STATISTICAL RECONSTRUCTION AND SIMULTANEOUS PARAMETER ESTIMATION FOR ITERATIVE X-RAY COMPUTED TOMOGRAPHY

A Dissertation

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Abstract

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Statistical iterative image reconstruction methods for X-ray computed tomography (CT) have, over the past few years, shown promise in maintaining diagnostic image quality across a wider range of dosage than has been routinely practical with standard deterministic methods. In this dissertation, iterative reconstruction techniques are modified to attack three limited-information problems: (i) photon starvation due to low signals; (ii) artifacts induced by high attenuation objects and (iii) spatially non-uniform sampling geometry.

Dose reduction in clinical X-ray CT causes low signal-to-noise ratio (SNR) in photon-sparse situations. All techniques meet their limits of practicality when significant portions of the sinogram are near photon starvation. The corruption of electronic noise leads to measured photon counts’ taking on negative values, posing a problem for the log() operation in pre-processing of data. We propose two categories of methods for extremely low-count sinogram pretreatment: an adaptive denoising filter and a pointwise Bayesian inference method. The denoising filter is easy to implement and preserves local statistics, but it introduces correlation between channels and may affect image resolution. The Bayesian inference is a pointwise estimate incorporating a prior model for Poisson rates. Both approaches achieve significant improvements in diagnostic image quality at dramatically reduced dosage.
High-attenuation materials pose significant challenges to CT imaging. Formed of high mass density and high atomic number elements, bones and metals, for example, have high resistance to transmission of photons. Scatter and beam hardening effects are more prominent, raising substantially the importance of the nonlinear relation between line-integral projection estimates and path lengths. As a result, streaking artifacts often appear in reconstructed images along high density directions. In this dissertation, two novel iterative approaches are proposed to reduce such artifacts. One method parameterizes scatter and beam hardening as a locally varying additive Poisson noise, and attempts to estimate the offset as part of the iterative reconstruction loop. The other method uses a prior image to guide both reconstruction and sinogram correction. Artifacts are significantly reduced at little cost in resolution loss.

Cone-beam geometry creates non-uniform spatial sampling rates, which becomes a more pronounced issue as scanners are extended to wider cone angles. While in new, wider-coverage detectors, noise non-uniformity can be mitigated by spatially adaptive regularization design, some data dependent systematic errors are difficult to model deterministically. The detector array is adjusted to accommodate detection efficiency issue at large cone angles. Array modules are compromised to be tilted towards the source, which causes systematic inconsistencies between two module boundaries. To combat the challenge of increased sampling non-uniformity, we propose a joint system parameter estimation with image reconstruction algorithm. Clustering and regularization are used to avoid overfitting and “DC” drift.
“Comfort breeds complacency,
complacency breeds stagnation.”

*Casey Neistat*
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1.1 Background of Computed Tomography

Since first introduced by Cormack [17] and Hounsfield [45, 46] in 1960’s, X-ray computed tomography (CT) has achieved increasing importance in tomographic imaging. It enables noninvasive imaging of the cross-sectional structures of the scanned objects. A CT scan combines a series of X-ray images taken from different view angles and uses reconstruction algorithms to compute the three-dimensional content of the object. Figure 1.1 shows the diagram of a modern clinical CT scanner gantry. The X-ray tube emits photon beams, passing through the field of view (FOV) and recorded by a curved two-dimensional detector array. The tube and the detector array synchronously rotate behind a circular shroud to capture transmission information at various angles. With modern CT modalities, three-dimensional image volumes can be rendered to assist examinations. CT scan images provide more detailed information than plain X-rays do. It is a powerful tool with many applications, including medical diagnosis, industrial inspection, transportation security, etc.

Recent trends in CT imaging have focused on lowering radiation dosage and improving temporal resolution. The multi-slice cone-beam acquisition geometry, as shown in Figure 1.1 is extended to both higher rotating speed and wider cone angle to have better coverage of the object in terms of timing and volume. Clinically, for example, this allows capture of the whole heart’s activities in a single beat with uniform IV contrast. Consequently, low-count signals and spatially non-uniform data sampling pose challenges for modern image reconstruction.
1.2 Fundamental Mathematics and Physics

Mathematical models and photon activity physics are the foundation to CT imaging. The CT raw data acquired by the scanner consists of multiple projections of the object being scanned. These projections are effectively measured by X-ray photon transmission processes. This section is to give a brief introduction of the fundamentals of the data acquisition process. More detailed information can be found in references [61] and [47].

1.2.1 Line-Integral Density

The mathematical theory behind computed tomographic reconstruction dates back to 1917 with the derivation of the Radon Transform [97], which proves that
a function could be reconstructed from an infinite set of its projections varying in continuous angles. CT imaging measures projection values of the object of interest by tracking the ratio of incident and transmitted photons. According to the Beer-Lambert law \[4, 7, 67\], photon attenuation of a material sample is directly related to its thickness (path length), as

\[
E[\Lambda_j] = E[\Lambda_{0,j}] e^{-p_j}, \quad (1.1)
\]

where \(\Lambda_{0,j}\) and \(\Lambda_j\) are incident and received photon counts of the \(j^{th}\) projection ray respectively, and \(p_j\) is the line-integral of a density image \(\mu(x, y)\) along path \(s_j\), denoted as

\[
p_j = \int_{s_j} \mu(x, y) ds_j. \quad (1.2)
\]

Figure 1.2 shows the diagram of a single-slice fan-beam X-ray transmission process. An object of interest is placed at the center of the scan FOV. A set of discrete detectors is to measure the line-integral densities of the object along various paths from source to detector row.

1.2.2 Quantum Noise

Because of the random nature of X-ray generation that occurs within the anode of the X-ray tube, the emitted photon counts follow a Poisson distribution with Poisson rate determined by X-ray tube voltage, current and exposure duration. Therefore, the signal comes with inherent quantum noise. Traversing the scanned object, the received random count at projection \(j\) can be modeled as

\[
\Lambda_j \sim \text{Poisson}\{\lambda_{0,j} e^{-p_j}\}. \quad (1.3)
\]
Figure 1.2. A single-slice fan-beam X-ray transmission diagram at some view angle. (The sketch is for illustration and not to scale.)

Here, we assume the realization of incident photon counting variables $\Lambda_{0,j}$ can be pre-measured as $\lambda_{0,j}$ for a particular scan. In a Poisson process, the number of observed occurrences fluctuates about its mean $m_j = \lambda_{0,j}e^{-p_j}$ with a standard deviation $\sigma_j = \sqrt{m_j}$. The SNR of these fluctuations is therefore

$$\text{SNR}_j = \frac{m_j}{\sigma_j} = \sqrt{\lambda_{0,j}e^{-p_j}},$$

which suggests that the raw measurement SNR decreases as the incident photon rate $\lambda_{0,j}$ reduces at low dosage settings, or the line-integrals $p_j$ increases with larger or denser objects.
1.2.3 Attenuation Coefficient

The linear attenuation coefficient $\mu$ describes the fraction of attenuated photons per unit thickness of a material. It depends on the material’s volumetric mass density $\rho$, the atomic numbers $Z$ of composed elements and the penetrating photon energy $E$. Since the mass density contributes to attenuation itself, mass attenuation coefficient $\frac{\mu}{\rho}$ is often used to characterize the material’s attenuation property. In the typical photon energy range from 10 to 140 keV in medical imaging, three mechanisms dominate the attenuation: photoelectric absorption, Compton scattering and coherent (Rayleigh) scattering. Figure 1.3 shows the relative strengths of these interactions for water material. The data points are from Table 5.5 of reference [55].

The photoelectric effect is the absorption of a photon by a bound electron. If an electron within some material absorbs the energy of one photon, it may eject when the energy is greater than the binding energy of the material. The mechanism usually follows an “all or nothing” principle. All of the energy from one photon must be absorbed and used to liberate one electron from atomic binding, or else the energy is re-emitted. If the photon energy is absorbed, some of the energy liberates the electron from the atom, and the rest contributes to the electron’s kinetic energy as a free particle [81]. Photoelectric absorption is a random phenomenon, and the probability of an absorption is higher when the photon energy is close to the binding energy of the electron. Tightly bound electrons in high Z-materials (e.g., $^{20}$Ca, $^{22}$Ti) are more likely to be involved in photoelectric absorption because the binding energies are closer to X-ray energies.

The most significant source of photon attenuation in medical CT imaging is Compton scattering. It occurs when incident photon energy is significantly higher than the electron binding energy. A collision between an X-ray photon and a loosely bound electron result in an ionizing event, where the electron is scattered, exiting with certain velocity, and the photon is deflected from its original path with some energy
loss [16]. The deflection angle $\phi$ of the photon is determined by the wavelength change before and after the interaction,

$$\Delta \nu = \nu_2 - \nu_1 = \frac{h}{m_e c} (1 - \cos \phi),$$

where $\nu_1$ and $\nu_2$ are the initial wavelength and the wavelength after scattering, $h$ is the Planck constant, $m_e$ is the electron rest mass, and $c$ is the speed of light. In practice, because Compton scattering is related to outer shell, i.e., loosely bound electrons, it is relatively independent of atomic number Z, but depends heavily on the density of the material. As an attenuation mechanism, the Compton scatter effect also serves as a data noise source when deflected photons pass through the object and are recorded by the detectors. Anti-scatter collimators (or grids) are often used to reduce such effects.

Coherent (Rayleigh) scattering happens to photon-matter interaction when parti-
cles (e.g., atoms) are much smaller than the photon wavelength. Coherent scattering does not change the state of the material, and the photon is re-emitted with the same energy after the interaction, but possibly traveling in a different direction \[47\]. As Figure 1.3 illustrates, coherent scattering is the least significant attenuation mechanism among the three in medical CT energy range.

1.2.4 Beam Hardening Effects

Depending on the X-ray generator technology, the emitted X-ray contains a continuous energy spectrum, typically from 10 to 140 keV for medical CT scanners. In Figure 1.4 an example of a spectrum is shown, acquired on a GE Discovery CT750 HD at tube voltage of 120 kVp.

A material’s attenuation coefficient \( \mu \) is generally a non-linear function of the photon energy \( E \). Low-energy photons are more likely to be attenuated than high-energy ones, resulting in a spectrum-shift phenomenon, referring to as the “beam hardening” effect. Compared to the input spectrum at the source, the post-scanning spectrum at a detector contains larger fraction of high-energy photons, and thus appears to be “harder”. The nature and effect of physical interactions between photon and matter change accordingly. For two distinct materials, water and Calcium, the energy-dependent mass attenuation coefficients are plotted in Figure 1.5. The data points are from the National Institute of Standards and Technology (NIST) \[51\]. The two curves decline dramatically as the photon energy increases, particularly in the range from 20 to 60 keV. Scanning at a higher kVp would result in less beam hardening effect. However, materials are also less distinguishable at higher keV, and these energies are therefore less desirable for practical use.

As a result of the X-ray polychromaticity, modeling of received photon counts is
thus modified \[23, 47\] as

\[
\Lambda_j \sim \text{Poisson}\left\{ \int_{\cal E} \lambda_{0,j}(\cal E) e^{-p_j(\cal E)} d\cal E \right\},
\]

(1.6)

where line-integral projections are computed at individual energy levels,

\[
p_j(\cal E) = \int_{s_j} \mu(x, y, \cal E) ds_j.
\]

(1.7)

Consequently, the estimated projection values by the Beer-Lambert law would no longer be linearly proportional to the line-integral densities of a fixed energy level, which leads to streaking or cupping artifacts \[10, 47\] in reconstructed images. Beam hardening correction based on the attenuation behavior of water is commonly applied. High density materials may require subsequent correction.
Figure 1.5. Mass attenuation coefficients of water and Calcium from 20 to 120 keV. (Data points courtesy of NIST.)

1.2.5 Electronic Noise

Solid state detectors utilize scintillating material to convert X-ray photons to light, followed by photodiodes transforming light into electrical current. The data acquisition system (DAS) converts these analog signals into signals intended to be proportional to photon accumulations, before mathematical operations to form projection estimates, illustrated by Figure 1.6. Photodiodes and electronic components of the DAS add corruption to the signal, in addition to Poisson characteristics of photon counting. We refer to this corruption as electronic noise due to its appearance in the electron stream (analog signal) of the DAS. It is modeled as additive and zero-mean Gaussian, independent of the information-laden Poisson variates, similarly to previous work in photon-limited optical imaging [106]. Then the photon transmission
Figure 1.6. Illustration of generic photon detection and data acquisition process.

statistical model would be modified as

\[ \Lambda_j \sim \text{Poisson}\left\{ \int_{\mathcal{E}} \lambda_{0,j}(\mathcal{E}) e^{-\mathcal{E}_j(\mathcal{E})} d\mathcal{E} \right\} + \mathcal{N}(0, \sigma_{e,j}^2), \quad (1.8) \]

where \( \sigma_{e,j}^2 \) is the variance of the electronic noise at projection \( j \). In well-designed systems, the electronic noise variance is sufficiently low compared to the Poisson mean of counts to allow it to be neglected in computing variance of projection measurements. With very limited dosage and penetration of X-rays, photon numbers may fall to the level of electronic noise variance and both noise sources are essential in modeling the statistics of the received photon counts [14].

1.2.6 Cone-Beam Geometry

Cone-beam CT is the most commercially available of CT geometry types. However, one of its limitations is non-uniform spatial sampling. As shown in Figure 1.1, because of the cone-shaped beams, voxels that are farther away from the source are sampled more coarsely. At the same time, the cone beam has less spatial coverage
Figure 1.7. Illustration of wide cone geometry with tilted detector modules compared to one with narrower cone angle. (The sketch is for illustration and not to scale.)

when nearing the source. In previous generation CT scanners with 64 detector rows, the cone angle $\varphi_{cone}$ spans only 4°, in which allows the approximation of cone-beam to multi-slice fan-beam geometry, as shown in Figure 1.7. Thus, only the fan-beam non-uniform sampling issue is considered in image reconstruction.

However, when the cone extends to larger angles, as in recently released systems, cone-induced sampling issues cannot be overlooked, especially in axial scan mode. For instance, the latest GE Revolution CT scanner has a cone angle $\varphi_{cone}$ of around 16°, leading to a significant portion of the scan volume visible only from some projection angles, but not from the opposite side. As a result, certain spatial frequency information is missing. Therefore, nonuniform sampling becomes a more pronounced issue. In addition, for detectors to achieve higher detection efficiency at large cone angles, the detector array is composed of tilted subarrays, which creates systematic
inconsistencies between two subarray boundaries. An illustration is shown in Figure 1.7. This increases the challenge of the non-uniform sampling problem for image reconstruction.

1.3 Image Reconstruction

CT image reconstruction solves a multi-dimensional inverse problem from sets of projection data. Two major categories of reconstruction techniques exist: analytical reconstruction and iterative reconstruction (IR). Typically, analytical reconstruction takes a single pass of projections to create 2D or 3D imagery, among which filtered back projection (FBP) is the most commonly used method in medical CT. IR approaches, on the contrary, require multiple iterations to refine the image estimate. They have emerged in commercial products in recent years.

1.3.1 Filtered Back Projection

Modeled as a continuous sampling, projection or forward projection is a linear transformation of the original object \( f(x, y) \). At each view angle, the 2D image is integrated onto a 1D detector row, as illustrated in Figure 1.2. Fan-beam reconstruction algorithms are derived from parallel-beam geometry, but they suffer from high computational cost due to pixel-dependent factoring. In practice, a fan-to-parallel rebinning process is adopted for reconstruction. In parallel-beam geometry, as Figure 1.8 shows, projections can be expressed as

\[
p_\varphi(t) = \int_{-\infty}^{\infty} f(x, y) ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(t - x \cos \varphi - y \sin \varphi) dx dy,
\]  

(1.9)
where $\varphi$ is the rotating angle, $t$ and $s$ form the corresponding rotated Cartesian coordinate system. Taking 1D Fourier transform of (1.9), we have

$$S_\varphi(\omega) = \int_{-\infty}^{\infty} p_\varphi(t) e^{-j2\pi\omega t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\omega(x \cos \varphi + y \sin \varphi)} dx dy, \quad (1.10)$$

which is a “slice” from the 2D Fourier transform of the image $f(x, y)$ in polar coordinates

$$S_\varphi(\omega) = F(\omega \cos \varphi, \omega \sin \varphi), \quad (1.11)$$

where the 2D Fourier transform $F(u, v)$ is defined as

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy. \quad (1.12)$$

This is the core of the Fourier Slice theorem [61]. It shows that the projection data contain all the frequency information of the original image if both $\varphi$ and $t$
are continuous variables. Then the original image can be reconstructed simply by performing a 2D inverse Fourier transform, given by

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(xu+vy)} du dv. \quad (1.13)$$

Rewriting (1.13) in polar variables, the original image can be computed as

$$f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} F(\omega \cos \varphi, \omega \sin \varphi)e^{-j2\pi\omega(x \cos \varphi+y \sin \varphi)} \omega d\omega d\varphi
= \int_{0}^{2\pi} \int_{-\infty}^{\infty} S_{\varphi}(\omega)|\omega|e^{j2\pi\omega(x \cos \varphi+y \sin \varphi)} d\omega d\varphi, \quad (1.14)$$

which is the conventional FBP formula in continuous space. A ramp filter $|\omega|$ is applied to 1D line-integrals before back projection.

However, in practice only a finite number of projections can be taken, and discrete projection samples are measured by the detector array. Therefore $F(u, v)$ is populated by sample points along radial lines of each rotating angle, as shown in Figure 1.8. Due to the non-uniform sampling of the discrete radial lines, interpolation would be required to convert $S_{\varphi}(\omega)$ into square grid before applying the inverse discrete Fourier transform (DFT) [61]. Sampling becomes sparse in high frequencies. Therefore interpolation is more likely to be inaccurate for high frequency information. This approach is seldom used in practice.

The most common reconstruction from samples of the Radon transform inverts the 1D Fourier space $S_{\varphi}(\omega)|\omega|$ in (1.14) and back-projects at each of the angles of capture,

$$f(x, y) = \frac{\pi}{K} \sum_{k=1}^{K} Q_{\varphi_k}(x \cos \varphi_k + y \sin \varphi_k), \quad (1.15)$$

where $K$ is the total number of projection angles and $Q_{\varphi_k}()$ is the discrete filtered projection of sampled view angle $\varphi_k$. FBP suffers from boosted high frequency noise due to the ramp filter. Also, interpolation in frequency domain may cause streaking.
artifacts in image domain in some circumstances [17].

1.3.2 Iterative Reconstruction

Alternatively, the IR approach pursues the inverse problem by solving an algebraic problem derived from discretization of (1.9). Iteratively calculating the difference between the projection data and forward projection of current estimate, an image can be updated to fit the model and data optimally. Conceptually it is simpler than analytical reconstruction, but is computationally more intensive. In 1970, Gordon et al. proposed the algebraic reconstruction techniques (ART) to solve three-dimensional image reconstruction [38]. Denote the measured angular projections as a vector $y$ with a total number of $M$ measurements, and the image to be estimated as $x$ with a total number of $N$ pixels. The imaging process or forward projection is a linear transformation, which can be described by a matrix $A$ of size $M \times N$, with its entry $A_{ji}$ specifying the contributing weight of pixel $i$ to projection $j$. ART computes the solution of the linear system as the following formula,

$$x^{(n+1)} = x^{(n)} + \frac{y_j - A_j x}{\|A_j x\|_2^2} A_j,$$

(1.16)

where $A_{j*}$ stands for the $j^{th}$ row of matrix $A$, and $\| \|$ is the $l2$-norm.

To model the forward projection $A$ and back projection $A^T$ for discrete system, three categories of methods have been proposed: pixel-driven, ray-driven and distance-driven. In pixel-driven models [43, 94, 127], the image is represented by 2D or 3D grid points corresponding to pixel centers, and the line-integral of a projection is calculated by summing pixels whose connection lines with the focal spot intersect with the detector surface. The thoroughly accurate geometric models suffer from high complexity, especially for fan-beam or cone-beam geometry. Ray-driven projection methods [52, 103], measure the line-integral as a weighted sum of all pixels whose
centers are sufficiently close to the line from focal spot to the detector cell center. Ray-driven performs well in forward modeling, but back projection produces Moiré pattern artifacts due to individual pixel weighting approximation [18]. Alternatively, the distance-driven method [18, 110] maps the boundaries of each pixel and detector cell onto a common axis, and computes the projection by the amount of overlapping. As illustrated in Figure 1.9, three-dimensional cone-beam distance-driven forward projection is approximated by decomposing it into separable two-dimensional fan-beam spaces, and the coefficient \( A_{ji} \) can be derived as

\[
A_{ji} = \left( \frac{\Delta_{xy}}{\cos \varphi_c} V_c(\delta_c) \ast S_c(\delta_c) \right) \left( \frac{1}{\cos \varphi_r} V_r(\delta_r) \ast S_r(\delta_r) \right),
\]

(1.17)

where \( V() \) is the voxel window, \( S() \) is the detector sensitivity function, and operator \( \ast \) denotes convolution [110]. Subscripts \( c \) and \( r \) denote the column and row indices of the detector respectively. The geometric relation between angles \( \varphi_{c,r}, \varphi_c \) and \( \varphi_r \) can

Figure 1.9. Three-dimensional forward projection using distance-driven model, decomposed onto detector row and column dimensions.
be described as
\[ \cos \varphi_{c,r} = \cos \varphi_c \cos \varphi_r. \] (1.18)

The distance-driven model has significant advantage in both accuracy and efficiency over the other two methods. In this dissertation, distance-driven model is selected to perform transformations \( A \) and \( A^T \).

Numerous IR methods have been proposed to maintain diagnostic image quality across a wider range of physical limitations, including low dose, truncation, sparse views, etc. \([96, 110, 128]\). These improvements have naturally led to experimentation with greatly reduced dosages in the clinic \([86, 95, 122]\). Extremely low photon count data also arise routinely in estimation of attenuation maps in positron emission tomography (PET) systems \([121]\). This naturally provides motivation to consider the lower limits of radiation exposure providing worthwhile information for image estimation.

Recent results in model-based iterative reconstruction (MBIR) have demonstrated its ability to improve the reconstructed image quality in the presence of noisy and/or sparse data \([96, 110, 128]\). MBIR algorithms formulate a weighted least square penalty function combined with a image prior term. The objective function can be written as
\[ \Phi(x) = \frac{1}{2} (y - Ax)^T W (y - Ax) + U(x), \] (1.19)

where \( W = \text{Diag}\{w_j\} \) is the data weighting term, and \( U(x) \) is a regularization term derived from a stochastic \textit{a priori} model for the image. The choice of \( W \) usually follows the statistical modeling of the data acquisition process, which sets \( w_j \) to be inversely proportional to the variance estimate of projection \( y_j \). \( U(x) \), on the other hand, penalizes the difference between neighborhood pixels to provide smooth texture in reconstructed image. A simple quadratic penalty is effective, but would compromise details with large gradients. Edge preserving prior models using various
penalties have been proposed to mitigate the concern \cite{8, 19, 108, 110, 118}. The relative strength of the prior model to the data term is critical in balancing the trade-off between image resolution and noise characteristics.

Another active researching area in IR is the optimization strategy. Gradient descent is widely used iterative estimation, but is often slow to converge. Ordered subsets, momentum and pre-conditioned gradient based IR algorithms have been proposed to target the convergence rate \cite{32, 64, 123}. Alternatively, the iterative coordinate descent (ICD) algorithm accelerates convergence by sequentially updating individual pixels using Newton-style optimization techniques \cite{102, 124}. With the state of the art computational power, IR methods can be implemented to reconstructed 256 slices of image in under 5 minutes.

1.3.3 Image Quality Overview

IR has become increasingly popular in medical CT diagnostics for its advantages over traditional FBP techniques. System features, including cone-beam geometry, focal spot wobbling and X-ray spectrum, can be easily incorporated into IR framework. IR images appear to have lower noise level and higher spatial resolution compared to FBP \cite{21}. Also, beam hardening artifacts and metal artifacts can be reduced by IR methods \cite{19, 23}.

By handling projection data differently, FBP and IR images have different appearances (e.g., textures) \cite{47}. One advantage of IR methods is that it is relatively easy to incorporate image prior models into the reconstruction process. Uniform resolution or uniform noise image property can be achieved by incorporating spatially adaptive regularization strength models \cite{13, 28}. A pair of reconstructed chest images using FBP and MBIR techniques is compared in Figure 1.10. In general, IR methods feature improved insensitivity to noise and capability of reconstructing an optimal image in the case of incomplete data.
1.4 Generic Data Processing Flow

The aim of X-ray transmission tomography is to measure the line-integral densities of an object at a single photon energy level, and use them to reconstruct the formation of the object. The process can be described in five stages: X-ray emission, X-ray transmission, data acquisition, data pre-processing and image reconstruction. In Figure 1.11, the flow diagram shows the generic procedures of individual stages.

In X-ray emission, tube voltage, current, rotation speed and scan mode (e.g., axial or helical) are specified. A bowtie shaped pre-object attenuation is applied, considering the usual round appearance of scan objects, to filter the incident spectrum in order to balance spatially non-uniform SNRs. After the photon transmission stage, the detectors convert X-ray signals into digital data stream via scintillator, photodiode and analog-to-digital converter steps. The digital data is then transferred from the scanner to the image generation computer. As a data pretreatment procedure,
scatter correction is applied to remove the additive noise caused by Compton scattering, where an estimate of scattered counts is subtracted from total measurements. In the case of low-dose exposure or large object scanning, measured data may need low-signal processing to reduce data noise and/or handle negative measurements. While “regular” detectors measure attenuated signals used for reconstruction, a few outside (at large fan angle) channels are used to record incident photon counts in absence of attenuation, with which reference normalization is performed before taking the negative log() operation to calculate line-integral estimates. Water-based beam hardening correction is routinely applied as spectral pretreatment for the projections. For MBIR, statistical weights are generated to be proportional to the inverse of data variance estimate as a measure of individual data credibility. In contrast, analytical reconstruction methods usually weight data equally. The beam hardening variance step is to handle the projection variances’ changes from beam hardening correction. Subsequently, with information about projection estimates and corresponding statis-
tical weights, three-dimensional imagery can be reconstructed.

1.5 Dissertation Organization

In this section, a brief outline of the dissertation is provided. Chapter 2 discusses the statistical modeling in photon-sparse or very low SNR scenarios. Compton scatter is modeled as additive Poisson noise, and electronic noise is considered zero-mean Gaussian distributed. Two methods are proposed to handle the increased noise level as well as model complexity in pre-processing steps. Both methods have their merits and have performed effectively for subsequent image reconstruction.

Spatially adaptive regularization design is discussed in Chapter 3 focusing on achieving uniform noise in reconstructed 3D imagery. An effective voxel dosage (EVD) metric is developed to automate the selection of regularization strength according to data variance and system sampling information. Results showed robust performance of the design across varied systems, dosages and object sizes.

In Chapter 4 non-linear spectral correction for dense objects is discussed. A multi-material pretreatment approach is proposed to compensate for beam hardening effects. In the extreme case of metal presence, two iterative metal artifacts reduction (MAR) algorithms are presented. The first approach uses unsupervised estimation to compensate for sinogram biases in metal projection traces. The second method takes a prior image as guidance for iterative image update, and performs sinogram correction in a more predictive sense. An alternating optimization technique is used in the simultaneous sinogram correction and image reconstruction process.

Chapter 5 covers system modeling for wide-cone CT imaging. While the geometric forward model can be improved by accurately calculating detector boundaries and source-to-detector angles, some data-dependent systematic errors are difficult to model deterministically. Therefore, a joint system parameter estimation with image reconstruction algorithm is proposed. Clustering and regularization are used to avoid
overfitting and “DC” drift.

Chapter 6 concludes with the contributions and improvements of the proposed algorithms in addressing various challenges in CT imaging, as well as potential future work.
CHAPTER 2

SINOGRAM PRETREATMENT FOR LOW-COUNT STATISTICAL X-RAY CT

Recent trends in CT imaging have focused on higher speed scanners to improve temporal resolution and lower radiation dosage for safety, both of which motivate interest in low-signal CT data processing.

Photon starvation in CT detectors has long been an issue, particularly in such low-flux-rate applications as attenuation compensation scanning systems in emission tomography [121], where extremely low SNR is routine. In conventional diagnostic CT, near-total photon starvation has generally been treated as an outlier event, with ad hoc treatments to allow such measurements to be treated in the conventional processing chain. Particularly in cases of metal or other nearly radio-opaque objects in the field of view, correction of “bad” data with various interpolation or data replacement approaches [62, 75, 80] reduces severe artifacts from highly localized X-ray blockage. Dramatically reduced overall dosage however, may cause a significant fraction of data to be photon-starved; to this situation the replacement methods are not particularly well-suited.

As indicated by its name, MBIR could allow us to model the data in such a way that detectors registering no photon energy, or even a negative number due to additive electronic noise, do not pose a serious statistical problem. Effects of polychromaticity in beam hardening, scatter, Poisson statistics, arbitrary detector response shapes, etc. can be included in a forward model that allows all readings to contribute appropriately, statistically, to the reconstructed imagery. However, for the sake of limiting the computational cost of iterative methods, simplifications are necessary, and we
will seek to treat photon-starved data such that reconstruction can be equally simple to the normal-dosage case. That is, we will consider approaches that can be applied as pre-treatment of data for application of conventional MBIR routines. A variety of sinogram preprocessing techniques has improved performance in generic reduced dosage settings [22, 47, 48, 60, 65, 72, 77, 83, 100, 117], primarily as a prelude to reconstruction by deterministic methods such as FBP. The fact that we plan to reconstruct via MBIR changes the requirements for sinogram data restoration. The aforementioned noise pre-processing methods are aimed primarily at reducing noise variance to enhance the limited control of FBP over that attribute of reconstructions. MBIR has the capability of suppressing noise according to locally varying statistics and spatially adaptive stochastic image models; therefore it is less sensitive to data variance and can tolerate sinogram correction methods aimed at eliminating bias at the expense of potentially increased sinogram variance.

We propose two methods for X-ray CT sinogram restoration under very low photon counts. The first is in the same vein as [60], for recovering local means in photon counts through spatially adaptive filtering. Our algorithm is based on non-stationary, signal-dependent noise and modeled after that of Kuan et al. [69], called the local linear minimum mean-squared error estimator (LLMMSE). Local estimates of statistics parameterize filter coefficients. The second alternative is designed to better preserve conditional independence among sinogram photon count data, and operates point-wise. It relies on a Bayesian scalar estimation model, with a weak \textit{a priori} density on the Poisson mean of each detector output, and will be referred to as pointwise Bayesian restoration (PBR).

2.1 Statistical Modeling

We begin with treatment of modeling issues that are especially pronounced at very low transmission rates, but merit inclusion in a large fraction of CT scans. As
discussed in Chapter 1, the transmission tomographic data follow Poisson distribution and the Beer-Lambert law, described in [1,3]. The diagram in Figure 1.2 illustrates the X-ray transmission at a given view angle. We denote the discretized Radon transform as $p = Ax$, where $x$ is the unknown image with total number of voxels $N$, and matrix $A$, of size $M \times N$, models the geometry of forward projection. Then, we can write $p_j = A_j x$, where $A_j$ stands for the $j^{th}$ row of matrix $A$. Let $\Lambda$ denote the random vector of photon counts of all projections and $\lambda$ be one of its realizations, under the usual assumption of conditional independence among detectors, the joint conditional probability mass function (PMF) of measured data is given by

$$f_\Lambda(\lambda|x) = \prod_{j=1}^{M} (\lambda_j!)^{-1} \exp(-\lambda_0 e^{-p_j})(\lambda_0 e^{-p_j})^\lambda_j. \quad (2.1)$$

The log-likelihood function for the simple, linear Poisson model is then reduced to

$$L(\lambda|x) = \sum_{j=1}^{M} L_j(\lambda_j|x) + \text{constant}(\lambda_j), \quad (2.2)$$

where the log-likelihood of each measurement

$$L_j(\lambda_j|x) = -\lambda_0 e^{-p_j} - \lambda_j p_j. \quad (2.3)$$

While (2.3) is not highly complex, greater simplicity in both computation and intuitive understanding of the estimation problem is available in a collection of one-dimensional Taylor series expansions [102] about the point $y$ in data space,

$$L_j(\lambda_j|p_j) \approx L_j(\lambda_j|y_j) - d_{1,j}(y_j - p_j) + \frac{d_{2,j}}{2}(y_j - p_j)^2, \quad (2.4)$$

in which $d_{k,j}$ is the $k$-th derivative of $L_j(\lambda_j|p)$ with respect to $p_j$. Probably the most useful and common choice of expansion point for the approximation of (2.4) is the
maximum-likelihood (ML) value of the integral density,

\[ y_j = -\log \frac{\lambda_j}{\lambda_{0,j}}, \quad (2.5) \]

derived from (1.1). This transformation is also the standard mapping of counts to projections in X-ray CT. The first and second derivatives at the ML value are evaluated as

\[ d_{1,j} = 0, \quad (2.6) \]

\[ d_{2,j} = -\lambda_j. \quad (2.7) \]

The end goal of this work is not estimation of sinogram values, but rather the discretized image, \( x \). We model each unknown, true integral projection as \( p_j = A_j \ast x \), which requires that the matrix \( A \) capture the geometry of the voxelized 3D image, the detector sensitivity profiles and a linearization of the response of materials to the generally polychromatic X-rays. Choosing a diagonal data weighting matrix \( W \) populated by entries \( w_j = -d_{2,j} \), the approximate ML estimation of \( x \) is given by

\[ \hat{x} = \arg \min_{x \in \Omega} \left\{ \frac{1}{2} (y - Ax)^T W (y - Ax) \right\}, \quad (2.8) \]

where \( \Omega \) is the feasible region of \( x \). This is the generic weighted least squares algebraic formulation in Chapter [1].

This approximation has provided a basis for a range of useful statistical approaches to tomographic estimation formulations and their optimizations [9, 26]. The entries in the matrix \( W \) may vary over several orders of magnitude in common scans, representing inverse variance of individual measurements. Exploitation of this varying data reliability is the primary advantage of statistical methods over deterministic methods. The Hessian of this formulation, \( A^T W A \), is an approximation to the
Fisher information matrix [27] for estimating voxel values from the sinogram data. The diagonal entries therefore provide a measure of the information available for the value of a voxel when the remainder of the image is fixed.

Figure 2.1 shows scalar log-likelihoods in \( p_j \) and their quadratic approximations for four values of the received photon count \( \lambda_j \). The higher counts on the left yield log-likelihoods that are well-matched by the Taylor series approximations. As counts decrease, representing greater attenuation due to greater densities, the deviation from quadratic becomes dramatic. Simultaneously, the curvature is reduced proportionally to \( \lambda_j \). In a vector estimation problem such as the one at hand for \( x \), the compromise made to minimize (2.8) will tend to accept approximately equal weighted costs among measurements. Thus low-count projection estimates in the vector optimal estimate for \( x \) may be forced far from their scalar maximum-likelihood values of (2.5), into the region where the simple quadratic approximation is quite inaccurate. When counts drop to starvation levels (2.5) and (2.8) may need adjustment from their simplest, high-dosage versions to allow useful interpretation of the full set of measurements.

2.1.1 Additive Poisson Noise

For clinical CT X-rays, the primary attenuation mechanism is Compton scatter [47]. This deflection of photons from their original path is accompanied by energy loss, but a large fraction remains within the sensitivity range and the physical capture surface of detectors. Most detectors are not capable of energy discrimination and cannot distinguish between scattered and unscattered photons. Scatter forms an additive offset to \( \Lambda \), sometimes constituting the majority of captured energy in uncollimated, or poorly collimated, systems. Because angular distribution is wide relative to detector size, this bias in counts is often modeled as independent of local image content, which allows it to be treated as an additive, independent Poisson,
Figure 2.1. Log-likelihood function $L_j(\lambda_j|p_j)$ in (2.3) (smooth curves) and corresponding quadratic approximation (curves with markers) in (2.4) for varying photon counts. Input flux $\lambda_{0,j} = 2 \times 10^5$.

whose rate we denote $\gamma_j$. Then (1.3) may be modified as

$$\Lambda_j \sim \text{Poisson}\{\lambda_{0,j} e^{-p_j} + \gamma_j\},$$

(2.9)

and the log-likelihood function is changed accordingly,

$$L_j(\lambda_j|x) = - (\lambda_{0,j} e^{-p_j} + \gamma_j) + \lambda_j \log(\lambda_{0,j} e^{-p_j} + \gamma_j).$$

(2.10)

The parameter $\gamma_j$ of these additive Poisson variates possess slowly varying spatial features, which can often be estimated with sufficient accuracy to be subtracted in preprocessing. Additional variance from the higher counts, however, remains.

The background rate may also make the log-likelihood non-convex [26, 101]. In
cases where $\lambda_j > \gamma_j$ the log-likelihood has its maximum at

$$y_j = -\log \frac{(\lambda_j - \gamma_j)}{\lambda_{0,j}},$$

(2.11)

the natural point for the Taylor expansion of (2.10) similarly to (2.4). Second derivatives result in data weighting coefficients

$$w_j = \frac{(\lambda_j - \gamma_j)^2}{\lambda_j},$$

(2.12)

demonstrating information loss in the background counts \[26\].

Figure 2.2 shows effect of $\gamma_j$ on the log-likelihood. In cases of very small numbers of unscattered, unattenuated photons such as the problems of interest here, scattered energy may be a major noise source. Incorrect scatter compensation may lead to bias in projection values, as illustrated in Figure 2.2. Failing to compensate for the 20% fraction of radiation received as scatter leads to a shift of 0.223 for the ML line integral estimates.

2.1.2 Electronic Noise

In the current X-ray transmission modality, the initial physical activity in detectors is not directly measured in photons. As illustrated in Section 1.2, solid state detectors utilize scintillating material to convert X-ray photons to light, followed by photodiodes transforming light into electrical current. Together with the data acquisition system (DAS), these processes add corruption to the signal in addition to Poisson characteristics of photon counting. It is modeled as additive and zero-mean Gaussian, independent of the information-laden Poisson variates. The electronic noise variance is sufficiently low compared to the Poisson mean of counts, but with very limited dosage and penetration of X-rays, photon numbers may fall to the level of electronic noise variance and both noise sources are essential in modeling the statistics.
Figure 2.2. Log-likelihood function $L_j(\lambda_j|p_j)$ for different noise models (with additive Poisson noise $\gamma_j$ and electronic noise $\sigma_j$ of Section 2.1.2) at $\lambda_j = 20$, and $\lambda_{0,j} = 2 \times 10^5$.

of the received $\lambda_j$.

We map the electronic noise into the photon domain, including it as the Gaussian component of the combined Poisson-Gaussian model,

$$\Lambda_j = K_j + N_j,$$

where

$$K_j \sim \text{Poisson}\{\lambda_{0,j}e^{-p_j} + \gamma_j\},$$

$$N_j \sim \mathcal{N}(0, \sigma_j^2),$$

where $\sigma_j$ is the variance of electronic noise at projection $j$. Define

$$\theta_j := \lambda_{0,j}e^{-p_j} + \gamma_j,$$
as the mean of the \( j^{th} \) photon count. The probability density function (PDF) of \( \Lambda_j \) is formed by the convolution of PDFs of \( K_j \) and \( N_j \), as

\[
f_{\Lambda_j}(\lambda_j|x) = (f_{K_j} * f_{N_j})(\lambda_j|x) = \sum_{k=0}^{\infty} (2\pi \sigma_j^2)^{-\frac{1}{2}} \exp\left\{-\frac{(\lambda_j - k)^2}{2\sigma_j^2}\right\} \frac{\theta_j^k e^{-\theta_j}}{k!}.
\]

(2.17)

The joint PDF of all received count measurements \( \Lambda \) is then given by

\[
f_\Lambda(\lambda|x) = \prod_{j=1}^{M} f_{\Lambda_j}(\lambda_j|x).
\]

(2.18)

Figure 2.2 shows the effect of this second noise source on the log-likelihood, which again fails to be concave. The nature of the PDF of (2.17) presents complex computation for derivatives of the log-likelihood. Closed-form evaluation of the ML projection value is therefore not practical; we employ a lookup table indexed by \( \lambda_j \) to record maxima from (2.17) in the variable \( p_j \). Given the same measurement \( \lambda_j \), the ML integral density estimates with different electronic noise models are practically unchanged, with a shift of 0.005 and 0.014 for \( \sigma_j = 2 \) and \( \sigma_j = 4 \) respectively.

The primary effect of electronic noise is increasing variance of photon counts and projection estimates. For the weights in (2.8), we use an approximation of variance,

\[
\text{Var}[\Lambda_j] \approx \frac{\lambda_j + \sigma_j^2}{(\lambda_j - \gamma_j)^2}.
\]

(2.19)

So the statistical weight of \( y_j \) is given by

\[
w_j = \frac{(\lambda_j - \gamma_j)^2}{\lambda_j + \sigma_j^2}.
\]

(2.20)

This shows a further decrease in weighting due to this second noise source. For reconstruction, we pursue a MAP estimation for image \( x \) with an \textit{a priori} model
Thus, the total objective function is defined as

$$\Phi(x|\lambda) = \frac{1}{2}(y - Ax)^TW(y - Ax) + U(x), \quad (2.21)$$

where $U(x)$ is generally a penalty function of neighboring pixels to achieve smooth texture. Throughout this dissertation, the $q$-generalized Gaussian Markov random field ($q$-GGMRF) [110] is used as a priori image model. This model results in the edge-preserving regularizing term

$$U(x) = \frac{1}{f(\zeta)} \sum_{i,k \in C} b_{ik} \frac{|x_i - x_k|^p}{1 + |\frac{x_i - x_k}{c}|^{p-q}}, \quad (2.22)$$

in which $\{i, k\}$ sum over all pairs of neighboring voxels in the collection $C$, $c$ is a scale parameter and $p, q$ shape the penalty function on local differences, $b_{i,k}$ weighs the penalty based on the relative location of $x_i$ and $x_k$, and $\frac{1}{f(\zeta)}$ determines the regularization strength to balance the image noise/resolution trade-off. The prior scaling strategy will be discussed in Chapter 3.

2.2 Processing of Photon-Starved Sinograms

We consider cases of extremely low photon counts, in which the Poisson mean is comparable to the standard deviation of electronic noise in terms of counts. Negative data outputs are then common occurrences due to Gaussian error, provided the system allows direct access to raw measurements without truncation such that underlying information may be recovered. The key counts-projection conversion of (2.11) then is invalid since the log() does not admit negative arguments. In standard pre-processing, this means no valid measurement exists for these points, and for ultra-low-dose cases, a significant number of data may be missing. However, these raw data hold information, and finding a valid way to extract it may be important.
in extending the range of statistical methods to unprecedentedly low exposure rates.

Non-positive measurements are commonly replaced by somewhat arbitrary, small positive numbers or simply discarded, which is equivalent to setting the corresponding $w_j$ to zero. Alternatively, values may be interpolated from neighboring detectors with positive values [62]. When negative and zero measurements are common, these replacement methods bias overall local count values positively and local projection values negatively. The result may be low-spatial-frequency shading reminiscent of beam-hardening effects.

Error diffusion methods offer an attractive option for preserving local means while correcting individual channels’ negative values. Floyd and Steinberg proposed a recursive filter that minimizes quantization errors [29], and that has a counterpart for processing of low-count regions of sinograms [109]. The filter’s output is a strictly positive set of sinogram counts, with error from negative-valued channels spatially dithered. However, a positive value, $\epsilon$, for the output floor must be chosen, and we have found local bias to be relatively sensitive to $\epsilon$ in the presence of large numbers of negatives.

2.2.1 Adaptive Denoising in Counts

Integral projections $A_jx$ are exponentiated to form the mean of the Poisson photon counts. Thus accurate estimation of the mean will yield accurate estimates of the projections through the log() transformation. Any averaging is best done among photon counts rather than projection values, since per Jensen’s inequality, averaging values after the convex − log() operation will bias measurements upward. This averaging is primarily intended for correcting estimates of means, but there may also be merit in smoothed maps of data weights [26] in these cases. A simple approach would be repeated linear filtering that is terminated at the point all negative count measurements have disappeared. Such filtering could well establish local count
means, but will be costly in terms of resolution [22, 77, 100].

Our goal is to adapt continuously and locally to the statistics of low-quality count data, similarly to the adaptive filters that have found success in FBP preprocessing for noise suppression [48], and suffering minimal resolution loss. Somewhat similarly to recursive, least-squares linear filtering, we model the counts data as a non-stationary process, with low-frequency variation in local means for larger structure, and higher-frequency variation representing edges and texture. The adaptation of the filter will depend on local estimates of statistics for both signal and noise. The local linear minimum mean-squared error (LLMMSE) filter [69] of Kuan et al., appears to capture the characteristics appropriate to our denoising problem. A continuously varying convex combination of original signal values and local averages, directed by local statistics, has shown itself quite efficient in removal of signal-dependent noise [3, 68].

Operation of the LLMMSE filter is summarized by

\[ \hat{\lambda}_j = \bar{\lambda}_j + \frac{\sigma^2_{s,j}}{\sigma^2_{s,j} + \sigma^2_{n,j}} (\lambda_j - \bar{\lambda}_j), \]  

where \( \bar{\lambda}_j \) is the local average count value, intended to provide the mean of the Poisson-Gaussian variates. Each raw measurement \( \lambda_j \) is decomposed into the local mean component and the residual prediction error component \( (\lambda_j - \bar{\lambda}_j) \). The parameters \( \sigma^2_{s,j} \) and \( \sigma^2_{n,j} \) are signal and electronic noise variances, respectively. The former is estimated directly from the counts, yielding the approximation \( \sigma^2_{s,j} \approx \bar{\lambda}_j \). The principal supplemental noise \( \sigma^2_{n,j} \) will be the electronics component, modeled as Gaussian, typically at the level of a handful of photons. The weighting of the residual in (2.23) is the approximate ratio of signal variance to observation variance.

The defining equation (2.23) may be rewritten as

\[ \hat{\lambda}_j = \eta_j \lambda_j + (1 - \eta_j)\bar{\lambda}_j, \]  

(2.24)
where $\eta_j \in [0, 1]$ is

$$
\eta_j = \frac{\sigma^2_{s,j}}{\sigma^2_{s,j} + \sigma^2_{n,j}} \approx \frac{\bar{\lambda}_j}{\bar{\lambda}_j + \sigma^2_e},
$$

(2.25)

$$
1 - \eta_j = \frac{\sigma^2_{n,j}}{\sigma^2_{s,j} + \sigma^2_{n,j}} \approx \frac{\sigma^2_e}{\bar{\lambda}_j + \sigma^2_e},
$$

(2.26)

When LLMMSE is applied as pre-processing for MBIR reconstruction in the formulation in (2.21), regularization strength in the log \textit{a priori} $U(x)$ may need adjustment to preserve resolution. Alternatively, the operation of the filter can be restricted to very low SNR regions by thresholding $\eta_j$ with

$$
\hat{\eta}_j = \begin{cases} 
\max\{\eta_j, 0\} & \bar{\lambda}_j \leq \lambda_{TH}, \\
1 & \bar{\lambda}_j > \lambda_{TH}.
\end{cases}
$$

(2.27)

This adaptation allows MBIR to be applied without change to overall regularization levels. Should we reach the case $\eta_j \leq 0$, then $\hat{\lambda}_j$ is fixed by $\bar{\lambda}_j$. The LLMMSE filter does not guarantee positivity in its output, but in practice, the small number of remaining negatives may be effectively zero-weighted in reconstruction. After filtering, the counts $\{\hat{\lambda}_j\}$ are fed into the processes in computing integral projections and corresponding weights described in Section 2.1.

Here the estimate of $\bar{\lambda}_j$ is computed as an average over a small neighborhood, which is sufficient to provide a good approximation of local mean for the LLMMSE filter without excessively degrading spatial resolution. To our knowledge, this type of filter has not previously been applied in CT, and may be desirable only in conjunction with statistical reconstruction, where the data weighting term $W$ in (2.20) helps control the residual noise not removed by the filter. Figure 2.3 shows an example of an LLMMSE denoised signal, with significantly reduced noise level and well-preserved local means. Compared to raw signal $\lambda_j$, few counts remain negative.
Figure 2.3. Low-signal processing of photon-starved measurements $\lambda_j$ using the LLMMSE filter $\hat{\lambda}_j$ and pointwise Bayesian restoration (PBR) $\hat{\theta}_j$ of Section 2.2.2 ($\lambda_j$ are realizations of a Poisson-Gaussian process with $\theta_j = 2$ and $\sigma_e = 4$, and $\hat{\lambda}_j$ and $\hat{\theta}_j$ are computed using a 9-point sliding window.)

2.2.2 Bayesian Inference for Means of Counts

LLMMSE sinogram preprocessing implicitly places a prior distribution on sinogram data that introduces correlation among neighboring channels. Most MAP formulations of the reconstruction problem rely on the independence of sinogram data (when conditioned on the image $x$) and apply prior beliefs for noise suppression only in the image domain. A preprocessing that could more effectively preserve the independence of sinogram data would better fit existing MAP reconstruction methods. To that end, we introduce an alternative approach that allows use of negative-valued measurements without sinogram filtering. This technique may also better prevent positive bias in photon counts than does the LLMMSE. The process results from a Bayesian view of sinograms, but its principal effects are administered pointwise to the data; we therefore entitle it pointwise Bayesian restoration (PBR).
The basis for our MBIR formulation has been to use the best quadratic approximation of the log-likelihood we can produce, such that it can be inserted into the many available optimization routines suited to that form. We may generate, perhaps numerically, a log likelihood from (2.17), and would normally choose the ML value of the projection for our expansion point, but for negative counts the ML value is unbounded, as the log-likelihood has no finite maximum point. We therefore augment this procedure with the application of an a priori density to the mean of the Poisson process. It is hoped that a weak prior model may stabilize the log-likelihood in ultra-low-dose cases sufficiently to utilize a much greater fraction of low SNR data by using the scalar MAP value as input to the remainder of the data preparation.

2.2.2.1 Conjugate Prior

The conjugate prior approach [98], in which the posterior is of the same family as the prior distribution, is natural for our problem, providing a relatively simple parameterization. The prior and posterior are called “conjugate” distributions in this case. If a conditionally Poisson distribution has rate $\theta > 0$,

$$f_K(k|\theta) = \frac{\theta^k e^{-\theta}}{k!}, \quad (2.28)$$

its conjugate is the gamma distribution, characterized by the shape parameter $\alpha > 0$ and the rate parameter $\beta > 0$,

$$f_\Theta(\theta; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}, \quad \theta > 0. \quad (2.29)$$
If the measurement is $K = k$ from the Poisson counts, $\theta$ will have the a posteriori PDF

$$f_{\Theta}(\theta|k) = \frac{f_K(k|\theta) f_{\Theta}(\theta)}{f_K(k)} = \frac{(1 + \beta)^{k+\alpha}}{\Gamma(k + \alpha)} \theta^{k+\alpha-1} e^{-(1+\beta)\theta}. \quad (2.30)$$

The posterior for the Poisson rate has the form of the gamma PDF, with shape parameter of $k + \alpha$ and rate parameter of $1 + \beta$. ML estimation from observation of the Poisson variate provides the unbiased

$$\hat{\theta}_{\text{MLE}} = k \quad (2.31)$$

for a single realization. The PBR approach gives

$$\hat{\theta}_{\text{MAP}} = \mathbb{E}[\Theta|K = k] = \frac{k + \alpha}{1 + \beta}. \quad (2.32)$$

### 2.2.2.2 Parameter Selection

For non-positive measurements, $\hat{\theta}_{\text{MLE},j} = \lambda_j$ poses a problem for the log() operation in converting counts to line-integral densities. Thus, informative prior knowledge is applied to provide a positivity constraint. The true scalar a priori distribution of $\theta_j$ may be derived from a model of the random image $X$. For the present purpose, though, such a distribution for all possible $X$ would likely be far less useful than one that adapts to some partial knowledge of the local characteristics of the projections. Similarly to the LLMMSE method in this regard, we use local statistics to select parameters for the density of a single detector count. Assuming an a priori distribution for $\theta_j$ in (2.17) is given to follow Gamma($\alpha_j, \beta_j$) as in (2.29), we choose a priori
expectation and variance based on the local data statistics so that

\[
E[\Theta_j] = \frac{\alpha_j}{\beta_j} := \bar{\theta}_j, \quad (2.33)
\]

\[
\text{Var}[\Theta_j] = \frac{\alpha_j}{\beta_j^2} := \bar{\theta}_j + \sigma^2_e, \quad (2.34)
\]

where \(\bar{\theta}_j\) is an approximation of the mean of \(\Lambda_j\) using its local average, similar to \(\bar{\lambda}_j\) in (2.23), but with a required positive minimum \(\epsilon_{\theta} > 0\),

\[
\bar{\theta}_j = \max\{\bar{\lambda}_j, \epsilon_{\theta}\}. \quad (2.35)
\]

Then, the parameters of the conjugate prior are calculated as

\[
\alpha_j = \frac{\bar{\theta}_j^2}{\bar{\theta}_j + \sigma^2_e}, \quad (2.36)
\]

\[
\beta_j = \frac{\bar{\theta}_j}{\bar{\theta}_j + \sigma^2_e}. \quad (2.37)
\]

Inserting \(\alpha_j\) and \(\beta_j\) into (2.29) and evaluating the \textit{a posteriori} mean of combined Poisson-Gaussian model of PDF in (2.17), the PBR estimation of Poisson mean is given by

\[
\hat{\theta}_{\text{MAP},j} = \frac{\int \theta_j f_{\Lambda_j}(\lambda_j|\theta_j) f_{\Theta_j}(\theta_j) \, d\theta_j}{\int f_{\Lambda_j}(\lambda_j|\theta_j) f_{\Theta_j}(\theta_j) \, d\theta_j} = \frac{\sum_{k=0}^{\infty} h(k; \alpha_j + 1, \beta_j, \lambda_j)}{\sum_{k=0}^{\infty} h(k; \alpha_j, \beta_j, \lambda_j)} > 0, \quad (2.38)
\]

where

\[
h(k; \alpha_j, \beta_j, \lambda_j) = \exp\left\{-\frac{(\lambda_j - k)^2}{2\sigma^2_e}\right\} \Gamma^{-1}(k + 1) \\
\Gamma(k + \alpha_j)(1 + \beta_j)^{-(k + \alpha_j)}. \quad (2.39)
\]
Compared to the LLMMSE denoising method, PBR has potential advantages in retaining greater channel independence and enforcing positivity, as suggested by Figure 2.3. Figure 2.4 shows that relatively high-count measurements associated with small local means $\bar{\lambda}_j$ are less suppressed with PBR than with LLMMSE denoising, illustrating the potential of the Bayesian method to preserve resolution better than LLMMSE by retaining more of these local deviations. Also, the Bayesian method appears more robust to inaccurate parameter selection in the design of the local sinogram transformation. As shown in Figure 2.4, with the same perturbation on the model parameter $\bar{\lambda}_j$, the resultant PBR estimates vary less than do the LLMMSE estimates. For cases in which the observed photon count value $\lambda_j$ is close to the chosen parameter $\bar{\lambda}_j$ ($\lambda_j \approx 2.5$ in Figure 2.4), the respective shifts in the two methods’ curves due to changing $\bar{\lambda}_j$ are approximately equivalent. However, for $\lambda_j \approx 15$, for example, the two curves for the LLMSE method have a greater distance between them, while the PBR curves show significantly less difference. To fix the regularization strength outside photon-starved regions, we apply PBR only to low-count regions when $\bar{\lambda}_j \leq \lambda_{TH}$, similar to the treatment used in LLMMSE denoising.

2.3 Results and Discussion

We applied the proposed methods to helical CT scan data acquired on a Discovery CT750 HD scanner (GE Healthcare, Waukesha, WI). All data were composed of 984 views per rotation, with a detector array made up of $888 \times 64$ sub-detectors, having maximum collimation of 40 mm at isocenter. A large bowtie filter was used. The electronic noise level was pre-calibrated from a scan with no tube current, yielding a standard deviation of $\sigma_e = 3.9$ photon counts, assumed constant across channels. All data were water-corrected for beam hardening, but not otherwise treated for polychromatic effects. Subsequently, model-based iterative reconstruction (MBIR)
results were achieved by applying the $q$-GGMRF as a priori image model \cite{13,110} and non-homogeneous iterative coordinated descent (ICD) for optimization \cite{124}. The reconstruction field of view was 50 cm. Resolution of reconstructed images was $512 \times 512$ pixels, with slice thickness of 0.625 mm.

2.3.1 Phantom Study

The “low-signal phantom” (LSP) provides a highly varying attenuation profile with sufficiently low minimum counts in some projection paths to test the techniques discussed in Section 2.2. The phantom is oval-shaped in cross-section with major axis of 45 cm and minor axis of 25 cm. Two Teflon rods were inserted parallel to the $z$-axis in imitation of large bones. The phantom was scanned at both high (120 kVp, 835 mA) and low (120 kVp, 20 mA) dosage settings, with speed of 1.0 sec/rotation, 20 mm collimation at normalized pitch of 31/32. Figure 2.5 shows comparison be-
between FBP and MBIR at high dosage, when low-count measurements are not encountered. The same pretreatment in converting counts to line integral projections has been applied for both algorithms. As shown, MBIR produces higher quality image than FBP with apparent reduction of noise, homogeneous texture across the plane and fine details.

The low dosage experiment begins in Figure 2.6 (a), where the FBP with the standard filtering kernel, employing an adaptive low-pass filter to fill negative measurements [48], is affected by both low SNR and significant bias in the horizontal, high-attenuation paths. The high dose reference Figure 2.5 (b) is essentially free of beam-hardening type artifacts using standard water-correction treatment, suggesting that local biases in low-dosage examples derive from low-signal photon-counting phenomena, when some detectors reach photon starvation level and register negative measurements. General interpolation or smoothing type approaches negatively bias projection estimates, as observed in the accompanying low-frequency dark shading artifacts along the horizontal rays of large attenuation. Statistical iterative methods, on the other hand, have the convenience of discarding negative measurements by putting zero weights on them. As shown in Figure 2.6 (b), drastic reduction of noise and artifacts is achieved, compared to Figure 2.6 (a), but there are remaining residual negative biases in horizontal high attenuation paths.

To improve projection estimation in low-count regions, we first considered the use of the LLMMSE filter as proposed in (2.24) to denoise low SNR signals. \( \tilde{\lambda}_j \) was computed as a local \( 3 \times 3 \) average in order to provide only the minimum amount of necessary local smoothing, and the threshold \( \lambda_{TH} = 3\sigma_e \) was used across the sinogram. Thus only areas with local average counts on the order of electronic noise standard deviation were affected by the adaptive filter. As a low-signal treatment, the LLMMSE filter can be applied for both FBP and MBIR preprocessing steps. For FBP, LLMMSE was applied before adaptive smoothing in counts domain; while for
MBIR, LLMMSE was followed by the log() operation, and few remaining negative channels were zero-weighted in reconstruction. For this low-dose LSP scan, 80% of the originally registered negative measurements became positive after denoising. In Figures 2.6 (c) and 2.6 (d), both FBP and MBIR results show significant improvement in terms of biases. For the FBP reconstruction approach, all measurements were equally weighted. Figure 2.6 (c) is considerably noisier than 2.6 (a), especially in the areas where high density projections passed, because the adaptive smoothing was less aggressive and thus those projections have higher variances. However, due to dynamic statistical weighting in (2.20) and the regularization model $U(x)$ in [13], MBIR obtained results with further reduced biases and relatively uniform noise behavior.

With the Bayesian inference method, the same threshold of $\lambda_{TH} = 3\sigma_e$ was used to apply the MAP estimate for low-count regions. A minimum value $\epsilon_\theta = 0.01$ was assumed for the floor of the $a\ priori$ means. The result in Figure 2.6 (f) shows the advantage of using all measurements for reconstruction. Both visual and quantitative assessments favor Figures 2.6 (d) and 2.6 (f) over Figure 2.6 (b). The statistical weights play an important role in balancing the information and noise carried by each
Figure 2.6. Low-dose LSP reconstructions with different algorithms applied on photon-starved regions. Input photon count $\lambda_{0,j}$ at center of bowtie was approximately $2.4 \times 10^4$. (a): FBP with adaptive smoothing; (b): MBIR with no low-signal processing and zero-weights for negative measurements; (c): FBP with LLMMSE before adaptive smoothing; (d): MBIR with LLMMSE and zero-weights for residual negative channels; (e): FBP with PBR before adaptive smoothing; (f): MBIR with PBR. Display window width 400 HU.
Figure 2.7. Low-dose LSP chart of means (HU) and standard deviations (error bars) at selected ROI’s in the images shown in Figure 2.6.

projection. On the other hand, plugging in PBR to conventional FBP preprocessing led to an increased non-homogenous noise pattern, shown in Figure 2.6 (e).

The chart in Figure 2.7 measures the quantitative improvements of results in Figures 2.6 (b), (d) and (f) at selected regions of interest (ROI’s). Both proposed methods consistently outperformed the conventional zero-weighting model. Reference values were measured from Figure 2.5 (b).

Line integral projection estimations at view angles with highest attenuations are plotted in Figure 2.8. Of the central 200 channels of measurements at this angle, 33% of registered counts are negative, shown as absent data points in Figure 2.8 (b). The adaptive smoothing filter used in FBP pretreatment is very aggressive and caused negative bias and loss of resolution. Both LLMMSE and PBR methods, however, better preserved local means as well as channel-to-channel variations. Combining LLMMSE or PBR with FBP’s adaptive smoothing is plausible, but needs further adjustment to control noise amplification.
Figure 2.8. Low-dose LSP line integral projection estimates with different algorithms applied on photon-starved regions. (a): adaptive smoothing; (b): no low-signal processing and discarding negative measurements; (c): concatenating LLMMSE and adaptive smoothing; (d): LLMMSE with residual negative channels discarded; (e): concatenating Bayesian inference and adaptive smoothing; (f): Bayesian inference.
2.3.2 Low-Dose Clinical Application

We also applied the proposed methods to clinical data, scanned at 120 kVp, 20 mA, with 40 mm collimation, at pitch ratio of 63/64. In this circumstance, the object was more rounded, and there is no particular path that has apparently higher attenuation than others. However, the input flux was low enough that there was a small portion of measurements falling into negative territory. In Figure 2.9 (b), although there is no localized bias, the overall image suffers from loss of density. LLMMSE denoising and PBR inference achieved similar performance in preserving local means from otherwise corrupted signals. Results in Figures 2.9 (c) and 2.9 (d) are substantially improved and closer to their true HU values, compared to the high dose reference in Figure 2.9 (a). The high dosage scan (120 kVp, 400 mA) was taken of the same patient but at a different time. (For reference purpose, some of the anatomies may not align perfectly.) Measured means and standard deviations are shown in Figure 2.10. To compute $\bar{\lambda}_j$ and $\bar{\theta}_j$ for the two methods, a $3 \times 3$ neighborhood was selected and processing was applied when $\bar{\lambda}_j \leq 5 \sigma_e$.

2.3.3 Ultra-Low-Dose Clinical Application

We extended the low-signal study with an ultra-low dose clinical scan, taken at 80 kVp, 10 mA, with 40 mm collimation, at pitch ratio of 33/64. A high dosage (120 kVp, 310 mA) reference scan of the same patient was taken in Figure 2.11 (a) with the illustrated image nearly co-located with low-signal reconstructions. Figure 2.11 (b) shows a heavily negatively biased image, suggesting that a significant number of negative measurements was registered across all view angles and zero-weighting negatives was no longer feasible to achieve informative reconstruction. Figures 2.11 (c) and 2.11 (d) used a $5 \times 5$ neighborhood to compute $\bar{\lambda}_j$ and $\bar{\theta}_j$, and the threshold for low-count region was determined when $\bar{\lambda}_j \leq 5 \sigma_e$. In spite of the nearly complete photon starvation in some projections, the ensemble of mea-
Figure 2.9. Reconstruction comparison of a low-dose abdomen scan. Input photon count $\lambda_{0,j}$ at center of bowtie was approximately $1.1 \times 10^4$. (a): high-dose MBIR as reference; (b): low-dose MBIR with zero-weights on negatives; (c): low-dose MBIR with LLMMSE filter for low-signal denoising; (d): low-dose MBIR with Bayesian inference for low-signal processing. Display window width 400 HU. (Data courtesy of M.K. Kalra and A. Padole of Massachusetts General Hospital.)
measurements still contained significant information from the chest anatomy. The PBR result in Figure 2.11 (d) outperformed LLMMSE in 2.11 (c) marginally, by preserving higher definition boundaries and less bias. Quantitative improvements of means and standard deviations at selected ROI’s are shown in Figure 2.12. A zoomed-in comparison is also available in Figure 2.13. In practice, such extremely low-count data may occur primarily in attenuation scans for PET/CT systems, in which case image quality may be secondary in importance to appropriate local mean values.

2.4 Conclusion

The two methods presented in this chapter have shown promise in dealing with very limited-count CT data. From preliminary results of handling non-positive measurements, it appears that LLMMSE denoising achieves comparable image quality to the more sophisticated Bayesian inference at moderately low-dose scenarios under statistical reconstruction framework. But in extreme cases of photon starvation,
Figure 2.11. Reconstruction comparison of an ultra-low-dose chest scan. Input photon count $\lambda_{0,j}$ at center of bowtie was approximately $1.4 \times 10^3$. (a): high-dose MBIR as reference; (b): low-dose MBIR with zero-weights on negatives; (c): low-dose MBIR with LLMMSE filter for low-signal denoising; (d): low-dose MBIR with Bayesian inference for low-signal processing. Display window width 600 HU. (Data courtesy of M.K. Kalra and A. Padole of Massachusetts General Hospital.)
PBR showed more capability in avoiding biases. As a pretreatment in projection estimation, both methods have the potential of being applied for analytical reconstruction algorithms, but will require further adjustment. Our subsequent work with these innovations will include more extensive, quantitative phantom studies and other clinical applications.

This work has proceeded from the point of view that maintaining the simplicity of a quadratic model is important for computational savings in evaluation of derivatives of the log-likelihood. Therefore, single-pass sinogram treatments were designed, assuming that the log() operation would be performed on counts-domain data before initiation of reconstruction. The degree of importance of the quadratic objective form may vary substantially with the type of optimization followed in achieving the MBIR reconstruction. We have oriented these techniques toward sequential update methods, in which pre-log data operations could be quite costly due to non-linear operations in computation of the derivative at each voxel update. For global update
methods, such as are typically used in ordered-subsets, gradient-type algorithms [64],
the relative cost of using data retaining the pre-log form in place of least squares is
potentially much lower than in the sequential case. There is likely additional benefit
to statistical modeling in MBIR in exploiting this pre-log formulation. Subsequent
publications address the comparison between the methods in this chapter and more
accurate, more costly pre-log modeling [31].
CHAPTER 3
UNIFORM NOISE REGULARIZATION DESIGN IN CT IMAGE RECONSTRUCTION

The quality of information available may vary widely across a CT reconstruction, as attenuation may reduce photon count levels by many orders of magnitude in the presence of dense materials or large objects. The noise modeling discussed in Chapter 2 associates statistical weight to each datum, as a measure of its credibility. However, with standard global regularization scaling, the maximum a posteriori probability (MAP) estimate in (2.21) will smooth noise to varying degrees and with varying spatial noise correlation structure across the volume to be estimated. While this non-stationarity in noise may be an inherent feature of the optimal estimate in the sense in which we have formulated the problem, spatial uniformity of noise characteristics may be desirable in practice. Within the framework of the prior weighting for uniform resolution of Fessler and Rogers [28], we propose an alternative set of prior weights to achieve approximately uniform noise covariance.

3.1 Background of Regularization Design

In statistical reconstruction, the image resolution and noise character are generally improved from analytical reconstruction methods [104, 110, 119]. However, the iterative ML estimate yields an undesirably noisy result. The MAP approach encourages smoothness between neighborhood pixels by incorporating an image prior as regularization term [5, 73]. The term $U(x)$ in (2.21) can be formulated as a simple quadratic penalty of neighboring pixel differences, or a stochastic model including
more flexibility in the sanction of image characteristics \cite{76}. The majority of common models is described by

\[ U(x) = \frac{1}{f(\zeta)} \sum_{(i, k) \in C} b_{i, k} \rho(x_i - x_k), \quad (3.1) \]

with \( \rho() \) a symmetric function of local pixel/voxel differences, \( b_{i, k} \) weights according to relative spatial location of neighbors, \( C \) is the collection of all pairs that contribute to the penalty, and \( f() \) is a scaling factor related to the expectation of local image differences. Traditionally, a global penalty strength \( \frac{1}{f(\zeta)} \) is chosen deterministically or interactively to balance the trade-off between noise and resolution \cite{33, 74, 113}; then the \textit{a priori} image model is stationary. But it is difficult to achieve consistent performances across different scan objects and settings, given the information non-uniformity. Although numerous options for design of \( \rho() \) are available in the literature \cite{5, 6, 8, 35, 40, 93, 110}, there are fewer available results and less consensus on the so-called “hyper-parameters” that control the balance between fidelity to data and faithfulness to \textit{a priori} modeling.

The Hessian of the log-likelihood, \( A^T W A \), may have several orders of magnitude of dynamic range in its diagonal entries, due to proportional variation in the weighting matrix \( W \). X-ray survival rates may range from 1 to \( 10^{-4} \) or less in common scans through dense tissue and large patients. All common Bayesian estimators adjust between adherence to prior modeling and fidelity to data as SNR varies. If we assume a stationary image model, with the scalar \( f(\zeta) \) set heuristically to a good compromise between resolution and noise suppression, those two qualities may vary widely in their balance across an image.

To generalize the regularization scaling selection as well as combat the signal-dependent spatially non-uniform noise concern, Fessler and Rogers \cite{28} provided a framework for spatially-adaptive control of the trade-off between fidelity to data and
image noise attribution with the modification of (3.1) to

\[ \hat{U}(x) = \frac{1}{f(\zeta)} \sum_{(i,k) \in C} b_{i,k} \hat{\kappa}_i \hat{\kappa}_k \rho(x_i - x_k), \]  

(3.2)

where each \( \hat{\kappa}_i \) is square-root of the \( i^{th} \) diagonal element of the approximate Fisher information matrix \( A^TWA \), as \( \hat{\kappa}_i = (\sum_{j=1}^{M} A^2_{ji} w_j)^{1/2} \). It is to associate individual pixel with a scalar for non-stationary image prior model. The goal of [28] was to achieve approximately constant resolution in the image estimate in the face of widely varying pixel-wise Fisher information. We will seek constant noise characteristics across the same variations [13]. This generalization is also the subject of a more recent paper [15]. Mathematically our approach to the problem is different from the above, but we arrive at a similar form for \( \hat{\kappa}_i \) in practice.

3.2 Uniform Noise Regularization

For the sake of tractability, we begin with the case in which \( U(x) \) is a quadratic function, or \( U(x) = \frac{\zeta}{2} x^T R x \), with \( R \) having unity on the diagonal and negative values totaling unity in each set of off-diagonal row or column entries, as

\[
R_{ik} = \begin{cases} 
1 & i = k, \\
-b_{ik} & i \neq k.
\end{cases}
\]  

(3.3)

This is the conventional Gaussian \( a \ priori \) image model when \( f(\zeta) = \frac{2}{\zeta} \). We then arrive at the MAP estimator from (2.21) as a linear function of sinogram data:

\[ \hat{x} = (A^TWA + \zeta R)^{-1} A^T Wy. \]  

(3.4)
Given that \( y \) contains additive (approximately) zero-mean noise \( n \) in the sinogram vector, we may also write the image-domain error due to that noise as

\[
\hat{e} = (A^TWA + \zeta R)^{-1}A^T W n.
\]  

(3.5)

The covariance of the noise in the reconstructed image has the form \([27]\)

\[
\text{cov}(\hat{e}) = (A^TWA + \zeta R)^{-1}A^T WE[nn^T]W(A^TWA + \zeta R)^{-1}
\]

\[
= (A^TWA + \zeta R)^{-1}A^T \tilde{W} A (A^TWA + \zeta R)^{-1}.
\]  

(3.6)

The matrix \( \tilde{W} \) is diagonal, with elements \( w_j^2 \xi_j^2 \), with \( \xi_j^2 \) the Poisson-Gaussian signal dependent variance of the \( j^{th} \) sinogram entry, as derived in Chapter \([2]\) If we use conventional statistical weighting, \( w_j \approx \xi_j^{-2} \) and \( \tilde{w}_j \approx w_j \). \( A^T W A \) is low-frequency, approximated in the 2D in-plane frequency domain by \( 1/\|\omega\| \) when \( W \) is constant on its diagonal and zero elsewhere. \( R \) is a high-frequency penalizing function and therefore will have its larger eigenvalues matched with high frequency eigenvectors. Figure \( 3.1 \) demonstrates the in-plane frequency responses of \( A^T W A \) and \( R \) respectively.

If we approximate the operator \( (A^TWA + \zeta R) \) as having its eigenvalues separated between the first and second terms into components at low and high frequencies, we may rewrite \( (3.6) \) as

\[
\text{cov}(\hat{e}) \approx (A^TWA)^{-1}(A^T \tilde{W} A)(A^TWA)^{-1} + \zeta^{-2}R^{-1}A^T \tilde{W} A R^{-1}.
\]  

(3.7)

Each of these terms represents a “tail” of an operator away from its passband. Should we use weighting in the likelihood term with \( W \approx \tilde{W} \) and the first term is recognizable as the inverse of the Fisher information matrix, which would supply a lower bound on covariance for the ML estimator. Since the \( a \ priori \) density assigns little penalty to low frequencies, it is to be expected that the noise covariance would have the given
Figure 3.1. In-plane frequency responses of $A^TWA$ and $R$.

form in this lightly regularized portion of the spectrum.

For visual quality of many CT images, the effects of noise having the greatest impact are those at higher spatial frequencies, and we consider the second term of (3.7). The penalty enforced by a conventional low-order Markov model described by $R$ is high-frequency (for encouragement of low-pass behavior) which makes $R^{-1}$ a low-pass operator. We approximate $R^{-1}$ as constant in its stop-band. The second term of (3.7) then simplifies to

$$c_2 \zeta^{-2} A^T \tilde{W} A,$$

an operator whose spatial characteristics are not difficult to visualize. The local covariance will have the familiar $1/|x|$ shape, but has strong tails in the direction of rays with large entries in $\tilde{W}$, corresponding to high count sinogram entries when $W$ has detector photon counts on its diagonal. This long spatial correlation may be the statistical characterization of elevated probability of noise streaks in low-attenuation
directions. Setting \( w_j = \xi_j^{-1} \) restores the isotropic covariance by making \( \tilde{W} = I \). In the X-ray CT problem, \( \xi_j^{-1} \) is approximated by \( \sqrt{\lambda_j} \), where \( \lambda_j \) is the observed \( j^{th} \) photon count. This modification is a substantial change to the log-likelihood function, departing from the Poisson counting model but previously shown to offer advantages in some CT nondestructive testing applications \[66\].

We are interested particularly in the diagonal entries of \( \text{cov}(\hat{e}) \) for local noise variance. For the sake of our next approximate analysis, we let \( \tilde{W} = W \) such that the first term of (3.7) reduces. We know from theory of positive definite matrices that

\[
[(A^TWA)^{-1}]_{ii} \geq \frac{1}{(A^TWA)_{ii}}. \tag{3.9}
\]

Since the spatial structure of the \( A \) matrix does not typically vary greatly over varying scans, we conjecture that (3.9) may, for local approximations, be made an equality with an appropriate scaling constant on one side. Let us furthermore approximate the low-pass filter \( R^{-1} \) as being flat across the high frequencies to which the second term in (3.7) corresponds, such that it may be replaced by a constant. The mean-squared noise error at voxel \( i \) can now be approximated by

\[
\text{Var}(\hat{e}_i) \approx \frac{c_1}{\sum_{j=1}^{M} w_j A_{ji}^2} + \frac{c_2 \sum_{j=1}^{M} \tilde{w}_j A_{ji}^2}{\zeta^2}. \tag{3.10}
\]

This rough analysis tells us that, with \( \zeta \) fixed, noise variance at low frequencies drops as the inverse of effective voxel dosage (EVD) (as measured by photon counts times sampling rates times projection weights squared). This is exactly the asymptotic behavior we would expect in an ML estimate. Somewhat surprisingly but consistent with empirical results, high frequency noise variance rises proportionally to EVD.

Should we wish to maintain constant noise variance across EVDs, a starting point may be to ignore the low frequency noise components, and attempt to fix the second
term in (3.10). In relatively fully sampled geometries, low frequencies are much more heavily sampled than high frequencies. Thus we may replace our current $\zeta$ with

$$
\zeta_{\text{EVD}} = \zeta \left( \sum_{j=1}^{M} \tilde{w}_j A_{ji}^2 \right)^{\frac{1}{2}}.
$$

(3.11)

Maintaining the variables $\{\kappa_i\}$ as prior scaling as in [28], this would correspond to

$$
\kappa_i = \left( \sum_{j=1}^{M} \tilde{w}_j A_{ji}^2 \right)^{\frac{1}{4}}.
$$

(3.12)

3.3 Results

As illustration of the effects of adaptive regularization with uniform noise levels as goal, we compare spatial noise variation of three a priori weighting schemes in scans of a human torso phantom, with modest tube current of 25 mA. This places our data in the lower end of practical dosage range, in keeping with the theme of this chapter. Additionally, a second dosage level of 200 mA shows the behavior of the method under varying global signal-to-noise levels.

The first of the models is the standard penalty of (3.1), in which the regularizing prior term is spatially invariant. This allows the balance between the log-likelihood and prior term to vary according to local Fisher information, with lower dosage areas, featuring greater uncertainty, being smoothed more severely. The second is the adaptive weighting of (3.2), intended to maintain nearly constant resolution while allowing noise variance to respond accordingly, becoming more prominent in low-count areas. Finally, we add our proposed modification of (3.12) in the hope of maintaining noise levels which vary little over both spatial displacement and overall dosage. The overall strength of regularization for each technique is adjusted manually to yield approximately equal noise levels among the methods in soft tissues near the surface of the abdomen at 25 mA.
Not surprisingly, since the proposed model is designed for the sake of noise uniformity, it shows superior noise control in this sense. In Figure 3.2, we compare axial images from the abdomen area, where the data-independent prior scaling produces over-smoothed texture at image central region, while the constant resolution kappa scaling leaves the central region under-regularized. The proposed method generates an image with far less spatial variation in noise levels. Measured means and standard deviations in selected ROI’s are listed in Table 3.1. Figures 3.4 and 3.6 show coronal and sagittal images, respectively, of the phantom. With data-independent prior scaling, we again observe that the region with lower counts is generally over-regularized, with attendant loss of spatial resolution. In contrast, the constant resolution kappa appears to under-regularize the low-counts area. Some anatomical structures are compromised in the consequently higher noise. In Figures 3.4 (a) and 3.6 (a), abdominal tissues near the spine are over-smoothed, and some high frequency information is attenuated, and the variance in myocardium tissue is significantly higher than in any other regions. Figures 3.4 (b) and 3.6 (b) show the opposite situation, where the standard deviation in the central abdominal region can be as high as triple that of cardiac tissue. The proposed prior scaling method again achieves a more spatially consistent balance between image spatial resolution and noise suppression. The noise strength throughout the body volume manages to stay relatively constant, shown in Figures 3.4 (c) and 3.6 (c). In practice, constant noise kappa scaling would appear to serve better for diagnostic purposes.

Raising the tube current to 200 mA reduces sinogram data variance by a factor of approximately 8 and would therefore be expected to reduce image noise standard deviation by approximately 2.8 in conventional reconstruction methods. Regularizing for constant resolution will amplify the difference yet more strongly. Our uniform noise approach limits the noise deviation reduction to roughly a factor of 1.5 between the two dosage levels, and channels the improved SNR primarily into resolution.
improvement at higher dosage, as illustrated in image (d) of Figures 3.2, 3.4, and 3.6. Various homogeneous ROIs across the image volume are selected to demonstrate the noise uniformity. ROI means and standard deviations are included in Tables 3.1, 3.2, and 3.3. The charts in Figures 3.3, 3.5, and 3.7 plot the ROI standard deviations, to demonstrate the quantitative improvements of proposed method.
TABLE 3.1

NOISE MEASUREMENTS (HU) AT SELECTED ROI’S IN TORSO AXIAL IMAGES IN FIGURE 3.2

<table>
<thead>
<tr>
<th>Regularization</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>data-indep. prior</td>
<td>-2.05</td>
<td>-1.79</td>
<td>-6.42</td>
<td>-3.65</td>
<td>mean</td>
</tr>
<tr>
<td>(25 mA)</td>
<td>8.50</td>
<td>10.85</td>
<td>12.63</td>
<td>12.24</td>
<td>std</td>
</tr>
<tr>
<td>const resolution $\kappa$</td>
<td>-2.87</td>
<td>-2.49</td>
<td>-6.33</td>
<td>-3.99</td>
<td>mean</td>
</tr>
<tr>
<td>(25 mA)</td>
<td>15.16</td>
<td>11.93</td>
<td>8.16</td>
<td>11.86</td>
<td>std</td>
</tr>
<tr>
<td>uniform noise $\kappa$</td>
<td>-2.73</td>
<td>-2.21</td>
<td>-6.28</td>
<td>-3.87</td>
<td>mean</td>
</tr>
<tr>
<td>(25 mA)</td>
<td>13.51</td>
<td>13.46</td>
<td>12.49</td>
<td>13.33</td>
<td>std</td>
</tr>
<tr>
<td>uniform noise $\kappa$</td>
<td>-2.81</td>
<td>-3.29</td>
<td>-5.35</td>
<td>-4.51</td>
<td>mean</td>
</tr>
<tr>
<td>(200 mA)</td>
<td>10.98</td>
<td>9.69</td>
<td>9.11</td>
<td>9.99</td>
<td>std</td>
</tr>
</tbody>
</table>

Figure 3.3. ROI standard deviation chart corresponding to Table 3.1.
Figure 3.4. Torso coronal images. (a): data-independent prior scaling at 25 mA; (b): constant resolution $\kappa_i$ at 25 mA; (c): uniform noise $\kappa_i$ at 25 mA; (d): uniform noise $\kappa_i$ at 200 mA. Window width 200 HU.
**TABLE 3.2**

NOISE MEASUREMENTS (HU) AT SELECTED ROI’S IN TORSO CORONAL IMAGES IN FIGURE 3.4

<table>
<thead>
<tr>
<th>ROI</th>
<th>ROI 1</th>
<th>ROI 2</th>
<th>ROI 3</th>
<th>ROI 4</th>
<th>ROI 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>data-indep. prior (25 mA)</td>
<td>-2.33</td>
<td>32.15</td>
<td>76.08</td>
<td>3.48</td>
<td>-3.60</td>
</tr>
<tr>
<td>mean</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>12.36</td>
<td>17.10</td>
<td>7.56</td>
<td>6.52</td>
<td>14.24</td>
</tr>
<tr>
<td>const resolution κ (25 mA)</td>
<td>-3.35</td>
<td>32.06</td>
<td>76.45</td>
<td>4.71</td>
<td>-5.24</td>
</tr>
<tr>
<td>mean</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>8.78</td>
<td>7.06</td>
<td>30.99</td>
<td>19.92</td>
<td>9.33</td>
</tr>
<tr>
<td>uniform noise κ (25 mA)</td>
<td>-2.63</td>
<td>31.81</td>
<td>76.96</td>
<td>4.20</td>
<td>-3.58</td>
</tr>
<tr>
<td>mean</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>12.69</td>
<td>12.21</td>
<td>15.93</td>
<td>13.41</td>
<td>13.37</td>
</tr>
<tr>
<td>uniform noise κ (200 mA)</td>
<td>-3.24</td>
<td>30.69</td>
<td>70.31</td>
<td>1.34</td>
<td>-7.77</td>
</tr>
<tr>
<td>mean</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>8.25</td>
<td>9.08</td>
<td>11.10</td>
<td>11.43</td>
<td>7.95</td>
</tr>
</tbody>
</table>

Figure 3.5. ROI standard deviation chart corresponding to Table 3.2.
Figure 3.6. Torso sagittal images. (a): data-independent prior scaling at 25 mA; (b): constant resolution $\kappa_i$ at 25 mA; (c): uniform noise $\kappa_i$ at 25 mA; (d): uniform noise $\kappa_i$ at 200 mA. Window width 200 HU.
### Table 3.3

*Noise Measurements (HU) at Selected ROIs in Torso Sagittal Images in Figure 3.6*

<table>
<thead>
<tr>
<th></th>
<th>ROI 1</th>
<th>ROI 2</th>
<th>ROI 3</th>
<th>ROI 4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>data-indep. prior</strong>&lt;br&gt;(25 mA)</td>
<td>4.27</td>
<td>36.32</td>
<td>2.68</td>
<td>-6.16</td>
<td>mean</td>
</tr>
<tr>
<td><strong>const resolution</strong>&lt;br&gt;κ&lt;br&gt;(25 mA)</td>
<td>1.63</td>
<td>35.18</td>
<td>5.35</td>
<td>-6.75</td>
<td>mean</td>
</tr>
<tr>
<td><strong>uniform noise</strong>&lt;br&gt;κ&lt;br&gt;(25 mA)</td>
<td>3.61</td>
<td>35.60</td>
<td>4.43</td>
<td>-6.07</td>
<td>mean</td>
</tr>
<tr>
<td><strong>uniform noise</strong>&lt;br&gt;κ&lt;br&gt;(200 mA)</td>
<td>0.62</td>
<td>37.39</td>
<td>1.66</td>
<td>-7.93</td>
<td>mean</td>
</tr>
</tbody>
</table>

Figure 3.7. ROI standard deviation chart corresponding to Table 3.3.
3.4 Discussion and Conclusion

The proposed, spatially variant regularization to achieve uniform noise relies on the separation of eigenvalues of the matrix \((A^TWA + R)\) into low and high frequency components. At medium to high dosage, the greater part of the noise power spectrum in the MBIR reconstruction is found in the upper spatial frequencies, in agreement with the assumptions made in our analysis. Asymptotically with increasing dosage, the \(A^TWA\) term dominates an ever larger fraction of the spectrum, as increasing information in the data overpowers the regularization kernel. When projection data SNR decreases, the noise power spectrum grows most rapidly at lower frequencies, and an ever larger fraction of noise power is concentrated near the cross-over point between the two plots in Figure 3.1, whose corresponding noise power spectrum is illustrated along the radial axis of the \((x, y)\) frequency plane in Figure 3.8. In this case, the separation of the noise into its high and low frequency components according to the respective dominance of the two terms is more questionable, and may require special adaptation. Ultra-low-dose noise uniformity control will therefore require further study.

Our approach does, however, yield relatively robust control of noise levels consistent across spatially varying X-ray penetration and overall clinical-range dosage adjustments. Such uniformity would appear preferable to radiologists for diagnostic purposes. Our subsequent work with these innovations will include clinical applications and more extensive, quantitative phantom studies.
Figure 3.8. Noise spectrum density illustration of a low-dose CT scan.
CHAPTER 4

NONLINEAR SINOGRAM CORRECTION FOR DENSE OBJECTS

4.1 Effects of Dense Objects

Chapter 2 focuses on modeling of monochromatic photon transmission process. Widely used commercial CT scanners, however, emit polychromatic X-ray beams. While most materials absorb low-energy X-rays at higher rates than they absorb high-energy X-ray photons [10, 47, 125], measured line-integral projections are not linearly proportional to the path lengths, which leads to beam hardening effects.

Due to the high percentage of water-like materials (e.g., soft tissues) in human body, water-based beam hardening correction is usually performed as data pretreatment on mainstream clinical CT scanners [42, 78]. However, in the presence of bones, iodine as contrast media, or other high density materials, it becomes insufficient to compensate for all the nonlinear effects of beam hardening. High density materials are usually composed of elements with high atomic number Z, such as \(^{20}\text{Ca},^{26}\text{Fe},^{53}\text{I},\) etc., whose attenuation coefficients’ energy dependencies have larger curvatures than does water, and thus heavier beam hardening effects, as shown in Figure 1.5.

To combat the nonlinearity, polynomial pre-correction methods are most commonly applied. Pre-correction techniques are based on the knowledge of incident X-ray spectrum, two or more base materials’ mass attenuation functions and the assumption that objects can be easily decomposed into these base materials [11, 50, 57]. Similarly, iterative post-processing techniques using the same set of information are also effective [56, 85]. However, in more general CT applications, the scanned objects
may be composed of a number of unknown materials, and these methods meet their limit.

Based on the MBIR framework, several methods were developed to address beam hardening issue \[1, 20, 23, 24, 53, 107\]. De Man et al. proposed a method to decompose the linear attenuation coefficient into a photoelectric component and a Compton scatter component, and use a known X-ray spectrum to model the polychromatic photon acquisition process \[20\]. Elbakri and Fessler presented a statistical polychromatic forward model by segmenting materials into soft tissue and bone, and used ordered-subsets algorithm to compute the iterative updates for image reconstruction \[23\]. They later augmented their approach to handle multiple materials by decomposing each material as a combination of water and bone \[24\]. All these methods require advanced knowledge of the system and are computationally inefficient, depending on the choices of spectrum discretization levels. More recently, Jin et al. proposed a simultaneous MBIR-BHC algorithm, which assumes no knowledge about the spectrum, and estimates a second degree BHC polynomial correction formula jointly with image reconstruction \[53\].

Perhaps the most challenging materials in X-ray CT are metals \[47\]. Compared to bone, which is mainly composed of \(^{20}\text{Ca}\), elements such as \(^{22}\text{Ti}, ^{56}\text{Fe}\) have not only higher attenuation coefficients but also higher volumetric mass densities. Combining the two factors, they absorb significantly more photons, and aggravate the nonlinear effect of beam hardening. In the extreme case of insufficient incident flux, photon starvation is also encountered.

Besides beam hardening effects, scattering is also more prominent in metal. According to the Klein-Nishina formula, many photons that penetrate an object are subject to Compton scatter, and deviate from their original direction. In order not to contaminate primary photon measurements, these amounts must be subtracted from total received photon counts, as described in Chapter \[2\]. The generic scatter
estimation model is largely based on system calibration and/or Monte Carlo simulation using water-like materials. It is usually effective for low density substances, such as soft tissue, but may under-estimate when it comes to dense materials like metal, where the scatter-to-primary ratio (SPR) is much higher. Therefore, a considerable portion of the generic “scatter-corrected” counts are still from scattering effect, and the projection estimates are negatively biased accordingly.

Beam hardening and scatter effects often cannot be distinguished from each other with current photon detection mechanism, and therefore collectively cause metal artifacts. Metal artifacts usually appear as strong streaks, most prominent in the directions with high attenuations (or low counts). The artifacts extend from metal objects through the reconstructed image, and significantly degrade its diagnostic quality. To minimize metal impact, materials with lower attenuation coefficients, or devices with smaller cross-sections are adopted for practical use [41, 99, 115]. The CT scan protocol can also be modified to reduce metal artifacts. Data noise can be reduced by using a higher tube current. Beam hardening can be mitigated using pre-filtered high energy input beams [41].

Figure 4.1. Illustration of metal artifacts from a hip implant. Display window [-100, 100] HU.
Many metal artifact reduction (MAR) methods have been proposed in recent years. The majority of them are based on the assumption that projections involving metal materials are “bad” data, and reconstruct the image with either synthetic replacement or omission of heavily corrupted rays. Inspired by Kalender (1987) [62], sinogram completion methods try to replace the “bad” metal projection data with interpolation [37, 44, 58, 79, 80, 91, 114], pattern recognition [84], or linear prediction methods [105]. Among them, normalized metal artifact reduction (NMAR), proposed by Meyer et al. [80] is most influential. It provides a two-pass framework by using the intermediate image from the first-pass sinogarm inpainting to create a normalized projection space, where the second-pass interpolation may avoid abrupt transition on metal trace boundaries. Alternatively, an iterative approach is used to estimate the “bad” sinogram data. La Rivière [75] proposed penalized-likelihood sinogram smoothing as a pre-processing method, which estimates the metal line-integrals by maximizing a penalized-likelihood objective function. In terms of metal artifacts, IR approaches generally outperform standard FBP reconstruction due to statistical data weighting and a prior image models [19, 70]. Combined with iterative reconstruction, maximum likelihood expectation maximization (ML-EM) algorithm and the algebraic reconstruction technique (ART) are modified to ignore the missing data [75, 90, 116]. Recently, Jin et at. [54] proposed a joint MAR and segmentation algorithm by using a dictionary-based image prior and continuous-relaxed Potts model. However, all of the methods have their limits and are not able to achieve consistency across varied iterative reconstruction applications.

In this chapter, a physics-based multi-material polynomial correction is incorporated into the MBIR framework, as a data pretreatment procedure. Secondly, to combat metal artifacts, two iterative methods are proposed. Metal-corrupted projection data are not completely discarded but rather utilized to estimate the sinogram biases from beam hardening and scatter effects.
4.2 Multi-Material Correction in Pre-Processing

Multi-material correction (MMC) is a physics-based artifacts reduction technique. It is derived from careful calibration and polynomial regression model training \[120\]. It provides flexibility of handling bone and iodine (from contrast injection) induced artifacts. Applying it directly to MBIR projection data would lead to changes in noise characteristics. Therefore, selection of the first-pass image and adjustment of statistical weights are analyzed, in order to fully incorporate MMC into MBIR pre-processing steps.

4.2.1 First-Pass Image

In order to perform MMC, a segmentation procedure in image space is done to divide dense materials into bone and iodine categories. Depending on the specific mass densities, these materials are then decomposed into a linear combination of two basis materials: water and iodine. Forward projecting of the re-mapped iodine image gives us an iodine sinogram, to which the MMC formula can be applied. The MMC technique is essentially a fifth degree polynomial correction with two input variables: total projection value from data pre-processing and the pre-computed iodine sinogram value. (See coefficients example in Appendix \[B\].) Naturally, the selection of the first-pass image, used for image segmentation and material decomposition, is the first factor to consider in adapting MMC to MBIR framework.

For analytical reconstruction, low resolution \((320 \times 320)\) full FOV images are used to perform MMC. But our past experience with polynomial bone beam hardening correction, derived from (7.24) of \[47\], a similar spectral correction technique for bone material, suggests that MBIR can be very sensitive to inaccurate corrections near dense object projection trace boundaries, caused by pixelization in low resolution images. This has required that conventional polynomial BHC be applied at later stages of multi-resolution MBIR, at the cost of greater computation due to
adjustment of first-pass images. To incorporate MMC economically, several first-pass image strategies are evaluated in terms of both BHC performance and computational efficiency.

Besides the option of low resolution FBP, two other choices are under consideration for the first-pass image: high resolution \((512 \times 512)\) FBP and high resolution \((512 \times 512)\) MBIR. We pick two of the most challenging scenarios for the performance test. In Figure 4.2, MBIR images without MMC appear non-homogeneous in white matter, especially near bone structure. Particularly a dark shading is formed between left and right inner ear bone tissues. These problems show the inadequacy of water-based correction.

We applied MMC with three different first-pass image settings in pre-processing, and compare reconstruction results in Figures 4.3 and 4.4. Overall, the three MMC projection corrections are very effective in reducing bone-induced artifacts. In Fig-

Figure 4.2. MBIR reconstructed images without MMC of head data scanned on a GE Discovery CT750 HD scanner in circumstances of (a) metal presence in white matter and (b) voluminous inner ear bone structures. Display window \([-60, 140]\) HU.
Figure 4.3. Applying MMC to MBIR projection data in presence of cortical bones and metal. Top row: MMC first pass images with (a) low resolution FBP, (b) high resolution FBP and (c) high resolution MBIR; Middle row: MBIR result with MMC (d) using low-res FBP, (e) using high-res FBP and (f) using high-res MBIR; Bottom row: (g) \( \text{diff} = (d) - (c) \); (h) \( \text{diff} = (e) - (d) \); (i) \( \text{diff} = (f) - (d) \); Display windows for images (a) through (f) are \([-60, 140]\) HU, for (g) through (i) are \([-50, 50]\) HU.
Figure 4.4. Applying MMC to MBIR projection data with voluminous inner ear bone structures. Top row: MMC first pass images with (a) low resolution FBP, (b) high resolution FBP and (c) high resolution MBIR; Middle row: MBIR result with MMC (d) using low-res FBP (d), (e) using high-res FBP and (f) using high-res MBIR; Bottom row: (g) diff = (d) − (c); (h) diff = (e) − (d); (i) diff = (f) − (d); Display windows for images (a) through (f) are [-60, 140] HU, for (g) through (i) [-50, 50] HU.
ures 4.3 (d) through (f), intensity and noise uniformity are both largely improved by applying MMC. Bones appear much sharper and well-defined compared to Figure 4.3 (c). Metal artifacts, though not intended to be fixed by MMC, are also mitigated to smaller impact regions. In Figures 4.4 (d) through (f), similar improved homogeneity in soft tissue is observed. The dark shading between two inner ear pieces is significantly reduced. No apparent drawbacks are noticed with any of the corrections. Detailed comparisons show that Figures 4.3 (f) and 4.4 (f) with MBIR first-pass images have the least amount of artifacts along the high attenuation paths. But considering the computational cost by running MBIR twice, either low or high resolution FBP images are perhaps the most practical solution. In the MMC regression modeling, the correction magnitudes are kept small enough to achieve line-integral density linearization, which manages to maintain the original image noise level and also makes it more forgiving of the errors in the material segmentation.

More experiments are conducted to study the impact of MMC on MBIR, particularly for the mean densities of objects. The Gammex phantom, made of water-like material with different levels of iodine concentration inserts, is scanned on a GE Discovery CT750 HD system of different tube voltage settings, at 140, 120 and 100 kVp respectively. Higher tube voltage determines the higher limit of the X-ray spectrum, and thus less beam hardening effect than lower ones. Tube current is set to 450 mA. System rotation speed is adjusted to achieve similar photon flux across the scans and therefore comparable image noise levels. Figure 4.5 shows the comparison of MBIR results with and without MMC for sinogram correction. In the absence of MMC, there appear to be dark streaking artifacts in between the five high-concentrated iodine inserts. Also, the iodine intensities themselves are not accurate, appearing to be over-estimated in high kVp and under-estimated low kVp scans, compared to their corresponding effective KeV attenuation coefficients. Applying MMC improves both uniformity and density issues.
Figure 4.5. Applying MMC to MBIR on Gammex phantom scanned at different tube settings, from left to right are 140, 120 and 100 kVp respectively. Top row: MBIR images without MMC. Middle row: MBIR images with MMC in pre-processing. Bottom row: corresponding difference images of middle row subtracting top row. Display windows for images (a) through (f) are [-50, 50] HU, for (g) through (i) are [-25, 25] HU.
4.2.2 Statistical Weights Adjustment

We model the MBIR statistical weights as inverse of the projection variance estimates. When sinogram correction is performed as pre-processing, the projection data variance also change. Therefore, statistical weights need be adjusted accordingly.

The MMC spectral correction is a polynomial function of two variables: total projection and iodine projection. We denote the relation as the following form,

\[ \tilde{y}_j = y_j + g(y_j, p_{io,j}), \]  

(4.1)

where \( y_j \) is the projection estimate after water-based BHC for projection \( j \), and \( p_{io,j} \) is the corresponding projection value of the iodine image from material decomposition. The function \( g() \) does not contain a constant term of \( p_{io,j} \), since the bias is zero when there is no iodine content in the path \( j \). (An explicit expression of \( g() \) can be found in Appendix B)

The iodine projections \( p_{io,j} \) contain little variance compared to total projections \( y_j \), because they are obtained by two stages of low-pass filtering: FBP reconstruction for material segmentation and subsequent forward projection of decomposed iodine image. Figure 4.6 shows profile plots of iodine projection against total projection of two dose levels: 120 kV, 450 mA and 120 kV, 100 mA. Despite the total projection noise surging by reducing tube current, iodine projections remain little changes in terms of spatial fluctuation. If we see them as stationary/error-free information for a particular scan, then the only variable in (4.1) subject to change is the total projection estimate \( y_j \). Thus, the statistical weights can be modified as

\[ \text{Var}(\tilde{Y}_j) = (1 + \frac{dg}{dy_j})^2 \text{Var}(Y_j), \]  

(4.2)
which is equivalent to pointwise scale weight $w_j$ by a factor of

$$
\xi_j = \frac{1}{(1 + \frac{d\theta}{dy_j})^2}.
$$

Figure 4.7 shows reconstruction results with modified statistical weights. The differences between with and without weights adjustment lie mostly in the high gradient regions in image space, where the changes near dense object projection trace boundaries are appropriately tailored. Because the MMC sinogram correction is modeled to make minimum changes for linearization, the weights adjustment does not make a big difference in terms of visual image quality. Should there be denser materials such as metal, the effect may be more prominent.

4.2.3 Verification on Revolution CT Data

A water cylinder with three dense insertions is scanned at GE Revolution CT system, with two different tube voltage settings at 80 kVp and 140 kVp. The three rods are in the iodine density range, but have slight HU differences. We applied
Figure 4.7. Applying MMC and weights adjustment to MBIR on Gammex phantom scanned at different tube settings, from left to right are 140, 120 and 100 kVp respectively at top row. Bottom row are corresponding difference images of top row subtracting no weights adjustment MBIR results from Figure 4.5 (d) to (f). Display windows for images (a) through (f) are [-50, 50] HU, for (g) through (i) are [-25, 25] HU.

MMC projection correction with weights adjustment on both scans, and compared reconstructed MBIR images to no MMC ones. Results are evaluated by three metrics.

First of all, the MMC spectrum correction removes the beam hardening effects caused by the iodine rods. Without MMC, Figures 4.8 (a) and 4.8 (d) show strong streaking artifacts along the highest attenuation paths in between any two rods. Applying MMC makes the water regions in the cylinder much more uniform, in terms of both intensity and noise characteristics, as suggested by Figures 4.8 (b) and 4.8 (e). In Figure 4.9 the image profile plot shows that not only the water uniformity
is improved, the pixel intensity in each rod is also flatter and quantitatively more accurate, according to the phantom ground-truth information.

Second, the MMC coefficients are spatially variant. Each individual detector is well pre-calibrated, which turns out to compensate for systematic inconsistencies at detector module boundaries. The inconsistencies appear in the reconstructed image domain as concentric “ringing” artifacts, captured by the difference images in Figures 4.8 (c) and 4.8 (f). The image thickness in Figure 4.8 is 0.625 mm per slice. The ringing artifacts without MMC would appear to be more disruptive when
Figure 4.9. Profile plot in center slice image of 140kVp scan. (a): a profile line is selected; (b): profile plot with/without including MMC.

Figure 4.10. Uniformity of rods along z direction. (a): three ROI’s are selected (color coded), one in each rod; (b): plots of mean values (shifted 1024 HU) of the ROI’s along z dimension (256 slices), dotted lines are of scenario without MMC.

reconstructing thicker slices due to noise reduction.

Last but not least, because of the X-ray generation mechanism, the input X-ray spectrum is substantially "harder" towards one direction along table movement (z
direction). The phenomenon is called the “heel effect”. As a result, the effective photon energies increase from one side to another in z, resulting in varying reconstructed image intensities. This non-ideal fact of the X-ray source is typically not desirable in CT imaging, as a homogeneous tissue would appear darker or brighter in some slices than others, and needs to be treated. Due to the spatially variant nature of MMC coefficients, the heel effect in the z direction can also be largely reduced. As shown in Figure 4.10, MBIR images with MMC have very uniform ROI means across the total 256 slices.

4.3 Modeling Metal Artifacts as Offset Poisson Rates

For metal and other dense objects, generic scatter estimation and water-based BHC in preprocessing is inadequate. To model the residual effect of metal, we parameterize the nonlinearity as additive offset Poisson counts, resembling the scatter model depicted in (2.9). This method does not require prior knowledge about input spectrum or materials’ attenuation properties. Instead, the offset parameters $\gamma$ are to be estimated in the iterative loop. The data acquisition can be described as

$$\Lambda_j \sim \text{Poisson}\{\lambda_0^j e^{-p_j} + \gamma_j\}, \quad (4.4)$$

where $p_j = A_{j,x}x$ is the line-integral density, and $\gamma_j$ is the offset rate parameter for projection $j$ respectively. Naturally, $\gamma$ shares the parameter dimension of the projection data $y \in \mathbb{R}^M$. But non-metal projection paths are generally considered free of metal effects, thus the number of non-zero components in $\gamma$ to be estimated is only $L$ (satisfying $L < M$), which is the total number of rays that pass through metal objects. Metal segmentation can be easily done in image space, which would determine the metal projection mask to specify $L$ via forward projection. The optimization
objective function is then modified as

$$
\Phi(x, \gamma) = \sum_{j=1}^{M} (\lambda_{0,j} e^{-A_{j}x} + \gamma_j) - \lambda_j \log(\lambda_{0,j} e^{-A_{j}x} + \gamma_j) + U(x) + R(\gamma),
$$

(4.5)

where the regularization term \( R(\cdot) \) is to encourage a smooth offset rate profile by

$$
R(\gamma) = \frac{\beta}{2} \sum_{(j,m) \in C} b_{jm} (\gamma_j - \gamma_m)^2,
$$

(4.6)

with \( \beta \) the penalty strength and \( b_{jm} \) the MRF coefficients. Alternating optimization methods are commonly used to solve such joint estimation problem \[53, 54\]. In the following, we design an algorithm to compute updates for the image \( x \in \mathbb{R}^{N} \) and the non-zero offset parameter \( \gamma_{nz} \in \mathbb{R}^{L} \) simultaneously. In each update, a quadratic surrogate function is used to approximate the direction in which the total cost function (4.5) is minimized.

### 4.3.1 Step 1: Image Update

First, we fix \( \gamma_j = \hat{\gamma}_j \) of current offset parameter estimate, and (4.5) becomes

$$
\Phi(x, \gamma) \big|_{\gamma=\hat{\gamma}} = \sum_{j=1}^{M} \phi(x, \hat{\gamma}_j) + U(x) + R(\hat{\gamma}),
$$

which is a convex problem with respect to \( x \). Expanding \( \phi() \) into a second-order Taylor series at point \( y_j = -\log(\frac{\lambda}{\lambda_{0,j}}) \) and denote \( p_j = A_{j}x \), we have

$$
\phi(p_j, \hat{\gamma}_j) \approx \phi(y_j, \hat{\gamma}_j) + \theta_{1,j}(p_j - y_j) + \frac{\theta_{2,j}}{2}(p_j - y_j)^2,
$$

(4.7)

and the derivatives are evaluated as

$$
\theta_{1,j} = \frac{\partial \phi}{\partial p_j}(y_j, \hat{\gamma}_j) = \lambda_{0,j} e^{-p_j}(\frac{\lambda_j}{\lambda_{0,j} e^{-p_j} + \hat{\gamma}_j} - 1)\bigg|_{p_j=y_j} = 0,
$$

(4.8)

$$
\theta_{2,j} = \frac{\partial^2 \phi}{\partial p_j^2}(y_j, \hat{\gamma}_j) = \frac{\lambda_j(\lambda_{0,j} e^{-p_j})^2}{(\lambda_{0,j} e^{-p_j} + \hat{\gamma}_j)^2} - \theta_{1,j}\bigg|_{p_j=y_j} = \frac{(\lambda_j - \hat{\gamma}_j)^2}{\lambda_j}.
$$

(4.9)
This yields the quadratic surrogate function for image update

$$\Phi'(x) = \sum_{j=1}^{M} \frac{\theta_{2,j}}{2} (y_j - A_j x)^2 + U(x). \quad (4.10)$$

The statistical weighting has the same form as the scatter correction model in (2.12), meaning that each projection datum under metal influence should carry lower weight than those not intersecting metal objects.

4.3.2 Step 2: Offset Rate Update

As a second step, we fix the image estimate $x = \hat{x}$, and update the offset parameter space. Equation (4.5) can be rewritten as $\Phi(x, \gamma)|_{x=\hat{x}} = \sum_{j=1}^{M} \phi(\gamma_j, \hat{x}) + R(\gamma) + U(\hat{x})$. We compute the second-order Taylor expansion of $\phi()$ w.r.t. $\gamma_j$ at its previously estimated value $\hat{\gamma}_j$,

$$\phi(\gamma_j, \hat{x}) = \phi(\hat{\gamma}_j, \hat{x}) + d_{1,j}(\gamma_j - \hat{\gamma}_j) + \frac{d_{2,j}}{2}(\gamma_j - \hat{\gamma}_j)^2, \quad (4.11)$$

and the derivatives are calculated as

$$d_{1,j} = \frac{\partial \phi}{\partial \gamma_j}(\hat{\gamma}_j, \hat{p}_j) = 1 - \frac{\lambda_j}{\lambda_{0,j} e^{-\hat{p}_j} + \gamma_j} \bigg|_{\gamma_j = \hat{\gamma}_j} = 1 - \frac{\lambda_j}{\lambda_{0,j} e^{-\hat{p}_j} + \hat{\gamma}_j}, \quad (4.12)$$

$$d_{2,j} = \frac{\partial^2 \phi}{\partial \gamma_j^2}(\hat{\gamma}_j, \hat{p}_j) = \frac{\lambda_j}{(\lambda_{0,j} e^{-\hat{p}_j} + \gamma_j)^2} \bigg|_{\gamma_j = \hat{\gamma}_j} = \frac{\lambda_j}{(\lambda_{0,j} e^{-\hat{p}_j} + \hat{\gamma}_j)^2}. \quad (4.13)$$

Then the surrogate function for offset rate update is

$$\Phi''(\gamma) = \sum_{j=1}^{M} (d_{1,j} + \frac{\partial R}{\partial \gamma_j}(\hat{\gamma})) (\gamma_j - \hat{\gamma}_j) + \frac{1}{2} (d_{2,j} + \frac{\partial^2 R}{\partial \gamma_j^2}(\hat{\gamma})) (\gamma_j - \hat{\gamma}_j)^2, \quad (4.14)$$

and the updating formula is given by

$$\gamma_j = \hat{\gamma}_j - \frac{d_{1,j} + \beta \sum_{(j,m) \in \mathcal{C}} b_{jm}(\hat{\gamma}_j - \hat{\gamma}_m)}{d_{2,j} + \beta}. \quad (4.15)$$
Note that the $\gamma_j$ estimate is converged when the first derivative $d_{1,j} + \frac{\partial R}{\partial \gamma_j}(\hat{\gamma}) = 0$ is achieved, which means the data fit well to the linear forward model under the assumed $R(\gamma)$ prior model, so that there is no need for further correction. Then, alternate between Step 1 and Step 2 to simultaneously estimate both image and the offset Poisson rates.

4.3.3 Clustering for Offset Rates

The offset Poisson rates increase the dimension of the parameter space. Without any limitation on $\gamma$, the alternating minimization technique would face an overfitting problem. The converged image parameter estimate may not be uniquely defined or may be trapped at local minima. One solution to avoid these problems is to reduce the dimensionality of random variables.

Considering the nature of beam hardening, the projection measurement is a non-linear combination of different materials’ path lengths. Suppose one projection passes only two materials: water and metal; then the effective line-integral density $p_j$ is computed as follows

$$p_j = -\log \sum_\mathcal{E} \bar{S}(\mathcal{E}) e^{-(\mu_w(\mathcal{E})l_{w,j} + \mu_m(\mathcal{E})l_{m,j})}$$  \hspace{1cm} (4.16)

which is a function of water path length $l_{w,j}$ and metal path length $l_{m,j}$. $\bar{S}(\mathcal{E})$ stands for the normalized X-ray spectrum, and $\mu_w(\mathcal{E})$ and $\mu_m(\mathcal{E})$ are linear attenuation coefficients of water and metal respectively. After applying water-based beam hardening correction, $\tilde{y}_j = \text{BHC}(y_j)$ is still a nonlinear function of both $l_{w,j}$ and $l_{m,j}$. Therefore we propose to cluster the offset rate parameters into 2D space as

$$\gamma_j := \gamma_{t,s},$$  \hspace{1cm} (4.17)

when $l_{w,j}$ falls in the $t^{th}$ water projection level and $l_{m,j}$ falls in the $s^{th}$ metal projection level. In other words, we assume the photon penetration rays that pass with same
lengths of water and metal share an additive bias $\gamma_{t,s}$, which can be updated by

$$
\gamma^{(n)}_{t,s} = \gamma^{(n-1)}_{t,s} - \frac{\sum \gamma_j = \gamma_{t,s}}{\sum \gamma_j = \gamma_{t,s}} d_{1,j} + \frac{\partial R}{\partial \gamma_j}(\gamma^{(n-1)}) d_{2,j} + \frac{\partial^2 R}{\partial \gamma_j^2}(\gamma^{(n-1)}) \sum \gamma_j = \gamma_{t,s} d_{1,j} + \frac{\partial^2 R}{\partial \gamma_j^2}(\gamma^{(n-1)}) \sum \gamma_j = \gamma_{t,s} d_{2,j}.
$$

where the superscripts $(n)$ and $(n-1)$ stand for the number of iterations.

### 4.3.4 Least Square Weights Adjustment

Due to the change of expansion point $y_j$ after offset rate update, the statistical weights for image reconstruction need to be updated as well to represent the local curvature of log-likelihood function in (4.5). Taking into account the electronic noise in the model, the weights are adjusted as

$$
w_j^{(n)} = \frac{(\lambda_j - \gamma_j^{(n)})^2}{\lambda_j + \sigma_j^2} = \frac{1}{(1 + r_j^{(n)})^2} w_j^{(0)},
$$

where $r_j := \frac{\gamma_j}{\lambda_j - \gamma_j}$ is defined as the ratio of offset-to-primary counts in projection $j$.

### 4.3.5 Algorithmic Implementation

In the pre-processing chain from Figure 1.11, a lot of pretreatments are applied to get the most sensible projection estimate. To preserve the gain of these data pretreatments, we compute $\lambda_{0,j}e^{-y_j}$ in iterative loop as equivalent total counts $\lambda_j$ in metal projection traces, which include primary photons, remaining scattered photons after generic scatter correction and remaining beam hardening effects after water-based BHC. $\gamma_j$ are initialized to 0, assuming no additional correction is applied yet. A pseudocode of the implementation details is described in Algorithm 1.
Algorithm 1: MBIR with simultaneous offset rate correction (ORC) for MAR

Require: initial projection estimation \( y(0) \), initial weights estimation \( W(0) \), initial image estimate \( x(0) \), incident X-ray photon \( \lambda_0 \) and number of iterations \( N_{\text{iter}} \).

1: for \( n = 1 \) to \( N_{\text{iter}} \) do
2:   image: \( x^{(n)} = \arg \min_x \sum_{j=1}^{M} w_j^{(n-1)}(y_j^{(n-1)} - A_j x)^2 + U(x) \)
3:   forward projection: \( p^{(n)} = Ax^{(n)} \)
4:   1st derivative: \( d_1^{(n)} = 1 - (\lambda_0 e^{-y_j^{(n-1)}} + \gamma_j^{(n-1)})/(\lambda_0 e^{-p_j^{(n)}} + \gamma_j^{(n-1)}) \)
5:   2nd derivative: \( d_2^{(n)} = (\lambda_0 e^{-y_j^{(n-1)}} + \gamma_j^{(n-1)})/(\lambda_0 e^{-p_j^{(n)}} + \gamma_j^{(n-1)})^2 \)
6:   offset: \( \gamma_{t,s}^{(n)} = \gamma_{t,s}^{(n-1)} - \left( \sum_{r_j = r_t,s} d_1^{(n)} + \frac{\partial R}{\partial \gamma_j^{(n-1)}} \right) / \left( \sum_{r_j = r_t,s} d_2^{(n)} + \frac{\partial^2 R}{\partial \gamma_j^{(n-1)^2}} \right) \)
7:   projection estimate: \( y_j^{(n)} = y_j^{(n-1)} + \log(1 + \hat{r}_j^{(n)}) - \log(1 + \hat{r}_j^{(n-1)}) \)
8:   weight: \( w_j^{(n)} = w_j^{(n-1)}(1 + \hat{r}_j^{(n-1)})^2 / (1 + \hat{r}_j^{(n-1)}) \)
9: end for

4.3.6 Results and Discussion

The proposed method is applied to axial CT scan data acquired on a Discovery CT750 HD scanner (GE Healthcare, Waukesha, WI), at X-ray tube voltage of 120 kVp. and tube current of 640 mA. All data are composed of 984 views per rotation, speed of 1.0 sec/rotation and 20 mm collimation. The objects of interest are two water cylinders of diameter 125 mm (5 inch). One of them has a Titanium rod inserted in the middle with size 9.5 mm (3/8 inch). The other is inserted with a stainless steel rod of size 4.8 mm (3/16 inch). Stainless steel is mainly composed of Fe element, which has higher linear attenuation coefficient than does Ti.

First, we reconstruct images using the standard MBIR method in (2.21). Though projections affected by metal have lower statistical weights, the nonlinearly biased data cause strong streaking artifacts traversing the image space. The dominant disturbances are along the projection directions with high attenuation or sharp gradients, as shown in Figure 4.11 (a). Using the offset rate correction method without parameter clustering, overfitting becomes a major issue in metal corrupted projection data. Both the image \( x \) and the offset rate \( \gamma \) attempt to compensate for the data nonlinearity. As a result, the image estimate is trapped into an undesired minimum,
Figure 4.11. MBIR images for the water-cylinder phantom with metal inserts scan data. (a): standard reconstruction; (b): alternating MBIR with pointwise offset rate estimation (no clustering); (c): alternating MBIR with 2D clustered offset rate estimation based on projection view and detector row indices; (d): alternating MBIR with 2D clustered offset rate estimation based on water and metal path lengths. Display window [-100, 100] HU.

where some metal artifacts were reduced, while new artifacts arose in other directions. Figure 4.11 (b) shows the converged result. In Figure 4.12 (b), the corresponding converged error sinogram suggested there are enough degrees of freedom to completely remove the discrepancies between forward model $Ax$ and projection measurements $y$ during offset rate estimation, which leaves room for the image update to move in any direction. Such unsupervised parameter estimation is ineffective.

Alternatively, if we assume a low frequency profile for the metal projection biases, and apply the MRF regularization $R(\gamma)$ to the objective function, as formulated in
Figure 4.12. Converged error sinograms ($Ax - y$) of corresponding MBIR reconstructions in Figure 4.11. (a): standard reconstruction; (b): alternating MBIR with pointwise offset rate estimation (no clustering); (c): alternating MBIR with 1D clustered offset rate estimation based on view indices; (d): alternating MBIR with 2D clustered offset rate estimation based on water and metal path lengths.

The 2D clustering method for $\gamma_j$ based on water and metal path lengths is the...
most effective. A $16 \times 16$ parameter grid is selected to fit the model. Presented in Figure 4.11(d), the metal artifacts are significantly reduced, while image texture and noise level remain the same level as standard MBIR in Figure 4.11(a). The converged error sinogram in Figure 4.12(d) suggests that the nonlinear discrepancies between $Ax$ and $y$ are captured by the additive parameter $\gamma$. But a lack of structural guidance in image domain leads to uncertainty of edge discrepancies and other remaining artifacts. Note that in difference images in Figure 4.12, the groups of three high-intensity errors are from table edges and have limited effect on the reconstruction ROI. As the 2D offset rate profile plot shown in Figure 4.13, individual cells $\gamma_{t,s}$ are estimated by (4.18), clustered by metal and non-metal projection path lengths. Empty or zero-valued cell parameter combinations did not appear in this data set.

In conclusion, the simultaneous offset rate sinogram correction is successful in reducing metal induced artifacts. The physics based 2D clustering essentially provides a way to apply polynomial-like sinogram correction without specifying polynomial
degrees and coefficients. No prior knowledge about the incident X-ray spectrum of materials’ attenuation information is used in this unsupervised parameter estimation.

4.4 Prior-Based Image Modeling for Metal Artifact Reduction

Many sinogram completion methods have been proposed to address metal artifacts for analytical reconstruction, where problematic projection data in metal traces are replaced by synthetic values. Such approaches often manage to reduce strong shading artifacts, but leave high frequency streaks tangential to metal boundaries. Image noise level usually increases from standard reconstruction in the regions absent of metal artifacts. Applying the same technique for MBIR as data pre-processing yields even worse results. Statistical methods weigh each datum by the approximate inverse of its variance, and sinogram inpainting undermines the integrity of the measurement statistics and cause failure of the model. In the meanwhile, high frequency discrepancies at metal projection boundaries would be enhanced by iteratively penalizing the mean square error between data and forward model.

In Figure 4.14, NMAR, as one of the state of the art metal correction techniques, is compared against standard FBP reconstruction. Major streaking artifacts are removed, but new radial noise textures arise in both cylinders, due to the imperfect data replacement. Simple incorporation of NMAR into MBIR as pretreatment by substituting metal affected data with forward projection of NMAR image yields Figure 4.14 (d). The replacement data disrupt the statistical noise model introduced in Chapter 2, resulting in a different appearance of the artifacts, and persistence of their effects well beyond the metal regions.

Although the image qualities, in terms of remaining artifacts and noise characteristics, have some compromises, the NMAR result as in Figure 4.14 (b) can be incorporated into MBIR to provide supervision in image estimate by biasing the result towards smoothness away from metal artifacts. FBP and NMAR approaches
Figure 4.14. Applying NMAR as sinogram pre-processing for both FBP and MBIR. (a): standard FBP; (b): NMAR result; (c): standard MBIR; (d): MBIR result with metal corrupted data replaced by forward projection of NMAR in (b). Display window [-100, 100] HU.

are generally computationally efficient, and therefore treated as prior image information for iterative estimation at essentially no cost. The formation of our proposed augmented objective function can be written as

\[ \Psi(x, \gamma) = \Phi(x, \gamma) + \frac{1}{2}(x - \bar{x})^T D(x - \bar{x}), \]  

(4.20)

where \( \Phi(x, \gamma) \) is defined in (4.5), \( \bar{x} \) is the prior image used to guide convergence of image \( x \), and \( D = \text{Diag}\{d_i\} \) provides a spatially varying prior scaling to accommodate the control of noise uniformity discussed in Chapter 3. A three-stage iterative scheme
is designed to estimate projection bias $\gamma$ and image volume $x$ jointly.

### 4.4.1 Stage 1: Guided Intermediate Image Reconstruction

Previous experiments from Figure 4.14 suggest that there are large discrepancies between the data $y$ and forward projections of NMAR image $Ax^{(0)}$ near metal projection trace boundaries, which lead to unsuccessful sinogram inpainting. The difference can have various causes: non-uniform water-like material intensities, inaccurate estimation of metal object sizes, shapes and densities in NMAR image. So the motivation of proposing a prior-guided image reconstruction stage is to approximately maintain the image quality of the prior while reducing the sinogram error between $Ax$ and $y$. The original sinogram data are not modified in this stage, so the objective function can be simplified as

$$
\Psi'(x) = \frac{1}{2} (y - Ax)^T W (y - Ax) + U(x) + \frac{1}{2} (x - \bar{x})^T D (x - \bar{x})
$$

(4.21)

where $D$ is a diagonal matrix entries populated by

$$
\hat{d}_i = \beta_1 \kappa_i^2,
$$

(4.22)

which is proportional to the EVD of Chapter 3 of each voxel for the sake of uniform noise, and $\beta_1$ is a scalar. By penalizing the distance from $x$ to $\bar{x}$, the prior helps prevent propagation of edge errors. At the end of this stage, the image is used to perform sinogram correction as in Section 4.3 rather than presented as final estimate; therefore details are less important than voxel means, and the prior term is heavily weighted. $x$ will be iteratively refined by minimizing (4.21), and we denote the converged result as

$$
x^{(1)} = \arg \min_x \Phi'(x).
$$

(4.23)
4.4.2 Stage 2: Regularized Sinogram Bias Estimate

By optimizing (4.20) w.r.t. the image \( x \), we arrive at a state where sinogram biases can be more accurately estimated. To calculate the bias term \( \gamma \), (4.20) can be reduced to

\[
\Psi''(\gamma) = \sum_{j=1}^{M} \left( \lambda_{0,j} e^{-A_{j} x^{(1)}} + \gamma_{j} \right) - \lambda_{j} \log(\lambda_{0,j} e^{-A_{j} x^{(1)}} + \gamma_{j}) + R(\gamma). \tag{4.24}
\]

A spatially variant MRF regularization term is introduced to take into account both the signal variance and correlation between neighbor detectors,

\[
R(\gamma) = \frac{\beta_{2}}{2} \sum_{(j,m) \in C} w_{j} w_{m} b_{jm} (\gamma_{j} - \gamma_{m})^{2}, \tag{4.25}
\]

where \( \beta_{2} \) is a global scaling factor to balance the data and prior strengths. By the end of this stage, an estimate of the offset rate \( \gamma \) is obtained

\[
\gamma^{(1)} = \arg\min_{\gamma} \Psi''(\gamma), \tag{4.26}
\]

then the sinogram is updated by

\[
y_{j}^{(1)} = -\log \left( \frac{\lambda_{0,j} e^{y_{j}} - \gamma_{j}^{(1)}}{\lambda_{0,j}} \right). \tag{4.27}
\]

Equivalently, if the offset-to-primary count ratio \( r^{(1)} \) is computed from (4.24), then

\[
y_{j}^{(1)} = y_{j} + \log(1 + r^{(1)}). \tag{4.28}
\]

4.4.3 Stage 3: Guided Final Image Reconstruction

Ideally, the metal corrected sinogram \( y^{(1)} \) from Stage 2 can be used to run a standard MBIR (2.21). But since \( x^{(1)} \) is not the ground truth image, \( y^{(1)} \) would not
be perfect either. To reduce the likelihood of subsequent reconstruction picking up potential artifacts, the same prior-guided objective function (4.21) is used but with a much smaller prior penalty strength $\beta'_{1}$, in order to achieve high resolution and refined details.

4.4.4 Results and Discussion

We first apply the prior-based iterative MAR solution on the two-cylinder phantom scan data. In Figure 4.15 (a), the NMAR image from Figure 4.14 (b) is passed to standard MBIR as a better initial condition. Naturally, the converged result looks very similar to Figure 4.11 (c) by using standard FBP initial image. The structural advantage of NMAR is not utilized while penalizing the weighted least square function in (2.21). However, if we use the NMAR as a prior image $\bar{x}$, and apply the three-stage prior-based sinogram correction strategy, a desirable result is obtained as shown in Figure 4.15 (b). The image is free of prominent streaking artifacts, and the noise texture in water substance is much more uniform than with NMAR.

In Figure 4.16, illustration in the sinogram domain is provided. As expected, since the initial image underwent a data inpainting process, the difference between its forward projection and measured data is relatively large, shown in Figure 4.16 (b). Metal intensities are underestimated, leaving the majority of the metal traces as dark. After heavily guided Stage 1 reconstruction, the error sinogram is noticeably reduced. Metal affected projections show transitional discrepancies, depending on sinogram locations in Figure 4.16 (b). Computing the sinogram biases using the surrogate function (4.24) in Stage 2 yields the correction sinogram $y^{(1)}$. Figure 4.16 (d) suggests error uniformity between $Ax^{(1)}$ and $y^{(1)}$ is largely improved. Then Stage 3 reconstructs the final image by replacing the data term with $y^{(1)}$ and performing the final round of MBIR.

Additionally, the prior-guided MBIR algorithm is applied on a clinical torso scan,
in which a metal implant is present in the femur. The data are acquired on a GE Discovery CT750 HD scanner (GE Healthcare, Waukesha, WI) at 120 kVp. 750 mA, with speed of 0.5 sec/rotation in helical mode of pitch ratio 63/64. As shown in Figures 4.17 (a) and 4.19 (a), standard FBP reconstruction suffers from major metal artifacts across the image. Streaks and shadings compromise the image’s diagnostic quality. The NMAR images in Figures 4.17 (b) and 4.19 (b) show significant reduction of artifacts, but some residue shadings and high-frequency noise persist. Compared to FBP, standard MBIR, as shown in Figures 4.17 (c) and 4.19 (c), contains largely reduced metal artifacts due to statistical weighting and the image prior model. In Figures 4.17 (d) and 4.19 (d), our proposed prior-guided iterative MAR combines the merits of NMAR and IR, resulting in reduced metal artifacts as well as improved noise characteristics. A zoomed-in comparison is also available in Figures 4.18 and 4.20.
Figure 4.16. Sinogram domain analysis. (a): Original sinogram data $y$; (b): initial sinogram difference ($Ax^{(0)} - y$); (c): sinogram difference after Stage 1 ($Ax^{(1)} - y$); (d): sinogram difference after Stage 2 ($Ax^{(1)} - y^{(1)}$).
Figure 4.17. Clinical scan data with metal implant in slice 16 of reconstructed volume. (a): standard FBP; (b): NMAR; (c): standard MBIR; (d): NMAR-guided MBIR. Display window [-100, 100] HU.

<table>
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TABLE 4.1

STANDARD DEVIATION MEASUREMENTS (HU) AT SELECTED ROIS IN SLICE 16 IMAGES IN FIGURE 4.17
Figure 4.18. Zoomed-in comparison of Figure 4.17 for clinical scan data with metal implant. (a): standard FBP; (b): NMAR; (c): standard MBIR; (d): NMAR-guided MBIR. Display window [-100, 100] HU.
Figure 4.19. Clinical scan data with metal implant in slice 32 of reconstructed volume. (a): standard FBP; (b): NMAR; (c): standard MBIR; (d): NMAR-guided MBIR. Display window [-100, 100] HU.

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TABLE 4.2

STANDARD DEVIATION MEASUREMENTS (HU) AT SELECTED ROIs IN SLICE 32 IMAGES IN FIGURE 4.19
Figure 4.20. Zoomed-in comparison of Figure 4.19 for clinical scan data with metal implant. (a): standard FBP; (b): NMAR; (c): standard MBIR; (d): NMAR-guided MBIR. Display window [-100, 100] HU.
A major trend in modern computed tomography (CT) scanners has been toward wider cone angles, allowing capture of the whole heart in a single beat with uniform IV contrast, and other applications in which the cone-beam aspect becomes important. Large detector arrays, composed of many focally-aligned rectangular sub-arrays, preclude a perfectly uniform spacing of detectors, which poses a modeling challenge for native geometry iterative image reconstruction. Small miscalibration or inaccuracies in data pretreatment, particularly when correlated in space and/or time, result in reduced diagnostic quality due to local biases in reconstructions.

We present a method of estimating a limited set of offset and gain parameters in X-ray CT systems simultaneously with the reconstruction of 3D imagery. Commencing with an initial, filtered back projection image, the synthesized forward projections of the current image estimate are compared to sinogram data, and before application in a new image update, data are corrected for systematic errors. Each offset or gain parameter is applied to a geometrically selected subset of the sinogram entries. Correction parameters are stabilized by both the nature of the data subsets chosen, and penalties for correction magnitudes. This process is somewhat similar to our joint reconstruction and offset rate estimation in Chapter 4.
5.1 Cone-Beam CT Geometry

Model-based methods of reconstruction have proven an important tool for improving image quality in X-ray CT systems [96, 110, 128]. Iterative solution of associated optimization problems, however, may be more sensitive to system modeling errors than conventional, single-pass, back-projection techniques [30, 88]. Previously, we have corrected for partial blockage of reference-normalization sensors [111] by estimating a relatively small number of detector gain parameters simultaneously with reconstruction. Each of these parameters scales photon count data at detectors and becomes an additive factor in a detector row after transformation by the log() operator. Modeling of this constant-valued offset, which we will refer to as “DC”, in projection data allows compensation for inconsistent photon count scaling among various projection angles and removal of low-frequency shading in iterative estimates. Somewhat similarly, parameters in polynomial beam-hardening correction may be estimated simultaneously [20, 23, 50, 53, 56, 71].

Modern CT scanners with wider cone angles and faster rotating speeds have the advantage of capturing of the whole heart in a single beat as well as other important diagnostic applications. As shown by the diagram in Figure 5.1, large detector arrays may be composed of many rectangular sub-arrays, with the geometry of their arrangement precluding a perfectly uniform angular spacing of detectors, and necessitating some variation in the relative positions of boundary detectors in adjacent sub-arrays, illustrated as Figure 5.2. This geometry aims to have focally-aligned sub-arrays to reduce distortions and other generic detection challenges. However, the module boundary detectors may have slightly different responses from those in the center of a block, and scatter level may be affected by the small gaps between modules. If the detector discrepancy is not accurately modeled, systematic local bias may arise. Other forms of detector response non-uniformity may cause similar distortions [34, 47].
Figure 5.1. Schematic diagram of spatial sampling and reconstruction volume of wide cone-beam CT system. (The sketch is for illustration and not to scale.)

Image slices at the boundaries of a wide-cone scan, particularly a full rotation at a single bed position (e.g., slice locations marked by points A, B and C in Figure 5.2), are sampled spatially quite differently from those near the center. With boundary image slices incompletely sampled and the estimation problem possibly underdetermined, problematic artifacts may arise from any systematic errors in data. Figure 5.3 (b) presents an example of cone-beam artifacts using native geometry iterative reconstruction (IR) methods. An analytic reconstruction specifically designed to address sampling inconsistencies in the cone-beam geometry [25, 39] is shown for reference, and remains free of cone-beam artifacts in Figure 5.3 (a). Cone-beam analytic reconstruction methods generally use rebinning and filtering techniques as pretreatments, and such row-dependent biases may not be reflected in image space. For native-geometry-based IR techniques, reconstruction, though having the advantages in noise control and resolution increase, may be very sensitive to systematic errors in
Figure 5.2. Cone-beam axial acquisition trajectories at two opposite view angles. Reconstruction volume is indicated by the blue box. (The sketch is for illustration and not to scale.)

projection data. The very purpose of iterative methods is to more precisely solve the finite-dimensional, ill-conditioned inverse problem than is possible with single-pass, back-projection techniques. Failure to accurately model the forward operator can therefore produce more amplified image artifacts, which propagate spatially as the algorithm “refines” its estimate to produce the best match to the inaccurate data. While some of this systematic bias in IR can be reduced by the simultaneous DC estimation process, the artifacts cannot be entirely compensated by the approach. In this chapter, we augment the offset estimation process with a second set of parameters representing signal-dependent gain in the projection domain.
Figure 5.3. Cone-beam reconstructions using (a) FBP with special cone-beam treatment and (b) statistical IR approach with native geometry model. Display window [-75, 125] HU.

5.2 Description of Methods

A quadratic model of the log-likelihood in CT for image $x$, projection data $y$ and vector of DC correction parameters $d$ may be written as

$$\ln p(y|x, d) \approx -\frac{1}{2}(y + d - Ax)^TW(y + d - Ax) + c(y), \quad (5.1)$$

where $A$ represents forward projection in native geometry, $W$ is a diagonal weighting matrix with entries proportional to received radiation strength, and $c(y)$, a function of the measurements but independent of the parameter vectors $x$ and $d$ [111]. For wider cone-beam CT, we find that DC compensation in $d$ cannot fully capture the localized inconsistencies between $y$ and $Ax$ of the rows at detector module boundaries, associated with points $A$, $B$ and $C$ in Figure 5.2.

According to the Beer-Lambert law, projections are estimated by

$$h(y_j) = -\log \left( \frac{\lambda_j - \gamma_j}{\tau_j \lambda_{0,j}} \right) = -\log \left( \frac{\lambda_j - \gamma_j}{\lambda_{0,j}} \right) + \log \tau_j, \quad (5.2)$$

where $\lambda_j$ and $\lambda_{0,j}$ are received and incident photon counts at projection ray $j$ re-
respectively, and $\gamma_j$ is the scatter estimate for corresponding projection. The scaling factor $\tau_j$ accounts for any inaccuracy of air scan measurements $\lambda_{0,j}$, or other spatially-correlated, multiplicative error in the counts domain. $h()$ stands primarily for beam hardening effects of polychromatic X-ray beams \cite{10, 17}, but here we will expand its definition to include the modeling errors discussed above. The operator $h^{-1()}$ is standard beam hardening correction (BHC). The second term in \eqref{5.2} associated with $\tau_j$ could be compensated with DC correction using \eqref{5.1}.

Commonly, a spatially variant, water-based BHC step is applied as part of CT data pretreatment procedure \cite{42, 59, 78, 92}. The sinogram estimation can then be written as

$$y_j = \hat{h}^{-1}(-\log\left(\frac{\lambda_j - \gamma_j}{\lambda_{0,j}}\right) + \log\tau_j),$$

\hspace{1cm} \text{(5.3)}

where $\hat{h}^{-1}$ denotes a polynomial BHC pre-correction. Should there be any mis-calibration of BHC coefficients or mis-estimation of scatter level, especially correlated in space, systematic error may exist in some measurements. In fact, due to detector efficiency discrepancy and tilting angle inconsistency at detector module boundaries in the wide cone-beam geometry, such estimation biases are likely to occur, because both BHC coefficients and scatter estimation generally assume low spatial frequency profiles \cite{2, 36, 56, 71, 79, 89, 126}. To combat the model mismatch, we introduce a geometry-dependent projection compensation technique to augment DC correction.

5.2.1 Gain Parameter

The amount of mismatch between the mean of the measured line-integral density $y$ and forward projection $Ax$ is a nonlinear function of the data itself, according to \eqref{5.3}. Thus, a simple DC offset model in \eqref{5.1} may be insufficient. Since the residual discrepancies are generally observed to be a very small fraction of the line integrals, we propose a simple affine model to approximate the mismatch, a more parsimonious
choice than, for example, a model similar to full, conventional BHC. The modified log-likelihood function is then given as

$$\ln p(y|x, g, d) \approx -\frac{1}{2} \sum_{j=1}^{M} w_j (g_j y_j + d_j - A_{j*} x)^2 + c(y),$$

(5.4)

with the weights \(\{w_j\}\) from the diagonal matrix \(W\) of (5.1), \(A_{j*}\) representing the \(j^{th}\) row of matrix \(A\), and \(c(y)\) constant as a function of the unknown parameters. The correction factors \(g_j\) and \(d_j\) will be constant along some variables, and may deviate from 1 and 0, respectively for only selected subsets of detector indices. The robustness of parameter estimation and avoidance of over-fitting of data depends on allowing parameters to vary only along variables for which there is physical justification. In the present, wide-cone application, boundaries of detector segments aligned in the row direction appear to produce the most noticeable inconsistencies in response. We define the entries of \(g\) as

$$g_j := 1 + \alpha_r, \quad j \in \Omega_r,$$

(5.5)

where \(\Omega_r\) is a non-overlapping sub-collection of data indices \((j = 1, 2, ..., M)\), divided by physical detector row indices. Specification of the parameter \(\alpha_r\) allows us to emphasize the deviation from unity in the linear term of correction, as well as highlighting the nature of the variation in the gain parameter as a function of detector row only. Detector channel independence, on the other hand, is preserved to avoid overfitting. The spatial correlation of channel-dependent sinogram correction often leads to concentric circle artifacts. Similarly, the elements of vector \(d\) are defined as

$$d_j := \beta_{v,r}, \quad j \in \Omega_{v,r},$$

(5.6)

where \(\Omega_{v,r}\) is a non-overlapping sub-collection of data indices divided by view and row indices, to take into account incident photon fluctuations from view to view.
We pursue a maximum \textit{a posteriori} probability (MAP) reconstruction for image \( x \) with an \textit{a priori} model \( U(x) \), and approximate maximum likelihood (ML) estimate for gain parameters \( \alpha \) and \( \beta \). Rewriting (5.4) in vector format, the objective function is formed as

\[
\Phi(x, G, d) = \frac{1}{2} (Gy + d - Ax)^T W (Gy + d - Ax) + U(x), \tag{5.7}
\]

where the diagonal matrix \( G = \text{diag}\{g_j\} \). To perform ML estimation of the parameters \( \alpha \) and \( \beta \) at a given image \( x \), we carry on the computation using Newton’s method for successive convex optimizations; the update equations are as follows:

\[
\beta_v^{(n)} = \beta_v^{(n-1)} - \frac{\sum_{j \in \Omega_{v,r}} w_j (g_j^{(n-1)} y_j + d_j^{(n-1)} - A_j x^{(n-1)})}{\sum_{j \in \Omega_{v,r}} w_j}, \tag{5.8}
\]

\[
\alpha_r^{(n)} = \alpha_r^{(n-1)} - \frac{\sum_{j \in \Omega_r} w_j y_j (g_j^{(n-1)} y_j + d_j^{(n-1)} - A_j x^{(n-1)})}{\sum_{j \in \Omega_r} w_j y_j^2}. \tag{5.9}
\]

To avoid introducing the row-dependent biases from preprocessed data \( y \) to the image domain, the DC and gain corrections as described in (5.8) and (5.9) are applied ahead of the image update in each iteration. Algorithm 2 shows a pseudocode of the implementation for a total iteration number of \( N_{\text{iter}} \). (Note Algorithm 1 refers to the offset rate estimation algorithm in Chapter 4.) This simultaneous sinogram correction is a generic technique that in theory should work well with any gradient-based IR methods [32, 63, 123].
Algorithm 2 Simultaneous sinogram correction for iterative cone-beam CT reconstruction

Require: preprocessed projection estimation $y$, statistical weights $W$, initial reconstruction image $x^{(0)}$ and number of iterations $N_{iter}$.

1: compute forward projection of initial condition $Ax^{(0)}$
2: for $n = 1$ to $N_{iter}$ do
3:    update DC vector $\beta^{(n)}$ with (5.8)
4:    update gain vector $\alpha^{(n)}$ with (5.9)
5:    update image: $x^{(n)} \leftarrow \arg \min_x \Phi(x^{(n-1)}, G^{(n)}, d^{(n)})$
6:    update forward projection $Ax^{(n)}$
7: end for

5.2.2 Parameter Constraints

A common concern with joint parameter estimation is overfitting and underspecification [53, 82]. While we intend only to introduce two sets of sinogram compensation parameters with limited degrees of freedom in (5.7), the objective function is not guaranteed to be convex in the full parameter space. Alternating sinogram correction and image updates both minimize the cost function, but the unconstrained optimization for DC parameter $\beta$ and gain parameter $\alpha$ do not prevent the global DC value from drifting. Therefore, we would ideally enforce linear constraints on DC parameters, as in

$$\sum_r \beta_{v,r} = 0, \quad \forall v.$$  

Thus, at each rotating view angle, the total projection would assume zero DC drift from the simultaneous sinogram correction, and the algorithm would reliably maintain accurate CT density. Rewriting the constraint in vector format, we have

$$B1 = 0,$$  

(5.11)
where $B$ is a two dimensional matrix with entries of $\beta_{r,v}$. Gain parameters, on the other hand, are assumed to carry only small values. To encourage the property, we penalize the square of each gain parameter by adding the following regularization term to the objective function

$$\sum_r \alpha_r^2$$

(5.12)

to stabilize the simultaneous sinogram correction. Rather than enforce (5.11) as hard constraints, which may slow convergence of our algorithm, we penalize the energy of DC drift. Accordingly, the objective function is augmented to

$$\Psi(x, \alpha, \beta) = \Phi(x, \alpha, \beta) + \frac{\pi_1}{2} \| B1 \|_2^2 + \frac{\pi_2}{2} \| \alpha \|_2^2,$$

(5.13)

where $\pi_1$ and $\pi_2$ are scaling factors of the two penalty terms, and operator $\| \|_2$ denotes the $l^2$-norm. The pseudocode with penalized DC drift terms is presented in Algorithm 3.

**Algorithm 3** Penalized simultaneous sinogram correction for iterative cone-beam CT reconstruction

**Require:** preprocessed projection estimation $y$, statistical weights $W$, initial reconstruction image $x^{(0)}$ and number of iterations $N_{iter}$.

1: compute forward projection of initial condition $Ax^{(0)}$
2: for $n = 1$ to $N_{iter}$ do
3: update DC: $\hat{\beta}^{(n)} \leftarrow \arg\min_{\beta} \Psi(x^{(n-1)}, \alpha^{(n-1)}, \beta^{(n-1)})$
4: update gain: $\hat{\alpha}^{(n)} \leftarrow \arg\min_{\alpha} \Psi(x^{(n-1)}, \alpha^{(n-1)}, \beta^{(n)})$
5: update image: $x^{(n)} \leftarrow \arg\min_x \Psi(x^{(n-1)}, \alpha^{(n)}, \beta^{(n)})$
6: update forward projection $A x^{(n)}$
7: end for
5.3 Results

We apply the proposed method to axial cone-beam CT scan data acquired on a GE Revolution CT scanner (GE Healthcare, Waukesha, WI). A water phantom is tested as uniformity check, without significant gradients along z, and four clinical scans are used for demonstration of the technique in different scenarios where sampling non-uniformity is more difficult to address given the gradient changes in patient’s anatomy. All data are composed of 984 views per rotation and 160 mm collimation for full 256-slice reconstruction with slices of thickness 0.625 mm. Subsequent MBIR results are reconstructed with the $q$-GGMRF as a priori image model [8, 110]. For line 5 in either Algorithm 2 or 3, we optimize using a pre-conditioned gradient-based IR algorithm that simultaneously updates all the voxels [32, 123]. Approximately 10 iterations were required for convergence of the images presented. Parameters $\pi_1 = \frac{1}{256} \sum_r \sum_{j \in \Omega} w_j$ and $\pi_2 = \frac{1}{2} \sum_{j \in \Omega} w_j y_j^2$ are used to balance the log-likelihood and the penalty terms.

5.3.1 Image Quality

The first experiment was conducted on an isotropic 20 cm water phantom, scanned at 120 kV, 680 mA and speed of 1.0 sec/rotation. The standard MBIR (without DC and gain correction) method creates interleaved bright/dark horizontal stripes in coronal views as shown in Figure 5.4 (a), and cupping or ringing shadings in axial slices, especially near locations marked by A, B and C in Figure 5.2, which correspond to detector module edges. The shadings are usually centered at the transaxial plane isocenter, caused by spatially correlated projection errors. Figure 5.6 (a) shows an example of artifacts from slice location B, with bright circular shading in the center and a dark ringing band surrounding it. For reference, we compare MBIR results to artifact-free FBP images, which undergo special cone-beam rebinning and filtering treatment, as shown in Figures 5.4 (d) and 5.6 (d). Applying only DC correction
Figure 5.4. Cone-beam reconstruction water phantom in coronal view. (a): standard MBIR; (b): MBIR with DC correction; (c): MBIR with DC and gain correction; (d): FBP with cone-beam treatment; (e): diff = (b) - (a); (f): diff = (c) - (a). Display windows for images (a) through (d) are [-50, 50] HU, for (e) and (f) are [-25, 25] HU.

to the iterative estimation by skipping step 4 in Algorithm 3 shows major improvement in terms of the wide-cone artifacts. Due to the homogeneous nature of the water phantom, its projection profiles are smooth and contain mostly low-frequency information. Therefore, by compensating the sinogram mismatch with only DC components, we are able to eliminate the majority of the artifacts, as Figures 5.4 (b) and 5.6 (b) indicate. With gain correction included, Algorithm 3 images in Figures 5.4 (c) and 5.6 (c) manage to remove residual artifacts, particularly in under-sampled outside axial slices and corner regions in reformatted coronal or sagittal views, and further improve homogeneity through the cylinder volume. At 100 HU display window width, the water phantom is, appropriately, much more uniform. A zoomed-in comparison of the coronal images is also available in Figure 5.5.

To illustrate the significance of the simultaneous parameter estimation, we plotted DC and gain profiles. In Figure 5.7, a converged DC parameter estimate profile
Figure 5.5. Zoomed-in comparison of cone-beam reconstruction water phantom in coronal view. (a): standard MBIR; (b): MBIR with DC correction; (c): MBIR with DC and gain correction; (d): FBP with cone-beam treatment; (e): diff = (b) - (a); (e): diff = (c) - (a). Display windows for images (a) through (d) are [-50, 50] HU, for (e) and (f) are [-25, 25] HU.

At view 200 shows high-frequency row-dependent sinogram compensation for multiplicative inaccuracies in the photon counts domain. The profile fluctuates around zero with a total of approximately zero DC drift, thanks to the the penalty $\|B\|_2^2$ in (5.13). DC parameters do not suggest abnormality on the module boundaries. The converged gain vector is plotted in Figure 5.8. All values are very small, but are able to capture additional row-by-row variations, especially in the most tilted detector module on the outside of the cone as well as module boundary detectors. For instance, in the vicinity of row numbers 32 and row 224, marked by arrows $r_B$ and $r_B'$ respectively, abrupt data discrepancies are measured and corrected, which contributes to further improvement of image quality. Also, at the widest cone angle regions, i.e., outermost detector rows, the gain profile suggests substantially more errors than center rows.
Figure 5.6. Cone-beam reconstruction of water phantom at slice location $B$ from Figure 5.2. (a): standard MBIR; (b): MBIR with DC correction; (c): MBIR with DC and gain correction; (d): FBP with cone-beam treatment; (e): diff = (b) - (a); (f): diff = (c) - (a). Display windows for images (a) through (d) are [-50, 50] HU, for (e) and (f) are [-25, 25] HU.

Having a large portion of high attenuation cortical bone structures and high gradient in z direction, head scan is one of most challenging cone-beam construction cases [87]. We apply the proposed algorithms on two head scan data. The first clinical head scan was taken at 140 kV, 270 mA and 1.0 sec/rotation. Due to the high density of the skull, any spatially correlated data error would cause heavier artifacts in the reconstructed image volume than in the water phantom. As shown in Figures 5.9 (b) and 5.10 (b), standard MBIR results have visible improvements on resolution and noise characteristics over FBP, but suffer from streaking artifacts in
reformatted sagittal images and shading artifacts in some axial planes, which correspond to locations marked by $B$ and $C$ from Figure 5.2. The artifacts are much more severe than those in the water phantom, and noticeably corrupt the images’ quality. FBP images with special cone-beam rebinning and filtering treatment are presented in Figures 5.9 (a) and 5.10 (a) for reference. Applying only DC correction, the artifacts are largely mitigated, but are still visible in the tight display window, especially in top skull tangential regions, as shown in Figures 5.9 (c) and 5.10 (c). With unconstrained DC and gain parameter estimation from Algorithm 2, the artifacts are mostly removed, in Figures 5.9 (d) and 5.10 (d), but noticeable DC drift also occurs. The difference images from Figures 5.9 (g) and 5.10 (g) suggest unevenly distributed density changes through the image volume. Notably, densities at the top and bottom of the sagittal image differ from standard MBIR results. Also, Figure
Figure 5.8. Water phantom converged gain parameter profile $\alpha$. (Dotted lines mark detector module boundaries.)

Figure 5.10 (d) shows a little over-correction near the center when DC and gain parameters are not regulated. Algorithm 3, however, achieves significant artifact reduction while retaining DC level consistent with standard MBIR, shown in Figures 5.9 (e) and 5.10 (e). In both white matter and skull high gradient regions, the images possess HU consistency, sharp edges and low noise level.

For quantitative assessment, we compare the converged DC and gain profiles of Algorithm 3 against Algorithm 2. In Figure 5.11, both DC profiles capture row-dependent sinogram error, but Algorithm 3 manages to make minimum corrections to prevent the global DC from drifting. Similarly, in the gain plots Figure 5.12, Algorithm 3 holds the magnitudes of gain parameters closer to zero, which helps to stabilize the joint image and parameter estimation. Additionally, Figure 5.12 indicates that unconstrained sinogram correction is prone to cause over-correction in some areas, which may lead to newly introduced artifacts in image space, such as the
Figure 5.9. First clinical cone-beam head reconstruction in sagittal view. (a): FBP with cone-beam treatment; (b): standard MBIR; (c): MBIR with DC correction; (d): MBIR with unconstrained DC and gain correction from Algorithm 2; (e): MBIR with constrained DC and gain correction from Algorithm 3. (f): diff = (c) - (b); (g): diff = (d) - (b); (h): diff = (e) - (b). Display windows are [-75, 125] HU for (a) through (e) and [-50, 50] HU for (f) through (h).

bright spot in the middle of Figure 5.10 (d). A few sharp spikes appear in the gain profile, corresponding to outer-cone detector module edge rows. For example, the correction at $r_C$ row would contribute to the removal of artifacts in the white matter regions as shown in Figures 5.9 (e) and 5.10 (e).

Another head scan data with standard MBIR shows similar wide-cone artifacts in Figures 5.13 (a) and 5.13 (g). Scanned at 120 kV, 320 mA and 1.0 sec/rotation,
Figure 5.10. First clinical cone-beam head reconstruction in slice location \( C \) from Figure 5.2. (a): FBP with cone-beam treatment; (b): standard MBIR; (c): MBIR with DC correction; (d): MBIR with unconstrained DC and gain correction from Algorithm 2; (e): MBIR with constrained DC and gain correction from Algorithm 3; (f): \( \text{diff} = (c) - (b) \); (g): \( \text{diff} = (d) - (b) \); (h): \( \text{diff} = (e) - (b) \). Display windows are [-75, 125] HU for (a) through (e) and [-50, 50] HU for (f) through (h).
this dataset is slightly off-center towards patient posterior direction. The artifacts corresponding to location $C$ from Figure 5.2 are closer the front of the skull. Also, due to the decrease of X-ray tube voltage, the effective attenuation coefficient of bone is higher and the artifacts appear to be more severe than those in Figures 5.9 (b) and 5.10 (b) respectively. The proposed Algorithm 3 manages to capture data errors in DC and gain parameter spaces, and leave image domain free of wide-cone artifacts, as shown in Figures 5.13 (b) and 5.13 (h).

Resolution and noise variance move together in IR methods such as those presented here [13, 110]. We therefore consider changes caused in noise levels as an indicator of resolution changes. The standard deviation measurements in the indicated region-of-interest (ROI) of Figures 5.13 (g) and 5.13 (h) are 8.07 HU and 8.03 HU, respectively. This statistic, in addition to the fact that, except for some relatively high-frequency artifacts, strong edge signals do not appear in our differ-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_11.png}
\caption{Clinical head scan converged DC parameter profile $\beta_{v,*}$ for an arbitrary view in the scale of integral density. (Dotted lines mark detector module boundaries.)}
\end{figure}
ence images indicate that the resolution/noise advancement of IR methods is not significantly affected by the simultaneous parameter estimation.

Compared to clinical head scans, axial thorax or abdomen scans do not contain such sharp gradients in the sinogram domain. Therefore, the spatially correlated errors translate to less disturbing shadings in reconstructed 3D imagery, especially in under-sampled outer-cone regions. Nonetheless, Figure 5.14 demonstrates the benefits of using the proposed simultaneous sinogram correction algorithm over standard MBIR on a thorax dataset, scanned at 120 kV, 180 mA and 0.5 sec/rotation. The coronal image of the standard MBIR in Figure 5.14 (a) shows unevenly distributed density in homogeneous soft tissue regions. For instance, the bottom right soft tissues around ribs, shown in zoomed Figure 5.14 (d), are overly dark compared to connected adjacent regions, while bottom left and top right soft tissues are negatively offset with bright shadings. Applying the proposed Algorithm 3 shows in Figures 5.14 (b) and 5.14 (h) improvement of image uniformity, particularly at large cone angle regions.
Figure 5.13. Second clinical cone-beam head reconstruction. (a), (d): standard MBIR in full and zoomed-in sagittal views; (b), (e): MBIR with DC and gain correction in full and zoomed-in sagittal views; (c): \( \text{diff} = (b) - (a) \); (f): \( \text{diff} = (e) - (d) \); (g): standard MBIR in axial view at slice location \( C \) from Figure 5.2; (h): MBIR with DC and gain correction in axial view at slice \( C \). (f): difference image of (h) - (g). Display windows are \([-75, 125]\) HU for (a), (b), (d), (e), (g), (h) and \([-50, 50]\) HU for (c), (f) and (i).

The last clinical dataset is an abdomen scan acquired at 100 kV, 320 mA and speed of 0.5 sec/rotation. Similarly to the thorax case, the abdomen data with standard MBIR does not suffer from severe wide-cone artifacts. But closer measurement shows potentially problematic inconsistency in local density. As an example, Figure 5.15 from axial location \( B \) of Figure 5.2 demonstrates this issue, as well as improvement.
using Algorithm 3. In Figures 5.15 (a) and 5.15 (d), the liver tissues apparently have higher density in locations closer to the FOV center and appear to be saturated at this display window, which contradicts the fact that liver is expected to be a relatively homogeneous organ.

The liver region of Figure 5.15 (d) provides an opportunity to evaluate some quantitative behavior of the algorithm. By direct measurement, the mean of the blue ROI in the middle is 9 HU higher than that of the red ROI on the left. In comparison, Figure 5.15 (e), using Algorithm 3, is able to take care of the biases caused by spatially correlated data errors and non-isotropic under-sampling in outer-cone regions. The mean measurements of two ROI’s here are within 1 HU. The standard deviation measurements of red ROI’s in Figures 5.15 (d) and 5.15 (e) are 20.32 HU and 20.16 HU respectively, again suggesting that the parameter estimation is not adversely affecting resolution.

### 5.3.2 Convergence of Parameters

The DC parameter $\beta$ and gain vector $\alpha$ are simultaneously estimated within the iterative loop. We cluster the extra parameters by the scanner’s physical detector layout. The convergence rates for each $\beta_{v,r}$ and $\alpha_r$ are expected to be fast due to the quadratic nature of the objective function (5.13) with regard to each parameter. The linear penalty terms $\|B1\|_2^2$ and $\|\alpha\|_2^2$ stabilize the alternating optimization by preventing the joint estimation from DC drifting and over-correction. As shown in Figures 5.16 and 5.17, the convergence rates are relatively fast and approximately uniform. Here we plotted the parameter at one of the “problematic” row locations, pointed by $r_C$ in Figures 5.11 and 5.12 of the first head scan, and compared it against its two neighborhood rows. In Figure 5.16, the three DC parameters of chosen rows consistently move in the same direction, implying that there are some DC mismatches between the data and forward model, but nothing particular about
Figure 5.14. Clinical cone-beam thorax reconstruction in coronal and axial views. (a), (d): standard MBIR in full and zoomed-in coronal views; (b), (e): MBIR with DC and gain correction in full and zoomed-in coronal views; (c): \( \text{diff} = (b) - (a) \); (f): \( \text{diff} = (e) - (d) \); (g): standard MBIR in axial view at slice location \( B \) from Figure 5.2; (h): MBIR with DC and gain correction in axial view at slice \( B \). (f): difference image of (h) - (g). Display windows are \([-100, 100]\) HU for (a), (b), (d), (e), (g), (h) and \([-50, 50]\) HU for (c), (f) and (i).

the specific row \( r_C \). However, the gain convergence plots in Figure 5.17 indicate a divergence of \( r_C \) row from two adjacent rows. The gap between them measures the amount of row-by-row discrepancies in sinogram data.
Figure 5.15. Clinical cone-beam abdomen reconstruction in coronal and axial views. Left column: standard MBIR; middle column: MBIR with DC and gain correction from Algorithm 3; right column: difference of middle - left. Top row: coronal views; middle row: axial location of $B$ from Figure 5.2; bottom row: zoomed-in axial views. Display windows are [-30, 170] HU for (a), (b), (d), (e), (g), (h) and [-50, 50] HU for (c), (f) and (i).

5.4 Discussion and Conclusion

The results of the parameter estimation and reconstruction in the previous section support the feasibility of correcting for deficiencies in forward system models
in wide-cone CT simultaneously with iterative reconstruction. The most dramatic artifacts, and visible improvements from our estimation, result when high-contrast edges appear in non-uniformly and/or incompletely sampled regions such as the end regions of the head scans illustrated in Figures 5.9, 5.10 and 5.13. Such edges, particularly those nearly perpendicular to the axis of rotation, have previously been cited as problematic even under conventional multi-slice scanning [112]. More generic body scans are less disturbed by the sort of wide-cone artifacts we attack here, but may still benefit from improved consistency in local density values achieved by the method. We do not presume to evaluate the clinical implications of our improvements to image quality. However, we would speculate that some of the artifacts shown in the clinical scans may mask significant content, and erroneous local shifts in mean could be misinterpreted as having physiologic implications.
Figure 5.17. Convergence plot of clinical head scan gain parameters $\alpha_r$ for row indices of $r_C$, $r_C - 1$ and $r_C + 1$.

An important component of this particular type of system parameter estimation is the availability of the initial image, $x^{(0)}$, included in the descriptions of Algorithms 2 and 3. Iterative methods in CT nearly always benefit in terms of reconstruction speed from a good starting point, usually the output of the conventional back-projection techniques [102]. This also gives us a consistent and relatively accurate set of projections to which we can compare the measured data to commence our parameter estimation. Without $x^{(0)}$, we may still effectively eliminate artifacts resulting from some types of DC offsets and data inconsistencies as in our previous work [111], but the more varied inaccuracies tackled here would be much more difficult. Thus we owe a debt to the designers of the back-projection algorithms for their innovations and corrections, already applied in the cone-beam reconstruction to produce proper local average densities.

The joint DC and gain offset estimation described in Algorithm 3 is highly par-
allelizable, and the computation is very efficient since it only involves addition and multiplication operations. In this paper, we use a gradient based IR algorithm with fast convergence [32, 123] to simultaneously update all the voxels. The extra cost of simultaneous parameter estimation is negligible compared to overall computational time of IR algorithm. If one applies the proposed simultaneous sinogram correction technique to other, slower-converging IR algorithms, we suggest decreasing the frequency of parameter estimation, which should further diminish the extra computational cost.

Our approach to simultaneously correct row-dependent sinogram mis-estimation in the iterative reconstruction framework yields relatively robust control of cone-beam related artifacts in the 3D volume. This improvement could extend the merit of IR methods in native geometry reconstruction to wider cone settings and other spatially-varying sampling structures. Our subsequent work with these innovations will include robustness testing on various clinical applications and more extensive convergence and image quality studies.
6.1 Low-Signal Bayesian Inference

The Bayesian inference work in Chapter 2 opens a new direction for projection estimation in photon sparse situation. The conjugate prior model provide flexibility in handling low-count data of different local statistics. Two degrees of freedom, a shape parameter and a rate parameter, are determined by input data. One solution proposed in this dissertation estimates the two parameters using a moving window on the data stream, which forms an underlying parameter state evolving gradually with the data. This guarantees the non-biased property of the PBR, which performs robustly and consistently across the experimental trials and particularly fits well with MBIR framework for its preservation of data variance. Currently a $3 \times 3$ or $5 \times 5$ window is used for parameter estimation, depending on the data noisy level. Adapting it to three-dimensional neighborhoods to combine correlation between views is plausible in the future. Also, further study could focus on automating the estimation process completely by exploring more variance information from local data. For instance, when data variances are higher, the Bayesian denoising aspect can be more aggressively. Non-local or patch-based statistical approaches may be also incorporated into Bayesian prior parameter estimation.

Additionally, further investigation could be conducted to adapt the proposed LLMMSE and PBR methods to analytical reconstructions (e.g., FBP). As shown in Figure 2.6 Both LLMMSE and PBR offers significant mean improvements for
FBP in low-count projection directions, but the image noise level surges in the same time. Because analytical reconstruction methods generally have equal data weighting, low-count regions have to be denoised more heavily than statistical methods. Current LLMMSE and PBR mainly concern about the non-biasedness of the estimation. To achieve low-variance, one may assume stronger local correlation when estimating the prior parameters. On the other hand, concatenating a standalone post smoothing/denoising step may also be feasible.

Finally, current projection data denoising is only limited to those with limited photon measurements. But proofs show that both LLMMSE and PBR are mean-preserving “filters”. One tentative experiment is to apply denoising on the entire data. Because the denoising strength of proposed methods is adaptive to data statistics, processing high-count data would not cause resolution loss. The denoised data may contribute to lower noise level in reconstructed images. More experiments and clinical evaluations are required to complete the studies.

6.2 High-Attenuation Material Treatments

We proposed two iterative solutions to reduce high attenuation materials (e.g., metal) induced artifacts. Preliminary results are encouraging. The work can be extended in several directions.

For the offset-rate-based MAR algorithm, advanced prior modeling for the extra parameter space can be essential. Currently the parameters are clustered based on path lengths of two basis materials, which seems to show its limit near high frequency boundaries. For future research, an edge-tolerant prior models can be used to improve upon current quadratic regularization. Different clustering schemes may also be explored.

Metal artifacts may also be subject to the “bowtie” filter [49], which leads to spatially variant input spectrum and thus beam hardening effect. In most of clinical CT
Figure 6.1. Schematic 2D diagram of the appearance of bowtie filter. (The sketch is for illustration and not to scale.)

applications, the scan objects (i.e., patients) have round or oval in-plane appearances. The bowtie filter is a bowtie shaped attenuator positioned between X-ray source and the object, aiming to achieve relatively uniform SNR across detector array, as the diagram illustrated in Figure 6.1. However, due to beam hardening effects of the bowtie filter itself, X-ray spectrum entering the patient become spatially variant. One of the advantages to estimate projection biases in count domain is to incorporate such bowtie effect natively. Given information about the bowtie shape and attenuation property, one could model it deterministically as a spatially variant scaling factor, and thus to improve the offset rate parameter estimation.

Last but not least, the prior-based image modeling MBIR framework generates a few potential research areas. As an algorithmic prototype, we took NMAR images as
both initial condition and as prior guidance. Penalizing the image estimate against the prior prevents image updates from developing major metal artifacts. Potentially a low-pass filtered prior image would be more preferable, since it carries the desired pixel means but with reduced noise level. Other existing MAR approaches can also be incorporated into this framework.

6.3 Cone-Beam Iterative Reconstruction

The geometric aspect becomes increasingly important as cone-beam CT scanner adopts wider cone angle. Our proposed spatially adaptive regularization model in Chapter 3 and linear regression data model in Chapter 5 manage to achieve uniform noise throughout the reconstructed volume and reduce data inconsistency caused wide-cone artifacts respectively. Some of the explorations can be pursued further. For instance, to compensate for the mismatch between data and forward model, higher degree polynomial regression models can be examined. To avoid overfitting, we augmented the objective function by incorporating regularization terms in (5.11) and (5.12). One may also consider using different penalty functions or even algebraic constraints.

Also, efforts can be taken to make such sinogram correction as a one- or two-step operation so that it can be incorporated into data pretreatment, which could in turn reduce the risk of overfitting and/or additional computational cost in iterative estimation.

6.4 Conclusion

In this dissertation, we have discussed three statistical iterative reconstruction problems for X-ray CT. Chapter 2 focuses on statistical modeling of low-count data in the presence of ultra-low-dose exposure or high-attenuation scans. In modeling photon transmission process, we included inherent photon quantum noise, Compton
scattering effect, thermal noise and electronic noise in the proposed combined Poisson-Gaussian model. For noise corrupted low-count data, we proposed two categories to methods to restore signal means: an adaptive filtering LLMMSE and a Bayesian inference method PBR. Both methods are designed for data pretreatment. Computational costs are negligible for iterative image reconstruction. The two methods have shown promise in dealing with very limited-count CT data. From preliminary results, it appears that the LLMMSE denoising and PBR achieve comparable results on broadly tested datasets. For more extreme cases when large amount of data are photon sparse measurements, PBR has more capability in avoiding biases.

The second problem we discussed in Chapter 4 is to reduce artifacts caused by high-attenuation objects. Well-calibrated MMC approach is applicable to MBIR on reducing bone and iodine induced beam hardening effects. For more challenging materials, e.g., metal, we proposed two iterative MAR approaches. The offset rate estimation method is very flexible in modeling either scatter or beam hardening effects as additive Poisson noise, and is able to achieve simultaneous artifacts reduction in MBIR iterative framework. But such unsupervised estimation has limitation in stability control. In comparison, the three-stage prior-based image modeling MAR method is more robust in mitigating the impact of metal objects. Not requiring any knowledge about the X-ray spectrum and material types, both approaches are expected to have wide applicabilities. Compared to analytical reconstruction-based sinogram completion methods, e.g., NMAR, prior-guided MBIR has great advantages in image resolution, noise texture and uniformity.

In Chapter 3 and Chapter 5 we addressed two different non-uniform sampling caused issues. Chapter 3 focused on fan-beam induced in-plane noise varying phenomenon, and we proposed a spatially adaptive regularization model in MAP estimation framework, based on evaluation of a metric EVD. In Chapter 5 we targeted a physical detector design compromise when adapting to large cone angles. A joint
parameter estimation with image reconstruction algorithm is presented to combat the systematic data errors. Preliminary results are very promising with improved diagnostic image quality. We expect the techniques to be adopted in industrial applications.
APPENDIX A

POINTWISE BAYESIAN RESTORATION DERIVATION

The Bayesian estimation with conjugate prior in (2.38) can be derived as follows:

\[ N_j = \int_0^\infty \theta_j f_{\Lambda_j}(\lambda_j|\theta_j) f_{\Theta_j}(\theta_j) \, d\theta_j \]
\[ = \int_0^\infty \theta_j \left( \sum_{k=0}^{\infty} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{ -\frac{(\lambda_j - k)^2}{2\sigma^2} \right\} \frac{\theta_j^k e^{-\theta_j}}{k!} \right) \]
\[ \left( \frac{\beta_{\alpha_j}}{\Gamma(\alpha_j)} \theta_j^{\alpha_j-1} e^{-\beta_j \theta_j} \right) \, d\theta_j \]
\[ = (2\pi\sigma^2)^{-\frac{1}{2}} \frac{\beta_{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} \exp\left\{ -\frac{(\lambda_j - k)^2}{2\sigma^2} \right\} \frac{(k!)^{-1}}{(k+1)} \]
\[ \int_0^\infty \theta_j^{k+\alpha} e^{-(1+\beta_j)\theta_j} \, d\theta_j \]
\[ = (2\pi\sigma^2)^{-\frac{1}{2}} \frac{\beta_{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} \exp\left\{ -\frac{1}{2\sigma^2} (\lambda_j - k)^2 \right\} \Gamma^{-1}(k+1) \]
\[ \Gamma(k + \alpha_j + 1)(1 + \beta_j)^{-(k+\alpha_j+1)} \]
\[ = (2\pi\sigma^2)^{-\frac{1}{2}} \frac{\beta_{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} h(k; \alpha_j + 1, \beta_j, \lambda_j) \]
\[ D_j = \int_0^\infty f_{\Lambda_j}(\lambda_j|\theta_j) f_{\Theta_j}(\theta_j) \, d\theta_j \]

\[ = \int_0^\infty \left( \sum_{k=0}^{\infty} (2\pi \sigma_e^2)^{-\frac{1}{2}} \exp\left\{ -\frac{(\lambda_j - k)^2}{2\sigma_e^2} \right\} \frac{\theta_j^k e^{-\theta_j}}{k!} \right) \]

\[ \left( \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \theta_j^{\alpha_j-1} e^{-\beta_j \theta_j} \right) \, d\theta_j \]

\[ = (2\pi \sigma_e^2)^{-\frac{1}{2}} \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} \left( \exp\left\{ -\frac{(\lambda_j - k)^2}{2\sigma_e^2} \right\} (k!)^{-1} \right) \]

\[ \int_0^\infty \theta_j^{k+\alpha-1} e^{-(1+\beta)\theta_j} \, d\theta_j \]

\[ = (2\pi \sigma_e^2)^{-\frac{1}{2}} \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} \left( \exp\left\{ -\frac{1}{2\sigma_e^2} (\lambda_j - k)^2 \right\} \Gamma^{-1}(k+1) \right) \]

\[ \Gamma(k + \alpha_j)(1 + \beta_j)^{-(k+\alpha_j)} \]

\[ = (2\pi \sigma_e^2)^{-\frac{1}{2}} \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \sum_{k=0}^{\infty} h(k; \alpha_j, \beta_j, \lambda_j) \]
APPENDIX B

MULTI-MATERIAL CORRECTION POLYNOMIAL COEFFICIENTS

MMC provides a spatially varying fifth degree polynomial correction to projection measurements. The polynomial takes two input variables: total projection $p_t$ and iodine projection $p_{io}$, given by

$$g(p_t, p_{io}) = c_0 p_{io} + c_1 p_{io}^2 + c_2 p_{io}^3 + c_3 p_{io}^4 + c_4 p_{io}^5$$

$$+ c_{11} p_{io}^2 p_t + c_{12} p_{io}^2 p_t^2 + c_{13} p_{io}^3 p_t + c_{14} p_{io}^3 p_t^2 + c_{15} p_{io}^5 p_t$$

$$+ c_{21} p_{io}^2 p_t^2 + c_{22} p_{io}^2 p_t^2 + c_{23} p_{io}^3 p_t^2 + c_{24} p_{io}^4 p_t^2 + c_{25} p_{io}^5 p_t^2$$

$$+ c_{31} p_{io}^3 p_t^3 + c_{32} p_{io}^3 p_t^3 + c_{33} p_{io}^3 p_t^3 + c_{34} p_{io}^4 p_t^3 + c_{35} p_{io}^5 p_t^3$$

$$+ c_{41} p_{io}^4 p_t^4 + c_{42} p_{io}^4 p_t^4 + c_{43} p_{io}^4 p_t^4 + c_{44} p_{io}^5 p_t^4 + c_{45} p_{io}^5 p_t^4$$

$$+ c_{51} p_{io}^5 p_t^5 + c_{52} p_{io}^5 p_t^5 + c_{53} p_{io}^5 p_t^5 + c_{54} p_{io}^5 p_t^5 + c_{55} p_{io}^5 p_t^5.$$

For each detector, 30 coefficients are obtained from pre-calibrated data. One example of such coefficients for the center detector of the array is listed in Table B.1.
TABLE B.1

MMC POLYNOMIAL COEFFICIENTS FOR CENTER DETECTOR OF THE DETECTOR ARRAY

<table>
<thead>
<tr>
<th>$c_{ij}$</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
<th>j=4</th>
<th>j=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>i=0</td>
<td>-0.2680</td>
<td>0.1501</td>
<td>-0.0143</td>
<td>0.0007</td>
<td>-0.0000</td>
</tr>
<tr>
<td>i=1</td>
<td>0.0471</td>
<td>-0.0200</td>
<td>0.0028</td>
<td>-0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>i=2</td>
<td>0.0047</td>
<td>0.0016</td>
<td>-0.0005</td>
<td>0.0000</td>
<td>-0.0000</td>
</tr>
<tr>
<td>i=3</td>
<td>-0.0014</td>
<td>0.0001</td>
<td>0.0000</td>
<td>-0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>i=4</td>
<td>0.0001</td>
<td>-0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-0.0000</td>
</tr>
<tr>
<td>i=5</td>
<td>-0.0000</td>
<td>0.0000</td>
<td>-0.0000</td>
<td>0.0000</td>
<td>-0.0000</td>
</tr>
</tbody>
</table>


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