm update to time $t$. One can conclude that the value of $\gamma$ characterizes the model discrepancy between $\Sigma$ and $\hat{\Sigma}$.

Remark 7.1. One can verify that for linear systems, $\gamma$ is an upper bound on the $H_\infty$ norm of the difference in the transfer functions of systems $\Sigma$ and $\hat{\Sigma}$.

Figure 7.1. Model Approximation
Assume that system $\Sigma$ is IF-OFP($\nu, \rho$)$^m$, and system $\hat{\Sigma}$ is an approximation of system $\Sigma$ which is IF-OFP($\hat{\nu}, \hat{\rho}$)$^m$. It can be shown that under some conditions, the supply rate of system $\Sigma$ is always bounded above by the supply rate of $\hat{\Sigma}$. Those conditions are summarized in Lemma 7.1.

**Lemma 7.1.** Consider system $\Sigma$ and system $\hat{\Sigma}$ as shown in Fig. 7.1, where $\Sigma$ is IF-OFP($\nu, \rho$)$^m$ and system $\hat{\Sigma}$ is IF-OFP($\hat{\nu}, \hat{\rho}$)$^m$. If (7.2) holds and there exists a $\xi > 0$ such that

$$
\nu - \hat{\nu} \geq \frac{\gamma^2}{\xi} + \gamma + b
$$

(7.3)

$$
\rho - \hat{\rho} \geq \xi \hat{\rho}^2
$$

(7.4)

where $b = 2 \max\{0, \hat{\rho}^2\}$, then

$$
u_T y - \rho y_T y - \nu u_T u \leq u_T \hat{y} - \hat{\rho} y_T \hat{y} - \nu u_T u.
$$

(7.5)

Proof. Let $\omega = u_T y - \rho y_T y - \nu u_T u$, $\hat{\omega} = u_T \hat{y} - \hat{\rho} y_T \hat{y} - \nu u_T u$, then

$$
\hat{\omega} - \omega = u_T \Delta y - \hat{\rho} y_T \Delta y - \nu u_T u - (u_T y - \rho y_T y - \nu u_T u)
$$

(7.6)

$$
= -u_T \Delta y - \hat{\rho} y_T \Delta y - \nu u_T u,
$$

with $-u_T \Delta y \geq -\gamma \|u\|_2^2$ and $-2\hat{\rho} y_T \Delta y \geq -\frac{\gamma^2}{\xi} \|u\|_2^2 - \xi \hat{\rho}^2 \|y\|_2^2$, based on (7.5), we can further get

$$
\hat{\omega} - \omega \geq (\nu - \hat{\nu} - \frac{\gamma^2}{\xi} - \gamma) \|u\|_2^2 + (\rho - \hat{\rho} - \xi \hat{\rho}^2) \|y\|_2^2 - \hat{\rho} \|\Delta y\|_2^2.
$$

(7.6)

So for the case $\hat{\rho} \leq 0$, if $\nu - \hat{\nu} \geq \frac{\gamma^2}{\xi} + \gamma$ and $\rho - \hat{\rho} \geq \xi \hat{\rho}^2$, then $\hat{\omega} \geq \omega$; for the case $\hat{\rho} > 0$, if $\nu - \hat{\nu} \geq \frac{\gamma^2}{\xi} + \gamma + \gamma^2 \hat{\rho}$ and $\rho - \hat{\rho} \geq \xi \hat{\rho}^2$, then $\hat{\omega} \geq \omega$. This completes the proof.

□

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Remark 7.2. Lemma 7.1 provides conditions under which the supply rate of system $\Sigma$ is upper bounded by the supply rate of $\hat{\Sigma}$. Those conditions are related to the bound on the model discrepancy between $\Sigma$ and $\hat{\Sigma}$ (which, in our case, is characterized by $\gamma$), and is also related to their passivity indices (as indicated by $\nu \geq \hat{\nu} + \gamma^2 + \gamma + b$ and $\rho \geq \hat{\rho} + \xi \hat{\rho}^2$). The conditions on the bound of passivity indices can be relaxed if the bound on the model discrepancy is small (i.e., $\gamma$ is small, thus $\hat{\Sigma}$ approximates the behavior of $\Sigma$ well for the same excitation function).

7.4 Robust Stabilizing Output Feedback NMPC

In this section, we first study the problem of stabilization of IF-OFP systems by using static output feedback gains. This result is important for us to derive the stabilizing condition in our proposed NMPC scheme. We then propose a NMPC scheme by using passivity and dissipativity when there is no model discrepancy between the nominal model and the real system. Finally, we extend this result to the case when model discrepancy between the nominal model and the real system can be characterized in the way as discussed in Section 7.3.

7.4.1 Stabilization by Using Static Output Feedback Gain

**Lemma 7.2.** If system $H$ is IF-OFP($\rho, \nu$), then there always exists an output feedback stabilizing control $u(t) = r(t) - Ky(t)$, where $K \in \mathbb{R}$, $r(t), y(t) \in \mathcal{L}_2$, such that the closed-loop system is $\mathcal{L}_2$ stable from $r(t)$ to $y(t)$. Moreover, if the system is also zero-state detectable, then with $r(t) = 0$, the closed-loop system is asymptotically stable.
Proof. Since \( u(t) = r(t) - Ky(t) \), we can get

\[
V(x_t) - V(x_0) \leq \int_0^t \left\{ \left[ r(\tau) - Ky(\tau) \right]^T y(\tau) - \rho y(\tau)^T y(\tau) - \nu \left[ r(\tau) - Ky(\tau) \right]^T \left[ r(\tau) - Ky(\tau) \right] \right\} d\tau
\]

\[
= \int_0^t \left[ (1 + 2\nu K)r^T(\tau)y(\tau) - (K + \rho + K^2\nu)y^T(\tau)y(\tau) - \nu r^T(\tau)r(\tau) \right] d\tau;
\]

thus

\[
V(x_t) - V(x_0) \leq \int_0^t \left( |1 + 2\nu K||r(\tau)||_2||y(\tau)||_2 \right)
\]

\[+ |\nu||r(\tau)||_2^2 - (K + \rho + K^2\nu)||y(\tau)||_2^2 \right) d\tau. \tag{7.7}
\]

If \( K + \rho + K^2\nu > 0 \), then we can obtain

\[
V(x_t) - V(x_0) \leq \int_0^t \left( \frac{|1 + 2K\nu|^2}{2(K + \rho + \nu K^2)} + |\nu| \right)||r(\tau)||_2^2 - \frac{K + \rho + \nu K^2}{2}||y(\tau)||_2^2 \right) d\tau, \tag{7.8}
\]

which further yields

\[
\int_0^t \frac{K + \rho + \nu K^2}{2}||y(\tau)||_2^2 d\tau \leq \int_0^t \left( \frac{|1 + 2K\nu|^2}{2(K + \rho + \nu K^2)} + |\nu| \right)||r(\tau)||_2^2 d\tau + V(x_0), \tag{7.9}
\]

which shows that the closed-loop system is \( \mathcal{L}_2 \) stable from \( r(t) \) to \( y(t) \). With \( r(t) = 0 \), we have \( \int_0^t \frac{K + \rho + \nu K^2}{2}||y(\tau)||_2^2 d\tau \leq V(x_0) \). With \( V(x_0) \) being bounded, we can further conclude that \( \lim_{t \to \infty} y(t) = 0 \). Asymptotic stability follows from that the system \( H \) is zero-state detectable. Now it remains to show that there always exists \( K \) such that \( K + \rho + \nu K^2 > 0 \). Assume that there does not exist a \( K \) such that \( K + \rho + \nu K^2 > 0 \). This can only happen when \( \nu < 0 \). Let \( p(K) = K + \rho + \nu K^2 \), then one can find that with \( \nu < 0 \), \( p(K) \) has a global maximum at \( K = -\frac{1}{2\nu} \), and
max_K \{p(K)\} = \frac{4\rho \nu - 1}{4\nu}. \] In view of Lemma 2.6 with \( \nu < 0 \), we have \( \rho \in \Omega_1 \), which yields \( \max_K \{p(K)\} = \frac{4\rho \nu - 1}{4\nu} > 0 \). This implies that there exists \( K \) such that \( p(K) > 0 \) when \( \nu < 0 \), which completes the proof.

**Remark 7.3.** It can be shown that:

- if \( \nu = 0 \), then we can choose \( K > -\rho \);
- if \( \nu > 0 \), then we can choose \( K > \frac{-1+\sqrt{1-4\rho \nu}}{2\nu} \) or \( K < \frac{-1-\sqrt{1-4\rho \nu}}{2\nu} \);
- if \( \nu < 0 \), we can choose \( \frac{-1+\sqrt{1-4\rho \nu}}{2\nu} < K < \frac{-1-\sqrt{1-4\rho \nu}}{2\nu} \).

So based on the passivity indices \( (\nu, \rho) \), we can find the range of stabilizing output feedback gains for the system.

### 7.4.2 Stabilizing Output Feedback NMPC with No Model Discrepancy

Motivated by the passivity-based NMPC scheme reported in [99], we extend this scheme to the more general cases, where the systems to be controlled are IF-OFP. In view of Lemma 7.2, we can conclude that it is always possible to find a range of stabilizing output feedback gains for an IF-OFP system based its passivity indices. We first consider the case when there is no model discrepancy between the system to be controlled and the nominal model being used for prediction. The scheme of stabilizing output feedback NMPC for IF-OFP systems with no model discrepancy is given by:

\[
\min_{u(t)} \int_{t_k}^{t_k+T_p} \left[ q(x(\tau)) + u(\tau)^T Ru(\tau) \right] d\tau
\]

s.t.

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\
y(t) &= h(x(t), u(t)) \\
u^T(t)y(t) - \rho y^T(t)y(t) - \nu u^T(t)u(t) &\leq -\frac{K+\rho+\nu K^2}{2} y^T(t)y(t),
\end{align*}
\]

(7.11)
where $R$ is a positive definite matrix, the stabilizing output feedback gain $K$ should be chosen based on the indices $(\nu, \rho)$ of the system such that $K + \rho + \nu K^2 > 0$, which has been discussed in Remark 7.3.

**Theorem 7.1.** The output feedback NMPC scheme proposed in (7.11) can asymptotically stabilize system (2.7)-(2.8) if it is IF-OFP($\nu, \rho)^m$ with a continuously differentiable storage function and is zero-state detectable.

**Proof.** The proof is very similar to the proof provided in [99]. First, we need to show that the NMPC scheme proposed in (7.11) for the system which is IF-OFP($\nu, \rho)^m$ is always feasible; second, we need to show that the NMPC scheme will stabilize the system asymptotically. Actually, feasibility is guaranteed due to the known output feedback stabilizing control law $u(t) = -Ky(t)$. Let $V$ be the storage function of the system. With the differentiable storage function $V$ and the stability constraint $u^T(t)y(t) - \rho y^T(t)y(t) - \nu u^T(t)u(t) \leq -\frac{K+\rho+\nu K^2}{2} y^T(t)y(t)$, one can obtain

$$\dot{V} \leq u^T(t)y(t) - \rho y^T(t)y(t) - \nu u^T(t)u(t) \leq -\frac{K+\rho+\nu K^2}{2} y^T(t)y(t).$$

Using the fact that the system is zero-sate detectable, asymptotic stability follows from the result shown in Lemma 7.2.

7.4.3 Stabilizing Output Feedback NMPC with Model Discrepancy

Consider system $\Sigma$ and system $\hat{\Sigma}$ as shown in Fig. 7.1, where $\hat{\Sigma}$ is an approximation of $\Sigma$. In NMPC, $\Sigma$ represents the real system to be controlled, and $\hat{\Sigma}$ represents the nominal model used in MPC for prediction. System $\Sigma$ is IF-OFP($\nu, \rho)^m$ and system $\hat{\Sigma}$ is IF-OFP($\hat{\nu}, \hat{\rho})^m$. Since $\hat{\Sigma}$ is an approximation of $\Sigma$, $(\hat{\nu}, \hat{\rho})$ is not necessarily equal to $(\nu, \rho)$. In this case, we need to rectify the stabilizing output feedback NMPC
scheme proposed in Section \[7.4.2\] as:

\[
\min_{u(t)} \int_{t_k}^{t_k+T_p} \left[ q(\hat{x}(\tau)) + u(\tau)^T R u(\tau) \right] d\tau
\]

\[
s.t. \begin{cases}
\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{g}(\hat{x}(t))u(t) \\
\dot{\hat{y}}(t) = \hat{h}(\hat{x}(t), u(t)) \\
u^T(t)\hat{y}(t) - \hat{\rho}\hat{y}(t) - \hat{\nu}u^T(t)u(t) \\
\leq -\frac{K + \hat{\rho} + \hat{\nu}K^2}{2}\hat{y}(t)\hat{y}(t),
\end{cases}
\]

\[\text{(7.12)}\]

where

\[
\begin{cases}
\dot{x}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) \\
y(t) = h(\hat{x}(t), u(t))
\end{cases}
\]

\[\text{(7.13)}\]

is the state-space model of \(\hat{\Sigma}\), and \(K\) is chosen based on \((\hat{\nu}, \hat{\rho})\) such that \(K + \hat{\rho} + \hat{\nu}K^2 > 0\) (see Remark \[7.3\] on how to choose the range of \(K\)).

Due to possible model mismatch between \(\Sigma\) and \(\hat{\Sigma}\), the control action generated through the NMPC \[\text{(7.12)}\] may not be able to stabilize the real system \(\Sigma\). Intuitively, stabilization results may still hold if \(\hat{\Sigma}\) is a good approximation of \(\Sigma\). Theorem \[7.2\] provides sufficient conditions under which the NMPC scheme provided in \(\text{(7.12)}\) can still stabilize the real system \(\Sigma\).

**Theorem 7.2.** Consider system \(\Sigma\) and system \(\hat{\Sigma}\) as shown in Fig. \[7.1\], where \(\Sigma\) is IF-OFP\((\nu, \rho)^m\) with a continuously differentiable storage function \(V\) and is zero-state detectable; system \(\hat{\Sigma}\) is IF-OFP\((\hat{\nu}, \hat{\rho})^m\) with a continuously differentiable storage function \(\hat{V}\) and is also zero-state detectable. If \(\text{(7.2)}\) holds and there exists a \(\xi > 0\) such that

\[
\nu - \hat{\nu} \geq \frac{\gamma^2}{\xi} + \gamma + b
\]

\[
\rho - \hat{\rho} \geq \xi \hat{\rho}^2
\]

where \(b = 2 \max\{0, \hat{\rho}\gamma^2\}\), then the NMPC scheme provided in \(\text{(7.12)}\) is also a stabi-
Remark 7.4. One can see that the basic idea behind this robust stabilizing NMPC scheme is that if the supply rate of the real system is upper bounded by the supply rate of the nominal model. Then by introducing the stabilizing condition provided in Theorem 7.2, we are able to stabilize the real system as well. And in view of the discussions provided in Section 7.3, this bound on the supply rate is related to
bound on the model discrepancy between the real systems and the nominal models, and their own passivity indices. This approach may appear to be conservative at the first look, but one should be aware that the nominal model used for prediction can always be adapted in order to meet those conditions.

7.5 Example

Example 7.1. In this example, the dynamics of the real system is given by

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-2 & 0.1 \\
0.2 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0.1 \\
1
\end{bmatrix} u
\]
\]
\[
y = 0.1 x_1 + x_2,
\]

while the nominal model is given by

\[
\begin{bmatrix}
\dot{\hat{x}}_1 \\
\dot{\hat{x}}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0.2 \\
0.3 & 0.95
\end{bmatrix} \begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix} + \begin{bmatrix}
0.12 \\
0.96
\end{bmatrix} u
\]
\]
\[
\hat{y} = 0.12 \hat{x}_1 + 0.96 \hat{x}_2.
\]

In this case, one can verify that \( \rho = -1.05, \ \nu = -0.04, \ \hat{\rho} = -1.2, \ \hat{\nu} = -0.2, \ \gamma = 0.0451 \), which satisfy the conditions provided in Lemma 7.1, and we choose \( K = 2.5 \) based on Remark 7.3. By using the proposed NMPC scheme provided in Theorem 7.2, we get the simulation results where the state measurements of the plant are sent to the NMPC at every 0.5s, while the prediction period of NMPC is 2s. In the cost function, \( q(\hat{x}) = 0.5 \hat{x}_1^2 + 0.3 \hat{x}_1^2 \), and \( R = 0.5 \). The simulation results are shown in Fig. 7.2-Fig. 7.4.
Figure 7.2. State of the plant and state of the nominal model
Figure 7.3. Output and control input
Fig. 7.2 compares the state of the plant and the state of the nominal model; Fig. 7.3 shows the output and the control input of the plant and the nominal model; Fig. 7.4 compares their supply rate and the cost, where $\omega$ denotes the supply rate of the real system, and $\hat{\omega}$ denotes the supply rate of the nominal model; one can see that $\omega$ is always upper bounded by $\hat{\omega}$.

7.6 Summary

In this chapter, we proposed a robust nonlinear model predictive control scheme by using passivity and dissipativity. Compared with the previous results on passivity-
based MPC reported in [99], our proposed scheme is more general because it can also be applied to a class of non-passive systems, and model discrepancy can be accommodated as well. By introducing specific stabilizing constraints based on the passivity indices of the nominal model into the MPC, we showed that our proposed NMPC scheme can guarantee the stability of the real system to be controlled.
CHAPTER 8

CONCLUSIONS

This dissertation provided a novel way to analyze and design CPS using passivity and dissipativity theory. It was shown that the proposed approaches, extended conventional passivity and dissipativity theory to be successfully applied to hybrid systems and networked control systems. Numerical simulations and illustrative examples were provided to motivate the theory or to demonstrate its practical usage.

Preliminaries and background on passivity and dissipativity theory were given in Chapter 2 with their properties and implications to stability. Chapter 3 introduced the notion of cyber-physical systems (CPS) and the design challenges. Emphasis was placed on passivity-based methods for hybrid systems and networked control systems. Results on passivity and passivation of the feedback interconnected systems were presented in Chapter 4. The results can be viewed as the extension of the well-known compositional property of passivity and do not require linearity of the systems as it is commonly assumed in the literature. A transformation-based passivation scheme for non-switched and switched systems was discussed in Chapter 5. The scheme centering on the use of an input-output coordinate transformation to passivate a non-switched system was first introduced for nonlinear systems and then applied to switched systems under quantization. Chapter 6 considered the problems of passivity analysis and passivation using passivity indices for interconnected event-triggered feedback systems. The event-triggering conditions proposed for two event-triggered control schemes, were used to guarantee passivity of interconnected systems. The trade off between performance (passivity levels) and communication
resource utilization was discussed. Chapter 7 covered a robust stabilizing output feedback nonlinear model predictive control (NMPC) scheme by using passivity and dissipativity. Particularly, model discrepancy between the nominal model and the real system was considered.

This dissertation presented a number of new results. However, it is worth pointing out some possible future directions and open questions.

1) The examples in Chapter 4 show that the provided conditions may be conservative. It would be interesting if the degree of conservatism could be studied in theory. It may be also possible to consider a parametric plant and to investigate how the conservatism changes depending on the parameter change.

2) The transformation scheme proposed in Chapter 5 requires a linear input-output mapping, which could significantly change the input-output dynamics. It is possible to impose additional control requirements other than passivity.

3) Two event-triggered schemes were considered respectively in Chapter 6. It is possible to propose a unified framework to include them together. Moreover, other network-induced factors, such as time-delay and quantization can also be included.

4) The physical interpretation of passivity indices is not very straightforward for certain complex systems. The study of the relation between passivity indices and other system performance metrics could be beneficial.


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