TOPICS IN ALGORITHMIC RANDOMNESS AND EFFECTIVE PROBABILITY

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Abstract

by

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This dissertation contains the results from three related projects, each within the fields of algorithmic randomness and probability theory.

The first project we undertake, which can be found in Chapter 2, contains the definition a natural, computable Borel probability measure on the space of Borel probability measures over $2^\omega$ that allows us to study algorithmically random measures. The main results here are as follows. Every (algorithmically) random measure is atomless yet mutually singular with respect to the Lebesgue measure. The random reals of a random measure are random for the Lebesgue measure, and every random real for the Lebesgue measure is random for some random measure. However, for a fixed Lebesgue-random real, the set of random measures for which that real is random is small. Relatively random measures, though mutually singular, always share a random real that is in fact computable from the join of the measures. Random measures fail Kolmogorov’s 0-1 law. The shift of a random real for a random measure is no longer random for that measure.

In our second project, which makes up Chapter 3, we study algorithmically random closed subsets of $2^\omega$, algorithmically random continuous functions from $2^\omega$ to $2^\omega$, and the algorithmically random Borel probability measures on $2^\omega$ from Chapter 2 especially the interplay among these three classes of objects. Our main tools are preservation of randomness and its converse, the “no randomness ex nihilo princi-
ple,” which together say that given an almost-everywhere defined computable map from $2^{\omega}$ to itself, a real is Martin Löf random for the pushforward measure if and only if its preimage is random with respect to the measure on the domain. These tools allow us to prove new facts, some of which answer previously open questions, and reprove some known results more simply.

The main results of Chapter 3 are the following. We answer an open question in [3] by showing that $\mathcal{X} \subseteq 2^{\omega}$ is a random closed set if and only if it is the set of zeros of a random continuous function on $2^{\omega}$. As a corollary, we obtain the result that the collection of random continuous functions on $2^{\omega}$ is not closed under composition. We construct a computable measure $Q$ on the space of measures on $2^{\omega}$ such that $\mathcal{X} \subseteq 2^{\omega}$ is a random closed set if and only if $\mathcal{X}$ is the support of a $Q$-random measure. We also establish a correspondence between random closed sets and the random measures studied in Chapter 2. Lastly, we study the ranges of random continuous functions, showing that the Lebesgue measure of the range of a random continuous function is always strictly between 0 and 1.

In Chapter 4 we effectivize a theorem of Erdős and Rényi [11], which says that for $c \geq 1$, if a fair coin is used to generate a length-$N$ string of 1’s and −1’s, which are interpreted as gain and loss, then the maximal average gain over $\lfloor c \log N \rfloor$-length substrings converges almost surely (in $N$) to the same limit $\alpha(c)$. We show that if the 1’s and −1’s are determined by the bits of a Martin Löf random, then the convergence holds.
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CHAPTER 1

INTRODUCTION

Algorithmic randomness was born out of an attempt to make precise the term “random” by distinguishing a set of random elements that satisfy every almost-sure property for the Lebesgue measure, $\lambda$, on $2^\omega$, the space of (one-way) infinite binary sequences (aka *reals*). However, being in the complement of a singleton is an almost-sure property, so satisfying every almost sure property is impossible. Thus, the theory of computation was brought into the picture, and the only properties considered were those that were sufficiently computable. This gives rise to a distinguished set $\text{MLR}_\lambda \subseteq 2^\omega$, called *Martin L"of randoms* or just *randoms*, with the property that $\lambda(\text{MLR}) = 1$.

One trend in algorithmic randomness has been to code other mathematical objects by infinite binary sequences, declare an object to be (algorithmically) random if it has a random code, and then to investigate what those random objects look like and how they behave. In Chapters 2 and 3, the objects are Borel probability measures on $2^\omega$, continuous functions from $2^\omega$ to itself, and closed subsets of $2^\omega$.

Another trend is the so-called effectivization of classical theorems of probability/measure theory. Many theorems of probability are “almost sure” results. To effectivize such a theorem, one assumes the objects in the hypothesis (e.g. functions/random variables) sufficiently computable and concludes that the result holds on the algorithmically randoms. In Chapter 4, we effectivize a 1970 result of Erdös and Rényi [11].
1.1 Summary of Chapter 2

In Chapter 2 the objects of study are measures\(^1\) over \(2^{\omega}\). This project started as a result of studying the Ergodic Decomposition Theorem (specifically the results in [14]), which can be viewed as a statement about measures on the space of measures on \(2^{\omega}\). We wondered:

**Question 1.1.1.** Is there a natural measure on the space of measures over \(2^{\omega}\)?

Here the word “natural” corresponds to the fact that Lesbesgue measure is considered the most natural measure on \(2^{\omega}\). Just as the randoms for the Lebesgue measure on \(2^{\omega}\) constitute the truly random real numbers, if there were a natural measure on the space of measures, then its random elements should constitute the truly random measures on \(2^{\omega}\).

Definition 2.3.2 defines a measure \(P\) on the space of measures on \(2^{\omega}\) that we see as answering Question 1.1.1 affirmatively. Essentially, the measure \(P\) says that the conditional probability of going left from a given node in the full binary tree is uniformly distributed and independent of other nodes. The measure \(P\) is natural in the sense that every measure on \(2^{\omega}\) comes from assigning said conditional probabilities according to *some* sequence of distributions, and taking that sequence to be IID-uniform is somehow most natural.

The measure \(P\) on determines a collection \(\text{MLR}_P\) of \(P\)-random measures. The remainder of this section is a synopsis of the results we prove in Chapter 2 about these \(P\)-random measures.

The Lebesgue measure \(\lambda\) is the so-called *barycenter* of \(P\); that is, \(\int \mu(A)\,dP(\mu) = \lambda(A)\) for any Borel set \(A\). By results of Hoyrup [14], this implies that Lebesgue randoms are exactly those that are random elements for some \(P\)-algorithmically-random measure: \(\text{MLR}_\lambda = \bigcup_{\mu \in \text{MLR}_P} \text{MLR}_\mu\).

\(^1\)Here the term *measure* is short for Borel probability measure.
Every $P$-random measure is atomless (i.e., it gives zero probability to singletons) yet mutually singular with respect to the Lebesgue measure $\lambda$. (A measure $\mu$ is mutually singular with respect to $\lambda$ if $\lambda(A) = 1$ and $\mu(A) = 0$ for some Borel set $A$.)

We conjectured initially that if $\mu$ and $\nu$ are relatively random measures, then they share no random reals. Relative randomness is the algorithmic analog of independence. So, this conjecture said that if measures $\mu$ and $\nu$ are generated independently, then they will not agree on any real’s being random. This conjecture is almost true: relatively random measures are mutually singular, and, hence, $\text{MLR}_\mu \cap \text{MLR}_\nu$ has both $\mu$ and $\nu$ measure zero. Therefore any agreement between $\mu$ and $\nu$ on what is random is rare. Surprisingly, however, there actually is a real that is random for both. Moreover, there is a uniform construction of such a real using $\mu$ and $\nu$ as oracles.

Whenever $\mu$ is $P$-random, $x \in \text{MLR}_\mu$, and $y$ differs from that of $x$ at only finitely many bits, then $y \notin \text{MLR}_\mu$; i.e., $\text{MLR}_\mu$ is an “anti-tailset”. Thus $P$-random measures badly fail Kolmogorov’s 0-1 law. We use this fact to prove that if $x \in \text{MLR}_\lambda$, then

$$P\{\mu : x \in \text{MLR}_\mu\} = 0.$$  

1.2 Summary of Chapter

Again, in this project, the focus is on algorithmic randomness in spaces other than $2^\omega$. Here, however we focus not just on how the random objects behave, but also on how they behave with each other. Our main tools are the randomness preservation principle and its converse, the no randomness ex nihilo principle. This work is joint with Chris Porter.

The objects in play here are Borel probability measures on $2^\omega$ (as in Chapter 2), nonempty closed subsets of $2^\omega$, and continuous functions from $2^\omega$ to $2^\omega$. In each of the three cases, there is a surjective map that assigns to each $x \in 2^\omega$ an object $O_x$ (a measure, a nonempty closed set, or a continuous function) and then the object $O_x$ is
said to be (Martin L"of) random if $x$ is (Martin L"of) random. So, we can talk about random measures, random closed sets (first defined in [2]), and random continuous functions (first defined in [3]).

In this project, we rely heavily on the preservation of randomness and no randomness ex nihilo principles. Together, they say that given a computable measure $\mu$ on $2^\omega$ and a $\mu$-a.e. defined computable map $\Phi: 2^\omega \to 2^\omega$, an element $y \in 2^\omega$ is Martin L"of random for the pushforward measure $\mu \circ \Phi^{-1}$ if and only if $y = \Phi(x)$ for some $x \in 2^\omega$ that is Martin L"of random for $\mu$. These are powerful tools because they often allow one to draw conclusions about Martin L"of randoms by showing that the pushforward measure is what was desired and then merely observing that the map at hand is computable. Using these tools, we reprove, in a much simpler way, the result (in [2]) that every random closed set contains an element that is Martin L"of random for the Lebesgue measure and that every element that is Martin L"of random for the Lebesgue measure is contained in some random closed set.

We also reprove the fact that if $F$ is a random continuous function for which the zero set $F^{-1}\{0\}$ is nonempty, then this set is, in fact, a random closed set. Our tools then give for free the previously-left-open converse, which says that every random closed set is realized as the zero set of some random continuous function. The fact that the composition of random continuous functions need not be random follows as a corollary.

For a measure $\mu$ on $2^\omega$, the $1/3$-support is defined to be the set of $x \in 2^\omega$ such that $\mu([x \upharpoonright n + 1] \mid [x \upharpoonright n]) > 1/3$, where $[x \upharpoonright n]$ denotes the set of all elements of $2^\omega$ that agree with $x$ on the first $n$ bits. We show that a closed subset of $2^\omega$ is random if and only if it is the $1/3$-support of some random measure. We also show that there is a different way of defining a “random measure” (i.e., a different measure on the space of measures) so that the (regular) supports of these random measures are exactly the random closed sets.
It was shown in [3] that random continuous functions are not necessarily injective nor surjective. We extend this by showing that random continuous functions are never injective and never surjective. Moreover, we show that the Lebesgue measure of the range of a random continuous function is strictly between 0 and 1.

1.3 Summary of Chapter 4

Erdős and Rényi [11] proved that for any \( c \geq 1 \), if a fair coin is used to generate a length-\( N \) string \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_N \) of 1’s and -1’s, then the maximal average

\[
\max_{0 \leq n \leq N - \lfloor c \log_2 N \rfloor} \frac{\sigma_{n+1} + \sigma_{n+1} + \cdots + \sigma_{n+\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor}
\]

converges almost surely (in \( N \)) to the same limit \( \alpha = \alpha(c) \), which is determined by the equation

\[
\frac{1}{c} = 1 - h \left( \frac{1 + \alpha}{2} \right),
\]

where \( h: [0,1] \to [0,1] \) is the binary entropy function

\[
h(x) = -x \log_2 x - (1-x) \log_2 (1-x).
\]

The 1’s and -1’s can be interpreted as the gain or loss of a player in a fair game, so this result says that the maximal average gain over appropriately-sized subgames converges almost surely to \( \alpha \).

This result is a threshold theorem. As Erdős and Rényi note, if \( K(N) \) is an integer-valued function of \( N \) such that \( \frac{K(N)}{\log N} \to \infty \), then the maximal average

\[
\max_{0 \leq n \leq N - K(N)} \frac{\sigma_{n+1} + \sigma_{n+2} + \cdots + \sigma_{n+K(N)}}{K(N)}
\]

converges almost surely to 0. If \( K(N) \) is an integer-valued function of \( N \) such that
$K(N) \leq c \log N$ for some $0 < c < 1$, then the maximal average is almost surely eventually 1 (and hence converges almost surely to 1). So, the theorem explains what happens in the only case left to consider, when $K(N)$ grows like $c \log N$ for some $c \geq 1$.

Chapter 4 contains an effectivization of this theorem: for any $c \geq 1$, not only does the maximal average converge almost surely to $\alpha$, but, in fact, the convergence holds on every infinite sequence of 1’s and −1’s that is Martin L¨of random.

1.4 A word on notation

Our notation is fairly standard. For various reasons though, we use some different notation and conventions in each chapter. In order to make each chapter more self-contained we also state some definitions more than once (but never more than once in a given chapter).
CHAPTER 2

ALGORITHMICALLY RANDOM MEASURES

2.1 Introduction

Algorithmic randomness attempts to make precise the notion of a random real number. Coding other objects (e.g., graphs) by real numbers allows for the study of the algorithmically random versions of these objects (e.g., algorithmically random graphs). This has been done, for example, in [6], [3], and [1]. Here we undertake a similar project, where the objects of study are Borel probability measures on $2^\omega$.

We define a natural, computable (in the sense of computable analysis) map from $2^\omega$ to the space $\mathcal{P}(2^\omega)$ of Borel probability measures on $2^\omega$. This map pushes the Lebesgue measure forward, yielding a natural, computable Borel probability measure $P$ on $\mathcal{P}(2^\omega)$. The construction of $P$ is a special (and, we think, the most natural) case of a construction in [18]. We investigate the algorithmically $P$-random Borel probability measures and the algorithmically random reals for those measures.

2.2 Preliminaries

2.2.1 Basics

The set of natural numbers is denoted by $\omega$. The function $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ is any computable bijection. The computably enumerable (c.e.) subsets of $\omega$ are effectively numbered as $\langle W_e \rangle_{e \in \omega}$.

The set of finite binary strings is denoted by $2^{<\omega}$. It is effectively numbered via $\sigma_0 = \emptyset$ (the empty string), $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = 00$, $\sigma_4 = 01$, etc. For strings $\sigma$ and
the notation $\sigma \prec \tau$ means that $\sigma$ is an initial segment of $\tau$ and $\sigma \perp \tau$ means that neither $\sigma \prec \tau$ nor $\tau \prec \sigma$.

Cantor space, the space of all one-way infinite binary strings, is denoted by $2^\omega$. For $\sigma \in 2^{<\omega}$, let $[\sigma] = \{x \in 2^\omega : x \succ \sigma\}$ (where $x \succ \sigma$ means that $\sigma$ is an initial segment of $x$); this is the cylinder set generated by $\sigma$. The collection of all cylinder sets forms a clopen basis for a topology on $2^\omega$. This topology is metrizable via $d(x, y) = 2^{-\min\{n : x(n) \neq y(n)\}}$.

2.2.2 General effective spaces

We assume familiarity with the basics of computability theory and computable analysis over $2^\omega$ and $\mathbb{R}$. In order to do computable analysis and probability theory in spaces other than $2^\omega$ and $\mathbb{R}$, we, following [13] and [8], work in an effective Polish space, which is a complete metric space $(X, d)$ with a countable dense subset $Q = \langle q_i \rangle$ such that $d(q_i, q_j)$ is a computable real number uniformly in $i$ and $j$. A representation of $X$ is a partial surjective function $\rho : 2^\omega \to X$. Given a representation $\rho$, a $\rho$-name for $x \in X$ is an element of $\rho^{-1}\{x\}$. Any effective Polish space is equipped with a representation called its standard fast Cauchy representation, $\rho_C : 2^\omega \to X$, defined by $\rho_C(0^{n_0}10^{n_1}10^{n_2}1\cdots) = x$ if $d(x, q_{n_i}) \leq 2^{-i}$. Note that different elements of $X$ cannot have the same $\rho_C$-name. We will simply say name when $\rho$ is clear from the context.

Any effective Polish space admits an effective basis for its topology; $B_k^X$, with $k = \langle i, j \rangle$, is the ball centered at $q_i$ with radius $2^{-j}$. When no confusion will be caused, $B_k^X$ will be written simply as $B_k$. A subset $U \subseteq X$ is then effectively open, or $\Sigma^0_1$, if $U = \bigcup_{i \in W_e} B_i$ for some c.e. $W_e$; and $C \subseteq X$ is effectively closed, or $\Pi^0_1$, if $X - C$ is effectively open. A compact subset $K \subseteq X$ is effectively compact if the set of (indices for) finite covers by the $B_i$’s is c.e.

1The authors there use the term computable metric space.
A function $f: X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is **left-** (resp. **right-**) **computable** if $f^{-1}(r, \infty]$ (resp. $f^{-1}[-\infty, r)$) is effectively open in $X$ for each $r \in \mathbb{Q}$. Let $X$ and $Y$ be effective Polish spaces. Then $f: X \to Y$ is **computable** if $f^{-1}(U)$ is effectively open in $X$ whenever $U$ is effectively open in $Y$, uniformly in $U$. In particular, computability implies continuity. The following proposition is straightforward

**Proposition 2.2.1.**

1. A function $f: X \to Y$, where $(Y, \rho, R)$ is an effective Polish space, is computable if and only if there is a computable function that outputs a name for $f(x) \in Y$ whenever given a name for $x \in X$.

2. $f: X \to \overline{\mathbb{R}}$ is left- (resp. right-) computable if and only if there’s a computable function that outputs an increasing (resp. decreasing) sequence of rationals converging to $f(x)$ whenever given a name for $x \in X$.

3. $f: X \to \overline{\mathbb{R}}$ is computable if and only if it is both left- and right-computable.

Where it makes sense, all notions are relativizable. Thus for $x \in 2^\omega$, we can speak of an $x$-computable function, an $x$-left-computable function, etc.

**Proposition 2.2.2** ([14]). Let $X$ and $Y$ be effective Polish spaces.

1. An effectively compact set is effectively closed.

2. If $f: X \to Y$ is computable, and $K \subseteq X$ is effectively compact, then $f(K)$ is effectively compact.

3. An effectively closed subset of $2^\omega$ is effectively compact.

2.2.3 The space of probability measures

For an effective Polish space $X$, let $\mathcal{B}(X)$ be its Borel $\sigma$-algebra; that is, $\mathcal{B}(X)$ is the smallest class of subsets of $X$ that contains the open sets and is closed under complementation and countable unions. A **Borel probability measure**, or just **measure** for short, on $X$ is a function $\mu: \mathcal{B}(X) \to [0, 1]$ such that $\mu(X) = 1$ and $\mu(\bigcup_{i \in \omega} A_i) = \sum_i \mu A_i$ whenever the $A_i$’s are pairwise disjoint elements of $\mathcal{B}(X)$. When $X = 2^\omega$, Carathéodory’s extension theorem [12] guarantees that the conditions
(a) $\mu([\emptyset]) = 1$ and
(b) $\mu([\sigma]) = \mu([\sigma 0]) + \mu([\sigma 1])$ for all $\sigma \in 2^{<\omega}$

uniquely determine a measure on $2^{\omega}$. Thus, a measure on Cantor space is identified with a function $\mu : 2^{<\omega} \to [0, 1]$ satisfying conditions (a) and (b). We may write $\mu(\sigma)$ instead of $\mu([\sigma])$. The **Lebesgue measure** $\lambda$ is defined by $\lambda(\sigma) = 2^{-|\sigma|}$ for each string $\sigma$.

The space of all Borel probability measures on an effective Polish space $X$ is denoted by $\mathcal{P}(X)$. It is itself an effective Polish space under the (metrizable) weak-∗ topology [16]. We do not need the details of the effective structure of $\mathcal{P}(X)$ but only the following proposition.

**Proposition 2.2.3** ([16]).

1. A measure $\mu \in \mathcal{P}(X)$ is computable if and only if $\mu(U)$ is uniformly left-computable on effectively open sets $U \subseteq X$.
2. A function $f : X \to \mathcal{P}(Y)$ is computable if and only if a name for $x \in X$ uniformly left-computes the value of $f(x)(B_k^Y)$.

2.2.4 Algorithmic randomness

Let $X$ be an effective Polish space endowed with a Borel probability measure $\mu$, and let $y \in 2^{\omega}$ be a name for $\mu$. A **$y$-Martin L"of test for $\mu$ randomness** is a uniformly $\Sigma^0_y$ sequence $\langle U_n \rangle_{n \in \omega}$ such that $\mu(U_n) \leq 2^{-n}$. An element $x \in X$ passes the $y$-Martin L"of test for $\mu$ randomness if $x \notin \bigcap U_n$. An element $x \in X$ is **$y$-Martin L"of random for $\mu$** if it passes all $y$-Martin L"of tests for randomness.

An element $x \in X$ is **Martin L"of random for $\mu$**, or just **$\mu$-random**, if it is $y$-Martin L"of random for $\mu$ for some name $y \in 2^{\omega}$ of $\mu$. We write $\text{MLR}_\mu$ for the set of all $\mu$-randoms. Note that because there are only countably many $\Sigma^0_y$ sets, $\mu(\text{MLR}_\mu) = 1$. Because the Lebesgue measure is special, we often write $\text{MLR}_\lambda$. 

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As noted in [8], for any name $y$ of $\mu$, there is a single $y$-Martin L"of test for $\mu$ randomness that suffices to define $y$-Martin L"of randomness for $\mu$. Such a $y$-Martin L"of test for $\mu$ randomness is called a \textbf{universal $y$-Martin L"of test for $\mu$ randomness}.

The following proposition allows us to work with a single name for $\mu$ in the cases we consider.

\textbf{Proposition 2.2.4.} If $\mu \in \mathcal{P}(X)$ has a name $y$ of least Turing degree, then $x \in X$ is $\mu$-random if and only if $x$ is $y$-Martin L"of random for $\mu$.

\textit{Proof.} Suppose $z \in 2^\omega$ is a name for $\mu$ with $z \geq_T y$. Then any $y$-Martin L"of test for $\mu$ randomness is also a $z$-Martin L"of test for $\mu$ randomness. Thus if $x$ is $z$-Martin L"of random for $\mu$ it is also $y$-Martin L"of random for $\mu$. \hfill \Box

When $\mu \in \mathcal{P}(X)$ has a name $y \in 2^\omega$ of least Turing degree, we will call the universal $y$-Martin L"of test for $\mu$ randomness simply a \textbf{universal $\mu$-test}.

An element $x \in X$ is \textbf{$\mu$-Kurtz random} if there is a name $y$ of $\mu$ such that $x \in U$ for every $U \in \Sigma^0_{1,y}$ with $\mu(U) = 1$. We write $\text{KR}_\mu$ for the collection of all $\mu$-Kurtz randoms.

Note that because there are only countably many effectively open sets, $\mu(\text{KR}_\mu) = 1$. Moreover, the following is true.

\textbf{Proposition 2.2.5 ([17])}. If $x \in \text{MLR}_\mu$ then $x \in \text{KR}_\mu$.

2.3 Random measures

Given $x \in 2^\omega$, the $n^{th}$ \textbf{column} $x_n \in 2^\omega$ of $x$ is defined by $x_n(k) = 1$ if and only if $x(\langle n, k \rangle) = 1$ (recall that $\langle n, k \rangle$ is a fixed computable bijection between $\omega^2$ and $\omega$).

We write $x = \oplus_{n \in \omega} x_n$; this is the infinite join operation in the Turing degrees.

Define the map $\Phi: 2^\omega \to \mathcal{P}(2^\omega)$, with $\Phi(x)$ written $\mu_x$, by $\mu_x(\emptyset) = 1$ and $\mu_x(\sigma \cap 0) = x_n \star \mu_x(\sigma)$, where $x_n$ is (the real number represented by) the $n^{th}$ col-
umn of \( x \). This map is essentially in [18], but is independently due to Chris Porter.

It is, as the next proposition shows, really just another representation of \( \mathcal{P}(2^\omega) \).

**Proposition 2.3.1.** The map \( \Phi \) is computable.

**Proof.** By Proposition [2.2.3] it suffices to show that given \( x \in 2^\omega \) we can uniformly compute \( \mu_x(\sigma) \). Write \( \mu_x(\sigma) = \prod_{i < |\sigma|} \mu_x(\sigma \upharpoonright i + 1 | \sigma \upharpoonright i) \), where \( \mu_x(\sigma | \tau) := \mu_x([\sigma] \cap [\tau]) / \mu_x([\tau]) \). Multiplication is computable, so it suffices to show that \( \mu_x(\sigma \upharpoonright i + 1 | \sigma \upharpoonright i) \) is uniformly computable from \( x \). But this is clear since \( \mu_x(\sigma \upharpoonright i + 1 | \sigma \upharpoonright i) \) is either a column or one minus a column of \( x \), and computably so.

Being computable implies being Borel (indeed continuous), so \( \Phi \) pushes \( \lambda \) forward to a Borel probability measure on the space \( \mathcal{P}(2^\omega) \) of measures.

**Definition 2.3.2.** The measure \( P \in \mathcal{P}(\mathcal{P}(2^\omega)) \) is the pushforward via \( \Phi \) of the Lebesgue measure; that is,

\[
P(B) := \lambda \circ \Phi^{-1} B
\]

for Borel \( B \subseteq \mathcal{P}(2^\omega) \).

**Proposition 2.3.3.** The measure \( P \) is computable.

**Proof.** By Proposition [2.2.3] it suffices to show that the measure of an effectively open set \( U \subseteq \mathcal{P}(2^\omega) \) is uniformly left-computable. Since \( \Phi \) is computable, \( \Phi^{-1} U \) is uniformly effectively open. Because \( \lambda \) is computable, \( P(U) = \lambda \circ (\Phi^{-1}(U)) \) is left-computable.

The measure \( P \) was defined with a goal that Martin Löf random elements of \( P \) are exactly the images under \( \Phi \) of random elements of \( 2^\omega \).

**Theorem 2.3.4** (Preservation of randomness and no randomness ex nihilo). \( \nu \in \text{MLR}_P \) if and only if \( \nu = \mu_x \) for some (unique) \( x \in \text{MLR}_\lambda \). 

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Proof. (Preservation of randomness) If $\nu \notin \text{MLR}_P$, then $\nu \in \bigcap_{n \in \omega} U_n$ for some $P$-test $\langle U_n \rangle_{n \in \omega}$. But, $\Phi$ is computable, so $\Phi^{-1}U_n$ is effectively open in $2^\omega$ for each $n$. Moreover $\lambda(\Phi^{-1}(U_n)) = P(U_n) \leq 2^{-n}$ for each $n$. Thus $\langle \Phi^{-1}(U_n) \rangle_{n \in \omega}$ is a $\lambda$-test, and, hence, $x \notin \text{MLR}_\lambda$ if $\nu = \Phi(x) = \mu_x$.

(No randomness ex nihilo) Now, we show, following Shen (see [4, Theorem 3.5]), that if $\nu \in \text{MLR}_P$, then $\nu = \Phi(x)$ for some $x \in \text{MLR}$. Uniqueness follows because $\Phi(x) = \Phi(y)$ for $x \neq y$ implies each of $x$ and $y$ has a dyadic rational column.

Fix a universal Martin L"{o}f test $\langle U_n \rangle_{n \in \omega}$ for $\lambda$ randomness, and set $K_n := 2^\omega - U_n$. Define $V_n = \mathcal{P}(2^\omega) - \Phi(K_n)$. Since $\Phi$ is computable, $\Phi(K_n) \in \Pi^0_1$ by Proposition 2.2.2 parts (2) and (3), so $V_n \in \Sigma^0_1$, and uniformly so. Now

$$P(V_n) = 1 - P(\Phi(K_n)) = 1 - \lambda(\Phi^{-1}(\Phi(K_n)))$$

$$\leq 1 - \lambda(K_n)$$

$$\leq 2^{-n}.$$ 

Thus, $\langle V_n \rangle_{n \in \omega}$ is a $P$-Martin L"{o}f test, so if $\nu \in \text{MLR}_P$, then $\nu \notin V_n$ for some $n$; i.e., $\nu \in \Phi(K_n)$. The proof is now complete since $K_n \subseteq \text{MLR}_\lambda$. \qed

The next proposition shows that $\Phi$ is a measure theoretic isomorphism (see [22]) between $(2^\omega, \mathcal{B}(2^\omega), \lambda)$ and $(\mathcal{P}(2^\omega), \mathcal{B}(\mathcal{P}(2^\omega)), P)$.

**Proposition 2.3.5.** For any Borel $A \subseteq 2^\omega$, $P(\Phi(A)) = \lambda(A)$.

*Proof.* Note that $\Phi(\text{MLR}_\lambda) \cap \Phi(A) = \Phi(\text{MLR}_\lambda \cap A)$ since $\Phi(x) = \Phi(x')$ and $x \in \text{MLR}.$
implies $x = x'$. Thus,

\[
P(\Phi(A)) = P(\text{MLR}_P \cap \Phi(A))
= P(\Phi(\text{MLR}_\lambda) \cap \Phi(A))
= P(\Phi(\text{MLR}_\lambda \cap A))
= \lambda \Phi^{-1}(\text{MLR}_\lambda \cap A)
= \lambda(\text{MLR}_\lambda \cap A)
= \lambda(A).
\]

We will need the next result, which is the same as Theorem 2.3.4 for Kurtz randomness.

Proposition 2.3.6. $\nu \in \mathsf{KR}_P$ if and only if $\nu = \mu_x$ for some (unique) $x \in \mathsf{KR}_\lambda$.

Proof. If $\nu \notin \mathsf{KR}_P$, then $\nu \in C$ for some $C \in \Pi^0_1$ with $P(C) = 0$. Because $\Phi$ is computable, $\Phi^{-1}C \in \Pi^0_1$ and by the definition of $P$, $\lambda \Phi^{-1}C = P(C) = 0$, so $\Phi^{-1}\nu \cap \mathsf{KR}_\lambda = \emptyset$.

Now, if $x \notin \mathsf{KR}_\lambda$, then $x \in C$ for some $C \in \Pi^0_1$ with $\lambda(C) = 0$. But then $C$ is effectively compact, so $\Phi(C) \in \Pi^0_1$. By Proposition 2.3.5, $P(\Phi(C)) = 0$, so $\Phi(x) \notin \mathsf{KR}_P$.

The last preliminaries we need regard relative randomness and a slight variation of Van Lambalgen’s Theorem.

Definition 2.3.7. A real $y$ is $\lambda$-random relative to a real $x$, written $y \in \text{MLR}_\lambda^x$, if $y \notin \bigcap U_n$ whenever $U_n$ is a uniformly $\Sigma^0_{1,x}$ sequence with $\lambda(U_n) \leq 2^{-n}$. We write $\text{MLR}_\lambda^\mu$ for $\text{MLR}_\lambda^x$, where $\mu = \mu_x$.

Theorem 2.3.8. In the product space $\mathcal{P}(2^\omega) \times 2^\omega$, with the product measure $P \otimes \lambda$, the pair $(\mu, y)$ is $(P \otimes \lambda)$-random if and only if $\mu \in \text{MLR}_\lambda^\mu$ and $y \in \text{MLR}_\lambda^\mu$. 

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Proof. By Theorem 2.3.4, \((\mu, y)\) is \((P \otimes \lambda)\)-random if and only if \(\mu = \mu_x\) and \((x, y)\) is \((\lambda \otimes \lambda)\)-random in \(2^\omega \times 2^\omega\). By Van Lambalgen’s Theorem [21], this happens if and only if \(y \in \text{MLR}^x_\lambda = \text{MLR}^\mu_\lambda\) and \(x \in \text{MLR}^y_\lambda = \text{MLR}^\mu_\lambda\).

2.4 Random measures and their randoms

Now, we begin an analysis of \(\text{MLR}_P\).

Proposition 2.4.1. If \(\mu \in \text{MLR}_P\), then \(\text{MLR}_\mu\) is dense in \(2^\omega\).

Proof. If \(\mu \in \text{MLR}_P\) (indeed if \(\mu \in \text{KR}_P\)), then \(\mu(\sigma) > 0\) for any \(\sigma \in 2^{<\omega}\).

So, \(\text{MLR}_\mu\) is, in some way, topologically large when \(\mu \in \text{MLR}_P\). Theorem 2.6.5 below shows, however, that \(\text{MLR}_\mu\) is (Lebesgue) measure theoretically small.

Lemma 2.4.2 below says that \(\lambda\) is the barycenter of \(P\). From work of Hoyrup [14], this gives Theorem 2.4.4 which says, in particular, that the \(\lambda\) randoms are exactly the the \(P\) randoms’ randoms.

Lemma 2.4.2. For each Borel \(A \subseteq 2^\omega\), \(\lambda(A) = \int_{\mathcal{P}(2^\omega)} \mu(A) dP(\mu)\).

Proof. The function \(A \mapsto \int_{\mathcal{P}(2^\omega)} \mu(A) dP(\mu)\) is a measure, so it suffices to consider sets of the form \(A = [\sigma]\), where \(\sigma \in 2^{<\omega}\).

\[
\int_{\mathcal{P}(2^\omega)} \mu(\sigma) dP(\mu) = \int_{\mathcal{P}(2^\omega)} \prod_{i < |\sigma|} \mu(\sigma(i)|\sigma \upharpoonright i) dP
\]

\[
= \prod_{i < |\sigma|} \int_{\mathcal{P}(2^\omega)} \mu(\sigma(i)|\sigma \upharpoonright i) dP \quad \text{(By independence.)}
\]

\[
= 2^{-|\sigma|}.
\]

Hoyrup proved the following result, which applies directly to the setting here.

Theorem 2.4.3 ([14] Theorem 3.1, relativized]). Let \(Q \in \mathcal{P}(\mathcal{P}(2^\omega))\) be computable
with barycenter \( \mu \). Then for any \( z \in 2^\omega \),

\[
\text{MLR}_\mu^z = \bigcup_{\nu \in \text{MLR}_\nu^z} \text{MLR}_\nu^z.
\]

Since our measure \( P \in \mathcal{P}(2^\omega) \) is computable and \( \lambda \) is its barycenter, the following holds.

**Corollary 2.4.4.** For any \( z \in 2^\omega \),

\[
\text{MLR}_\lambda^z = \bigcup_{\mu \in \text{MLR}_\mu^z} \text{MLR}_\mu^z.
\]

### 2.5 Random measures are atomless

We show now that every random measure assigns each singleton set measure zero; i.e., random measures are atomless. An **atom** of a measure \( \mu \in \mathcal{P}(2^\omega) \) is \( x \in 2^\omega \) such that \( \mu(\{x\}) > 0 \). Define \( \mathcal{A} = \{\mu : \mu \text{ has an atom}\} \) so that \( \mathcal{A} = \bigcup_n \mathcal{A}_n \) where \( \mathcal{A}_n := \{\mu : \mu \text{ has an atom with measure } \geq 1/n\} \).

**Lemma 2.5.1.** \( \mathcal{A}_n \) is effectively closed.

**Proof.** Let \( A(n, \sigma) = \{\mu : \mu(\sigma) \geq 1/n\} \). By the proof of Proposition 2.3.1, the map \( \varphi_\sigma(\mu) = \mu(\sigma) \) is computable, so \( 2^\omega - A(n, \sigma) = \varphi_\sigma^{-1}[0, 1/n) \in \Sigma^0_1 \). Thus, \( \bigcup_{\sigma \in 2^k} A(n, \sigma) \in \Pi^0_1 \), and, hence, \( \mathcal{A}_n = \bigcap_k \bigcup_{\sigma \in 2^k} A(n, \sigma) \in \Pi^0_1 \).

The notation \( x =^* y \) for \( x, y \in 2^\omega \) means that \( x \) and \( y \) differ on only finitely many bits; i.e. \( x =^* y \) if and only if \( |\{i : x(i) \neq y(i)\}| < \infty \).

**Lemma 2.5.2** (Kolmogorov’s 0-1 Law [10]). If \( A \in \mathcal{B}(2^\omega) \) is closed under \( =^* \) (i.e. for all \( x \in A, y =^* x \Rightarrow y \in A \)), then \( \lambda(A) = 0 \) or \( \lambda(A) = 1 \).
Corollary 2.5.3. If \( A \in \mathcal{B}(2^\omega) \) is almost closed under \( =^* \) (i.e. for \( \lambda \)-almost every \( x \), \( x \in A \Rightarrow y \in A \) whenever \( x =^* y \)), then \( \lambda(A) = 0 \) or \( \lambda(A) = 1 \).

**Proof.** The set \( \hat{A} := \{ x \in A : \forall y [x =^* y \Rightarrow y \in A] \} \) is closed under \( =^* \) and has the same measure as \( A \) since \( A \) was already almost closed under \( =^* \). Applying Lemma 2.5.2 to \( \hat{A} \) then gives the result. \( \square \)

Lemma 2.5.4. \( P(A) = 0 \) or 1.

**Proof.** By Proposition 2.3.5 and Corollary 2.5.3, it suffices to show that the set \( \tilde{A} = \{ x : \mu_x \text{ has an atom} \} \) is almost closed under \( =^* \). To that end, we prove that if \( x \in \text{MLR} \cap \tilde{A} \), then \( x' \in \tilde{A} \). The key here is to notice that \( y \in 2^\omega \) is an atom for \( \mu_x \) if and only if \( \prod_{i \in \omega} \mu_x(y(i)|y \upharpoonright i) > 0 \). If \( x' =^* x \), then there is \( N \) such that \( \mu_x(y(i)|y \upharpoonright i) = \mu_{x'}(y(i)|y \upharpoonright i) \) for all \( i > N \). Thus, unless \( \mu_{x'}(y(i)|y \upharpoonright i) = 0 \) for some \( i \leq N \), \( y \) is also an atom of \( x' \). But \( \mu_{x'}(y(i)|y \upharpoonright i) \) is either a column of \( x' \) or one minus a column of \( x' \), and since \( x \in \text{MLR} \), so is \( x' \), which means that \( \mu_{x'}(y(i)|y \upharpoonright i) = 0 \) is impossible. \( \square \)

For \( \sigma \in 2^{<\omega} \), we define a map \( T_\sigma : \mathcal{P}(2^\omega) \rightarrow \mathcal{P}(2^\omega) \), and write \( \mu_\sigma \) for \( T_\sigma(\mu) \), by \( T_\sigma(\mu)(\tau) = \mu_\sigma(\tau) = \mu(\tau|\sigma) := \frac{\mu(\sigma \tau)}{\mu(\sigma)} \). This map is like a shift map. It takes a measure \( \mu \), which can be thought of as a tree of conditional probabilities and outputs a new measure \( \mu_\sigma \) whose tree of probabilities is the same as that of \( \mu \)’s above \( \sigma \).

**Lemma 2.5.5.** For each \( \sigma \in 2^{<\omega} \), \( T_\sigma \) preserves \( P \); i.e. \( P(A) = P(T_\sigma^{-1} A) \) for every Borel \( A \subseteq \mathcal{P}(2^\omega) \).

**Proof.** Since \( T_\sigma = T_{\sigma(n-1)} \circ T_{\sigma(n-2)} \circ \cdots \circ T_{\sigma(1)} \circ T_{\sigma(0)} \) for \( \sigma \in 2^n \), it suffices to consider only the maps \( T_0 \) and \( T_1 \). We prove only that \( T_0 \) preserves \( P \), since the proof that \( T_1 \) does is essentially the same.

Because \( \Phi \) is a measure isomorphism, it suffices to show that the map \( \check{T}_0 := \)
Φ−1 ∘ T_0 ∘ Φ : 2^ω → 2^ω preserves λ. For then
\[ P(B) = \lambda(Φ^{-1}(B)) = \lambda(T_0^{-1}(Φ^{-1}(B))) \]
\[ = \lambda(Φ^{-1}(T_0^{-1}(Φ(Φ^{-1}(B)))))) \]
\[ = \lambda(Φ^{-1}(T_0^{-1}(B))) \]
\[ = P(T_0^{-1}(B)) \].

To show that \( \tilde{T}_0 \) preserves \( \lambda \), we first get a nice description of \( \tilde{T}_0 \). With \( x = \oplus_{i \in \omega} x_i \), we can write \( \tilde{T}_0(x) = x_1 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_8 \oplus x_9 \oplus x_{10} \oplus \cdots \). There is a 1-1 (computable) function \( f : \omega \to \omega \) such that \( \tilde{T}_0(x)(n) = x(f(n)) \). To show \( \tilde{T}_0 \) preserves \( \lambda \), it suffices to show that \( \lambda(\sigma) = \lambda(\tilde{T}_0^{-1}(\sigma)) \) for each \( \sigma \in 2^{<\omega} \). But \( \tilde{T}_0^{-1}(\sigma) = \{ x : \forall i < |\sigma| [x(f(i)) = \sigma(i)] \} \), so clearly \( \lambda(\sigma) = \lambda(\tilde{T}_0^{-1}(\sigma)) \).

Lemma 2.5.6. \( P(A) = 0 \)

Proof. Define \( m : \mathcal{P}(2^\omega) \to [0,1] \) by
\[ m(\mu) = \max\{ r \in [0,1] : \exists y \in 2^\omega(\mu\{y\} = r) \} \].

We want to show that \( m(\mu) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) \).

Notice that \( m(\mu) = \max\{ \mu(0)m(T_0(\mu)), \mu(1)m(T_1(\mu)) \} \). Let
\[ M = \{ \mu : \mu(0)m(T_0(\mu)) \geq \mu(1)m(T_1(\mu)) \} \],
so \( M \) is the set where the maximum-mass atom is to the left in the tree. Then using standard facts about the integral and the fact that the functions \( \mu \mapsto \mu(0) \) and
\[ \mu \mapsto m(T_0(\mu)) \] are \( P \)-independent gives

\[
\int_{\mathcal{P}(2^\omega)} m(\mu) \, dP(\mu) = \int_M \mu(0)m(T_0(\mu)) \, dP + \int_{\mathcal{P}(2^\omega)-M} \mu(1)m(T_1(\mu)) \, dP
\]

\[
\leq \int_{\mathcal{P}(2^\omega)} \mu(0)m(T_0(\mu)) \, dP + \int_{\mathcal{P}(2^\omega)-M} \mu(1)m(T_1(\mu)) \, dP
\]

\[
= \int_{\mathcal{P}(2^\omega)} \mu(0) \, dP \int_{\mathcal{P}(2^\omega)} m(T_0(\mu)) \, dP + \int_{\mathcal{P}(2^\omega)-M} \mu(1)m(T_1(\mu)) \, dP
\]

\[
= \frac{1}{2} \int_{\mathcal{P}(2^\omega)} m(T_0(\mu)) \, dP + \frac{1}{2} \int_{\mathcal{P}(2^\omega)-M} \mu(1)m(T_1(\mu)) \, dP
\]

\[
= \frac{1}{2} \int_{\mathcal{P}(2^\omega)} m(\mu) \, dP + \frac{1}{2} \int_{\mathcal{P}(2^\omega)-M} \mu(1)m(T_1(\mu)) \, dP
\]

\[
- \int_M \mu(1)m(T_1(\mu)) \, dP
\]

\[
= \int_{\mathcal{P}(2^\omega)} m(\mu) \, dP - \int_M \mu(1)m(T_1(\mu)) \, dP.
\]

Thus, either \( P(M) = 0 \) or \( m(T_1(\mu)) = 0 \) for \( P \)-a.e. \( \mu \in M \) (because \( \mu(1) \) is \( P \)-a.s. positive). Symmetric computations show that either \( P(\mathcal{P}(2^\omega) - M) = 0 \) or \( m(T_0(\mu)) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) - M \).

In the case where \( P(M) = 0 \), we have \( P(\mathcal{P}(2^\omega) - M) = 1 \), so \( m(T_0(\mu)) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) \). Similarly, in the case where \( P(\mathcal{P}(2^\omega) - M) = 0 \), we have \( m(T_1(\mu)) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) \). In either case, because \( T_0 \) and \( T_1 \) both preserve \( P \), \( m(\mu) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) \).

In the case where \( 0 < P(M) < 1 \), Lemma 2.5.4 implies that \( P(A) = 0 \) or \( P(A) = 1 \), so that \( m(\mu), m(T_0(\mu)), \) and \( m(T_1(\mu)) \) are all \( P \)-a.s. 0 or all \( P \)-a.s. strictly positive. Thus in this case \( m(T_1(\mu)) = 0 \) on a \( P \)-positive-measure set, and hence on \( P \)-almost all of \( \mathcal{P}(2^\omega) \) as well. Therefore \( m(\mu) = 0 \) for \( P \)-a.e. \( \mu \in \mathcal{P}(2^\omega) \).

Now, we arrive at the main result of this section; random measures are atomless.

**Theorem 2.5.7.** Every \( \mu \in \text{KR}_P \) is atomless. In particular every \( \mu \in \text{MLR}_P \) is...
atomless.

\textit{Proof.} By Lemma 2.5.6, $A = \bigcup A_n$ is $P$-null. Hence, $A_n$ is $P$-null for each $n$. By Lemma 2.5.1 each $A_n$ is $\Pi^0_1$.

\[2.6\] Random measures are mutually singular (with respect to the Lebesgue measure)

A measure $\mu$ is \textit{absolutely continuous} with respect to another measure $\nu$, written $\mu \ll \nu$ if $\nu(A) = 0 \Rightarrow \mu(A) = 0$. The property $\mu \ll \lambda$ implies atomlessness (recall that $\lambda$ is the Lebesgue measure), so it is natural to ask if $\mu \in \text{MLR}_P$ implies $\mu \ll \lambda$. This is far from the case. We show now, in fact, that every $\mu \in \text{MLR}_P$ is \textit{mutually singular} with respect to $\lambda$, in symbols $\mu \perp \lambda$, which means that $\mu(A) = 1$ for some $A \in \mathcal{B}(X)$ with $\lambda(A) = 0$. We will actually prove a stronger result: that $\text{MLR}_\mu \cap \text{MLR}_{\lambda}^\mu = \emptyset$.\footnote{The results in this section were proven with Laurent Bienvenu.}

To show that $\text{MLR}_\mu \cap \text{MLR}_{\lambda}^\mu = \emptyset$, we will employ the well-studied notion of selection functions. A \textit{selection function} is a partial function $f : 2^{<\omega} \rightarrow \{\text{select, exclude}\}$; $f$ determines which bits are selected for entry into a subsequence. The next lemma tells us that a certain selection function we use later will select infinitely often.

\textbf{Lemma 2.6.1.} Let $\mu \in \text{MLR}_P$, let $x \in \text{MLR}_{\lambda}^\mu$, and let $0 < \alpha < 1$. Then there are infinitely many $n$ such that $\mu(0|x \upharpoonright n) > \alpha$.

\textit{Proof.} In the product space, $\mathcal{P}(2^\omega) \times 2^\omega$, the set

$$\mathcal{E}_N = \{ (\mu, x) : \forall n \geq N [\mu(0|x \upharpoonright n) \leq \alpha] \}$$

is $\Pi^0_1$, so it suffices to prove that $(P \otimes \lambda)(\mathcal{E}_N) = 0$, where $P \otimes \lambda$ denotes the product measure, because then for every $(\mu, x) \in \text{MLR}_{P \otimes \lambda}$ (actually for every $(\mu, x) \in \text{KR}_{P \otimes \lambda}$)
there are infinitely many $n$ such that $\mu(0|x \upharpoonright n) > \alpha$. By Theorem 2.3.8 $(\mu, x) \in \text{MLR}_{P \otimes \lambda}$ if and only if $\mu \in \text{MLR}_P$ and $x \in \text{MLR}_\lambda^\mu$.

Now, with $\mathcal{F}_N$ the complement (in $\mathcal{P}(2^\omega) \times 2^\omega$) of $\mathcal{E}_N$, it is clear that for a fixed $x$, $P$-a.e. $\mu$ has the property that $(\mu, x) \in \mathcal{F}_N$. Thus

\[
(P \otimes \lambda)(\mathcal{F}_N) = \int_{\mathcal{P}(2^\omega) \times 2^\omega} 1_{\mathcal{F}_N} \, d(P \otimes \lambda) \\
= \int_{2^\omega} \int_{\mathcal{P}(2^\omega)} 1_{\mathcal{F}_N} \, dP \, d\lambda \quad \text{(By Fubini’s Theorem [12].)} \\
= \int_{2^\omega} 1 \, d\lambda \\
= 1.
\]

\[\square\]

**Lemma 2.6.2.** Let $\mu \in \text{MLR}_P$, let $x \in \text{MLR}_\lambda^\mu$, let $0 < \alpha < 1$ be rational, and let $n_1 < n_2 < \cdots$ be the sequence of all $n_i$ such that $\mu(0|x \upharpoonright n_i) > \alpha$ for all $i$, which is infinite by Lemma 2.6.1. Then $y_x \in 2^\omega$ defined by $y_x(i) = x(n_i)$ satisfies the law of large numbers; i.e.

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i<n} y_x(i) = \frac{1}{2}.
\]

**Proof.** This proof is a relativization of the proof of Theorem 7.4.2 in Downey & Hirschfeldt’s book [10]. The point is that $y$ is the result of a $\mu$-computable selection strategy that simply selects the bit $x(n)$ from $x$ whenever $\mu(0|x \upharpoonight n) > \alpha$. Since $x \in \text{MLR}_\lambda^\mu$ and the $\text{MLR}_\lambda^\mu$ sequences are among those from which it is impossible to $\mu$-computably select a subsequence that violates the law of large numbers, $y_x$ must satisfy the law of large numbers. \[\square\]

The next lemma gives us an effective bound for the proof of Lemma 2.6.4. We state it in slightly simplified form.

**Lemma 2.6.3** (Hoeffding’s inequality [23]). Let $y(1), \ldots, y(n)$ be independent ran-
dom variables on the probability space \((2^\omega, \mu)\) taking values in \([0, 1]\). Then
\[
\mu \left( \frac{1}{n} \sum_{i=1}^{n} y(i) - \int_{2^\omega} \frac{1}{n} \sum_{i=1}^{n} y(i) \, d\mu \geq \epsilon \right) \leq e^{-2n\epsilon^2}
\]
for every \(\epsilon > 0\).

**Lemma 2.6.4.** Let \(\mu \in \text{MLR}_P\), let \(x \in \text{MLR}_\mu\), let \(0 < \alpha < 1\) be rational, and let \(n_1 < n_2 < \cdots\) be the sequence of all \(n_i\) such that \(\mu(0 \mid x \upharpoonright n_i) > \alpha\) for all \(i\). If \(\langle n_i \rangle_{i \in \omega}\) is infinite, then \(y_x \in 2^\omega\) defined by \(y_x(i) = x(n_i)\) satisfies
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i<n} y_x(i) \leq 1 - \alpha.
\]

**Proof.** Let \(\beta = 1 - \alpha\) and \(W_\epsilon^k = \{x : \frac{1}{k} \sum_{i<k} y_x(i) - \beta > \epsilon\}\) for \(\epsilon \in \mathbb{Q}^+\). Then \(W_\epsilon^k\) is uniformly \(\Sigma^0_{1,\mu}\) and
\[
\left\{ x : \frac{1}{n} \sum_{i<n} y_x(i) - \beta > \epsilon \text{ for infinitely many } n \right\} = \bigcap_{N} \bigcup_{k>N} W_\epsilon^k.
\]
Thus it suffices to show that \(\mu \left( \bigcup_{k>N} W_\epsilon^k \right) \to 0\) effectively in \(N\).

Noticing that
\[
\int_{2^\omega} \frac{1}{k} \sum_{i<k} y_x(i) \, d\mu = \frac{1}{k} \sum_{i<k} \int_{2^\omega} y_x(i) \, d\mu \leq \frac{1}{k} \sum_{i<k} \beta = \beta
\]
and
\[
W_\epsilon^k \subseteq \{x : \frac{1}{k} \sum_{i<k} y_x(i) - \int_{2^\omega} \frac{1}{k} \sum_{i<k} y_x(i) \, d\mu > \epsilon\},
\]
we can apply Lemma 2.6.3 to conclude that \(\mu \left( \bigcup_{k>N} W_\epsilon^k \right) \leq \sum_{k>N} e^{-2k\epsilon^2} \to 0\) effectively in \(N\).

**Theorem 2.6.5.** If \(\mu \in \text{MLR}_P\), then \(\text{MLR}_\mu \cap \text{MLR}_\lambda^\mu = \emptyset\), so \(\mu \perp \lambda\).

**Proof.** Lemmas 2.6.2 and 2.6.4 together imply that \(\text{MLR}_\mu \cap \text{MLR}_\lambda^\mu = \emptyset\). But, since
\[ \lambda(\text{MLR}_\lambda) = 1, \text{ it must be the case that } \lambda(\text{MLR}_\mu) = 0. \]

2.7 Relatively random measures

Recall that \( \mu \in \text{MLR}_\nu \) means that \( y \in \text{MLR}_\lambda^\nu \) where \( \mu = \Phi(y) \) and \( \nu = \Phi(x) \). If both \( \mu \in \text{MLR}_\nu^\lambda \) and \( \nu \in \text{MLR}_\mu^\lambda \), we say that \( \mu \) and \( \nu \) are relatively random.

It was conjectured initially that relatively random measures would share no randoms. In fact an immediate consequence of Theorems \ref{thm:2.4.4} and \ref{thm:2.6.5} is the following, which shows the conjecture almost true.

**Theorem 2.7.1.** If \( \mu \) and \( \nu \) are relatively random, then \( \text{MLR}_\mu^\nu \cap \text{MLR}_\nu^\mu = \emptyset = \text{MLR}_\nu^\mu \cap \text{MLR}_\mu^\nu \). In particular, \( \mu \perp \nu \).

Interestingly though, relatively random measures do share a random real, and in a rather strong way. Before proving this, we need a lemma about a universal test for \( \mu \)-randomness.

**Lemma 2.7.2.** Each \( \mu \in \text{MLR}_P \) has a name of least Turing degree and hence admits a universal \( \mu \)-test.

**Proof.** The point here is that \( x := \Phi^{-1}(\mu) \) is essentially a name for \( \mu \), and the one of least Turing degree. The proof of Proposition \ref{prop:2.3.1} shows that \( x \) computes a name for \( \mu \). Also any name for \( \mu \) computes \( \mu(\sigma) \) for any \( \sigma \in 2^{<\omega} \) and hence also must be able to compute \( \mu(0|\sigma) \) for each \( \sigma \in 2^{<\omega} \); this is the same as computing \( x \).

**Theorem 2.7.3.** There is a computable function \( G: \mathcal{P}(2^\omega) \times \mathcal{P}(2^\omega) \to 2^\omega \) such that if \( \mu \) and \( \nu \) are relatively random measures, then \( G(\mu, \nu) \in \text{MLR}_\mu \cap \text{MLR}_\nu \).

**Proof.** The algorithm we are about to define is a greedy one. It builds the common random by asking the two measures “Which of you cares the most which direction I go?” and then acting accordingly.

\[ ^3 \text{This result was proven with Joe Miller and Uri Andrews.} \]
Define the function $G : \mathcal{P}(2^\omega) \times \mathcal{P}(2^\omega) \to 2^\omega$ by

$$G(\mu, \nu)(i) = 0 \iff \mu(0 \mid G(\mu, \nu) \upharpoonright i) > \nu(1 \mid G(\mu, \nu) \upharpoonright i).$$

Note that also

$$G(\mu, \nu)(i) = 1 \iff \mu(1 \mid G(\mu, \nu) \upharpoonright i) > \nu(0 \mid G(\mu, \nu) \upharpoonright i).$$

The key fact here is that $P\{\nu : G(\mu, \nu) \succ \sigma\} = \mu(\sigma)$ for each $\mu \in \mathcal{P}(2^\omega)$ and $\sigma \in 2^{<\omega}$. Indeed, by induction, if $P\{\nu : G(\mu, \nu) \succ \sigma\} = \mu(\sigma)$, then by independence

$$P\{\nu : G(\mu, \nu) \succ \sigma \upharpoonright i\} = P\{\nu : G(\mu, \nu) \succ \sigma\} \cdot P\{\nu : G(\mu, \nu)(|\sigma|) = i \mid G(\mu, \nu) \succ \sigma\}$$

$$= \mu(\sigma) \cdot P\{\nu : \nu(1 - i \mid \sigma) < \mu(i \mid \sigma)\}$$

$$= \mu(\sigma) \cdot \mu(i \mid \sigma)$$

$$= \mu(\sigma \upharpoonright i).$$

Let $\langle U^\mu_n \rangle_{n \in \omega}$ be a universal $\mu$-test. The sets $V_n := \{\nu : G(\mu, \nu) \in U^\mu_n\}$ are uniformly $\Sigma^0_1$ and, with $S_n$ a prefix-free set of generators of $U_n$, the above calculation gives

$$P(V_n) = P\left(\bigcup_{\sigma \in S_n} \{\nu : G(\mu, \nu) \succ \sigma\}\right) = \sum_{\sigma \in S_n} P(\{\nu : G(\mu, \nu) \succ \sigma\}) = \cdots$$

$$\cdots = \sum_{\sigma \in S_n} \mu(\sigma) = \mu(U_n) \leq 2^{-n}.$$

Thus $\langle V_n \rangle_{n \in \omega}$ is a $P$-test for Martin L"of randomness relative to $\mu$ and so if $\nu$ is relatively random to $\mu$, $G(\mu, \nu) \in \text{MLR}_\mu$. By symmetry, $G(\mu, \nu) = G(\nu, \mu) \in \text{MLR}_\nu$ as well.

The random element shared by the relatively random measures in Theorem 2.7.3.
was derandomized (indeed computed) by the join of those measures. Theorem 2.7.1 shows that in fact any \( x \in \text{MLR}_\mu \cap \text{MLR}_\nu \) must be derandomized (with respect to either measure) by \( \mu \oplus \nu \). This raises the following question.

**Question 2.7.4.** If \( \mu \) and \( \nu \) are relatively random and \( x \in \text{MLR}_\mu \cap \text{MLR}_\nu \), must \( x \) be computed by \( \mu \oplus \nu \)?

Theorem 2.7.1 also leads to Theorem 2.7.7, a special case of which says that the randoms for a random measure \( \mu \) form an “anti-tailset”: change even a single bit and the real loses its \( \mu \)-randomness.

Recall that for \( \sigma \in 2^{<\omega} \), \( \mu_\sigma \) is defined by \( \mu_\sigma(\tau) = \mu(\tau|\sigma) \).

**Lemma 2.7.5.** \( \nu \) and \( \xi \) are relatively random if and only if \( \nu = \mu_\sigma \) and \( \xi = \mu_\tau \) for some \( \mu \in \text{MLR}_P \) and incompatible \( \tau, \sigma \in 2^{<\omega} \).

**Proof.** Clearly \( \mu_\sigma \) and \( \mu_\tau \) are relatively random whenever \( \mu \in \text{MLR}_P \) and \( \sigma \perp \tau \). Given two relatively random measure \( \nu \) and \( \xi \), by taking \( p \in \text{MLR}_\nu \) and defining \( \mu \) by \( \mu(0) = p \), \( \mu_0 = \nu \), and \( \mu_1 = \xi \), we have \( \mu \in \text{MLR}_P \). \(\square\)

**Lemma 2.7.6.** If \( \sigma x \in \text{MLR}_\mu \), then \( x \in \text{MLR}_{\mu_\sigma} \).

**Proof.** If \( x \notin \text{MLR}_{\mu_\sigma} \), then \( x \in U_{\mu_\sigma}^n \) for each \( n \), where \( \langle U_{\mu_\sigma}^n \rangle_{n\in\omega} \) is a universal \( \mu_\sigma \)-test. Then \( V_n := \{ \sigma \tau : \tau \in U_{\mu_\sigma}^n \} \) is \( \Sigma_1^{0,\mu} \) since \( \mu \succeq_T \mu_\sigma \) and \( \mu(V_n) = \mu[\sigma]u_\sigma(U_{\mu_\sigma}^n) \leq 2^{-n} \). Therefore, \( \langle V_n \rangle_{n\in\omega} \) is a \( \mu \)-test capturing \( \sigma x \), and, whence, \( \sigma x \notin \text{MLR}_\mu \). \(\square\)

The next result shows that \( \mu \)-random elements in one part of the full binary-branching tree look much different from those in another part. Contrast this with the Lebesgue measure, where randomness does not depend on prefixes (i.e. randomness is a “tail event”).

**Theorem 2.7.7.** If \( \sigma x \in \text{MLR}_\mu \) and \( \tau \perp \sigma \), then \( \tau x \notin \text{MLR}_\mu \).
Proof. Suppose $\sigma x \in \text{MLR}_\mu$ and $\tau x \in \text{MLR}_\mu$. Then $x \in \text{MLR}_\mu \cap \text{MLR}_\mu$ by Lemma 2.7.6. Since $\sigma \perp \tau$, the measures $\mu_\sigma$ and $\mu_\tau$ are relatively random and hence by Theorem 2.7.1, $x \not\in \text{MLR}_{\mu_\sigma}$. Since $\mu \geq_T \mu_\sigma$, $x \not\in \text{MLR}_\mu$, so by Lemma 2.7.6, $\tau x \not\in \text{MLR}_\mu = \text{MLR}_\mu$, a contradiction. \qed

We used Kolmogorov’s 0-1 law earlier (Lemma 2.5.2). It seems that random measures should fail to satisfy Kolmogorov’s 0-1 law, since changing finitely many bits of a real puts it in a part of the tree whose conditional probabilities are wildly different (for a fixed random $\mu$). We now confirm this intuition.

**Corollary 2.7.8.** Random measures fail to satisfy Kolmogorov’s 0-1 law.

**Proof.** Let $\mu \in \text{MLR}_P$. By Theorem 2.7.7, closing the set $\text{MLR}_\mu \cap [0]$ under tails adds no randoms and hence no measure. The result is therefore a tailset of measure $\mu(0) \in (0, 1)$. \qed

We can also use Theorem 2.7.7 to show that given a random $x$, the probability of choosing a measure that thinks $x$ is random is zero.

**Corollary 2.7.9.** If $x \in \text{MLR}_\lambda$, then $P(\{\mu : x \in \text{MLR}_\mu\}) = 0$.

**Proof.** Changing only finitely many bits of (the preimage under $\Phi$ of) any $P$-random $\mu$ does not affect whether $x \in \text{MLR}_\mu$, so by Kolmogorov’s 0-1 law, either

$$P(\{\mu : x \in \text{MLR}_\mu\}) = 0$$

or

$$P(\{\mu : x \in \text{MLR}_\mu\}) = 1.$$  

But, $P(\{\mu : x \in \text{MLR}_\mu\}) = 1$ if and only if $P(\{\mu : x' \in \text{MLR}_\mu\}) = 1$, where $x'(i) = x(i)$ for $i > 0$ and $x'(0) = 1 - x(0)$. So, if $P(\{\mu : x \in \text{MLR}_\mu\}) = 1$, then there is $\mu$ such that $x, x' \in \text{MLR}_\mu$, contrary to Theorem 2.7.7. \qed
Let $T: 2^\omega \to 2^\omega$ be the shift map; so $Tx(i) = x(i + 1)$. This map preserves the Lebesgue measure. Clearly, no element of $\text{MLR}_P$ is preserved by $T$, since that would introduce dependence amongst the conditional probabilities of $\mu$. Moreover, the following holds.

**Theorem 2.7.10.** If $\mu \in \text{MLR}_P$ and $x \in \text{MLR}_\mu$, then $Tx \notin \text{MLR}_\mu$.

*Proof.* If $x \in \text{MLR}_\mu$ and $Tx \in \text{MLR}_\mu$, then there are incompatible strings $\sigma \prec x$ and $\tau \prec Tx$ such that $x = \sigma y$ and $Tx = \tau y$. But $\mu_\sigma$ and $\mu_\tau$ are relatively random with $y \in \text{MLR}_\mu^\mu \cap \text{MLR}_\mu^\sigma$ contradicting Theorem 2.7.1. 

We believe Theorem 2.7.10 can be generalized: Recall (see [19]) that if $f: \omega \to \omega$ is 1-1 and computable, then $x \circ f \in \text{MLR}_\lambda$ whenever $x \in \text{MLR}_\lambda$. Our final conjecture states that this fails for random elements of random measures.

**Conjecture 2.7.11.** Suppose $\mu \in \text{MLR}_P$ and $x \in \text{MLR}_\mu$. If $f: \omega \to \omega$ is 1-1, computable and non-identity, then $x \circ f \notin \text{MLR}_\mu$. 

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CHAPTER 3

THE INTERPLAY OF CLASSES OF ALGORITHMICALLY RANDOM OBJECTS

Work in this chapter was done jointly with Chris Porter.

3.1 Introduction

In this chapter, we have two primary goals: (1) to study the interplay between algorithmically random closed sets on $2^\omega$, algorithmically random continuous functions on $2^\omega$, and algorithmically random measures on $2^\omega$; and (2) to apply two central results, namely the preservation of randomness principle and the no randomness ex nihilo principle, to the study of the algorithmically random objects listed above.

Barmpalias, Brodhead, Cenzer, Dashti and Weber initiated the study of algorithmically random closed subsets of $2^\omega$ in [2]. Algorithmically random closed sets were further studied in, for instance, [1], [9], and [7]. In the spirit of their definition of algorithmically random closed set, Barmpalias, Brodhead, Cenzer, Dashti and Weber also defined a notion of algorithmically random continuous function on $2^\omega$ in [3]. The connection between random closed sets and effective capacities was explored in [6]. Algorithmically random measures on $2^\omega$ were studied first in Chapter 2.

One of the central results in [3] is that the set of zeroes of a random continuous function of $2^\omega$ is a random closed subset of $2^\omega$. Inspired by this result, we here investigate similar “bridge results,” which allow us to transfer information about one class of algorithmically random objects to another.
Two tools that are central to our investigation, mentioned in (2) above, are the preservation of randomness principle and the no randomness ex nihilo principle. In $2^\omega$, the space of infinite binary sequences, the preservation of randomness principle tells us that if $\Phi: 2^\omega \to 2^\omega$ is an effective map and $\mu$ is a computable probability measure on $2^\omega$ such that the domain of $\Phi$ has $\mu$ measure 1, then $\Phi$ maps $\mu$-random members of $2^\omega$ to members of $2^\omega$ that are random with respect to the measure $\nu$ obtained by pushing $\mu$ forward via $\Phi$. Furthermore, the no randomness ex nihilo principle tells us that any sequence that is random with respect to $\nu$ is the image of some $\mu$-random sequence under $\Phi$. Used in tandem, these two principles allow us to conclude that the image of the $\mu$-random sequences under $\Phi$ is precisely the $\nu$-random sequences.

With the exception of our work in Chapter 2, the studies listed above do not make use of these two tools used in tandem. As we will show, they not only allow for the simplification of a number of proofs in the above-listed studies, but they also allow us to answer a number of questions that were left open in the above studies.

The outline of the remainder of this chapter is as follows. In Section 3.2, we provide the requisite background for the rest of the chapter. In Section 3.3, we review the basics of algorithmic randomness, including preservation and the no randomness ex nihilo principle. We also provide the definitions of algorithmic randomness for closed sets in $2^\omega$, random continuous functions on $2^\omega$, and measures on $2^\omega$ and we list some basic properties of these objects. Section 3.4 contains simplified proofs of some previously obtained results from 2 and 3, as well as a proof of a conjecture in 3 that every random closed subset of $2^\omega$ is the set of zeros of a random continuous function on $2^\omega$. We study the support of a certain class of random measures in Section 3.5 and we establish a correspondence between between random closed sets and the random measures studied in Chapter 2. Lastly, in Section 3.6, we prove that the Lebesgue measure of the range of a random continuous function on $2^\omega$ is always non-
zero, from which it follows that no random continuous function is injective (which
had not been previously established). We also strengthen a result in [3] (namely,
that not every random continuous function is surjective) by proving that no random
continuous function is surjective, from which it follows that the Lebesgue measure of
the range of a random continuous function is never equal to one.

3.2 Background

3.2.1 Some topological and measure-theoretic basics

For \( n = \{0, 1, \ldots, n - 1\} \in \omega \), the set of all finite strings over the alphabet \( n \) is
denoted \( n^{<\omega} \). When \( n = 2 \), we let \( \sigma_0, \sigma_1, \sigma_2, \ldots \) be the canonical length-lexicographic
enumeration of \( 2^{<\omega} \), so that \( \sigma_0 = \epsilon \) (the empty string), \( \sigma_1 = 0, \sigma_2 = 1 \), etc.

The space of all infinite sequences over the alphabet \( n \) is denoted \( n^\omega \). The elements
of \( n^\omega \) are also called reals. The product topology on \( n^\omega \) is generated by the clopen sets

\[ [\sigma] = \{ x \in n^\omega : x \succ \sigma \}, \]

where \( \sigma \in n^{<\omega} \) and \( x \succ \sigma \) means that \( \sigma \) is an initial segment of \( x \). When \( x \) is a real
and \( k \in \omega \), \( x \upharpoonright k \) denotes the initial segment of \( x \) of length \( k \).

For \( \sigma, \tau \in n^{<\omega} \), \( \sigma \tau \) denotes the concatenation of \( \sigma \) and \( \tau \). In some cases, we will
write this concatenation as \( \sigma \tau \).

A tree is a subset of \( n^{<\omega} \) that is closed under initial segments; i.e. \( T \subseteq n^{<\omega} \) is
a tree if \( \sigma \in T \) whenever \( \tau \in T \) and \( \sigma \preceq \tau \). A path through a tree \( T \subseteq n^{<\omega} \) is a
real \( x \in n^\omega \) satisfying \( x \upharpoonright k \in T \) for every \( k \). The set of all paths through a tree \( T \) is
denoted \([T]\). Recall the correspondence between closed sets and trees.

Proposition 3.2.1. A set \( C \subseteq n^\omega \) is closed if and only if \( C = [T] \) for some tree
\( T \subseteq n^{<\omega} \). Moreover, \( C \) is nonempty if and only if \( T \) is infinite.

A measure \( \mu \) on \( n^\omega \) is a function that assigns to each Borel subset of \( n^\omega \) a number
in the unit interval \([0, 1]\) and satisfies \(\mu(\bigcup_{i \in \omega} B_i) = \sum_{i \in \omega} \mu(B_i)\) whenever the \(B_i\)'s are pairwise disjoint. Carathéodory’s extension theorem guarantees that the conditions

- \(\mu([\ell]) = 1\) and
- \(\mu([\sigma]) = \mu([\sigma0]) + \mu([\sigma1]) + \ldots + \mu([\sigma^{-}(n-1)])\) for all \(\sigma \in n^{<\omega}\)

uniquely determine a measure on \(n^\omega\). Thus, a measure is identified with a function \(\mu: n^{<\omega} \to [0, 1]\) satisfying the above conditions, and \(\mu(\sigma)\) is often written instead of \(\mu([\sigma])\). The Lebesgue measure \(\lambda\) on \(n^\omega\) is defined by \(\lambda(\sigma) = n^{-|\sigma|}\) for each string \(\sigma \in n^{<\omega}\).

3.2.2 Some computability theory

A \(\Sigma^0_1\) class \(S \subseteq n^\omega\) is an effectively open set, i.e., an effective union of basic clopen subsets of \(n^\omega\). \(P \subseteq n^\omega\) is a \(\Pi^0_1\) class if \(2^\omega \setminus P\) is a \(\Sigma^0_1\) class.

A partial function \(\Phi: \subseteq n^\omega \to m^\omega\) is computable if the preimage of a \(\Sigma^0_1\) subset of \(m^\omega\) is a \(\Sigma^0_1\) subset of the domain of \(\Phi\), uniformly; that is, if for every \(\Sigma^0_1\) class \(U \subseteq m^\omega\), there is a \(\Sigma^0_1\) class \(V \subseteq n^\omega\) such that \(\Phi^{-1}(U) = V \cap \text{dom}(\Phi)\), and an index for \(V\) can be uniformly computed from an index for \(U\). Equivalently, \(\Phi: \subseteq n^\omega \to m^\omega\) is computable if there is an oracle Turing machine that when given \(x \in n^\omega\) (as an oracle) and \(k \in \omega\) outputs \(\Phi(x)(k)\). We can relativize the notion of a computable function \(\Phi: \subseteq n^\omega \to m^\omega\) to any oracle \(z \in 2^\omega\) to obtain a \(z\)-computable function.

A measure \(\mu\) on \(n^\omega\) is computable if \(\mu(\sigma)\) is a computable real number, uniformly in \(\sigma \in n^{<\omega}\). Clearly, the Lebesgue measure \(\lambda\) is computable.

If \(\mu\) is a computable measure on \(n^\omega\) and \(\Phi: \subseteq n^\omega \to m^\omega\) is a computable function
defined on a set of $\mu$-measure one, then the pushforward measure $\mu_\Phi$ defined by

$$\mu_\Phi(\sigma) = \mu(\Phi^{-1}(\sigma))$$

for each $\sigma \in m^{<\omega}$ is a computable measure.

3.3 Algorithmically random objects

3.3.1 Algorithmically random sequences

**Definition 3.3.1.** Let $\mu$ be a computable measure on $n^\omega$ and $z \in m^\omega$. Then $\text{MLR}^z_\mu$ is the set of all $x \in n^\omega$ such that $x \notin \bigcap_n U_n$ whenever $U_0, U_1, \ldots$ is a uniformly $\Sigma^0_1$ sequence of subsets of $n^\omega$ with $\mu U_n \leq 2^{-n}$. Such an $x$ is said to be $\mu$-random relative to $z$ and such a sequence $U_0, U_1, \ldots$ is called a $\mu$-test relative to $z$. When $z$ is computable, we simply write $\text{MLR}_\mu$, say $x$ is $\mu$-random, and call $U_0, U_1, \ldots$ a $\mu$-test.

The following is well-known and straightforward.

**Proposition 3.3.2.** Let $\mu$ be a computable measure on $n^\omega$ and let $z \in m^\omega$. If $C \subseteq n^\omega$ is $\Pi^0_1, z$ and $\mu(C) = 0$, then $C \cap \text{MLR}^z_\mu = \emptyset$.

The following is likely folklore, but it was at least observed in [4].

**Proposition 3.3.3.** Let $\mu$ be a computable measure on $n^\omega$. If $T: \subseteq n^\omega \to m^\omega$ is computable with $\lambda(\text{dom}(T)) = 1$, then $\text{MLR}_\mu \subseteq \text{dom}(T)$.

**Lemma 3.3.4** (Folklore). Let $T: \subseteq 2^\omega \to 2^\omega$ be computable, and suppose $C$ is a $\Pi^0_1$ subset of $\text{dom}(T)$. Then $T(C) \in \Pi^0_1$, uniformly.

The next theorem represents our main tool here.

**Theorem 3.3.5** (Preservation of Randomness and No Randomness Ex Nihilo). Let $T: \subseteq 2^\omega \to 2^\omega$ be computable with $\lambda(\text{dom}(T)) = 1$. 

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(i) If \( x \in \text{MLR}_\lambda \) then \( T(x) \in \text{MLR}_{\lambda \circ T^{-1}} \).

(ii) If \( y \in \text{MLR}_{\lambda \circ T^{-1}} \), then there exists \( x \in \text{MLR}_\lambda \) such that \( T(x) = y \).

**Proof.**

(i) If \( T(x) \notin \text{MLR}_{\lambda \circ T^{-1}} \), then \( T(x) \in \bigcap_n V_n \) for some \( \lambda \circ T^{-1} \) test. Then \( x \in \bigcap_n T^{-1} V_n \) and \( \lambda(T^{-1} V_n) \leq 2^{-n} \). Moreover, because \( T \) is computable (on its domain), \( T^{-1} V_n = U_n \cap \text{dom}(T) \) for some \( \Sigma_1^0 \) class \( U_n \). Since \( \lambda(\text{dom}(T)) = 1 \), \( \lambda(T^{-1} U_n) \leq 2^{-n} \). Thus, \( x \notin \text{MLR}_\lambda \).

(ii) Let \( U_n \) be a universal test for \( \lambda \) randomness, and set \( K_n = X - U_n \). Then \( T(K_n) \) is uniformly \( \Pi_1^0 \) by Lemma 3.3.4, so \( Y - T(K_n) \) is uniformly \( \Sigma_1^0 \). Because \( \lambda \circ T^{-1}(Y - T(K_n)) = 1 - \lambda \circ T^{-1}(T(K_n)) \leq 1 - \lambda(K_n) \leq 2^{-n} \), the sets \( Y - T(K_n) \) form a test for \( \lambda \circ T^{-1} \) randomness. So if \( y \in \text{MLR}_{\lambda \circ T^{-1}} \), then \( y \notin Y - T(K_n) \) for some \( n \); i.e. \( y \in T(K_n) \). The proof is now complete, since \( K_n \subseteq \text{MLR}_\lambda \). \( \square \)

We will also use a relativization of Theorem 3.3.5.

**Corollary 3.3.6.** Let \( T : \subseteq 2^\omega \to 2^\omega \) be computable relative to \( z \in 2^\omega \) with \( \lambda(\text{dom}(T)) = 1 \).

(i) If \( x \in \text{MLR}_\lambda^z \), then \( T(x) \in \text{MLR}_{\lambda \circ T^{-1}}^z \).

(ii) If \( y \in \text{MLR}_{\lambda \circ T^{-1}}^z \), then there is \( x \in \text{MLR}_\lambda^z \) such that \( T(x) = y \).

Lastly, the following result, known as van Lambalgen’s Theorem, will be useful to us.

**Theorem 3.3.7 ([21]).** Let \( \mu \) and \( \nu \) be computable measures on \( m^\omega \) and \( n^\omega \), respectively. Then for \( (x, y) \in m^\omega \times n^\omega \), \( (x, y) \in \text{MLR}_{\mu \otimes \nu} \) if and only if \( x \in \text{MLR}_\mu^y \) and \( y \in \text{MLR}_\nu^x \).

3.3.2 **Algorithmically random closed subsets of \( 2^\omega \)**

Let \( \mathcal{C}(2^\omega) \) denote the collection of all nonempty closed subsets of \( 2^\omega \). As noted in Proposition 3.2.1, these are the sets of paths through infinite binary trees. Thus, to randomly generate a nonempty closed set, it suffices to randomly generate an infinite
tree. We’ll code infinite trees by reals in $3^\omega$, so we can reduce the process of randomly generating infinite trees to randomly generating reals.

Given $x \in 3^\omega$, define a tree $T_x \subseteq 2^{<\omega}$ inductively as follows. First, $\emptyset$, the empty string is automatically in $T_x$. Now, suppose $\sigma_i \in T_x$. Then

- $\sigma_i \cdot 0 \in T_x$ and $\sigma_i \cdot 1 \notin T_x$ if $x(i) = 0$;
- $\sigma_i \cdot 0 \notin T_x$ and $\sigma_i \cdot 1 \in T_x$ if $x(i) = 1$;
- $\sigma_i \cdot 0 \in T_x$ and $\sigma_i \cdot 1 \in T_x$ if $x(i) = 2$.

Under this coding $T_x$ has no dead ends and hence is always infinite. This coding can be thought of as a labeling of the nodes of $2^\omega$ by the digits of $x$; a 0 at a node means branch only left, a 1 means branch only right, and a 2 means branch both ways. Note that every tree without dead ends except $2^{<\omega}$ itself has infinitely many codes.

**Definition 3.3.8.** A nonempty closed set $C \in C(2^\omega)$ is a random closed set if $C = [T_x]$ for some $x \in MLR_{\lambda}$.

The main facts about random closed sets that we will use in the sequel are as follows.

**Theorem 3.3.9 ([2]).** Every random closed set has Lebesgue measure zero.

**Theorem 3.3.10 ([2]).** Every random closed set is perfect.

### 3.3.3 Algorithmically random continuous functions on $2^\omega$

Let $\mathcal{F}(2^\omega)$ denote the collection of all continuous $F : \subseteq 2^\omega \to 2^\omega$. To define a random continuous function, we code each element of $\mathcal{F}(2^\omega)$ by a real $x \in 3^\omega$. The coding is a labeling of the edges of $2^\omega$ (or equivalently, all nodes in $2^{<\omega}$ except $\epsilon$) by the digits of $x$. Having labeled the edges according to $x$, the function $F_x$ coded by $x$ is defined by $F_x(y) = z$ if $z$ is the element of $2^\omega$ left over after following $y$ through
the labeled tree and removing the 2’s. (In the case where only finitely many 0’s and 1’s remain after removing the 2’s, \(F_x(y)\) is undefined.)

Formally, define a labeling function \(\ell_x : 2^{<\omega} \setminus \{\epsilon\} \to 3\) by \(\ell_x(\sigma_i) = x_{i-1}\). Now \(F_x \in \mathcal{F}(2^\omega)\) is defined by \(F_x(y) = z\) if and only if \(z\) is the result of removing the 2’s from the sequence \(\ell_x(y \upharpoonright 1), \ell_x(y \upharpoonright 2), \ell_x(y \upharpoonright 3), \ldots\).

**Definition 3.3.11.** A function \(F \in \mathcal{F}(2^\omega)\) is a **random continuous function** if \(F = F_x\) for some \(x \in \text{MLR}_\lambda\).

**Remark 3.3.1.** \(F_x\) is continuous (on its domain), because it is computable relative to some oracle; namely \(x\). Since \(2^\omega\) is compact and Hausdorff, it follows that \(F_x\) is a closed map and, hence, that \(\text{ran}(F)\) is \(\Pi^0_{1,F}\).

We will make use of the following facts about random continuous functions.

**Theorem 3.3.12** (\cite{3}). If \(F \in \mathcal{F}(2^\omega)\) is random and \(x \in 2^\omega\) is computable, then \(F(x) \in 2^\omega\) is random.

**Theorem 3.3.13** (\cite{3}). If \(F \in \mathcal{F}(2^\omega)\) is random, then \(F\) is total.

### 3.3.4 Algorithmically random measures on \(2^\omega\)

Let \(\mathcal{P}(2^\omega)\) be the space of probability measures on \(2^\omega\). Given \(x \in 2^\omega\), the \(n\)th column \(x_n\) of \(x\) is defined by \(x_n(k) = 1\) if and only if \(x(\langle n, k \rangle) = 1\), where \(\langle n, k \rangle\) is some fixed computable bijection between \(\omega^2\) and \(\omega\). We write \(x = \bigoplus_{n \in \omega} x_n\). Let \((\sigma_i)_{i \in \omega}\) be the canonical enumeration of \(2^{<\omega}\) in the length-lexicographical order. We define a map \(\Psi : 2^\omega \to \mathcal{P}(2^\omega)\) that sends a real \(x\) to the measure \(\mu_x\) satisfying (i) \(\mu_x(\epsilon) = 1\) and (ii) \(\mu_x(\sigma_n0) = x_n \cdot \mu_x(\sigma_n)\), where \(x_n\) is the real number corresponding to the \(n\)th column of \(x\).

**Definition 3.3.14.** A measure \(\mu \in \mathcal{P}(2^\omega)\) is a **random measure** if \(\mu = \mu_x\) for some \(x \in \text{MLR}_\lambda\).
Let $P$ be the pushforward measure on $\mathcal{P}(2^\omega)$ induced by $\lambda$ and $\Psi$. Then we have the following.

**Theorem 3.3.15.** Let $\nu \in \mathcal{P}(2^\omega)$. Then $\nu \in \text{MLR}_P$ if and only if $\nu = \mu_x$ for some $x \in \text{MLR}_\lambda$.

The support of a measure $\mu$ on $2^\omega$ is defined to be

$$\text{Supp}(\mu) = \{x \in 2^\omega : (\forall n)[\mu(x|n) > 0]\}$$

It is not hard to see that $\text{Supp}(\mu) = 2^\omega$ for every random measure $\mu$.

In Chapter 2, it was shown that random measures are atomless and that the reals that are random with respect to some random measure are precisely the reals in $\text{MLR}_\lambda$.

### 3.4 Applications of Randomness Preservation and No Randomness Ex Nihilo

In this section, we demonstrate the usefulness of preservation of randomness and the no randomness ex nihilo principle in the study of algorithmically random objects such as closed sets and continuous functions.

The following is a new, simpler proof of a known result from [2].

**Theorem 3.4.1.** Every random closed set contains an element of $\text{MLR}_\lambda$, and every element of $\text{MLR}_\lambda$ is contained in some random closed set.

**Proof.** We define a computable map $T : \mathcal{C}(2^\omega) \times 2^\omega \rightarrow 2^\omega$ that pushes forward the product measure $\lambda_C \otimes \lambda$ to $\lambda$ and satisfies $T(C, x) \in C$ for every $(C, x) \in \mathcal{C}(2^\omega) \times 2^\omega$.

Once we have done this, preservation of randomness and no randomness ex nihilo imply that the image of a $\lambda_C \otimes \lambda$-random pair is $\lambda$-random and any $\lambda$-random is the image of some $\lambda_C \otimes \lambda$-random pair. The result then follows because by Van
Lambalgen’s Theorem (Theorem 3.3.7), a pair \((C, x)\) is \(\lambda_C \otimes \lambda\)-random if and only if \(C\) is \(\lambda_C\)-random and \(x\) is \(\lambda\)-random relative to \(C\).

The map works by using \(x\) to tell us which way to go through \(C\) (viewed as a tree) when we have a choice to make. Specifically, having \(T(C, x) \upharpoonright n = \sigma\) such that \([\sigma] \cap C \neq \emptyset\), we define \(T(C, x)(n) = 0\) if \([\sigma 1] \cap C = \emptyset\) and \(T(C, x)(n) = 1\) if \([\sigma 0] \cap C = \emptyset\). If neither \([\sigma 0] \cap C = \emptyset\) nor \([\sigma 1] \cap C = \emptyset\), then \(T(C, x)(n) := x(n)\).

The map \(T\) is clearly computable. It pushes \(\lambda_C \otimes \lambda\) forward to \(\lambda\) because if \(T\) has output \(\sigma \in 2^n\), then \(T\) outputs a next bit of 0 if and only if either \([\sigma 1] \cap C = \emptyset\) or both \([\sigma 1] \cap C \neq \emptyset\) \(\neq [\sigma 0] \cap C\) and \(x(n) = 0\). The former happens with probability \(1/3\), and the latter happens with probability \(1/3 \cdot 1/2\) by independence. The proof is now complete since \(1/3 + 1/6 = 1/2\). \(\square\)

Let \(F \in \mathcal{F}(2^\omega)\). We define \(Z_F = \{x : F(x) = 0\}\). This is clearly a closed subset of \(2^\omega\). In \([3]\), the following was shown.

**Theorem 3.4.2** ([3]). Let \(F \in \mathcal{F}(2^\omega)\) be random. Then \(Z_F\) is a random closed set provided it is nonempty.

In \([3]\), it was conjectured that the converse also holds, but this was left open. We prove this conjecture. To do so, we provide a new proof of Theorem 3.4.2 from which the converse follows immediately. We also make use of an alternative characterization of random closed sets, due to Diamondstone and Kjøs-Hanssen \([9]\).

Just as a binary tree with no dead ends is coded by a sequence in \(3^\omega\) (see the paragraph preceding Definition 3.3.8), an arbitrary binary tree is coded by a sequence in \(4^\omega\), except now a 3 at a node indicates that the tree is dead above that node. That is, given \(x \in 4^\omega\), we define a tree \(S_x \subseteq 2^{<\omega}\) inductively as follows. First \(\epsilon\), the empty string, is included in \(S_x\) by default. Now suppose that \(\sigma_i \in S_x\). Then

- \(\sigma_i 0 \in S_x\) and \(\sigma_i 1 \notin S_x\) if \(x(i) = 0\);
- \(\sigma_i 0 \notin S_x\) and \(\sigma_i 1 \in S_x\) if \(x(i) = 1\);
• $\sigma_i \uparrow 0 \in S_x$ and $\sigma_i \uparrow 1 \in S_x$ if $x(i) = 2$;
• $\sigma_i \uparrow 0 \notin S_x$ and $\sigma_i \uparrow 1 \notin S_x$ if $x(i) = 3$.

This coding can be thought of as a labeling of the nodes of $2^\omega$ by the digits of $x$; a 0 at a node means that only the left branch is included, a 1 means that only the right branch is included, a 2 means that both branches are included, and a 3 means that neither branch is included. Note that every tree except $2^{<\omega}$ itself has infinitely many codes.

Let $\mu_{GW}$ be the measure on $4^\omega$ induced by setting, for each $\sigma \in 4^{<\omega}$,

$$
\mu_{GW}(\sigma_0 | \sigma) = \mu_{GW}(\sigma_1 | \sigma) = 2/9, \quad \mu_{GW}(\sigma_2 | \sigma) = 4/9, \quad \text{and} \quad \mu_{GW}(\sigma_3 | \sigma) = 1/9
$$

Via this coding, we can also think of $\mu_{GW}$ as a measure on Tree, the space of binary trees. Then the probability of extending a string in a tree by only 0 is $2/9$, by only 1 is $2/9$, by both 0 and 1 is $4/9$, and by neither is $1/9$. We call a tree $T$ GW-random if it has a random code; i.e., there is $x \in \MLR_{\mu_{GW}}$ such that $T = S_x$.

**Lemma 3.4.3** (Diamondstone and Kjøs-Hanssen [9]). A closed set $C$ is random if and only if $C$ is the set of paths through an infinite GW-random tree.

**Theorem 3.4.4.**  
(i) For every random $F \in \mathcal{F}(2^\omega)$, $Z_F$ is a random closed set provided that it is nonempty.
(ii) For every random $C \in \mathcal{C}(2^\omega)$, there is some random $F \in \mathcal{F}(2^\omega)$ such that $C = Z_F$.

**Proof.** We define a computable map $\Psi: \mathcal{F}(2^\omega) \to \text{Tree}$ that pushes forward $\lambda_F$ to $\mu_{GW}$ such that the set of paths through $\Psi(F) \cap \text{dom}(F)$ is exactly $Z_F$. Given our representation of functions as members of $3^\omega$ and binary trees as members of $4^\omega$, we are really defining a computable map $\hat{\Psi}: 3^\omega \to 4^\omega$ that pushes forward $\lambda$ to $\mu_{GW}$.

Given $F \in \mathcal{F}(2^\omega)$, which we think of as a $\{0,1,2\}$-labeling of the edges of the full binary tree, we build the desired tree by declaring that $\sigma \in \Psi(F)$ if and only
if the labels by $F$ of the edges of $\sigma$ consists only of 0’s and 2’s. More formally, as in the paragraph preceding Definition 3.3.11, $F$ comes with a labeling function $\ell_F: 2^{<\omega} \setminus \{\epsilon\} \to 3$ defined by $\ell_F(\sigma_i) = j$ if and only if $x(i) = j$ where $x$ is the given code for $F$. So, $\sigma \in \Psi(\sigma)$ if and only if $\ell_F(\sigma|k) \in \{0, 2\}^{<\omega}$ for every $0 < k \leq |\sigma|$. Clearly, this map is computable.

Now we show that the map $\Psi$ pushes $\lambda_F$ forward to $\mu_{GW}$. Suppose $\sigma \in \Psi(F)$, which, as stated above, means that $\ell_F(\sigma|k) \in \{0, 2\}^{<\omega}$ for every $0 < k \leq |\sigma|$. Then

$$\sigma 0 \in \Psi(F) \ & \ & \sigma 1 \notin \Psi(F) \ \iff \ \ell_F(\sigma 0) \in \{0, 2\} \ & \ \ell_F(\sigma 1) = 1.$$  

The right-hand side of the equivalence occurs with probability $(2/3)(1/3) = 2/9$.

Similarly,

$$\sigma 0 \notin \Psi(F) \ & \ & \sigma 1 \in \Psi(F) \ \iff \ \ell_F(\sigma 0) = 1 \ & \ \ell_F(\sigma 1) \in \{0, 2\},$$

where this latter event also occurs with probability $2/9$. Next,

$$\sigma 0 \in \Psi(F) \ & \ & \sigma 1 \in \Psi(F) \ \iff \ \ell_F(\sigma 0) \in \{0, 2\} \ & \ \ell_F(\sigma 1) \in \{0, 2\},$$

with the latter event occurring with probability $(2/3)(2/3) = 4/9$. Lastly,

$$\sigma 0 \notin \Psi(F) \ & \ & \sigma 1 \notin \Psi(F) \ \iff \ \ell_F(\sigma 0) = \ell_F(\sigma 1) = 1,$$

where the event on the right-hand side occurs with probability $(1/3)(1/3) = 1/9$.

Now, by construction, it follows immediately that any path through the tree $\Psi(F)$ is a sequence $X$ such that either $F(X) = 0^\omega$ (in the case that $\ell(X|n) = 0$ for infinitely many $n$) or $F(X) \uparrow$ (in the case that $\ell(X|n) = 0$ for only finitely many $n$).

By preservation of randomness and no randomness ex nihilo, a tree is GW-random
if and only if it is the image of some random continuous function $F$. The conclusion then follows, by Lemma 3.4.3.

One consequence of Theorem 3.3.12 and Theorem 3.4.4(ii), not noted in [3], is that the composition of two random continuous functions need not be random.

**Corollary 3.4.5.** For every random $F \in \mathcal{F}(2^\omega)$, there is some random $G \in \mathcal{F}(2^\omega)$ such that $G \circ F$ is not random.

*Proof.* By Theorem 3.3.12, there is some $R \in \text{MLR}$ such that $F(0^\omega) = R$. By Theorem 3.4.1, there is some random $C \in \mathcal{C}(2^\omega)$ containing $R$. By Theorem 3.4.4(ii), there is a $G \in \mathcal{F}(2^\omega)$ such that $G^{-1}(\{0^\omega\}) = C$. It follows that $G(F(0^\omega)) = 0^\omega$. This, together with Theorem 3.3.12 implies that $G \circ F$ is not random.

Another consequence of Theorem 3.4.4 lets us answer an open question from [3] involving random pseudo-distance functions. Given a closed set $C \in \mathcal{C}(2^\omega)$, a function $\delta : 2^\omega \to 2^\omega$ is a **pseudo-distance function** for $C$ if $C$ is the set of zeroes of $\delta$. In [3] it was shown that if $\delta$ is a random pseudo-distance function for some $C \in \mathcal{C}(2^\omega)$, then $C$ is a random closed set, but the converse was left open. By Theorem 3.4.4, the converse immediately follows.

**Corollary 3.4.6.** Let $C \in \mathcal{C}(2^\omega)$. Then $C$ has a random pseudo-distance function if and only if $C$ is a random closed set.

3.5 The support of a random measure

In the previous section, we established a correspondence between random closed sets and and random continuous functions: a closed set $C$ is random if and only if it is the set of zeroes of some random continuous function. In this section, we establish similar correspondences between random closed sets and random measures.
Since the support of a measure \( \mu \), i.e., the set \( \text{Supp}(\mu) = \{ x \in 2^\omega : \forall n \mu(x|n) > 0 \} \) is a closed set, one might hope to establish such a correspondence by considering the supports of random measures. However, it is not hard to see that for each random measure \( \mu \), \( \text{Supp}(\mu) = 2^\omega \).

If we consider a computable measure on \( \mathcal{P}(2^\omega) \) different from the measure \( P \) defined in Section 3.3.4, then such a correspondence can be given. In the first place, we want a measure \( Q \) on \( \mathcal{P}(2^\omega) \) with the property that no \( Q \)-random measure has full support. In fact, we can choose a measure \( Q \) such that each \( Q \)-random measure is supported on a random closed set.

**Theorem 3.5.1.** There is a computable measure \( Q \) on \( \mathcal{P}(2^\omega) \) such that

(i) every \( Q \)-random measure is supported on a random closed set, and

(ii) for every random closed set \( C \subseteq 2^\omega \), there is a \( Q \)-random measure \( \mu \) such that \( \text{Supp}(\mu) = C \).

**Proof.** We will define the measure \( Q \) so that each \( Q \)-random measure is obtained by restricting Lebesgue measure to a random closed set. That is, each \( Q \)-random measure will be uniform on all of the branching nodes of its support.

We define \( Q \) in terms of an almost total functional \( \Phi : 3^\omega \to 2^\omega \). On input \( x \in 3^\omega \), \( \Phi \) will treat \( x \) as the code for a closed set and will output the sequence \( y = \bigoplus_{i \in \omega} y_i \) defined as follows. For each \( i \in \omega \), we set

\[
y_i = \begin{cases} 
1^\infty & \text{if } x_i = 0 \\
0^\infty & \text{if } x_i = 1 \\
10^\infty & \text{if } x_i = 2 
\end{cases}.
\]

If we think of the columns of \( y \) as encoding the conditional probabilities of a measure \( \mu_y \), then if \( (\sigma_i)_{i \in \omega} \) is the standard enumeration of \( 2^{<\omega} \), these conditional probabilities
are given by
\[
p_{\sigma_i} = \begin{cases} 
1 & \text{if } x_i = 0 \\
0 & \text{if } x_i = 1 \\
1/2 & \text{if } x_i = 2 
\end{cases}
\]
That is, \( \Phi(x) = y \), where \( y \) represents the unique measure \( \mu_y \) such that \( \mu_y(\sigma_0 \mid \sigma) = p_{\sigma} \) for each \( \sigma \in 2^{<\omega} \). Let \( Q \) be the measure on \( \mathcal{P}(2^{\omega}) \) induced by the composition of \( \Phi \) and the representation map \( \Psi : 2^{\omega} \rightarrow \mathcal{P}(2^{\omega}) \) defined in Section 3.3.4.

We now verify (i) by showing that \( \Phi \) maps each \( x \in \text{MLR} \) to a \( Q \)-random measure supported on a random closed set. Let \( x \in \text{MLR} \) and set \( \Phi(x) = y \). By preservation of randomness, \( \Psi(\Phi(x)) = \mu_y \) is \( Q \)-random.

Next, since \( x \in \text{MLR} \), \([T_x]\) is a random closed set. We claim that \( \text{Supp}(\mu_y) = [T_x] \). Suppose that \( \sigma \in 2^{<\omega} \) is the \((n+1)\)-st extendible node of \( T_x \). Then one of the following holds:

(a) \( \sigma_0 \in T_x \) and \( \sigma_1 \notin T_x \);
(b) \( \sigma_0 \notin T_x \) and \( \sigma_1 \in T_x \); or
(c) \( \sigma_0 \in T_x \) and \( \sigma_1 \in T_x \).

Moreover, we have

- Condition (a) holds iff \( x = 0 \) iff \( \mu_y(\sigma_0 \mid \sigma) = 1 \) and \( \mu_y(\sigma_1 \mid \sigma) = 0 \).
- Condition (b) holds iff \( x = 1 \) iff \( \mu_y(\sigma_0 \mid \sigma) = 0 \) and \( \mu_y(\sigma_1 \mid \sigma) = 1 \).
- Condition (c) holds iff \( x = 2 \) iff \( \mu_y(\sigma_0 \mid \sigma) = \mu_y(\sigma_1 \mid \sigma) = 1/2 \).

One can readily verify that \( \mu_y(\sigma \upharpoonright i \mid \sigma) > 0 \) if and only if \( \sigma \upharpoonright i \in T_x \). Thus,

\[
Z \in \text{Supp}(\mu_y) \iff \mu_y(Z \mid n) > 0 \text{ for every } n \\
\iff \mu_y(Z \mid (n + 1) \mid Z \mid n) > 0 \text{ for every } n \\
\iff Z \mid (n + 1) \in T_x \text{ for every } n \\
\iff Z \in [T_x].
\]
We have thus established that \(\mu_y\) is supported on a random closed set.

To show (ii), let \(C \subseteq 2^\omega\) be a random closed set. By no randomness ex nihilo, there is some Martin-Löf random \(z \in 3^\omega\) such that \(C = [T_z]\). Hence, \(\Psi(\Phi(z))\) is a \(Q\)-random measure \(\nu\). By the definition of \(\Phi\), \(\nu\) has support \([T_z] = C\), which establishes the claim. \(\square\)

Instead of changing the measure on \(\mathcal{P}(2^\omega)\) we can also establish a correspondence between random closed sets and random measures by considering, not the support of a random measure, but what we refer to as its 1/3-support.

**Definition 3.5.2.** Let \(\mu \in \mathcal{P}(2^\omega)\) and set

\[
T_\mu = \{ \sigma : (\forall i < |\sigma|) [ \mu(\sigma|(i + i) \cup \sigma[i]) > 1/3] \} \cup \{\epsilon\}.
\]

Then the **1/3-support** of the measure \(\mu\) is the closed set \([T_\mu]\).

**Theorem 3.5.3.** A closed set \(C \in \mathcal{C}(2^\omega)\) is random if and only it is the 1/3-support of some random measure \(\mu \in \mathcal{P}(2^\omega)\).

**Proof.** We define an almost-total, computable, and Lebesgue-measure-preserving map \(\Phi : 2^\omega \to 3^\omega\) that induces a map \(\tilde{\Phi} : \mathcal{P}(2^\omega) \to \mathcal{C}(2^\omega)\) such that \(\tilde{\Phi}(\mu) = [T_\mu]\). Suppose \(x = \oplus x_i \in 2^\omega\) such that \(\mu(\sigma_i^{-0} | \sigma_i) = x_i\) for each \(i\). Then for \(\sigma \in T_\mu\) (which must exist since \(\epsilon \in T_\mu\)),

- if \(\mu(\sigma_0) \in [0, 1/3)\), then \(\sigma 1 \in T_\mu\) and \(\sigma 0 \not\in T_\mu\);
- if \(\mu(\sigma_0) \in (2/3, 1]\), then \(\sigma 0 \in T_\mu\) and \(\sigma 1 \not\in T_\mu\);
- if \(\mu(\sigma_0) \in (1/3, 2/3)\), then \(\sigma 0 \in T_\mu\) and \(\sigma 1 \in T_\mu\); and
- if \(\mu(\sigma_0) = 1/3\) or \(\mu(\sigma_0) = 2/3\), then \(\Phi(x)\) is undefined.

Clearly \(\Phi\) is defined on a set of measure one, since it is defined on all sequences \(x\) such that \(x_i \neq 1/3\) and \(x_i \neq 2/3\). Observe that each \(\sigma \in T_\mu\) extends to an infinite path in \([T_\mu]\). Thus, if \(\sigma\) is the \((n+1)\)-st extendible node in \(T_\mu\), then each of the events
• $\sigma_0 \in T_\mu$ and $\sigma_1 \notin T_\mu$,
• $\sigma_0 \notin T_\mu$ and $\sigma_1 \in T_\mu$, and
• $\sigma_0 \in T_\mu$ and $\sigma_1 \in T_\mu$,

occurs with probability 1/3, since each event corresponds to whether $\mu(\sigma_0) \in [0, 1/3)$, $\mu(\sigma_0) \in (2/3, 1]$, or $\mu(\sigma_0) \in (1/3, 2/3)$, respectively. It thus follows that the pushforward measure induced by $\lambda$ and $\Phi$ is the Lebesgue measure on $3^\omega$. By preservation of randomness, each random measure $\mu$ is mapped to a random closed set, and by no randomness ex nihilo, each random closed set is the image of a random measure under $\Phi$. This establishes the theorem.

3.6 The range of a random continuous function

In [3], it was shown that for each $y \in 2^\omega$

$$\lambda(\{x \in 2^\omega : y \in \text{ran}(F_x)\}) = 3/4.$$ 

From this, it follows that every $y \in 2^\omega$ is in the range of some random $F \in \mathcal{F}(2^\omega)$. In this section, we prove that $\lambda(\text{ran}(F)) \in (0, 1)$ for every random function $F$. First, we will prove that $\lambda(\text{ran}(F)) > 0$ for each random function. This implies that no random function is injective and that the range of a random function is never a random closed set. These improve two results of [3] according to which (i) not every random function is injective and (ii) the range of a random function is not necessarily a random closed set. Our proof requires us to prove some auxiliary facts about the measure induced by a random function.

To prove that $\lambda(\text{ran}(F)) < 1$ for every $F \in \mathcal{F}(2^\omega)$, we will show that no random function is surjective, from which the result immediately follows. Our result on surjectivity also improves a result of [3] according to which not every random function is surjective.
We begin by proving the following, which is similar to Lemma 2.4.2 in Chapter 2 for random measures.

**Lemma 3.6.1.** Let $\lambda_F$ be the natural measure on $F(2^\omega)$. Then the measure $P_F$ on $\mathcal{P}(2^\omega)$ induced by the map $F \mapsto \lambda \circ F^{-1}$ has barycenter $\lambda$; i.e.

$$
\lambda(\sigma) = \int_{F(2^\omega)} \mu(\sigma) dP_F(\mu)
$$

for each $\sigma \in 2^{<\omega}$.

*Proof.* By change of variables, it suffices to show that

$$
2^{-|\sigma|} = \int_{F(2^\omega)} \lambda(F^{-1}[\sigma]) d\lambda_F
$$

(3.1)

for each $\sigma \in 2^{<\omega}$. Without loss of generality, we assume $\sigma = 0^n$. We proceed then by induction on $n$.

Equation (3.1) holds when $\sigma = \epsilon$, since each random $F$ is total by Theorem 3.3.13.

Now supposing that equation (3.1) holds for $0^n$, we show it also holds for $0^{n+1}$. Suppose that \( \int_{F(2^\omega)} \lambda(F^{-1}[0^n]) d\lambda_F = 2^{-n} \). To compute \( \int_{F(2^\omega)} \lambda(F^{-1}[0^{n+1}]) d\lambda_F \), we note that by symmetry \( \int_{F(2^\omega)} \lambda(F^{-1}[0^{n+1}]) d\lambda_F = 2 \cdot \int_{F(2^\omega)} \lambda([0] \cap F^{-1}[0^{n+1}]) d\lambda_F \) and we proceed to compute \( s_{n+1} := \int_{F(2^\omega)} \lambda([0] \cap F^{-1}[0^{n+1}]) d\lambda_F \).

Recall that any $F \in F(2^\omega)$ can be viewed as a labeling by 0’s, 1’s, and 2’s of the nodes of the full binary branching tree (where the root node is unlabeled). We compute \( \int_{F(2^\omega)} \lambda([0] \cap F^{-1}[0^{n+1}]) d\lambda_F \) by considering the three equiprobable cases for the label of the node 0 for an arbitrary $F \in F(2^\omega)$. The point is that the label 0 contributes to producing an output beginning with $0^{n+1}$, the label 1 rules out the possibility of producing an output beginning with $0^{n+1}$, and the label 2 neither contributes to nor rules out the possibility of producing an output beginning with $0^{n+1}$.
Case 1: If the node 0 is labeled with a 0, then the measure of all sequences extending
the node 0 that (after removing 2’s) yield an output extending \(0^{n+1}\) is equal
to the measure of all sequences that yield an output extending \(0^n\) times \(1/2\)
(the measure determined by the initial label 0), i.e., \(1/2 \cdot 2^{-n}\).

Case 2: If the node 0 is labeled with a 1, then the measure of all sequences extending
the node 0 that (after removing 2’s) yield an output extending \(0^{n+1}\) is equal
to 0.

Case 3: If the node 0 is labeled with a 2, then the measure of all sequences extending
the node 0 that (after removing 2’s) yield an output extending \(0^{n+1}\) is equal
to the measure of all sequences that yield an output extending \(0^{n+1}\) times \(1/2\)
(the measure determined by the initial label 2), i.e., \(1/2 \cdot s_{n+1}\).

Putting this all together gives

\[
s_{n+1} = \frac{1}{3} \cdot \frac{1}{2} \cdot 2^{-n} + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} \cdot 2s_{n+1},
\]

which yields \(s_{n+1} = 2^{-n}/4\), as desired.

\[\square\]

**Lemma 3.6.2** (Hoyrup [15], relativized). Let \(Q\) be a computable measure on \(\mathcal{P}(2^\omega)\)
with barycenter \(\mu\). Then for any \(z \in 2^\omega\),

\[
\text{MLR}_\mu^z = \bigcup_{\nu \in \text{MLR}_Q^z} \text{MLR}_\nu^z.
\]

**Theorem 3.6.3.** If \(F \in \mathcal{F}(2^\omega)\) is random, then \(\lambda(\text{ran}(F)) > 0\).

**Proof.** Fix a random \(F \in \mathcal{F}(2^\omega)\). We show that \(\text{ran}(F)\) always contains an element
of \(\text{MLR}_{}^F\). Since \(\text{ran}(F)\) is \(\Pi_1^{0,F}\) by Remark 3.3.1 it follows by Proposition 3.3.2 that
\(\lambda(\text{ran}(F)) > 0\).
By preservation of randomness relative to $F$, if $x \in \text{MLR}_F^F$, then $F(x) \in \text{MLR}_F^F$. By Lemmas 3.6.1 and 3.6.2, $\text{MLR}_F^F \subseteq \text{MLR}_{F^{-1}}^F$, so $F(x) \in \text{MLR}_F^F$, as desired.

**Corollary 3.6.4.** If $F \in \mathcal{F}(2^\omega)$ is random, then $F$ is not injective.

**Proof.** For any $y \in 2^\omega$, a relativization of Theorem 3.4.4(i) shows that $F^{-1}(\{y\})$, if nonempty, is a random closed set relative to $y$ provided that $F$ is random relative to $y$. Since ran($F$) has positive Lebesgue measure, there is $y \in \text{ran}(F)$ that is random relative to $F$. Then by Van Lambalgen’s Theorem, $F$ is also random relative to $y$. So, $F^{-1}(\{y\})$ is a nonempty random closed set and, hence, has size continuum by Theorem 3.3.10. Thus, $F$ is not injective.

**Corollary 3.6.5.** If $F \in \mathcal{F}(2^\omega)$ is random, then ran($F$) is not a random closed set.

**Proof.** By Theorem 3.3.9, every random closed set has Lebesgue measure 0. But by Theorem 3.6.3, the range of a random $F \in \mathcal{F}(2^\omega)$ has positive Lebesgue measure. This gives the conclusion.

From the proof of Corollary 3.6.4, we can also obtain the following.

**Corollary 3.6.6.** The measures induced by random functions are atomless.

**Proof.** Let $F \in \mathcal{F}(2^\omega)$ be random and suppose that $z \in 2^\omega$ is an atom of $\lambda_F$, i.e., $\lambda_F(\{z\}) > 0$. It follows that $z \in \text{MLR}_F^F$, since $z$ is not contained in any $\lambda_F$-nullsets. As we argued in the proof of Corollary 3.6.4, $F^{-1}(\{z\})$ is a nonempty random closed set and, thus, has Lebesgue measure zero, by Theorem 3.3.9. This contradicts our assumption.

We now turn to showing that $\lambda(\text{ran}(F)) < 1$ for every random $F \in \mathcal{F}(2^\omega)$. Instead of proving this directly, we will first prove the following.

**Theorem 3.6.7.** If $F \in \mathcal{F}(2^\omega)$ is surjective, then $F$ is not random.
To prove Theorem 3.6.7, we provide a careful analysis of the result from [3] stated at the beginning of this section; namely, that for each \( y \in 2^{\omega} \),
\[
\lambda(\{ x \in 2^{\omega} : y \in \text{ran}(F_x) \}) = 3/4.
\]

This result is obtained by showing that the strictly decreasing sequence \((q_n)_{n \in \omega}\) defined by
\[
q_n = \lambda(\{ x \in 2^{\omega} : \text{ran}(F_x) \cap [0^n] \})
\]
converges to 3/4 and using the fact that
\[
\lambda(\{ x \in 2^{\omega} : \text{ran}(F_x) \cap [0^n] \}) = \lambda(\{ x \in 2^{\omega} : \text{ran}(F_x) \cap [\sigma] \})
\]
for each \( \sigma \in 2^{<\omega} \) of length \( n \). The sequence \((q_n)_{n \in \omega}\) is obtained by using a case analysis to derive the following recursive formula:
\[
q_{n+1} = \frac{3}{2} \sqrt{1 + 4q_n} - \frac{3}{2} - q_n. \tag{3.2}
\]

For details, see [3, Theorem 2.12].

For \( F \in \mathcal{F}(2^{\omega}) \) and \( \sigma \in 2^{<\omega} \), let us say that \( F \) hits \([\sigma]\) if \( \text{ran}(F) \cap [\sigma] \neq \emptyset \). Thus, \( q_n \) is the probability that a random \( F \in \mathcal{F}(2^{\omega}) \) hits \([\sigma]\) for some fixed \( \sigma \in 2^{<\omega} \) such that \( |\sigma| = n \). It is worth noting that the function \( T([\sigma]) = q_n \) for each \( \sigma \) of length \( n \) induces an effective capacity on \( \mathcal{C}(2^{\omega}) \); see [6] for details on effective capacities.

We will proceed by proving a series of lemmas. First, for each \( n \in \omega \), let \( \epsilon_n \) satisfy \( q_n = 3/4 + \epsilon_n \). Since
\[
(i) \quad q_n > q_{n+1} \text{ for every } n, \text{ and } \\
(ii) \quad \lim_{n \to \infty} q_n = 3/4,
\]
we know that each \( \epsilon_n \) is non-negative and \( \lim_{n \to \infty} \epsilon_n = 0 \). Moreover, we have the
Lemma 3.6.8. For each $n \geq 1$, 

(a) $\epsilon_{n+1} \leq \frac{1}{2} \epsilon_n$, 
(b) $\epsilon_n \leq 2^{-(n+2)}$, 
(c) $\epsilon_{n+1} \geq \frac{1}{2} \epsilon_n - 2^{-(2n+5)}$, and 
(d) $\epsilon_n \geq \frac{1}{2^{n+5} - 1}$. 

Proof. First, let $n \geq 1$. If we substitute $3/4 + \epsilon_{n+1}$ and $3/4 + \epsilon_n$ for $q_{n+1}$ and $q_n$, respectively, into Equation (3.2), we obtain (after simplification)

$$\epsilon_{n+1} = 3\sqrt{1 + \epsilon_n} - 3 - \epsilon_n. \quad (3.3)$$

Since $\sqrt{1 + x} \leq 1 + \frac{x}{2}$ on $[0, 1]$, from (3.3) we can conclude

$$\epsilon_{n+1} \leq 3\left(1 + \frac{\epsilon_n}{2}\right) - 3 - \epsilon_n = \frac{1}{2} \epsilon_n,$$

thereby establishing (a). To show (b), we proceed by induction. Using the fact from [3] that $q_1 = \frac{\sqrt{45} - 5}{2}$, it follows by direct calculation that

$$\epsilon_1 = \frac{\sqrt{45} - 5}{2} - \frac{3}{4} \leq 2^{-3}.$$

Next, assuming that $\epsilon_n \leq 2^{-(n+2)}$, it follows from (a) that

$$\epsilon_{n+1} \leq \frac{1}{2} \epsilon_n \leq \frac{1}{2} 2^{-(n+2)} = 2^{-(n+3)}.$$

To show (c), for each fixed $n \geq 1$, we use a different approximation of $\sqrt{1 + x}$ from below. By (b), since $\epsilon_n \leq 2^{-(n+2)}$, we use the Taylor series approximation $1 + \frac{x}{2}$ of
$\sqrt{1 + x}$ on $[0, 2^{-(n+2)}]$, with error term

$$\max_{c \in [0, 2^{-(n+2)}]} \frac{1}{4(1 + c)^{3/2}} \frac{x^2}{2} = \frac{x^2}{8}.$$ 

Thus,

$$\sqrt{1 + x} \geq 1 + \frac{x}{2} - \left(2^{-(n+2)}\right)^2/8 = 1 + \frac{x}{2} - 2^{-(2n+7)}$$

on $[0, 2^{-(n+2)}]$. Combining this with Equation (3.3) yields

$$\epsilon_{n+1} \geq 3(1 + \frac{\epsilon_n}{2} - 2^{-(2n+7)}) - 3 - \epsilon_n \geq \frac{1}{2} \epsilon_n - 2^{-(2n+5)}.$$ 

Lastly, to prove (d), first observe that

$$\epsilon_1 = \frac{\sqrt{45} - 5}{2} - \frac{3}{4} \geq 2^{-4}$$  \hspace{1cm} (3.4)

and thus it certainly follows that

$$\epsilon_1 \geq \frac{1}{2^6 - 1}.$$ 

Next, using (c), we verify by induction that for $n \geq 2$,

$$\epsilon_n \geq \frac{1}{2^{n-1}} \epsilon_1 - \left(2^{-(n+5)} + \ldots + 2^{-(2n+3)}\right).$$  \hspace{1cm} (3.5)

For $n = 2$, by part (c) we have

$$\epsilon_2 \geq \frac{1}{2} \epsilon_1 - 2^{-7}.$$ 

Supposing that

$$\epsilon_n \geq \frac{1}{2^{n-1}} \epsilon_1 - \left(2^{-(n+5)} + \ldots + 2^{-(2n+3)}\right),$$
again, by part (c), we have

\[ \epsilon_{n+1} \geq \frac{1}{2} \epsilon_n - 2^{-(2n+5)} \geq \frac{1}{2} \left( \frac{1}{2n-1} \epsilon_1 - \left( 2^{-(n+5)} + \ldots + 2^{-(2n+3)} \right) \right) - 2^{-(2n+5)} \]
\[ = \frac{1}{2n} \epsilon_1 - \left( 2^{-(n+6)} + \ldots + 2^{-(2n+4)} \right) - 2^{-(2n+5)} \]
\[ = \frac{1}{2n} \epsilon_1 - \left( 2^{-(n+6)} + \ldots + 2^{-(2n+5)} \right), \]

which establishes Equation (3.5). Combining Equations (3.4) and (3.5) yields

\[ \epsilon_n \geq \frac{1}{2n-1} 2^{-4} - \left( 2^{-(n+5)} + \ldots + 2^{-(2n+3)} \right). \]
\[ = \frac{1}{2(n+3)} - 2^{-(n+4)} \left( 2^{-1} + \ldots + 2^{-(n-1)} \right) \]
\[ = \frac{1}{2(n+3)} - 2^{-(n+4)} \left( 1 - 2^{(n-1)} \right) \]
\[ \geq 2^{-(n+3)} - 2^{-(n+4)} \]
\[ \geq 2^{-(n+4)} \]
\[ \geq \frac{1}{2n+5 - 1}. \]

\[ \square \]

**Lemma 3.6.9.** For \( n \geq 1 \), we have

\[ \frac{q_{n+1}}{q_n} \leq 1 - 2^{-(n+6)}. \]

**Proof.** By Lemma 3.6.8(d),

\[ \epsilon_n \geq \frac{1}{2n+5 - 1} = \frac{2^{-(n+5)}}{1 - 2^{-(n+5)}}, \]

which implies

\[ (1 - 2^{-(n+5)}) \epsilon_n \geq 2^{-(n+5)} = 4 \cdot 2^{-(n+7)} \geq 3 \cdot 2^{-(n+7)}. \]

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Multiplying both sides by $1/2$ yields

$$\frac{1}{2}(1 - 2^{-(n+5)}) \epsilon_n \geq \frac{3}{4} 2^{-(n+6)}.$$  

Expanding the left hand side and using the fact from Lemma 3.6.8(a) that $\frac{1}{2} \epsilon_n \geq \epsilon_{n+1}$, we have

$$\frac{1}{2} \epsilon_n + \left(\frac{1}{4} + \ldots + 2^{-(n+6)}\right) \epsilon_n \geq \frac{3}{4} 2^{-(n+6)} + \epsilon_{n+1},$$

which is equivalent to

$$(1 - 2^{-(n+6)}) \epsilon_n + \frac{3}{4} (1 - 2^{-(n+6)}) \geq \frac{3}{4} + \epsilon_{n+1}.$$  

This yields the inequality

$$(1 - 2^{-(n+6)}) q_n \geq q_{n+1},$$

from which the conclusion follows.

\[\square\]

**Lemma 3.6.10.** For $n \geq 1$, we have

$$\left(2 \left(\frac{q_{n+1}}{q_n}\right) - 1\right)^{2^n} \leq \frac{1}{\sqrt[2^n]{e}} < 1.$$  

**Proof.** First, it follows from Lemma 3.6.9 that

$$2 \left(\frac{q_{n+1}}{q_n}\right) - 1 \leq 1 - 2^{-(n+5)}$$

and hence

$$\left(2 \left(\frac{q_{n+1}}{q_n}\right) - 1\right)^{2^n} \leq \left(1 - 2^{-(n+5)}\right)^{2^n}. \quad (3.6)$$

Next, it is straightforward to verify by cross-multiplication that

$$\frac{2^{n+5} - 1}{2^{n+5}} \leq \frac{2^{n+6} - 1}{2^{n+6}}.$$
and
\[ \frac{2^{n+5} - 1}{2^{n+5}} \leq \left( \frac{2^{n+6} - 1}{2^{n+6}} \right)^2, \]
from which it follows that
\[ \left( \frac{2^{n+5} - 1}{2^{n+5}} \right)^2 \leq \left( \frac{2^{n+6} - 1}{2^{n+6}} \right)^{2n+1}. \]
Lastly, we have
\[ \lim_{n \to \infty} \left( 1 - 2^{-(n+5)} \right)^{2n} = \frac{1}{32\sqrt{e}}, \]
From Equation (3.6) and the fact that the sequence \( \left( (1 - 2^{-(n+5)})^{2n} \right)_{n \in \omega} \) is non-decreasing and converges to \( 1/ \sqrt{e} \), the claim immediately follows.

The proof of following result is essentially the proof of the effective Choquet Capacity Theorem in [6]. We reproduce the proof here for the sake of completeness.

**Lemma 3.6.11.** The probability that a random continuous function \( F \) hits both \([0]\) and \([1]\) is \( 2q_1 - 1 \), and the probability that \( F \) hits both \([\sigma 0]\) and \([\sigma 1]\) for a fixed \( \sigma \in 2^{<\omega} \) of length \( n \geq 1 \), given that \( F \) hits \([\sigma]\), is equal to \( 2 \left( \frac{q_{n+1}}{q_n} \right) - 1 \).

**Proof.** We write the probability that \( F \) hits \([\sigma]\) for some fixed \( \sigma \) as \( \mathbb{P}(F \in H_\sigma) \). Now since \( \mathbb{P}(F \in H_0) = q_1 \), it follows that \( \mathbb{P}(F \in H_1 \setminus H_0) = 1 - q_1 \) (here we use the fact that every random function is total). By symmetry, we have \( \mathbb{P}(F \in H_0 \setminus H_1) = 1 - q_1. \)

Since \( F \) is total with probability one, it follows that
\[ \mathbb{P}(F \in H_0 \cap H_1) = 1 - (\mathbb{P}(F \in H_0 \setminus H_1) + \mathbb{P}(F \in H_1 \setminus H_0)) \]
and thus
\[ \mathbb{P}(F \in H_0 \cap H_1) = 1 - ((1 - q_1) + (1 - q_1)) = 2q_1 - 1. \]
Next, let \( \sigma \) be a string of length \( n \geq 1 \) and let \( i \in \{0, 1\} \). Since \( \mathbb{P}(F \in H_\sigma) = q_n \) and
\[ P(F \in H_{\sigma^{-i}}) = q_{n+1}, \] it follows that

\[ P(F \in H_{\sigma^{-i}} \mid F \in H_{\sigma}) = \frac{P(F \in H_{\sigma^{-i}} \& F \in H_{\sigma})}{P(F \in H_{\sigma})} = \frac{P(F \in H_{\sigma^{-i}})}{P(F \in H_{\sigma})} = \frac{q_{n+1}}{q_n}. \]

Consequently,

\[ P(F \in H_{\sigma_1} \setminus H_{\sigma_0} \mid F \in H_{\sigma}) = P(F \in H_{\sigma_0} \setminus H_{\sigma_1} \mid F \in H_{\sigma}) = 1 - \frac{q_{n+1}}{q_n} \]

Thus,

\[ P(F \in H_{\sigma_0} \cap H_{\sigma_1} \mid F \in H_{\sigma}) = 1 - \left( \frac{q_{n+1}}{q_n} \right) + \frac{q_{n+1}}{q_n} - 1 \]

\[ = 2 \left( \frac{q_{n+1}}{q_n} \right) - 1. \]

\[ \square \]

To complete the proof of Theorem 3.6.7, we now define a Martin-Löf test on \( F(2^\omega) \) that covers all surjective functions. We say that a function \( F \in F(2^\omega) \) is **onto up to level** \( n \) if \( F \in H_\sigma \) for every \( \sigma \in 2^n \). By Lemma 3.6.11, the probability that a function is onto up to level \( n \) is

\[ (2q_1 - 1) \prod_{i=1}^{n-1} \left( 2 \left( \frac{q_{i+1}}{q_i} \right) - 1 \right)^{2^i} \leq \left( \frac{1}{\sqrt[3]{e}} \right)^n. \]

Thus, if we set

\[ \mathcal{U}_n = \{ F \in F(2^\omega) : F \text{ is onto up to level } n \}, \]

and

\[ f(n) = \min \{ k : \left( \frac{32\sqrt{e}}{3} \right)^{-k} \leq 2^{-n} \}, \]

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which is clearly computable, then \((U_{f(n)})_{n \in \omega}\) is a Martin-Löf test with the property that \(F \in \mathcal{F}(2^\omega)\) is onto if and only if \(F \in \bigcap_{n \in \omega} U_{f(n)}\). This completes the proof.

**Corollary 3.6.12.** If \(F \in \mathcal{F}(2^\omega)\) is random, then \(\lambda(\text{ran}(F)) < 1\).

**Proof.** Suppose \(\lambda(\text{ran}(F)) = 1\). Since \(\text{ran}(F)\) is closed, it follows that \(\text{ran}(F) = 2^\omega\). Then \(F\) is onto, so it cannot be random.

We also have the following corollary.

**Theorem 3.6.13.** No measure induced by a random function is a random measure in the sense of Definition 3.3.14.

**Proof.** Let \(F \in \mathcal{F}(2^\omega)\) be random. Then by Corollary 3.6.12, \(\lambda(\text{ran}(F)) < 1\). It follows that \(2^\omega \setminus \text{ran}(F)\) is non-empty and open, so \([\sigma] \subseteq 2^\omega \setminus \text{ran}(F)\) for some \(\sigma \in 2^{<\omega}\). Thus, \(\lambda_F(\sigma) = 0\). By contrast, for every random measure \(\mu\), we have \(\mu(\sigma) > 0\), and the result follows.
CHAPTER 4

A NEW LAW OF LARGE NUMBERS EFFECTIVIZATION

4.1 Introduction

To effectivize a classical theorem of mathematics roughly means to make all the objects mentioned in the theorem (sufficiently) computable and gauge the effectivity of the conclusion. When the classical theorem comes from probability, the effectivity of the conclusion can usually be gauged by whether the conclusion holds of algorithmically randoms. For example, Birkhoff’s Ergodic Theorem (a generalization of the Strong Law of Large Numbers) says that if \( f \) is an integrable function on the probability space \((X, \mu)\) and \(T: X \to X\) is measure-preserving, then

\[
\frac{1}{n} \sum_{i<n} f(T^i(x)) \to \int f \, d\mu
\]

for \( \mu \)-almost-every \( x \in X \). This theorem was effectivized in [5], where it was concluded that if \( X = 2^\omega \) and \( \mu, f, \) and \( T \) are all computable, then the convergence holds on every Martin L"of random.

This chapter contains an effectivization of a theorem of Erd"os and R"enyi [11], which says that the maximal average winnings of short subgames of a fair game converges almost-surely to a certain constant depending on the length of the subgame. We basically follow their proof, injecting effectivity wherever necessary.
4.2 Stirling’s Formula

We recall Stirling’s formula, which is a crucial part of the proof the main theorem of this chapter.

**Theorem 4.2.1** ([20]).

\[ n! = (1 + o(1))\sqrt{2\pi nn^n e^{-n}}. \quad (4.1) \]

In other words, there is a function \( R(n) \) such that \( R(n) \to 0 \) as \( n \to \infty \) and

\[ n! = (1 + R(n))\sqrt{2\pi nn^n e^{-n}}. \quad (4.2) \]

The next lemma is a consequence of Stirling’s formula and gives an effective bound for the probability that a Bernoulli-1/2 random variable has at least \( \gamma n \) successes where \( \gamma \) is some parameter between 1/2 and 1.

**Lemma 4.2.2.** For all \( \gamma \in (1/2, 1) \) there are positive reals \( A \) and \( B \), uniformly computable from \( \gamma \), such that for all \( n \)

\[ An^{-1/2}2^{n(h(\gamma)-1)} \leq 2^{-n} \sum_{n\gamma \leq k \leq n} \binom{n}{k} \leq Bn^{-1/2}2^{n(h(\gamma)-1)}, \quad (4.3) \]

where \( h(\gamma) := -\gamma \log_2 \gamma - (1 - \gamma) \log_2(1 - \gamma) \).

**Proof.** First we prove the upper bound in (4.3). Let \( m = \lceil n\gamma \rceil \). For every \( m \leq k \leq n \),

\[ \binom{n}{k} = \binom{n}{m} \prod_{i=m}^{k-1} \frac{n-i}{i+1} \leq \binom{n}{m} \left( \frac{n-m}{m+1} \right)^{k-m}. \]

Note that \( \frac{n-m}{m+1} < 1 \) since \( n \leq 2m \). This allows for a first bound using a geometric
series:

\[ 2^{-n} \sum_{m \leq k \leq n} \binom{n}{k} \leq 2^{-n} \binom{n}{m} \sum_{m \leq k \leq n} \binom{n-m}{m+1}^{k-m} \]
\[ < 2^{-n} \binom{n}{m} \sum_{m \leq k \leq n} \binom{n-m}{m+1}^{k-m} \]
\[ = 2^{-n} \binom{n}{m} \frac{m+1}{2m+1-n} \]

Now we use Stirling’s formula to bound \( \binom{n}{m} \) from above. Since \( R(n) \to 0 \), there is a natural number \( R \) such that \( R \geq R(n) \) for all \( n \). We will use the fact that \( \gamma \leq m/n \), together with the following claim.

**Claim 1.** For any \( \delta \in (\gamma, 1) \), there is an \( N \), computable from \( \delta \) and \( \gamma \), such that \( m/n < \delta \) whenever \( n \geq N \).

**Proof of claim.** Since \( n\gamma \leq m \leq n\gamma + 1 \), \( \gamma \leq m/n \leq \gamma + 1/n \). So, with \( \gamma \) and \( \delta \) as oracles, we can compute \( N \) such that \( \gamma + 1/n < \delta \) whenever \( n \geq N \). This proves the claim.

Fix such a \( \delta \) computable from \( \gamma \) (e.g. \( \delta = (\gamma + 1)/2 \)) and the corresponding \( N \). Then for \( n \geq N \),

\[
\binom{n}{m} = \frac{(1 + R(n))\sqrt{2\pi mn^n e^{-n}}}{(1 + R(m))\sqrt{2\pi mm^m e^{-m}(1 + R(n - m))} \sqrt{2\pi (n-m)(n-m)e^{-(n-m)}}} \\
= \frac{1}{\sqrt{n} \sqrt{2\pi}} \frac{1}{(1 + R(n))} \frac{1}{\sqrt{(m/n)(1 - m/n)}} 2^{nh(m/n)} \tag{4.4}
\]
\[
\leq \frac{1}{\sqrt{n} \sqrt{2\pi}} (1 + R) \frac{1}{\sqrt{(m/n)(1 - m/n)}} 2^{nh(\gamma)}
\]
\[
\leq \frac{1}{\sqrt{n} \sqrt{2\pi}} (1 + R) \frac{1}{\sqrt{\gamma \sqrt{(1 - m/n)}}} 2^{nh(\gamma)}
\]
\[
\leq \frac{1}{\sqrt{n} \sqrt{2\pi}} (1 + R) \frac{1}{\sqrt{\gamma \sqrt{(1 - \delta)}}} 2^{nh(\gamma)} \tag{4.5}
\]
It remains only to bound $\frac{m+1}{2m+1-n}$. We again use the fact that $\gamma \leq m/n < \delta$ for $n \geq N$:

\[
\frac{m+1}{2m+1-n} = \frac{m/n + 1/n}{2m/n + 1/n - 1} \leq \frac{\delta + 1}{2\gamma - 1}.
\]

Thus $B := \frac{1}{\sqrt{2\pi}}(1 + R)\frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{(1-\delta)(2\gamma-1)}}$ works, where $\delta = (1 + \gamma)/2$ and $R \geq R(n)$ for all $n$. (Actually $B$ only works for $n \geq N$, but we can compute $B_1, B_2, \ldots, B_{N-1}$ that work for $n < N$, and then compute $\max\{B, B_1, B_2, \ldots, B_{N-1}\}$.)

Now to finding $A$. This is a bit easier since we’ll actually show that $\left(\frac{n}{[n\gamma]}\right) \geq An^{-1/2}2^{nh(\gamma)}$ for some positive constant $A$. We’ll use (4.4), still with $m = [n\gamma]$, and the following facts:

- there is $N$ such that $\frac{1+R(n)}{(1+R(m))(1+R(n-m))} \geq 1/2$ whenever $n \geq N$,
- $\frac{1}{m/n} \geq \frac{1}{\gamma + 1}$ for $n \geq 1$,
- $\frac{1}{1-m/n} \geq \frac{1}{1-\gamma}$ for all $n$.

These facts, together with (4.4), yield

\[
\left(\frac{n}{[n\gamma]}\right) \geq \frac{1}{2\sqrt{2\pi(\gamma + 1)(1-\gamma)}} n^{-1/2}2^{n\gamma/[n\gamma]} \geq An^{-1/2}2^{nh([n\gamma]/n)}.
\]

It is sufficient then to show that $2^{nh([n\gamma]/n)} \geq C2^{nh(\gamma)}$ for some positive constant $C$ computable from $\gamma$. To do this, we start by noting that $2^{nh([n\gamma]/n)} \geq 2^{nh(\gamma + 1/n)}$, since $[n\gamma]/n \leq \gamma + 1/n$ and $h$ is decreasing on $[1/2, 1]$. Using the Taylor expansion of $h$ centered at $\gamma$, evaluated at $\gamma + 1/n$, we get

\[
h(\gamma + 1/n) = h(\gamma) + \frac{h'(\gamma)}{n} + \frac{h''(\gamma)}{2n^2} + \cdots.
\]
so that
\[ nh(\gamma + 1/n) = nh(\gamma) + h'(\gamma) + \frac{h''(\gamma)}{2n} + \cdots. \]
The tail of that series,
\[ \frac{h''(\gamma)}{2n} + \cdots \]
goes to 0 as \( n \to \infty \); and, moreover, we can use Taylor’s inequality to compute \( N \) after which
\[ \frac{h''(\gamma)}{2n} + \cdots \geq -1. \]

Putting this all together, we have
\[ 2^{nh([n\gamma]/n)} \geq 2^{nh(\gamma+1/n)} = 2^{nh(\gamma)2^{h'(\gamma)/2}(\gamma) + \cdots} \geq 2^{h'(\gamma)}2^{nh(\gamma)}. \]

So, in summary, \( A = \frac{2^{h'(\gamma)-1}}{2\sqrt{2\pi(\gamma+1)(1-\gamma)}} \) works. (Again, \( A \) actually only works for sufficiently large \( n \) (computable from \( \gamma \)), but we can check the first finitely many \( n \)'s to find an \( A \) that works for all.) \( \Box \)

### 4.3 Maximal average gains over short subgames of a fair game

Let \( X_1, X_2, \ldots \) be a sequence of IID random variables taking on the values \( \pm 1 \) with probability 1/2. Each \( X_n \) represents the winnings in a fair game. Let \( S_n = \sum_{i \leq n} X_i \) and
\[
\vartheta(N, k) = \max_{0 \leq n \leq N-k} \frac{S_{n+k} - S_n}{k} = \max_{0 \leq n \leq N-k} \frac{X_{n+1} + X_{n+2} + \cdots + X_{n+k}}{k}
\]
So \( \vartheta(N, k) \) represents the maximal average gain over length-\( k \) subgames of the length-\( N \) game.

For those who prefer the non-probabilistic point-of-view, we are working in the Cantor space \( \{-1, 1\}^\omega \) with the Lebesgue (fair coin) measure. \( S_n \) and \( \vartheta(N, k) \) are
then both functions from $\{-1, 1\}^\omega$ to $\mathbb{R}$; thus we will write $S_n(x)$ and $\vartheta(N, k)(x)$, where $x \in \{-1, 1\}^\omega$.

**Lemma 4.3.1.** Let $c \geq 1$ and define $\alpha \in (0, 1]$ via $1/c = 1 - h \left(\frac{1+\alpha}{2}\right)$. Then for any $\epsilon > 0$, with $\alpha' = \alpha + \epsilon$, there are positive constants $B$ and $\delta$, depending only on and uniformly computable from $c$ and $\epsilon$, such that

$$P(\vartheta(N, \lfloor c \log_2 N \rfloor) \geq \alpha') \leq BN^{-\delta},$$

where $P(\ldots)$ denotes the probability of the event in parenthesis.

**Proof.** We begin by noticing that $\vartheta(N, \lfloor c \log_2 N \rfloor) \geq \alpha'$ if and only if $S_{n+\lfloor c \log_2 N \rfloor} - S_n \geq \alpha'$ for some $n \in \{0, 1, \ldots, N - \lfloor c \log_2 N \rfloor\}$. When there are exactly $d$ 1’s among $X_{n+1}, X_{n+2}, \ldots, X_{n+\lfloor c \log_2 N \rfloor}$,

$$S_{n+\lfloor c \log_2 N \rfloor} - S_n = \frac{d - ([c \log_2 N] - d)}{\lfloor c \log_2 N \rfloor}.$$

This is $\geq \alpha'$ if and only if $d \geq \lfloor c \log_2 N \rfloor \frac{\alpha'+1}{2}$. The probability that $d \geq \lfloor c \log_2 N \rfloor \frac{\alpha'+1}{2}$ can then be bounded above using (4.3) with $\gamma = \frac{\alpha'+1}{2}$:

$$P(\vartheta(N, \lfloor c \log_2 N \rfloor) \geq \alpha') = P\left(\bigcup_{0 \leq n \leq N - \lfloor c \log_2 N \rfloor} \frac{S_{n+\lfloor c \log_2 N \rfloor} - S_n}{\lfloor c \log_2 N \rfloor} \geq \alpha'\right) \leq \sum_{0 \leq n \leq N - \lfloor c \log_2 N \rfloor} P\left(\frac{S_{n+\lfloor c \log_2 N \rfloor} - S_n}{\lfloor c \log_2 N \rfloor} \geq \alpha'\right) \leq \sum_{0 \leq n \leq N - \lfloor c \log_2 N \rfloor} B\lfloor c \log_2 N \rfloor^{-1/2} 2^{\lfloor c \log_2 N \rfloor (h(\gamma) - 1)} \leq NB\lfloor c \log_2 N \rfloor^{-1/2} 2^{\lfloor c \log_2 N \rfloor (h(\gamma) - 1)} \leq NB2^{\lfloor c \log_2 N \rfloor (h(\gamma) - 1)} \leq NB2^{c \lfloor \log_2 N \rfloor (h(\gamma) - 1)} = BN^{c(h(\gamma) - 1) + 1}$

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Since \( h \) is decreasing on \([1/2, 1]\), \( h((1 + \alpha + \epsilon)/2) < h((1 + \alpha)/2) \), which is equivalent to \( c(h(\gamma) - 1) + 1 < 0 \), so \( \delta = -(c(h(\gamma) - 1) + 1) \) and the \( B \) from Lemma 4.2.2 work.

In Chapters 2 and 3 we primarily used the Martin L"of test definition of Martin L"of randomness. Here, however, we use the Solovay test definition. Recall also from those chapters that \( S \subseteq \{-1, 1\}^\omega \) is effectively open if it is a c.e. union of cylinder sets. Also, the preimage of an effectively open set via a computable map is effectively open.

**Definition 4.3.2 ([10]).** A sequence \( x \in \{-1, 1\}^\mathbb{N} \) is **Martin L"of random relative to** \( c \) if \( x \) is not in infinitely many \( A_i \)'s when \( \langle A_i \rangle_{i \in \mathbb{N}} \) is a sequence of subsets of \( \{-1, 1\}^\mathbb{N} \), uniformly effectively open relative to \( c \), such that \( \sum P(A_i) < \infty \).

**Lemma 4.3.3.** Let \( c \geq 1 \) be Martin L"of random (relative to \( \emptyset \)) and define \( \alpha \in (0, 1) \) via \( 1/c = 1 - h \left( \frac{1+\alpha}{2} \right) \). Then

\[
\limsup_{N \to \infty} \vartheta(N, \lfloor c \log_2 N \rfloor)(x) \leq \alpha
\]

for every \( x \in \{-1, 1\}^\mathbb{N} \) that is Martin L"of random relative to \( c \).

**Proof.** Let \( \epsilon > 0 \) be a (small) computable number, and set \( \alpha' = \alpha + \epsilon \). It is straightforward to show that \( \lfloor c \log_2 (2^{(j+1)/c} - 1) \rfloor = j \), so by Lemma 4.3.1, the series

\[
\sum_{j=1}^{\infty} P(\vartheta(2^{(j+1)/c} - 1, j) > \alpha')
\]

converges.

The hypothesis that \( c \) is random means that the random variables \( \vartheta(2^{(j+1)/c} - 1, j) \) are uniformly computable relative to \( c \) (note that if \( c \) were not random, determining \( [2^{(j+1)/c} - 1] \) could be noncomputable relative to \( c \), since the floor function is not computable). Thus, the events \( \{ \vartheta(2^{(j+1)/c} - 1, j) > \alpha' \} \) are effectively open relative
to \( c \). It follows that for every \( x \) that is Martin L"of random relative to \( c \), the inequality 
\[
\vartheta(2^{(j+1)/c} - 1, j)(x) \leq \alpha'
\]
holds for all but finitely many \( j \).

Since \( \lfloor c \log_2 N \rfloor = j \) for \( 2^{j/c} \leq N \leq 2^{(j+1)/c} - 1 \), the random variables \( \vartheta(N, \lfloor c \log_2 N \rfloor) \) and \( \vartheta(2^{(j+1)/c} - 1, j) \) are looking at the same length of windows to take the max, and since the latter has more windows,

\[
\vartheta(N, \lfloor c \log_2 N \rfloor) \leq \vartheta(2^{(j+1)/c} - 1, j)
\]

when \( 2^{j/c} \leq N \leq 2^{(j+1)/c} - 1 \). Thus, we know now that for any \( x \) that is Martin L"of random relative to \( c \),

\[
\vartheta(N, \lfloor c \log_2 N \rfloor)(x) \leq \alpha'
\]

for all but finitely many \( N \). Since \( \epsilon \) is arbitrary, the proof is complete. \( \square \)

**Lemma 4.3.4.** Let \( c \geq 1 \) be Martin L"of random (relative to \( \emptyset \)) and define \( \alpha \in (0, 1] \)
via \( 1/c = 1 - h \left( \frac{1+\alpha}{2} \right) \). Then

\[
\liminf_{N \to \infty} \vartheta(N, \lfloor c \log_2 N \rfloor)(x) \geq \alpha
\]

for every \( x \in \{-1, 1\}^\mathbb{N} \) that is Martin L"of random relative to \( c \).

**Proof.** Let \( 0 < \epsilon < \alpha \) be computable and set \( \alpha'' = \alpha - \epsilon \).

If \( \vartheta(N, k) \leq \alpha'' \), then \( \frac{S_{(r+1)k}-S_{rk}}{k} \leq \alpha'' \) for each \( 0 \leq r \leq n/k - 1 \). The random
variables $S_{(r+1)k} - S_{rk}$ are IID for different $r$'s, so

\[ P(\vartheta(N, \lfloor c \log_2 N \rfloor) \leq \alpha'') \leq P \left( \frac{S_{(r+1)\lfloor c \log_2 N \rfloor} - S_{r\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor} \leq \alpha'' \right), \]

\[ 0 \leq r \leq \frac{N}{\lfloor c \log_2 N \rfloor} - 1 \]

\[ = P \left( \frac{S_{\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor} \leq \alpha'' \right) \]

\[ = P \left( \frac{S_{\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor} \leq \alpha'' \right) \]

\[ \leq P \left( \frac{S_{\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor} \leq \alpha'' \right) \]

If there are $d$ many 1’s among $X_1, \ldots, X_{\lfloor c \log_2 N \rfloor}$, then

\[ \alpha'' \geq \frac{S_{\lfloor c \log_2 N \rfloor}}{\lfloor c \log_2 N \rfloor} = \frac{2d - \lfloor c \log_2 N \rfloor}{\lfloor c \log_2 N \rfloor} \iff d \leq \lfloor c \log_2 N \rfloor \frac{1 + \alpha''}{2}. \]

Now, we use the lower bound in (4.3) to estimate $P \left( d \leq \lfloor c \log_2 N \rfloor \frac{1 + \alpha''}{2} \right)$, with $\gamma := \frac{1 + \alpha''}{2}$.

\[ P \left( d \leq \lfloor c \log_2 N \rfloor \frac{1 + \alpha''}{2} \right) = 1 - P \left( d > \lfloor c \log_2 N \rfloor \frac{1 + \alpha''}{2} \right) \]

\[ = 1 - P \left( d \geq \lfloor c \log_2 N \rfloor \frac{1 + \alpha''}{2} \right) \]

\[ \leq 1 - A \lfloor c \log_2 N \rfloor^{-1/2} \left( \lfloor c \log_2 N \rfloor (h(\gamma) - 1) \right) \]

\[ \leq 1 - A \lfloor c \log_2 N \rfloor^{-1/2} \left( \lfloor c \log_2 N \rfloor - 1 \right) (h(\gamma) - 1) \]

\[ = 1 - A \lfloor c \log_2 N \rfloor^{-1/2} (\lfloor c \log_2 N \rfloor h(\gamma) - 1) \]

\[ = 1 - A \lfloor c \log_2 N \rfloor^{-1/2} N c(\gamma) - 1 \]

\[ \leq 1 - A \lfloor c \log_2 N \rfloor^{-1/2} N^c(\gamma) - 1 \]

Because the function $h$ is decreasing on $[1/2, 1]$, $h(\gamma) > h((1 + \alpha)/2)$, so $c(h(\gamma) - 1) >
\[ c(h((1 + \alpha)/2) - 1) = -1, \text{ say } c(h(\gamma) - 1) = -1 + \delta. \] Thus,

\[
1 - A[c \log_2 N]^{-1/2} N^{c(h(\gamma) - 1)} = 1 - A[c \log_2 N]^{-1/2} N^{1+\delta} \\
\leq 1 - AN^{-1+\delta/2}.
\]

The last inequality holds because \(|c \log_2 N| \leq N^\delta\) for sufficiently large \(N\), so that \(|c \log_2 N|^{-1/2} \geq N^{-\delta/2}\).

Putting these inequalities together, with \(\delta_1 := \delta/2\), and then using the inequality \(1 - x \leq e^{-x}\), we get

\[
P(\vartheta(N, [c \log_2 N]) \leq \alpha'') \leq \left(1 - \frac{AN^{\delta_1}}{N}\right)^{N/|c \log_2 N| - 1} \\
\leq \left(e^{-AN^{\delta_1}/N}\right)^{N/|c \log_2 N| - 1} \\
= e^{-AN^{\delta_1}/c \log_2 N} \\
\leq e^{-AN^{\delta_1}/c \log_2 N} \\
\leq e^{-N^{\delta_1/2}} \quad \text{(eventually)} \\
\leq N^{-2} \quad \text{(eventually)}.
\]

Thus,

\[
\sum_{N=1}^{\infty} P(\vartheta(N, [c \log_2 N]) \leq \alpha'')
\]

converges.

Because \(c\) is random, the random variables \(\vartheta(N, [c \log_2 N])\) are uniformly computable relative to \(c\) (note that if \(c\) weren’t random, the floor function could be problematic), so the sets \(\{x \in \{-1, 1\}^\omega : \vartheta(N, [c \log_2 N])(x) < \alpha''\}\) are uniformly effectively open. Thus, if \(x\) is Martin Löf random relative to \(c\), it is in only finitely many of those sets; i.e., there is \(M\) such that \(\vartheta(N, [c \log_2 N])(x) \geq \alpha'' = \alpha - \epsilon\).
whenever $N \geq M$. Since $\epsilon$ is arbitrary, we are done.

Putting Lemmas 4.3.3 and 4.3.4 together, we have so far shown:

**Lemma 4.3.5.** For any Martin L"of random $c \geq 1$,

$$\vartheta(N, \lfloor c \log_2 N \rfloor)(x) \to \alpha$$

for every $x$ that is Martin L"of random relative to $c$.

We strengthen this now by showing that in fact the convergence holds for any $c \geq 1$ and any Martin L"of random $x$, even when $x$ is not random relative to $c$.

**Theorem 4.3.6.** Let $c \geq 1$, and define $\alpha = \alpha(c) \in (0, 1]$ via $1/c = 1 - h(1 + \alpha^2)$. Then

$$\lim_{N \to \infty} \vartheta(N, \lfloor c \log_2 N \rfloor)(x) = \alpha$$

for every $x \in \{-1, 1\}^\mathbb{N}$ that is Martin L"of random (relative to $\emptyset$).

**Proof of Theorem 4.3.6.** Fix $c \geq 1$, a Martin L"of random $x \in \{-1, 1\}^\mathbb{N}$, and let $\epsilon > 0$. By Van Lambalgen’s Theorem, Lemma 4.3.5 and the continuity of $\alpha(c)$, there is $c_1 \geq c$ that is Martin L"of random relative to $x$, and there is $M_1 \in \omega$ such that $\vartheta(N, \lfloor c_1 \log_2 N \rfloor)(x) > \alpha(c) - \epsilon$ whenever $N \geq M_1$. Further, we assume $M_1$ is large enough to guarantee that for every integer $n \geq \lfloor c_1 \log_2 M_1 \rfloor$, there is $N \geq M_1$ such that $\lfloor c_1 \log_2 N \rfloor = n$.

Let $M$ be such that $\lfloor c \log_2 M \rfloor \geq \lfloor c_1 \log_2 M_1 \rfloor$ and let $N \geq M$. By the hypothesis on $M_1$, there is $N_1 \in [M_1, N]$ such that $\lfloor c \log_2 N \rfloor = \lfloor c_1 \log_2 N_1 \rfloor$. Then

$$\vartheta(N, \lfloor c \log_2 N \rfloor) \geq \vartheta(N_1, \lfloor c_1 \log_2 N_1 \rfloor) > \alpha(c) - \epsilon.$$

Since $\epsilon$ is arbitrary, we have that

$$\liminf_{N \to \infty} \vartheta(N, \lfloor c \log_2 N \rfloor)(x) \geq \alpha(c).$$
For the lim sup direction, let \( c_2 \leq c \) be random relative to \( x \) and \( M_2 \in \omega \) be such that \( \vartheta(N, \lfloor c_2 \log_2 N \rfloor)(x) < \alpha(c) + \epsilon \) whenever \( N \geq M_1 \). Again we assume \( M_2 \) is large enough to guarantee that for every integer \( n \geq \lfloor c_2 \log_2 M_2 \rfloor \), there is \( N \geq M_2 \) such that \( \lfloor c_2 \log_2 N \rfloor = n \).

If \( N \geq M_2 \), then \( \lfloor c \log_2 N \rfloor = \lfloor c \log_2 K \rfloor \) for some \( K \geq N \). But then

\[
\vartheta(N, \lfloor c \log_2 N \rfloor) \leq \vartheta(K, \lfloor c_2 \log_2 K \rfloor) < \alpha(c) + \epsilon.
\]

Since \( \epsilon \) is arbitrary, we have that

\[
\limsup_{N \to \infty} \vartheta(N, \lfloor c \log_2 N \rfloor)(x) \leq \alpha(c),
\]

and the proof is now complete.

We do not know if this Theorem 4.3.6 holds for other notions of randomness. It would be interesting to know, for example, if the theorem is satisfied by all Kurtz randoms (recall from Chapter 2 that a real is Kurtz random if it avoids all null effectively closed sets). It would be even more interesting to know whether Theorem 4.3.6 characterizes a certain notion of randomness, so that the theorem holds on a real \( x \) if and only if \( x \) is that type of random.


