A VERTEX SUPERALGEBRA VIA SPIN FACTORIZATION ALGEBRAS WITH 
POINT DEFECTS

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Abstract

by

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Factorization algebras are one “approximation” to physicists’ quantum field theories, and spin factorization algebras with point defects are a generalization of factorization algebras which allow us to take spin structures into account. In this thesis, we construct the spin factorization algebra with point defects of quantum observables for a particular free BV theory. Taking cohomology yields a spin prefactorization algebra with point defects. We investigate the structure maps of this cohomology prefactorization algebra and use them to give a geometric description of the free fermion vertex superalgebra and a geometric description of a twisted module over a vertex superalgebra.
## CONTENTS

<table>
<thead>
<tr>
<th>FIGURES</th>
<th>ix</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHAPTER 1: INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER 2: FACTORIZATION ALGEBRAS - THE BASICS</td>
<td>4</td>
</tr>
<tr>
<td>2.1 Prefactorization Algebras</td>
<td>4</td>
</tr>
<tr>
<td>2.1.1 An Example</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Factorization Algebras</td>
<td>6</td>
</tr>
<tr>
<td>2.3 A More General Factorization Algebra</td>
<td>7</td>
</tr>
<tr>
<td>CHAPTER 3: ONE DIMENSIONAL COMPLEX SPIN MANIFOLDS</td>
<td>8</td>
</tr>
<tr>
<td>3.1 Constructing Spin Structures</td>
<td>8</td>
</tr>
<tr>
<td>3.2 How Many Spin Structures?</td>
<td>10</td>
</tr>
<tr>
<td>3.2.1 The Spin Structure on $\mathbb{C}$</td>
<td>16</td>
</tr>
<tr>
<td>3.2.2 Spin Structures on $\mathbb{C}^\times$</td>
<td>17</td>
</tr>
<tr>
<td>3.2.3 The Annulus and the Disc</td>
<td>20</td>
</tr>
<tr>
<td>3.3 Maps Between Spin Manifolds</td>
<td>21</td>
</tr>
<tr>
<td>CHAPTER 4: SPIN FACTORIZATION ALGEBRAS WITH POINT DEFECTS</td>
<td>22</td>
</tr>
<tr>
<td>CHAPTER 5: FREE BV THEORIES</td>
<td>26</td>
</tr>
<tr>
<td>5.1 An Example</td>
<td>26</td>
</tr>
<tr>
<td>5.2 A Factorization Algebra</td>
<td>30</td>
</tr>
<tr>
<td>CHAPTER 6: DIFFERENTIAL FORMS ON SPIN MANIFOLDS WITH POINT DEFECTS</td>
<td>32</td>
</tr>
<tr>
<td>6.1 The Disk</td>
<td>33</td>
</tr>
<tr>
<td>6.2 In General</td>
<td>35</td>
</tr>
<tr>
<td>CHAPTER 7: THE SPIN FACTORIZATION ALGEBRA WITH POINT DEFECTS</td>
<td>36</td>
</tr>
<tr>
<td>CHAPTER 8: THE COHOMOLOGY PREFACTORIZATION ALGEBRA</td>
<td>37</td>
</tr>
<tr>
<td>CHAPTER 9: A CIRCLE ACTION</td>
<td>40</td>
</tr>
<tr>
<td>FIGURES</td>
<td>PAGE</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>2.1 The subsets $W^1_1, W^2_1 \subseteq U_1$ and $W^1_2, W^2_2, W^3_2 \subseteq U_2$, with $U_1, U_2 \subseteq V$</td>
<td>5</td>
</tr>
<tr>
<td>10.1 Multiplication of Neveu-Schwarz annuli</td>
<td>50</td>
</tr>
<tr>
<td>11.1 Inclusion of a Neveu-Schwarz annulus into a disk</td>
<td>56</td>
</tr>
<tr>
<td>11.2 The action of a Neveu-Schwarz annulus on a Neveu-Schwarz disk</td>
<td>59</td>
</tr>
<tr>
<td>11.3 A sub-annulus of $\mathbb{D}_2$</td>
<td>60</td>
</tr>
<tr>
<td>11.4 Multiplication of $A_1$ and $A_2$</td>
<td>61</td>
</tr>
<tr>
<td>11.5 Inclusion of $A$ into $\mathbb{D}$</td>
<td>62</td>
</tr>
<tr>
<td>12.1 Inclusion of a Neveu-Schwarz disk into a Neveu-Schwarz annulus</td>
<td>66</td>
</tr>
<tr>
<td>12.2 Multiplication of two Neveu-Schwarz disks</td>
<td>70</td>
</tr>
<tr>
<td>12.3 Using an annulus, $A$, to multiply two disks</td>
<td>71</td>
</tr>
<tr>
<td>13.1 The disk $\mathbb{D}' \subseteq \mathbb{D}$</td>
<td>74</td>
</tr>
<tr>
<td>13.2 The translated disk $y\mathbb{D}'$</td>
<td>75</td>
</tr>
<tr>
<td>13.3 The setup for defining the map $T$</td>
<td>77</td>
</tr>
<tr>
<td>15.1 Multiplication of Ramond annuli</td>
<td>95</td>
</tr>
<tr>
<td>16.1 Inclusion of a Ramond annulus into a Ramond disk</td>
<td>101</td>
</tr>
<tr>
<td>17.1 Multiplication of a Neveu-Schwarz disk and a Ramond disk</td>
<td>104</td>
</tr>
<tr>
<td>17.2 Using an annulus, $A'$, to multiply a Neveu-Schwarz disk and a Ramond disk</td>
<td>105</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

Physicists have been studying matter and how it interacts for centuries. In the eighteenth century, mathematical physicists studying Newton’s theory of gravitation found it convenient to describe gravitation as a gravitational field (though the term field did not emerge until 1849). For each point in space, the gravitational field gives a vector describing the gravitational force acting on any particle at that point. This became the first successful classical field theory. Fields made another appearance in the study of electromagnetism in the nineteenth century. By the end of the nineteenth century, physicists had started thinking of fields as more than just a mathematical curiosity and in fact, many now viewed matter as a manifestation of electric and magnetic fields.

At this point, physicists discovered a problem with classical theories. Classical theories predicted the energy released by an oscillating particle to be infinite at very high frequencies. In 1900, Planck proposed that energy is quantized; only certain allowed amounts of energy can produce oscillation. Planck’s ideas lead to quantum mechanics, which took the quantized nature of energy into account, fixing the problem with infinite energy. However, quantum mechanics applied to particles rather than fields. The ideas of quantum mechanics were first applied to fields in 1926 by Born, Heisenberg, and Jordan, and quantum field theory emerged. [10]

Though the ideas of quantum field theory were introduced almost a century ago and many examples of quantum field theories have been studied by physicists since, we have yet to come up with a rigorous mathematical definition for a general quan-
tum field theory. We do understand classical field theories mathematically, and we know these classical field theories should yield quantum field theories through a process called quantization, which mimics the process of obtaining quantum mechanics from classical mechanics. However, quantization in general is also poorly understood mathematically.

Factorization algebras and functorial field theories are two mathematical “approximations” to quantum field theory. In this thesis, we discuss factorization algebras and then use the factorization algebra of quantum observables of a particular classical field theory to describe a vertex superalgebra and a twisted module over that vertex superalgebra geometrically. More specifically, using ideas we will define later, we do the following.

The classical field theory we will use arises from a particular co-chain complex,

\[(\mathcal{E}, Q) := (\Pi \Omega^{1/2,*}(M), \bar{\partial}),\]

over a complex one dimensional spin manifold \(M\), where \(\Omega^{1/2,0}(M)\) is the space of sections of a complex line bundle, \(T^{1/2,0}(M)\), which is a square root of \(T^{1,0}(M)\) determined by the spin structure on \(M\), and \(\Omega^{1/2,k}(M)\) for \(k \neq 0\) is the space of sections of the bundle \(T^{1/2,0}(M) \otimes T^{0,k}(M)\). See Chapter 3 for more on spin manifolds. The co-chain complex \((\mathcal{E}, Q)\) is a variation of the Dolbeault complex \((\Omega^{1,*}(M), \bar{\partial})\), which can be applied to complex one dimensional manifolds with spin structures. Equipping \((\mathcal{E}, Q)\) with a pairing gives us an example of a free BV theory. See Definition 5.0.8 for details.

In Section 5.2 we construct the factorization algebra, \(\text{Obs}^q\), of quantum observables of the free BV theory \((\mathcal{E}, Q)\), which is defined to be

\[\text{Obs}^q(U) := (\text{Sym}(\mathcal{E}^1(U))[\hbar], Q + \hbar \Delta)\]
for open subsets $U \subseteq M$, where $\Delta$ is the BV Laplacian, as defined in [4]. For the definition of a factorization algebra, see Chapter 2.

Taking cohomology gives us a cohomology prefactorization algebra, $H^\ast(\text{Obs}^q)$. We want to use this cohomology prefactorization algebra to give a geometric description of an associated vertex superalgebra (see Chapter 14 for a discussion of vertex superalgebras). Such a description is given in [4] for a different free BV theory, the $\beta\gamma$ system, in [11] for a free BV theory that yields the Virasoro algebra, and in [3] for a general free BV theory over $\mathbb{C}$ satisfying certain conditions. With some minor changes, the construction in [3] gives a geometric description of a vertex superalgebra using the prefactorization algebra $H^\ast(\text{Obs}^q)$ we’ve discussed so far if $M = \mathbb{C}$. However, we want to do something more general.

The free BV theory, $(\mathcal{E}, Q)$, we’ve defined depends on the spin structure on the manifold $M$. This means the cohomology prefactorization algebra we get, and therefore, the vertex superalgebra, depends on both our choice of a manifold $M$ and on our choice of spin structure on $M$. Since $\Omega^{1/2,\ast}(M)$ makes sense for any complex one dimensional manifold with any spin structure, we can define the factorization algebra of quantum observables, $\text{Obs}^q$, more generally, as a rule we can apply to any one dimensional complex spin manifold. We will, in fact, define a notion of factorization algebra which can be applied to any spin manifold with point defects, any one dimensional complex spin manifold equipped with a finite set of points in the manifold and a spin structure on the complement of the set of points. Such a factorization algebra will be called a spin factorization algebra with point defects (see Chapter 4).

Our spin factorization algebra with point defects will give rise to a vertex superalgebra, the free fermion vertex superalgebra (Chapter 14), and a twisted module over a vertex superalgebra (Chapter 18).
CHAPTER 2

FACTORIZATION ALGEBRAS - THE BASICS

We begin by defining a prefactorization algebra. A factorization algebra is a prefactorization algebra that satisfies extra conditions. Our definitions for prefactorization algebras (and for factorization algebras) will follow [3] with some slight modifications.

2.1 Prefactorization Algebras

Definition 2.1.1. A prefactorization algebra, $\mathcal{F}$, on a Hausdorff space, $M$, consists of the following data:

- For each open $U \subseteq M$, a chain complex, $\mathcal{F}(U)$;
- For any finite collection, $U_1, U_2, \ldots, U_k$, of pairwise disjoint opens inside an open, $V$, a linear map

$$m_{U_1, \ldots, U_k}^V : \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \cdots \otimes \mathcal{F}(U_k) \rightarrow \mathcal{F}(V)$$

called a structure map;

such that

- For $U_1, \ldots, U_k$ and $V$ as above, and a pairwise disjoint collection of open subsets, $W_i^1, \ldots, W_i^{m_i}$, of $U_i$ for each $i$, a commutative diagram

$$\bigotimes_{i=1}^k (\mathcal{F}(W_i^1) \otimes \cdots \otimes \mathcal{F}(W_i^{m_i})) \xrightarrow{\bigotimes_{i=1}^k (m_{U_i}^{w_i^1, \ldots, w_i^{m_i}})} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k)$$

$$\xrightarrow{m_{U_1, \ldots, U_k}^V} \mathcal{F}(V)$$
For example, in figure 2.1 below, including $W_1^1$ and $W_1^2$ into $U_1$ and $W_1^1$, $W_2^2$, and $W_2^3$ into $U_2$ and then including $U_1$ and $U_2$ into $V$ yields the same structure map as the one arising from including $W_1^1$, $W_1^2$, $W_2^1$, $W_2^2$, and $W_2^3$ directly into $V$.

Figure 2.1. The subsets $W_1^1$, $W_1^2 \subseteq U_1$ and $W_2^1$, $W_2^2$, $W_2^3 \subseteq U_2$, with $U_1, U_2 \subseteq V$.

- $\mathcal{F}(\emptyset) = \mathbb{C}$

2.1.1 An Example

Given the name prefactorization algebra, one might expect these objects to be related somehow to algebras. This is in fact the case; given an associative algebra, we can construct a prefactorization algebra on $\mathbb{R}$.

Let $A$ be an associative algebra. Then we can define $\mathcal{F}_A$ to be the prefactorization
algebra on $\mathbb{R}$ where

- For open intervals $(a, b) \subset \mathbb{R}$, $\mathcal{F}((a, b)) = A$;
- For disjoint open intervals $U_1 = (a_1, b_1), U_2 = (a_2, b_2), \ldots, U_n = (a_n, b_n)$ contained in a larger open interval $V = (a, b)$ the structure map is given by

$$m_{U_1, \ldots, U_n} \colon \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \rightarrow \mathcal{F}(V)$$

$$A \otimes \cdots \otimes A \rightarrow A$$

$$x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdot x_2 \cdots \cdot x_n$$

where $\cdot$ denotes multiplication in the algebra $A$.

The fact that $A$ is an associative algebra ensures $\mathcal{F}_A$ is a prefactorization algebra. For details see [4], Section 4.2.

2.2 Factorization Algebras

A factorization algebra is a prefactorization algebra that satisfies a locality condition which says that we can determine what the prefactorization algebra does to large open subsets by what it does on smaller open subsets. That is, we want to be able to determine what a prefactorization algebra does on an open subset $U$ of $M$ by what it does on an open cover of $U$. However, we will need a slightly different notion of open cover.

**Definition 2.2.1.** For an open subset $U \subset M$, a Weiss cover of $U$ is a collection $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets of $M$ such that for any finite collection, $\{x_1, \ldots, x_n\}$, of $M$, there is some $i \in I$ such that $\{x_1, \ldots, x_n\} \subseteq U_i$.

A Weiss cover is much finer (contains more open subsets) than an ordinary open cover.

**Definition 2.2.2.** A factorization algebra is a prefactorization algebra that is determined on larger open subsets of $M$ by what it does on a Weiss cover of $M$. For a more technical definition, see Chapter 6 of [3].
Our example, $\mathcal{F}_A$ of a prefactorization algebra from above is a factorization algebra, as shown in [4] Section 4.2.

2.3 A More General Factorization Algebra

Our goal will be to examine a factorization algebra obtained from a classical field theory and use it to describe a vertex algebra. This has been done for some examples including a field theory that leads to a factorization algebra giving rise to the $\beta\gamma$ vertex algebra [4] and another field theory that leads to a factorization algebra that gives rise to the Virasoro vertex algebra [11]. In fact, in [3] it is shown that any factorization algebra satisfying certain conditions gives rise to a vertex algebra.

The classical field theory we will be interested in gives rise to a more general version of a factorization algebra. We will need to be able to apply our factorization algebra to any manifold of a certain type rather than restricting ourselves to subsets of a given Hausdorff space. The manifolds we will be interested in are one dimensional complex spin manifolds.
Definition 3.0.1. If $M$ is a one dimensional complex manifold, a spin structure on $M$ is a pair, $(S, \alpha)$, where $S$ is a complex line bundle over $M$ and $\alpha : S^{\otimes 2} \to T^{1,0}(M)$ is a complex line bundle isomorphism. We want to think of $S$ as a square root of the cotangent bundle.

We will need the following proposition about complex line bundles.

Proposition 3.0.1. Let $M$ be a complex manifold. If we denote by $\text{Vect}^1_\mathbb{C}(M)$ the group of isomorphism classes of complex line bundles over $M$, then the function

$$c_1 : \text{Vect}^1_\mathbb{C}(M) \to H^2(M; \mathbb{Z}),$$

which takes the first Chern class of a complex line bundle, is an isomorphism.

Proof. See Proposition 3.10 in [6]. □

3.1 Constructing Spin Structures

Let $X$ be a one dimensional complex manifold with $T^{1,0}(X)$ isomorphic to the trivial bundle, and let $\beta \in \Omega^{1,0}(X)$ be a nowhere vanishing section of $T^{1,0}(X)$. We can construct a spin structure on $X$ as follows. Let $S$ be the trivial complex line bundle, $\mathbb{C} := X \times \mathbb{C} \to X$. Define a section, $\sigma$, of $S$, as follows

$$\sigma : X \to X \times \mathbb{C}$$

$$x \mapsto (x, 1).$$
Notice that any section, $\gamma : X \to S = X \times \mathbb{C}$ can be written in terms of $\sigma$. In particular,

$$\gamma(x) := (x, z_\gamma(x)) = (x, 1)z_\gamma(x) = \sigma(x)z_\gamma(x).$$

where $z_\gamma$ is the image of $\gamma$ in $\mathbb{C}$. Similarly, any section $\gamma_1 \otimes \gamma_2 : X \to S^{\otimes 2}$ can be written in terms of $\sigma^{\otimes 2}$, that is,

$$(\gamma_1 \otimes \gamma_2)(x) = (x, z_{\gamma_1}) \otimes (x, z_{\gamma_2}) = (x, 1)^{\otimes 2}z_{\gamma_1}z_{\gamma_2} = \sigma^{\otimes 2}z_{\gamma_1}z_{\gamma_2}.$$

Thus, we can define the map of sections $\tilde{\alpha} : \Gamma(S^{\otimes 2}) \to \Gamma(T^{1,0}(X)) = \Omega^{1,0}(X)$ by requiring that $\tilde{\alpha}(\sigma^{\otimes 2}) = \beta$ and extending linearly to all of $\Gamma(S^{\otimes 2})$.

Now, define $\alpha : S^{\otimes 2} \to T^{1,0}(X)$ such that the induced map, $\alpha^* : \Gamma(S^{\otimes 2}) \to \Gamma(T^{1,0}(X))$ on sections is equal to $\tilde{\alpha}$. That is, for a section $s : X \to S^{\otimes 2}$ of $S^{\otimes 2}$ and $x \in X$,

$$(\tilde{\alpha}(s))(x) = (\alpha^*(s))(x) = \alpha(s(x)),$$

and in particular,

$$(\tilde{\alpha}(\sigma^{\otimes 2}))(x) = \alpha(\sigma^{\otimes 2}(x)).$$

Let $(x, z_1) \otimes (x, z_2) \in S^{\otimes 2}$. Since

$$(x, z_1) \otimes (x, z_2) = (x, 1)^{\otimes 2}z_1z_2$$

$$= (\sigma(x))^{\otimes 2}z_1z_2$$

$$= \sigma^{\otimes 2}(x)z_1z_2,$$

we have

$$\alpha((x, z_1) \otimes (x, z_2)) = \alpha(\sigma^{\otimes 2}(x)z_1z_2)$$

$$= z_1z_2\alpha(\sigma^{\otimes 2}(x))$$
\[
\begin{align*}
&= z_1 z_2 (\tilde{\alpha}(\sigma^{\otimes 2}))(x) \\
&= z_1 z_2 \beta(x).
\end{align*}
\]

Since \( \beta \) is a nowhere vanishing section of \( T^{1,0}(X) \), we can write any section of \( T^{1,0}(X) \) as \( f \beta \) where \( f \in C^\infty(X) \). Thus, we can write any element of \( T^{1,0}(X) \) as \( (f \beta)(x) \) for some \( f \in C^\infty(X) \) and some \( x \in X \). Thus, we can define

\[
\alpha^{-1} : T^{1,0}(X) \to S^{\otimes 2}
\]

\[
(f \beta)(x) \mapsto (x, f(x)) \otimes (x, 1).
\]

Since

\[
(\alpha \circ \alpha^{-1})(f \beta)(x) = \alpha((x, f(x)) \otimes (x, 1)) = f(x) \beta(x) = (f \beta)(x)
\]

and

\[
(\alpha^{-1} \circ \alpha)((x, z_1) \otimes (x, z_2)) = \alpha^{-1}(z_1 z_2 \beta(x)) = (x, z_1 z_2) \otimes (x, 1) = (x, z_1) \otimes (x, z_2),
\]

\( \alpha \) is an isomorphism of vector spaces. Since \( \alpha \) preserves fibers, it is a complex line bundle isomorphism.

Thus, \((S, \alpha)\) is a spin structure on \( X \). The isomorphism \( \alpha \) identifies the section \( \sigma^{\otimes 2} \) of \( S^{\otimes 2} \) with the section \( \beta \) of \( T^{1,0}(X) \). Thus, we write \( \beta_\frac{1}{2} := \sigma \) since \( (\beta_\frac{1}{2})^{\otimes 2} = \sigma^{\otimes 2} \) is identified with \( \beta \) under \( \alpha \). Notice that any section of \( S \) can be written as \( f \beta_\frac{1}{2} \) for some \( f \in C^\infty(X) \).

3.2 How Many Spin Structures?

We would like a way to determine the number of isomorphism classes of spin structures on a complex 1-manifold, \( M \). First, we must describe when two spin structures are isomorphic.
Definition 3.2.1. Let $M$ be a complex 1-manifold, and let $(S_1, \alpha_1)$ and $(S_2, \alpha_2)$ be spin structures on $M$. We say $(S_1, \alpha_1)$ and $(S_2, \alpha_2)$ are isomorphic if there exists an isomorphism of complex vector bundles, $\theta : S_1 \to S_2$, such that the following diagram commutes,

$$
\begin{array}{ccc}
S_1 \otimes^2 & \xrightarrow{\alpha_1} & T^{1,0}M \\
\downarrow{\theta^\otimes^2} & & \downarrow{id} \\
S_2 \otimes^2 & \xrightarrow{\alpha_2} & T^{1,0}M.
\end{array}
$$

Let $M$ be a complex 1-manifold with $H^2(M; \mathbb{Z}) = 0$. By Proposition 3.0.1, all complex line bundles over $M$ are isomorphic to the trivial bundle. Therefore, for any spin structure $(S, \alpha)$ on $M$, $S$ must be isomorphic to the trivial bundle, and $T^{1,0}(M)$ must be isomorphic to the trivial bundle as well. This means the number of possible spin structures on $M$ is determined by the number of isomorphisms, $\alpha : S^\otimes^2 \to T^{1,0}M$. We have shown that when $S$ is the trivial bundle, the isomorphism, $\alpha$, is completely determined by choosing a nowhere vanishing section of $T^{1,0}M$. This space of nowhere vanishing sections consists of maps $f : M \to \mathbb{C}^\times$. The following results give us some information about when two such maps yield isomorphic spin structures.

Lemma 3.2.1. Let $M$ be a complex one dimensional manifold with $H^2(M; \mathbb{Z}) = 0$, and assume the map $f : M \to \mathbb{C}^\times$ is homotopic to the constant map $1 : M \to \mathbb{C}^\times$. Then $f$ and $1$ give rise to isomorphic spin structures.

Proof. Assume $f$ is null homotopic. Then there is a homotopy

$$H : M \times I \to \mathbb{C}^\times,$$

where $H(m,0) = 1(m) = 1$ and $H(m,1) = f(m)$. Notice that the maps $f$ and $1$ determine nowhere vanishing sections, $\beta$ and $\beta_0$ of $T^{1,0}(M) = M \times \mathbb{C}$ given by
\( \beta(m) = (m, f(m)) \) and \( \beta_0(m) = (m, 1) \) respectively. This gives us two spin structures, \((S, \alpha)\) and \((S, \alpha_0)\), where \( S \) is the trivial bundle, \( \alpha^*(\sigma^{\otimes 2}) = \beta \) and \( \alpha_0^*(\sigma^{\otimes 2}) = \beta_0 \).

Recall that defining \( \alpha^* \) and \( \alpha_0^* \) on \( \sigma^{\otimes 2} \) determines \( \alpha \) and \( \alpha_0 \). To show \((S, \alpha)\) and \((S, \alpha_0)\) are isomorphic spin structures, we must find an automorphism, \( \theta \), of \( S \) such that \( \theta^{\otimes 2} \) commutes with \( \alpha \) and \( \alpha_0 \).

Let \( D = \{(z, w) \in \mathbb{C}^\times \times \mathbb{C}^\times | z = w^2 \} \), and consider the double cover

\[
\begin{array}{ccc}
D & \xrightarrow{\pi} & \mathbb{C}^\times \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{C}^\times & \xrightarrow{z} & z.
\end{array}
\] (3.1)

We have a lift, \( \tilde{1} : M \to D \) where \( \tilde{1}(m) = (1, 1) \). By the homotopy lifting property (see Theorem 11.13 of [7]), there is a lift, \( \tilde{H} : M \times I \to D \), of \( H \) such that \( \tilde{H}(m, 0) = 1(m) = 1 \). Define the map \( \gamma : D \to \mathbb{C}^\times \) to be \( \gamma(z, w) = w \), and note that

\[
(\gamma(z, w))^2 = w^2 = z = \pi(z, w).
\] (3.2)

Define \( f^{\frac{1}{2}} := \gamma \circ \tilde{H}(-, 1) : M \to \mathbb{C}^\times \), and define the isomorphism \( \theta : S \to S \) as follows. First, notice that all sections of \( S \) can be written as \( g\sigma \) for some \( g \in \mathcal{C}^\infty(M) \). Thus, we can obtain a map of sections \( \theta' : \Gamma S \to \Gamma S \) by giving the image of \( \sigma \) under \( \theta' \). Let \( \theta'(\sigma) = f^{\frac{1}{2}}\sigma \). This determines the map \( \theta \) by requiring that the induced map on sections, \( \theta^* \), be \( \theta' \). That is, for a section, \( s \), of \( S \),

\[
(\theta'(s))(m) = (\theta^*(s))(m) = \theta(s(m)).
\]
Notice that since $S$ is trivial, we can write any element of $S$ as

$$(m, z) = z(m, 1) = z\sigma(m)$$

where $m \in M$ and $z \in \mathbb{Z}$. Thus, we have

$$\theta((m, z)) = \theta(z\sigma(m)) = z\theta(\sigma(m)) = z(\theta'(\sigma))(m)$$

$$= z(f^{\frac{1}{2}}\sigma)(m) = zf^{\frac{1}{2}}(m)(m, 1) = (m, zf^{\frac{1}{2}}(m)).$$

Since $\theta$ preserves fibers and we can define

$$\theta^{-1} : S \rightarrow S$$

$$(m, z) \mapsto \left( m, \frac{z}{f^{\frac{1}{2}}(m)} \right)$$

where

$$(\theta \circ \theta^{-1})(m, z) = \theta \left( m, \frac{z}{f^{\frac{1}{2}}(m)} \right) = \left( m, \frac{z}{f^{\frac{1}{2}}(m)} f^{\frac{1}{2}}(m) \right) = (m, z)$$

$$(\theta^{-1} \circ \theta)(m, z) = \theta^{-1}(m, zf^{\frac{1}{2}}(m)) = \left( m, \frac{zf^{\frac{1}{2}}(m)}{f^{\frac{1}{2}}(m)} \right) = (m, z),$$

$\theta$ is an isomorphism.

Now consider the map $\theta^{\otimes 2} : S^{\otimes 2} \rightarrow S^{\otimes 2}$. We have

$$\theta^{\otimes 2}((m, z_1) \otimes (m, z_2)) = (m, z_1 f^{\frac{1}{2}}(m)) \otimes (m, z_2 f^{\frac{1}{2}}(m))$$

$$= (f^{\frac{1}{2}}(m))^2((m, z_1) \otimes (m, z_2)).$$

Notice that

$$(f^{\frac{1}{2}}(m))^2 = (\gamma \circ \tilde{H}(m, 1))^2$$
\[ = \pi(\widetilde{H}(m, 1)) \]  
\[ = H(m, 1) \]  
\[ = f(m), \]  

where in (3.3) we used (3.2), so

\[ \theta^{\otimes 2}((m, z_1) \otimes (m, z_2)) = f(m)((m, z_1) \otimes (m, z_2)). \]

Since

\[ (id \circ \alpha)((m, z_1) \otimes (m, z_2)) = z_1 z_2 \beta(m) \]
\[ = z_1 z_2(m, f(m)) \]

and

\[ (\alpha_0 \circ \theta^{\otimes 2})((m, z_1) \otimes (m, z_2)) = \alpha_0(f(m)((m, z_1) \otimes (m, z_2))) \]
\[ = f(m) z_1 z_2 \beta_0(m) \]
\[ = f(m) z_1 z_2(m, 1) \]
\[ = z_1 z_2(m, f(m)), \]

the diagram in Definition 3.2.1 commutes, and therefore, \((S, \alpha)\) and \((S, \alpha_0)\) are isomorphic.

\[ \square \]

**Proposition 3.2.1.** Let \(M\) be a complex one dimensional manifold with \(H^2(M; \mathbb{Z}) = 0\), and assume the maps \(f_1, f_2 : M \to \mathbb{C}^\times\) are homotopic. Then \(f_1\) and \(f_2\) give rise to isomorphic spin structures.
Proof. Assume $f_1$ and $f_2$ are homotopic. Then there is a homotopy

$$H : M \times I \to \mathbb{C}^\times$$

where $H(m, 0) = f_1(m)$ and $H(m, 1) = f_2(m)$. The maps $f_1$ and $f_2$ determine nowhere vanishing sections $\beta_1$ and $\beta_2$ of $T^{1,0}(M) = M \times \mathbb{C}$ given by $\beta_1(m) = (m, f_1(m))$ and $\beta_2(m) = (m, f_2(m))$ respectively. This gives us two spin structures, $(S, \alpha_1)$ and $(S, \alpha_2)$, where $S$ is the trivial bundle, $\alpha_1^*(\sigma^{\otimes 2}) = \beta_1$, and $\alpha_2^*(\sigma^{\otimes 2}) = \beta_2$.

To show $(S, \alpha_1)$ and $(S, \alpha_2)$ are isomorphic spin structures, we must find an automorphism, $\theta$, of $S$ such that $\theta^{\otimes 2}$ commutes with $\alpha_1$ and $\alpha_2$.

Since $\beta_1$ is a nowhere vanishing section of the complex line bundle $T^{1,0}(M)$, any other section of $T^{1,0}(M)$ can be written as $g\beta_2$ for some $g \in C^\infty(M)$. Choose $g \in C^\infty(M)$ such that $\beta_1 = g\beta_2$. Then $\beta_1(m) = (g\beta_2)(m)$ and therefore $(m, f_1(m)) = g(m)(m, f_2(m)) = (m, g(m)f_2(m))$. Thus, $f_1(m) = g(m)f_2(m)$.

Define

$$H' : M \times I \to \mathbb{C}^\times$$

$$(m, t) \mapsto \frac{f_1(m)}{H(m, t)}.$$ 

Since

$$H'(m, 0) = \frac{f_1(m)}{H(m, 0)} = \frac{f_1(m)}{f_1(m)} = 1 = 1(m)$$

and

$$H'(m, 1) = \frac{f_1(m)}{H(m, 1)} = \frac{f_1(m)}{f_2(m)} = \frac{g(m)f_2(m)}{f_2(m)} = g(m),$$

$H'$ is a homotopy from $1$ to $g$. Thus, as we did in the proof of Lemma 3.2.1 we can lift $H'$ to a homotopy $\widetilde{H}' : M \times I \to D$ and define $g^{\frac{1}{2}} := \gamma \circ \widetilde{H}'(-, 1) : M \to \mathbb{C}^\times$. Still following Lemma 3.2.1 we can define the vector bundle isomorphism $\theta : S \to S$ such that

$$\theta(m, z) = (m, zg^{\frac{1}{2}}(m)).$$
As in Lemma 3.2.1, \((g^1(m))^2 = g(m)\), so

\[
(id \circ \alpha_1)((m, z_1) \otimes (m, z_2)) = z_1 z_2 \beta_1(m)
= z_1 z_2(m, f_1(m)),
\]

and

\[
(\alpha_2 \circ \theta^2)((m, z_1) \otimes (m, z_2)) = \alpha_2(g(m)((m, z_1) \otimes (m, z_2)))
= g(m) z_1 z_2 \beta_2(m)
= z_1 z_2(m, g(m) f_2(m))
= z_1 z_2(m, f_1(m)).
\]

Thus, the diagram in Definition 3.2.1 commutes, and \((S, \alpha_1)\) and \((S, \alpha_2)\) are isomorphic spin structures.

3.2.1 The Spin Structure on \(\mathbb{C}\)

We claim there is only one possible spin structure on \(\mathbb{C}\). Notice that \(H^2(\mathbb{C}; \mathbb{Z}) = 0\).

We have shown this implies that the spin structures on \(\mathbb{C}\) are completely determined by maps \(f : \mathbb{C} \to \mathbb{C}^x\). Now, we must determine when two such maps give rise to isomorphic spin structures. The following proposition gives us more information about these maps.

Proposition 3.2.2. For an abelian group \(G\) and a CW complex \(X\), there are natural bijections

\[
T : [X, K(G, n)] \to H^n(X; G)
\]

for \(n > 0\) where \(K(G, n)\) is an Eilenberg-MacLane space and \([X, K(G, n)]\) denotes the homotopy classes of maps from \(X\) to \(K(G, n)\).
Proof. See Theorem 4.57 in [5]

By Proposition 3.2.2 with $X = \mathbb{C}$ and $G = \mathbb{Z}$, $[\mathbb{C}, \mathbb{C}^\times] \cong H^1(\mathbb{C}; \mathbb{Z}) = 0$. Thus, there is only a single homotopy class of maps from $\mathbb{C}$ to $\mathbb{C}^\times$, meaning every map $f : \mathbb{C} \to \mathbb{C}^\times$ is homotopic to the constant map that sends every element of $\mathbb{C}$ to $1 \in \mathbb{C}^\times$. Thus, by Lemma 3.2.1 every map $f : \mathbb{C} \to \mathbb{C}^\times$ gives rise to the same spin structure up to isomorphism, meaning, up to isomorphism, there is only one spin structure on $\mathbb{C}$. In discussing this isomorphism class of spin structures, we will use as the representative $(S = \mathbb{C}, \alpha)$ where $\alpha$ is determined by the nowhere vanishing section $dz \in T^{1,0}(\mathbb{C})$. Thus, we will write the canonical section of $S$ as $(dz)^{\frac{1}{2}}$.

3.2.2 Spin Structures on $\mathbb{C}^\times$

We claim there are two isomorphism classes of spin structures on $\mathbb{C}^\times$. Again, we have $H^2(\mathbb{C}^\times; \mathbb{Z}) = 0$, so the spin structures on $\mathbb{C}^\times$ are completely determined by maps $f : \mathbb{C}^\times \to \mathbb{C}^\times$. By Proposition 3.2.2 $[\mathbb{C}^\times, \mathbb{C}^\times] \cong [\mathbb{C}^\times, K(\mathbb{Z}, 1)] \cong H^1(\mathbb{C}^\times; \mathbb{Z}) \cong \mathbb{Z}$. We know from Proposition 3.2.1 that homotopic maps yield isomorphic spin structures, so we need to determine whether or not there are any distinct homotopy classes of maps that yield isomorphic spin structures.

For each element $n \in \mathbb{Z} \cong [\mathbb{C}^\times, \mathbb{C}^\times]$, the function $f_n(z) := z^n$ is a representative in $[\mathbb{C}^\times, \mathbb{C}^\times]$ (see [7], chapter 8). We want to determine whether the functions $f_n$ and $f_m$ give isomorphic spin structures for $n \neq m$.

First, assume $n$ and $m$ are both even. Then $n = 2k_n$ and $m = 2k_m$ for integers $k_n$ and $k_m$. We claim that $f_n$ and $f_m$ determine isomorphic spin structures.

Let $(S, \alpha_n)$ and $(S, \alpha_m)$ be the spin structures determined by $f_n$ and $f_m$ respectively. Let $\theta : S \to S$ be the function such that

$$\theta(w, z) = (w, zw^{k_n-k_m}).$$
Then since $\theta$ preserves the fibers of $S$ and we can define an inverse for $\theta$,

$$\theta^{-1}(w, z) = (w, zw^{k_m-k_n}),$$

$\theta$ is a vector bundle isomorphism. Since

$$\theta^\otimes 2((w, z_1) \otimes (w, z_2)) = (w, z_1w^{k_n-k_m} \otimes (w, z_2w^{k_n-k_m}))$$

$$= w^{2(k_n-k_m)}((w, z_1) \otimes (w, z_2)),$$

we have

$$(\text{id} \circ \alpha_n)((w, z_1) \otimes (w, z_2)) = z_1z_2(w, f_n(w))$$

$$= z_1z_2(w, w^n)$$

and

$$(\alpha_m \circ \theta^\otimes 2)((w, z_1) \otimes (w, z_2)) = \alpha_m(w^{2(k_n-k_m)}((w, z_1) \otimes (w, z_2))))$$

$$= w^{2(k_n-k_m)}z_1z_2(w, w^m)$$

$$= w^{2(k_n-k_m)}z_1z_2(w, w^{2k_m})$$

$$= z_1z_2(w, w^{2k_n})$$

$$= z_1z_2(w, w^n).$$

Thus, the diagram in Definition 3.2.1 commutes, and $(S, \alpha_n)$ and $(S, \alpha_m)$ are isomorphic spin structures.

Now, assume $n$ and $m$ are both odd. Then $n = 2k_n + 1$ and $m = 2k_m + 1$ for integers $k_n$ and $k_m$. We again claim $f_n$ and $f_m$ determine isomorphic spin structures. If $(S, \alpha_n)$ and $(S, \alpha_m)$ are the spin structures determined by $f_n$ and $f_m$ respectively,
defining

$$\theta(w, z) = (w, zw^{k_n-k_m})$$

and following the argument above for the even case shows that \((S, \alpha_n)\) and \((S, \alpha_m)\) are isomorphic.

Now assume, without loss of generality, that \(n\) is even and \(m\) is odd. In this case, we will show that we do not get isomorphic spin structures. Let \(n = 2k_n\) and \(m = 2k_m + 1\) for \(k_n, k_m\) integers, and let \((S, \alpha_n)\) and \((S, \alpha_m)\) be the spin structures determined by \(f_n\) and \(f_m\) respectively. By way of contradiction, assume there is an isomorphism, \(\theta\), which makes the diagram in Definition 3.2.1 commute. Then for \((x, z) \in S\), we have

$$\text{id} \circ \alpha_n((x, z) \otimes (x, z)) = (\alpha_m \circ \theta^\otimes 2)((x, z) \otimes (x, z)).$$

Since \(\theta\) must preserve fibers, we can think of the image of \(\theta\) as being in \(\mathbb{C}\), and we get

$$\alpha_n((x, z) \otimes (x, z)) = \alpha_m((x, \theta(x, z)) \otimes (x, \theta(x, z)))$$

where here we regard \(\theta(x, z)\) as an element of \(\mathbb{C}\). Applying \(\alpha_n\) and \(\alpha_m\) gives

$$z^2(x, x^n) = (\theta(x, z))^2(x, x^m) = (\theta(x, z))^2 x^{m-n}(x, x^n),$$

and therefore,

$$z^2 = (\theta(x, z))^2 x^{m-n}.$$

Solving for \(\theta(x, z)\) gives

$$\theta(x, z) = (z^2 x^{n-m})^{1/2}$$

$$= z(x^{2k_n-2k_m-1})^{1/2}$$
Since we cannot define the function $x^{1/2}$ on all of $\mathbb{C}^*$, there is no isomorphism, $\theta$ that makes the diagram in Definition 3.2.1 commute, and therefore, $f_n$ and $f_m$ give non-isomorphic spin structures.

Thus, we have seen that there are two isomorphism classes of spin structures on $\mathbb{C}^*$, one obtained from functions of even degree and the other obtained from functions of odd degree.

For the isomorphism class obtained from even degree functions, we will use $(S_N = \mathbb{C}, \alpha_N)$ as our spin structure representative, where $\alpha_N$ is determined by the nowhere vanishing section $dz \in T^{1,0}(\mathbb{C}^*)$. We will write the canonical section of $S_N$ as $(dz)^{1/2}$, and we will call this the Neveu-Schwarz spin structure (see Section 1.1 of [1]). Notice that this is the spin structure obtained by restricting the spin structure $S$ on $\mathbb{C}$ to $\mathbb{C}^*$ and therefore, $S_N$ extends to all of $\mathbb{C}$.

For the isomorphism class obtained from odd degree functions, we will use $(S_R = \mathbb{C}, \alpha_R)$ as our spin structure representative, where $\alpha_R$ is determined by the nowhere vanishing section $zdz \in T^{1,0}(\mathbb{C}^*)$. We will write the canonical section of $S_R$ as $(zdz)^{1/2} = z^{1/2}(dz)^{1/2}$, and we will call this the Ramond spin structure (see Section 1.1 of [1]).

3.2.3 The Annulus and the Disc

If $D \subseteq \mathbb{C}$ is a disc, we can restrict the Neveu-Schwarz spin structure on $\mathbb{C}$ to the disc. Abusing notation, we will call this restricted spin structure on $D$ the Neveu-Schwarz spin structure and denote it $(S_n, \alpha_N)$.

If $A \subseteq \mathbb{C}^*$ is an annulus, we can restrict both the Neveu-Schwarz and Ramond spin structures on $\mathbb{C}^*$ to $A$. We will call the spin structures restricted to $A$ the Neveu-Schwarz and Ramond spin structures and denote them by $(S_N, \alpha_N)$ and $(S_R, \alpha_R)$.
respectively.

3.3 Maps Between Spin Manifolds

When we consider a map between two spin manifolds, we would like to say what it means for such a map to preserve the spin structure.

\textbf{Definition 3.3.1.} Let $X$ and $Y$ be complex one dimensional manifolds with spin structures $(S_X, \alpha_X)$ and $(S_Y, \alpha_Y)$ respectively. A spin embedding is a pair $(f, \theta)$ where $f : X \to Y$ is an embedding and $\theta : f^* S_Y \to S_X$ is a vector bundle isomorphism such that the following diagram commutes

\[
\begin{array}{ccc}
(f^* S_Y)^{\otimes 2} & \xrightarrow{\theta^{\otimes 2}} & S_X^{\otimes 2} \\
\downarrow \alpha_Y \circ (f^*)^{\otimes 2} & & \downarrow \alpha_X \\
T^{1,0} Y & \xrightarrow{f^*} & T^{1,0} X.
\end{array}
\]
CHAPTER 4

SPIN FACTORIZATION ALGEBRAS WITH POINT DEFECTS

We are now ready to define a more general notion of a factorization algebra which can be applied to any complex one dimensional manifold equipped with a finite set of points and a spin structure on the complement of the set of points.

To do this, we first need to look more closely at the spin structures on arbitrary one dimensional complex manifolds and relate them to the spin structures on annuli and discs in \( \mathbb{C} \). We know there are two spin structures on a punctured disc (an annulus), the Neveu-Schwarz spin structure which extends to the disc, and the Ramond spin structure which does not extend to the disc. We would like to extend this terminology to punctured neighborhoods of arbitrary complex one dimensional manifolds.

**Definition 4.0.2.** Let \( M \) be a complex one dimensional manifold, and let \( P \subseteq M \) be a finite set of points such that \( M \setminus P \) has a spin structure \( (S, \alpha) \). Let \( p \in M \) and let \( U \) be a neighborhood of \( p \) which is small enough that \( U \setminus \{ p \} \) contains no points in \( P \). Then we obtain a spin structure \( (S', \alpha') \) on \( U \setminus \{ p \} \). If \( (S', \alpha') \) extends over \( p \) to a spin structure on all of \( U \), call \( (S', \alpha') \) the Neveu-Schwarz spin structure on \( U \setminus \{ p \} \). If \( (S', \alpha') \) does not extend over \( p \) to a spin structure on all of \( U \), call \( (S', \alpha') \) the Ramond spin structure on \( U \setminus \{ p \} \).

**Remark 4.0.1.** Notice that if \( p \notin P, U \subseteq M \setminus P \), and \( (S', \alpha') \) will always extend over \( p \) to all of \( U \). Therefore, if \( p \notin P, U \setminus \{ p \} \) has the Neveu-Schwarz spin structure.

**Definition 4.0.3.** A spin manifold with point defects is a triple \( (M, P, (S, \alpha)) \) where \( M \) is a complex one dimensional manifold, \( P \subseteq M \) is a finite set of points, \( (S, \alpha) \) is
a spin structure on $M \setminus P$, and for each $p \in P$ and each neighborhood, $U$, of $p$ such that $U \setminus \{p\}$ contains no point of $P$, $(S, \alpha)$ restricts to the Ramond spin structure on $U \setminus \{p\}$.

**Remark 4.0.2.** Notice that if $P = \emptyset$, then $(S, \alpha)$ is a spin structure on $M$, and $(M, P, (S, \alpha))$ is just an ordinary complex one dimensional spin manifold.

**Remark 4.0.3.** Our reason for introducing point defects and defining them in this way is so we have a way to think of a disc with a Ramond spin structure. If $M = \mathbb{D}$, a disc centered at the origin, and $P = \{0\}$, $\mathbb{D}$ must have the Neveu-Schwarz spin structure, but the punctured disc, $\mathbb{D} \setminus \{0\}$ can have the Ramond spin structure. Thus, we think of $(\mathbb{D}, \{0\}, (S_R, \alpha_R))$ as a disc, but we associate the Ramond spin structure with it by only looking at a spin structure on the punctured disc.

To simplify notation, we will write the triple $(M, P, (S, \alpha))$ as $M^P_S$. We will not write the $P$ and/or $S$ if $P$ and $S$ are understood. Now we can define a category of spin manifolds with point defects.

**Definition 4.0.4.** The category $\text{Man}^P$, of spin manifolds with point defects, consists of

- **Objects:** Spin manifolds with point defects;
- **Morphisms from** $M^P_S$ **to** $N^Q_T$: Embeddings
  \[
  \phi : M \rightarrow N
  \]

such that

1. $\phi(P) \subseteq Q$
2. $\text{im}(\phi_{M \setminus P}) \subseteq N \setminus Q$
3. The map
   \[
   \phi_{M \setminus P} : M \setminus P \rightarrow N \setminus Q
   \]
   is a spin embedding.

Call such a map a *spin embedding with point defects.*
The category $\text{Man}^P$ is a symmetric monoidal category with product the disjoint union, where $M^P \sqcup N^Q = (M \sqcup N)^{P \sqcup Q}$. Let $\text{CoCh}$ be the category of co-chain complexes, which has

- Objects: Co-chain complexes of complex vector spaces;
- Morphisms from $N_\bullet$ to $Q_\bullet$: Co-chain maps $N_\bullet \to Q_\bullet$,

and notice that $\text{CoCh}$ has a symmetric monoidal structure under the tensor product of co-chain complexes. The following definitions are variations of Definition 3.0.2 in Chapter 6 of [3].

**Definition 4.0.5.** A spin prefactorization algebra with point defects is a symmetric monoidal functor

$$\mathcal{F} : \text{Man}^P \to \text{CoCh}.$$ 

This definition allows us to apply a spin prefactorization algebra with point defects to any spin manifold with point defects. Now, we would like to define a spin factorization algebra with point defects. We first need to extend the notion of a Weiss cover to spin manifolds with point defects.

**Definition 4.0.6.** Let $M^P_S$ be a spin manifold with point defects. A Weiss cover on $M^P_S$ is a collection, $\mathcal{U} = \{ \phi_i : U_i \to M \mid i \in I \}$, of spin embeddings with point defects such that for any finite collection, $\{x_1, \ldots, x_k\}$, of points in $M$, there is some $i \in I$ such that $\{x_1, \ldots, x_k\} \subseteq \phi_i(U_i)$.

Now, we can define a spin factorization algebra with point defects as follows.

**Definition 4.0.7.** A spin factorization algebra with point defects is a spin prefactorization algebra with point defects that is a homotopy co-sheaf in the Weiss topology.

This condition of being a homotopy co-sheaf in the Weiss topology ensures that what we get when we apply the spin factorization algebra with point defects to a spin
manifold with point defects, $M^P_S$, arises from applying the spin factorization algebra with point defects to a Weiss cover of $M^P_S$. 
CHAPTER 5

FREE BV THEORIES

We would like to obtain a spin factorization algebra with point defects from a classical field theory. The field theory we are interested in is a more specific type of field theory called a free BV theory. Our definition comes from [4].

Definition 5.0.8. A free BV theory on a manifold $M$ consists of the following data:

1. A finite rank $\mathbb{Z}$-graded super vector bundle, $E$, on $M$.
2. A vector bundle map, $\langle -, - \rangle_{\text{loc}} : E \otimes E \rightarrow \text{Dens}_M$ that is fiberwise non-degenerate, (super) anti-symmetric, and of cohomological degree $-1$ which induces a pairing on compactly supported sections of $E$,

\[
\langle -, - \rangle : \mathcal{E}_c \otimes \mathcal{E}_c \rightarrow \mathbb{C} \\
\quad s_1 \otimes s_2 \mapsto \int_{x \in M} \langle s_1(x), s_2(x) \rangle_{\text{loc}}
\]

3. A differential operator $Q : \mathcal{E} \rightarrow \mathcal{E}$ of cohomological degree 1 such that

(a) $(\mathcal{E}, Q)$ is an elliptic complex;

(b) $Q$ is (super) skew self-adjoint with respect to the pairing, i.e.,

\[
\langle s_1, Qs_2 \rangle = -(-1)^{|s_1|+p(s_1)p(Q)}\langle Qs_1, s_2 \rangle = -(-1)^{|s_1|}\langle Qs_1, s_2 \rangle.
\]

5.1 An Example

Let $M$ be a manifold with a spin structure $(S, \alpha)$. Then we have a free BV theory defined as follows:

1. Let $E = \Pi T^{1/2,0}(M) \oplus \Pi T^{1/2,1}(M)$ where $\Pi T^{1/2,0}(M) := \Pi S$ has degree 0, and $\Pi T^{1/2,1}(M) := \Pi (S \otimes T^{0,1}(M)) = \Pi S \otimes \Pi T^{1,0}(M)$ has degree 1.
(2) Define the pairing

\[ \langle , \rangle : \Pi \Omega^{1/2,*}(M) \otimes \Pi \Omega^{1/2,*}(M) \to \mathbb{C} \]

on \( \mathcal{E} = \Pi \Omega^{1/2,*}(M) \) where on homogeneous elements \( \alpha \in \Pi \Omega^{1/2,1}(M) \) and \( \gamma \in \Pi \Omega^{1/2,0}(M) \),

\[ \langle \alpha, \gamma \rangle = \int_M \alpha \wedge \gamma, \]

\[ \langle \gamma, \alpha \rangle := \langle \alpha, \gamma \rangle \], and if homogeneous elements \( \tau, \tau' \in \mathcal{E} \) have the same cohomological degree, then \( \langle \tau, \tau' \rangle = \langle \tau', \tau \rangle = 0 \). We define \( \langle , \rangle \) on other elements of \( \mathcal{E} \) by requiring that the pairing be bilinear. For \( \alpha_0, \gamma_0 \in \Pi \Omega^{1/2,0}(M) \) and \( \alpha_1, \gamma_1 \in \Pi \Omega^{1/2,1}(M) \), we have

\[ \langle \alpha_0 + \alpha_1, \gamma_0 + \gamma_1 \rangle = \langle \alpha_0 + \alpha_1, \gamma_0 \rangle + \langle \alpha_0 + \alpha_1, \gamma_1 \rangle \]
\[ = \langle \alpha_0, \gamma_0 \rangle + \langle \alpha_1, \gamma_0 \rangle + \langle \alpha_0, \gamma_1 \rangle + \langle \alpha_1, \gamma_1 \rangle \]
\[ = 0 + \int_{\Sigma} \alpha_1 \wedge \gamma_0 + \int_{\Sigma} \gamma_1 \wedge \alpha_0 + 0 \]
\[ = \int_{\Sigma} \alpha_1 \wedge \gamma_0 + \gamma_1 \wedge \alpha_0. \]

The pairing \( \langle , \rangle \) has cohomological degree \(-1\).

(3) To define the differential, \( Q \), first let \( U \subseteq M \) be an open subset of \( M \). Consider the coordinate chart \( z : U \to V \subseteq \mathbb{C} \) on \( M \). Then we can define

\[ \overline{\partial} : \Omega^{1,0}(U) \to \Omega^{1,1}(U) \]
\[ gz \quad \mapsto \quad (\overline{\partial} g)dz. \]

Notice that \( \overline{\partial} \) does not depend on our choice of chart. To see this, let \( f : U \to V \subseteq \mathbb{C} \) be a smooth chart on \( M \). Then \( f \) must be a holomorphic function, and

\[ df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \]
\[ = \frac{\partial f}{\partial \overline{z}} dz. \]  \hspace{1cm} (5.1)

Now define

\[ \overline{\partial} : \Omega^{1,0}(U) \to \Omega^{1,1}(U) \]
\[ gdf \quad \mapsto \quad (\overline{\partial} g)df. \]

Since

27
\[ \partial(gdf) = \partial \left( g \frac{\partial f}{\partial z} dz \right) \]
\[ = \partial \left( g \frac{\partial f}{\partial z} \right) dz \]
\[ = (\partial g) \frac{\partial f}{\partial z} dz + g \partial \left( \frac{\partial f}{\partial z} \right) dz \]
\[ = (\partial g) df \]
\[ = \partial'(gdf), \]

we have \( \partial = \partial' \), and therefore, \( \partial \) does not depend on our choice of chart. Thus, we now have \( \partial \) defined for any \( U \subseteq M \). We can use partitions of unity to extend the sections in \( \Omega^{1,0}(U) \) to sections of \( \Omega^{1,0}(M) \). Thus, since any section of \( \Omega^{1,0}(M) \) can be written as a sum of sections of \( \Omega^{1,0}(U) \), we get a definition for \( \partial \) on all of \( \Omega^{1,0}(M) \). A similar argument allows us to define \( Q = \partial \) where

\[ \partial : \Pi \Omega^{1/2,0}(U) \to \Pi \Omega^{1/2,1}(U) \]
\[ f(dz)^{1/2} \mapsto (\partial f)(dz)^{1/2} \]

using the coordinate section \( dz \) on \( T^{1,0}(U) \). These maps can also be extended to define

\[ \partial : \Pi \Omega^{1/2,0}(M) \to \Pi \Omega^{1/2,1}(M). \]

We show \( \partial \) is skew self-adjoint as follows. We need to show for \( \alpha, \gamma \in \mathcal{E}_c \),

\[ \langle \alpha, \partial \gamma \rangle = -(-1)^{|\alpha|} \langle \partial \alpha, \gamma \rangle. \]

Notice that the only interesting cases occur when \( \alpha, \gamma \in \Pi \Omega^{1/2,0}_c(M) \) since otherwise we get zero. We need to show \( \langle \alpha, \partial \gamma \rangle + \langle \partial \alpha, \gamma \rangle = 0 \). Notice that locally, on an open subset, \( U \subseteq M \), we can write \( \alpha = f(dz)^{1/2} \) and \( \gamma = g(dz)^{1/2} \). On \( U \), we have

\[ \langle \alpha, \partial \gamma \rangle + \langle \partial \alpha, \gamma \rangle = \langle \partial \gamma, \alpha \rangle + \langle \partial \alpha, \gamma \rangle \]
\[ = \int_U \partial \gamma \wedge \alpha + \partial \alpha \wedge \gamma \]
\[ = \int_U \partial(g(dz)^{1/2}) \wedge f(dz)^{1/2} + \partial(f(dz)^{1/2}) \wedge g(dz)^{1/2} \]

28
\[
= \int_U \frac{\partial g}{\partial z} f d\bar{z} dz + \frac{\partial f}{\partial z} g d\bar{z} dz \\
= \int_U d(f g dz) \\
= \int_{\partial U} f g dz \\
= 0
\]

(5.2)

(5.3)

where in the first line, we used that the pairing is (super) skew-symmetric, in (5.2), we used that

\[
d(f g dz) = \frac{\partial}{\partial z} (f g dz) + \frac{\partial}{\partial \bar{z}} (f g dz) = \frac{\partial f}{\partial z} g d\bar{z} dz + f \frac{\partial g}{\partial z} d\bar{z} dz,
\]

and in (5.3), we used that \(U\) is open.

Given an open cover of \(M\), we can use a partition of unity to write global forms \(\alpha\) and \(\gamma\) on \(M\) as sums of forms supported on open subsets, \(U\), of \(M\). Thus, the local result we just proved gives the global result on \(M\), and \(\bar{\partial}\) is skew self-adjoint.

Notice that the parity reversal of \(\Pi \Omega^{1/2,*}(M)\) gave us the (super) skew symmetry as

\[
\langle \alpha, \gamma \rangle = \langle \gamma, \alpha \rangle.
\]

If we had not given elements of \(\Omega^{1/2,*}(M)\) odd parity, the skew symmetry condition would be \(\langle \alpha, \gamma \rangle = -\langle \gamma, \alpha \rangle\) while the skew self adjoint condition on \(\bar{\partial}\) would have remained the same. It would be impossible to satisfy both conditions. This is why we need the parity reversal.

This gives the free BV theory determined by the elliptic complex

\[
(\mathcal{E}, Q) = (\Pi \Omega^{1/2,*}(M), \bar{\partial}).
\]

Notice that we can define a map

\[
\mathcal{E}_c[1] \to \mathcal{E}^\vee \\
\alpha \mapsto (\beta \mapsto \langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta)
\]

where \(\mathcal{E}_c[1]\) is just the complex \(\mathcal{E}_c\) shifted by 1, so that \(\mathcal{E}_i = \mathcal{E}[1]_{i-1}\). This allows us to view elements of \(\mathcal{E}_c[1]\) as linear observables of \(\mathcal{E}\).
5.2 A Factorization Algebra

Following [4], we can define the factorization algebra, $\text{Obs}^q$, of quantum observables of the free BV theory $(\mathcal{E}, Q)$ on a manifold $M$ as follows.

For each open $U \subseteq M$,

$$\text{Obs}^q(U) := (\text{Sym}(\mathcal{E}_c[1](U))[\hbar], Q + \hbar \Delta)$$

where $\Delta$ is the BV Laplacian, which is defined as follows. If $\alpha \in \text{Sym}^0(\mathcal{E}_c[1](U))$ or $\alpha \in \text{Sym}^1(\mathcal{E}_c[1](U))$, then $\Delta(\alpha) = 0$. For $\alpha_1 \otimes \alpha_2 \in \text{Sym}^2(\mathcal{E}_c[1](U))$,

$$\Delta(\alpha_1 \otimes \alpha_2) = \langle \alpha_1, \alpha_2 \rangle.$$

To define $\Delta$ on the rest of $\text{Sym}(\mathcal{E}_c[1](U))$, we use the relation

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|p(x)} x(\Delta y) + \langle x, y \rangle.$$

It turns out that $\text{Obs}^q$ is in fact a factorization algebra. For a proof, see Section 5.3 of [4].

Let us look at the factorization algebra we get from the free BV theory in the example from Section 5.1. We have

- On open subsets $U \subseteq M$,

$$\text{Obs}^q(U) = (\text{Sym}(\mathcal{E}_c[1](U))[\hbar], Q + \hbar \Delta) = (\text{Sym}(\Pi\Omega^{1/2,*}_c(M)[1])[\hbar], \mathcal{D} + \hbar \Delta)$$

- For disjoint open subsets $U_1, \ldots, U_n$ of $M$ contained in a larger open subset, $V$, of $M$, the structure map

$$m^{U_1,\ldots,U_n}_V : \text{Obs}^q(U_1) \otimes \cdots \otimes \text{Obs}^q(U_n) \rightarrow \text{Obs}^q(V)$$

takes compactly supported sections on $U_1, \ldots, U_n$ to a compactly supported section on $V$ by extending the sections on $U_1, \ldots, U_n$ by 0 to all of $V$. 
If we let $M = \mathbb{C}$, this factorization algebra allows us to describe a vertex algebra as shown in [3]. However, if we use the factorization algebra we’ve just constructed, we’re missing a lot of data. Since we’re looking at a factorization algebra on $\mathbb{C}$, which only has one spin structure, all of the open subsets of $\mathbb{C}$ to which we apply our factorization algebra have the same spin structure, given by restricting the spin structure on $\mathbb{C}$. We would like to be able to apply our factorization algebra to manifolds with different spin structures, and in particular, to spin manifolds with point defects. Clearly, we can apply $\text{Obs}^0(U)$ to any complex one dimensional spin manifold, since we can apply $\Omega^{1/2,*}(M)$ to any complex one dimensional spin manifold. However, we must specify how we will define $\Omega^{1/2,*}(M)$ if $M = M^P$, a spin manifold with point defects.
CHAPTER 6

DIFFERENTIAL FORMS ON SPIN MANIFOLDS WITH POINT DEFECTS

Let $M$ be a complex manifold with spin structure $(S, \alpha)$. Then

$$\Omega^{1/2,k}(M) := \Gamma(S \otimes T^{0,k}(M)).$$

We must now extend this to complex spin manifolds with point defects, that is, for a spin manifold with point defects, $(M, P, (S, \alpha))$, we want to define

$$\Omega^{1/2,0}(M^P_S).$$

We begin by defining

$$\Omega^{1/2,0}(M^P_S)$$

since the point defects only effect the spin structure. We would like to define $\Omega^{1/2,0}(M^P_S)$ using the spin structure $(S, \alpha)$, and we would like it to be $\Gamma(S)$. However, the spin structure $(S, \alpha)$ is only a spin structure on $M \setminus P$ and need not extend to all of $M$, so sections in $\Gamma(S)$ only need to be supported on $M \setminus P$. We will define $\Omega^{1/2,0}(M^P_S)$ to be the subspace of $\Omega^{1/2,0}(M \setminus P) = \Gamma(S)$ consisting of the sections which, in a way we will define shortly, extend to all of $M$. To determine what this subspace should be, we first look at a simpler case, where $M$ is a disc.
6.1 The Disk

Consider the complex spin manifold with point defects, $\mathbb{D}_S^{(0)}$, where $S$ denotes the Ramond spin structure. Define $\Omega^{1/2,0}(\mathbb{D}_S^{(0)})$ to be the subset, $X$, of $\Omega^{1/2,0}(\mathbb{D} \setminus \{0\}) = \Gamma(S)$ defined as follows. Notice that we can write elements of $\Omega^{1/2,0}(\mathbb{D} \setminus \{0\})$ as $f\omega$ where $f$ is a smooth function on $\mathbb{D} \setminus \{0\}$ and $\omega = z^{1/2}(dz)^{1/2}$. We define $X$ to be the subset of $\Omega^{1/2,0}(\mathbb{D} \setminus \{0\})$ consisting of elements $f\omega$ where the function $f$ extends over the puncture at the origin to all of $\mathbb{D}$.

6.2 In General

Let $M_P^S$ be a complex spin manifold with point defects. We assume $P$ is non-empty since if $P = \emptyset$, $M$ has no point defects, and we already know how to define $\Omega^{1/2,0}(M)$. Let $p \in P$, and choose a neighborhood, $U$, of $p$ which contains no other point of $P$. We obtain a spin structure, $S'$, on $U \setminus \{p\}$ by restricting $S$, and $S'$ is the Ramond spin structure. Then $U_S^{(p)}$ is a complex spin manifold with point defects. Let $\mathbb{D}$ be the complex unit disk centered at the origin, and let $T$ be the Ramond spin structure on $\mathbb{D} \setminus \{0\}$. Then $\mathbb{D}_T^{(0)}$ is a complex spin manifold with point defects.

Let $\phi : U_S^{(p)} \rightarrow \mathbb{D}_T^{(0)}$ be a smooth chart on $U$ such that $\phi(p) = 0$, and $\phi|_{U \setminus \{p\}} : U \setminus \{p\} \rightarrow \mathbb{D} \setminus \{0\}$ is a spin embedding. Note that we may need to use a smaller neighborhood of $p$ to get such a chart, but abusing notation, we call this neighborhood $U$ as well. Now, pull the map, $\phi$, back to get

$$\phi^* : \Omega^{1/2,0}(\mathbb{D} \setminus \{0\}) \rightarrow \Omega^{1/2,0}(U \setminus \{p\}).$$

Then, we define

$$\Omega^{1/2,0}(U_S^{(p)}) := \phi^*(\Omega^{1/2,0}(\mathbb{D}_T^{(0)})).$$

To be sure this definition doesn’t depend on our choice of smooth chart, $\phi$, we
must check that by choosing a different smooth chart, we still land in the appropriate subspace.

Notice that if we choose a different chart, it can be obtained by composing \( \phi \) with a diffeomorphism, \( f \), of \( \mathbb{D} \) which fixes the origin and is a spin embedding on \( \mathbb{D} \setminus \{0\} \).

Since \( f \) may only be a diffeomorphism on subspaces of \( \mathbb{D} \), we write \( f : (\mathbb{D}_1)^{0}_T \to (\mathbb{D}_2)^{0}_T \), where \( \mathbb{D}_1 \) and \( \mathbb{D}_2 \) are subspaces of \( \mathbb{D} \) and we’ve restricted the spin structure, \( T \), to \( \mathbb{D}_1 \setminus \{0\} \) and \( \mathbb{D}_2 \setminus \{0\} \).

We obtain the pullback of the map \( f \),

\[
f^* : \Omega^{1/2,0}(\mathbb{D}_2 \setminus \{0\}, T) \to \Omega^{1/2,0}(\mathbb{D}_1 \setminus \{0\}, T).
\]

We need to show the image of the map \( f^* \) restricted to \( \Omega^{1/2,0}(\mathbb{D}_2 \setminus \{0\}, T) \) is contained in \( \Omega^{1/2,0}(\mathbb{D}_1 \setminus \{0\}, T) \). Notice that since \( f \) is a transition function for a complex manifold, \( f \) is holomorphic, so \( f = \sum_{i=1}^{\infty} a_i z^i \), where the indexing starts at 1 because \( f(0) = 0 \).

Let \( g z^{1/2} (dz)^{1/2} \in \Omega^{1/2,0}(\mathbb{D}_2)^{0}_T \). Then

\[
f^* (g z^{1/2} (dz)^{1/2}) = (f^* g)(f^* z^{1/2})(f^* (dz)^{1/2})
\]

\[
= (g \circ f) \left( z^{1/2} \sum_{i=1}^{\infty} a_i z^i \right) \left( \sum_{i=1}^{\infty} i a_i z^{i-1} \right)^{1/2} (dz)^{1/2}
\]

\[
= (g \circ f) \left( \sum_{i=1}^{\infty} a_i z^i \right)^{1/2} \left( \sum_{i=1}^{\infty} i a_i z^{i-1} \right)^{1/2} (dz)^{1/2}
\]

\[
= (g \circ f) \left( z \sum_{i=1}^{\infty} a_i z^{i-1} \right)^{1/2} \left( \sum_{i=1}^{\infty} i a_i z^{i-1} \right)^{1/2} (dz)^{1/2}
\]

\[
= (g \circ f) \left( \sum_{i=1}^{\infty} a_i z^{i-1} \right)^{1/2} \left( \sum_{i=1}^{\infty} i a_i z^{i-1} \right)^{1/2} z^{1/2} (dz)^{1/2}.
\]

Since \( f \) is holomorphic on \( \mathbb{D}_1 \), and \( g \) extends to the origin, \( g \circ f \) does as well. Since both sums have no negative powers of \( z \), both extend over the origin. We only need to show that we can take the square roots of these sums. The first sum, \( \sum_{i=1}^{\infty} a_i z_{i-1} \), has
constant term $a_1$. Since $f$ is invertible, $a_1 \neq 0$, and therefore, for $z$ near 0, $\sum_{i=1}^{\infty} a_i z^{i-1}$ is in a neighborhood (not containing 0) of $a_1$. Thus, we can choose a branch cut away from this neighborhood, and $(\sum_{i=1}^{\infty} a_i z^{i-1})^{1/2}$ is defined and extends over the origin. A similar argument gives us that we can take the square root of the second term as well, and it also extends over the origin. Therefore, $f^*(gz^{1/2}(dz)^{1/2}) \in \Omega^{1/2,0}((\mathbb{D}_1)^{(0)})$.

Now we can define $\Omega^{1/2,0}(M^P_S)$.

**Definition 6.2.1.** Let $M^P_S$ be a complex spin manifold with point defects. For each $p \in P$, choose a small neighborhood, $U_p$ of $p$ which contains no other point in $P$. We define

$$\Omega^{1/2,0}(M^P_S) \subseteq \Omega^{1/2,0}(M \setminus P)$$

as follows. Let $\sigma \in \Omega^{1/2,0}(M \setminus P) = \Gamma(M \setminus P, S)$. Notice that $\sigma|_{U_p \setminus \{p\}} \in \Gamma(U_p \setminus \{p\}, S_p) = \Omega^{1/2,0}(U_p \setminus \{p\})$ where $S_p$ is the spin structure on $U_p \setminus \{p\}$ obtained by restricting $S$ to $U_p \setminus \{p\}$. We say $\sigma \in \Omega^{1/2,0}(M^P_S)$ if for each $p \in P$, $\sigma|_{U_p \setminus \{p\}} \in \Omega^{1/2,0}((U_p)_{S_p}^{\{p\}})$.

We would like to define $\Omega^{1/2,1}(M^P_S)$ as a subspace of

$$\Omega^{1/2,1}(M \setminus P) = \Gamma(S \otimes T^{0,1}(M \setminus P)).$$

If we let $M^P_S = \mathbb{D}^{(0)}_S$ where $S$ is the Ramond spin structure, we see that elements of $\Omega^{1/2,1}(\mathbb{D} \setminus \{0\})$ must be of the form $f \omega$ where $f$ is a smooth function on $\mathbb{D} \setminus \{0\}$ and $\omega = z^{1/2}dz(dz)^{1/2}$. Then, as in the previous case, we let $\Omega^{1/2,1}(\mathbb{D}^{(0)}_S)$ consist of the elements of $\Omega^{1/2,1}(\mathbb{D} \setminus \{0\})$ such that $f$ extends to all of $\mathbb{D}$. An argument similar to that given above allows us to define $\Omega^{1/2,1}(M^P_S)$. 

35
CHAPTER 7

THE SPIN FACTORIZATION ALGEBRA WITH POINT DEFECTS

Now, we will define the factorization algebra with point defects as follows.

- For a complex one dimensional manifold with point defects, $M^P$,

  $$\text{Obs}^q(M^P) := (\text{Sym}(\Pi\Omega^{1/2,*}(M^P)[1])[\hbar], \bar{\partial} + \hbar\Delta);$$

- For complex one dimensional spin manifolds $U_1^P, \ldots, U_n^P, V^P$ and a spin embedding

  $$U_1^P \sqcup \cdots \sqcup U_n^P \to V,$$

the structure map

$$m_{U_1^P, \ldots, U_n^P} : \text{Obs}^q(U_1) \otimes \cdots \otimes \text{Obs}^q(U_n) \to \text{Obs}^q(V)$$

is given by using the spin embedding to produce a compactly supported form on $V$ given by extending to all of $V$ by zero.

One can prove this is in fact a factorization algebra by making appropriate modifications to the proof of Theorem 5.2.1 in Chapter 6 of [3].
CHAPTER 8

THE COHOMOLOGY PREFACTORIZATION ALGEBRA

Given the factorization algebra of quantum observables of $\mathcal{E}$ we defined above, we can obtain a prefactorization algebra $H^*(\text{Obs}^q)$ by taking cohomology. Given a spin manifold with point defects, $M^P$, the prefactorization algebra $H^*(\text{Obs}^q)$ assigns the co-chain complex

$$H^*(\text{Obs}^q(M^P)) := H^*(\text{Sym}(\Pi\Omega^{1/2,*}(M^P)[1])[h, \bar{\partial} + h\Delta]).$$

The structure maps of $H^*(\text{Obs}^q)$ are just the induced maps in cohomology of the structure maps of $\text{Obs}^q$.

The prefactorization algebra $H^*(\text{Obs}^q)$ is the one we will be working with, so we will first discuss the spaces $H^*(\text{Obs}^q(M^P))$ explicitly. We’ll need the following propositions.

**Proposition 8.0.1.** For a spin manifold $M \subseteq \mathbb{C}$,

$$H^k(\Pi\Omega^{1/2,*}_c(M)[1], \bar{\partial}) \cong \begin{cases} (\Pi\Omega^{1/2,0}_\text{hol}(M))^\vee & k = 0 \\ 0 & k \neq 0. \end{cases}$$

**Proof.** Note that the complex $\mathcal{E}_c[1](M)$ is chain homotopy equivalent to the complex $\mathcal{E}^\vee(M)$ (see Lemma 5.2.13 and the proof of Lemma 5.3.7 of [4]). Thus,

$$H^k(\Pi\Omega^{1/2,*}_c(M)[1], \bar{\partial}) = H^k(\mathcal{E}_c[1](M), \bar{\partial}) \cong H^k(\mathcal{E}^\vee(M), \bar{\partial})$$

37
\[ \cong (H^k(\mathcal{E}(M), \overline{\partial}))^\vee \]
\[ = (H^k(\Pi\Omega^{1/2,*}(M), \overline{\partial}))^\vee. \]

Now, we will compute
\[ H^k(\Pi\Omega^{1/2,*}(M), \overline{\partial}) = H^k \left( \Pi\Omega^{1/2,0}(M) \xrightarrow{\overline{\partial}} \Pi\Omega^{1/2,1}(M) \right). \]

Let \( f_\gamma \in \Pi\Omega^{1/2,0}(M) \) where \( \gamma = (dz)^{1/2} \) if the spin structure on \( M \) is Neveu-Schwarz, and \( \gamma = z^{1/2}(dz)^{1/2} \) if the spin structure on \( M \) is Ramond. Then \( \overline{\partial}(f_\gamma) = \frac{df}{dz}dz\gamma \). This means \( \ker(\overline{\partial}) = \Pi\Omega^{1/2,0}_{\text{hol}}(M) \) and \( \text{im}(\overline{\partial}) = \Pi\Omega^{1/2,1}(M) \). Thus,

\[ H^0(\Pi\Omega^{1/2,*}(M), \overline{\partial}) \cong \ker(\overline{\partial})/0 = \Pi\Omega^{1/2,0}_{\text{hol}}(M) \]
\[ H^1(\Pi\Omega^{1/2,*}(M), \overline{\partial}) \cong \Pi\Omega^{1/2,1}(M)/\text{im}(\overline{\partial}) = \Pi\Omega^{1/2,1}(M)/\Pi\Omega^{1/2,1}(M) = 0. \]

Clearly, we get zero for all other \( k \), proving the proposition. \( \square \)

Proposition 8.0.1 shows that we have an isomorphism,

\[ \Phi_M : H^0(\Pi\Omega^{1/2,*}_c(M)[1], \overline{\partial}) \rightarrow (\Pi\Omega^{1/2,0}_{\text{hol}}(M))^\vee. \]

If \( M \) is a complex annulus centered at the origin with the Ramond spin structure, \( \Phi_M \) is defined as follows.

\[ \Phi_M : H^0(\Pi\Omega^{1/2,*}_c(M)[1], \overline{\partial}) \rightarrow (\Pi\Omega^{1/2,0}_{\text{hol}}(M))^\vee \]
\[ [fz^{1/2}d\bar{z}(dz)^{1/2}] \quad \mapsto \quad \Pi\Omega^{1/2,0}_{\text{hol}}(M) \quad \rightarrow \quad \mathbb{C} \]
\[ gz^{1/2}(dz)^{1/2} \quad \mapsto \quad \int_M fz^{1/2}d\bar{z}(dz)^{1/2} \wedge gz^{1/2}(dz)^{1/2} \quad (8.1) \]

**Proposition 8.0.2.** For \( \mathbb{D}_S^{(0)} \), where \( \mathbb{D} \subset \mathbb{C} \) is a disk centered at the origin and \( S \)}

38
denotes the Ramond spin structure on \( \mathbb{D} \setminus \{0\} \),

\[
H^k(\Pi\Omega^{1/2,*}_{c}(\mathbb{D}_S^{(0)})[1], \overline{\partial}) \cong \begin{cases} 
(\Pi\Omega^{1/2,0}_{\text{hol}}(\mathbb{D}_S^{(0)}))^\vee & k = 0 \\
0 & k \neq 0.
\end{cases}
\]

**Proof.** Recall that \( \Pi\Omega^{1/2,*}(\mathbb{D}_S^{(0)}) \subset \Pi\Omega^{1/2,*}(\mathbb{D} \setminus \{0\}) \). Where \( f(d\bar{z})(dz)^{1/2} \in \Pi\Omega^{1/2,i}(\mathbb{D} \setminus \{0\}) \) is in \( \Pi\Omega^{1/2,i}(\mathbb{D}_S^{(0)}) \) if \( f \) extends over the origin. Notice that Proposition 8.0.1 shows that

\[
H^k(\Pi\Omega^{1/2,*}_{c}(\mathbb{D} \setminus \{0\}), \overline{\partial}) \cong \begin{cases} 
(\Pi\Omega_{\text{hol}}(\mathbb{D} \setminus \{0\}))^\vee & k = 0 \\
0 & k \neq 0.
\end{cases}
\]

Now, if we restrict to \([fd\bar{z}(dz)^{1/2}] \in H^0(\Pi\Omega^{1/2,*}_{c}(\mathbb{D} \setminus \{0\}), \overline{\partial})\) where \( fd\bar{z}(dz)^{1/2} \in \Pi\Omega^{1/2,*}_{c}(\mathbb{D}_S^{(0)}) \), that is, \( f \) extends over the origin, the image of the restricted \( \Phi_{\mathbb{D} \setminus \{0\}} \) becomes \((\Pi\Omega^{1/2,0}_{\text{hol}}(\mathbb{D}_S^{(0)}))^\vee\), proving the proposition. \( \square \)

Notice that if \( M \) is a complex disc or annulus with the Neveu-Schwarz spin structure, we can define

\[
\Phi_M : H^0(\Pi\Omega^{1/2,*}_{c}(M)[1], \overline{\partial}) \to (\Pi\Omega^{1/2,0}_{\text{hol}}(M))^\vee \\
[fd\bar{z}(dz)^{1/2}] \mapsto \Pi\Omega^{1/2,0}_{\text{hol}}(M) \to \mathbb{C} \\
g(dz)^{1/2} \mapsto \int_M fd\bar{z}(dz)^{1/2} \wedge g(dz)^{1/2}.
\]

(8.2)
CHAPTER 9

A CIRCLE ACTION

The last ingredient we will need before describing the vertex algebra arising from our spin factorization algebra with point defects is an $S^1$ action on $\Omega^{1/2,1}(\mathbb{C}^\times)$. Choose a spin structure, $(S, \alpha)$, on $\mathbb{C}^\times$. Let $w \in S^1$, and write $w$ as a function $w : \mathbb{C}^\times \to \mathbb{C}^\times$ where $w(z) = wz$ for all $z \in \mathbb{C}^\times$. Consider the pullback $W^* : \Omega^{1/2,1}(\mathbb{C}^\times, S) \to \Omega^{1/2,1}(\mathbb{C}^\times, S)$. Since $dz = ((dz)^{1/2})^2$,

$$w^*((dz)^{1/2}) = (w^*(dz))^{1/2}$$
$$= (dw)^{1/2}$$
$$= \left( \frac{dw}{dz}dz + \frac{dw}{d\bar{z}}d\bar{z} \right)^{1/2}$$
$$= (wdz + 0)^{1/2}$$
$$= w^{1/2}(dz)^{1/2}.$$  

If $(S, \alpha) = (S_N, \alpha_N)$, we have

$$w^*(fd\bar{z}(dz)^{1/2}) = (w^*(f^*d\bar{z}))(w^*(dz)^{1/2})$$
$$= (f \circ w)(d(w^*\bar{z}))w^{1/2}(dz)^{1/2}$$
$$= f(wz)d(wz)w^{1/2}(dz)^{1/2}$$
$$= f(wz) \left( \frac{d(wz)}{dz}dz + \frac{d(wz)}{d\bar{z}}d\bar{z} \right)w^{1/2}(dz)^{1/2}$$
$$= f(wz)(0 + w\bar{z}d\bar{z})w^{1/2}(dz)^{1/2}$$
$$= f(wz)w^{-1}d\bar{z}w^{1/2}(dz)^{1/2}.$$
Notice that this doesn’t quite give us an $S^1$ action, since we have to choose a square root of $w$. However, we do have a right $D$ action on $\Omega^{1/2,1}(M)$, where

$$D = \{(w,u) \in S^1 \times S^1 \mid w = u^2\}$$

is a double cover on $S^1$ analogous to the double cover on $\mathbb{C}^\times$ we defined previously in (3.1). The right $D$-action is given by

$$\Omega^{1/2,1}(\mathbb{C}^\times) \times D \rightarrow \Omega^{1/2,1}(\mathbb{C}^\times)$$

$$(f \, dz(dz)^{1/2}) \cdot (w, w^{1/2}) \mapsto w^*(f \, dz(dz)^{1/2}) = f(wz)w^{-1/2}d\overline{z}(dz)^{1/2}.$$ 

This action allows us to write $\Omega^{1/2,1}(\mathbb{C}^\times)$ as a half-integer graded vector space,

$$\Omega^{1/2,1}(\mathbb{C}^\times) = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} K_n$$

where $K_n$ consists of elements of $\Omega^{1/2,1}(\mathbb{C}^\times)$ on which $(w, w^{1/2}) \in D$ acts by multiplication by $w^n$.

If $(S, \alpha) = (S_R, \alpha_R)$, we have

$$w^*(f \, z^{1/2}d\overline{z}(dz)^{1/2}) = (w^* f)(w^* z^{1/2})(w^*(d\overline{z}))(w^*(dz)^{1/2})$$

$$= f(wz)(wz)^{1/2}w^{-1}d\overline{z}w^{1/2}(dz)^{1/2}$$

$$= f(wz)z^{1/2}d\overline{z}(dz)^{1/2},$$
giving a right $S^1$ action

\[ \Omega^{1/2,1}(\mathbb{C}^\times) \times S^1 \to \Omega^{1/2,1}(S) \]

\[ (f z^{1/2} d\bar{z}(dz)^{1/2}) \cdot w \to w^* (f z^{1/2} d\bar{z}(dz)^{1/2}) = f(wz) z^{1/2} d\bar{z}(dz)^{1/2}. \]

This $S^1$ action allows us to write $\Omega^{1/2,1}(\mathbb{C}^\times)$ as a $\mathbb{Z}$-graded vector space,

\[ \Omega^{1/2,1}(\mathbb{C}^\times) = \bigoplus_{n \in \mathbb{Z}} K_n \]

where $K_n$ consists of elements of $\Omega^{1/2,1}(\mathbb{C}^\times)$ on which $w \in S^1$ acts by multiplication by $w^n$. 
In the next few chapters, we will investigate the structure maps of our spin factorization algebra with point defects on the simplest observables. In doing so, we roughly follow Section 6.2 of [4].

Our first task in obtaining a vertex algebra from our cohomology spin prefactorization algebra with point defects, $H^*(\text{Obs}^q)$, will be to define a multiplication on a dense subspace of $H^*(\text{Obs}^q(\mathbb{A}))$, where $\mathbb{A}$ is an annulus with the Neveu-Schwarz spin structure. We will do this by investigating what the structure maps of $H^*(\text{Obs}^q)$ do on the simplest observables.

**Definition 10.0.2.** Consider the spin manifold with point defects, $\mathbb{A}_N^\emptyset(x)$, where $\mathbb{A}$ is a complex annulus centered at $x \in \mathbb{C}$, and the $N$ means that $\mathbb{A} \setminus \emptyset$ has the Neveu-Schwarz spin structure. For each $n \in \mathbb{Z} + \frac{1}{2}$, define the simplest observable, $b_n(x) \in (\Omega_{hol}^{1/2,0}(\mathbb{A}_N^\emptyset(x)))' \subseteq H^*(\text{Obs}^q(\mathbb{A}_N^\emptyset(x)))$, to be the linear functional

$$b_n(x) : \Omega_{hol}^{1/2,0}(\mathbb{A}_N^\emptyset(x)) \rightarrow \mathbb{C}$$

$$\beta(dz)^{1/2} \mapsto (-n - \frac{1}{2}) \text{Laurent coefficient of } \beta$$

where we’re looking at the Laurent expansion of $\beta$ about $x$. For simplicity of notation, set $b_n := b_n(0)$.

**Remark 10.0.1.** One might ask why we define the $b_n(x)$’s to be the $(-n - \frac{1}{2})$ Laurent coefficient rather than some other Laurent coefficient. We’ll see shortly that the choice we’ve made results in indexing that agrees with that typically used for vertex algebras,
and this indexing arises naturally from the $S^1$ action we defined on differential forms earlier.

Remark 10.0.2. For the rest of this section, all annuli will have the Neveu-Schwarz spin structure and will be of the form $A_0^0$, that is, the annulus is equipped with no point defects and $A \setminus \emptyset$ has the Neveu-Schwarz spin structure. Therefore, to simplify notation, such an annulus will be denoted by $A$ without the decorations.

10.1 A Co-Cycle for $b_n$

For each $n$, the observable

$$b_n \in (\Omega^{1/2,0}_{\text{hol}}(A))^\vee \cong H^*(\Pi \Omega^{1/2,*}(A)[1], \overline{\partial})$$

$$= H^*(\text{Sym}^1(\Pi \Omega^{1/2,*}(A)[1]), \overline{\partial})$$

$$\subseteq H^*(\text{Sym}(\Pi \Omega^{1/2,*}(A)[1]), \overline{\partial} + h\Delta)$$

$$= H^*(\text{Obs}^q(A)),$$

where $A$ is an annulus centered at the origin with inner radius $r$ and outer radius $R$, is a cohomology class, so we would like to come up with a co-cycle representative for $b_n$. To do so, we first choose a bump function, $\phi(\rho^2)$, where $\rho$ is the radius, $\rho = \sqrt{z\overline{z}}$, such that

1. $\int_0^\infty \phi(\rho^2)\rho d\rho = 1$, and
2. $\phi(\rho^2) \geq 0$ with $\phi$ supported on $\rho \in (r, R)$.

We will use this bump function to define our co-cycle,

$$\tilde{b}_n = -\phi(\rho^2)z^{n+\frac{1}{2}} \frac{d\overline{z}(dz)^{\frac{1}{2}}}{4\pi i}.$$

To show $\tilde{b}_n$ is in fact a co-cycle representative, we will need the following lemma.
Lemma 10.1.1. If $X$ is a complex annulus centered at the origin, $n \in \mathbb{Z}$, and $\phi(\rho^2)$ is a bump function as defined above,

$$\int_X \phi(\rho^2) z^n d\bar{z} \frac{dz}{4\pi i} = \delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker delta function.

Proof. Assume the annulus $X$ has inner radius $r$ and outer radius $R$. Then

$$\int_X \phi(\rho^2) z^n d\bar{z} \frac{dz}{4\pi i} = \int_r^R \int_0^{2\pi} \phi(\rho^2) \rho^n e^{i\theta n} 2\rho i d\theta d\rho \frac{1}{4\pi i}$$

$$= \delta_{n,0} \int_r^R \int_0^{2\pi} \phi(\rho^2) \rho d\theta d\rho \frac{1}{2\pi}$$

$$= \delta_{n,0} \int_r^R \phi(\rho^2) \rho d\rho$$

$$= \delta_{n,0}.$$

Remark 10.1.1. The argument used in Lemma 10.1.1 also works if $X$ is a complex disc centered at the origin.

Now, we’re ready to show $\tilde{b}_n$ is a co-cycle representative.

Proposition 10.1.1. For $n \in \mathbb{Z} + \frac{1}{2}$,

$$\tilde{b}_n = -\phi(\rho^2) z^{n+\frac{1}{2}} \frac{d\bar{z}(dz)^{\frac{1}{2}}}{4\pi i}$$

is a co-cycle representative for $b_n(x) \in H^*(\text{Obs}^8(A))$.

Proof. We will use the isomorphism, $\Phi_A$, defined in Section 8 to send $[\tilde{b}_n]$ to an element of $(\Omega_{hod}^{1/2,0}(A))^\vee$ and show this element sends a holomorphic $\frac{1}{2}$ form $\beta(dz)^{\frac{1}{2}}$ to the $-n - \frac{1}{2}$ Laurent coefficient of $\beta$, that is, we will show $\Phi_A([\tilde{b}_n]) = b_n$. We have
\((\Phi_A(\tilde{b}_n))(z^k(dz)^{1/2}) = \int_{A} \tilde{b}_n \wedge (z^k(dz)^{1/2}) \)

\[
= \int_{A} \left(-\phi(\rho^2)z^{n+1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i}\right) \wedge (z^k(dz)^{1/2}) \\
= \int_{A} \phi(\rho^2)z^{k+n+1/2} \frac{dz d\bar{z}}{4\pi i} \\
= \delta_{k+n+\frac{1}{2},0} = \delta_{k,-n-\frac{1}{2}}
\]

where for the last line, we used Lemma 10.1.1. Thus, applying \(\Phi_A(\tilde{b}_n)\) to \(\beta(dz)^{1/2} \in \Omega^{1/2,0}(\mathbb{A})\) gives the \(-n - \frac{1}{2}\) Laurent coefficient of \(\beta\), and therefore, \(\Phi_A(\tilde{b}_n) = b_n\), and \(\tilde{b}_n\) is a co-cycle representative for \(b_n\). \(\square\)

One might wonder why we’ve indexed the \(b_n\)’s so that they pick out the \(-n - \frac{1}{2}\) Laurent coefficient. This choice of index comes from the action of the double cover of \(S^1\) we defined in Section 9. We found a lift

\[
\tilde{b}_n = -\phi(\rho^2)z^{n+1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i}
\]

for \(b_n\) where \(\phi\) is a suitable bump function. Applying the circle action described in Section 9 to \(\tilde{b}_n\) (keeping in mind all of our annuli currently have the Neveu-Schwarz spin structure) gives

\[
\tilde{b}_n \cdot w = \left(-\phi(\rho^2)z^{n+1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i}\right) \cdot w \\
= -\phi((wz)(\bar{w}\bar{z}))(wz)^{n+1/2}w^{-1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i} \\
= -\phi(\rho^2)w^n z^{n+1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i} \\
= w^n \tilde{b}_n,
\]
which shows that $\tilde{b}_n \in K_n$, where $K_n$ consists of elements of $\Omega^{1/2,1}(A)$ on which $(w, w^{1/2})$ in the double cover defined in Chapter 9 acts by multiplication by $w^n$ for $n \in \mathbb{Z} + \frac{1}{2}$.

10.2 The Multiplication

Given disjoint annuli $A = A_{r<R}(x)$, $A_1 = A_{r_1<R_1}(x)$, and $A_2 = A_{r_2<R_2}(x)$ with

$$0 < r \leq r_2 < R_2 \leq r_1 < R_1 \leq R,$$

we have a structure map from the factorization algebra $\text{Obs}^q$,

$$m_{A_{1},A_{2}}^{A_{1},A_{2}} : \text{Obs}^q(A_1) \otimes \text{Obs}^q(A_2) \to \text{Obs}^q(A),$$

which induces a structure map from the prefactorization algebra $H^*(\text{Obs}^q)$

$$(m_{A_{1},A_{2}}^{A_{1},A_{2}})^* : H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(A_2)) \to H^*(\text{Obs}^q(A))$$

$$b_n(x) \otimes b_m(x) \mapsto b_n(x) \cdot b_m(x) := [\tilde{b}_n(x) \cdot \tilde{b}_m(x)].$$

These structure maps give us a way to “multiply” elements of $H^*(\text{Obs}^q(A))$ as long as the elements are defined on disjoint annuli. Notice that $\tilde{b}_n(x) \cdot \tilde{b}_m(x)$ is guaranteed to be a co-cycle because $\tilde{b}_n(x)$ and $\tilde{b}_m(x)$ are supported on disjoint annuli, and therefore, $\langle \tilde{b}_n(x), \tilde{b}_m(x) \rangle = 0$ making $\partial + h\Delta$ into a derivation.

What if we want to "multiply" $b_{n_1}(x), b_{n_2}(x), \ldots, b_{n_k}(x) \in H^*(\text{Obs}^q(A))$ in this way? The $b_{n_i}(x)$ are all defined on the same annulus, so we can’t use the structure map above directly. Choose disjoint annuli $A_{n_i}$ for $i \in \{1, \ldots, k\}$ all centered at $x$ and contained in $A$ such that for $i < j$, $A_{n_i}$ is farther from $x$ that $A_{n_j}$. Then for each $i$, we can define a co-cycle $\tilde{b}_{n_i}(x)$ for $b_{n_i}(x)$ which is supported on $A_{n_i}$. Thus, we have
the structure map

$$(m_{\mathbb{A}}^{\mathbb{A}_1 \ldots \mathbb{A}_{n_k}})^*: H^*(\text{Obs}^q(\mathbb{A}_{n_1})) \otimes \ldots \otimes H^*(\text{Obs}^q(\mathbb{A}_{n_k})) \rightarrow H^*(\text{Obs}^q(\mathbb{A}))$$

$$[\tilde{b}_{n_1}(x)] \otimes \ldots \otimes [\tilde{b}_{n_k}(x)] \mapsto [\tilde{b}_{n_1}(x) \cdots \tilde{b}_{n_k}(x)]$$

where we define $b_{n_1}(x) \cdots b_{n_k}(x) := [\tilde{b}_{n_1}(x) \cdots \tilde{b}_{n_k}(x)]$.

As above, $\tilde{b}_{n_1}(x) \cdots \tilde{b}_{n_k}(x)$ is guaranteed to be a co-cycle because the $\tilde{b}_{n_i}$’s are supported on disjoint annuli. Notice that due to linearity of the structure map, this “multiplication” satisfies distributivity and compatibility with scalars. Also notice that in defining this multiplication, it did not matter which annuli $\mathbb{A}_{n_1}, \ldots, \mathbb{A}_{n_k}$ we chose as long as they were centered at $x$, contained in $\mathbb{A}$, disjoint, and nested with $\mathbb{A}_{n_k}$ closest to $x$; the size of each individual annulus didn’t matter. We show this for $k = 2$ and $x = 0$ as follows.

For $b_{n_1}, b_{n_2} \in H^*(\text{Obs}^q(\mathbb{A}))$, we take disjoint annuli $\mathbb{A}_1$ and $\mathbb{A}_2$ centered at zero and contained in $\mathbb{A}$ such that $\mathbb{A}_2$ is closer to zero. Define co-cycles, $\tilde{b}_{n_1}$ on $\mathbb{A}_1$ and $\tilde{b}_{n_2}$ on $\mathbb{A}_2$. Now take annuli $\mathbb{A}_1'$ and $\mathbb{A}_2'$ satisfying the same conditions, and define co-cycles $\tilde{b}_{n_1}'$ and $\tilde{b}_{n_2}'$ on $\mathbb{A}_1'$ and $\mathbb{A}_2'$ respectively. We will assume without loss of generality that $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_1'$, and $\mathbb{A}_2'$ are all disjoint. If they are not, we can choose annuli $\mathbb{A}_1''$ and $\mathbb{A}_2''$ which have the desired properties and do not intersect any of $\mathbb{A}_1$, $\mathbb{A}_2, \mathbb{A}_1'$, and $\mathbb{A}_2'$. The argument that follows will then apply to $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_1''$, and $\mathbb{A}_2''$ as well as to $\mathbb{A}_1', \mathbb{A}_2', \mathbb{A}_1', \text{ and } \mathbb{A}_2''$.

We must show that $[\tilde{b}_{n_1} \cdot \tilde{b}_{n_2}] = [\tilde{b}_{n_1}' \cdot \tilde{b}_{n_2}']$. We do so by showing $\tilde{b}_{n_1} \cdot \tilde{b}_{n_2} - \tilde{b}_{n_1}' \cdot \tilde{b}_{n_2}'$ is a boundary. Since $\tilde{b}_{n_1}$ and $\tilde{b}_{n_1}'$ are both co-cycles for $b_{n_1}$, in $\mathbb{A}_1 \cup \mathbb{A}_1'$, $[\tilde{b}_{n_1}] = [\tilde{b}_{n_1}']$, so $\tilde{b}_{n_1} - \tilde{b}_{n_1}'$ is a boundary, and there is an element $B_{n_1}$ supported on $\mathbb{A}_1 \cup \mathbb{A}_1'$ such that $(\partial + h\Delta)(B_{n_1}) = \tilde{b}_{n_1} - \tilde{b}_{n_1}'$. Similarly, there is an element, $B_{n_2}$, supported on $\mathbb{A}_2 \cup \mathbb{A}_2'$ such that $(\partial + h\Delta)(B_{n_2}) = \tilde{b}_{n_2} - \tilde{b}_{n_2}'$. We have
(\bar{\partial} + \hbar \Delta)(B_{n_1} \tilde{b}_{n_2} - B_{n_2} \tilde{b}_{n_1}') = (\bar{\partial} + \hbar \Delta)(B_{n_1} \tilde{b}_{n_2}) - (\bar{\partial} + \hbar \Delta)(B_{n_2} \tilde{b}_{n_1}') \\
= ((\bar{\partial} + \hbar \Delta)(B_{n_1}))\tilde{b}_{n_2} \pm B_{n_1}(\bar{\partial} + \hbar \Delta)(\tilde{b}_{n_2}) \\
- ((\bar{\partial} + \hbar \Delta)(B_{n_2}))\tilde{b}_{n_1}' + B_{n_2}(\bar{\partial} + \hbar \Delta)(\tilde{b}_{n_1}') \\
(10.1) \\
= (\tilde{b}_{n_1}' - \tilde{b}_{n_1})\tilde{b}_{n_2} - (\tilde{b}_{n_2}' - \tilde{b}_{n_2})\tilde{b}_{n_1}' \\
(10.2) \\
= \tilde{b}_{n_1} \tilde{b}_{n_2} - \tilde{b}_{n_1}' \tilde{b}_{n_2} - \tilde{b}_{n_2} \tilde{b}_{n_1}' + \tilde{b}_{n_2}' \tilde{b}_{n_1}' \\
= \tilde{b}_{n_1} \tilde{b}_{n_2} - \tilde{b}_{n_1}' \tilde{b}_{n_2} + \tilde{b}_{n_2} \tilde{b}_{n_1}' - \tilde{b}_{n_1}' \tilde{b}_{n_2}' \\
(10.3) \\
= \tilde{b}_{n_1} \tilde{b}_{n_2} - \tilde{b}_{n_1}' \tilde{b}_{n_2}' \\

where in (10.1), we used that since $B_{n_1}$ and $\tilde{b}_{n_2}$ are supported on disjoint annuli and $B_{n_2}$ and $\tilde{b}_{n_1}'$ are supported on disjoint annuli, $\bar{\partial} + \hbar \Delta$ acts as a derivation on the two products, in (10.2), we used that $(\bar{\partial} + \hbar \Delta)(\tilde{b}_{n_2}) = 0$ and $(\bar{\partial} + \hbar \Delta)(\tilde{b}_{n_1}') = 0$ since $\tilde{b}_{n_2}$ and $\tilde{b}_{n_1}'$ are co-cycles, and in (10.3), we used that $\tilde{b}_{n_1}'$, $\tilde{b}_{n_2}$, and $\tilde{b}_{n_2}'$ all have odd parity. Thus, this multiplication of $b_{n_1}$ and $b_{n_2}$ does not depend on our choice of the annuli $A_1$ and $A_2$ given that these annuli satisfy the conditions given above. This argument can be extended to any $k$ and to any $x$. In the following proposition, we will see when we compute the super commutator for the $b_n$’s, why the order of the annuli does matter.

**Proposition 10.2.1.** If $\mathbb{A} = \mathbb{A}_{r<R(0)}$, and $b_n, b_m \in H^*(\text{Obs}^n(\mathbb{A}))$,

$$[b_m, b_n] = b_m \bullet b_n - (-1)^{p(b_m)p(b_n)}b_n \bullet b_m = b_m \bullet b_n + b_n \bullet b_m = -\frac{\hbar}{2\pi i} \delta_{m,-n}$$

where $\delta_{m,-n}$ is the Kronecker delta.

**Proof.** Let $\mathbb{A} = \mathbb{A}_{r<R(0)}$, $\mathbb{A}_1 = \mathbb{A}_{r_1<R_1(0)}$, $\mathbb{A}_2 = \mathbb{A}_{r_2<R_2(0)}$, and $\mathbb{A}_3 = \mathbb{A}_{r_3<R_3(0)}$ with

$$0 < r \leq r_3 < R_3 \leq r_2 < R_2 \leq r_1 < R_1 \leq R$$

49
as shown in Figure 10.1 below.

Figure 10.1. Multiplication of Neveu-Schwarz annuli

The factorization algebra $\text{Obs}^q$ gives us two structure maps

$$m_{12} : \text{Obs}^q(\hat{A}_1) \otimes \text{Obs}^q(\hat{A}_2) \to \text{Obs}^q(\hat{A})$$

$$m_{23} : \text{Obs}^q(\hat{A}_2) \otimes \text{Obs}^q(\hat{A}_3) \to \text{Obs}^q(\hat{A}).$$

Pick co-cycle representatives

$$b_m^1 = -\phi_1(\rho^2)z^{m+1/2}\frac{d\bar{z}(dz)^{1/2}}{4\pi i}$$
\[ b_m^3 = -\phi_3(\rho^2)z^{m+1/2}d\pi(dz)^{1/2}/4\pi i \]

for \( b_m \) in \( A_1 \) and \( A_3 \) respectively, and pick a co-cycle representative

\[ b_n^2 = -\phi_2(\rho^2)z^{n+1/2}d\pi(dz)^{1/2}/4\pi i \]

for \( b_n \) in \( A_2 \) where the \( \phi_i \) are bump functions as defined in Section 10.1. We need to show

\[ [m_{12}(b_m^1 \otimes b_n^2) + m_{23}(b_n^2 \otimes b_m^3)] = -\hbar/2\pi i \delta_{m,-n} \]

in cohomology, \( H^*(\text{Obs}^q(A)) \). Recall that

\[ \text{Obs}^q(A) = (\text{Sym}(\Pi\Omega_c^{1/2,*}(A)[1])[\hbar], \overline{\partial} + h\Delta) \]

\[ = \text{Sym}(\Pi\Omega_c^{1/2,0}(A) \overset{\overline{\partial} + h\Delta}{\rightarrow} \Pi\Omega_c^{1/2,1}(A))[\hbar]. \]

We will show

\[ [\sigma] := \left[ m_{12}(b_m^1 \otimes b_n^2) + m_{23}(b_n^2 \otimes b_m^3) + \frac{\hbar}{2\pi i} \delta_{m,-n} \right] = 0. \]

If \( \sigma \in \text{im}(\overline{\partial} + h\Delta) \), then clearly, \( [\sigma] = 0 \). Our next task will be to show \( \sigma \in \text{im}(\overline{\partial} + h\Delta) \).

Let

\[ \Psi(\rho^2) := \int_0^{\rho^2} \phi_3(s) - \phi_1(s)ds. \]

We will need the following lemma.

**Lemma 10.2.1.** If \( \rho \in (r_2, R_2) \), then \( \Psi(\rho^2) = 2 \).

**Proof.** Let \( \rho \in (r_2, R_2) \). Then

\[ \Psi(\rho^2) = \int_0^{\rho^2} \phi_3(s) - \phi_1(s)ds = \int_{r_3}^{R_3^2} \phi_3(s)ds. \]
and using u-substitution with \( u = \rho^2 \) gives
\[
1 = \int_0^\infty \phi_3(\rho^2) \rho \, d\rho = \int_0^\infty \frac{1}{2} \phi_3(u) \, du = \frac{1}{2} \Psi(\rho^2).
\]

We also know
\[
\frac{\partial \Psi}{\partial (\rho^2)} = \phi_3(\rho^2) - \phi_1(\rho^2).
\]

Let
\[
A := \Psi(\rho^2) z^{m-1/2} \left( \frac{dz}{4\pi i} \right)^{1/2}.
\]

Then
\[
\overline{\partial} A = \overline{\partial} \left( \Psi(\rho^2) z^{m-1/2} \left( \frac{dz}{4\pi i} \right)^{1/2} \right).
\]
\[
= \frac{\partial}{\partial z} \left( \Psi(z \bar{z}) z^{m-1/2} \left( \frac{1}{4\pi i} \right)^{1/2} \right) dz(dz)^{1/2}
\]
\[
= \frac{\partial \Psi}{\partial (\rho^2)} z^{m-1/2} \left( \frac{1}{4\pi i} \right)^{1/2} dz(dz)^{1/2}
\]
\[
= (\phi_3(\rho^2) - \phi_1(\rho^2)) z^{m+1/2} \left( \frac{dz}{4\pi i} \right)^{1/2}
\]
\[
= \left( -\phi_1(\rho^2) z^{m+1/2} \left( \frac{dz}{4\pi i} \right)^{1/2} \right) - \left( -\phi_3(\rho^2) z^{m+1/2} \left( \frac{dz}{4\pi i} \right)^{1/2} \right)
\]
\[
= b_m^1 - b_m^3.
\]

Let
\[
\alpha = A \cdot b_n^2 = \left( \Psi(\rho^2) z^{m-1/2} \left( \frac{dz}{4\pi i} \right)^{1/2} \right) \cdot b_n^2.
\]

We will show \((\overline{\partial} + \hbar \Delta)(\alpha) = \sigma\). We have
\[
\overline{\partial}(\alpha) = \overline{\partial}(A \cdot b_n^2)
\]
\[
= \overline{\partial}(A) \cdot b_n^2 \pm A \cdot \overline{\partial}(b_n^2)
\]
\[
\begin{align*}
&= \bar{\partial}(A) \cdot b^2_n \\
&= (b^1_m - b^3_m) \cdot b^2_n \\
&= b^1_m \cdot b^2_n - b^3_m \cdot b^2_n \\
&= b^1_m \cdot b^2_n - (-1)^p(b^l_m) \cdot b^2_m \cdot b^3_m \\
&= b^1_m \cdot b^2_n + b^2_n \cdot b^3_m \\
&= m_{12}(b^1_m \otimes b^2_n) + m_{23}(b^2_n \otimes b^3_m)
\end{align*}
\]

and

\[
(h\Delta)(\alpha) = h(\Delta(\alpha)) = h\{A, b^2_n\} = h\{b^2_n, A\} = h \int_A b^2_n \wedge A
\]

\[
= h \int_A \left( -\phi_2(z\bar{z})z^{n+1/2} \frac{d\bar{z}(dz)^{1/2}}{4\pi i} \right) \wedge \left( \Psi(z\bar{z})z^{m-1/2} \frac{(dz)^{1/2}}{4\pi i} \right)
\]

\[
= \frac{h}{4\pi i} \int_{A_2} \Psi(z\bar{z})\phi_2(z\bar{z})z^{n+m} \frac{dzd\bar{z}}{4\pi i}
\]

\[
= \frac{h}{4\pi i} \delta_{n+m,0} = \frac{h}{2\pi i} \delta_{m,-n}
\]

(10.4)

(10.5)

where in (10.4) we used that \(\phi_2\) is only supported on \(A_2\) and \(\Psi(\rho^2) = 2\) on \(A_2\) by Lemma 10.2.1 and in (10.5), we used Lemma 10.1.1.

Thus, \((\bar{\partial} + h\Delta)(\alpha) = \sigma\) proving the proposition.  

Remark 10.2.1. We’ve shown that we can multiply elements \(b_m\) and \(b_n\) in \(H^*(\text{Obs}^3(A))\) by lifting \(b_m\) and \(b_n\) to co-cycles on smaller disjoint annuli, \(A_1\) and \(A_2\), respectively.
and then using the structure map

\[ m_{A_1, A_2}^A : \text{Obs}^q(A_1) \otimes \text{Obs}^q(A_2) \to \text{Obs}^q(A) \]

to define the multiplication

\[ H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(A_2)) \to H^*(\text{Obs}^q(A)) \]

\[ b_m \otimes b_n \implies b_m \cdot b_n = [m_{A_1, A_2}^{A_1, A_2}(\tilde{b_m} \otimes \tilde{b_n})]. \]

Notice that it doesn’t matter how large the annuli \( A_1 \) and \( A_2 \) are. All that matters is that \( A_1 \) and \( A_2 \) are disjoint and that \( A_1 \) is larger than \( A_2 \).

10.3 An Algebra

We now know how to multiply certain elements (the \( b_n \)'s) of \( H^*(\text{Obs}^q(A)) \). Let \( A = \text{span}(b_{n_1} \cdot b_{n_2} \cdot \cdots \cdot b_{n_k}) \). Then \( A \subseteq H^*(\text{Obs}^qA) \) as a vector space. The product, \( \cdot \), we defined above from the structure maps makes \( A \) into an algebra.

Notice that as a complex vector space, elements of the form \( b_{m_1}b_{m_2} \cdots b_{m_k} \) with \( m_i \leq m_j \) for \( i < j \) span \( A \). This is because if the \( b_n \)'s are not in this order, we can reorder them using the commutator to get a sum of ordered products of \( b_n \)'s.

10.3.1 Associativity

Since the multiplication in \( A \) arises from a structure map, it is associative. To see this, let \( u_1, u_2, u_3 \in A \). Then the natural coherences of the structure maps arising from the symmetric monoidal structure give

\[ (u_1 \cdot u_2) \cdot u_3 = (m_{(A_1 \sqcup A_2), A_3}^{A_1, A_2} \cdot (m_{(A_1 \sqcup A_2), A_3}^{A_1, A_2} \cdot (u_1 \otimes u_2) \otimes u_3) \]

\[ = (m_{A_1, A_2, A_3}^{A_1, A_2, A_3})^*(u_1 \otimes u_2 \otimes u_3) \]

\[ = (m_{A_1, (A_2 \sqcup A_3)}^{A_1, (A_2 \sqcup A_3)})^*(u_1 \otimes (m_{A_2, A_3}^{A_2, A_3})^*(u_2 \otimes u_3)) \]

54
\[ u_1 \cdot (u_2 \cdot u_3). \]  

(10.6)

10.3.2 The Unit

Recall that for an annulus, \( A \),

\[ \text{Obs}^q(A) = (\text{Sym}(\Pi \Omega^{1/2} \ast (A)[1])[\hbar], \overline{\partial} + \hbar \Delta). \]

Consider the element \( 1 \in \mathbb{C} \cong \text{Sym}^0(\Pi \Omega^{1/2} \ast (A)[1])[[\hbar]]. \) We clearly have \((\overline{\partial}+\hbar \Delta)(1) = 0\), so \( 1 \) is a co-cycle, and we have the cohomology class \([1] \in H^*(\text{Obs}^q(A)).\)

Let \( A_1 \) and \( A_2 \) be nested annuli as in the previous section. Then we have

\[ (m_{A_1,A_2}^\star) : H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(A_2)) \rightarrow H^*(\text{Obs}^q(A)) \]

\[ b_n \otimes [1] \quad \mapsto \quad [\tilde{b}_n \cdot 1] = [\tilde{b}_n] = b_n. \]

That is, \( b_n \cdot [1] = b_n \), and similarly, \([1] \cdot b_n = b_n\). Thus, we see that \([1] \) is the unit in the algebra \( A \). Let \( 1 := [1]. \)

Notice that if \( A(x) \) is an annulus centered at a point \( x \in \mathbb{C} \), we can define the co-cycle \( 1 \in \mathbb{C} \cong \text{Sym}^0(\Pi \Omega^{1/2} \ast (A(x))[1])[[\hbar]] \) giving a cohomology class \([1] \in H^*(\text{Obs}^q(A(x))). \) If we let \( 1_x := [1] \in H^*(\text{Obs}^q(A(x))), \) we can show using the argument above that \( b_n(x) \cdot 1_x = b_n(x) = 1_x \cdot b_n(x). \)
Let $A_\emptyset^N$ be a complex annulus centered at the origin where $A \setminus \emptyset$ is equipped with the Neveu-Schwarz spin structure, and let $D_\emptyset^N$ be a complex disk containing $A$ centered at the origin where $D \setminus \emptyset$ is equipped with the Neveu-Schwarz spin structure (the only spin structure possible) as shown in Figure 11.1. Throughout this section, we will refer to $A_\emptyset^N$ as $A$ and $D_\emptyset^N$ as $D$.

Figure 11.1. Inclusion of a Neveu-Schwarz annulus into a disk
The factorization algebra \( \text{Obs}^q \) gives us a structure map

\[
m^\wedge_A \colon \text{Obs}^q(\mathbb{A}) \to \text{Obs}^q(\mathbb{D}),
\]

which induces the structure map

\[
(m^\wedge_A)^* : H^*(\text{Obs}^q(\mathbb{A})) \to H^*(\text{Obs}^q(\mathbb{D}))
\]
on the cohomology prefactorization algebra. Denote by \(|0\rangle\) the image of \([1]\) under this map, that is, \(|0\rangle := (m^\wedge_D)^*([1])\). Let’s investigate what this map does on the simplest observables, the \(b_n\)'s.

For \(b_n \in H^*(\text{Obs}^q(\mathbb{A}))\), we can lift \(b_n\) to a co-cycle,

\[
\tilde{b}_n = -\phi(\rho^2)z^{n+1/2} \frac{dz}{4\pi i},
\]
as we showed in Lemma 10.1.1. Applying \((m^\wedge_D)^*\) to \(b_n\) gives us

\[
H^*(\text{Obs}^q(\mathbb{A})) \to H^*(\text{Obs}^q(\mathbb{D}))
\]

\[
b_n = [\tilde{b}_n] \mapsto [\tilde{b}'_n],
\]

where \(\tilde{b}'_n\) is the co-cycle \(\tilde{b}_n\) extended by 0 to all of \(\mathbb{D}\). We would like to describe \([\tilde{b}'_n] \in H^*(\text{Obs}^q(\mathbb{D}))\) in terms of the \(b_n\)'s. We use the map \(\Phi_D\) defined in Chapter 8 to obtain an element of \((\Omega^{1/2,0})^\vee\). For \(k \geq 0\), we have

\[
(\Phi_D([\tilde{b}'_n]))(z^k(dz)^{1/2}) = \int_D \left(-\phi(\rho^2)z^{n+1/2} \frac{dz}{4\pi i}\right)^{\wedge} z^k(dz)^{1/2}
\]

\[
= \int_D \phi(\rho^2)z^{n+k+1/2} \frac{dz d\overline{z}}{4\pi i}
\]

\[
= \delta_{n+k+\frac{1}{2},0} = \delta_{k,-n-\frac{1}{2}},
\]

(11.1)
where in (11.1), we used Lemma 10.1.1. Thus, if we apply $\Phi_D([\tilde{b}_n])$ to $\beta(dz)^{\frac{1}{2}} \in \Omega^{1/2,0}_\text{hol}(D)$, we get the $-n-\frac{1}{2}$ Taylor coefficient of $\beta$. Notice that since $\beta$ is holomorphic on a disc, it only has a $-n-\frac{1}{2}$ Taylor coefficient when $-n-\frac{1}{2} \geq 0$, or $n < 0$. Thus,

$$(m^\text{A}_D)^*(b_n) = \begin{cases} b_n & \text{if } n < 0 \\ 0 & \text{if } n > 0, \end{cases}$$

where the $b_n$ on the right hand side is understood to be defined on $D$.

11.1 An $A$-Module

Let $V = \text{im}((m^\text{A}_D)^*_A) \subseteq H^*(\text{Obs}^q(D))$. One can show that $V$ is a dense subspace of $H^*(\text{Obs}^q(D))$. We can use the structure maps from $\text{Obs}^q$ to define an $A$-action on $V$.

Let $D = (D_R(0))^0_N$ be a complex disk with radius $R$ centered at the origin, let $A_1 = (A_{r_1 < R_1(0)})^0_N$, and let $D_2 = (D_{R'}(0))^0_N$ where

$$0 < R' \leq r_1 < R_1 \leq R$$

as shown in Figure 11.2 below.

We have a structure map

$$m^{A_1,D_2}_D : \text{Obs}^q(A_1) \otimes \text{Obs}^q(D_2) \to \text{Obs}^q(D)$$

which induces the structure map

$$(m^{A_1,D_2}_D)^* : H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(D_2)) \to H^*(\text{Obs}^q(D))$$

on the prefactorization algebra $H^*(\text{Obs}^q)$. What happens when we apply $(m^{A_1,D_2}_D)^*$
Figure 11.2. The action of a Neveu-Schwarz annulus on a Neveu-Schwarz disk
to elements of the subspaces $A$ and $V$? We will use additional structure maps to find out.

Let $A_2 = (A_{r_2 < R_2}(0))^\theta_N$ where

$$0 < r_2 < R_2 \leq R'$$

as shown in Figure 11.3 below

![Figure 11.3. A sub-annulus of $\mathbb{D}_2$](image)

We have a second structure map

$$(m_{\mathbb{D}_2}^{A_2})^*: H^*(\text{Obs}^q(A_2)) \to H^*(\text{Obs}^q(\mathbb{D}_2)).$$

Let $A = (A_{r_A < R_A}(0))^\theta_N$ such that $0 < r_A \leq r_2$ and $R_1 \leq R_A \leq R$ as shown in Figure 11.4 below.
Then we have a third structure map

$$(m_{A_1,A_2}^A)^* : H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(A_2)) \to H^*(\text{Obs}^q(A)).$$

We can also view $A$ as a subspace of $D$ (see Figure 11.5 below).

This gives us a fourth structure map

$$(m_D^A)^* : H^*(\text{Obs}^q(A)) \to H^*(\text{Obs}^q(D)).$$

Since $H^*(\text{Obs}^q)$ is a prefactorization algebra, we must have all natural coherences on the structure maps. In particular, we have the commutative diagram below.

$$
\begin{array}{ccc}
H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(A_2)) & \xrightarrow{(m_{A_1,A_2}^A)^*} & H^*(\text{Obs}^q(A)) \\
\downarrow \text{id} \otimes (m_{A_2}^A)^* & & \downarrow (m_D^A)^* \\
H^*(\text{Obs}^q(A_1)) \otimes H^*(\text{Obs}^q(D_2)) & \xrightarrow{(m_{A_1,D_2}^A)^*} & H^*(\text{Obs}^q(D))
\end{array}
$$
Recall that $|0⟩ := (m_0^A|A⟩)^*(1) ∈ V$. Then

$$b_m ⊗ |0⟩ ∈ A ⊗ V ⊆ H^*(\text{Obs}^q(A_1)) ⊗ H^*(\text{Obs}^q(D_2)),$$

and we can use the commutative diagram above to compute

$$(m_{D_2}^{A_1,D_2})^*(b_m ⊗ |0⟩) = ((m_{D_2}^A)^* ⋙ (m_{A_1}^{A_1,A_2})^* ⋙ (\text{id} ⊗ (m_{D_2}^{A_2})^*)^{-1})(b_m ⊗ |0⟩)$$

$$= ((m_{D_2}^A)^* ⋙ (m_{A_1}^{A_1,A_2})^*)(b_m ⊗ 1)$$

$$= (m_{D_2}^A)^*(b_m ⋙ 1)$$

$$= (m_{D_2}^A)^*(b_m).$$

We write $b_m|0⟩ := (m_{D_2}^{A_1,D_2})^*(b_m ⊗ |0⟩) = (m_{D_2}^A)^*(b_m)$. This same argument with $A$ and $D$ replaced with $A_2$ and $D_2$ respectively gives us (abusing notation on the last equality) $(m_{D_2}^{A_2})^*(b_m) = (m_{D_2}^{A_1,D_2})(b_m ⊗ |0⟩) = b_m|0⟩$ where $A'_1$ and $D'_2$ are the
appropriate annulus and disc sitting inside $\mathbb{D}_2$.

Let $b_{n_1} b_{n_2} \ldots b_{n_k} \in A \subseteq H^*(\text{Obs}^q \mathbb{A}_1)$. Using the commutative diagram above, we have

\[
(m^A_{\mathbb{D}})(b_{n_1} b_{n_2} \ldots b_{n_k} \otimes |0\rangle) = (m^A_{\mathbb{D}})(b_{n_1} b_{n_2} \ldots b_{n_k} \otimes |0\rangle).
\]

Since any element of $A$ can be written as a finite sum of elements of the form $b_{n_1} b_{n_2} \ldots b_{n_k}$, the structure maps are linear, and $V = \text{im}((m^A_{\mathbb{D}})|_A)$, we’ve shown that for any $v \in V$, $v = (m^A_{\mathbb{D}})(u \otimes |0\rangle)$ for some $u \in A$, and we write $u|0\rangle := (m^A_{\mathbb{D}})(u \otimes |0\rangle) = v$.

In particular, notice that if $u_1|0\rangle, u_2|0\rangle \in V$, since all of the structure maps are linear, we have

\[
u_1|0\rangle + u_2|0\rangle = (m^A_{\mathbb{D}})(u_1 \otimes |0\rangle) + (m^A_{\mathbb{D}})(u_2 \otimes |0\rangle) = (u_1 + u_2)|0\rangle.
\]

When we write $u|0\rangle$, we want to think of $u \in A$ acting on $|0\rangle \in V$, and the structure map $(m^A_{\mathbb{D}})^*$ does in fact give us an action of $A$ on $V$.

**Proposition 11.1.1.** The vector space $V$ is a left $A$-module.
Proof. We define the map

\[ - \cdot - : \quad A \otimes V \rightarrow V \]

\[ u_1 \otimes u_2 |0\rangle \mapsto (u_1 \cdot u_2) |0\rangle \]

where \( u_1 \cdot u_2 \) is multiplication in \( A \). We have \( A \subseteq H^*(\text{Obs}^q(A_1)) \) and \( V \subseteq H^*(\text{Obs}^q(D_2)) \), and \( (u_1 \cdot u_2) |0\rangle = (m_{A_1,D_2}^A)^*((u_1 \cdot u_2) \otimes |0\rangle) \).

Let \( u \in A \) and \( u_1|0\rangle, u_2|0\rangle \in V \). From (11.2) above and since the structure maps are linear and multiplication in \( A \) arises from a structure map, we have

\[ u \cdot (u_1|0\rangle + u_2|0\rangle) = u \cdot ((u_1 + u_2)|0\rangle) \]

\[ = (u \cdot (u_1 + u_2)) |0\rangle \]

\[ = (u \cdot u_1 + u \cdot u_2) |0\rangle \]

\[ = (u \cdot u_1) |0\rangle + (u \cdot u_2) |0\rangle \]

\[ = u \cdot u_1|0\rangle + u \cdot u_2|0\rangle. \quad (11.3) \]

Let \( w \in A \). Then since the structure maps are linear, and multiplication in \( A \) arises from a structure map and by (11.2) above, we have

\[ (w + u) \cdot u_1|0\rangle = ((w + u) \cdot u_1)|0\rangle \]

\[ = (((w \cdot u_1) + (u \cdot u_1)) |0\rangle \]

\[ = (w \cdot u_1)|0\rangle + (u \cdot u_1)|0\rangle \]

\[ = w \cdot u_1|0\rangle + u \cdot u_1|0\rangle. \]

Since multiplication in \( A \) is associative,

\[ (w \cdot u) \cdot (u_1|0\rangle) = ((w \cdot u) \cdot u_1)|0\rangle \]

\[ = (w \cdot (u \cdot u_1))|0\rangle \]
\[ w \cdot ((u \bullet u_1)|0\rangle) = w \cdot (u \cdot u_1|0\rangle). \]

We also have
\[ 1 \cdot (u_1|0\rangle) = (1 \bullet u_1)|0\rangle = u_1|0\rangle. \]

Thus, we’ve shown \( V \) is a left \( A \)-module.

We saw in Section 10.3 that elements of \( A \) of the form \( b_{m_1} \ldots b_{m_k} \) where \( m_i \leq m_j \) for \( i < j \) span \( A \) as a complex vector space. Similarly, elements of the form \( b_{m_1} \ldots b_{m_k}|0\rangle \) where \( m_i \leq m_j \) for \( i < j \) span \( V \) as a complex vector space.
CHAPTER 12

INCLUSION OF A DISK INTO A NEVEU-SCHWARZ ANNULUS

Let $A = (A_{r_A < R_A})_N^0$ and let $D_x = (D_r(x))_N^0$ where $x \in A$, and $|x| + r \leq R_A$ and $|x| - r \geq r_A$ as shown in Figure 12.1 below.

![Figure 12.1. Inclusion of a Neveu-Schwarz disk into a Neveu-Schwarz annulus](image)

Let $A_x = (A_{s_A < S_A} (x))_N$ such that $0 < s_A < S_A \leq r$, that is, $A_x$ is an annulus centered
at \( x \) which is contained in \( \mathbb{D}_x \). Define

\[
b_m(x)|0\rangle := (m_{\mathbb{D}_x}^*b_m(x)) \in H^*(\text{Obs}^q(\mathbb{D}_x)).
\]

We have a structure map

\[
(m_{\mathbb{A}}^D)^* : H^*(\text{Obs}^q(\mathbb{D}_x)) \to H^*(\text{Obs}^q(\mathbb{A}))
\]

\[
b_m(x)|0\rangle = [\tilde{b}_m(x)] \mapsto [\tilde{b}_m],
\]

where on the right hand side, \( \tilde{b}_m \) is taken to be an element of \( \text{Obs}^q(\mathbb{A}) \) by extending \( \tilde{b}_m \in \text{Obs}^q(\mathbb{D}_x) \) to all of \( \mathbb{A} \) by zero. We will determine what \([ \tilde{b}_m ]\) is as an element of \( H^*(\text{Obs}^q(\mathbb{A})) \) in terms of the \( b_n \)'s. To do this, we first need to find an explicit formula for \( \tilde{b}_m \). Let \( \mathbb{D}'_x = (\mathbb{D}_r(0))^0_\mathbb{N} \), that is, \( \mathbb{D}'_x \) is simply \( \mathbb{D}_x \) centered at the origin. Choose a bump function, \( \gamma(\rho^2) \), where \( \rho = \sqrt{zz} \), such that

- \( \int_0^\infty \gamma(\rho^2)\rho d\rho = 1 \), and
- \( \gamma(\rho^2) \geq 0 \) with \( \gamma \) supported on \( \rho \in [0, r] \).

For \( b_m(x)|0\rangle \in H^*(\text{Obs}^q(\mathbb{D}_x)) \), define a co-cycle

\[
\tilde{b}_m = -\gamma(|z - x|^2)(z - x)^{m+1/2} \frac{dz dz}{4\pi i}.
\]

To show that \( b_m(x)|0\rangle = [\tilde{b}_m] \), we use \( \Phi_{\mathbb{D}_x} \) as defined in Chapter 8 to compute

\[
(\Phi_{\mathbb{D}_x}([\tilde{b}_m]))((z - x)^i (dz)^{1/2}) = \int_{\mathbb{D}_x} b_m \wedge (z - x)^i (dz)^{1/2}
\]

\[
= \int_{\mathbb{D}_x} \gamma(|z - x|^2)(z - x)^{m+i+1/2} \frac{dz dz}{4\pi i}
\]

\[
= \int_{\mathbb{D}'_x} \gamma(|u|^2)u^{m+i+1/2} \frac{du d\mathbb{A}}{4\pi i}
\]

\[
= \delta_{m+i+1/2,0} = \delta_{i,-m-1/2}
\]
where in (12.1), we used $u$-substitution with $u = z - x$ and in (12.2) we used Lemma [10.1.1] Thus, if $\beta$ is a holomorphic function on $\mathbb{D}_x$ (which means its Taylor series is $\sum_{j=0}^{\infty} a_j (z - x)^j$), $(\Phi_{\mathbb{D}_x}([\tilde{b}_m]))(\beta(dz)^{1/2})$ picks out the $(-m - 1/2)$ Taylor coefficient of $\beta$, so $b_m(x)|0 = [\tilde{b}_m]$.

Now, we want to write $[\tilde{b}_m] \in H^*(\text{Obs}^q(A))$ in terms of the $b_n$’s.

**Proposition 12.0.2.** In $H^*(\text{Obs}^q(A))$,

$$[\tilde{b}_m] = \sum_{n \in \mathbb{Z}} \left( \frac{n}{m + n + \frac{1}{2}} \right) x^{m+n+\frac{1}{2}} b_{-n-\frac{1}{2}}.$$  

**Proof.** First, we compute

$$\left( \Phi_A([\tilde{b}_m]) \right)(z^j(dz)^{1/2}) = \int_A \tilde{b}_m \wedge z^j(dz)^{1/2},$$

$$= \int_{\mathbb{D}_x} \gamma(|z - x|^2)(z - x)^{m+1/2} z^j dz d\bar{z} \frac{4\pi i}{4\pi i}$$

$$= \int_{\mathbb{D}_x} \gamma(|u|^2) u^{m+1/2} (x + u)^j du d\bar{u} \frac{4\pi i}{4\pi i}$$

$$= \sum_{k=0}^{j} \left( \frac{j}{k} \right) x^k \left( \int_{\mathbb{D}_x} \gamma(|u|^2) u^{m+j+1/2-k} du d\bar{u} \frac{4\pi i}{4\pi i} \right)$$

$$= \delta_{m+j+\frac{1}{2}-k,0} \sum_{k=0}^{j} \left( \frac{j}{k} \right) x^k$$

$$= \left( \frac{j}{m + j + \frac{1}{2}} \right) x^{m+j+\frac{1}{2}},$$

where in (12.3) we used $u$-substitution with $u = z - x$, in (12.4) we used the binomial theorem, and in (12.5), we used Lemma [10.1.1]. Since
\[
\sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}} b_{-n-\frac{1}{2}} (z^j (dz)^{\frac{1}{2}})
\]
\[
= \sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}} \delta_{j,-(n-\frac{1}{2})-\frac{1}{2}}
\]
\[
= \sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}} \delta_{j,n}
\]
\[
= \left( m + j + \frac{1}{2} \right) x^{m+j+\frac{1}{2}}
\]

we have \( \Phi_A([\tilde{b}_m]) = \sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}} b_{-n-\frac{1}{2}} \) proving the proposition. □

By Proposition 12.0.2,

\[
(m_A^{D_x})^* (b_m(x)|0) = \sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}} b_{-n-\frac{1}{2}}.
\]

12.1 Multiplication of Two Neveu-Schwarz Disks

Now, with \( \mathbb{D}_x \) and \( \mathbb{D}'_x \) as in the previous section, let \( \mathbb{D} = (\mathbb{D}_R)_N^0 \), and let \( \mathbb{D}_0 = (\mathbb{D}_{r_0})_N^0 \) such that \( r_0 \leq |x| - r \) and \( R \geq |x| + r \) as shown in Figure 12.2 below.

We have a structure map

\[
(m_A^{D_x,\mathbb{D}_0})^* : H^*(\text{Obs}^q(\mathbb{D}_x)) \otimes H^*(\text{Obs}^q(\mathbb{D}_0)) \to H^*(\text{Obs}^q(\mathbb{D})).
\]

Let \( \mathbb{A} = (\mathbb{A}_{r_A < R_A})_N^0 \) where \( r_0 \leq r_A \leq |x| - r \) and \( |x| + r \leq R_A \leq R \) as shown in Figure 12.3 below. Then, as in the previous section, we get a structure map

\[
(m_A^{D_x})^* : H^*(\text{Obs}^q(\mathbb{D}_x)) \to H^*(\text{Obs}^q(\mathbb{A})).
\]

To determine what the map \( (m_A^{D_x,\mathbb{D}_0})^* \) does on the simplest observables, notice that since \( H^*(\text{Obs}^q) \) is a prefactorization algebra, the following diagram must commute.
Figure 12.2. Multiplication of two Neveu-Schwarz disks

\[
H^*(\text{Obs}^q(D_x)) \otimes H^*(\text{Obs}^q(D_0)) \xrightarrow{(m_{D_x,D_0})^*} H^*(\text{Obs}^q(D))
\]

Thus, for \( b_m(x)|0\rangle \otimes b_k|0\rangle \in H^*(\text{Obs}^q(D_x)) \otimes H^*(\text{Obs}^q(D_0)) \),

\[
(m_{D_x,D_0}^*)^* (b_m(x)|0\rangle \otimes b_k|0\rangle)
\]

\[
= ((m_{D_x,D_0}^*)^* \circ ((m_{A,D_0}^*)^* \otimes \text{id})) (b_m(x)|0\rangle \otimes b_k|0\rangle)
\]

\[
= (m_{D_x,D_0}^*)^* \left( \sum_{n \in \mathbb{Z}} \left( \binom{n}{m+n+\frac{1}{2}} x^{m+n+\frac{1}{2}b_{-n-\frac{1}{2}}} \right) \otimes b_k|0\rangle \right)
\]

\[
= \sum_{n \in \mathbb{Z}} \left( m + n + \frac{1}{2} \right) x^{m+n+\frac{1}{2}b_{-n-\frac{1}{2}} b_k|0\rangle}.
\]
The map \((m^{D_x,D_0}_D)^*\) gives us a map

\[
Y(\cdot, x) : V \rightarrow (\text{End}V)[[x, x^{-1}]]
\]

\[
b_m \mapsto Y(b_m, x) : V \rightarrow V[[x, x^{-1}]]
\]

\[
b_k \mapsto (m^{D_x,D_0}_D)^*(b_m(x)|0\rangle \otimes b_k|0\rangle).
\]
For $x \in \mathbb{C}$, $A(x)$ an annulus centered at $x$, and $D(x)$ a disc centered at $x$ containing $A(x)$, let

$$|0\rangle_x := (m_{D(x)}^{A(x)})^*(1_x) \in H^*(\text{Obs}^q(D_x)).$$

**Definition 13.0.1.** Given $D$, a disc with possible point defects, centered at $y \in \mathbb{C}$, write $x\!\!D$ for the disc $D$ centered at $y + x$. Define the map

$$\tau_x : H^*(\text{Obs}^q(D)) \rightarrow H^*(\text{Obs}^q(x\!\!D))$$

$$b_{n_1}(y)b_{n_2}(y)\ldots b_{n_k}(y)|0\rangle_y \mapsto b_{n_1}(x + y)b_{n_2}(x + y)\ldots b_{n_k}(x + y)|0\rangle_{x+y}.$$

Notice that $\tau_x$ translates a disc by $x$, and since $\tau_x(b_m|0\rangle) = b_m(x)|0\rangle_x$, we can write

$$Y(b_m|0\rangle, x) = (m_{D(x)}^{D_x,0})^*(b_m(x)|0\rangle_x \otimes b_k|0\rangle) = (m_{D}^{D_x,0})^*(\tau_x(b_m|0\rangle) \otimes b_k|0\rangle).$$

We want to show that the map $\tau_x$ commutes with the structure maps of $\text{Obs}^q$, that is, for complex one dimensional manifolds with defects, $U_1, U_2, \ldots, U_k$ and $V$ with a spin embedding $\iota : U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k \rightarrow V$, the following diagram commutes.
$H^* (\text{Obs}^q(U_1) \otimes \cdots \otimes H^* (\text{Obs}^q(U_k))) \xrightarrow{\tau_x} H^* (\text{Obs}^q(xU_1) \otimes \cdots \otimes H^* (\text{Obs}^q(xU_k)))$

$$\xrightarrow{(m_{xU_1 \cdots xU_k})^*} H^* (\text{Obs}^q(V)) \xrightarrow{\tau_x} H^* (\text{Obs}^q(xV))$$

(13.1)

To see that this diagram commutes, we must examine our structure maps. Notice that $(m_U^i)_{i=1}^k \otimes \cdots \otimes (m_U^k)_{i=1}^k$ is the observable $\alpha_i$ on the image of $U_i$ under $\iota$ for each $i$ and is extended by zero to all of $V$. Thus, $\tau_x ((m_U^i)_{i=1}^k \otimes \cdots \otimes (m_U^k)_{i=1}^k)$ is $\alpha_i$ on the image of $xU_i$ under $\iota$ for each $i$ and is extended by zero to all of $xV$. The other composition yields the same thing, so the diagram commutes.

We want to use the structure maps and the maps $\tau_x$ we just defined to define a map $T : V \to V$ that captures what happens to $H^* (\text{Obs}^q(D))$ when we move $D$ slightly.

To define $T$, let $D$ be a disc centered at the origin, and let $D'$ be a disc centered at the origin which is contained in $D$, as shown in Figure 13.1 below.

Now notice that for $y \in C$ small enough, $yD' \subseteq D$, as shown in Figure 13.2, so we have a structure map

$$(m_D^{yD'})^* : H^* (\text{Obs}^q(yD')) \to H^* (\text{Obs}^q(D)).$$

Composing this structure map with $\tau_y$ gives us a map

$$(m_D^{yD'})^* \circ \tau_y : H^* (\text{Obs}^q(D')) \to H^* (\text{Obs}^q(D))$$

that depends on $y$ and takes quantum observables on a disc at the origin to quantum observables on the same disc translated to the point $y$. We can take the derivative
of this map at zero to define

\[ T : H^*(\text{Obs}^q(D')) \rightarrow H^*(\text{Obs}^q(D)) \]

\[ v \mapsto \left. \frac{d}{dy} \left( \left( (m^{yD'})^* \circ \tau_y \right)(v) \right) \right|_{y=0}. \]

What does the map \( T \) do on the \( b_n \)'s? We saw in Section 12 that

\[ \tilde{b}_m = -\gamma(|z - y|^2)(z - y)^{m+1/2}\frac{d\zeta(dz)^{1/2}}{4\pi i} \]

is a co-cycle for \( b_m(y) \in H^*(\text{Obs}^q(yD')) \) where, as in Section 12, \( \gamma(\rho^2) \) is a bump function supported for \( \rho \in [0, r) \) where \( r \) is the radius of \( D' \).

We know

\[ (m_{D'}^y)^*(b_m(y)|0)_{y} = [\tilde{b}_m] \in H^*(\text{Obs}^q(D)) \]

so we need to compute \([\tilde{b}_m] \)
in terms of the $b_n|0\rangle$'s. We have

\[(\Phi_\mathcal{D}([\tilde{b}_m]))(z^j(dz)^{1/2}) = \int_\mathcal{D} \left( -\gamma(|z - y|^2)(z - y)^{m+1/2} \frac{dz(dz)^{1/2}}{4\pi i} \right) \land z^j(dz)^{1/2} \]

\[= \int_{\mathcal{D}} \gamma(|z - y|^2)(z - y)^{m+1/2} \frac{dzdz}{4\pi i} \]

\[= \int_{\mathcal{D}_0} \gamma(|u|^2)u^{m+1/2}(u + y)^j \frac{dud\pi}{4\pi i} \quad (13.2) \]

\[= \int_{\mathcal{D}_0} \gamma(|u|^2)u^{m+1/2} \left( \sum_{n=0}^{j} \binom{j}{n} w^{-n} y^n \right) \frac{dud\pi}{4\pi i} \quad (13.3) \]

\[= \sum_{n=0}^{j} \binom{j}{n} y^n \delta_{n,m+j+\frac{1}{2}} \quad (13.4) \]

\[= \left( m + j + \frac{1}{2} \right) y^{m+j+\frac{1}{2}}, \]
where in (13.2) we used $u$-substitution with $u = z - y$, in (13.3) we used the binomial theorem, and in (13.4), we used Lemma 10.1.1. By an argument similar to that used in Proposition 12.0.2, we find

$$(m_D^{y})^*(b_m(y)|0\rangle_y) = \sum_{n \in \mathbb{Z}} \left( \begin{array}{c} n \\ m + n + \frac{1}{2} \end{array} \right) y^{m + n + \frac{1}{2}} b_{-n - \frac{1}{2}} |0\rangle \in H^*(\text{Obs}_q(\mathbb{D})).$$

Thus, we have

$$T(b_m|0\rangle) = \frac{d}{dy} \left( \left( (m_D^{y})^* \circ \tau_y \right)(b_m|0\rangle) \right) \bigg|_{y=0} = \frac{d}{dy} \left( (m_D^{y})^*(b_m(y)|0\rangle) \right) \bigg|_{y=0} = \frac{d}{dy} \left( \sum_{n \in \mathbb{Z}} \left( \begin{array}{c} n \\ m + n + \frac{1}{2} \end{array} \right) y^{m + n + \frac{1}{2}} b_{-n - \frac{1}{2}} |0\rangle \right) \bigg|_{y=0} = \left( \sum_{n \in \mathbb{Z}} \left( \begin{array}{c} n \\ m + n + \frac{1}{2} \end{array} \right) y^{m + n + \frac{1}{2}} b_{-n - \frac{1}{2}} |0\rangle \right) \bigg|_{y=0} = \left( \frac{1}{2} - m \right) b_{m-1}|0\rangle.$$

This gives us an endomorphism of $V$,

$$T : V \rightarrow V \quad b_m|0\rangle \mapsto \left( \frac{1}{2} - m \right) b_{m-1}|0\rangle.$$

We know what $T$ does on the simplest observables, the $b_n|0\rangle$'s, but what does $T$ do on other elements of $V$? To determine this, we will need the following proposition about $T$ and the $Y$.

**Proposition 13.0.1.** For $b_m|0\rangle, b_k|0\rangle \in V$,

$$[T, Y(b_m|0\rangle, x)](b_k|0\rangle) = \frac{d}{dx} Y(b_m|0\rangle, x)(b_k|0\rangle).$$
Proof. Let $\mathbb{D}$, $\mathbb{D}_0$, and $\mathbb{D}'_0$ be discs centered at the origin such that $\mathbb{D}'_0 \subset \mathbb{D}_0 \subset \mathbb{D}$. Choose $x \in \mathbb{C}$ such that $x \in \mathbb{D} \setminus \mathbb{D}_0$, and let $\mathbb{D}_x$ and $\mathbb{D}'_x$ be discs centered at $x$ such that $\mathbb{D}'_x \subset \mathbb{D}_x \subset \mathbb{D}$ and $\mathbb{D}_x$ and $\mathbb{D}_0$ are disjoint. Let $\mathbb{D}'$ be a disc centered at the origin such that $\mathbb{D}_x \subset \mathbb{D}' \subset \mathbb{D}$. See the left side of Figure 13.3 picture below. Notice that for small enough $y \in \mathbb{C}$, $y\mathbb{D}'_0 \subseteq \mathbb{D}_0$, $y\mathbb{D}'_x \subseteq \mathbb{D}_x$, and $y\mathbb{D}' \subseteq \mathbb{D}$ as shown for $\mathbb{D}_0$ on the right side of Figure 13.3.

Figure 13.3. The setup for defining the map $T$

Then for $b_m|0\rangle, b_k|0\rangle \in V \subseteq H^*(\text{Obs}^q(\mathbb{D}'_0))$, if we let $A$ be an annulus centered at the origin such that $\mathbb{D}_x \subseteq A \subset \mathbb{D}'$ and $A$ and $\mathbb{D}_0$ are disjoint, we have

$$T(Y(b_m|0\rangle, x)(b_k|0\rangle))$$
\[ T \left( (m_{D_y}^{(x,D)} \star (\tau_x(b_m|0) \otimes b_k|0)) \right) \]

\[ = \frac{d}{dy} \left( ((m_{D_y}^{(x,D)} \star \tau_y (m_{D_y}^{(x,D)} \star (\tau_x(b_m|0) \otimes b_k|0))) \right|_{y=0} \]

\[ = \frac{d}{dy} \left( ((m_{D_y}^{(x,D)} \star \tau_y) (\tau_x(b_m|0) \otimes b_k|0)) \right|_{y=0} \]

\[ = \frac{d}{dy} \left( (m_{D_y}^{(x,D)} \star (\tau_x(b_m|0) \otimes b_k|0)) \right) \]

\[ = \frac{d}{dy} \left( (m_{D_y}^{A,D_0})^*(\tau_x(b_m|0) \otimes b_k|0)) \right) \]

\[ = \frac{d}{dy} \left( \sum_{n \in \mathbb{Z}} (n + m + \frac{1}{2})(n + m + \frac{1}{2}) \left( j + k + \frac{1}{2} \right) b_{-n-\frac{1}{2}} \right) \]

\[ = \frac{d}{dy} \left( \sum_{n,j \in \mathbb{Z}} M_n K_j (x + y)^{n + m + \frac{1}{2}} y^{j + k + \frac{1}{2}} b_{-n-\frac{1}{2}} \right) \]

\[ = \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) \left( \frac{1}{2} - k \right) x^{n + m + \frac{1}{2}} b_{-n-\frac{1}{2}} b_{k-1} \right) \]

\[ \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) \left( n + m + \frac{1}{2} \right) x^{n + m + \frac{1}{2}} b_{-n-\frac{1}{2}} b_k \right) \]

\[ (13.5) \]

where

\[ M_n = \left( \begin{array}{c} n \\ n + m + \frac{1}{2} \end{array} \right), \quad K_j = \left( \begin{array}{c} j \\ j + k + \frac{1}{2} \end{array} \right), \]

in (13.5) we used the commutative diagram (13.1), and in (13.6) we used the sym-
metric monoidal structure on the prefactorization algebra \( \text{Obs}^q \). Notice that

\[
Y(b_m|0), x)(T(b_k|0)) = Y(b_m|0), x) \left( \left( \frac{1}{2} - k \right) b_{k-1}|0\right)
= \left( \frac{1}{2} - k \right) Y(b_m|0), x)(b_{k-1}|0)
= \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} - k \right) \left( \frac{n}{n + m + \frac{1}{2}} \right) b_{n-m-\frac{1}{2}}b_{-n-\frac{1}{2}b_{k-1}|0},
\]

which is the first term of (13.7), and

\[
\left( \frac{d}{dx} Y(b_m|0), x) \right) (b_k|0)
= \frac{d}{dx} \left( \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) x^{n+m+\frac{1}{2}}b_{-n-\frac{1}{2}} \right) (b_k|0))
= \left( \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) \left( \frac{n}{n + m + \frac{1}{2}} \right) x^{n+m-\frac{1}{2}}b_{-n-\frac{1}{2}}b_{k|0},
\]

which is the second term of (13.7). Therefore,

\[
T(Y(b_m|0), x)(b_k|0)) = Y(b_m|0), x)(T(b_k|0)) + \left( \frac{d}{dx} Y(b_m|0), x) \right) (b_k|0)).
\]

Rearranging gives

\[
T(Y(b_m|0), x)(b_k|0)) - Y(b_m|0), x)(T(b_k|0)) = \left( \frac{d}{dx} Y(b_m|0), x) \right) (b_k|0)),
\]

and therefore,

\[
[T, Y(b_m|0), x])(b_k|0)) = \frac{d}{dx} Y(b_m|0), x)(b_k|0))
\]

\[\square\]
Now, notice that
\[
T(Y(b_m|x)(b_k|0))) = T\left(\sum_{n \in \mathbb{Z}} \left(\frac{n}{n+m+\frac{1}{2}}\right)x^{n+m+\frac{1}{2}}b_{-n-\frac{1}{2}}|b_k|0\right) \]
\[
= \sum_{n \in \mathbb{Z}} \left(\frac{n}{n+m+\frac{1}{2}}\right)x^{n+m+\frac{1}{2}}T(b_{-n-\frac{1}{2}}|b_k|0). \]

By using this and taking the constant coefficient on both sides of (13.9) gives us
\[
T(b_mb_k|0)) = \left(\frac{1}{2} - k\right)b_mb_{k-1}|0\right) + \left(\frac{1}{2} - m\right)b_{m-1}b_k|0). \quad (13.10)
\]

Now, we’ll show that the equality we proved in Proposition 13.0.1 for the \(b_k|0\)'s holds for all elements of \(V\).

**Proposition 13.0.2.**

\[
[T, Y(b_m|0), x)] = \frac{d}{dx} Y(b_m|0), x)
\]

We’ll need a few lemmas to prove Proposition 13.0.2.

**Lemma 13.0.1.** *The expression*

\[
(T \circ Y(b_m|0), x) \circ Y(b_m|0), x_1) \circ \cdots \circ Y(b_{m_k}|0), x_{k-1})) (b_{m_k}|0))
\]

*is a polynomial in \(x, x_1, \ldots, x_{k-1}\) with coefficients in \(V\). The term with no \(x_1, \ldots, x_{k-1}\) is*

\[
(T \circ Y(b_m|0), x))(b_{m_1}b_{m_2} \cdots b_{m_k}|0)).
\]

**Proof.** If we let \(x = x_0, m = m_0\), and \(n = n_0\), we have

\[
(T \circ Y(b_m|0), x) \circ Y(b_m|0), x_1) \circ \cdots \circ Y(b_{m_k}|0), x_{k-1})) (b_{m_k}|0))
\]
\[ \begin{align*}
&= T(Y(b_m|0), x)(\cdots (Y(b_{m_{k-1}}|0), x_{k-1})(b_{m_k}|0)))
&= T \left( \sum_{n_{k-1} \in \mathbb{Z}} N_k \cdots N_{k-1} \sum_{x_{k-1}} x_{k-1} Y(b_m|0) \right) \\
&= \sum_{n_{k-1} \in \mathbb{Z}} N_k \cdots N_{k-1} x_{k-1} Y(b_m|0)
&= T \left( \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) x^{n+m+\frac{1}{2}} T(b_m b_{m_1} \cdots b_{m_k}|0) \right) \\
&= T \left( Y(b_m, x) (b_{m_1} \cdots b_{m_k}|0) \right)
\end{align*} \]

where
\[ N_i = \binom{n_i}{n_i + m_i + \frac{1}{2}} \]
for each \( i \).

\( \square \)

**Lemma 13.0.2.** Let \( x_1, \ldots, x_{k-1} \in \mathbb{C}^* \) such that \( |x_1| > |x_2| > \cdots > |x_{k-1}| > 0 \), and for each \( i \in \{1, \ldots, k-1\} \), \( \mathbb{D}_i \) is a disc centered at \( x_i \) such that for every pair \( z_i \in \mathbb{D}_i \) and \( z_{i-1} \in \mathbb{D}_{i-1} \), \( 0 < |z_i| < |z_{i-1}| \). Let \( \mathbb{D}_0 \) and \( \mathbb{D} \) be discs centered at the origin such that \( \mathbb{D}_i \subset \mathbb{D} \) for all \( i \) and \( \mathbb{D}_0 \) and \( \mathbb{D}_{k-1} \) are disjoint. Then

\[ \begin{align*}
(Y(b_{m_1}|0), x_1) \circ Y(b_{m_2}|0), x_2) \circ \cdots \circ Y(b_{m_{k-1}}|0), x_{k-1}) (b_{m_k}|0))
&= (m_{\mathbb{D}_{x_1}, \ldots, \mathbb{D}_{x_{k-1}}, \mathbb{D}_0}^* (\tau_{x_1}(b_{m_1}|0) \otimes \tau_{x_2}(b_{m_2}|0)) \otimes \cdots \otimes \tau_{x_{k-1}}(b_{m_{k-1}}|0)) \otimes b_{m_k}|0) \) \\
\end{align*} \]

**Proof.** For \( z_1, z_2 \in \mathbb{C} \) and discs \( \mathbb{D}_{z_1} \) centered at \( z_1 \) and \( \mathbb{D}_{z_2} \) centered at \( z_2 \), let \( \mathbb{D}_{z_1z_2} \) be the smallest disk centered at the origin containing \( \mathbb{D}_{z_1} \) and \( \mathbb{D}_{z_2} \). We have

\[ \begin{align*}
(Y(b_{m_1}|0), x_1) \circ Y(b_{m_2}|0), x_2) \circ \cdots \circ Y(b_{m_{k-1}}|0), x_{k-1}) (b_{m_k}|0))
\end{align*} \]

81
\[= Y(b_{m_1} | 0), x_1 \]

\[
\left( \cdots (m_{D;D_{x_k-2}}^{D_{x_k-1}})^*(\tau_{x_k-2}(b_{m_2} | 0) \otimes (m_{D;D_{x_k-1}}^{D_{x_k-1}})^*(\tau_{x_{k-1}}(b_{m_{k-1}} | 0) \otimes b_{m_k} | 0)) \cdots \right)
\]

\[= (m_{D;D_{x_k-1}}^{D_{x_k-1}})^*(\tau_{x_1}(b_{m_1} | 0)) \otimes \cdots \otimes \tau_{x_{k-1}}(b_{m_{k-1}} | 0) \otimes b_{m_k} | 0)\]

where in the last line, we used the natural coherences of the structure maps. \hfill \Box

Now, we are ready to prove Proposition \ref{13.0.2}.\hfill Proof. (Proposition \ref{13.0.2}) Let \( b_{m_1}, b_{m_2} \ldots b_{m_k} | 0 \in V \). We will first show

\[
(T \circ Y(b_m | 0, x))(b_{m_1} \ldots b_{m_k} | 0))
\]

\[
= \left( \frac{d}{dx} Y(b_m | 0, x) \right) (b_{m_1} \ldots b_{m_k} | 0)) + Y(b_m | 0, x)(T(b_{m_1} \ldots b_{m_k} | 0))
\]

\[
(13.11)
\]

Choose disks \( D_{x_i} \) for \( i \in \{1, \ldots, k - 1\} \) and \( D_0 \) as in Lemma \ref{13.0.2} Let \( x \in \mathbb{C} \) and let \( D_x \) be a disc centered at \( x \) such that \(|z_x| > |z_1|\) for all \( z_x \in D_x \) and \( z_1 \in D_{x_1} \), and let \( D \) be a disc centered at zero containing \( D_x \). Define \( x_0 := x \) and \( x_k := 0 \). For each disk, \( D_{x_i} \), let \( D'_{x_i} \) be a slightly smaller disk centered at \( x_i \). Define nested, disjoint annuli, \( A_{x_i} \) centered at the origin such that \( D_{x_i} \subseteq A_{x_i} \). Then we have

\[
(T \circ Y(b_{m_1} | 0, x) \circ Y(b_{m_1} | 0, x_1) \circ \cdots \circ Y(b_{m_{k-1}} | 0, x_{k-1}))(b_{m_k} | 0))
\]

\[
= T \left( (m_{D_{x_k-1},D_{x_k-2}}^{D_{x_k-1},D_{x_k-2}})^*(\tau_{x_k-2}(b_{m_2} | 0) \otimes \tau_{x_1}(b_{m_1} | 0) \otimes \cdots \otimes \tau_{x_{k-1}}(b_{m_{k-1}} | 0) \otimes b_{m_k} | 0)) \right)
\]

\[
(13.12)
\]

\[
= \frac{dy}{dy} \left( ((m_{D}^{y_{D}})^*(\tau_{y}(b_{m_1} | 0)) \otimes \cdots \otimes b_{m_k} | 0)) \right) \bigg|_{y=0}
\]

\[
= \frac{dy}{dy} \left( ((m_{D}^{y_{D}})^*(m_{D}^{y_{D}})^*(\tau_{y}(b_{m_1} | 0)) \otimes \cdots \otimes b_{m_k} | 0)) \right) \bigg|_{y=0}
\]

\[
= \frac{dy}{dy} \left( (m_{D}^{y_{D}})^*(\tau_{x+y}(b_{m_1} | 0)) \otimes \cdots \otimes \tau_{y}(b_{m_k} | 0)) \right) \bigg|_{y=0}
\]

\[
= \frac{dy}{dy} \left( (m_{D}^{y_{D}})^*(\tau_{x+y}(b_{m_1} | 0)) \otimes \cdots \otimes (m_{D}^{y_{D}})^*(\tau_{y}(b_{m_k} | 0))) \right) \bigg|_{y=0}
\]

82
where in 13.13, we used Lemma 13.0.2, and in 13.14, we used (13.1).

We’ve just shown (13.12)=(13.15). If on both sides of this equality, we take the term with no \( x_{m_1}, \ldots x_{m_{k-1}} \) using Lemma 13.0.1 we get

\[
T(Y(b_m|0), x)(b_{m_1} \ldots b_{m_k}|0))
\]

\[
= \left( \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) \left( n + m + \frac{1}{2} \right) x^{n+m-\frac{1}{2}} b_{-n-\frac{1}{2}} \right) (b_{m_1} \ldots b_{m_k}|0))
\]

\[
+ \left( \sum_{n \in \mathbb{Z}} \left( \frac{n}{n + m + \frac{1}{2}} \right) x^{n+m+\frac{1}{2}} b_{-n-\frac{1}{2}} \right) T(b_{m_1} \ldots b_{m_k}|0))
\]

\[
= \frac{d}{dx} Y(b_m|0), x)(b_{m_1} \ldots b_{m_k}) + Y(b_m|0), x)T(b_{m_1} \ldots b_{m_k}|0))
\]
Rearranging gives,

\[ [T, Y(b_m|0, x)](b_{m_1} \ldots b_{m_k}|0) = \frac{d}{dx} Y(b_m|0, x)(b_{m_1} \ldots b_{m_k}|0). \]

Thus, since every element of \( V \) can be written as \( b_{m_1} \ldots b_{m_k}|0 \) for \( m_i \in \mathbb{Z} + \frac{1}{2} \) and some \( k \geq 0 \),

\[ [T, Y(b_m|0, x)] = \frac{d}{dx} Y(b_m|0, x). \]

Notice that Proposition 13.0.2 also gives us a way to define \( T \) on all of \( V \) (rather than just the \( b_k|0 \)'s). To compute \( T(b_{m_1} \ldots b_{m_k}|0) \), we first notice that Proposition 13.0.2 gives us

\[ [T, Y(b_{m_1}|0, x)](b_{m_2} \ldots b_{m_k}|0) = \frac{d}{dx} Y(b_{m_1}|0, x)(b_{m_2} \ldots b_{m_k}|0), \]

and therefore,

\[ T(Y(b_{m_1}|0, x)(b_{m_2} \ldots b_{m_k}|0)) - Y(b_{m_1}|0, x)(T(b_{m_2} \ldots b_{m_k}|0)) \]
\[ = \frac{d}{dx} Y(b_{m_1}|0, x)(b_{m_2} \ldots b_{m_k}). \]

This means

\[
T \left( \sum_{n \in \mathbb{Z}} \left( n + m_1 + \frac{1}{2} \right) x^{n+m_1+\frac{1}{2}} b_{-n-\frac{1}{2}} b_{m_2} \ldots b_{m_k} \right) \\
= \frac{d}{dx} Y(b_{m_1}|0, x)(b_{m_2} \ldots b_{m_k}|0) + Y(b_{m_1}|0, x)(T(b_{m_2} \ldots b_{m_k}|0)) \\
= \sum_{n \in \mathbb{Z}} \left( n + m_1 + \frac{1}{2} \right) \left( n + m_1 + \frac{1}{2} \right) x^{n+m_1+\frac{1}{2}} b_{-n-\frac{1}{2}} b_{m_2} \ldots b_{m_k|0} \\
+ \sum_{n \in \mathbb{Z}} \left( n + m_1 + \frac{1}{2} \right) x^{n+m_1+\frac{1}{2}} b_{-n-\frac{1}{2}} T(b_{m_2} \ldots b_{m_k}|0).
\]
Taking the constant term on both sides gives

\[ T(b_{m_1} \ldots b_{m_k} |0\rangle) = \left( \frac{1}{2} - m_1 \right) b_{m_1-1} b_{m_2} \ldots b_{m_k} |0\rangle + b_{m_1} T(b_{m_2} \ldots b_{m_k} |0\rangle). \] (13.16)

Thus, we’ve defined \( T \) on all of \( V \).
Now, we would like to use the prefactorization algebra, $H^*(\text{Obs}^0)$, its simplest observables, the $b_n$'s, and its structure maps discussed above to describe the free fermion vertex superalgebra. We’ll need a few definitions, which come, with slight modification, from [2].

**Definition 14.0.2.** If $V$ is a super vector space, a field acting on $V$ is a series

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1},$$

where $\alpha_n \in \text{End}V$, and for each $v \in V$, $\alpha_n(v) = 0$ for $n$ sufficiently large. The field $\alpha(z)$ has parity $p(\alpha) \in \mathbb{Z}_2$ if $\alpha_n V^{(i)} \subset V^{(i+p(\alpha))}$ for all $i \in \mathbb{Z}_2$, $n \in \mathbb{Z}$.

**Definition 14.0.3.** A vertex superalgebra consists of the following data

- **state space** - A complex super vector space, $V = V^{(0)} \oplus V^{(1)}$;
- **vacuum vector** - A distinguished vector $|0\rangle \in V^{(0)}$;
- **translation operator** - A linear operator, $T : V \rightarrow V$, with even parity;
- **state-field correspondence** - A parity preserving linear map

$$Y(-, x) : \quad v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1},$$

which takes $v \in V$ to a field acting on $V$;

subject to the following conditions
• vacuum axiom - \( Y(\langle 0 \rangle, x) = \text{id}_V \) and \( Y(v, z)\langle 0 \rangle|_{z=0} = v \);
• translation axiom - For any \( v \in V \),
\[
[T, Y(v, x)] = \partial_x Y(v, x)
\]
and \( T\langle 0 \rangle = 0 \);
• locality axiom - For \( v, w \in V \),
\[
(x - y)^N Y(v, x)Y(w, y) = (-1)^{p(v)p(w)}(x - y)^N Y(w, y)Y(v, x)
\]
for sufficiently large \( N \in \mathbb{Z} \).

A vertex algebra is \( \frac{1}{2}\mathbb{Z}\)-graded if \( V \) is a \( \frac{1}{2}\mathbb{Z}\)-graded super vector space,

\[
V = \coprod_{m \in \frac{1}{2}\mathbb{Z}} V_m,
\]

\( \langle 0 \rangle \in V_0 \), \( T \) has degree 1 (i.e., \( T : V_m \to V_{m+1} \)), and for \( v \in V_m \), the field \( Y(v, x) \) has conformal dimension \( m \), that is, \( \deg v_n = -n + m - 1 \).

Using the structure maps we discussed above, we can now show our prefactorization algebra \( H^*(\text{Obs}^9) \) gives us a vertex algebra.

**Theorem 14.0.3.** We have a \( \frac{1}{2}\mathbb{Z}\)-graded vertex superalgebra with

- state space \( V \) as defined in Section 11.1 with the \( \frac{1}{2}\mathbb{Z}\)-grading given by \( b_m\langle 0 \rangle \in V_{-m} \) and with the \( \mathbb{Z}_2 \) grading given by \( b_{n_1}b_{n_2} \ldots b_{n_i}\langle 0 \rangle \in V(i \text{ mod } 2) \),
- vacuum vector \( \langle 0 \rangle \in V \) as defined in Chapter 11,
- translation operator \( T \) as defined in Section 13, and
- vertex operator, \( Y(\cdot, x) \), as defined on the \( b_n\langle 0 \rangle \)'s in Section 12.1.

This \( \frac{1}{2}\mathbb{Z}\)-graded vertex super algebra is the free fermion vertex superalgebra.

To prove Theorem 14.0.3 we will use the following “reconstruction” theorem (Theorem 4.4.1 in [2]), based on Hai-sheng Li’s theory of local systems ([9]), which allows us to construct a vertex superalgebra from a generating set of fields and relations.
Theorem 14.0.4 ([2], [9]). Let \( V = V^{(0)} \oplus V^{(1)} \) be a super vector space, let \( |0\rangle \in V^{(0)} \), and let \( T \) be a parity preserving endomorphism of \( V \). Let \( \{\alpha^i\}_{i \in I} \) be a collection of vectors in \( V \), and suppose \( \{\alpha^i(x) = \sum_{n \in \mathbb{Z}} \alpha^i_{(n)} x^{-n-1}\}_{i \in I} \) is a collection of fields (the generating set of fields) such that

(1) \([T, \alpha^i(x)] = \frac{d}{dx} \alpha^i(x)\) for each \( i \in I \),

(2) \( T|0\rangle = 0 \) and \( \alpha^i(x)|0\rangle|_{x=0} = \alpha^i \) for all \( i \in I \),

(3) the fields \( \alpha^i(x) \) and \( \alpha^j(x) \) are mutually local, that is, \( \alpha^i \) and \( \alpha^j \) satisfy the locality axiom from Definition 14.0.3,

(4) the vectors \( \alpha^i_{(j_1)} \ldots \alpha^i_{(j_n)} |0\rangle \), where \( j_t \in \mathbb{Z}, j_t < 0 \), and \( i_s \in I \), span \( V \).

Then \( V \) is a vertex superalgebra with vacuum vector \( |0\rangle \), translation operator \( T \), and state field correspondence given by the formula

\[
Y(\alpha^i_{(j_1)} \ldots \alpha^i_{(j_n)} |0\rangle, x) = \frac{1}{(-j_1 - 1)! \ldots (-j_n - 1)!} : \partial_x^{-j_1 - 1} \alpha^i_1(x) \ldots \partial_x^{-j_n - 1} \alpha^i_n(x) : \]

where \( : - : \) denotes the normally ordered product of fields, which on fields

\[
A(z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}, \quad B(w) = \sum_{m \in \mathbb{Z}} w^{-m-1},
\]

gives the formal power series

\[
: A(z) B(w) := \sum_{n \in \mathbb{Z}} \left( \sum_{m < 0} A_{(m)} B_{(n)} z^{-m-1} + \sum_{m \geq 0} B_{(n)} A_{(m)} z^{-m-1} \right) w^{-n-1}.
\]

Furthermore, this is the unique vertex superalgebra structure on \( V \) satisfying conditions (1)-(4) and

\[
Y(\alpha^i, x) = \alpha^i(x).
\]

Proof. (Theorem 14.0.3) Let \( V \), \( |0\rangle \), and \( T \) be as in the statement of Theorem 14.0.3 Define the generating set of fields to be the set containing the single field
\[ b_{-\frac{1}{2}}(x) = Y(b_{-\frac{1}{2}}|0), x) = \sum_{n \in \mathbb{Z}} \binom{n}{n} x^n b_{n-\frac{1}{2}} = \sum_{k \in \mathbb{Z}} b_{k+\frac{1}{2}} x^{-k-1}. \]

To simplify notation, we let \( \alpha(x) := b_{-\frac{1}{2}}(x) \), so we have \( \alpha(k) = b_{k+\frac{1}{2}} \).

(1) By Proposition 13.0.2 we have
\[ [T, \alpha(x)] = [T, Y(b_{-\frac{1}{2}}|0), x] = \frac{d}{dx} Y(b_{\frac{1}{2}}|0), x) = \frac{d}{dx} \alpha(x). \]

(2) First we show \( T|0) = 0 \). For discs \( \mathbb{D} \) and \( \mathbb{D}' \) centered at the origin with \( \mathbb{D}' \subset \mathbb{D} \), for small \( y \in \mathbb{C} \), we have
\[
T|0) = \frac{d}{dy} ((m_{\mathbb{D}'}^y v_\mathbb{D})^* \circ \tau_y)|0)\bigg|_{y=0}
= \frac{d}{dy} ((m_{\mathbb{D}'}^y v_\mathbb{D})^*|0)\bigg|_{y=0}
= \frac{d}{dy} (|0)\bigg|_{y=0}
= 0.
\]

Now we show \( (\alpha(x)|0)\big|_{x=0} = b_{-\frac{1}{2}} \). We have
\[
(\alpha(x)|0)\big|_{x=0} = \left( \sum_{n \in \mathbb{Z}} b_{n+\frac{1}{2}} x^{-n-1} \right)|0)\bigg|_{x=0}
= \left( \sum_{n \in \mathbb{Z}} (b_{n+\frac{1}{2}}|0) x^{-n-1} \right)|_{x=0}
= b_{-\frac{1}{2}}|0).
\]

(3) Since there is only one field, we only need to show it is local with itself. We must show that
\[
(x - y)^N \alpha(x) \alpha(y) = (-1)^{p(b_{-\frac{1}{2}}) p(b_{-\frac{1}{2}})} (x - y)^N \alpha(y) \alpha(x).
\]

Since \( p(b_{-\frac{1}{2}}) = 1 \), this becomes
\[
(x - y)^N (\alpha(x) \alpha(y) + \alpha(y) \alpha(x)) = 0.
\]
To show this, we compute

\[ \alpha(x)\alpha(y) + \alpha(y)\alpha(x) = \left( \sum_{n \in \mathbb{Z}} b_{n+\frac{1}{2}} x^{-n-1} \right) \left( \sum_{k \in \mathbb{Z}} b_{k+\frac{1}{2}} y^{-k-1} \right) + \left( \sum_{b_{k+\frac{1}{2}} y^{-k-1}} \right) \left( \sum_{n \in \mathbb{Z}} b_{n+\frac{1}{2}} x^{-n-1} \right) + \sum_{n,k \in \mathbb{Z}} \left( \frac{-h}{2\pi i} \right) \delta_{n+\frac{1}{2},-k-\frac{1}{2}} x^{-n-1} y^{-k-1} \]

(14.1)

where in (14.1), we used Theorem 10.2.1. Thus,

\[ (x - y)(\alpha(x))\alpha(y) + \alpha(y)\alpha(x) = (x - y) \left( \sum_{n \in \mathbb{Z}} \frac{-h}{2\pi i} x^{-n-1} y^n \right) = \sum_{n \in \mathbb{Z}} \frac{-h}{2\pi i} x^{-n} y^n - \sum_{n \in \mathbb{Z}} \frac{-h}{2\pi i} x^{-n-1} y^{n+1} = \sum_{n \in \mathbb{Z}} \frac{-h}{2\pi i} x^{-n-1} y^{n+1} - \sum_{k \in \mathbb{Z}} \frac{-h}{2\pi i} x^{-k} y^k = 0, \]

so \( \alpha(x) \) is local with respect to itself.

(4) Notice that \( \alpha(j) = b_{j+\frac{1}{2}} \) for each \( j \in \mathbb{Z} \). Thus, we only need to show that vectors \( b_{m_1} b_{m_2} \ldots b_{m_k} |0\rangle \) for \( m_i \in \mathbb{Z} + \frac{1}{2} \) span \( V \). We showed this in Section 11.1

Thus, by Theorem 14.0.4, we have a vertex superalgebra with \( V, |0\rangle \), and \( T \) as in the statement of Theorem 14.0.3 and with state field correspondence

\[ Y(\alpha(j_1)\alpha(j_2) \ldots \alpha(j_n)|0\rangle, x) = \frac{1}{(-j_1 - 1)! \ldots (-j_n - 1)!} : \partial_x^{-j_1-1} \alpha(x) \ldots \partial_x^{-j_n-1} \alpha(x) :. \]
Since Theorem 14.0.4 tells us this is the unique vertex superalgebra satisfying these conditions, the vertex superalgebra $V$ is the free fermion vertex superalgebra (see [1] Section 3.2).

\[\square\]

Remark 14.0.1. Since $\alpha(j) = b_{j+\frac{1}{2}}$ and $\alpha(x) = Y(b_{-\frac{1}{2}}|0\rangle, x)$, the state field correspondence becomes

$$Y(b_{m_1} \ldots b_{m_k}|0\rangle, x)$$

$$= \frac{1}{(-m_1 - \frac{1}{2})! \ldots (-m_k - \frac{1}{2})!} : \partial_x^{-m_1 - \frac{1}{2}}Y(b_{-\frac{1}{2}}|0\rangle, x) \ldots \partial_x^{-m_k - \frac{1}{2}}Y(b_{-\frac{1}{2}}|0\rangle, x) : .$$

It is straightforward to show

$$Y(b_{m_1} \ldots b_{m_k}|0\rangle, x) =: Y(b_{m_1}|0\rangle, x) \ldots Y(b_{m_k}|0\rangle, x) : .$$
CHAPTER 15
MULTIPLICATION OF RAMOND ANNULI

We have used annuli with the Neveu-Schwarz spin structure above to define the free fermion vertex superalgebra, \( V \). Now, we will take look at annuli with the Ramond spin structure and use them to define a twisted module over this vertex superalgebra. As before, we start by defining a multiplication on \( H^*(\text{Obs}^q(A)) \), for \( A \) a Ramond annulus. Again, we will consider structure maps of \( H^*(\text{Obs}^q) \) on the simplest observables. This section will be analogous to the Neveu-Schwarz case.

**Definition 15.0.4.** Consider the spin manifold with point defects, \( \mathbb{A}^0_R(x) \), where \( \mathbb{A} \) is a complex annulus centered at \( x \in \mathbb{C} \), and the \( R \) means that \( \mathbb{A} \setminus \emptyset \) has the Ramond spin structure. For each \( n \in \mathbb{Z} \), define the simplest observables, \( w_n(x) \in (\Omega^{1/2,0}_{\text{hol}}(\mathbb{A}^0_R(x)))^\vee \subseteq H^*(\text{Obs}^q(\mathbb{A}^0_R(x))) \), to be the linear functionals

\[
    w_n(x) : \Omega^{1/2,0}_{\text{hol}}(\mathbb{A}^0_R(x)) \rightarrow \mathbb{C}
\]

\[
    \beta z^{1/2}(dz)^{1/2} \mapsto (-n - 1) \text{ Laurent coefficient of } \beta.
\]

Set \( w_n := w_n(0) \).

**Remark** 15.0.2. Again, we choose to index the \( w_n(x) \)'s using the circle action on \( \Omega^{1/2,1}(\mathbb{A}^0_R) \).

We again use ideas from Section 6.2 of [4] to investigate the structure maps on the simplest observables.
15.1 A Co-cycle for \( w_n \)

As we did in the Neveu-Schwarz case for \( b_n \), for a Ramond Annulus \( A \) centered at the origin with inner radius \( r \) and outer radius \( R \), we want to find a co-cycle representative for \( w_n \in (\Omega^{1/2,0}(A))^\vee \subseteq H^*(\text{Obs}^9(A)) \). Choose a bump function, \( \phi(\rho^2) \) on \( A \) as we did in Section 10.1.

**Proposition 15.1.1.** For \( n \in \mathbb{Z} \) and \( A \) as defined above,

\[
\tilde{w}_n = -\phi(\rho^2) z^n \frac{z^{1/2} d\bar{z}(dz)^{1/2}}{4\pi i}
\]

is a co-cycle representative for \( w_n \in H^*(\text{Obs}^9(A)) \).

**Proof.** Using the isomorphism \( \Phi_A \) defined in Section 8, we have

\[
\left( \Phi([\tilde{w}_n]) \right)(z^j z^{1/2}(dz)^{1/2}) = \int_A \left( -\phi(\rho^2) z^n \frac{z^{1/2} d\bar{z}(dz)^{1/2}}{4\pi i} \right) \wedge z^j z^{1/2}(dz)^{1/2}
\]

\[
= \int_A \phi(\rho^2) z^{n+j+1} \frac{dz d\bar{z}}{4\pi i}
\]

\[
= \delta_{n+j+1,0} = \delta_{j,-n-1}, \quad (15.1)
\]

where to get (15.1), we used Proposition 10.1.1. Thus, since applying \( \Phi_A([\tilde{w}_n]) \) to \( \beta z^{1/2}(dz)^{1/2} \in \Omega^{1/2,*}_{\text{hol}}(A) \) gives the \(-n-1\) Laurent coefficient of \( \beta \), \( \Phi_A([\tilde{w}_n]) = w_n \), and \( \tilde{w}_n \) is a co-cycle representative for \( w_n \).

As in the Neveu-Schwarz case, applying \( x \in S^1 \) to \( w_n \) using the \( S^1 \) action defined in Section 9 gives

\[
x \cdot w_n = [x^*(\tilde{w}_n)]
\]

\[
= [x^* \left( -\phi_2(\rho^2) z^n \frac{z^{1/2} d\bar{z}(dz)^{1/2}}{4\pi i} \right)]
\]

\[
= -\phi_2((xz)(\bar{z}x))(xz)^n \frac{z^{1/2} d\bar{z}(dz)^{1/2}}{4\pi i}
\]

93
so $w_n \in K_n$.

## 15.2 The Multiplication

Consider the annulus $(\mathbb{A}_i)_{R_i}^0$ with inner radius $r_i$ and outer radius $R_i$ for $i \in \{0, 1, 2, 3\}$ where

$$0 < r_0 \leq r_3 < R_3 \leq r_2 < R_2 \leq r_1 < R_1 \leq R_0,$$

as shown in Figure 15.1 below. Throughout, we will shade Ramond annuli and disks with a checkerboard pattern.

As in the Neveu-Schwarz case, we have a structure map

$$m_{\mathbb{A}_0^{\mathbb{A}_1, \mathbb{A}_2}} : \text{Obs}^q(\mathbb{A}_1) \otimes \text{Obs}^q(\mathbb{A}_2) \to \text{Obs}^q(\mathbb{A}_0)$$

on $\text{Obs}^q$ which induces a structure map on $H^*(\text{Obs}^q)$,

$$(m_{\mathbb{A}_0^{\mathbb{A}_1, \mathbb{A}_2}})^* : H^*(\text{Obs}^q(\mathbb{A}_1)) \otimes H^*(\text{Obs}^q(\mathbb{A}_2)) \to H^*(\text{Obs}^q(\mathbb{A}_0))$$

which gives a multiplication, $\cdot$, on the subspace of $H^*(\text{Obs}^q(\mathbb{A}_0))$ generated by the simplest observables, the $w_n$’s. As we did in Proposition 10.2.1, we compute the commutator for this multiplication.

**Proposition 15.2.1.** If $w_m, w_n \in H^*(\text{Obs}^q(\mathbb{A}_0))$, we have
Figure 15.1. Multiplication of Ramond annuli

\[ [w_m, w_n] = w_m \bullet w_n - (-1)^{p(w_m)p(w_n)} w_n \bullet w_m \]

\[ = w_m \bullet w_n + w_n \bullet w_m \]

\[ = -\delta_{m,-n} \frac{\hbar}{2\pi i}. \]

Proof. Since \( \text{Obs}^q \) is a factorization algebra, we have structure maps

\[ m_{12} : \text{Obs}^q(\mathbb{A}_1) \otimes \text{Obs}^q(\mathbb{A}_2) \rightarrow \text{Obs}^q(\mathbb{A}_0) \]

\[ m_{23} : \text{Obs}^q(\mathbb{A}_2) \otimes \text{Obs}^q(\mathbb{A}_2) \rightarrow \text{Obs}^q(\mathbb{A}_0). \]
Choose lifts

\[ w_1^m = -\phi_1(\rho^2)z^m z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \]
\[ w_3^m = -\phi_3(\rho^2)z^m z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \]

supported on \( A_1 \) and \( A_3 \) respectively, and a lift

\[ w_2^n = -\phi_2(\rho^2)z^n z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \]

supported on \( A_2 \). We need to show

\[ [m_{12}(w_1^m \otimes w_3^n) + m_{23}(w_2^n \otimes w_3^m)] = -\delta_{m,-n}\frac{\hbar}{2\pi i}. \]

We do as we did in Proposition [10.2.1] by showing for

\[ \sigma := [m_{12}(b_1^m \otimes b_2^n) - m_{23}(b_2^n \otimes b_3^m) + \delta_{m,-n}\frac{\hbar}{2\pi i}] \]

\( \sigma \in \text{im}(\overline{\partial} + \hbar\Delta) \) which implies \( [\sigma] = 0 \), proving the proposition.

Let

\[ \Psi(\rho^2) := \int_0^{\rho^2} \phi_3(s) - \phi_1(s) ds. \]

Then

\[ \frac{\partial \Psi}{\partial (\rho^2)} = \phi_3(\rho^2) - \phi_1(\rho^2). \]

If

\[ A := \Psi(\rho^2)z^{m-1} z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \]

we have

\[ \overline{\partial}A = \overline{\partial} \left( \Psi(\rho^2)z^{m-1} z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \right) \]

96
\[
\begin{align*}
\frac{\partial}{\partial \bar{z}} \left( \Psi(\rho^2) z^{m-1} \frac{z^{\frac{1}{2}}}{4\pi i} \right) d\bar{z} dz^{\frac{1}{2}} \\
= \frac{\partial \Psi}{\partial (\rho^2)} \bar{z} z^{m-1} \frac{z^{\frac{1}{2}}}{4\pi i} d\bar{z} dz^{\frac{1}{2}} \\
= (\phi_3(\rho^2) - \phi_1(\rho^2)) \bar{z} z^{m} z^{\frac{1}{2}} d\bar{z} dz^{\frac{1}{2}} \left( -\frac{4\pi i}{4\pi i} \right) \\
= -\phi_1(\rho^2) z^m z^{\frac{1}{2}} d\bar{z} dz^{\frac{1}{2}} - (-\phi_3(\rho^2) z^m z^{\frac{1}{2}} d\bar{z} dz^{\frac{1}{2}}) \\
= w^1_m - w^3_m.
\end{align*}
\]

Now, let
\[
\alpha = A \cdot w^2_n = \left( \Psi(\rho^2) z^{m-1} \frac{z^{\frac{1}{2}}}{4\pi i} \right) \cdot w^2_n.
\]

We will show \( \sigma \in \text{im}(\overline{\partial} + \hbar \Delta) \) by showing \((\overline{\partial} + \hbar \Delta)(\alpha) = \sigma \). We have

\[
\overline{\partial}(\alpha) = \overline{\partial}(A \cdot w^2_n) \\
= \overline{\partial}(A) w^2_n \pm A \overline{\partial}(w^2_n) \\
= \overline{\partial}(A) w^2_n \pm A(0) \\
= \overline{\partial}(A) w^2_n = (w^1_m - w^3_m) w^2_n \\
= w^1_m w^2_n - w^3_m w^2_n \\
= w^1_m w^2_n - (-1)^{p(w^2_n) + p(w^3_m)} w^2_n w^3_m \\
= m_{12}(w^1_m \otimes w^2_n) + m_{23}(w^2_n \otimes w^3_m)
\]

and

\[
(h\Delta)(\alpha) = h(\Delta \alpha) \\
= h\{A, w^2_n\} \\
= h\{w^2_n, A\}
\]
\begin{align*}
&= \hbar \int_{A_0} w_n^2 \wedge A \\
&= \hbar \int_{A_0} \left( -\phi_2(\rho^2) z^n \frac{z^{\frac{i}{2}} dz (dz)^{\frac{i}{2}}}{4\pi i} \right) \wedge \left( \Psi(\rho^2) z^{m-1} \frac{z^{\frac{i}{2}} (dz)^{\frac{i}{2}}}{4\pi i} \right) \\
&= \frac{\hbar}{4\pi i} \int_{A_2} \phi_2(\rho^2) \Psi(\rho^2) z^{n+m} \frac{dz d\bar{z}}{4\pi i} \\
&= \frac{\hbar}{2\pi i} \int_{A_2} \phi_2(\rho^2) z^{n+m} \frac{dz d\bar{z}}{4\pi i} \\
&= \frac{\hbar}{2\pi i} \delta_{n+m,0} \\
&= \frac{\hbar}{2\pi i} \delta_{n,-m},
\end{align*}

where in (15.2), we have \( \Psi(\rho^2) = 2 \) by the same argument as in Lemma 10.2.1 and to get (15.3), we used Proposition 10.1.1. Thus,

\((\overline{\partial} + h \Delta)(\alpha) = m_{12}(w_m^1 \otimes w_n^2) + m_{23}(w_n^2 \otimes w_m^3) + \frac{\hbar}{2\pi i} \delta_{n,-m}\)

proving the proposition.

15.3 An Algebra

As we did for the \( b_n \)'s, let \( A' = \text{span}(w_n \bullet w_n \bullet \cdots \bullet w_n) \). Then \( A \subseteq H^*(\text{Obs}^1(\mathbb{A})) \) as a vector space, and since a similar argument to that used for the \( b_n \)'s shows the product, \( \bullet \), we defined from the structure maps is well-defined, we have that the product \( \bullet \) makes \( A' \) into an algebra.

Notice that as in the Neveu-Schwarz case, elements of the form \( w_{m_1} w_{m_2} \cdots w_{m_k} \) with \( m_i \leq m_j \) for \( i < j \) span \( A' \) since if the \( w_n \)'s are not in this order, we can reorder them using the commutator to get a sum of ordered products of \( w_n \)'s.
15.3.1 Associativity

Since the multiplication in $A'$ arises from a structure map, as did the multiplication in $A$, the same argument we used in Section 10.3.1 shows that the multiplication in $A'$ is associative.

15.3.2 The Unit

The unit in $A'$ is $1'$ defined analogously to the unit $1$ in $A$ in Section 10.3.2.
CHAPTER 16

INCLUSION OF A RAMOND ANNULUS INTO A RAMOND DISK

Now, as we did for Neveu-Schwarz annuli in Section [1], we will investigate the structure map that includes a Ramond annulus into a Ramond disk. Let $A'$ be an annulus centered at the origin with inner radius $s$ and outer radius $S$, where $A' \setminus \emptyset$ has the Ramond spin structure, and let $D'$ be an disk centered at the origin with radius $r \geq S$ where $D' \setminus \{0\}$ has the Ramond spin structure as shown in Figure 16.1 below.

We have a structure map on $\text{Obs}^{q}$,

$$m_{D'}^{A'} : \text{Obs}^{q}(A') \rightarrow \text{Obs}^{q}(D')$$

which induces a map

$$(m_{D'}^{A'})^* : H^*(\text{Obs}^{q}(A')) \rightarrow H^*(\text{Obs}^{q}(D'))$$

on cohomology. Define $|0\rangle' := (m_{D'}^{A'})^*(1')$. We will compute $(m_{D'}^{A'})^*(w_n)$. We know we can lift $w_n \in H^*(\text{Obs}^{q}(A'))$ to a co-cycle,

$$\tilde{w}_n = -\phi(\rho^2)z^n z^\frac{1}{2} d\bar{z}(dz)^\frac{1}{2} \over 4\pi i$$

100
Figure 16.1. Inclusion of a Ramond annulus into a Ramond disk

as we showed in Proposition 15.1.1. Applying \((m_{D'})^*\) gives us

\[
H^*(\text{Obs}^q(A')) \to H^*(\text{Obs}^q(D'))
\]

\[
w_n = [\tilde{w}_n] \mapsto [\tilde{w}'_n].
\]

where \(\tilde{w}'_n\) is the co-cycle \(\tilde{w}_n\) extended by 0 to all of \(D'\). We would like to describe

\([\tilde{w}'_n] \in H^*(\text{Obs}^q(D'))\) in terms of the \(w_n\)'s. Using the isomorphism \(\Phi_{D'}\) we defined in

Chapter 8 we have

\[
(\Phi_{D'}([\tilde{w}'_n]))(z^j z^{\frac{1}{2}} d\bar{z}(dz)^{\frac{1}{2}} = \int_{D'} \phi(\rho^2) z^{n+j+1} \frac{dz d\bar{z}}{4\pi i} = \delta_{n+j+1,0}
\]

\[
= \delta_{n+j+1,0}
\]

\[
= \delta_{j,-n-1}
\]

(16.1)
where to get (16.1) we used Proposition 10.1.1.

Thus, applying $\Phi_{D'}([\bar{w}'_n])$ to an element $\beta z^{\frac{1}{2}} d\bar{z} (dz)^{\frac{1}{2}} \in \Omega^{1/2,0}_{hol}(D')$ where $\beta$ is a holomorphic function on $D'$, we get the $(-n-1)$st Taylor coefficient of $\beta$. Since $\beta$ only has a non-zero $(-n-1)$st Taylor coefficient if $-n-1 \geq 0$, or $n < 0$, we have

$$(m_{D'}^{A'})^*(w_n) = \begin{cases} w_n & \text{if } n < 0 \\ 0 & \text{if } n \geq 0 \end{cases}$$

where the right hand side is understood to be defined on $D'$.

16.1 An $A'$-Module

We proceed as in Section 11.1. Let $W = \text{im}((m_{D'}^{A'})^*|_{A'}) \subseteq H^*(\text{Obs}_q(D'))$. Then $W$ is a dense subspace of $H^*(\text{Obs}_q(D'))$, and we can use the structure maps from $\text{Obs}_q$ to define an $A'$-action on $W$. The only difference between our current situation and Section 11.1 is that we are now applying the same structure maps to Ramond annuli and discs rather than the Neveu-Schwarz versions. Since the results of Section 11.1 only depend on the structure maps and not on whether the discs and annuli involved are Neveu-Schwarz or Ramond, our results from Section 11.1 apply here as well, and $W$ is an $A'$-module.
CHAPTER 17

MULTIPLICATION OF A RAMOND DISK AND A NEVEU-SCHWARZ DISK

Now we reach the biggest difference between the Neveu-Schwarz case we already did and the Ramond case. Rather than multiplying two Neveu-Schwarz discs or two Ramond discs, we will multiply a Ramond disc and a Neveu-Schwarz disc. We will see momentarily that it would actually be impossible to multiply two Ramond discs.

Let $D' = (D^{(0)}_R)_{r(0)}$ be a disk of radius $r$ centered at the origin with the Ramond spin structure on $D'_\emptyset$, let $D'_0 = (D^{(0)}_R)_{r_0(0)}$ be a disk of radius $r_0$ centered at the origin with the Ramond spin structure on $D'_0 \emptyset$, and let $x \in \mathbb{C}^\times$ and $D'_x = (D^0_N)_{r_x}(x)$ be a disk of radius $r_x$ centered at the origin with the Neveu-Schwarz spin structure on $D'_x \emptyset$ where $r_0 < |x| - r_x$ and $|x| + r_x < r$ as seen in Figure 17.1 below.

We have a structure map

$$(m_{D'_x D'_0})^*: H^*(\text{Obs}^q(D'_x)) \otimes H^*(\text{Obs}^q(D'_0)) \rightarrow H^*(\text{Obs}^q(D')),$$

on the prefactorization algebra $H^*(\text{Obs}^q)$. Notice that if $D'_x$ were a Ramond disc, we would not have a spin embedding $D'_x \sqcup D'_0 \rightarrow D'$, since the puncture $x$ would not map to a puncture, meaning we wouldn’t have a structure map. Let $A' = (A^0_R)_{r_A < R_A(0)}$ be an annulus, centered at 0, with inner radius $r_A$ and outer radius $R_A$, where $A' \emptyset$ has the Ramond spin structure and where $r_0 \leq r_A \leq |x| - r_x$ and $|x| + r_x \leq R_A \leq r$ as shown in Figure 17.2 below.
We have a structure map

$$(m_{\mathcal{A}'})^*: H^*(\text{Obs}^q(\mathbb{D}')) \to H^*(\text{Obs}^q(\mathbb{A}')).$$

To compute $(m_{\mathcal{A}'})^*(b_m(x)|0\rangle)$, we note that

$$b_m(x)|0\rangle = [\tilde{b}_m(x)] = \left[ -\gamma(|z-x|^2)(z-x)^{m+\frac{1}{2}} \frac{dz(dz)^{\frac{1}{2}}}{4\pi i} \right],$$

so $(m_{\mathcal{A}'})^*(b_m(x)|0\rangle) = [\tilde{b}_m(x)]$, where we now think of $\tilde{b}_m(x) \in \Omega^{1/2,1}(\mathbb{A}')$ where we’ve extended $\tilde{b}_m(x)$ to all of $\mathbb{A}'$ by defining it to be zero outside of $\mathbb{D}'$. We will write $[\tilde{b}_m(x)] \in H^*(\text{Obs}^q(\mathbb{A}'))$ in terms of the $w_n$’s. We do so by using the map $\Phi_{\mathcal{A}'}$ defined in Chapter 8 to compute

$$(\Phi_{\mathcal{A}'}([\tilde{b}_m(x)]))(z^j z^{\frac{1}{2}} (dz)^{\frac{1}{2}})$$

104
Figure 17.2. Using an annulus, $A'$, to multiply a Neveu-Schwarz disk and a Ramond disk

\[
\begin{align*}
&= \int_{A'} \tilde{b}_m(x) \wedge z^j z^{\frac{1}{2}} (dz)^{\frac{1}{2}} \\
&= \int_{\tilde{D}'_x} \gamma(|z - x|^2)(z - x)^{m+\frac{1}{2}} z^{j+\frac{1}{2}} \frac{dz d\bar{z}}{4\pi i} \\
&= \int_{\tilde{D}'_x} \gamma(|u|^2)u^{m+\frac{1}{2}} (u + x)^{j+\frac{1}{2}} \frac{dud\bar{u}}{4\pi i} \\
&= \sum_{n \in \mathbb{Z}} \binom{j + \frac{1}{2}}{n} x^{j+\frac{1}{2}-n} \int_{\tilde{D}'_x} \gamma(|u|^2)u^{m+\frac{1}{2}+n} \frac{dud\bar{u}}{4\pi i} \\
&= \sum_{n \in \mathbb{Z}} \binom{j + \frac{1}{2}}{n} x^{j+\frac{1}{2}-n} \delta_{n,-m-\frac{1}{2}} \\
&= \left( \frac{j + \frac{1}{2}}{-m - \frac{1}{2}} \right) x^{j+m+1}
\end{align*}
\]

where in (17.1) we used $u$-substitution where $u = z - x$ and $\tilde{D}'_x$ is the disc $D'_x$
translated to be centered at zero, in (17.2), we used the binomial theorem, and in (17.3), we used Proposition [10.1.1]. Since

\[
\sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1} \left( z^j z^{\frac{1}{2}} (dz)^{\frac{1}{2}} \right)
\]

we have

\[
(\Phi_{A'}([b_m(x)]))(z^j z^{\frac{1}{2}} (dz)^{\frac{1}{2}}) = \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1},
\]

and therefore,

\[
(m_{A'}^{D'})^* : H^*(\text{Obs}^q(\mathbb{D}'_{x})) \to H^*(\text{Obs}^q(\mathbb{A}'))
\]

\[b_m(x)|0\rangle \mapsto \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1}.
\]

Since \(H^*(\text{Obs}^q)\) is a prefactorization algebra, we have the following commutative diagram.

\[
\begin{array}{ccc}
H^*(\text{Obs}^q(\mathbb{D}'_{x})) \otimes H^*(\text{Obs}^q(\mathbb{D}'_{0})) & \xrightarrow{(m_{\mathbb{D}'_{x},\mathbb{D}'_{0}}^*)} & H^*(\text{Obs}^q(\mathbb{D}')) \\
(m_{\mathbb{A}'},\mathbb{D}'_{0})^* \otimes \text{id} & \downarrow & (m_{\mathbb{A}'},\mathbb{D}'_{0})^* \\
H^*(\text{Obs}^q(\mathbb{A}')) \otimes H^*(\text{Obs}^q(\mathbb{D}'_{0})) & \xrightarrow{(m_{\mathbb{D}'_{x},\mathbb{D}'_{0}}^*)} & H^*(\text{Obs}^q(\mathbb{D}'))
\end{array}
\]

This means

\[
(m_{\mathbb{D}'_{x},\mathbb{D}'_{0}}^*)^* (b_m(x)|0\rangle \otimes w_k|0\rangle') = ((m_{\mathbb{A}',\mathbb{D}'_{0}}^* \circ (m_{\mathbb{A}',\mathbb{D}'_{0}}^* \otimes \text{id}))(b_m(x)|0\rangle \otimes w_k|0\rangle')
\]

106
\[
\begin{align*}
= (m_{\mathbb{D}'}^{p',p_0'})^* \left( \left( \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1} \right) \otimes w_k|0\rangle' \right) \\
= \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1} w_k|0\rangle'.
\end{align*}
\]

This gives us a map

\[
Y_W(-,x) : V \rightarrow (\text{End}W)[[x, x^{-1}]]
\]

\[
b_m|0\rangle \mapsto Y_W(b_m|0\rangle, x) : W \rightarrow W[[x, x^{-1}]]
\]

\[
w_k|0\rangle' \mapsto Y_W(b_m|0\rangle, x)(w_k|0\rangle')
\]

where

\[
Y_W(b_m|0\rangle, x)(w_k|0\rangle') := (m_{\mathbb{D}'}^{p',p_0'})^* (b_m(x)|0\rangle \otimes w_k|0\rangle')
\]

\[
= \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{2} \right) x^{n+m+1} w_{-n-1} w_k|0\rangle'
\]

\[
= \sum_{j \in \mathbb{Z} + \frac{1}{2}} \left( -m - j - \frac{3}{2} \right) x^{-j-1} w_{m+j+1} w_k|0\rangle'.
\]
We will now show that the super vector space, \( W \), we defined in Section 16.1 is a twisted module over the vertex superalgebra \( V \) from Theorem 14.0.3. Our module, \( W \), will be twisted by an automorphism, \( \sigma \), of the vertex superalgebra \( V \), so we first define what we mean by an automorphism of a vertex superalgebra. The definitions below come, with slight modification, from Chapter 5 Section 6 of [2].

**Definition 18.0.1.** Let \( (V, |0\rangle, T, Y) \) be a vertex superalgebra. An automorphism of \( V \) is a parity preserving linear automorphism 

\[
g : V \rightarrow V
\]

of the super vector space \( V \) such that

1. \( g(|0\rangle) = |0\rangle \)
2. \( [g, T] = 0 \)
3. \( Y(g(v), x) = g \circ Y(v, x) \circ g^{-1} \) for all \( v \in V \).

Consider the vertex superalgebra \( V \) we defined above in Section 14. Define the parity automorphism, \( \sigma \), of the super vector space \( V \) as follows

\[
\sigma : \ V \rightarrow \ V \\
v \mapsto (-1)^j v
\]

for \( v \in V^{(j)} \). Notice that \( \sigma \) has order 2 (so \( \sigma = \sigma^{-1} \)), and on the \( b_m \)'s we have \( \sigma(b_m) = -b_m \) since \( b_m \in V^{(1)} \).
**Proposition 18.0.1.** The automorphism, $\sigma$, of the super vector space $V$ we defined above is an automorphism of the vertex superalgebra $(V, |0\rangle, T, Y)$.

**Proof.** Clearly, $\sigma$ is parity preserving. Since $|0\rangle \in V^{(0)}$, $\sigma(|0\rangle) = |0\rangle$, so $\sigma$ satisfies (1). For (2), recall that $T$ is parity preserving. Then for $v \in V^{(0)}$, we must have $T(v) \in V^{(0)}$, so

$$[\sigma, T]v = (\sigma \circ T)v - (T \circ \sigma)v = \sigma(T(v)) - T(\sigma(v)) = T(v) - T(v) = 0.$$  

For $v \in V^{(1)}$, $T(v) \in V^{(1)}$, so since $T$ is linear, we have

$$[\sigma, T]v = (\sigma \circ T)v - (T \circ \sigma)v = \sigma(T(v)) - T(\sigma(v)) = -T(v) - (-v) = -T(v) - (-T(v)) = 0,$$

proving (2). For (3), recall that $Y$ is linear. By definition of a vertex superalgebra, for $v \in V^{(i)}$,

$$Y(v, x) = \sum_{n \in \mathbb{Z}} v(n)x^{-n-1}$$

where $v(n) \in (\text{End}V)^{(i)}$, that is, $v(n)$ is a parity preserving endomorphism of $V$ for $v \in V^{(0)}$, and $v(n)$ is a parity reversing endomorphism of $V$ for $v \in V^{(1)}$, for all $n \in \mathbb{Z}$.

Therefore, for $v \in V^{(i)}$ and $u \in V^{(j)}$, $v(n)u$ has parity $j + i$, and

$$Y(\sigma(v), x)u = Y((-1)^i v, x)u = (-1)^i \sum_{n \in \mathbb{Z}} v(n)ux^{-n-1} = (-1)^j(-1)^i \sum_{n \in \mathbb{Z}} v(n)ux^{-n-1}$$

109
\[
\begin{align*}
&= \sum_{n \in \mathbb{Z}} (-1)^{j+i} (v(n)((-1)^{j}u)x^{-n-1}) \\
&= \sum_{n \in \mathbb{Z}} \sigma(v(n)(\sigma(u))x^{-n-1}) \\
&= \sigma \left( \sum_{n \in \mathbb{Z}} v(n) (\sigma^{-1}(u))x^{-n-1} \right) \\
&= \sigma(Y(v, x)(\sigma^{-1}(u)))
\end{align*}
\]
proving (3). \qed

Now, we can define a \(g\)-twisted module over a vertex superalgebra.

**Definition 18.0.2.** Let \((V, |0\rangle, T, Y)\) be a vertex superalgebra, and let \(g\) be an order \(N\) vertex superalgebra automorphism of \(V\). A \(g\)-twisted \(V\)-module is a super vector space \(W\) equipped with a linear map

\[
Y_W : V \to (\text{End}W)[[x^\frac{1}{N}, x^{-\frac{1}{N}}]]
\]

\[
v \mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_W^{n} x^{-n-1}
\]
such that the following conditions hold

\begin{enumerate}
    \item For all \(w \in W\), \(v_W^n w = 0\) for large enough \(n\).
    \item \(Y_W(|0\rangle, x) = \text{id}_W\).
    \item For \(v, u \in V\) and \(w \in W\), there exists an element
        \[
f_w \in W[[x^\frac{1}{N}, y^\frac{1}{N}][x^{-\frac{1}{N}}, y^{-\frac{1}{N}}, (x - y)^{-1}]\]
        such that the formal power series
        \begin{enumerate}
            \item \(Y_W(v, x)Y_W(u, y)w\)
            \item \(Y_W(u, y)Y_W(v, x)w\)
            \item \(Y_W(Y(v, x - y)u, y)w\)
        \end{enumerate}
        are expansions of \(f_w\) in
        \begin{enumerate}
            \item \(W((x^\frac{1}{N}))(y^\frac{1}{N}))\)
        \end{enumerate}
\end{enumerate}
If \( v \in V \) such that \( g(v) = e^{2\pi im/N} v \) for some \( m \in \mathbb{Z} \), then \( v^W_{(n)} = 0 \) unless \( n = \frac{m}{N} + \mathbb{Z} \).

We would like to say that the map \( Y_W \) defined in Chapter 17 makes the vector space \( W \) defined in Section 16.1 into a \( \sigma \)-twisted \( V \)-module, for \( V \) the vertex superalgebra defined in Chapter 14 where \( \sigma \) is the parity involution we discussed earlier in this chapter.

This seems probable, but we do not have a complete proof. The difficult part of the proof is condition (3). This is a similar condition to locality. In our proof of Theorem 14.0.3 we got around proving locality for all elements of \( V \) by using the reconstruction theorem, Theorem 14.0.4 which was based on Li’s theory of local systems. Li has also developed a theory of local systems for twisted modules (see [8]) which gives us a way to construct a twisted module using generators. It is clear from Li’s work that \( W \) is a \( \sigma \)-twisted module over some vertex superalgebra (see Corollary 3.15 in [8]). It only remains to show that \( W \) is a \( \sigma \)-twisted module over our vertex superalgebra, \( V \).

This certainly seems likely given that applying \( Y_W \) to \( b_{-\frac{1}{2}} \) gives

\[
Y_W(b_{-\frac{1}{2}}|0\rangle, x) = \sum_{n \in \mathbb{Z}} \left( \frac{n}{2} + \frac{1}{2} \right) x^{n-\frac{1}{2}+1} w_{-n-1} \\
= \sum_{n \in \mathbb{Z}} x^{n+\frac{1}{2}} w_{-n-1} \\
= \sum_{k \in \mathbb{Z}} w_k x^{-k-\frac{1}{2}}.
\]

This matches up with equation 4.18 in [1] which is what one gets for the \( \sigma \)-twisted module over the free fermion vertex algebra.


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