BLOWUP IN NONLINEAR HEAT EQUATIONS

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

Shuangcai Wang

Israel Michael Sigal, Director

Graduate Program in Mathematics
Notre Dame, Indiana
December 2006
BLOWUP IN NONLINEAR HEAT EQUATIONS

Abstract

by

Shuangcai Wang

In this paper we study the blowup problem for nonlinear heat equation

\[ u_t = \partial^2_x u + |u|^{p-1} u \]
\[ u(x, 0) = u_0(x). \] \hspace{1cm} (1)

where \( u(t) : x \in \mathbb{R} \to u(x, t) \in \mathbb{R} \) and \( p > 1 \). We show that if the initial data is close enough to a 2-dimensional manifold of approximately homogenous solutions, the solution blows up in a finite time and the asymptotical profile is an approximate solution with parameters evolving according to a certain dynamical system plus a small fluctuation in \( L^\infty \).

Main Theorem. Assume the initial data \( u_0 \in L^\infty(\mathbb{R}) \) is even and satisfies

\[
\left\| \frac{2c_0}{p - 1 + b_0 x^2} \right\|_\infty^{\frac{1}{p-1}} < \delta_0 c_0^{\frac{1}{p-1}}
\]
\[
\left\| \frac{2c_0}{p - 1 + b_0 x^2} \right\|_\infty^{\frac{1}{p-1}} \lesssim b_0^2 c_0^{\frac{2}{2} + \frac{1}{p-1}}
\]

where \( \langle x \rangle = \sqrt{1 + x^2} \) with \( b_0 \geq 0 \) and \( c_0 > 0 \) constant. If \( 0 \leq \frac{b_0}{c_0} \ll 1 \) and \( 0 \leq \delta_0 \ll 1 \), then

(1) There exists a finite time \( t^* \in (0, \infty) \) such that the solution \( u(x, t) \) blows up at \( t = t^* \).
2. When $t < t^*$, there exist unique positive, $C^1$ functions $\lambda(t)$, $b(t)$ and $c(t)$ with $b(t) \lesssim b(0)$ such that $\lambda(0) = \sqrt{2c_0 + \frac{2}{p-1}b_0}$ and $u(x, t)$ can be decomposed as

$$u(x, t) = \lambda^{\frac{2}{p-1}}(t) \left[ \left( \frac{2c(t)}{p - 1 + b(t)\lambda^2(t)x^2} \right)^{\frac{1}{p-1}} + \eta(x, t) \right]$$

with the fluctuation part, $\eta$, admitting the estimate $\| (\lambda(t)x)^{-3}\eta(x, t) \|_{\infty} \leq Cb^2(t)$.

3. As $t \to t^*$, $\lambda(t)$, $a(t)$, $b(t)$ and $c(t)$ have the following approximations,

$$\begin{align*}
\lambda(t) &= \lambda_0^{-1}(t^* - t)^{-\frac{1}{2}}(1 + o(1)) \\
b(t) &= -\frac{(p-1)^2}{4p\ln(t^* - t)} \left[ 1 + O\left( \frac{1}{\ln(t^* - t)} \right)^{1/2} \right] \\
c(t) &= \frac{1}{2} - \frac{p-1}{4p|\ln(t^* - t)|} \left[ 1 + O\left( \frac{1}{\ln(t^* - t)} \right) \right].
\end{align*}$$

with $\lambda_0 = \sqrt{2c_0 + \frac{2}{p-1}b_0}$.

Furthermore, we have not only the asymptotic expressions for the parameters $b$ and $c$ determining the leading term and the size of the reminder, but also the dynamical equations for the parameters.

Finally, we point out that we can easily construct an initial data with more than one local maximum while it still satisfies our constraints. So we have a completely different result from that by Herrero and Velazquez. We also demonstrated that our result is not same as the theorem proved by Bricmont and Kupiainen.
To Fei

with all my love
CONTENTS

ACKNOWLEDGMENTS ......................................................... v

CHAPTER 1: INTRODUCTION ............................................. 1

CHAPTER 2: LOCAL EXISTENCE AND A BLOWUP TEST ............... 10
  2.1 Local Existence .................................................. 10
  2.2 A Blow-Up Test .................................................. 12

CHAPTER 3: FORMULATION OF THE PROBLEM ...................... 13
  3.1 Blow-Up Variables and Almost Solutions ....................... 13
  3.2 "Gauge" Transform ................................................ 15
  3.3 Reparametrization of Solutions .................................. 16

CHAPTER 4: A PRIORI ESTIMATES .................................. 20
  4.1 Statement of A Priori Estimates ................................. 20
  4.2 Lyapunov-Schmidt Splitting (Effective Equations) ............ 22
  4.3 Proof of Estimates (4.11)-(4.13) ............................... 27
  4.4 Rescaling of Fluctuations on a Fixed Time Interval .......... 28
  4.5 Estimate on the Propagators .................................... 30
  4.6 Estimate of $M_1(\tau)$ (4.14) .................................. 35
  4.7 Estimate of $M_2$ (4.15) ....................................... 39

CHAPTER 5: PROOF OF MAIN THEOREM 1 ......................... 43

APPENDIX A: BLOW-UP DYNAMICS ................................. 47

APPENDIX B: SPECTRUM OF THE LINEAR OPERATOR $\mathcal{L}_{abc}$ . 50

APPENDIX C: PROOF OF THE FEYNMANN-KAC FORMULA ........ 52
BIBLIOGRAPHY
I would like to express my thanks to my advisor, Israel Michael Sigal, for his guidance and patience throughout my graduate study. I also want to thank Zhou Gang and Steven Dejak for their friendship and valuable discussions.

I am grateful to Mark Alber, Qing Han, Bei Hu, and Gerard Misiolek for reading this dissertation and their comments. I own my gratitude to many other professors in the department for their kindly assistance.

I am also indebted to the our wonderful administrative assistants, Carole Martin, Betsy Karnes, Judy E. Hygema, Patti J. Strauch, Beth VerVerlde and Robin Lockhart for their efforts to help me.

My heartfelt thanks goes to my wife, Dr. Fei Zhou, for her endless help, love and encouragement. She received her Ph.D. degree at Purdue University, West Lafayette in 2006. I’m really proud of her. She has counseled me through innumerable emotional strains. Most importantly, she has built with me a life worth living.

I am externally grateful to my parents and my family for all their love and support and for their faith in me and allowing me to be as ambitious as I wanted over the years.
CHAPTER 1

INTRODUCTION

We study the blow-up problem for the one-dimensional nonlinear heat equations (or reaction-diffusion equations) of the form

\[
\begin{align*}
  u_t &= \partial_x^2 u + |u|^{p-1} u \\
  u(x,0) &= u_0(x)
\end{align*}
\]  

(1.1)

where \( p > 1 \). A solution \( u(t) \) to (1.1) blows up in finite time \( T \) if \( u(t) \in L^\infty \) exists for \( t \in [0,T) \) but

\[
\lim_{t \to T} \| u(t) \|_\infty = \infty.
\]

\( T \) is called the blowup time of \( u \). A point \( x_0 \in \mathbb{R} \) is said to be a blowup point if \( u \) is unbounded in any neighborhood of \((x_0, T)\). Equation (1.1) arises in the problem of heat flow[15] and the theory of chemical reactions. Similar equations also appeared in the motion by mean curvature (see [53]), vortex dynamics in superconductors (see [10, 42]), surface diffusion (see [4]) and chemotaxis (see [5, 3]). Equation (1.1) has the following properties:

- (1.1) is invariant under the scaling transformation,

\[
\begin{align*}
  u(x,t) &\to \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)
\end{align*}
\]  

(1.2)

for any constant \( \lambda > 0 \), i.e. if \( u(x,t) \) is a solution, so is \( \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \).
• (1.1) has $x$–independent(homogeneous) solutions:

$$u_{\text{hom}} = [ |u_0|^{-p+1} - (p-1)t ]^{-\frac{1}{p-1}}.$$  \hfill (1.3)

These solutions blow up in finite time $t^* = \left( (p-1)|u_0|^{p-1} \right)^{-1}$ for $p > 1$.

• (1.1) is an $L^2$-gradient system $\partial_t u = -\text{grad} \mathcal{E}(u)$, with the energy

$$\mathcal{E}(u) = \int \frac{1}{2} u_x^2 - \frac{1}{p+1} |u|^{p+1}.$$  \hfill (1.4)

with the $L^2(\mathbb{R})$ inner product $\langle \cdot , \cdot \rangle$; grad $\mathcal{E}$ is defined by the relation $\partial \mathcal{E}(u) \xi = \langle \text{grad} \mathcal{E}(u), \xi \rangle$, so that grad $\mathcal{E}(u) = -(\partial_x^2 u + |u|^{p-1}u)$. We immediately have that the energy $\mathcal{E}$ decreases under the flow of (1.1).

The linearization of (1.1) around $u_{\text{hom}}$ shows that the solution $u_{\text{hom}}$ is unstable. Moreover, it is shown in [23] that if either $n \leq 2$ or $p < \frac{n+2}{n-2}$, then the analogue problem to (1.1) in dimension $n$ has no other self-similar solutions of the form $(T-t)^{-\frac{1}{p-1}} \phi \left( x/\sqrt{T-t} \right)$, $\phi \in L^\infty$, besides $u_{\text{hom}}$.

Many local existence theorems in various spaces for (1.1) have been proved (see, e.g. [2] for $H^\alpha$, $0 \leq \alpha < 2$). We already knew that for some initial data $u_0(x)$, the solutions $u(x,t)$ might blow up in finite time $T > 0$. For given $0 \leq u_0 \in L^\infty, u_0 \neq 0$, it has been shown that if $1 < p < 3$, for any nontrivial solution of (1.1), $u$ blows up in a finite time (See [16, 31]). When $p \geq 3$, the problem has a unique global solution $u \in C([0, \infty); H^s \cap L^1) \cap C^1([0, \infty); H^{s-2}) \cap C((0, \infty); H^1), u_t \in L^2((0, \infty); H^s)$, provided $\|u_0\|_{H^s \cap L^1} = \|u_0\|_{H^s} + \|u_0\|_{L^1}$ is sufficiently small for $s > 5$ (see [58]). In other words, we can have a global solution which is uniformly bounded under $L^\infty$ norm. We can also define the blowup in other norms. For instance, if the maximal existence time for a solution $u$ in $L^p$ space is finite, we say $u$ blows up in $L^p$ norm. For given $u_0 \geq 0, u_0 \neq 0$, $u$ blows up in a finite time in $L^1$ when $p = 3$. While $p > 3$, if the $L^{\frac{p+1}{p-1}}$ norm of $u_0(x)$ is large enough, the solution blows up in $L^{\frac{p+1}{p-1}}$ norm(see [57]). Thus, two key problems about (1.1) for blowup are:
1. Describe initial conditions for which solutions of Equation (1.1) blowup in finite time;
2. Describe the blowup profile of such solutions.

It is expected (see e.g. [8]) that the blowup profile is universal — it is independent of lower power perturbations of the nonlinearity and of initial conditions within certain spaces.

There is rich literature regarding the blowup problem for Equation (1.1). We review quickly relevant results. Starting with [16], various criteria for blow-up in finite time were derived, see e.g. [16, 2, 9, 14, 35, 36, 46, 48, 52]. For example, if \( u_0 \in H^1 \cap L^{p+1} \) and \( E(u_0) < 0 \), where \( E(u) \) is the energy functional for (1.1) defined in (1.4), then it can be shown that \( u(t) \) blows up in finite time \( t^* \). In this paper, we’ll demonstrate a class of initial data whose solutions blow up in finite time while their energy functional may not be finite.

The first result on the asymptotics of blowup was found in the pioneering paper [23] where the authors showed that under the conditions

\[
|u(x, t)|(t_* - t)^{\frac{1}{p-1}} \text{ is bounded on } B_1 \times (0, t_*),
\]

where \( B_1 \) is the unit ball in \( \mathbb{R}^n \) centred at the origin, and either \( p < \frac{n+2}{n-2} \) or \( n \leq 2 \) and assuming the blowup takes place at \( x = 0 \), one has

\[
\lim_{\lambda \to 0} \lambda^{\frac{2}{p+1}} u(\lambda x, t_* + \lambda^2(t_* - t)) = \pm \left( \frac{1}{p-1} \right)^{\frac{1}{p-1}} (t_* - t)^{-\frac{1}{p-1}} \text{ or } 0.
\]

This result was further studied in several papers (see e.g. [24, 25, 32, 28, 54, 8, 43, 44, 45]). A blowup solution satisfying the bound (1.5) is said to be the singularity of type I and type II otherwise. This bound was proven under various conditions in [24, 43, 44, 56, 20]. Furthermore, the limits of \( H^1 \)-blowup solutions \( u(x, t) \) as \( t \uparrow T \), outside the blowup sets were established in [32, 28, 54, 8, 45, 18].
Herrero and Velázquez [29] proved that if the initial condition $u_0$ is continuous, nonnegative, bounded, even and has only one local maximum at 0, and if the corresponding solution blows up, then

$$\lim_{t \to t^*} (t^* - t)^{\frac{1}{p-1}} u \left( y \left( (t^* - t) |\ln(t^* - t)| \right)^{1/2}, t \right) = (p-1)^{-\frac{1}{p-1}} \left[ 1 + \frac{p-1}{4p} y^2 \right]^{-\frac{1}{p-1}}$$  \hspace{1cm} (1.6)

uniformly on sets $|y| \leq R$ with any $R > 0$. Further extensions of this result are achieved in [28, 54].

Later Bricmont and Kupiainen [8] constructed a co-dimension 2 submanifold in the space of initial conditions such that (1.6) is satisfied on the whole domain. More specifically, given a small function $g$ and a small constant $b > 0$, they find constants $d_0$ and $d_1$ depending on $g$ and $b$ such that the solution to (1.1) with the datum

$$u_0^*(x) = (p - 1 + bx^2)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0 + d_1 x}{p - 1 + bx^2} \right) \frac{1}{p-1} + g(x)$$ \hspace{1cm} (1.7)

has the convergence (1.6) uniformly in $y \in (-\infty, +\infty)$. The result of [8] was generalized in [41, 34] (see also [19]), where it is shown that there exists a neighborhood $U$, in the space $\mathcal{H} = L^{p+1} \cap \mathcal{H}^1$, of $u_0^*$, given in (1.7), such that if $u_0 \in U$, then the solution $u(x,t)$ blows up in a finite time $t^*$ and satisfies (1.6) for all $x \in \mathbb{R}$. They conjectured that this asymptotic behavior is generic for any blow-up solution.

The starting point in the above works, which goes back to Giga and Kohn [23], is passing to the similarity variables $y = x/\sqrt{t^* - t}$ and $s = -\ln(t^* - t)$, where $t^*$ is the blowup time, and to the rescaled function $w(y, s) = (t^* - t)^{\frac{1}{p-1}} u(x, t)$. Then one studies the resulting equation for $w$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1}{p - 1} w + |w|^{p-1} w.$$  \hspace{1cm} (1.8)

Most of the work above uses relations involving the energy functional

$$S(w) = \frac{1}{2} \int \left( |\nabla w|^2 + \frac{1}{p-1} |w|^2 - \frac{2}{p+1} |w|^{p+1} \right) e^{-\frac{1}{4} y^2} dy,$$ \hspace{1cm} (1.9)
introduced in [23], and related functionals. In particular, one uses the relation

\[
\partial_s S(w) = -\int |\partial_s w|^2 e^{-\frac{1}{4}y^2} \ dy. \tag{1.10}
\]

**Remark 1.** Equation (1.8) is the gradient system \(\partial_s w = -\text{grad} S(u)\) in the metric space \(L^2(e^{-\frac{1}{4}y^2} \ dx)\). In other words, \(\text{grad} S(u)\) is defined by the equation \(\partial S(u)\xi = \langle \text{grad} S(u), \xi \rangle_{L^2(e^{-\frac{1}{4}y^2} \ dy)}\). Hence \(S\) decreases under the flow of (1.1) and so (1.10) implies that \(\partial_s w \to 0\) as \(s \to \infty\).

Blowup as a single point was studied as early as [55]. In 1992, Merle [32] proved that given an finite number of points \(x_1, x_2, \ldots, x_k\) in \(I = (-1, 1)\) (or any other domain \(I\) in \(\mathbb{R}\)), there is a positive solution to the nonlinear heat equation which blowups up at time \(T\) with blowup points \(x_1, x_2, \ldots, x_k\). This theorem can be generalized to allow the sign \((+\infty\) or \(-\infty\) to be chosen at each blowup point \(x_i\).

In this paper, we consider (1.1) with initial conditions which are even, have a maximum at the origin after modulo a small perturbation, are slowly varying near the origin and are sufficiently small, but not necessarily vanishing, for large \(|x|\). In particular, the energy \(\mathcal{E}(u)\) for such initial conditions might be infinite. We show that the solutions of (1.1) for such initial conditions blow up in a finite time and we characterize asymptotic dynamics of these solutions. As it turns out, the leading term is given by the expression

\[
\lambda(t) \frac{2c(t)}{p - 1 + b(t)\lambda(t)^2 x^2} \tag{1.11}
\]

(cf (1.6)) where the parameters \(\lambda(t), b(t)\) and \(c(t)\) obey certain dynamical equations whose solutions give

\[
\begin{align*}
\lambda(t) &= \lambda_0^{-1}(t^* - t)^{-\frac{1}{2}}(1 + o(1)) \\
b(t) &= -\frac{(p - 1)^2}{4p \ln(t^* - t)} \left[1 + O\left(\frac{1}{\ln(t^* - t)}\right)^{1/2}\right] \\
c(t) &= \frac{1}{2} - \frac{p - 1}{4p \ln(t^* - t)} \left[1 + O\left(\frac{1}{\ln(t^* - t)}\right)^{1/2}\right].
\end{align*} \tag{1.12}
\]
with \( \lambda_0 = \sqrt{2c_0 + \frac{2}{p-1}b_0} \) (see \( b_0 \) and \( c_0 \) below), and \( o(1) \) is in \( t^* - t \). Moreover, we estimated the remainder, the difference between \( u(x,t) \) and (1.11). Our techniques are different from the papers mentioned above, the closest to our approach is [8]. Our main point is that we do not fix the time-dependent scale in the self-similarity (blowup) variables but let its behaviour, as well as behaviour of other parameters (\( b \) and \( c \)) to be determined by the equation. This approach is analogous to one used in bifurcation theory and our techniques can be regarded as a time-dependent version of the Lyapunov-Schmidt decomposition.

**Theorem 1.** Suppose the initial datum \( u_0 \in L^\infty(\mathbb{R}) \) in (1.1) is even and

\[
\| u_0(x) - \left( \frac{2c_0}{p - 1 + b_0x^2} \right)^{\frac{1}{p-1}} \|_\infty \leq \delta_0 \sqrt{c_0}
\]

\[
\| \langle x \rangle^{-3} \left[ u_0(x) - \left( \frac{2c_0}{p - 1 + b_0x^2} \right)^{\frac{1}{p-1}} \right] \|_\infty \leq b_0^2 c_0^{\frac{-1}{2} + \frac{1}{p-1}}
\]

with \( 0 \leq \frac{b_0}{c_0} \ll 1 \) and \( 0 \leq \delta_0 \ll 1 \). Then

1. There exists a time \( t^* \in (0, \infty) \) such that the solution \( u(x,t) \) blows up at time \( t^* \).

2. When \( t < t^* \), there exist unique positive, \( C^1 \) functions \( \lambda(t) \), \( b(t) \) and \( c(t) \) with \( b(t) \ll b(0) \) such that \( \lambda(0) = \sqrt{2c_0 + \frac{2}{p-1}b_0} \) and \( u(x,t) \) can be decomposed as

\[
u(x,t) = \lambda^{\frac{2}{p-1}}(t) \left[ \left( \frac{2c(t)}{p - 1 + b(t)c(t)x^2} \right)^{\frac{1}{p-1}} + \eta(x,t) \right]
\]

with the fluctuation part, \( \eta \), admitting the estimate \( \| \langle \lambda(t) \rangle^{-3} \eta(x,t) \|_\infty \leq Cb^2(t) \) and \( \| \eta(x,t) \| \leq C(b(t)^{\frac{1}{2}} + \delta_0) \)

3. The functions \( \lambda(t) \), \( a(t) \), \( b(t) \) and \( c(t) \) are of the form (1.12).

The proof is given in Chapter 5. Thus our result shows the blow-up at 0 for a certain neighborhood of the homogeneous solution, (1.3), with a detailed description of the leading term and an estimate of the remainder in \( L^\infty \). In fact, we have not only
the asymptotic expressions for the parameters $b$ and $c$ determining the leading term and the size of the remainder, but also dynamical equations for these parameters:

\[
\begin{align*}
  b_\tau &= - \frac{4p}{(p-1)^2} b^2 + c^{-1} c_\tau b + R_b(\eta, b, c), \\
  c^{-1} c_\tau &= 2 \left( \frac{1}{2} - c \right) - \frac{2}{(p-1)} b + R_c(\eta, b, c),
\end{align*}
\]

(1.14)

where the new time scale $\tau(t) = \int_0^t \lambda^2(s) \, ds$, the remainders have the estimates

\[
R_b(\eta, b, c), R_c(\eta, b, c) = O \left( b^2 + \left[ |c + \frac{1}{2}| + |c^{-1} c_\tau| \right] b^2 + |b_\tau| b \\
+ c^\frac{p-2}{p-1} \|\eta(\cdot, t)\|_X + c^\frac{p-3}{p-1} \|\eta(\cdot, t)\|_X^2 + c^{-\frac{1}{p-1}} \|\eta(\cdot, t)\|_X \right),
\]

(1.16)

with the norm $\|\eta(\cdot, t)\|_X = \|\langle \lambda(t) x \rangle^{-3} \eta\|_\infty$.

**Remark 2.**

(a) The restriction (1.13) on the initial condition $u_0(x)$ states roughly that mod $O(b_0^2 c_0^{\frac{n+2}{n-2}})$ $u_0(x)$ (after initial rescaling if necessary) has a form $\phi(\sqrt{b_0} x)$ for $|x| \lesssim \frac{1}{\sqrt{b}}$ with an absolute maximum at $x = 0$ and is of the size $\delta_0 c_0^{\frac{1}{p-1}}$ for $|x| \gg \frac{1}{\sqrt{b}}$.

(b) The theorem can not be true for a neighborhood with size independent of $b_0$ because of the existence of homogenous solutions. Therefore the initial conditions are optimal in this sense.

(c) We expect our approach can be extended to general data, to more general nonlinearities and to dimensions $\geq 2$. Most recent blowup rate estimate by Giga, Matsui and Sasayama [22] makes us believe that the similar results can be proved in higher dimensions for subcritical $p < \frac{n+2}{n-2}$ or $n \leq 2$.

Following the same argument with slight different seminorms, we have

**Theorem 2.**

\[
\begin{align*}
\| u_0(x) - \left( \frac{2c_0}{p - 1 + b_0 x^2} \right)^\frac{1}{p-1} \|_\infty \lesssim & b_0 c_0^{\frac{1}{p-1}} \\
\| \langle x \rangle^{-3} \left[ u_0(x) - \left( \frac{2c_0}{p - 1 + b_0 x^2} \right)^\frac{1}{p-1} \right] \|_\infty \lesssim & b_0^2 c_0^{-\frac{3}{p} + \frac{1}{p-1}}
\end{align*}
\]

(1.17)

and $0 \leq \frac{b_0}{c_0} \ll 1$. Then the results in the main theorem still hold and

\[
\| \eta(x, t) \| \leq Cb(t)^{\frac{1}{2}}.
\]
It’s interesting to note that Theorem 2 can be used to find $d_0$ and $d_1$ if we only consider even initial data as those Bricmont and Kupianen studied. We first recall the result they had proved. Let $f(y) = \left( p - 1 + \frac{(p-1)^2}{4p} y^2 \right)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0 + d_1 y}{p-1 + \frac{(p-1)^2}{4p} y^2} \right) + g(y)$, where $d_0$ and $d_1$ are two parameters to be fixed. They proved that there exists a $T_0 > 0$ such that for each $0 < T < T_0$ and $g \in C^0(\mathbb{R})$ with $\|g\|_{\infty} < (\ln T_0)^{-2}$, one can find $d_0$ and $d_1$ such that the equation (1.1) with initial data $u_0(x) = T^{-\frac{1}{p-1}} f \left( (T|\ln T|)^{-\frac{1}{2}} x \right)$ has a unique classical solution $u(x,t)$ on $\mathbb{R} \times [0,T)$ and

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u \left( ((T-t)|\ln(T-t)|)^{\frac{1}{2}} y,t \right) = \left( p - 1 + \frac{(p-1)^2}{4p} y^2 \right)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0 + d_1 y}{p-1 + \frac{(p-1)^2}{4p} y^2} \right) + \text{g}(y) \quad (1.18)$$

uniformly in $y$ on $\mathbb{R}$. Lemma 8, which is proved in Chapter 4, allows us to reduce two parameters to one. For this reason, we only present the proof for Theorem 2 under condition

$$\left\| u_0(x) - \left( \frac{1 - \frac{2}{p-1}b_0}{p - 1 + b_0x^2} \right)^{\frac{1}{p-1}} \right\|_{\infty} \lesssim b_0 \quad (1.19)$$

and

$$\left\| \langle x \rangle^{-3} \left[ u_0(x) - \left( \frac{1 - \frac{2}{p-1}b_0}{p - 1 + b_0x^2} \right)^{\frac{1}{p-1}} \right] \right\|_{\infty} \lesssim b_0^2. \quad (1.20)$$

The symmetry of initial data about 0 implies $d_1 = 0$. It is easy to identify that $b_0 = \frac{(p-1)^2}{4p} |\ln T|^{-\frac{1}{2}}$ and

$$u_0(x) = \left( p - 1 + b_0y^2 \right)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0}{p - 1 + b_0y^2} \right) + g(y).$$

If $0 < T < T_0$ and $T_0$ is sufficiently small, which means $b_0$ is so small. By Taylor expansion,

$$\left( p - 1 + b_0y^2 \right)^{-\frac{1}{p-1}} \left( 1 + \frac{d_0}{p - 1 + b_0y^2} \right) = \left( \frac{1 - \frac{2}{p-1}b_0}{p - 1 + b_0y^2} \right)^{\frac{1}{p-1}} \left( 1 + \frac{2}{(p-1)^2} b_0 + \frac{d_0}{p - 1 + b_0y^2} - \frac{b_0 d_0 y^2}{p - 1 + b_0y^2} + O(\Xi) \right),$$
where $\Xi = b_0d_0 + b_0^2 + \frac{b_0d_0^2y^2}{p-1+b_0y^2}$, $d_0 = -\frac{2}{p-1}b_0 + O(b_0^2)$ will assure that the initial data $u_0$ satisfies the conditions in Theorem 2. Therefore we proved the following result.

**Theorem 3.** Let $f(y)$ be as above and $u_0(x) = T^{-\frac{1}{p-1}}f\left((T|\ln T|)^{-\frac{1}{p}}x\right)$. There is a $T_0$ such that if $d_0 = -\frac{2}{p-1}b_0 + O(b_0^2)$ and $d_1 = 0$, then for any $\|g\|_{\infty} \leq C(\ln T)^{-2}$ with $T \leq T_0$ the equation (1.1) with the initial data has a unique classical solution $u(x,t)$ on $\mathbb{R} \times [0,t^*)$ for some $t^*$ and

$$
\lim_{t \to t^*} (t^* - t)^{\frac{1}{p-1}} u\left(((t^* - t)|\ln(t^* - t)|)^{\frac{1}{2}} y, t\right) = \left(p - 1 + \frac{(p - 1)^2}{4p} y^2\right)^{-\frac{1}{p-1}} (1.21)
$$

uniformly in $y$ on $\mathbb{R}$.

The next corollary is easy to implement and the proof is obvious.

**Corollary 4.** Assume that $u_0(x)$ is even and

$$
\left\| \left(u_0(x) - \left(\frac{2c_0}{p - 1 + b_0x^2}\right)^{\frac{1}{p-1}}\right) \right\|_{\infty} \lesssim b_0^2 c_0^{-\frac{3}{2} + \frac{1}{p-1}} (1.22)
$$

with $0 \leq \frac{b_0}{c_0} \ll 1$. Then the results in Theorem 1 still hold.

we want to remark that we can easily construct an initial data with more than one local maximum while it still satisfies the constraints in the Theorem 1. So our results are different from those Herrero and Velazquez had. From Theorem 3, our result is also not same as the theorem proved by Bricmont and Kupiainen.
2.1 Local Existence

In this section we prove the local well-posedness of (1.1) in $C([0,T], L^\infty)$. The proof is standard and is presented for the reader’s convenience as we did not find it in the literature.

**Theorem 5.** Let $u_0 \in L^\infty$. For $T = \frac{1}{2} \min \left( (p^p \|u_0\|_{L^\infty})^{-\frac{1}{p-1}}, 1 \right)$ there exists a unique function $u \in C([0,T], L^\infty)$ satisfying the nonlinear heat equation (1.1). The solution $u$ depends continuously on the initial condition $u_0$. Moreover, the solution $u$ satisfies the estimate

$$\|u\|_{C([0,T], L^\infty)} \leq \max \left[ 2^\frac{1}{p} p \|u_0\|_{L^\infty}, 2^\frac{1}{p} \|u_0\|_{L^\infty}^{\frac{1}{p}} \right].$$

Furthermore, either the solution is global in time or blows up in $L^\infty$ in a finite time.

**Proof.** Using Duhamel’s principle, Equation (1.1) can be written as the fixed point equation $u = H(u)$, where

$$H(u) = e^{t \partial_x^2} u_0 + \int_0^t e^{(t-s) \partial_x^2} |u|^{p-1} u(s) \, ds. \quad (2.1)$$

Thus, the proof of existence and uniqueness will be complete if we can show that the map $H$ has a unique fixed point in the ball

$$B_R = \{ u \in X, \|u\|_X \leq R \},$$

where $X = C([0,T], L^\infty)$ and $R = (T^{-1} \|u_0\|_{L^\infty})^{\frac{1}{p}}$. We prove this statement via the contraction mapping principle.

We begin by proving that $H$ is a well-defined map from $B_R$ to $B_R$. The estimate

$$\left\| e^{t \partial_x^2} u_0 \right\|_X \leq \|u_0\|_{L^\infty} \quad (2.2)$$
is obtained by using the integral kernel of $e^{t\partial^2_x}$, $e^{t\partial^2_x}(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$, defined for $t > 0$ and its property that $\int e^{t\partial^2_x}(x, y) dy = 1$. Similarly, we find that if $t < T$, then

$$\left\| \int_0^t e^{(t-s)\partial^2_x} |u|^{p-1} u(s) \, ds \right\|_X \leq T \|u\|^p_X.$$  \hspace{1cm} (2.3)

Estimates (2.2) and (2.3) imply that for $T < \infty$, $H : B_R \to B_R$.

We prove that $H : B_R \to B_R$ is a strict contraction. Recall the definition of $T$ in the statement of the theorem. Consider

$$\|H(u_1) - H(u_2)\|_X \leq \left\| \int_0^t \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2t}} |u_1|^{p-1} u_1(y, s) - |u_2|^{p-1} u_2(y, s) | \, dy \, ds \right\|_X.$$  

Using that $u_1, u_2 \in B_R$, we obtain the estimate $\|u_1|^{p-1} u_2 - |u_2|^{p-1} u_2| \leq p|u_1 - u_2|^p$. Thus,

$$\|H(u_1) - H(u_2)\|_X \leq p \sup_{[0, T]} \sup_{|x| < R} \int_0^t \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2t}} |u_1 - u_2| \, dy \, ds \|u_1 - u_2\|^p_{X R^{p-1}} \leq p \|u_1 - u_2\|^p_{X R^{p-1}}.$$  

Therefore, if $T < \frac{1}{2} \min\{\left(\frac{p}{p-1}\right)\|u_0\|_{\infty}^{p-1}, 1\}$, then $H$ is a strict contraction in $B_R$.

Substituting the choice $T = \frac{1}{2} \min\{\left(\frac{p}{p-1}\right)\|u_0\|_{\infty}^{p-1}, 1\}$ into the expression for $R$ completes the proof of existence and uniqueness of $u$ and the estimate on it.

It remains to prove that solution to the initial value problem is continuous with respect to changes in the initial condition $u_0$. Let $u$ and $v$ be the solutions with initial conditions $u_0$ and $v_0$. We estimate

$$\|u - v\|_X \leq \left\| e^{t\partial^2_x}(u_0 - v_0) \right\|_X + \left\| \int_0^t e^{(t-s)\partial^2_x} (u^{p}(s) - v^{p}(s)) \, ds \right\|_X.$$  

The estimate of these terms proceeds as above (take $u_1 = u$ and $u_2 = v$) and if $u, v \in B_R$, then

$$\|u - v\|_X \leq \|u_0 - v_0\|_{\infty} + \frac{1}{2} \|u - v\|_X.$$  

Thus, if $T$ is as above, then $\|u - v\|_X \leq 2\|u_0 - v_0\|_{\infty}$ completing the proof of continuity.

Finally, assume $[0, t_\ast)$ is the maximal interval of existence of $u$ and

$$\sup_{0 \leq t \leq t_\ast} \|u(t)\|_{\infty} = M < \infty.$$  

Let $T = \frac{1}{2} \min\{\left(\frac{p}{p}M^{p-1}\right)^{-1}, 1\}$. Then taking $u(t_\ast - \frac{1}{2}T)$ as a new initial condition, we see that the solution exists in the interval $[0, t_\ast + \frac{1}{2}T)$, a contradiction. This proves the dichotomy claimed in the theorem. □
2.2 A Blow-Up Test

In this section we present a blowup test for $L^\infty$ norm. This criterion gives many blowup examples for (1.1) using the Lyapunov functional $S(w)$ defined in (1.9) and it is sharp for initial data close to a constant solution in $L^\infty$ norm. Finally, we mention some regularity properties of the solutions.

Let $T > 0$ be any finite number and $w(y,s) = (T-t)^{1/p-1} u(x,t)$ with $x = \sqrt{T-t} \ y$ and $T-t = e^{-s}$. Then $S(w)(-\ln T) = S_T(u)$, where

$$S_T(u) = \frac{1}{2} \int \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) \rho(x) \ dx + \frac{1}{2} \frac{1}{p-1} T^{-\frac{1}{p+1}} \int |u|^2 \rho(x) \ dx$$

and $\rho(y) = e^{-\frac{1}{4}y^2}$.

**Theorem 6.** Let the initial condition $u_0$ satisfy $S_T(u_0) < 0$, modulo a shift, for some $T > 0$. Then (1.1) blows up in a finite time $t^* \leq T$.

**Proof.** Assume (1.1) has a global solution $u$, for an initial condition $u_0$ as in the theorem. Let $w$ be as defined in the paragraph preceding the theorem with $T$ as in the theorem. The time derivative of the functional $I(w) = \frac{1}{2} \int_{-\infty}^{\infty} w^2(y,s) \rho(y) \ dy$ along solutions to (1.8) is

$$\frac{d}{ds} I(w) = -2S(w) + \frac{p-1}{p+1} \int_{-\infty}^{\infty} |w|^{p+1} \rho \ dy.$$

We use Hölder’s inequality to obtain the estimate

$$\int_{-\infty}^{\infty} |w|^2 \rho \ dy \leq (4\pi)^{1/p+1} \left( \int_{-\infty}^{\infty} |w|^{p+1} \rho \right)^{\frac{2}{p+1}}.$$

This and the fact that $S$ is monotonically decreasing (see (1.10)) result in the inequality

$$\frac{d}{ds} I(w) \geq -2S(w_0) + \frac{p-1}{p+1} (4\pi)^{1/p} I(w)^{\frac{p+1}{p}} I(w)^{\frac{p+1}{p}}.$$

and hence if $S(w_0)$ is negative, $I(w)$ blows up in finite time and therefore so does $w$. This contradicts our assumption that $u$ and, consequently, $w$ exist globally. To complete the proof, we write $S(w_0)$ in terms of $S_T(u_0)$. 

Finally we point out that $u \in C^{2+\beta}$ for some $0 < \beta < 1$ due to parabolic $L^p$ estimate and Schauder’s estimate.
CHAPTER 3

FORMULATION OF THE PROBLEM

3.1 Blow-Up Variables and Almost Solutions

In this section we pass from the original variables $x$ and $t$ to the blowup variables $y = \lambda(t)(x - x_0(t))$ and $\tau = \int_0^t \lambda^2(s) \, ds$. The point here is that we do not fix $\lambda(t)$ and $x_0(t)$ but consider them as free parameters to be found from the evolution of (1.1). Assume for simplicity that 0 is a maximum point of $u_0$ and that $u_0$ is even with respect to $x = 0$. In this case $x_0$ can be taken to be 0. Suppose $u(x,t)$ is a solution to (1.1) with an initial condition $u_0(x)$. We define the new function

$$v(y, \tau) = \lambda^{-\frac{2}{p-1}}(t)u(x,t)$$

with $y = \lambda(t)x$ and $\tau = \int_0^t \lambda^2(s) \, ds$. The function $v$ satisfies the equation

$$v_\tau = \left( \partial_y^2 - a y \partial_y - \frac{2a}{p-1} \right) v + |v|^{p-1}v.$$  \hspace{1cm} (3.2)

where $a = \lambda^{-3} \partial_t \lambda$. The initial condition is $v(y,0) = \lambda_0 u_0(y/\lambda_0)$, where $\lambda_0$ is an initial condition for the scaling parameter $\lambda$.

If the parameter $a$ is a constant, then (3.2) has the following homogeneous, static (i.e. $y$ and $\tau$-independent) solutions

$$v_a = \left( \frac{2a}{p-1} \right)^{\frac{1}{p-1}}.$$  \hspace{1cm} (3.3)

In the original variables $t$ and $x$, this family of solutions corresponds to the homogeneous solution (1.3) of the nonlinear heat equation with the parabolic scaling
\( \lambda^{-2} = 2a(T - t) \), where the blowup time, \( T = [u_0^{p-1}(p - 1)]^{-1} \), is dependent on \( u_0 \), the initial value of the homogeneous solution \( u_{hom}(t) \).

If the parameter \( a \) is \( \tau \) dependent but \( |a_\tau| \) is small, then the above solutions are good approximations to the exact solutions. Another approximation is the solution of the equation \( ayv_y + \frac{2a}{p-1} v = v^p \), obtained from (3.2) by neglecting the \( \tau \) derivative and second order derivative in \( y \). This equation has the general solution

\[
v_{ab} = \left( \frac{2a}{p - 1 + by^2} \right)^{\frac{1}{p-1}} \tag{3.4}
\]

for all \( b \in \mathbb{R} \). (The above equation is equivalent to the equation \( \partial_y \left( y^{\frac{2}{p-1}} v \right) = \frac{1}{ay} \left( y^{\frac{2}{p-1}} v \right)^p \). In what follows we take \( b \geq 0 \) so that \( v_{ab} \) is nonsingular. Note that \( v_{0a} = v_a \).

To understand the emergence of the above scheme, we consider the following dilating which is called the renormalization group transformation [8],

\[
v_L(y, \tau) = v(Ly, L^2 \tau)
\]

and \( v_L \) satisfies the following equation,

\[
ay\partial_y v_L + \frac{2a}{p-1} v - |v|^{p-1} v = L^{-2} (-\partial_\tau v_L + v_{Lyy}). \tag{3.5}
\]

The asymptotical behavior of \( v \) at \( \tau = \infty \) is equivalent to the behavior of \( v_L \) when we first fix \( \tau \) and let \( L \) be sufficiently large. As \( L \to \infty \), the right side of (3.5) vanishes. In other words, \( v_L \), for \( L \) large, should be approximately equal to the solution of

\[
az\partial_z \psi + \frac{2a}{p-1} \psi - \psi^p = 0 \tag{3.6}
\]

which has a solution

\[
\psi(z) = \left( \frac{2a}{p - 1 + b^*z^2} \right)^{\frac{1}{p-1}} \tag{3.7}
\]
where \( b^* \geq 0 \) is a constant. By setting \( \tau = 1 \), \( L = \sqrt{\tau} \) and \( z = Ly \), \( v \) for \( \tau \) large is approximately equal to

\[
\psi_b(y, \tau) = \left( \frac{2a}{p - 1 + b^*y^2/\tau} \right)^{\frac{1}{p-1}}.
\]

Therefore, we expect that \( v_{ab} \) would be a better approximation to the asymptotical solutions of the nonlinear heat equations if \( b \) is almost zero.

### 3.2 "Gauge" Transform

We assume that the parameter \( a \) depends slowly on \( \tau \) and treat \( |a_\tau| \) as a small parameter in a perturbation theory for Equation (3.2). In order to convert the global non-self-adjoint operator \( ay\partial_y \) appearing in this equation into a more tractable local and self-adjoint operator we perform a gauge transform. Let

\[
w(y, \tau) = e^{-\frac{a^2}{4}y} v(y, \tau).
\]

(3.8)

Then we have

\[
w_\tau = \left( \partial_y^2 - \frac{1}{4} \omega^2 y^2 - \left( \frac{2}{p - 1} - \frac{1}{2} \right) a \right) w + e^{\frac{a}{4}(p-1)y^2} |w|^{p-1}w,
\]

(3.9)

where \( \omega^2 = a^2 + a_\tau \). The approximate solution \( v_{ab} \) to (3.2) transforms to \( v_{abc} \) where

\[
v_{abc} = v_{bc}e^{-\frac{a^2}{4}y}.
\]

(3.10)

Equation (3.9) is the \( L^2 \)-gradient system with the energy

\[
\mathcal{E}(w) = -\frac{1}{2} \int w \left( \partial_y^2 - \frac{1}{4} \omega^2 y^2 - \left( \frac{2}{p - 1} - \frac{1}{2} \right) a \right) w - \frac{1}{p + 1} e^{\frac{a}{4}(p-1)y^2} |w|^{p+1} dy.
\]

(3.11)

This energy is related to the functional (1.9). It satisfies the relation

\[
\partial_\tau \mathcal{E}(w)(\tau) = -\int |\partial_\tau w|^2 e^{-\frac{a}{4}y^2} dy.
\]

Indeed, multiplying (3.9) by \( w_\tau \), integrating over space and then using that the linear operator in (3.9) is self-adjoint gives this relation.
3.3 Reparametrization of Solutions

In this section we split the solution of (3.9) into a leading term - the almost solution $v_{abc}$ - and a fluctuation $\xi$ around it. More precisely, we would like to parameterize the solution by a point on the manifold $M_{as} = \{v_{abc} \mid a, b, c \in \mathbb{R}^+, b \leq \epsilon, a = a(b, c)\}$ of almost solutions and the fluctuation orthogonal to this manifold (large slow moving and small fast moving parts of the solution). Here $a = a(b, c)$ is a twice differentiable function of $b$ and $c$. For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold $M_{as}$ as

$$a = a(b, c)$$

whic is the almost tangent vectors to the above manifold, provided $b$ is sufficiently small. Note that $\xi$ is already orthogonal to $\phi_{1a} = \sqrt{ay}e^{-\frac{ay}{2}y^2}$ since our initial value is even, and therefore, the solution is even in $x$.

In this section and the rest of the paper except Appendix A we fix the relation between the parameters $a$, $b$ and $c$ as

$$2c = a + \frac{1}{2}.$$ 

In Appendix A we prove that under some conditions different functions of $a = a(c, b)$ can be used.

Let $V_{ab} = \left(\frac{2c}{p-1+by^2}\right)^{\frac{1}{p-1}}$ with $c = \frac{1}{2}a + \frac{1}{4}$. We define a neighborhood of $M_{as}$:

$$U_{\epsilon_0} = \{v \in L^\infty(\mathbb{R}) \mid \|e^{-\frac{ay}{2}y^2}(v - V_{a_0b_0})\|_\infty = o(b_0) \text{ for some } a_0 \in [1/4, 1], b_0 \in [0, \epsilon_0]\}.$$ 

**Proposition 1.** There exist an $\epsilon_0 > 0$ and a unique $C^1$ functional $g : U_{\epsilon_0} \to \mathbb{R}^+ \times \mathbb{R}^+$, such that any function $v \in U_{\epsilon_0}$ can be uniquely written in the form

$$v = V_{g(v)} + \eta,$$ 

(3.12)

with $\eta \perp e^{-\frac{ay}{2}y^2}\phi_{0a}$, $e^{-\frac{ay}{2}y^2}\phi_{2a}$ in $L^2(\mathbb{R})$, $g(v) = (a, b)$. Moreover, for all $(a_0, b_0) \in \left[\frac{1}{2}, 1\right] \times [0, \epsilon_0]$ and $v \in U_{\epsilon_0}$,

$$|g(v) - (a_0, b_0)| \lesssim \|e^{-\frac{ay}{2}y^2}(v - V_{a_0b_0})\|_\infty.$$ 

(3.13)
Proof. The orthogonality conditions on the fluctuation are equivalent to $G(\mu, v) = 0$, where $\mu = (a, b)$ and $G : [\frac{1}{4}, 1] \times (0, \infty) \times L^2(\mathbb{R}) \rightarrow \mathbb{R}^2$ is defined as

$$G(\mu, v) = \begin{pmatrix}
\langle V_\mu - v, e^{-\frac{a^2}{4}} \phi_{0a} \rangle \\
\langle V_\mu - v, e^{-\frac{a^2}{4}} \phi_{2a} \rangle
\end{pmatrix},$$

where and also in what follows, all inner products $\langle \cdot, \cdot \rangle$ are the $L^2$ inner products. Let $X = e^{\frac{1}{2}y^2} L^\infty$ with the corresponding norm. Using the implicit function theorem we will prove that for any $\mu = (a_0, b_0) \in \left[\frac{1}{4}, 1\right] \times (0, \epsilon_0]$ there exists a unique $C^1$ function $g : X \rightarrow \Lambda$ defined in a neighborhood $U_{\mu_0} \subset X$ of $V_{\mu_0}$ such that $G(g(v), v) = 0$ for all $v \in U_{\mu_0}$. Let $B_\epsilon(V_{\mu_0})$ and $B_\delta(\mu_0)$ be the balls in $X$ and $\mathbb{R}^2$ around $V_{\mu_0}$ and $\mu_0$ and of the radii $\epsilon$ and $\delta$, respectively.

Note first that the mapping $G$ is $C^1$ and $G(\mu_0, V_{\mu_0}) = 0$ for all $\mu_0$. We claim that the linear map $\partial_\mu G(\mu_0, V_{\mu_0})$ is invertible. Indeed, we compute

$$\partial_\mu G(\mu, v) = A_1 + A_2$$

where

$$A_1 = \begin{pmatrix}
\langle \partial_a V_\mu, e^{-\frac{a^2}{4}} y^2 \rangle & \langle \partial_b V_\mu, e^{-\frac{a^2}{4}} y^2 \rangle \\
\langle \partial_a V_\mu, (1 - ay^2) e^{-\frac{a^2}{4}} y^2 \rangle & \langle \partial_b V_\mu, (1 - ay^2) e^{-\frac{a^2}{4}} y^2 \rangle
\end{pmatrix},$$

and

$$A_2 = -\frac{1}{4} \begin{pmatrix}
\langle V_\mu - v, y^2 e^{-\frac{a^2}{4}} y^2 \rangle & 0 \\
\langle V_\mu - v, (1 - ay^2) y^2 e^{-\frac{a^2}{4}} y^2 \rangle & 0
\end{pmatrix}.$$ 

For $b > 0$ and small, we expand the matrix $A_1$ in $b$ to get $A_1 = G_1 G_2 + o(b)$, where the matrices $G_1$ and $G_2$ are defined as

$$G_1 = \begin{pmatrix}
-\frac{y^2 e^{-\frac{a^2}{4}} y^2}{a + \frac{1}{2}} e^{-\frac{a^2}{4}} y^2 & \frac{1}{a + \frac{1}{2}} (e^{-\frac{a^2}{4}} y^2, e^{-\frac{a^2}{4}} y^2) \\
-\frac{y^2 e^{-\frac{a^2}{4}} y^2}{1 - ay^2} (1 - ay^2) e^{-\frac{a^2}{4}} y^2 & 0
\end{pmatrix},$$

and

$$G_2 = \left(\frac{a + 1/2}{p - 1}\right)^{\frac{1}{p-1}} \frac{1}{p - 1} \begin{pmatrix}
\frac{p^{-1}}{4} & 1 \\
1 & 0
\end{pmatrix}.$$ 

Obviously the matrices $G_1$ and $G_2$ have uniformly (in $a \in \left[\frac{1}{4}, 1\right]$) bounded inverses. Furthermore, by the Schwarz inequality

$$\|A_2\| \lesssim \|v - V_{a_0 b_0}\|_X.$$ 

Therefore there exist $\epsilon_0$ and $\epsilon_1$ s.t. the matrix $\partial_\mu G(\mu, v)$ has a uniformly bounded inverse for $\mu \in [\frac{1}{4}, 1] \times [0, \epsilon_0]$ and $v \in \bigcup_{\mu \in [\frac{1}{4}, 1] \times [0, \epsilon_0]} B_\epsilon(V_\mu)$. Hence by the implicit function theorem, the equation $G(\mu, v) = 0$ has a unique solution $\mu = g(v)$ on a neighborhood of every $V_\mu, \mu \in [\frac{1}{4}, 1] \times [0, \epsilon_0]$, which is $C^1$ in $v$. Our next goal is to determine these neighborhoods.
To determine a domain of the function $\mu = g(v)$, we examine closely a proof of the implicit function theorem. Proceeding in a standard way, we expand the function $G(\mu, v)$ in $\mu$ around $\mu_0$:

$$ G(\mu, v) = G(\mu_0, v) + \partial_\mu G(\mu_0, v)(\mu - \mu_0) + R(\mu, v), $$

where $R(\mu, v) = O(1)$ uniformly in $v \in X$. Here $|\mu|^2 = |a|^2 + |b|^2$ for $\mu = (a, b)$. Inserting this into the equation $G(\mu, v) = 0$ and inverting the matrix $\partial_\mu G(\mu_0, v)$, we arrive at the fixed point problem $\alpha = \Phi_v(\alpha)$, where $\alpha = \mu - \mu_0$ and $\Phi_v(\alpha) = -\partial_\mu G(\mu_0, v)^{-1}[G(\mu_0, v) + R(\mu, v)]$. By the above estimates there exists an $\varepsilon_1$ such that the matrix $\partial_\mu G(\mu_0, v)^{-1}$ is bounded uniformly in $v \in B_{\varepsilon_1}(V_{\mu_0})$. Hence we obtain from the remainder estimate above that

$$ |\Phi_v(\alpha)| \lesssim |G(\mu_0, v)| + |\alpha|^2. \quad (3.15) $$

Furthermore, using that $\partial_\alpha \Phi_v(\alpha) = -\partial_\mu G(\mu_0, v)^{-1}[G(\mu, v) - G(\mu_0, v)]$ we obtain that there exist $\varepsilon \leq \varepsilon_1$ and $\delta$ such that $\|\partial_\alpha \Phi_v(\alpha)\| \leq \frac{1}{2}$ for all $v \in B_\varepsilon(V_{\mu_0})$ and $\alpha \in B_\delta(0)$. Pick $\varepsilon$ and $\delta$ so that $\varepsilon \ll \delta \ll b_0 \ll 1$. Then, for all $v \in B_\varepsilon(V_{\mu_0})$, $\Phi_v$ is a contraction on the ball $B_\varepsilon(0)$ and consequently has a unique fixed point in this ball. This gives a $C^1$ function $\mu = g(v)$ on $B_\varepsilon(V_{\mu_0})$ satisfying $|\mu - \mu_0| \leq \delta$. An important point here is that since $\varepsilon \ll b_0$ we have that $b > 0$ for all $V_{ab} \in B_\varepsilon(V_{\mu_0})$. Now, clearly, the balls $B_\varepsilon(V_{\mu_0})$ with $\mu_0 \in [\frac{1}{2}, 1] \times [0, \varepsilon_0]$ cover the neighbourhood $U_{\varepsilon_0}$. Hence, the map $g$ is defined on $U_{\varepsilon_0}$ and is unique, which implies the first part of the proposition.

Now we prove the second part of the proposition. The definition of the function $G(\mu, v)$ implies $G(\mu_0, v) = G(\mu_0, v - V_{\mu_0})$ and

$$ |G(\mu_0, v)| \lesssim \|e^{-\frac{1}{2}g^2}(v - V_{\mu_0})\|_\infty. \quad (3.16) $$

This inequality together with the estimate (3.15) and the fixed point equation $\alpha = \Phi_v(\alpha)$, where $\alpha = \mu - \mu_0$ and $\mu = g(v)$, implies $|\alpha| \lesssim \|e^{-\frac{1}{2}g^2}(v - V_{\mu_0})\|_\infty + |\alpha|^2$ which, in turn, yields (3.13).

\[ \square \]

**Proposition 7.** In the notation of Proposition 1, if $\|\langle y \rangle^{-n}(v - V_{a_0b_0})\|_\infty \leq \delta_n$ with $n = 0, 3, \delta_3 = O(b_0^2)$ and $\delta_0$ small, then

$$ |g(v) - (a_0, b_0)| \lesssim b_0^2, \quad (3.17) $$

$$ \|\langle y \rangle^{-3}(v - V_{g(v)})\|_\infty \lesssim \|\langle y \rangle^{-3}(v - V_{a_0b_0})\|_\infty \quad (3.18) $$

and

$$ \|v - V_{g(v)}\|_\infty \lesssim \delta_0 + b_0. \quad (3.19) $$

**Proof.** Let $g(v) = (a, b)$ and $\mu = (a_0, b_0)$. By (3.15) and the fixed point equation $\alpha = \Phi_v(\alpha)$, we have $\alpha \lesssim |G(\mu_0, v)| + |\alpha|^2$ which, in turn, yields $|\mu - \mu_0| \lesssim |G(\mu_0, v)|.$

\[ 18 \]
By (3.16) and one of the conditions of the proposition, \( G(\mu_0, v) = O(b_0^2) \) if \( a_0 \in \left[ \frac{1}{4}, 1 \right] \). The last two estimates imply (3.17). Using Equation (3.13) we obtain
\[
\| \langle y \rangle^{-3}(v - V_{g(v)}) \|_{\infty} \leq \| \langle y \rangle^{-3}(v - V_{\mu_0}) \|_{\infty} + \| \langle y \rangle^{-3}(V_{g(v)} - V_{\mu_0}) \|_{\infty}
\]
\[
\leq \| \langle y \rangle^{-3}(v - V_{\mu_0}) \|_{\infty} + |g(v) - \mu_0|
\]
which is (3.18). Finally, to prove Equation (3.19), we write
\[
\| v - V_{g(v)} \|_{\infty} \leq \| v - V_{a_0,b_0} \|_{\infty} + \| V_{g(v)} - V_{a_0,b_0} \|_{\infty}.
\]
A straightforward computation gives \( \| V_{ab} - V_{a_0,b_0} \|_{\infty} \leq |a - a_0| + \frac{|b - b_0|}{b_0} \). Since by (3.17), \( |a - a_0| + |b - b_0| = O(b_0^2) \), we have \( \| V_{ab} - V_{a_0,b_0} \|_{\infty} \leq b_0 \). This together with the fact \( \| v - V_{a_0,b_0} \|_{\infty} \leq \delta_0 \) completes the proof of (3.19).
4.1 Statement of A Priori Estimates

Because of the scaling invariance of the equation (1.1), we break the symmetry by choosing a fixed gauge. We prove our results for this particular gauge and scale it back to get the general cases. The next lemma uses a scale which makes the asymptotic profile in the form of \( \left( \frac{1}{p-1+\frac{2}{p}x^2} \right)^{\frac{1}{p-1}} \).

**Lemma 8.** If \( u_0 \) satisfies the condition (1.13), then there exists some scalar \( k_0 > 0 \) such that

\[
\|k_0^{\frac{2}{p-1}} u_0(k_0 x) - \left( \frac{1 - \frac{2}{p-1}\beta_0}{p - 1 + \beta_0 x^2} \right)^{\frac{1}{p-1}} \|_\infty \leq \delta_0 \quad (4.1)
\]

and

\[
\|\langle x \rangle^{-3} (k_0^{\frac{2}{p-1}} u_0(k_0 x) - \left( \frac{1 - \frac{2}{p-1}\beta_0}{p - 1 + \beta_0 x^2} \right)^{\frac{1}{p-1}}) \|_\infty \lesssim \beta_0^2. \quad (4.2)
\]

for some \( \beta_0 > 0 \).

**Proof.** Define \( k_0 = \frac{1}{\sqrt{c_0 + \frac{2}{p-1} \beta_0}} \) and \( \beta_0 = b_0 k_0^2 \). It is straightforward to verify that the function \( k_0^{\frac{2}{p-1}} u_0(k_0 x) \) has all the properties above. \( \square \)

So we only focus on the initial data satisfying the following conditions,

\[
\|u_0(x) - \left( \frac{1 - \frac{2}{p-1}b_0}{p - 1 + b_0 x^2} \right)^{\frac{1}{p-1}} \|_\infty \leq \delta_0 \text{ and } \|\langle x \rangle^{-3} (u_0(x) - \left( \frac{1 - \frac{2}{p-1}b_0}{p - 1 + b_0 x^2} \right)^{\frac{1}{p-1}}) \|_\infty \lesssim b_0^2. \quad (4.3)
\]

we assume that (1.1) has a unique solution, \( u(x,t) \), \( 0 \leq t \leq t_* \), such that \( v(y,\tau) = \lambda^{-\frac{2}{p-1}}(t)u(x,t) \) is in the neighborhood \( U_{\epsilon_0} \) determined in Proposition 1. Then
by Proposition 1 there exist $C^1$ functions $a(\tau)$ and $b(\tau)$ such that $v(y, \tau)$ can be represented as
\[
v(y, \tau) = \left( \frac{2c}{p - 1 + by^2} \right)^{\frac{1}{p-1}} + e^{\frac{ay^2}{4}} \xi(y, \tau)
\] (4.4)
where $\xi(\cdot, \tau) \perp \phi_0, \phi_2$ (see (3.12)), and $c = \frac{1}{2}a + \frac{1}{4} \lambda$ (and $\|\langle y \rangle^{-3}\xi\|_\infty \lesssim b(\tau)^2$). Now we set
\[
\lambda^{-3}(t) \partial_t \lambda(t) = a(\tau(t)).
\]
In this section we present a priori bounds on the fluctuation $\xi$ which are proved in later sections.

We begin with defining convenient estimating functions. Denote by $\chi_{\geq D}$ and $\chi_{\leq D}$ the characteristic functions of the sets $\{|x| \geq D\}$ and $\{|x| \leq D\}$:
\[
\chi_{\geq D}(x) = \begin{cases} 1 \text{ if } |x| \geq D \\ 0 \text{ otherwise} \end{cases} \quad \text{and} \quad \chi_{\leq D} = 1 - \chi_{\geq D}.
\] (4.5)
We take $D = \frac{C}{\sqrt{\beta}}$ where $C$ is a large constant to be specified in Section 4.7. Let the function $\beta(\tau)$ and the constant $\kappa$ be defined as
\[
\beta(\tau) = \frac{1}{b(0) + \frac{4p}{p-1} \tau^2} \quad \text{and} \quad \kappa = \min\left\{\frac{1}{2}, \frac{p-1}{2}\right\}.
\] (4.6)
For the functions $\xi(\tau), b(\tau)$ and $a(\tau)$ we introduce the following estimating functions (families of semi-norms)
\[
M_1(T) = \max_{\tau \leq T} \beta^{-2}(\tau) \|\langle y \rangle^{-3} e^{\frac{ay^2}{4}} \xi(\tau)\|_\infty,
\] (4.7)
\[
M_2(T) = \max_{\tau \leq T} \|e^{\frac{ay^2}{4}} \chi_{\geq D} \xi(\tau)\|_\infty,
\] (4.8)
\[
A(T) = \max_{\tau \leq T} \beta^{-2}(\tau) \left| a(\tau) - \frac{1}{2} + \frac{2b(\tau)}{p-1} \right|,
\] (4.9)
\[
B(T) = \max_{\tau \leq T} \beta^{-(1+\kappa)}(\tau) |b(\tau) - \beta(\tau)|.
\] (4.10)

**Proposition 9.** Let $\xi$ be defined in (4.4) and assume $M_1(0), M_2(0), A(0) \lesssim 1$. Assume there exists an interval $[0, T]$ such that for $\tau \in [0, T]$,
\[
A(\tau), B(\tau) \leq \beta^{-\kappa/2}(\tau).
\]
Then in the same time interval the parameters \( a, b \) and the function \( \xi \) satisfy the following estimates

\[
\left| b_\tau(\tau) + \frac{4p}{(p-1)^2}b^2(\tau) \right| \lesssim \beta^3(\tau) + \beta^3(\tau)M_1(\tau)(1 + A(\tau)) + \beta^4(\tau)M_1^2(\tau) + \beta^{2p}M_1^p(\tau), \tag{4.11}
\]

and

\[
B(\tau) \lesssim \beta(0)^{\min\left(\frac{3(p-1)}{2}, 1-\kappa\right)} \left[ 1 + M_1(\tau)(1 + A(\tau)) + M_1^2(\tau) + M_1^p(\tau) \right], \tag{4.12}
\]

\[
A(\tau) \lesssim A(0) + 1 + \beta(0)M_1(\tau)(1 + A(\tau)) + \beta(0)M_1^2(\tau) + \beta^{2p-2}(0)M_1^p(\tau), \tag{4.13}
\]

\[
M_1(\tau) \lesssim M_1(0) + \beta^2(0)[1 + M_1(\tau)A(\tau) + M_1^2(\tau) + M_1^p(\tau)] + [M_2(\tau)M_1(\tau) + M_1(\tau)M_2^{p-1}(\tau)], \tag{4.14}
\]

\[
M_2(\tau) \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + M_2^2(\tau) + M_2^p(\tau) + \beta^{2/3}(0)[1 + M_2(\tau) + M_1(\tau)A(\tau) + M_1^2(\tau) + M_1^p(\tau)]. \tag{4.15}
\]

Equations (4.11)-(4.13) will be proved in Section 4.3. Equations (4.14) and (4.15) will be proved in Sections 4.6 and 4.7 respectively.

### 4.2 Lyapunov-Schmidt Splitting (Effective Equations)

In this section we derive the equations for the parameters \( a(\tau), b(\tau) \) and \( c(\tau) \) and the fluctuation \( \xi(y, \tau) \).

According to Proposition 1 the solution \( w(y, \tau) \) of (3.9) can be decomposed as (4.4), with the parameters \( a, b \) and \( c \) and the fluctuation \( \xi \) depending on time \( \tau \):

\[
w = v_{abc} + \xi, \quad \xi \perp \phi_{0,a}, \quad \psi_{2a}, \tag{4.16}
\]

where \( v_{abc} = v_{bc}e^{-\frac{2}{a^2}y^2} \) and \( c = \frac{1}{2}a + \frac{1}{4} \). Plugging the decomposition (4.16) into (3.9) gives the equation

\[
\xi_\tau = -\mathcal{L}_{abc}\xi + \mathcal{N}(\xi, a, b, c) + \mathcal{F}(a, b, c), \tag{4.17}
\]
where $\mathcal{L}_{abc}, \mathcal{N}(\xi, a, b, c)$ and $\mathcal{F}(a, b, c)$ are defined as

$$
\mathcal{L}_{abc} = -\partial_y^2 + \frac{1}{4} (a^2 + ar)y^2 - \frac{a}{2} + \frac{2a}{p-1} - \frac{2pc}{p-1 + by^2},
$$

(4.18)

$$
\mathcal{N}(\xi, b, c) = \left[|\xi + v_{abc}|^{p-1}(\xi + v_{abc}) - v_{abc}^p - pv_{abc}^{p-1}\xi\right] e^{\frac{a}{2}(p-1)y^2},
$$

(4.19)

$$
\mathcal{F}(a, b, c) = \frac{1}{p-1} \left[\Gamma_0 + \Gamma_1 + \Gamma_1 p(ay^2 - 1) - \frac{4pb^3y^4}{(p-1)^2(p-1 + by^2)^2}\right] v_{abc},
$$

(4.20)

with

$$
\Gamma_0 = -\frac{cr}{c} + 2(c - a) - \frac{2}{p-1} b,
$$

(4.21)

$$
\Gamma_1 = \frac{1}{a(p-1)} \left( b_r - 2b(c - a) + \frac{2(3p - 1)b^2}{(p-1)^2}\right).
$$

(4.22)

**Proposition 10.** Assume $A(\tau), B(\tau) \leq \beta^{-\frac{\tau}{2}}(\tau)$ and $1/4 \leq c(0) \leq 1$, then

$$
\|\langle y\rangle^{-3} e^{\frac{a}{2}y^2} \mathcal{F}\|_{\infty} = O\left(|\Gamma_0| + |\Gamma_1| + \beta^2\right) \text{ and } \|e^{\frac{a}{2}y^2} \mathcal{F}\|_{\infty} = O\left(|\Gamma_0| + \frac{1}{\beta} |\Gamma_1| + \beta\right).
$$

(4.23)

**Proof.** Rearranging the leading terms in the expression for $\mathcal{F}$ so that $y^2$ appears in the form $ay^2 - 1$ gives the more convenient expression

$$
\mathcal{F} = \frac{1}{p-1} \left[\Gamma_0 + \Gamma_1 + \Gamma_1 (ay^2 - 1) - \Gamma_1 \frac{aby^4}{p-1 + by^2} + G_1\right] v_{abc}
$$

(4.24)

with $G_1 = -\frac{4pb^3y^4}{(p-1)^2(p-1 + by^2)^2}$. Using this form of $\mathcal{F}$ and the estimates

$$
\|\langle y\rangle^{-3} e^{\frac{a}{2}y^2} v_{abc}\|_{\infty}, \|\langle y\rangle^{-3} e^{\frac{a}{2}y^2} (ay^2 - 1)v_{abc}\|_{\infty}, \|e^{\frac{a}{2}y^2} v_{abc}\|_{\infty} \lesssim 1,
$$

we conclude

$$
\|\langle y\rangle^{-3} e^{\frac{a}{2}y^2} \mathcal{F}\|_{\infty} \lesssim |\Gamma_0| + (1 + b^\frac{1}{2}) |\Gamma_1| + b^{\frac{5}{2}}.
$$

(4.25)

The estimate of $\|e^{\frac{a}{2}y^2} \mathcal{F}\|$ can be proved similarly. Recall the expression of $\mathcal{F}$ in Equation (4.20). We use the estimates

$$
\left\|e^{\frac{a}{2}y^2} v_{abc}\right\|_{\infty}, \left\|e^{\frac{a}{2}y^2} \frac{1}{p-1 + by^2} v_{abc}\right\|_{\infty}, \left\|e^{\frac{a}{2}y^2} \frac{by^2}{(p-1 + by^2)^2} v_{abc}\right\|_{\infty} \lesssim 1,
$$

thus

$$
\left\|e^{\frac{a}{2}y^2} \frac{y^2}{p-1 + by^2} v_{abc}\right\|_{\infty} \lesssim \frac{1}{b}.
$$
So we arrive at
\[ \|e^{\frac{-y^2}{2}}F\|_\infty \lesssim |\Gamma_0| + \frac{1}{b}|\Gamma_1| + b. \tag{4.26} \]

Lastly, we estimate \( b \) in terms of \( \beta \) and \( B \) to complete the proof of the first bound. The assumption that \( B \leq \beta^{-\frac{2}{7}} \) implies that \( b = \beta + O\left(\beta^{1+\frac{2}{7}}\right) \), which together with estimates (4.25) and (4.26), implies the first estimate (4.23). \( \square \)

**Proposition 11.** Suppose that \( A(\tau), B(\tau), M_1(\tau), M_2(\tau) \leq \beta^{-\frac{2}{7}} \) and \( 1/4 \leq c(0) \leq 1 \) for \( 0 \leq \tau \leq T \). Let \( w = v_{abc} + \xi \) be a solution to (3.9) with \( \xi \perp \phi_{0a}, \phi_{2a} \). Recall that \( a = 2c - \frac{1}{2} \). Over times \( 0 \leq \tau \leq T \), the parameters \( b \) and \( c \) satisfy
\[ b_{\tau} = -\frac{2(3p-1)}{(p-1)^2} b^2 + 2b(c-a) + R_b(\xi, b, c), \tag{4.27} \]
\[ c_{\tau} = 2(c-a) - \frac{2}{p-1} b + R_c(\xi, b, c), \tag{4.28} \]
where the remainders \( R_b \) and \( R_c \) are of the order
\[ O(\beta^3 + \beta^3 M_1(1 + A) + \beta^4 M_1^2 + \beta^5 M_1^p) \] and satisfy \( R_b(0, b, c), R_c(0, b, c) = O(b^3) \).

**Proof.** We take inner product of the equation (4.17) with \( \phi_{ja} \) to get
\[ \langle \xi_{\tau}, \phi_{ja} \rangle = -\langle \mathcal{L}_{abc} \xi + N(\xi, a, b, c) + F(a, b, c), \phi_{ja} \rangle. \]
We use the orthogonality conditions \( \phi_{ja} \perp \xi \) to derive (4.27) (4.28). We start with analyzing the \( F \) term. The inner product of (4.24) with \( \phi_{0a} \) and \( \phi_{2a} \) gives the expression
\[ (p-1) \langle F, \phi_{ia} \rangle = (\Gamma_0 + \Gamma_1) \langle v_{abc}, \phi_{ia} \rangle + \Gamma_1 \langle v_{abc}, (ay^2 - 1)\phi_{ia} \rangle \]
\[ -\Gamma_1 \left( \frac{aby^4}{p-1 + by^2} v_{abc}, \phi_{ia} \right) + \langle G_1 v_{abc}, \phi_{ia} \rangle \tag{4.29} \]
where \( i = 0 \) or \( 2 \). By rescaling the variable of integration so that the exponential term does not contain the parameter \( a \), expanding \( v_{abc} \) to the constant term in \( \frac{b}{a} \) and estimating the remainder by \( O\left(a^{-\frac{3}{2}}by^2e^{-y^2/2}\right) \) we obtain the following estimates
\[ \langle v_{abc}, \phi_{0a} \rangle = \left( \frac{2c}{p-1} \right)^{-1} \sqrt{\frac{2\pi}{a}} + O\left(b\right), \]
\[ \langle v_{abc}, \phi_{2a} \rangle = O\left(b\right), \]
\[ \langle v_{abc}, (ay^2 - 1)\phi_{2a} \rangle = \left( \frac{2c}{p-1} \right)^{-1} \sqrt{\frac{8\pi}{a}} + O\left(b\right), \]
\[ \langle v_{abc}, (y)^3 \phi_{0a} \rangle, \langle v_{abc}, (y)^3 \phi_{2a} \rangle \lesssim 1. \]
Substituting these estimates into Equations (4.29) we have

\begin{align}
\langle \mathcal{F}, \phi_{0a} \rangle &= \frac{1}{p-1} \left( \frac{2c}{p-1} \right)^{1/p-1} \sqrt{\frac{2\pi}{a}} (\Gamma_0 + \Gamma_1) + R_1, \\
\langle \mathcal{F}, \phi_{2a} \rangle &= \frac{1}{p-1} \left( \frac{2c}{p-1} \right)^{1/p-1} \sqrt{\frac{8\pi}{a}} \Gamma_1 + R_2,
\end{align}

where both remainders $R_1$ and $R_2$ are bounded by $O(b|\Gamma_0| + b|\Gamma_1| + b^3)$.

To estimate the projection of $\partial_\tau \xi$ onto $\phi_{0a}$ and $\phi_{2a}$, we differentiate the orthogonality conditions $\langle \xi, \phi_{0a} \rangle = 0$ and $\langle \xi, \phi_{2a} \rangle = 0$, obtaining the relations $\langle \xi_\tau, \phi_{0a} \rangle = -\langle \xi, \partial_\tau \phi_{0a} \rangle$ and $\langle \xi_\tau, \phi_{2a} \rangle = -\langle \xi, \partial_\tau \phi_{2a} \rangle$. When simplified using the orthogonality conditions on $\xi$, these relations give

$$
\langle \xi_\tau, \phi_{0a} \rangle = 0 \quad \text{and} \quad |\langle \xi_\tau, \phi_{2a} \rangle| \leq \left| \frac{1}{4} a^{-1} a_\tau \left( \langle y \rangle^{-3} e^{\frac{2}{3} y^2} \xi, \alpha^2 \langle y \rangle^4 e^{-\frac{2}{3} y^2} \right) \right|.
$$

Applying Hölder’s inequality to the right hand side of the second inequality, we find that for any $\tau$, $0 \leq \tau \leq T$

$$
\langle \xi_\tau, \phi_{2a} \rangle = O \left( |a_\tau| \beta^2 M_1 \right).
$$

Next we replace $a_\tau$ with the expression involving $\Gamma_0$ in (4.21), then

$$
c_\tau = O \left( \Gamma_0 + \beta^2 A \right)
$$

for times $0 \leq \tau \leq T$. Note that $a_\tau = 2c_\tau$ and substitute the above estimate into $a_\tau$ to find,

$$
a_\tau = O \left( |\Gamma_0| + \beta^2 A \right)
$$

and hence

$$
\langle \xi_\tau, \phi_{2a} \rangle = O \left( \beta^2 M_1 (|\Gamma_0| + \beta^2 A) \right).
$$

We next move to estimate the terms involving the linear operator $\mathcal{L}_{abc}$. Write the operator $\mathcal{L}_{abc}$ as

$$
\mathcal{L}_{abc} = \mathcal{L}^* + \frac{1}{4} a_\tau y^2 - \frac{2pc}{p-1 + by^2},
$$

where $\mathcal{L}^*$ is self-adjoint and satisfies $\mathcal{L}^* \phi_{0a} = -2a_\phi_{0a}$ and $\mathcal{L}^* \phi_{2a} = 0$. Projecting $\mathcal{L}_{abc} \xi$ onto the eigenvectors $\phi_{0a}$ and $\phi_{2a}$ of $\mathcal{L}^*$ gives the equations

$$
\langle \mathcal{L}_{abc} \xi, \phi_{0a} \rangle = \frac{1}{4} a_\tau \left( \langle x, ay^2 e^{-\frac{2}{3} y^2} \rangle + \frac{2pc}{p-1} \langle x, \frac{by^2}{p-1 + by^2 e^{-\frac{2}{3} y^2}} \rangle \right)
= \frac{2pc}{p-1} \left( \langle x, \frac{by^2}{p-1 + by^2 e^{-\frac{2}{3} y^2}} \rangle \right),
$$

$$
\langle \mathcal{L}_{abc} \xi, \phi_{2a} \rangle = \frac{1}{4} a_\tau \left( \langle x, ay^2 (ay^2 - 1) e^{-\frac{2}{3} y^2} \rangle + \frac{2pc}{p-1} \langle x, \frac{(ay^2 - 1)by^2}{p-1 + by^2 e^{-\frac{2}{3} y^2}} \rangle \right).
$$
Hölder’s inequality again gives the following estimates
\[
\frac{M}{r_{abc}} |\langle \mathcal{L}_{abc} \xi, \phi_0 a \rangle| \lesssim b \| (y)^{-3} e^{x y^2} \|_\infty \\
\frac{M}{r_{abc}} |\langle \mathcal{L}_{abc} \xi, \phi_2 a \rangle| \lesssim |a_\tau| + b \| (y)^{-3} e^{x y^2} \|_\infty.
\]
In terms of the estimating functions $\beta$ and $M_1$, these estimates, after using the above estimate of $a_\tau$, become
\[
\langle \mathcal{L}_{abc} \xi, \phi_0 a \rangle \lesssim \beta^2 M_1 \\
\langle \mathcal{L}_{abc} \xi, \phi_2 a \rangle \lesssim \beta^2 M_1 (\beta + |\Gamma_0| + \beta^2 A).
\]
Lastly, we estimate the inner products involving the nonlinearity. Let $\mathcal{N} = \mathcal{N}(\xi, b, c)$. We claim that
\[
|\mathcal{N}| \lesssim e^{\frac{2n^2}{2}} |\xi|^2 + e^{(p-1)\frac{2}{p} y^2} |\xi|^p.
\] (4.35)
Indeed, if $v_{abc} \leq |\xi|$ we find $|\mathcal{N}| \leq e^{(p-1)\frac{2}{p} y^2} |\xi|^p$. If $v_{abc} \geq |\xi|$, then we use the formula $\mathcal{N} = e^{(p-1)\frac{2}{p} y^2} p \int_0^1 [(v_{abc} + s\xi)^r \frac{3}{p}] s \, ds$ and consider the cases $1 < p \leq 2$ and $p > 2$ separately to obtain (4.35). Due to (4.35), both $\langle \mathcal{N}, \phi_0 a \rangle$ and $\langle \mathcal{N}, \phi_2 a \rangle$ are bounded by $O\left(\| (y)^{-3} e^{x y^2} \xi \|^2_\infty + \| (y)^{-3} e^{x y^2} \xi \|^p_\infty\right)$. Writing them in terms of $\beta$ and $M_1$ and simplifying give the estimate
\[
|\langle \mathcal{N}, \phi_0 a \rangle| \lesssim \beta^4 M_1^2 + \beta^{2p} M_1^p.
\] (4.36)
Estimates (4.30), (4.31), (4.32), (4.33), (4.34) and (4.36) imply that $\Gamma_0 + \Gamma_1 = R_1$ and $\Gamma_2 = R_2$, where $R_1$ and $R_2$ are of the order
\[
O\left(\beta(|\Gamma_0| + |\Gamma_1|) + 2M_1 (\beta + |\Gamma_0| + \beta^2 A) + M_1^2 + M_1^p \right).
\]
By the facts that $\beta(\tau) \leq b_0 \ll 1$ and $A, M_1 \leq \beta^{-\frac{p}{2}}$, we obtain the estimate
\[
|\Gamma_0| + |\Gamma_1| \lesssim \beta^3 + \beta^3 M_1 (1 + A) + \beta^4 M_1^2 + \beta^{2p} M_1^p
\] (4.37)
for the times $0 \leq \tau \leq T$. 

Equations (4.23) and (4.37) yield the following corollary.

**Corollary 12.**
\[
\| (y)^{-n} e^{x y^2} \xi \|_\infty \lesssim \beta^{k_0} (\tau) [1 + M_1 (1 + A) + M_1^2 + M_1^p]
\] (4.38)
with $n = 0, 3$ and $k_0 = \min\{1, 2p - 1\}$, $k_3 = \min\{5/2, 2p\}$.

**Remark 3.** Equation (4.17) for the unknowns $a$, $b$, $c$ and $\xi$ is invariant under the transformation
\[
(a(\tau), b(\tau), c(\tau), \xi(\tau)) \mapsto (\mu a(\mu \tau), \mu^2 b(\mu \tau), \mu^2 c(\mu \tau), \mu^{\frac{3}{2}} \xi(\mu y, \mu^2 \tau)).
\]
This symmetry is related to the symmetry (1.2) of (1.1). Consequently, Equations (4.27) and (4.28) have the same symmetry.

**Remark 4.** Dynamical equations (4.27) and (4.28) have static solutions $(b, c, \xi) = (0, 0, 0)$ and $(b, c, \xi) = (0, a, 0)$ with $a$ a constant (the latter implies $a = \frac{1}{2}$).
4.3 Proof of Estimates (4.11)-(4.13)

In this section we will prove the inequalities for $A$ and $B$. Recall that $a = 2c - \frac{1}{2}$.

**Lemma 13.** Assume $B(\tau) \leq \beta^{-\frac{2}{p}}(\tau)$ for $\tau \in [0, T]$. Then (4.11) holds for all times $\tau \in [0, T]$.

**Proof.** We rewrite equation (4.27) as $b_\tau = -\frac{4p}{(p-1)^2}b^2 + b\left(\frac{1}{2} - a - \frac{2b}{p-1}\right) + \mathcal{R}_b$. By the definition of $A$, the second term on the right hand side is bounded by $b\beta^2 A \lesssim \beta^3 A$. Thus, using the bound for $\mathcal{R}_b$ given in Proposition 11, we obtain (4.11).

**Lemma 14.** If $B(\tau) \leq \beta^{-\frac{2}{p}}(\tau)$ for $\tau \in [0, T]$, then (4.12) holds.

**Proof.** We begin by dividing (4.11) by $b^2$ and use the inequality $b \lesssim \beta$ to obtain the estimate

$$|\partial_\tau \frac{1}{b} + \frac{4p}{(p-1)^2}| \lesssim \beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p.$$ (4.39)

Since $\beta$ is a solution to $-\partial_\tau \beta^{-1} + 4p(p-1)^{-2} = 0$, $\beta(0) = b(0)$, Equation (4.39) implies that

$$|\partial_\tau \left(\frac{1}{b} - \frac{1}{\beta}\right)| \lesssim \beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p.$$

Integrating this equation over $[0, \tau]$, multiplying the result by $\beta^{-1-\kappa}$ and using that $b \lesssim \beta$ gives the estimate

$$\beta^{-1-\kappa}|\beta - b| \lesssim \beta^{1-\kappa} \int_0^\tau \left(\beta + \beta M_1 (1 + A) + \beta^2 M_1^2 + \beta^{2p-2} M_1^p\right) \, ds.$$

Recall that the constant $\kappa = \min\{\frac{1}{2}, \frac{p-1}{2}\} < 1$. Hence, by the definition of $\beta$ and $B$, (4.12) follows.

**Lemma 15.** Suppose $A(\tau), B(\tau) \leq \beta^{-\frac{2}{p}}(\tau)$ for $\tau \in [0, T]$, then (4.13) holds.

**Proof.** Let $\Gamma = \frac{1}{2} - a - \frac{2}{p-1}b$. Differentiating $\Gamma$ with respect to $\tau$ and substituting for $b_\tau$ and $c_\tau$ Equations (4.27) and (4.28) we obtain

$$\partial_\tau \Gamma = -2c(\Gamma + \mathcal{R}_c) - \frac{2}{p-1} \left(\frac{-2(3p-1)}{(p-1)^2}b^2 + 2b(c - a) + \mathcal{R}_b\right).$$

Replacing $2b(c - a)$ by $b\Gamma + \frac{2}{p-1}b^2$ and rearranging the resulting equation gives that

$$\partial_\tau \Gamma + \left[a + \frac{1}{2} - \frac{2}{p-1}b\right] \Gamma = \frac{8p}{(p-1)^3}b^2 - (a + \frac{1}{2})\mathcal{R}_c - \frac{2}{p-1} \mathcal{R}_b.$$
Let \( \mu = \exp \left( \int_0^\tau \left( a + \frac{1}{2} - \frac{2}{p-1} b \right) ds \right) \). Then the above equation implies that
\[
\partial_\tau (\mu \Gamma) = \frac{8p}{(p-1)^3} \int_0^\tau \mu b^2 ds - \int_0^\tau \left( a + \frac{1}{2} \right) \mu \mathcal{R}_c ds - \int_0^\tau \frac{2}{p-1} \mu \mathcal{R}_b ds.
\]
We now integrate the above equation over \([0, \tau] \subseteq [0, T]\) and use the inequality \( b \lesssim \beta \) and the estimate of \( \mathcal{R}_b \) and \( \mathcal{R}_c \) to obtain
\[
|\Gamma| \lesssim \mu^{-1} \Gamma(0) + \mu^{-1} \int_0^\tau \mu b^2 ds + \mu^{-1} \int_0^\tau \mu \left( \beta^3 + \beta^3 M_1(1 + A) + \beta^4 M_1^2 + \beta^{2p} M_1^p \right) ds.
\]
For our purpose, it is sufficient to use the less sharp inequality
\[
|\Gamma| \lesssim \mu^{-1} \Gamma(0) + (1 + \beta(0)M_1(1 + A) + \beta(0)M_1^2 + \beta^{2p-2}(0)M_1^p) \mu^{-1} \int_0^\tau \mu b^2 ds.
\]
The assumption that \( A(\tau), B(\tau) \leq \beta^{-\frac{2}{p}}(\tau) \) implies that \( a + \frac{1}{2} - \frac{2}{p-1} b = 1 - \frac{4b}{p-1} \) and therefore \( \beta^{-2} \mu^{-1} \lesssim \beta^{-2}(0) \) and \( \int_0^\tau \mu(s) \beta^2(s) ds \lesssim \mu(\tau) \beta^2(\tau) \). The last two inequalities and the relation \( \max_{s \leq \tau} \beta^{-2}(s) \| \Gamma(s) \| = A(\tau) \) lead to (4.13).

4.4 Rescaling of Fluctuations on a Fixed Time Interval

The coefficient of \( y^2 \) in (4.18) is time dependent, complicating the estimation of the semigroup generated by this operator. In this section we re-parameterize the fluctuation \( \xi \) in such a way that the coefficient of \( y^2 \) in the new operator is constant (cf [6, 7, 47]).

Let \( T \) be given and let \( t(\tau) \) be the inverse of the function \( \tau(t) = \int_0^t \lambda^2(s) ds \).

We approximate the scaling parameter \( \lambda(t) \) over the time interval \([0, t(T)]\) by a new parameter \( \lambda_1(t) \). We choose \( \lambda_1(t) \) to satisfy

\[
\partial_t \left( \lambda_1^{-3} \partial_t \lambda_1 \right) = 0 \text{ with } \lambda_1 \left( \tau^{-1}(T) \right) = \lambda \left( \tau^{-1}(T) \right) \text{ and } \partial_t \lambda_1(\tau^{-1}(T)) = \partial_t \lambda \left( \tau^{-1}(T) \right).
\]

We define \( \alpha = \lambda_1^{-3} \partial_t \lambda_1 = a(T) \). This is an analog of the parameter \( a \) and it is constant. The last two conditions imply that \( \lambda_1 \) is tangent to \( \lambda \) at \( t = t(T) \). We then define a new function \( \eta(z, \sigma) \) by the equality
\[
\lambda_1^\frac{\alpha}{2} e^{\frac{\alpha}{4} z^2} \eta(z, \sigma) = \lambda^\frac{2}{p-1} e^{\frac{2}{p-1} y^2} \xi(y, \tau), \quad (4.40)
\]

28
where
\[ z = \frac{\lambda_1}{\lambda} y \] and \( \sigma = \sigma(t(\tau)) \) with \( \sigma(t) = \int_0^t \lambda_1^2(s) \, ds \).

Denote by \( t(\sigma) \) the inverse of the function \( \sigma(t) \). In the equation for \( \eta(z,\sigma) \) derived below and in what follows the symbols \( \lambda, a \) and \( b \) stand for \( \lambda(t(\sigma)), a(t(\sigma)) \) and \( b(t(\sigma)) \), respectively. Substituting this change of variables into (4.17) gives the governing equation for \( \eta \):

\[
\partial_\sigma \eta = -L_{\alpha\beta} \eta + W(a,b,\alpha) \eta + F(a,b,\alpha) + N(\eta,a,b,\alpha),
\]

(4.41)

where
\[
L_{\alpha\beta} = L_0 + V, \quad L_0 = -\partial^2_z + \frac{\alpha^2}{4} z^2 - \frac{5}{2} \alpha, \quad V = \frac{2p\alpha}{p-1} - \frac{2p\alpha}{p-1 + \beta z^2}.
\]

(4.42)

\[
W(a,b,\alpha) = \left( \frac{\lambda_1}{\lambda_1} \right)^{2p-1} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} z^2} \frac{a}{\lambda_1} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} y^2} F(a,b,c)
\]

and
\[
N(\eta,a,b,\alpha) = \left( \frac{\lambda_1}{\lambda_1} \right)^{2p-1} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} z^2} \frac{a}{\lambda_1} \frac{\alpha}{\lambda_1} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} y^2} N \left( \left( \frac{\lambda_1}{\lambda} \right)^{2p-1} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} z^2} e^{-\frac{2}{p} \frac{\alpha}{\lambda_1} y^2} \eta, b, c \right),
\]

where, recall, \( c \) and \( a \) are related as \( 2c = a + \frac{1}{2} \) and \( \beta \) is defined in (4.6).

In the next statement we prove that the new parameter \( \lambda_1(t) \) is a good approximation of the old one, \( \lambda(t) \), and we estimate the original fluctuation \( \xi(y,\tau) \) in terms of the new one, \( \eta(z,\sigma) \). Let \( S = \sigma(t(T)) \). We have

**Proposition 16.** Assume \( A(\tau) \leq \beta^{-\frac{2}{p}}(\tau) \) and \( b(0) \ll 1 \), then

\[
\left| \frac{\lambda}{\lambda_1}(t(\tau)) - 1 \right| \lesssim \beta(\tau) \leq b(0).
\]

(4.43)

**Proof.** Differentiating \( \frac{\lambda}{\lambda_1} - 1 \) with respect to \( \tau \) (recall that \( \frac{d}{d\tau} = \frac{1}{\lambda} \)) gives the expression

\[
\frac{d}{d\tau} \left( \frac{\lambda}{\lambda_1} - 1 \right) = \frac{\lambda}{\lambda_1} a - \frac{\lambda_1}{\lambda} \alpha
\]

29
or, after some manipulations

\[
\frac{d}{d\tau} \left[ \frac{\lambda}{\lambda_1} - 1 \right] = 2a \left( \frac{\lambda}{\lambda_1} - 1 \right) + \Lambda \tag{4.44}
\]

with

\[
\Lambda = a - \alpha + a \left( \frac{\lambda}{\lambda_1} + \frac{\lambda_1}{\lambda} - 2 \right) \left( \frac{\lambda}{\lambda_1} - 1 \right) + (a - \alpha) \left( \frac{\lambda_1}{\lambda} - 1 \right).
\]

Observe that \( \frac{\lambda}{\lambda_1}(t(T)) - 1 = 0 \). Integrating Equations (4.44) over \([\tau, T]\), we conclude that

\[
\frac{\lambda_1}{\lambda}(t(\tau)) - 1 = -\int_{\tau}^{T} e^{-\int_{\tau}^{\sigma} 2a(\rho) d\rho} \Lambda(\sigma) d\sigma. \tag{4.45}
\]

By the definition of \( A(\tau) \) and the definition \( \alpha = a(T) \) we have that, if \( A(\tau) \leq \beta - \frac{\pi}{2}(\tau) \), then

\[
|a(\tau) - \alpha|, \quad \left| a(\tau) - \frac{1}{2} \right| \leq 2\beta(\tau) \tag{4.46}
\]

in the time interval \( \tau \in [0, T] \). Thus

\[
|\Lambda| \lesssim \beta + \left( 1 + \frac{\lambda_1}{\lambda} \right) \left( \frac{\lambda}{\lambda_1} - 1 \right)^2 + \beta \left| \frac{\lambda}{\lambda_1} - 1 \right|. \tag{4.47}
\]

which together with (4.45) and (4.46) implies (4.43).

4.5 Estimate on the Propagators

Let \( \bar{P}^\alpha \) be the projection onto the space spanned by the first three eigenvectors of \( L_0 \) and \( P^\alpha = 1 - \bar{P}^\alpha \), i.e.

\[
\bar{P}^\alpha = \frac{1}{\sqrt{2\pi \alpha}} \sum_{m=0}^{2} \langle \phi_{m,\alpha}, \cdot \rangle \phi_{m,\alpha}. \tag{4.48}
\]

where the functions of \( \phi_{m,\alpha} \) are defined in (B.3) below. Denote by \( U_{\alpha\beta}^{(1)}(\tau, \sigma) \) the propagator generated on \( \text{Ran} \ P^\alpha \) by the operator \( -P^\alpha L_{\alpha\beta} P^\alpha \), where, recall, the definitions of the operator \( L_{\alpha\beta} \) and the function \( \beta \) are given in Equations (4.42) and (4.6) respectively.

**Proposition 17.** For any function \( g \in \text{Ran} \ P^\alpha \) and for \( \epsilon = \alpha - \epsilon \) with some \( \epsilon > 0 \) small we have

\[
\left\| \langle z \rangle^{-3} e^{\frac{\alpha^2}{4}} U_{\alpha\beta}^{(1)}(\tau, \sigma) g \right\|_\infty \lesssim e^{-C^*(\tau-\sigma)} \left\| \langle z \rangle^{-3} e^{\frac{\alpha^2}{4}} g \right\|_\infty.
\]

30
The proof of this proposition is given after Lemma 20. Here we just observe that in the $L^2$ norm $P_\alpha L_{\alpha,\beta}P_\alpha \geq (-\partial_x^2 + \frac{\alpha}{4}z^2 - \frac{5}{2}\alpha)P_\alpha \geq \frac{1}{4\alpha}P_\alpha$. However, this does not help in proving the weighted $L^\infty$ bound above. We start with an estimate for the propagator $U_{\alpha,\beta}(\tau, \sigma)$, generated by the operator $-L_{\alpha,\beta}$. A version of the following lemma is proved in [8].

**Lemma 18.** For any function $g$ and positive constants $\sigma$ and $r$, we have

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha}{4}z^2} U_{\alpha,\beta}(\sigma + r, \sigma) P_\alpha g \right\|_{\infty} \lesssim \left[ e^{2\alpha r}(1 + r)^{\frac{\sqrt{\beta}}{\sqrt{\alpha}}} + e^{-\alpha r} \right] \left\| \langle z \rangle^{-3} e^{\frac{\alpha}{4}z^2} g \right\|_{\infty}. $$

**Proof.** The spatial variables in this proof will be denoted by $x$, $y$ and $z$. Recall the definitions of the operators $L_0$ and $V$ in (4.42). Denote the integral kernel of $e^{-\frac{\alpha}{4}z^2} U_{\alpha,\beta}(\sigma + r, \sigma) e^{\frac{\alpha}{4}z^2}$ by $U(x, y)$. By Theorem 27, given in Appendix B, we have the representation

$$U(x, y) = U_0(x, y) \langle e^V \rangle(x, y), $$

(4.49)

where $U_0(x, y)$ is the integral kernel of the operator $e^{-\frac{\alpha}{4}z^2} e^{-rL_0} e^{\frac{\alpha}{4}z^2}$ and

$$\langle e^V \rangle(x, y) = \int_\sigma^{\sigma + r} e^{\int_\sigma^{\sigma + r} V(\sigma + s, \omega(s)) + \omega_0(s))ds} d\mu(\omega). $$

(4.50)

Here $\omega_0(s)$ is defined in Theorem 27 of Appendix C and $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma, \sigma + r]$ with the boundary condition $\omega(\sigma) = \omega(\sigma + r) = 0$. By a standard formula (see [51, 21]) we have

$$U_0(x, y) = 4\pi(1 - e^{-2\alpha r})^{-1/2} \sqrt{\alpha e^{2\alpha r}} e^{-\alpha \frac{(x-y)^2}{2(1-e^{-2\alpha r})}}. $$

Define a new function $f = e^{-\frac{\alpha}{4}z^2} P_\alpha g$. The definitions above imply

$$U_{\alpha,\beta}(\sigma + r, \sigma) P_\alpha g = \int e^{\frac{\alpha}{4}z^2} U_0(x, y) \langle e^V \rangle(x, y) f(y) dy. $$

(4.51)

Integrating by parts on the right hand side of (4.51), we obtain

$$U_{\alpha,\beta}(\sigma + r, \sigma) P_\alpha g = \sum_{k=0}^{2} e^{\frac{\alpha}{4}z^2} \int \delta^k_y U_0(x, y) \partial_y \langle e^V \rangle(x, y) f^{(-k-1)}(y) dy $$

$$+ e^{\frac{\alpha}{4}z^2} \int \delta^3_y U_0(x, y) \langle e^V \rangle(x, y) f^{(-3)}(y) dy. $$

(4.52)

where $f^{(-m-1)}(x) = \int_{-\infty}^{x} f^{(-m)}(y) dy$ and $f^{(-0)} = f$. We next estimate each term on the right hand side of Equation (4.52).
By the facts that \( f = e^{-\frac{\alpha x^2}{4}} P^\alpha g \) and \( P^\alpha g \perp y^n e^{-\frac{\alpha y^2}{4}}, \ n = 0, 1, 2, 3 \), we have that \( f \perp 1, \ y, \ y^2, \ y^3 \). Therefore by integrating by parts we have

\[
f_{(-m)}(y) = \int_{-\infty}^{y} f_{(-m+1)}(x)dx - \int_{y}^{\infty} f_{(-m+1)}(x)dx, \ m = 0, 1, 2, 3.
\]

Moreover by the definition of \( f_{(-m)} \) and the formula above we have

\[
|f_{(-m)}(y)| \lesssim \langle y \rangle^{-3} e^{-\frac{\alpha y^2}{2}} \parallel g \parallel_{\infty}.
\]

Using the explicit formula for \( U_0(x, y) \) given above we find

\[
|\partial_y \langle e^V \rangle(x, y)| \lesssim e^{-\alpha x^2} 2^n (1 - e^{2\alpha r}) \langle |x| + |y| \rangle^n |U_0(x, y)|
\]

By an estimate from Appendix C (see also [8]) we have that

\[
|\partial_y \langle e^V \rangle(x, y)| \leq \sqrt{\beta r}. \tag{4.53}
\]

Collecting the estimates (A)-(C) above and using Equation (4.52), we have the following result

\[
\langle x \rangle^{-3} e^{\frac{\alpha x^2}{4}} |U_{\alpha\beta}(\sigma + r, \sigma) P^\alpha g(x)|
\]

\[
\lesssim \frac{br(1 + r)}{(1 - e^{-2\alpha r})^3} \langle y \rangle^{-3} e^{\frac{\alpha y^2}{2}} \sum_{n=0}^{3} \int \langle |x| + |y| \rangle^n |U_0(x, y)| f_{(-3)}(y)dy
\]

This plus the estimate (4.55) in Lemma 19 below give the estimate of Lemma 18.

Recall the definition of the operator \( L_0 \) in (4.42). We next have a \( L^\infty \) estimate for \( L_0 \).

**Lemma 19.** Let \( n \) be a nonnegative integer, \( g \) any function and \( r > 0 \). we have

\[
\langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} e^{-L_0 r} g \parallel_{\infty} \lesssim \langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} g \parallel_{\infty} \tag{4.54}
\]

or equivalently

\[
e^{\frac{\alpha x^2}{4}} \int \langle x \rangle^{-n} U_0(x, y) e^{-\frac{\alpha y^2}{2}} \langle y \rangle^n dy \lesssim e^{2\alpha r}. \tag{4.55}
\]
Proof. By Mehler’s formula (Ref. [51]) we have
\[ e^{-tL_0(x, y)} = e^{-\frac{\alpha x^2}{4}} K(x, y, t) e^{\frac{\alpha y^2}{4}}, \] (4.56)
where
\[ K(x, y, t) = (2\pi)^{-1/2} \sqrt{\alpha}(1 - e^{-2\alpha t})^{-1/2} e^{2\alpha t} e^{-\frac{\alpha(x^2 - yt)^2}{2(1 - e^{-2\alpha t})}}. \]
Thus
\[ \| \langle x \rangle^{-n} e^{\frac{\alpha x^2}{4}} e^{-tL_0} g \|_\infty \]
\[ \leq \| \langle x \rangle^{-n} \int e^{\frac{\alpha x^2}{4}} e^{-tL_0(x, y)} e^{-\frac{\alpha y^2}{4}} \langle y \rangle^n e^{\frac{\alpha y^2}{4}} |g(y)| dy \|_\infty \]
\[ \leq \| \int \langle x \rangle^{-n} K(x, y, t) \langle y \rangle^n dy \|_\infty \| \langle y \rangle^{-n} e^{\frac{\alpha y^2}{4}} g \|_\infty \]
\[ \leq \| \int \langle x \rangle^{-n}(2\pi)^{-1/2} \sqrt{\alpha}(1 - e^{-2\alpha t})^{-1/2} e^{2\alpha t} e^{-\frac{\alpha(x^2 - yt)^2}{2(1 - e^{-2\alpha t})}} \langle y \rangle^n dy \|_\infty \| \langle y \rangle^{-n} e^{\frac{\alpha y^2}{4}} g \|_\infty \]
\[ \leq \| \int \langle x \rangle^{-n}(2\pi)^{-1/2} e^{2\alpha t} e^{-\frac{\alpha y^2}{4}} \langle y \rangle^n dy \|_\infty \| \langle y \rangle^{-n} e^{\frac{\alpha y^2}{4}} g \|_\infty \]
\[ \leq C e^{2\alpha t} \| \langle y \rangle^{-n} e^{\frac{\alpha y^2}{4}} g \|_\infty, \]
\[ \]  

The next result comes from the comparison of two relevant kernels.

**Lemma 20.** Let \( g \) be any function. Then we have
\[ \| \langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} U_{\alpha\beta}(\tau, \sigma) g \|_\infty \leq e^{2\alpha(\tau - \sigma)} \| \langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} g \|_\infty \] (4.57)
for any \( n \geq 0 \).

**Proof.** By the formulas (4.50) and (4.51), we have that
\[ |U_{\alpha\beta}(\tau, \sigma)(x, y)| \leq U_{\alpha\beta=0}(\tau, \sigma)(x, y) = e^{-L_0(\tau - \sigma)}(x, y). \]
Thus we have
\[ \| \langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} U_{\alpha\beta}(\tau, \sigma) g \|_\infty \leq \| \langle z \rangle^{-n} e^{\frac{\alpha z^2}{4}} e^{-L_0(\tau - \sigma)} |g| \|_\infty. \] (4.58)
Now we use Lemma 19 to estimate the right hand side to complete the proof.  

**Proof of Proposition 17.** Recall that \( \bar{P}_\alpha \) is the projection on the span of the three first eigenfunctions of the operator \( L_0 \) and \( P^\alpha = 1 - \bar{P}_\alpha \). We write
\[ \mathcal{L}_{\alpha\beta} = P^\alpha \mathcal{L}_{\alpha\beta} P^\alpha + E_1 + \bar{P}_\beta \mathcal{L}_{\alpha\beta} \bar{P}_\alpha, \] (4.59)
where the operator $E_1$ is defined as $E_1 = \bar{P}^\alpha \mathcal{L}_{\alpha \beta} P^\alpha + P^\alpha \mathcal{L}_{\alpha \beta} \bar{P}^\alpha$. Using that $\bar{P}^\alpha P^\alpha = 0$, we transform $E_1$ to

$$E_1 = -P^\alpha \frac{(2\alpha + 1)\beta z^2}{1 + \beta z^2} P^\alpha - P^\alpha \frac{(2\alpha + 1)\beta z^2}{1 + \beta z^2} \bar{P}^\alpha.$$  

Using $P^\alpha = 1 - \sum_{i=0}^3 \langle \cdot, \phi_{i,\alpha} \rangle \phi_{i,\alpha}$ and Hőlder’s inequality, we obtain

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} U_1 \eta(\sigma) \right\|_\infty \lesssim \beta(\tau(\sigma)) \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \right\|_\infty. \quad (4.60)$$

We use Duhamel’s principle to rewrite the propagator $U^{(1)}_{\alpha \beta}(\tau, \sigma)$ on $\text{Ran} P^\alpha$ as

$$U^{(1)}_{\alpha \beta}(\tau, \sigma) P^\alpha = U_{\alpha \beta}(\tau, \sigma) P^\alpha + \int_{\sigma}^\tau U_{\alpha \beta}(\tau, s) E_1 U^{(1)}_{\alpha \beta}(s, \sigma) P^\alpha ds. \quad (4.61)$$

Let $r = \tau - \sigma$, $g \in \text{Ran} P^\alpha$ and $\eta(\tau) = U^{(1)}_{\alpha \beta}(\tau, \sigma) g$. We next proceed to estimate the two terms on the right hand side of (4.61). We also assume $e^{\alpha r} \leq \beta^{-\frac{1}{\beta^2}}(\tau)$.

(A) Notice that $P^\alpha \eta(s) = \eta(s)$. We use Lemma 18 to obtain, for $e^{\alpha r} \leq \beta^{-1/32}(\tau)$,

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} U_{\alpha \beta}(\tau, \sigma) g \right\|_\infty \lesssim e^{-\alpha r} \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} g \right\|_\infty. \quad (4.62)$$

(B) By Lemma 20 and (4.60) we obtain

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \int_{\sigma}^\tau U_{\alpha \beta}(\tau, s) E_1 \eta(s) ds \right\|_\infty \lesssim \int_{\sigma}^\tau e^{\alpha(\tau - s)} \beta(s) \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(s) \right\|_\infty ds.$$

Using the condition $e^{\alpha r} \leq \beta^{-1/32}(\sigma)$ and the relation $\beta(s) \leq \beta(\sigma)$ for $s \geq \sigma$ again, we find

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \int_{\sigma}^\tau U_{\alpha \beta}(\tau, s) E_1 \eta(s) ds \right\|_\infty \lesssim \int_{\sigma}^\tau e^{-\alpha(\tau - s)} \beta^{1/2}(s) \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(s) \right\|_\infty ds. \quad (4.63)$$

Equations (4.61), (4.62) and (4.63) imply that if $e^{\alpha r} \leq \beta^{-1/32}(\sigma)$ then (remember that $\eta(\sigma) = g$)

$$\left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\tau) \right\|_\infty \lesssim e^{-\alpha r} \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \right\|_\infty$$

$$+ \int_{\sigma}^\tau e^{-\alpha(\tau - s)} \beta^{1/2}(s) \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(s) \right\| ds. \quad (4.64)$$
Now we claim that if \( e^{\alpha r} \leq \beta^{-1/32}(\sigma) \) then we have

\[
\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\tau) \|_{\infty} \lesssim e^{-\alpha r} \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_{\infty}.
\]  

(4.65)

Indeed, we define a function \( K(z) \) by

\[
K(z) = \max_{0 \leq z \leq r} e^{\alpha z} \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma + z) \|.
\]

(4.66)

From (4.64), we get

\[
K(\tau) \lesssim \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_{\infty} + \int_{\sigma}^{\tau} e^{-\alpha(\tau-s)} e^{-\alpha(s-\sigma)} \beta^{1/2}(s) ds K(\tau).
\]

We observe that

\[
\int_{\sigma}^{\tau} e^{-\alpha(\tau-s)} e^{-\alpha(s-\sigma)} \beta^{1/2}(s) ds \leq 1/2
\]

if \( \beta(0) \) and, therefore, \( \beta(s) = \frac{1}{\beta(0)} \frac{1}{(s-1)^3} \) are small. Thus we have

\[
K(\tau) \lesssim \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_{\infty},
\]

which together with Equation (4.66) implies (4.65). Iterating (4.65) over the time at step length \( r \) completes the proof of the proposition.

\[\square\]

4.6 Estimate of \( M_1(\tau) \) (4.14)

In this section we derive the estimate (4.14) for \( M_1 \). We first get the estimate for \( \eta \). Finally we use the relation between \( \xi \) and \( \eta \) in (4.40) to obtain the estimate for \( \xi \). Observe that the function \( \eta \) is not orthogonal to the first three eigenvectors of the operator \( L_0 \) defined in (4.42). Note that \( P^\alpha \) is \( \tau \)-independent. Thus we apply the projection \( P^\alpha \) to Equation (4.41) to get

\[
\frac{d}{d\sigma} P^\alpha \eta = -P^\alpha L_{\alpha \beta} P^\alpha \eta + P^\alpha \sum_{n=1}^{4} D_n.
\]

(4.67)
where $D_n = D_n(\sigma)$, $n = 1, 2, 3, 4$, are given by

$$D_1 = -P^\alpha V \eta + P^\alpha V P^\alpha \eta, \quad D_2 = W(a, b, \alpha) \eta,$$

$$D_3 = F(a, b, \alpha), \quad D_4 = N(\eta, a, b, \alpha).$$

The functions $V, W, F$ and $N$ were defined after (4.42).

In the following lemma, we prove some estimates for $D_1, \ldots, D_4$.

**Lemma 21.** Let $A(\tau)$, $B(\tau) \leq \beta^{-\frac{2}{5}}(\tau)$ and $b_0 \ll 1$. Then we have

$$\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_1(\sigma)\|_{\infty} \lesssim \beta^{5/2}(\tau(\sigma)) M_1(T),$$

$$\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_2(\sigma)\|_{\infty} \lesssim \beta^{2+\frac{2}{5}}(\tau(\sigma)) M_1(T),$$

$$\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_3(\sigma)\|_{\infty} \lesssim \beta^{\min\{5/2, 2p\}}(\tau(\sigma))[1 + M_1(T)(1 + A(T)) + M_1^2(T) + M_1^p(T)],$$

$$\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_4\|_{\infty} \lesssim \beta^{2}(\tau(\sigma)) M_1(T)[\beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T) + \beta^{\frac{1}{1-2}}(\tau(\sigma)) M_1^{p-1}(T) + M_2^{p-1}(T)].$$

**Proof.** In our proof we sometimes use the following estimates derived from (4.43). Recall that $\frac{\lambda_1}{\lambda}(t(\tau)) - 1 = O(\beta(\tau))$. Thus

$$\frac{\lambda_1}{\lambda}(t(\tau)), \quad \frac{\lambda}{\lambda_1}(t(\tau)) \leq 2, \quad \langle z \rangle^{-3} \lesssim \langle y \rangle^{-3}$$

(4.72)

where $z = \frac{\lambda_1}{\lambda} y$. Next we prove two inequalities which will be used frequently later. They are

$$\|e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_{\infty} \lesssim \beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T),$$

$$\|\langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_{\infty} \lesssim \beta^2(\tau(\sigma)) M_1(T).$$

By the relation between $\xi$ and $\eta$ in (4.40) and Estimate (4.72) we have

$$\|e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_{\infty} \lesssim \|e^{\frac{\alpha (\tau(\sigma)) y^2}{4}} \xi(\tau(\sigma))\|_{\infty} \leq \|e^{\frac{\alpha y^2}{4}} (1 - \chi_{\geq D}) \xi(\tau(\sigma))\|_{\infty} + \|e^{\frac{\alpha y^2}{4}} \chi_{\geq D} \xi(\tau(\sigma))\|_{\infty}$$

Since $1 - \chi_{\geq D} \lesssim \beta^{-3/2}(\tau) \langle y \rangle^{-3}$, we have

$$\|e^{\frac{\alpha z^2}{4}} \eta(\sigma)\|_{\infty}, \|e^{\frac{\alpha (\tau(\sigma)) y^2}{4}} \xi(\tau(\sigma))\|_{\infty} \lesssim \beta^{-3/2}(\tau(\sigma)) \langle y \rangle^{-3} e^{\frac{\alpha (\tau(\sigma)) y^2}{4}} \xi(\tau(\sigma))\|_{\infty} + \|e^{\frac{\alpha y^2}{4}} \chi_{\geq D} \xi(\tau(\sigma))\|_{\infty} \lesssim \beta^{1/2}(\tau(\sigma)) M_1(T) + M_2(T)$$

(4.75)
which is (4.73). Similarly we have
\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_\infty \lesssim \| \langle y \rangle^{-3} e^{\frac{\alpha (\tau(\sigma)) y^2}{4}} \xi(\tau(\sigma)) \|_\infty \lesssim \beta^2(\tau(\sigma)) M_1(T). \]

Thus we have (4.74).

Now we start to prove the lemma. First we rewrite \( D_1 \) in the form
\[ D_1(\sigma) = -P^\alpha \frac{2p\alpha}{(p - 1)(p - 1 + \beta(\tau(\sigma)) z^2)} \beta(\tau(\sigma)) z^2 (1 - P^\alpha) \eta(\sigma). \]

Using \( \langle z \rangle^{-1} \frac{\tau(\sigma) z^2}{1 + b(\tau(\sigma)) z^2} \lesssim b^\frac{1}{2}(\tau) \) and \( b(\tau) \lesssim \beta(\tau) \), we have
\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_1(\sigma) \|_\infty \lesssim \| \langle z \rangle^{-1} \frac{\beta(\tau(\sigma)) z^2}{1 + b z^2} \|_\infty \left\| \langle z \rangle^{-2} e^{\frac{\alpha z^2}{4}} (1 - P^\alpha) \eta(\sigma) \right\|_\infty \lesssim \beta^\frac{1}{2}(\tau) \left\| \langle z \rangle^{-2} e^{\frac{\alpha z^2}{4}} (1 - P^\alpha) \eta(\sigma) \right\|_\infty \].

Use the explicit form of \( 1 - P^\alpha \) in (4.48) and the properties of the eigenfunctions \( \phi_{m,\alpha} \) and we obtain that for any function \( g \)
\[ \| \langle z \rangle^{-2} e^{\frac{\alpha z^2}{4}} (1 - P^\alpha) g \|_\infty \lesssim \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} g \|_\infty. \tag{4.76} \]

The estimates above allow us to make such conclusion,
\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_1(\sigma) \|_\infty \lesssim \beta^{1/2}(\tau(\sigma)) \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_\infty \lesssim \beta^{5/2}(\tau(\sigma)) M_1(T). \]

To prove (4.69) we rewrite \( D_2 \) as
\[ D_2 = \left[ \frac{\lambda^2}{\lambda^2_1} - 1 \right] \frac{2a(\tau(\sigma)) + 1}{p - 1 + by^2} + \frac{2(a(\tau(\sigma)) - \alpha)}{p - 1 + by^2} \frac{b(2\alpha + 1) \left( \frac{\lambda^2}{\lambda^2_1} - 1 \right) y^2}{(p - 1 + b(\tau(\sigma)) z^2)(p - 1 + b(\tau(\sigma)) y^2)} + \frac{p - 1 - 2\alpha}{p - 1 + b(\tau(\sigma)) z^2} + \frac{2p\alpha \beta z^2}{\beta - b}. \]

Then Equations (4.43), (4.46) and the definition of \( B \) in (4.7) imply
\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_2(\sigma) \|_\infty \leq \beta^\frac{3}{2}(\tau(\sigma)) \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(\sigma) \|_\infty. \]

From (4.75) we obtain (4.69), noting \( \kappa = \min\{\frac{1}{2}, \frac{p - 1}{2} \} \).

Next let us prove (4.70). By (4.72) and the relation between \( D_3 \), \( F \) and \( \mathcal{F} \) we have
\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} D_3(\sigma) \|_\infty \lesssim \| \langle y \rangle^{-3} e^{\frac{\alpha (\tau(\sigma)) y^2}{4}} \mathcal{F}(a, b, c)(\tau(\sigma)) \|_\infty \]
which implies (4.70) under the aid of the inequality (4.38).
Finally we prove (4.71). By the relation between $D_4$, $N$ and $\mathcal{N}$ and the estimate in (4.35) we have

\[
\| \langle z \rangle^{-3} e^{\frac{\sigma^2}{4}} D_4(\sigma) \|_\infty \lesssim \| \langle y \rangle^{-3} e^{\frac{\alpha(\tau(y))}{4} y^2} \mathcal{N}(\xi(\tau(\sigma)), b(\tau(\sigma)), c(\tau(\sigma))) \|_\infty \\
\lesssim \| \langle y \rangle^{-3} e^{\frac{\alpha^2}{4}} \xi \|_\infty [ \| e^{\frac{\alpha^2}{4} \xi} \|_\infty + \| e^{\frac{\alpha^2}{4} \xi} \|_{\infty}^{p-1} ].
\]

combining (4.75) and the definition of $M$ we have

\[M \leq \beta^{-\frac{\eta}{4}}(\tau)\]

Below we prove a very simple result.

**Lemma 22.** Let $S = \sigma(t(T))$. If $A(\tau) \leq \beta^{-\frac{\eta}{4}}(\tau)$ and $b(0)$ is sufficiently small, then for any $C_1, C_2 > 0$ there exists a constant $C(C_1, C_2)$ such that

\[\int_0^S e^{-C_1(S-\sigma)} \beta^{C_2}(\tau(t(\sigma))) d\sigma \leq C(C_1, C_2) \beta^{C_2}(T). \quad (4.77)\]

**Proof.** We use the shorthand notation $\tau(\sigma) \equiv \tau(t(\sigma))$, where $t(\sigma)$ is the inverse of $\sigma(t) = \int_0^t \lambda^2(k) dk$ and $\tau(t) = \int_0^t \lambda^2(k) dk$. By Proposition 16 we have $\frac{1}{2} \leq \frac{\eta}{\lambda_1} \leq 2$ provided that $A(\tau) \leq \beta^{-\frac{\eta}{4}}(\tau)$. Hence

\[\frac{1}{4} \sigma \leq \tau(\sigma) \leq 4\sigma \quad (4.78)\]

which implies $\frac{1}{\tau(\sigma) + \frac{1}{p-1} \tau(\sigma)} \lesssim \frac{1}{\tau(\sigma)}$. By a direct computation we have

\[\int_0^S e^{-C_1(S-\sigma)} \beta^{C_2}(\tau(\sigma)) d\sigma \leq C(C_1, C_2) \frac{1}{b(0) + \frac{1}{p-1} S} \beta^{C_2}. \quad (4.79)\]

Notice that $4S \geq \tau(S) = T \geq \frac{1}{4} S$ from (4.78) and then (4.79) implies (4.77). \qed

Recall that $S = \sigma(t(T))$ and $U_{\alpha\beta}^{(1)}(t, s)$ is the propagator generated by the operator $-P^\alpha \mathcal{L}_{\alpha\beta} P^\alpha$. To order to estimate $P^\alpha \eta$ we rewrite Equation (4.67) in its integral form

\[P^\alpha \eta(S) = U_{\alpha\beta}^{(1)}(S, 0) \eta(0) + \sum_{n=1}^4 \int_0^S U_{\alpha\beta}^{(1)}(S, \sigma) P^\alpha D_n(\sigma) d\sigma.\]

which implies

\[\| \langle z \rangle^{-3} e^{\frac{\alpha^2}{4}} \sigma^2 P^\alpha \eta(S) \|_\infty \leq K_1 + K_2 \quad (4.80)\]

with

\[K_1 = \| \langle z \rangle^{-3} e^{\frac{\alpha^2}{4}} U_{\alpha\beta}^{(1)}(S, 0) \eta(0) \|_\infty;\]

\[K_2 = \| \langle z \rangle^{-3} e^{\frac{\alpha^2}{4}} \sum_{n=1}^4 U_{\alpha\beta}^{(1)}(S, \sigma) P^\alpha D_n(\sigma) d\sigma \|_\infty.\]
\[ K_2 = \left\| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \sum_{n=1}^{4} \int_0^S U_{\alpha \beta}^{(1)}(S, \sigma) P^n D_n(\sigma) d\sigma \right\|_\infty. \]

Using Proposition 17, Equation (4.74) and the slow decay property of \( \beta(\tau) \) we obtain

\[ K_1 \lesssim e^{-C^s S} \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} \eta(0) \|_\infty \lesssim \beta^2(T) M_1(0). \quad (4.81) \]

Proceeding in the similar way, Proposition 17, Equations (4.68)-(4.71) and \( \int_0^S e^{-C^s (S-\sigma)} \beta^2(\sigma) d\sigma \lesssim \beta^2(T) \) will give

\[ K_2 \lesssim \beta^2(T) \left\{ \beta \tilde{\xi}(0) \left[ 1 + M_1(T) A(T) + M^2_1(T) + M^{2p}_1(T) \right] + \left[ M_2(T) M_1(T) + M_1(T) M^{p-1}_2(T) \right] \right\} \quad (4.82) \]

Equation (4.40) and the definitions of \( S \) and \( T \) imply that \( \lambda_1(t(S)) = \lambda(t(T)) \), \( z = y \), \( \eta(S) = \xi(T) \), and \( P^\alpha \xi = \xi \), consequently

\[ \| \langle z \rangle^{-3} e^{\frac{\alpha z^2}{4}} P^\alpha \eta(S) \|_\infty = \| \langle y \rangle^{-3} e^{\frac{\alpha y^2}{4}} \xi(T) \|_\infty. \quad (4.83) \]

Based on the estimates (4.80)-(4.83) and the definition of \( M_1 \) in (4.7) we conclude

\[ M_1(T) = \sup_{\tau \leq T} \beta^{-2}(\tau) \| \langle y \rangle^{-3} e^{\frac{\alpha y^2}{4}} \xi(\tau) \|_\infty \lesssim M_1(0) + \beta \tilde{\xi}(0) \left[ 1 + M_1(T) A(T) + M^2_1(T) + M^{2p}_1(T) \right] + M_2(T) M_1(T) + M_1(T) M^{p-1}_2(T). \]

Because \( T \) is arbitrary, we proved the estimate (4.14).

\[ \square \]

4.7 Estimate of \( M_2 \) (4.15)

The following lemma can be proved in the same way to the corresponding parts of Lemma 21 and therefore no detail is provided.
Lemma 23. If $A(\tau), B(\tau) \leq \beta^{-\frac{n}{2}}(\tau)$ and $b_0 \ll 1$, then

$$\|e^{\frac{\kappa}{2}z^2}D_2(\sigma)\|_\infty \lesssim \beta^\frac{n}{2}(\tau(\sigma)) \left[\beta^{1/2}(\tau(\sigma))M_1(T) + M_2(T)\right];$$

$$\|e^{\frac{\kappa}{2}z^2}D_3(\sigma)\|_\infty \lesssim \beta^{\min\{1, 2p-1\}}(\tau(\sigma)) \left[1 + M_1(T)(1 + A(T)) + M_1(T) + M_2(T)\right];$$

$$\|e^{\frac{\kappa}{2}z^2}D_4(\sigma)\|_\infty \lesssim \beta(\tau(\sigma))M_1^2(T) + M_2^2(T) + \beta^{p/2}(\tau(\sigma))M_1^p(T) + M_2^p(T).$$

The following proposition implies the estimate (4.15) for $M_2$.

Proposition 24. Let $C$ be the constant appeared in the definition of $M_2$. Assume that $C$ is large enough, then Equation (4.15) holds for any time $\tau$.

Proof. To estimate $M_2$, it is convenient to treat the $z$-dependent part of the potential in (4.42) as a perturbation. Let the operator $L_0$ be the same as in (4.41). For convenience, we introduce in $L_\alpha = L_0 + \frac{2p\alpha}{p-1}$. Rewrite (4.41) to have

$$\eta(S) = e^{-L_\alpha S}\eta(0) + \int_0^S e^{-L_\alpha (S-\sigma)}(V_2\eta(\sigma) + \sum_{n=2}^4 D_n(\sigma))d\sigma;$$

where, recall $S = \sigma(t(T))$, $V_2$ is the operator given by

$$V_2 = \frac{2p\alpha}{p-1 + \beta(\tau(\sigma))z^2},$$

and $D_n$, $n = 2, 3, 4$, are the same as those in (4.67). Lemma 19 implies that

$$\|e^{\frac{\alpha z^2}{4}}e^{-L_\alpha^s}g\|_\infty = e^{-\frac{2p\alpha}{p-1}s}\|e^{\frac{\alpha z^2}{4}}e^{-L_0^s}g\|_\infty \lesssim e^{-\frac{2\alpha z^2}{p-1}s}\|e^{\frac{\alpha z^2}{4}}g\|_\infty$$

for any function $g$ and time $s \geq 0$. Hence we have

$$\|e^{\frac{\alpha z^2}{4}}\eta(S)\|_\infty \lesssim K_0 + K_1 + K_2$$

where $K_n$ are given by

$$K_0 = e^{-\frac{2p\alpha}{p-1}s}\|e^{\frac{\alpha z^2}{4}}\eta(0)\|_\infty;$$

$$K_1 = \int_0^S e^{-\frac{2p\alpha}{p-1}(S-\sigma)}\|e^{\frac{\alpha z^2}{4}}V_2\eta(\sigma)\|_\infty d\sigma;$$

$$K_2 = \sum_{n=2}^4 \int_0^S e^{-\frac{2p\alpha}{p-1}(S-\sigma)}\|e^{\frac{\alpha z^2}{4}}D_n\|_\infty d\sigma.$$

Let us now begin to estimate $K_n$’s, $n = 0, 1, 2$ and we start with $K_0$.

(K0) By (4.73) and the decay of $e^{-\frac{2p\alpha}{p-1}s}$ we have

$$K_0 \lesssim M_2(0) + \beta^{1/2}(0)M_1(0).$$
(K1) By the definition of $V_2$ we have

$$\| e^{\frac{az^2}{4}V_2\eta}(\sigma) \|_\infty \lesssim \left\| \frac{1}{p - 1 + \beta(\tau(\sigma))} e^{\frac{az^2}{4}V_2\eta(\sigma)} \right\|_\infty .$$

Moreover by the relation between $\xi$ and $\eta$ in Equation (4.40) and Proposition 16 we have

$$\max_{0 \leq \sigma \leq S} \left\| e^{\frac{az^2}{4}V_2\eta}(\sigma) \right\|_\infty \lesssim \max_{0 \leq \tau \leq T} \left\| \frac{1}{p - 1 + \beta y^2} e^{\frac{a(\tau)y^2}{4}} \xi(\tau) \right\|_\infty .$$

Since $D = C / \sqrt{\beta}$, we find

$$\frac{1}{p - 1 + \beta y^2} \chi_D(y) \leq \epsilon(C) = \frac{1}{p - 1 + C^2},$$

which implies

$$\left\| \frac{1}{p - 1 + \beta y^2} e^{\frac{az^2}{4}} \xi(s) \right\|_\infty \leq \epsilon(C) \left\| \chi_D e^{\frac{az^2}{4}} \xi(s) \right\|_\infty + \left\| \chi_D e^{\frac{az^2}{4}} \xi(s) \right\|_\infty .$$

By the definition of the function $\chi_D$ in Equation (4.5) we have that for any $\tau \leq T$, $\chi_D(y)^3 \lesssim \beta^{-3/2}(\tau)$, which implies

$$\left\| \chi_D e^{\frac{az^2}{4}} \xi(s) \right\|_\infty \lesssim \beta^{-3/2}(s) \left\| \chi_D e^{\frac{az^2}{4}} \xi(s) \right\|_\infty .$$

in view of the estimates above, the definitions of $M_n, n = 1, 2$, in (4.7) and (4.77), we obtain

$$K_1 \lesssim \max_{S \geq s \geq 0} \left\| e^{\frac{az^2}{4}V_2\eta} \right\|_\infty \int_0^S e^{-\frac{az^2}{4}(S-s)} ds \lesssim \epsilon(C)M_2(T) + \beta^{1/2}(0)M_1(T).$$

(K2) As for $K_2$, by the definitions of $D_n, n = 2, 3, 4$ and Equations (4.84)-(4.86) we have

$$\sum_{n=2}^4 \left\| e^{\frac{az^2}{4}D_n(\sigma)} \right\|_\infty \lesssim \beta^2(\tau(\sigma)) \left[ 1 + M_2(T) + M_1(T)A(T) + M_1^0(T) + M_1^p(T) \right] + M_2^3(T) + M_2^p(T).$$

consequently

$$K_2 \lesssim \beta^2(0) \left[ 1 + M_2(T) + M_1(T)A(T) + M_1^0(T) + M_1^p(T) \right] + M_2^3(T) + M_2^p(T).$$
Because of the estimates (4.88)-(4.92) we have

\[ \|e^{\frac{\alpha z}{4}} \eta(S)\|_\infty \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + \epsilon(C)M_2(T) + \beta^{1/2}(0)M_1(T) \]
\[ + \beta^{\frac{2}{3}}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)] \]
\[ + M_2^p(T) + M_2^p(T). \]  

(4.93)

The relation between \( \xi \) and \( \eta \) in Equation (4.40) implies

\[ \| \chi \geq D e^{\frac{\alpha z}{4}} \xi(T) \|_\infty \leq \| e^{\frac{\alpha z}{4}} \xi(T) \|_\infty = \| e^{\frac{\alpha z}{4}} \eta(S) \|_\infty \]

which combined with (4.93) gives

\[ M_2(T) \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + \epsilon(C)M_2(T) + \beta^{1/2}(0)M_1(T) + M_2^2(T) \]
\[ + M_2^p(T) + \beta^{\frac{2}{3}}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)]. \]

Choosing \( C \) so large that \( \epsilon(C) \) in Equation (4.90) is sufficiently small, we obtain

\[ M_2(T) \lesssim M_2(0) + \beta^{1/2}(0)M_1(0) + M_2^2(T) + M_2^p(T) \]
\[ + \beta^{\frac{2}{3}}(0)[1 + M_2(T) + M_1(T)A(T) + M_1^2(T) + M_1^p(T)]. \]

Since \( T \) is an arbitrary time, the proof is complete. \( \Box \)
In this chapter we prove the main theorem 1.

Thanks to Lemma 8, it suffices to show Theorem 1 under the condition
\[
\|u_0(x) - \left(\frac{1 - \frac{2}{p-1}b_0}{p-1 + b_0x^2}\right)^\frac{1}{p-1}\|_\infty \leq \delta_0 \text{ and } \|\langle x \rangle^{-3}(u_0(x) - \left(\frac{1 - \frac{2}{p-1}b_0}{p-1 + b_0x^2}\right)^\frac{1}{p-1})\|_\infty \leq Cb_0^2.
\]
(5.1)

Note \(c_0 = \frac{1}{2} - \frac{1}{p-1}b_0\), then this implies \(a_0 = \frac{1}{2} - \frac{2}{p-1}b_0\).

By Theorem 5, there exists \(0 < t_* \leq \infty\) such that Equation (1.1) has a unique solution \(u(x,t)\) for \(0 \leq t < t_*\), but has no solutions on a larger time interval. Moreover, if \(t_* < \infty\), then \(\|u_0(\cdot, t)\|_\infty \to \infty\) as \(t \to t_*\). Recall that the solutions \(u(x,t), v(y,\tau)\) and \(w(y,\tau)\) and the corresponding initial conditions are related by the scaling and gauge transformations (see (3.1) and (3.8)). Take \(\lambda(0) = 1\). Then we have that \(u_0(x) = v_0(y)\).

Choose \(b_0\) so that \(Cb_0^2 = \frac{1}{2}\epsilon_0\) with \(C\) the same as in (5.1) and with \(\epsilon_0\) given in Propositions 1. Then \(v_0 \in U_{\frac{1}{2}\epsilon_0}\), by the second inequality of (4.2) on the initial value. By continuity there is a (maximal) time \(t_\# \leq t_*\) such that \(v \in U_{\epsilon_0}\) for \(t < t_\#\).

For this time interval Propositions 1 and 7 hold for \(v\) and we also have the splitting (4.4). Recall that we assume \(a = 2c - \frac{1}{2}\) in the decomposition (3.12). In particular, this implies that the initial condition can be written in the form
\[
v_0(y) = V_{a(0)\xi(0)}(y) + e^{\frac{a(0)y^2}{4}}\xi_0(y),
\]
(5.2)
where \((a(0), b(0)) = g(v_0)\) and \(\xi_0 \perp e^{-\frac{a(0)}{4} y^2}, (1 - a(0)y^2)e^{-\frac{a(0)}{4} y^2}\).

By the relation \(\beta(0) = b(0)\), Equation (4.3) and Proposition 7, we then have \(A(0), M_1(0) \lesssim 1, M_2(0) \ll 1\), while \(B(0) = 0\) by the definition. Since \(\beta(\tau) \leq \beta(0) \ll 1\), we have, by the continuity, that for a sufficient small time interval

\[ M_1(\tau), M_2(\tau), B(\tau), A(\tau) \leq \beta^{-\frac{\kappa}{2}}(\tau), \quad (5.3) \]

where, recall, the definitions of \(\beta(\tau)\) and \(\kappa\) are given in (4.6). Then Equations (4.12)-(4.15) imply that for the same time interval there exists a constant \(C\) depending on \(A(0), M_1(0)\) and \(M_2(0)\) such that

\[ M_1(\tau), B(\tau), A(\tau) \leq C \text{ and } M_2(\tau) \ll 1. \quad (5.4) \]

In fact, we have that \(M_1 \lesssim M_1(0) + \beta^\frac{\kappa}{2}(0)\), and \(M_2 \lesssim M_2(0) + \beta^\frac{\kappa}{2}(0)\). Indeed, set \(M = A + B + M_1 + \frac{M_2}{M_1(0) + \beta^\frac{\kappa}{2}(0)}\) and we have the inequality that \(M \leq C + \beta^\gamma(0)(M_1^2 + M_2^p)\) for some small \(\gamma = \gamma(p)\) by using (4.12)-(4.15). Then \(\beta(0) \ll 1\) implies that \(A, B, M_1 \lesssim 1, M_2 \lesssim M_2(0) + \beta^\frac{\kappa}{2}(0)\). By continuity, (5.3) holds on a larger interval which in turn implies (5.4) on this large time interval and so forth. Hence, (5.4) holds for \(t < t_\# = t_*\).

By the definitions of \(A(\tau)\) and \(B(\tau)\) in (4.7) and the facts that \(A(\tau), B(\tau) \leq C\) proved above and the relation \(2c = a + \frac{1}{2}\), we have that

\[ a(\tau) - \frac{1}{2} = -\frac{2}{p - 1}b(\tau) + O(\beta^2(\tau)), \quad b(\tau) = \beta(\tau)(1 + O(\beta^\kappa(\tau))). \quad (5.5) \]

Hence \(a(\tau) - \frac{1}{2} = O(\beta(\tau))\). Recall that \(a = \lambda^{-3}\frac{d}{dt}\lambda\), which can be rewritten as \(\lambda(t)^{-2} = 1 - 2\int_0^t a(\tau(s))ds\) or

\[ \lambda(t) = \left[1 - 2\int_0^t a(\tau(s))ds\right]^{-1/2} \quad (5.6) \]

where, recall that \(\lambda(0) = 1\). Since \(|a(\tau(t)) - \frac{1}{2}| = O(b(\tau(t)))\), there exists a time \(t^*\) such that \(1 = 2\int_0^{t^*} a(\tau(s))ds\), i.e. \(\lambda(t) \to \infty\) as \(t \to t^*\). Furthermore, by the
definition of $\tau$ and the property of $a$ we have that $\tau(t) \to \infty$ as $t \to t^*$. We will show below that $t^*$ is the blow-up time.

Equation (5.5) implies $b(\tau(t)) \to 0$ and $a(\tau(t)) \to \frac{1}{2}$ as $t \to t^*$. By the analysis above and the definitions of $a$, $\tau$ and $\beta$ (see (4.6)) we have

$$
\lambda(t) = (t^* - t)^{-1/2}(1 + o(1)), \quad \tau(t) = -\ln |t^* - t|(1 + o(1)),
$$

and

$$
\beta(\tau(t)) = -\frac{(p - 1)^2}{4p \ln |t^* - t|}(1 + o(1)).
$$

By (5.5) we have

$$
b(\tau(t)) = -\frac{(p - 1)^2}{4p \ln(t^* - t)} \left[ 1 + O \left( \frac{1}{\ln |t^* - t|} \right) \right]
$$

and

$$
a(\tau(t)) = \frac{1}{2} - \frac{p - 1}{2p |\ln(t^* - t)|} \left[ 1 + O \left( \frac{1}{\ln |t^* - t|} \right) \right].
$$

Therefore,

$$
c(\tau(t)) = \frac{1}{2} - \frac{p - 1}{4p |\ln(t^* - t)|} \left[ 1 + O \left( \frac{1}{\ln |t^* - t|} \right) \right].
$$

Now, using the relation between the functions $u(x, t)$ and $v(y, \tau)$ and the splitting result (Proposition 1) we obtain the following a priori estimate on the (non-rescaled) solution $u(x, t)$ of equation (1.1):

$$
\|u(t)\|_{\infty} \lesssim \lambda(t)^{\frac{2}{p-1}} [1 + M_1(\tau) + M_2(\tau)],
$$

where $\tau = \tau(t)$ is defined above. By the estimate above the majorants $M_j(\tau)$ are uniformly bounded and therefore

$$
\|u(t)\|_{\infty} \lesssim \lambda(t)^{\frac{2}{p-1}}. \quad (5.7)
$$

Recall that if $t_* < \infty$, then $\|u_0(\cdot, t)\|_{\infty} \to \infty$ as $t \to t_*$. Hence, $t_* \geq t^*$, by the bounds (5.7) and $\lambda(t) < \infty$ for $t < t^*$. On the other hand, if $t_* > t^*$, then by (3.18)
and (4.4)

\[ |u(0, t)| \geq \lambda(\tau(t))^{\frac{2}{p-1}} \left[ \left( \frac{2c(\tau(t))}{p-1} \right)^{\frac{1}{p-1}} - Cb(\tau(t))^2 \right] \to \infty, \quad (5.8) \]

as \( t \uparrow t^* \), which contradicts the existence of \( u(x, t) \) on \([0, t^*)\). Hence \( t_* = t^* \). Thus we have shown the existence of the solution \( u \) up to the time \( t^* \) having \( v \in U_{e_0} \) and obeying the estimates (5.4). Then the results above describing the dynamics of the parameters \( a, b, c \) and \( \lambda \) as well as (5.4) imply that \( u \) blows up at the time \( t^* \), see Equation (5.8), and that all other statements of Theorem 1 hold. This completes the proof of Theorem 1.

\[ \square \]
APPENDIX A

BLOW-UP DYNAMICS

In this appendix we investigate the dynamics of the parameters $a$, $b$ and $c$ described by Equations (4.27) and (4.28) if we neglect the remainder terms determined by the fluctuations $\xi$. In other words we consider the truncated dynamical system for the parameters $b$ and $c$ which reads

\begin{align*}
\dot{b} &= -\frac{2}{p-1} \left(1 + \frac{2p}{p-1}\right) b^2 + 2(c-a)b + O(b^3), \quad (A.1) \\
\dot{c} &= 2c(c-a) - \frac{2}{p-1} bc + O(b^3). \quad (A.2)
\end{align*}

This system is obtained if we set $\xi = 0$ in (4.27)-(4.28). Thus $(b^*, c^*)$ is a static solution (equilibrium) for this system iff $(b^*, c^*, 0)$ is an equilibrium for (4.27)-(4.28).

First we observe the following key fact: if $(a, b, c, \xi)$, $a = f(b, c)$, is a stationary solution to (1.1) satisfying the estimate $\|\langle y \rangle^{-3} e^{\frac{2}{p-1} \nu^2 \xi} \| \lesssim b^2$, then

$$f(0, \frac{1}{2}) = \frac{1}{2}. \quad (A.3)$$

Indeed, if $b = 0$, then the estimate above gives that $\xi = 0$ and therefore $v(y, \tau) = (\frac{2c}{p-1})^{\frac{1}{p-1}}$. Since $v(y, \tau)$ satisfies (3.2), this implies (A.3).

In order to simplify our argument, we assume that $f(b, c)$ is of the form $lc + k$ for some constant $l, k$. By (A.3) we have that $k = \frac{1}{2} - \frac{1}{2}l$. Thus we have $a = lc + \frac{1}{2} - \frac{1}{2}l$.

Before we start proving the main result, we drop $O(b^3)$ and rewrite Equations
(A.1) and (A.2) and rewrite the resulting equations as

\[ b_{\tau} = -\frac{4p}{(p-1)^2} b^2 + c_{\tau} b, \]  
\[ (A.4) \]

\[ c_{\tau} = 2c(1-l) \left( c - \frac{1}{2} \right) - \frac{1}{p-1} bc. \]  
\[ (A.5) \]

**Proposition 25.** If \( l > 1 \), then the point \((b, c) = (0, \frac{1}{2})\) is marginally stable for (A.4) and (A.5). Moreover, for \( l > 1 \), the different functions \( a = lc + \frac{1}{2} - \frac{1}{2} l \) lead to dynamics equivalent up to rescaling of (1.1).

**Proof.** A simple computation shows that \( l > 1 \) implies the stability of the point \((0, \frac{1}{2})\). Moreover we have that

\[ 2c = 1 + \frac{2}{(1-l)(p-1)} b + O(b^2). \]

Next, we prove that all the functions \( a = lc + \frac{1}{2} - \frac{1}{2} l \) are equivalent up to rescaling. First we recall that following key points when we prove the case \( a = 2c - \frac{1}{2} \), i.e. \( l = 2 \). We decompose the solution of (1.1) as

\[ u_{l=2}(x, t) = \lambda_{p-1}^{\frac{2}{p-1}}(t) \left[ \left( \frac{2c(\tau)}{p - 1 + b(\tau)y} \right)^{\frac{1}{p-1}} + \eta(y, \tau) \right] \]  
\[ (A.6) \]

with \( \eta \) satisfying \( \| \langle x \rangle^{-3} \eta(x, 0) \| = o(b(0)) \) and some orthogonality conditions, and \( \tau \) and \( y \) as defined in (4.4). And for any \( l \) we define

\[ a(t(\tau)) = \lambda^{-3}(t) \frac{d}{dt} \lambda(t). \]  
\[ (A.7) \]

We require \( 2c(0) = c_{l=2}(0) = 1 - \frac{2}{p-1} b(0) + O(b^2(0)) \). Using Equations (A.4) and (A.5) we get that \( 2c_{l=2}(\tau) = 1 - \frac{2}{p-1} b(\tau) + O(b^2) \) and \( b(\tau) \rightarrow 0^+ \). After \( \frac{dc_{l=2}(\tau)}{d\tau} = O(b^3) \). On the other hand we have that if \( 2c(0) = 2c_{l}(0) = 1 + \frac{2}{(1-l)(p-1)} b(0) + O(b^2(0)) \) and \( \| \langle x \rangle^{-3} \eta(x, 0) \| = o(b(0)) \), we fix the function as

\[ a = lc_l + \frac{1}{2} - \frac{1}{2} l, \]  
\[ (A.8) \]

after going through the same procedure we prove that \( 2c_l(\tau) = 1 + \frac{2}{(1-l)(p-1)} b(\tau) + O(b^2) \), \( b(\tau) \rightarrow 0^+ \). The two equations are related to each other in the following sense.

If \( c(0) \) in (A.6) satisfies the condition that \( c(0) = c_l(0) = 1 + \frac{2}{(1-l)(p-1)} b(0) + O(b^2) \) for \( l > 1 \) then we rewrite

\[ u_{l=2}(y, \tau) = \lambda_{l}^{\frac{2}{p-1}}(t) \left[ \left( \frac{2c_1(\tau)}{p - 1 + b(\tau)y} \right)^{\frac{1}{p-1}} + \eta_1(y_1, \tau) \right] \]  

48
with \( \lambda_1(t) = \lambda(t) \sqrt{\frac{c_{l=2}(\tau(t))}{c_{l=0}(\tau(t))}} \), \( y_1 = \lambda_1(t)x \) and \( \beta(\tau) = b(\tau) \frac{c_{l=1}(\tau)}{c_{l=2}(\tau)} \) and \( \eta_2 \) from \( \eta(y, \tau) = o(b) \). We compute to get

\[
a_1 = \lambda_1^{-3}(t) \frac{d}{dt} \lambda_1(t) = a_{l=2} \left( \frac{c_{l=2}(\tau(t))}{c_{l}(\tau(t))} \right)^2 + O(b^3) = a_l + O(b^2)
\]

\[
\frac{d}{d\tau} \beta = - \frac{4p}{(p-1)^2} \beta^2 + O(b^3)
\]

thus \( a_1 = l c_l + \frac{1}{2} - \frac{1}{2} l + O(b^2) \) which is consistent with (A.4), (A.7) and (A.8) (the remainder \( O(b^2) \) in the function of \( a_1 \) can be erased by adding some correction on \( c_l \)). Thus the case \( l = 2 \) can be transformed into the other \( l > 1 \) cases. By similar argument we prove that all these are equivalent. \(\square\)
APPENDIX B

SPECTRUM OF THE LINEAR OPERATOR $\mathcal{L}_{abc}$

We assume that the $|a_\tau|$ term is negligible in comparison with $a$ and consider the operator $\tilde{\mathcal{L}}_{abc}$, which differs from $\mathcal{L}_{abc}$ by the term $\frac{1}{4}a_\tau y^2$:

$$
\tilde{\mathcal{L}}_{abc} = -\partial_y^2 + \frac{1}{4}a^2 y^2 - \frac{a}{2} + \frac{2a}{p-1} - \frac{2pc}{p-1+by^2}.
$$

Due to the quadratic term $\frac{1}{4}ay^2$, the operator $\tilde{\mathcal{L}}_{abc}$ has a purely discrete spectrum. We can obtain a better understanding of its eigenvalues by comparing it to the harmonic oscillator

$$
\mathcal{L}_0 = -\partial_y^2 + \frac{1}{4}a^2 y^2 - \frac{a}{2}.
$$

(B.1)

Then $\mathcal{L}_0 + \frac{2}{p-1}(a - pc)$ and $\mathcal{L}_0 + \frac{2a}{p-1}$ approximate $\tilde{\mathcal{L}}_{abc}$ near zero and at infinity, respectively. The spectrum of the operator $\mathcal{L}_0$ is

$$
\sigma(\mathcal{L}_0) = \{na\mid n = 0, 1, 2, \ldots\}.
$$

(B.2)

The first three normalized eigenvectors of $\mathcal{L}_0$, which are used below, are

$$
\phi_{0a} = \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{a}{2}y^2}, \quad \phi_{1a} = \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} \sqrt{a} ye^{-\frac{a}{2}y^2}, \quad \phi_{2a} = \left(\frac{a}{8\pi}\right)^{\frac{1}{4}} (1 - ay^2) e^{-\frac{a}{2}y^2}.
$$

(B.3)

**Proposition 26.** If $p > 1$, $c \geq 0$ and $b \geq 0$, then the eigenvalues $\lambda_n$ of $\tilde{\mathcal{L}}_{abc}$ satisfy the bounds

$$
n a + \frac{2a}{p-1} \geq \lambda_n \geq na + \frac{2}{p-1}(a - pc).
$$

(B.4)
Proof. First we show that
\[
\mathcal{L}_0 + \frac{2a}{p-1} > \tilde{\mathcal{L}}_{abc} > \mathcal{L}_0 + \frac{2}{p-1}(a - pc). \tag{B.5}
\]
Since \( p > 1, b \geq 0 \) and \( c \geq 0, 0 < \frac{2pc}{p-1+by^2} \leq \frac{2pc}{p-1} \), and hence (B.5). The \( n \)-th eigenvalue of \( \tilde{L}_{abc} \) (starting from \( n = 0 \)) is by the MinMax principle
\[
\lambda_n = \sup_{\dim X = n} \inf_{\{\psi \in X^+ ||\psi|| = 1\}} \langle \psi, \tilde{\mathcal{L}}_{abc} \psi \rangle. \tag{B.6}
\]
Using the inequality \( \langle \psi, \tilde{\mathcal{L}}_{abc} \psi \rangle \geq \langle \psi, \mathcal{L}_0 \psi \rangle \) and the characterization of the spectrum of \( \mathcal{L}_0 \) we obtain
\[
\lambda_n + \frac{2}{p-1}(a - pc) \geq \sup_{\dim X = n} \inf_{\{\psi \in X^+ ||\psi|| = 1\}} \langle \psi, \mathcal{L}_0 \psi \rangle = na \tag{B.7}
\]
and similarly for the upper bound. \( \square \)

Equation (4.28) and the relation \( a = 2c - \frac{1}{2} \) suggests that \( c = a + O(b) \) where \( b \) is small. In this case Equation (B.4) shows that the operator \( \tilde{\mathcal{L}}_{abc} \) has at most three non-positive eigenvalues. The second eigenvalue corresponds to an odd eigenfunction and therefore drops out if we assume that the initial condition \( u_0(x) \) is even (so that \( x_0 = 0 \), otherwise one has to use the parameter \( x_0 \)). The two parameters \( b \) and \( c \) are chosen so that the fluctuation \( \xi \) is orthogonal to the other two eigenfunctions. Hence on the space of \( \xi \)'s the linear operator \( \mathcal{L}_{abc} \) has strictly positive spectrum.
APPENDIX C

PROOF OF THE FEYNMANN-KAC FORMULA

In this appendix we present, for the reader’s convenience, a proof of the Feynman-Kac formula (4.49)-(4.50) and the estimate (4.53) (cf. [8]). For stochastic calculus proofs of similar formulae see [11, 21, 27, 33, 51].

Let $L_0 = -\partial_y^2 + \frac{\alpha^2}{4} y^2 - \frac{\alpha}{2}$ and $L = L_0 + V$ where $V$ is a multiplication operator by the function $V(y, \tau)$, which is bounded and Lipschitz continuous in $\tau$. Let $U(\tau, \sigma)$ and $U_0(\tau, \sigma)$ be the propagators generated by the operators $-L$ and $-L_0$, respectively. The integral kernels of these operators will be denoted by $U(\tau, \sigma)(x, y)$ and $U_0(\tau, \sigma)(x, y)$.

**Theorem 27.** The integral kernel of $U(\tau, \sigma)$ can be represented as

$$U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{\int_{\sigma}^{\tau} V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)$$  \hspace{1cm} (C.1)

where $d\mu(\omega)$ is a probability measure (more precisely, a conditional harmonic oscillator, or Ornstein-Uhlenbeck, probability measure) on the continuous paths $\omega : [\sigma, \tau] \rightarrow \mathbb{R}$ with $\omega(\sigma) = \omega(\tau) = 0$, and $\omega_0(\cdot)$ is the path defined as

$$\omega_0(s) = e^{-\alpha s} \left[ \frac{e^{2\alpha \sigma} - e^{2\alpha s}}{e^{2\alpha \sigma} - e^{2\alpha \tau}} (e^{\alpha \tau} x - y) + y \right].$$  \hspace{1cm} (C.2)

**Remark 5.** $d\mu(\omega)$ is the Gaussian measure with mean zero and covariance $(-\partial_s^2 + \frac{1}{4} \alpha^2)^{-1}$, normalized to 1. The path $\omega_0(s)$ solves the boundary value problem

$$(-\partial_s^2 + \frac{1}{4} \alpha^2)\omega_0 = 0 \quad \text{with} \quad \omega(\sigma) = y \quad \text{and} \quad \omega(\tau) = x.$$  \hspace{1cm} (C.3)

Below we will also deal with the normalized Gaussian measure $d\mu_{xy}(\omega)$ with mean $\omega_0(s)$ and covariance $(-\partial_s^2 + \frac{1}{4} \alpha^2)^{-1}$. This is a conditional Ornstein-Uhlenbeck probability measure on continuous paths $\omega : [\sigma, \tau] \rightarrow \mathbb{R}$ with $\omega(\sigma) = y$ and $\omega(\tau) = x$ (see e.g. [21, 27, 51]).
Now, assume in addition that the function $V(y, \tau)$ satisfies the estimates

$$V \leq 0 \text{ and } |\partial_y V(y, \tau)| \leq C \beta^{-(\frac{1}{2})}(\tau),$$

where $\beta(\tau)$ is a positive function. Then Theorem 27 implies Equation (4.53) by the following corollary.

**Corollary 28.** Under (C.4),

$$\left| \partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega) \right| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau).$$

**Proof.** By Fubini’s theorem

$$\partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)$$

$$= \int \partial_y \left[ \int_0^\tau V(\omega_0(s) + \omega(s), s)ds \right] e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega)$$

Equation (C.4) implies

$$\left| \partial_y \int_\sigma^\tau V(\omega_0(s) + \omega(s), s)ds \right| \leq |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau), \text{ and } e^{\int_\sigma^\tau V(\omega_0(s) + \omega(s), s)ds} \leq 1.$$

Thus

$$\left| \partial_y \int e^{\int_0^\tau V(\omega_0(s) + \omega(s), s)ds} d\mu(\omega) \right| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau) \int d\mu(\omega) = |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{1/2}(\tau)$$

to complete the proof.

**Proof of Theorem 27.** We begin with the following extension of the Ornstein-Uhlenbeck process-based Feynman-Kac formula to time-dependent potentials:

$$U(\tau, \sigma)(x, y) = U_0(\tau, \sigma)(x, y) \int e^{-\int_\sigma^\tau V(\omega(s), s)ds} d\mu_{xy}(\omega).$$

where $d\mu_{xy}(\omega)$ is the conditional Ornstein-Uhlenbeck probability measure described in Remark 5 above. This formula can be proven in the same way as the one for time independent potentials (see [21], Equation (3.2.8)), i.e. by using the Kato-Trotter formula and evaluation of Gaussian measures on cylindrical sets. Since its proof contains a slight technical wrinkle, for the reader’s convenience we present it below.

Now changing the variable of integration in (C.5) as $\omega = \omega_0 + \tilde{\omega}$, where $\tilde{\omega}(s)$ is a continuous path with boundary conditions $\tilde{\omega}(\sigma) = \tilde{\omega}(\tau) = 0$, using the translational change of variables formula $\int f(\omega) d\mu_{xy}(\omega) = \int f(\omega_0 + \tilde{\omega}) d\mu(\tilde{\omega})$, which can be proven by taking $f(\omega) = e^{i\langle \omega, \xi \rangle}$ and using (C.3) (see [21], Equation (9.1.27)) and omitting the tilde over $\omega$ we arrive at (C.1).
There are at least three standard ways to prove (C.5): by using the Kato-Trotter formula, by expanding both sides of the equation in $V$ and comparing the resulting series term by term and by using Ito’s calculus (see [33, 51, 50, 21]). The first two proofs are elementary but involve tedious estimates while the third proof is based on a fair amount of stochastic calculus. For the reader’s convenience, we present the first elementary proof of (C.5).

Before starting proving (C.5) we establish the following result. We define the operator $\mathcal{K}$ as

$$
\mathcal{K}(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s)V(\sigma + s, \cdot) U_0(\sigma + s, \sigma) ds - U_0(\sigma + \delta, \sigma) \int_0^\delta V(\sigma + s, \cdot) ds
$$

(C.6)

**Lemma 29.** For any $\sigma \in [0, \tau]$ and $\xi \in C_0^\infty$ we have, as $\delta \to 0^+$,

$$
\sup_{0 \leq \sigma \leq \tau} \left\| \frac{1}{\delta} \mathcal{K}(\sigma, \delta) U(\sigma, 0) \xi \right\|_2 \to 0.
$$

(C.7)

**Proof.** If the potential term, $V$, is independent of $\tau$, then the proof is standard (see, e.g. [50]). We use the property that the function $V$ is Lipschitz continuous in time $\tau$ to prove (C.7). The operator $\mathcal{K}$ can be further decomposed as

$$
\mathcal{K}(\sigma, \delta) = \mathcal{K}_1(\sigma, \delta) + \mathcal{K}_2(\sigma, \delta)
$$

with

$$
\mathcal{K}_1(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s)V(\sigma, \cdot) U_0(\sigma + s, \sigma) ds - \delta U_0(\sigma + \delta, \sigma) V(\sigma, \cdot)
$$

and

$$
\mathcal{K}_2(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s)[V(\sigma + s, \cdot) - V(\sigma, \cdot)] U_0(\sigma + s, \sigma) ds
$$

$$
- U_0(\sigma + \delta, \sigma) \int_0^\delta [V(\sigma + s, \cdot) - V(\sigma, \cdot)] ds.
$$

By the facts that $U(\tau, \sigma)$ is uniformly $L^2$-bounded and the function $V(\tau, y)$ is Lipschitz continuous in $\tau$ we have that

$$
\|\mathcal{K}_2(\sigma, \delta)\|_{L^2 \to L^2} \lesssim 2 \int_0^\delta s ds = \delta^2.
$$
We rewrite $K_1(\sigma, \delta)$ as

\[
K_1(\sigma, \delta) = \int_0^\delta U_0(\sigma + \delta, \sigma + s)\{V(\sigma, \cdot)[U_0(\sigma + s, \sigma) - 1] - [U_0(\sigma + s, \sigma) - 1]V(\sigma, \cdot)\}ds.
\]

Let $\xi(\sigma) = U(\sigma, 0)\xi$. We claim that for a fixed $\sigma \in [0, \tau]$,

\[
\|K_1(\sigma, \delta)\xi(\sigma)\|_2 = o(\delta). \tag{C.8}
\]

Indeed, the fact that $\xi_0 \in C_0^\infty$ implies $L_0\xi(\sigma), L_0V(\sigma)\xi(\sigma) \in L^2$, consequently (see [49])

\[
\lim_{s \to 0^+} \frac{(U_0(\sigma + s, \sigma) - 1)g}{s} \to L_0g,
\]

for $g = \xi(\sigma)$ or $V(\sigma, y)\xi(\sigma)$ which implies our claim. Since the set of functions $\{\xi(\sigma) | \sigma \in [0, \tau]\} \subseteq L_0L^2$ is compact and $\|\frac{1}{2}K_1(\sigma, \delta)\|_{L^2 \to L^2}$ is uniformly bounded, we have (C.8) as $\delta \to 0$ uniformly in $\sigma \in [0, \tau]$.

Collecting the estimates on the operators $K_i, i = 1, 2$, we arrive at (C.7).

**Lemma 30.** Equation (C.5) holds.

**Proof.** In order to simplify our notation, in the proof that follows we assume, without losing generality, that $\sigma = 0$. We divide the proof into two parts. First we prove that for any fixed $\xi \in C_0^\infty$ the following Kato-Trotter type formula is valid:

\[
U(\tau, 0)\xi = \lim_{n \to \infty} \prod_{0 \leq k \leq n-1} \left[U_0(\frac{k + 1}{n}\tau, \frac{k}{n}\tau) e^{\int_{\frac{k}{n}\tau}^{\frac{k+1}{n}\tau} V(y, s)ds}\right] \xi \tag{C.9}
\]

in the $L^2$ space. We start with the formula

\[
U(\tau, 0) - \prod_{0 \leq k \leq n-1} \left[U_0(\frac{k + 1}{n}\tau, \frac{k}{n}\tau) e^{\int_{\frac{k}{n}\tau}^{\frac{k+1}{n}\tau} V(y, s)ds}\right] = \prod_{0 \leq k \leq n-1} \left[U(\frac{k + 1}{n}\tau, \frac{k}{n}\tau) - \prod_{0 \leq k \leq n-1} \left[U_0(\frac{k + 1}{n}\tau, \frac{k}{n}\tau) e^{\int_{\frac{k}{n}\tau}^{\frac{k+1}{n}\tau} V(y, s)ds}\right] \right]
\]

\[
= \sum_{0 \leq j \leq n} \prod_{0 \leq k \leq n-1} U_0(\frac{k + 1}{n}\tau, \frac{k}{n}\tau) e^{\int_{\frac{k}{n}\tau}^{\frac{k+1}{n}\tau} V(y, s)ds} A_j U(\frac{j}{n}\tau, 0)
\]

with the operator

\[
A_j = U_0(\frac{j + 1}{n}\tau, \frac{j}{n}\tau) e^{\int_{\frac{j}{n}\tau}^{\frac{j+1}{n}\tau} V(y, s)ds} - U(\frac{j + 1}{n}\tau, \frac{j}{n}\tau).
\]
Using that the operators $U_0(\tau, \sigma)$ and $U(\tau, \sigma)$ are uniformly bounded in $\tau$ and $\sigma$ in any compact set, we obtain

$$
\| [U(\tau, 0) - \prod_{0 \leq k \leq n-1} U_0(\frac{k+1}{n} \tau, \frac{k}{n} \tau) e^{\int_{\frac{k}{n} \tau}^{\frac{(k+1)}{n} \tau} V(y, s) ds}] \xi \|_2 
\leq \max_j n \| \prod_{j \leq k \leq n-1} U_0(\frac{k+1}{n} \tau, \frac{k}{n} \tau) e^{\int_{\frac{k}{n} \tau}^{\frac{(k+1)}{n} \tau} V(y, s) ds} A_j U(\frac{j}{n} \tau, 0) \xi \|_2 \tag{C.10}
$$

\[
\lesssim n \max_j \| A_j + \mathcal{K}(\frac{j}{n} \tau, \frac{1}{n} \tau) \|_{L^2 \to L^2} + \max_j \| \mathcal{K}(\frac{j}{n} \tau, \frac{1}{n} \tau) U(\frac{j}{n}, 0) \xi \|_2
\]

where, recall the definition of $\mathcal{K}$ from (C.6). Now we claim that

$$
\| A_j + \mathcal{K}(\frac{j}{n} \tau, \frac{1}{n} \tau) \|_{L^2 \to L^2} \lesssim \frac{1}{n^2}. \tag{C.11}
$$

Indeed, by Duhamel’s principle we have

$$
U(\frac{j+1}{n} \tau, \frac{j}{n} \tau) = U_0(\frac{j+1}{n} \tau, \frac{j}{n} \tau) + \int_0^{\frac{j}{n} \tau} U_0(\frac{j+1}{n} \tau, s) V(y, s) U(s, \frac{j}{n} \tau) ds.
$$

Iterating this equation we find

$$
\| U(\frac{j+1}{n} \tau, \frac{j}{n} \tau) - U_0(\frac{j+1}{n} \tau, \frac{j}{n} \tau) - \int_0^{\frac{j}{n} \tau} U_0(\frac{j+1}{n} \tau, s) V(y, s) U(s, \frac{j}{n} \tau) ds \|_{L^2 \to L^2} \lesssim \frac{1}{n^2}.
$$

On the other hand we expand $e^{\int_{\frac{j}{n} \tau}^{\frac{(j+1)}{n} \tau} V(y, s) ds}$ to get

$$
\| U_0(\frac{j+1}{n} \tau, \frac{j}{n} \tau) e^{\int_{\frac{j}{n} \tau}^{\frac{(j+1)}{n} \tau} V(y, s) ds}
- U_0(\frac{j+1}{n} \tau, \frac{j+1}{n} \tau) e^{\int_{\frac{(j+1)}{n} \tau}^{\frac{j}{n} \tau} V(y, s) ds} \|_{L^2 \to L^2} \lesssim \frac{1}{n^2}.
$$

By the definition of $\mathcal{K}$ and $A_j$ we complete the proof of (C.11). Equations (C.7), (C.10) and (C.11) imply (C.9). This completes the first step.

In the second step we compute the integral kernel, $G_n(x, y)$, of the operator

$$
G_n = \prod_{0 \leq k \leq n-1} U_0(\frac{k+1}{n} \tau, \frac{k}{n} \tau) e^{\int_{\frac{k}{n} \tau}^{\frac{(k+1)}{n} \tau} V(\cdot, s) ds}
$$

in (C.9). By the definition, $G_n(x, y)$ can be written as

$$
G_n(x, y) = \int \cdots \int U_n(x_{k+1}, x_k) e^{\int_{\frac{k}{n} x_k}^{\frac{(k+1)}{n} x_k} V(x_k, s) ds} dx_1 \cdots dx_{n-1} \tag{C.12}
$$
with \( x_n = x \), \( x_0 = y \) and \( U_\tau(x, y) \equiv U_0(0, \tau)(x, y) \) is the integral kernel of the operator \( U_0(\tau, 0) = e^{-L_0 \tau} \). We rewrite (C.12) as

\[
G_n(x, y) = U_\tau(x, y) \int e^{\sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} V(x_k, s) ds} d\mu_n(x_1, \ldots, x_n),
\]

where

\[
d\mu_n(x_1, \ldots, x_n) = \prod_{0 \leq k \leq n-1} \frac{U_n^\tau(x_{k+1}, x_k)}{U_\tau(x, y)} dx_k \ldots dx_{k-1}.
\]

Since \( G_n(x, y)|_{V=0} = U_\tau(x, y) \) we have that \( \int d\mu_n(x_1, \ldots, x_n) = 1 \). Let \( \Delta = \Delta_1 \times \ldots \times \Delta_n \), where \( \Delta_j \) is an interval in \( \mathbb{R} \). Define a cylindrical set

\[
P_\Delta^n = \{ \omega : [0, \tau] \to \mathbb{R} \mid \omega(0) = y, \omega(\tau) = x, \omega(k\tau/n) \in \Delta_k, 1 \leq k \leq n-1 \}.
\]

By the definition of the measure \( d\mu_{xy}(\omega) \), we have \( \mu(P_\Delta^n) = \int_\Delta d\mu_n(x_1, \ldots, x_n) \). Thus, we can rewrite (C.13) as

\[
G_n(x, y) = U_\tau(x, y) \int e^{\sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} V(x_k, s) ds} d\mu_{xy}(\omega),
\]

By the dominated convergence theorem the integral on the right hand side of (C.14) converges in the sense of distributions as \( n \to \infty \) to the integral on the right hand side of (C.5). Since the left hand side of (C.14) converges to the left hand side of (C.5), also in the sense of distributions (which follows from the fact that \( G_n \) converges in the operator norm on \( L^2 \) to \( U(\tau, \sigma) \)), (C.5) follows.

Note that on the level of finite dimensional approximations the change of variables formula can be derived as follows. It is tedious, but not hard, to prove that

\[
\prod_{0 \leq k \leq n-1} U_n(x_{k+1}, x_k) = e^{-\alpha (y - e^{-\alpha \tau} y)^2 / 2(1 - e^{-2\alpha \tau})} \prod_{0 \leq k \leq n-1} U_n(y_{k+1}, y_k)
\]

with \( y_k = x_k - \omega_0(k/n \tau) \). By the definition of \( \omega_0(s) \) and the relations \( x_0 = y \) and \( x_n = x \) we have

\[
G_n(x, y) = U_\tau(x, y) G_n^{(1)}(x, y)
\]

where

\[
G_n^{(1)}(x, y) = \frac{1}{4\pi \sqrt{\alpha (1 - e^{-2\alpha \tau})}} \times \int \cdots \int \prod_{0 \leq k \leq n-1} U_n(y_{k+1}, y_k) e^{\int_{y_k}^{y_{k+1}} V(y_k + \omega_0(k/n \tau), s) ds} dy_1 \cdots dy_{k-1}.
\]
Since \( \lim_{n \to \infty} G_n \xi \) exists by (C.7), we have \( \lim_{n \to \infty} G_n^{(1)} \xi \) (in the weak limit) exists also. As shown in [21], \( \lim_{n \to \infty} G_n^{(1)} = \int e^{\int_0^T V(\omega_0(s)+\omega(s), s)ds} d\mu(\omega) \) with \( d\mu \) being the (conditional) Ornstein-Uhlenbeck measure on the set of path from 0 to 0. This completes the derivation of the change of variables formula.

**Remark 6.** In fact, Equations (C.9), (C.15) and (C.16) suffice to prove the estimate in Corollary 28.
BIBLIOGRAPHY


