PREDICTIVE MODELING OF HEALTHY AND AMPUTEE WALKING USING
A SIMPLE PLANAR MODEL

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Abstract

by

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It is generally accepted that the design of the foot can influence gait for both robots and amputees. However, the nonlinear nature of bipedal systems combined with the intermittent contacts required for walking makes rigorous investigation of the influence of foot design difficult. Further, it is not known how to predict even normal human walking across a range of speeds. This work extends a robot modeling and control technique to create highly accurate, yet computationally inexpensive, predictive models of human walking. The models were validated using new robotic hardware experiments and existing data from human subject studies. The results have implications both for the design and control of robots and for models of amputee gait.

The models are grounded on a hybrid zero dynamics (HZD)-based control technique originally developed for planar bipedal robots with point feet and instantaneous transfer of support. To account for the function of the human foot, curved feet were incorporated into the HZD framework, which required rederiving the model of the transfer of support and the stability criteria. The effects of foot design on gait were systematically investigated in both simulation and hardware. Due to interaction effects between foot radius and center of curvature location, there are two distinct design strategies that yield energetically efficient gaits.
In addition, models of human walking must also include the effects of toe-off. This can be achieved either with an impulsive force at the hip during the impact phase or with ankle joints. Using a six-link planar model with ankle joints and curved feet, normal human joint kinematics and energy expenditures can be predicted very accurately at walking speeds ranging from very slow to very fast using a torque-squared-based objective function. By removing one of the ankle joints from the symmetric human model, asymmetric amputee gait can be investigated. It is shown that small reductions in amputated/contralateral leg symmetry lead to large improvements in energetic efficiency.
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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

1.1.1 Background on Amputees

As of 2005, there were an estimated 623,000 people living with major lower-limb loss in the United States (Ziegler-Graham et al., 2008). The majority (81%) of the amputations were due to complications from dysvascular disease, particularly diabetes (Ziegler-Graham et al., 2008). The two most common types of major lower-limb amputation are transtibial (below-knee) and transfemoral (above-knee) (Dillingham et al., 2002). Regardless of the amputation type, most amputees use a prosthesis for ambulation at least some of the time (Raichle et al., 2008).

Lower-limb prostheses are composed of multiple components which can be adjusted relative to one another. The process of adjusting the relative positions of the components to find the optimal configuration is called alignment. Currently, prosthesis alignment depends almost exclusively on the qualitative observations of the prosthetist and amputee, although there have been attempts to make it more quantitative (Blumentritt, 1997; Hansen et al., 2003; Srinivasan et al., 2009). It appears that there is not a unique, ideal alignment, but rather several discrete alignments that all produce acceptable gait (Geil, 2002; Zahedi et al., 1986). It has been suggested that it can take up to one week for an amputee to fully adapt to a change of component in his/her prosthetic foot (English et al., 1995), but surprisingly, alignment perturbations produce relatively little variation in gait patterns. There have been
multiple studies demonstrating that walking speed, gait cadence and peak ground reaction forces do not change with alignment perturbations (Neumann, 2009), suggesting that amputees compensate for improperly aligned prostheses. Because of all of these factors, even experienced prosthetists often cannot repeat a given alignment (Zahedi et al., 1986).

Beyond amputation, amputees are at increased risk for numerous secondary physical conditions (Gailey et al., 2008). Over 50% of amputees report at least one fall per year, which is significantly higher than the general population (Kulkarni et al., 1996). In elderly adults, over 50% of all falls result in at least minor injury (Nevitt et al., 1991), and it seems likely that the risk of injury is similar for amputees. Back pain is very common among lower-limb amputees (Ehde et al., 2001), likely due in part to altered spinal loading (Hendershot and Wolf, 2014), which in turn arises from altered pelvis and trunk motion (Goujon-Pillet et al., 2008). Similarly, amputees are at increased risk for osteoarthritis (Kulkarni et al., 1998; Struyf et al., 2009), with improperly aligned prostheses increasing the risk (Hurley et al., 1990). In addition, improperly aligned prostheses are thought to increase both the axial and shear loading on the stump (Pinzur et al., 1995; Sanders et al., 1998), which presumably increases user discomfort. Increased stump loading is also positively correlated with an increase in stump skin problems (Levy, 1980).

While many of the secondary conditions are exacerbated by poorly aligned prostheses, it is likely that some conditions are caused in part by the lost function of the joints. Most current prostheses fail to completely restore the lost function of the foot and ankle, leading to reduced amputee walking performance as compared to the general population. Amputees tend to walk more slowly than their healthy counterparts (Hermodsson et al., 1994). Generally, amputees expend approximately the same amount of metabolic energy per unit time as that of healthy humans, but because of their slower speed, they expend more energy per unit distance (Waters and
Mulroy, 1999). It is likely that at least part of the reduced performance is due to the fact that most commercially available prosthetic feet are passive (Martin et al., 2010), which means the amputated ankle cannot provide positive work as the physiological ankle does (DeVita et al., 2007). It is well documented that unilateral amputee gait is asymmetric between the amputated and contralateral sides (Neumann, 2009; Sagawa et al., 2011), with some of the key differences including step length, step period, knee flexion range, and peak ground reaction forces. While achieving amputated/contralateral kinematic symmetry is often considered to be the objective of prosthesis design (Bamberg et al., 2010; Hannah et al., 1984; Sjödahl et al., 2002), it is unclear if this actually produces the best gait. Because both the musculature and the mass properties of the prosthetic limb are different from those of the intact limb, the kinetics required to generate symmetric kinematics will be different. Thus, there is likely a trade-off between kinematically symmetric amputee gait and energetically efficient amputee gait. Understanding the effects of each strategy on amputee gait could allow physical therapists to make more informed decisions regarding patient care and biomechanists to make more realistic modeling assumptions, specifically in regard to the choice of objective function used to predict amputee gait.

Because most prior experimental studies of amputee gait have had small sample sizes, it has been difficult to obtain statistically significant results (Hafner et al., 2002; Neumann, 2009; Sagawa et al., 2011). Predictive models of amputee gait could be used to provide some much needed quantitative results. Further, such models could be used to systematically evaluate and optimize prosthesis designs, thereby reducing the development time and cost while improving the design.

1.1.2 Background on Models of Human Gait

At one extreme of modeling human walking are very basic models (Kuo, 2007) that typically approximate one aspect of gait using a simple mechanical system.
Examples include modeling the swing leg as a pendulum (Doke et al., 2005) and modeling the step-to-step transition with a two-link, point-foot biped (Donelan et al., 2002b). Such models clearly demonstrate the key underlying gait mechanics, and their straightforward mathematics tend to lead to easily verifiable predictions that, when tested with human subject experiments, are consistently accurate (Kuo, 2007).

At the other end of the complexity spectrum are detailed, often three-dimensional, forward dynamics models (Anderson and Pandy, 2001; Liu et al., 2008). When driven by muscles, such models can help determine how muscles affect the energy flow among the limbs (Zajac et al., 2002) and how muscle forces change with differing conditions (Neptune et al., 2008), ultimately providing insight into muscle synergies and coordination. Model control either determines muscle forces to reproduce motion capture data (e.g. Thelen and Anderson, 2006) or assumes an objective function for forward simulation (e.g. Ackermann and van den Bogert, 2010). The complicated nature of these systems limits the number of cases examined because both the optimization and simulation are time consuming.

Models between the two extremes can be complex enough to capture multiple gait features simultaneously, yet simple enough to enable rapid motion optimization. Planar models with controllers based on the hybrid zero dynamics (HZD) framework (Westervelt et al., 2007) are one example. HZD-based control parameterizes the desired joint trajectories as functions of a phase variable that measures step progression, and not of time explicitly. Prior work has shown that an HZD-inspired model can accurately match human hip and knee kinematics at one walking speed (Srinivasan et al., 2008). From a different prospective, human gait has been translated to robotic gait using an HZD-based controller (Sinnet and Ames, 2012).

The ability to predict human gait, rather than simply mimic measured experimental gait, could make intermediate complexity models powerful design tools. Despite the many possible options for walking gaits at a particular speed, humans consis-
tently choose the same gait patterns for a particular speed (Winter, 1983), indicating that an appropriate optimization should be able to predict human gait. In addition, it appears that humans minimize a convex cost function to determine gait speed and cadence (Bertram and Ruina, 2001; Bertram, 2005). It has been suggested that the cost function is metabolic cost, possibly normalized by speed and distance traveled (Ackermann and van den Bogert, 2010; Bertram, 2005; Zarrugh et al., 1974). Unfortunately, metabolic cost is not a quantity that can be directly calculated within a mathematical model, although it can be estimated as the square of the muscle activations (Silder et al., 2012). While it is tempting to equate mechanical power with metabolic cost, this appears to be inaccurate, although there is some evidence that the two quantities are correlated (Adamczyk et al., 2006; Burdett et al., 1983; Foerster et al., 1995). Further, it has been shown that minimum energy objective functions do not match experimental results particularly well (Ackermann and van den Bogert, 2010). Thus, in order to accurately predict a typical human gait, an optimization function needs to be identified and validated using human subject data.

To date, predictive simulations of normal human walking that capture more than one aspect of gait have typically been used only to model the self-selected walking speed with step length constrained (for example, Anderson and Pandy, 2001; Ren et al., 2007; Xiang et al., 2011). Simulated joint kinematics generally show reasonable agreement with human data, although quantitative results are rarely provided. A variety of frameworks for these predictive models has been used, including two- (Ogihara and Yamazaki, 2001; Sellers et al., 2005) and three-dimensional (Ackermann and van den Bogert, 2010; Anderson and Pandy, 2001) musculoskeletal models, three-dimensional zero-moment-point (ZMP) models (Kim et al., 2008; Xiang et al., 2009), and planar models with ideal actuators at the joints (Peng and Ono, 2003; Ren et al., 2007).

The ability to predict gait at the self-selected speed, however, does not necessarily
translate into the ability to predict gait over a range of speeds because there are well
documented speed-related changes in gait (Kirtley et al., 1985; Lythgo et al., 2011;
Oberg et al., 1994). The most significant changes are increases in the step length, the
ranges of motion of all joints, and the required mechanical energy as walking speed
increases. The ability to predict gait over a range of speeds is necessary for the design
and selection of mechanical interventions, such as prostheses.

From the existing modeling frameworks, the two with complexity most appropri-
ate for a predictive study are the ZMP-based models and the planar models with ideal
actuators. ZMP-based control was originally developed to control bipedal robots. It
exploits the fact that for an arbitrary system, it is possible to find a point, called
the ZMP, about which there is no net external moment (Vukobratovic and Borovac,
2004). To keep the biped from falling over, ZMP-based control requires the ZMP to
be within the support polygon of the biped at every instant in time. During dou-
ble support, when both feet are on the ground, the ZMP shifts from under the old
stance foot to under the new stance foot. While robots controlled in this manner
are typically quite stable, they are energetically inefficient. It has been estimated
that Honda’s Asimo (ASIMO, 2014) requires approximately ten times the energy of
a human to walk when scaled for size and speed (Collins et al., 2005). More trou-
bling from a modeling standpoint is that humans do not keep their ZMP within the
support polygon as ZMP-based control methods require (Dasgupta and Nakamura,
1999). Thus, a planar model with ideal actuators is the most promising option to
develop a predictive model of human gait. The primary challenges are in determining
which features must be included and how to control the model.

The model in this work is grounded in HZD-based control because it offers an
appealing compromise between complexity and simulation speed, and prior work
has suggested that humans control their motion based on phase variables (Grasso
et al., 1998; Gregg et al., 2014). Unfortunately, HZD-based control has only been
formally developed for point-foot bipeds. Since it has been estimated that human feet reduce energy expenditure by approximately 50% (Adamczyk et al., 2006), a means of accounting for the feet must be determined. Due to the deformation of the stance foot, modeling the foot as a flat, rigid plate jointed to the shank is unlikely to yield accurate results with an HZD-based model. Rather, a curved foot is favored since the center of pressure (CoP) trace forms a roughly circular arc through most of stance when plotted in a shank-fixed coordinate frame (Hansen et al., 2004a). This effective curved foot is called the ankle-foot rollover shape and is found in at least healthy adults (Hansen et al., 2004a), healthy children (Hansen and Meier, 2010), and amputees (Hansen et al., 2000). At least for healthy adults, the effective foot radius is remarkably invariant to changes in walking speed (Hansen et al., 2004a), ground incline (Hansen et al., 2004b), and shoe heel height (Hansen and Childress, 2004), although the center of curvature location relative to the shank shifts in all three cases. For prosthetic feet, the effective radius depends on both the style of the foot (Curtze et al., 2009) and the alignment (Hansen, 2008).

Beyond helping to form the ankle-foot rollover shape (Wang and Hansen, 2010), the stance ankle performs significant positive work, particularly toward the end of the step (DeVita et al., 2007). This work helps to control the whole body center of mass (CoM) acceleration (Anderson and Pandy, 2003) and to redirect the CoM velocity in an energetically optimal manner during the step-to-step transition (Donelan et al., 2002b). This phase of the gait cycle is called toe-off.

1.1.3 Background on HZD-Based Control

Under HZD-based control, the trajectories of the actuated degrees of freedom (DOF) during the single support phase are constrained to be functions of the monotonically increasing, unactuated DOF (the phase variable), rather than functions of time (Westervelt et al., 2007). These virtual constraints are enforced by encoding
the desired joint trajectories into the output functions to be zeroed when the system is written in zero dynamics coordinates (Isidori, 1995). Foot impacts and transfer of support cause the system to be hybrid. At impact, an algebraic map is used to determine the post-impact state of the biped. If the post-impact state is identical for every step, the gait is periodic, and local stability can be determined by analyzing this fixed point of the corresponding Poincaré map (Westervelt et al., 2004). For a simple, passive, planar two-link biped, it is possible to numerically determine the basin of attraction for orbital stability (Garcia et al., 1998), but this is not practical for significantly more complicated systems. Because of this, there has been some work using Lyapunov functions to estimate the basin of attraction (Tedrake et al., 2010). Recent work has also enabled formal stability analysis of gaits that are not periodic (Yang et al., 2009; Manchester et al., 2011).

HZD-based control has been successfully implemented on point-foot bipedal robots with both rigid (Chevallereau et al., 2003) and compliant drivetrains (Sreenath et al., 2011), as well as on point-foot bipedal robots with parallel knee compliance (Yang et al., 2008). HZD-based controllers have also been developed for point-foot biped running (Sreenath et al., 2013). Theoretical work has shown that HZD-based control can be implemented on three-dimensional, point-foot bipeds with two degrees of underactuation (Chevallereau et al., 2009) and that an HZD-like control method can be utilized to control and stabilize three-dimensional bipeds with more than two degrees of underactuation (Hamed and Grizzle, 2014; Ramezani et al., 2014). The feasibility of the three-dimensional controllers has primarily been demonstrated in simulation, although some hardware experiments have been conducted (Grizzle, 2012; Grizzle et al., 2013).

HZD-based control has only been formally developed for point-foot bipeds. However, the point-foot controller has been used to achieve stable walking, with some difficulty, for curved foot bipedal robots (McCandless, 2008). Also, Grizzle (2012)
has demonstrated walking and running with passive feet in sneakers on the robotic bi
ed MABEL using a point-foot controller, and Ames et al. (2012) achieved stable walk-
ing with the flat-footed NAO humanoid robot using a point-foot controller opti-
mized to approximate human data. The ability to truly design HZD-based controllers
for bipeds with curved feet is needed to allow accurate investigation of human gait.
Previous work applying HZD-based control to curved foot bipeds has either been
robot-specific (Kinugasa et al., 2008) or incomplete (Srinivasan et al., 2008).

1.1.4 Previous Research on Curved Feet

As mentioned previously, it appears that human feet, which function as rigid cir-
cular arcs in terms of the control of the CoP, might improve the efficiency of gait,
although the effects of artificially enforcing foot radius on human gait are unclear.
In prior studies (Adamczyk et al., 2006; Wang and Hansen, 2010), healthy sub-
jects wore a shoe or rigid boot with a constant radius arc rigidly attached, and the effects
on gait were quantified. In the shoe experiments, the foot radii were 25%LL, 40%LL,
55%LL, and infinite (flat shoe bottom) (Hansen and Wang, 2011; Wang and Hansen,
2010). To account for the varying biped sizes, the radii are normalized by leg length
LL, which is defined as the distance from the ground to the hip when the leg is vertical
with the knee straight. The joint kinematics at the hip and knee remained relatively
constant regardless of foot radius, while the range of motion of the stance ankle
decreased as foot radius decreased. Pairwise comparisons indicated larger metabolic
cost with the infinite foot radius but no statistical differences in metabolic cost among
the three finite foot radii. In the rigid boot experiments, the foot radii ranged from
2 to 43%LL (Adamczyk et al., 2006). As foot radii increased, mechanical work
decreased. Metabolic cost, however, exhibited a quadratic relationship, with the
minimum occurring at about 30%LL. The design of prostheses could potentially
benefit from a better understanding of the effects of foot radius on human joint
kinematics and metabolic cost.

In addition, it appears that curved feet can help improve both the efficiency and robustness of robotic biped walking gaits as compared to point feet. Curved feet with radii up to $100\%LL$ were shown to be more efficient than point feet for biped robots without knees (Asano and Luo, 2007; Kwan and Hubbard, 2007; McGeer, 1990). For bipeds with knees, similar work demonstrated the benefit of feet with radii up to $45\%LL$ (Martin and Schmiedeler, 2012; Ono et al., 2004; Post, 2013). All of the studies used simulation to investigate the effect of foot radius on energy efficiency, while Post (2013) and Ono et al. (2004) also performed hardware experiments. In addition, the robustness of the simplest walking model (Garcia et al., 1998) was shown to improve with increasing foot radii up to $90\%LL$ in terms of both local stability (Asano and Luo, 2007; Hobbelen and Wisse, 2007; Jeon et al., 2013) and the ability to reject finite-sized disturbances (Hobbelen and Wisse, 2007). Beyond radius, the other most significant geometric parameter of a curved foot is the location of the center of curvature, referred to herein as ankle offset ($X_F$). Positive ankle offset is known to yield superior results in minimally actuated bipeds (Wisse and van Frankenhuyzen, 2003), but the parameter has otherwise received relatively little attention in the robotics literature.

To date, most robots with curved feet have been passive or minimally actuated (Table 1.1). Passive bipeds are mechanisms capable of walking down a shallow slope powered only by gravity. Minimally actuated bipeds are similar to passive bipeds, but have actuators controlled with simple control laws to enable walking on level ground. The absolute simplest physically realizable biped capable of walking is a two-link, planar passive biped. The one built by McGeer (1990) had a foot radius of $40\%LL$ and an ankle offset of $1.5\%LL$. A substantially more complicated, three-dimensional passive biped (Collins et al., 2001) had curved feet consisting of an inner rail of radius $16\%LL$ and an outer rail of radius $37\%LL$ over most of the rail
TABLE 1.1

FOOT PROPERTIES FOR PASSIVE AND MINIMALLY ACTUATED BIPEDS

<table>
<thead>
<tr>
<th>Reference</th>
<th>Type</th>
<th>Knees</th>
<th>Radius (%LL)</th>
<th>X&lt;sub&gt;F&lt;/sub&gt; (%LL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>McGeer (1990)</td>
<td>P 2D</td>
<td>No</td>
<td>40</td>
<td>1.5</td>
</tr>
<tr>
<td>Collins et al. (2001)</td>
<td>P 3D</td>
<td>Yes</td>
<td>16, 37</td>
<td>6</td>
</tr>
<tr>
<td>Bhounsule (2012)</td>
<td>MA 2D</td>
<td>No</td>
<td>17</td>
<td>9.5</td>
</tr>
<tr>
<td>Ono et al. (2004)</td>
<td>MA 2D</td>
<td>Yes</td>
<td>3, 16, 31</td>
<td>0</td>
</tr>
<tr>
<td>Wisse and van Frankenhuyzen (2003)</td>
<td>MA 2D</td>
<td>Yes</td>
<td>37</td>
<td>9</td>
</tr>
<tr>
<td>Wisse et al. (2007)</td>
<td>MA 3D</td>
<td>Yes</td>
<td>37</td>
<td>9</td>
</tr>
</tbody>
</table>

NOTE: ‘P’ stands for passive, ‘MA’ stands for minimally actuated, and X<sub>F</sub> stands for ankle offset.

(Ruina, 2013). Both rails had an ankle offset of 6%<i>LL</i> (Ruina, 2013). The minimally actuated Cornell Ranger has a foot radius of 17%<i>LL</i> and an ankle offset of 9.5%<i>LL</i> (Bhounsule, 2012). The Delft Biorobotics lab built a series of two- (Wisse and van Frankenhuyzen, 2003) and three- (Wisse et al., 2007) dimensional minimally actuated bipeds with curved feet (<i>R</i> ≈ 37%<i>LL</i>) and positive ankle offsets (<i>X</i><sub>F</sub> ≈ 9%<i>LL</i>). In all of these cases, the foot radius and ankle offset were typically chosen based on trial and error (e.g. (Collins et al., 2001)) and/or experience with what worked for simpler bipeds (e.g. (Wisse et al., 2007)). A notable exception is the minimally actuated biped in Ono et al. (2004), which had three sets of feet (<i>R</i> = 3%<i>LL</i> to 31%<i>LL</i>) so that the effect of foot radius on energy efficiency could be investigated.
1.2 Overview of the Five-Link Biped ERNIE

To validate the results in Chapters 2 and 3, hardware experiments using the existing bipedal robot ERNIE were conducted. This section provides an overview of the ERNIE hardware as currently configured.

ERNIE is a five-link planar biped with curved feet and actuated hip and knee joints (Fig. 1.1, Table 1.2). A brief description of the physical ERNIE system is provided here; additional details are available in Post (2013) and Yang et al. (2008). The torso angle $q_5$ is measured counterclockwise (CCW) from the vertical. The stance hip angle $q_1$ is measured CCW from the torso, as is the swing hip angle $q_2$. The stance knee angle $q_3$ is measured CCW from the stance thigh. The swing knee angle $q_4$ is measured CCW from the swing thigh. As shown in Fig. 1.1(a), the perpendicular distance between the center of curvature of the foot and the line passing through the center of rotation of the knee and the shank’s CoM is called the ankle offset $X_F$.

The robot is designed so that the foot radius and ankle offset can be easily changed between experimental trials while keeping the length of the leg constant.

ERNIE walks overground with lateral stability provided by a carbon fiber boom, which in turn is attached to a rotating center support also containing the electronics. ERNIE has four Maxon EC45 brushless 48 volt servomotors (#136212) and GP 52 C planetary gearheads (#223094) with a 91:1 reduction ratio. All four motors are located in the torso to keep the legs light. ERNIE does not have mechanical knee stops to prevent hyperextension.

The experimental controller consists of a high-gain PD controller that depends only on the robot configuration and approximates the full zero dynamics controller. For some experiments, a heuristic controller that modulates the torso and swing hip motion based on the error in hip velocity (Post, 2013) was also used. The heuristic controller helps to reduce consistent speed variations around the room likely caused
Figure 1.1. (a) The generalized coordinates and geometric parameters for the biped ERNIE. The four actuated angles are given by $q_1$, $q_2$, $q_3$, and $q_4$. The unactuated angle is $q_5$. (b) ERNIE configured for overground walking with curved feet.
## TABLE 1.2

GEOMETRIC AND MASS PARAMETERS OF THE BIPED ERNIE.

<table>
<thead>
<tr>
<th></th>
<th>Length (m)</th>
<th>COM (m)</th>
<th>Mass (kg)</th>
<th>Inertia (kg·m(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torso</td>
<td>0.279</td>
<td>0.132</td>
<td>14.5</td>
<td>0.100</td>
</tr>
<tr>
<td>Thigh</td>
<td>0.356</td>
<td>0.125</td>
<td>1.47</td>
<td>0.024</td>
</tr>
<tr>
<td>Shank</td>
<td>0.378</td>
<td>0.136</td>
<td>1.07</td>
<td>0.024</td>
</tr>
<tr>
<td>Foot Radius</td>
<td>variable</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Ankle Offset</td>
<td>variable</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Boom</td>
<td>2.89</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Center Support</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4.48</td>
</tr>
<tr>
<td>Reflected Rotor Inertia</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.173</td>
</tr>
</tbody>
</table>

NOTE: The COM location is measured from the proximal joint. The inertia is measured about the COM.
by the center support not being absolutely vertical or small differences in floor height.

1.3 Contributions and Organization

This work makes four main contributions to the literature.

- The HZD-based modeling and control method has been extended to correctly account for the effects of foot radius, and the results have been validated in experiment (Chapter 2; Martin and Schmiedeler, 2011; Martin et al., 2011, 2014a).

- The effects of foot radius and ankle offset on gait kinematics and energy efficiency for symmetric bipeds have been investigated in both simulation and hardware (Chapter 3; Martin and Schmiedeler, 2012; Martin et al., 2014b).

- A validation study was performed to demonstrate that the extended HZD-based model can be used to predict normal human walking across a range of speeds (Chapter 4; Martin and Schmiedeler, 2013a,b, 2014a).

- The effects of varying the importance of kinematic symmetry and energetic efficiency when designing asymmetric gait has been quantified (Chapter 5; Martin and Schmiedeler, 2014b).

The ability to model and control bipeds with feet is necessary to systematically investigate the effects of foot design on both robots and amputee gait and for the development of an HZD-based model of human gait. Curved feet are of particular interest because prior research has indicated that such feet can reduce the energy required to walk, particularly for robots. In addition, the human foot and ankle effectively function like a circular arc, which means a circular foot is appropriate to model human gait.
Using the extended model, the effects of foot radius and center of curvature location on robot gait were systematically investigated to provide insights into the optimal design of robot feet. The results may also provide insights into the effects of prosthetic foot design and alignment on amputee gait.

While human gait has been predicted at the self-selected speed previously, there have been very few attempts to predict gait at other speeds. Since human gait changes with changing speed, a truly predictive model should be able to capture these differences. This work serves to both validate the use of an HZD-based model with curved feet in modeling human gait and finds an objective function capable of predicting normal human gait across speeds.

By further extending the model, asymmetric amputee gait can be investigated. The first step in developing a predictive objective function for amputee gait is understanding the impact of optimizing for kinematic symmetry as opposed to energetic efficiency when using an asymmetric model. This trade-off has been quantified.
CHAPTER 2

MODELING AND CONTROL OF CURVED FOOT BIPEDS

This chapter details how to formally design walking controllers based on hybrid zero dynamics (HZD) for an $N$-link planar curved foot biped and provides experimental validation of the approach using the five-link planar biped ERNIE (Yang et al., 2008). Section 2.1 contains the general curved foot biped model and details the derivation of the new impact map for curved feet. Section 2.2 validates this map by comparing the results of simulation to hardware results for McGeer’s two-link, passive dynamic walker (McGeer, 1990) traversing a decline and the five-link, actuated biped ERNIE walking on a treadmill with a supporting boom. Section 2.3 constructs the HZD-based controller of the model, and Section 2.4 proves its stability. Section 2.5 discusses how to find optimal periodic gaits. Section 2.6 validates the control method in hardware via experiments on the biped ERNIE.

2.1 The Model

A planar, left-right symmetric biped was assumed with $N$ links connected by revolute joints and $N-1$ actuators. Because the foot/ground interface is unactuated, the number of degrees of freedom (DOF) of the biped, $N$, is one greater than the number of actuators. While the model presented in this section can be used for bipeds with more than one degree of underactuation, the control method presented in Sections 2.3 and 2.4 and used throughout this dissertation cannot. Thus, unless otherwise specified, it will be assumed that there is a single degree of underactuation.
In this chapter, the absolute angle of the unactuated link is $q_N$, and the relative angles of the actuated links are $q_1...N-1$. A representative biped is shown in Fig. 2.1.

For underactuated bipedal robots, each step can be modeled by a finite time single support phase during which the swing leg moves from behind to in front of the stance foot and an instantaneous, inelastic collision during which the stance foot switches (Fig. 2.2). The single support phase is modeled with dynamic equations that are continuous second-order ODE’s. The impact is modeled with an algebraic map relating the state of the robot immediately before impact to the state immediately after impact. In order for a gait to be one-step periodic, the state of the system just before impact must be the same for every step. In other words, the system must have a fixed point at the end of the single support phase.

Prior attempts (Kinugasa et al., 2008; McCandless, 2008; McGeer, 1990) at modeling the impact phase have not yielded validated models. The methods proposed by McCandless (2008) and McGeer (1990) do not match experimental results, while the method proposed by Kinugasa et al. (2008) does not provide experimental validation.
Figure 2.2. Diagram of a biped walking from left to right. The swing leg (dashed) swings from behind the stance leg (solid) to in front of the stance leg. The swing leg then collides with the ground. For clarity during the impact event, the swing leg is called the impacting leg, and the stance leg is called the trailing leg.

2.1.1 Single Support

For the single support phase, the development of the dynamic equations follows the procedure described in Westervelt et al. (2007), but takes into account the curved feet. Thus, the dynamic equations are

\[ D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu, \]  

(2.1)

where \( q \) is a vector of joint angles, \( D \) is the \( N \times N \) inertia matrix, \( C \) is the \( N \times N \) matrix containing Coriolis and centripetal terms, \( G \) is the \( N \times 1 \) vector containing gravity terms, \( u \) is the \( N - 1 \times 1 \) vector of actuator torques and \( B \) is the \( N \times N - 1 \) matrix mapping actuator torques to joint torques. The effect of the curved feet appears in \( D \), \( C \) and \( G \). The matrix terms can be determined by multiple methods, including a variation on the Denavit-Hartenberg convention (McCandless, 2008), screw theory (Yang, 2007), or the method of Lagrange (Westervelt et al.,
2007). Regardless of the method used, the effect of the curved feet can be captured by considering the no-slip condition between the foot and the ground. The motion of the biped during single support can be modeled by integrating Eq. 2.1 forward in time from the instant after impact until the instant before the next impact.

2.1.2 Impact Map

When the swing foot contacts the ground, an impact occurs, and support is transferred from the pre-impact stance leg to the pre-impact swing leg. For clarity, the pre-impact stance leg will be referred to as the trailing leg, and the pre-impact swing leg will be referred to as the impacting leg, as indicated in Fig. 2.2. The impact map has two parts. In the first part, an algebraic mapping relating the state of the biped immediately before impact to the state of the biped immediately after impact is applied. This mapping is modified from the one presented in Westervelt et al. (2007) to account for the rolling contact between the foot and the ground that occurs with curved feet. In the second part, the role of each leg is swapped so that the same set of angles always refers to the stance side and the swing side. This is possible because the biped is left-right symmetric. The resulting equations are

\[
\begin{align*}
q^+ &= S q^- \quad \text{(2.2a)} \\
\dot{q}^+ &= A(q^-) \dot{q}^- \quad \text{(2.2b)}
\end{align*}
\]

where \(S\) is an \(N \times N\) matrix that switches the angle definitions and \(A\) is an \(N \times N\) matrix that relates the pre-impact velocities to the post-impact velocities. The superscript ‘\(^+\)’ refers to the instant after impact, and the superscript ‘\(^-\)’ refers to the instant before impact. The effect of the curved feet appears in \(A\).

To find \(A\), the biped’s trailing leg is mathematically detached from the ground, and the equations of motion are integrated over the instantaneous duration of im-
pact. After impact, the biped is then reattached to the ground by assuming that
the impacting foot rolls without slip. The key difference between the impact map
presented here and previous impact maps (Kinugasa et al., 2008; McCandless, 2008;
Westervelt et al., 2007) is how the biped is reattached to the ground.

For the impact map to be valid, the impact must satisfy six key assumptions
(Westervelt et al., 2007):

1. there is an impulsive force at the impacting foot,
2. the impact occurs over an infinitesimal period of time,
3. the position of the biped does not change over the duration of the impact,
4. the trailing foot lifts off the ground without interaction,
5. the impacting foot rolls without slip, and
6. the moments at the joints generated by the impulsive impact force are much
   larger than the moments generated by gravity and the actuators.

These assumptions result in three contact conditions between the foot and the ground
over the time of interest. Just before impact, the trailing foot rolls without slip.
During impact, the contact condition is unknown. Just after impact, the impacting
foot rolls without slip.

Derivation of the impact map begins with Lagrange’s equations of motion,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q_e} \right) - \frac{\partial L}{\partial q_e} - E^T F = B_e u, \tag{2.3}
\]

where \( L = T - V \), \( T \) is the kinetic energy of the system, \( V \) is the potential energy of
the system, \( q_e \) are the generalized coordinates, \( B_e \) is a mapping from the actuator
torques to the generalized forces, \( E \) is a matrix of terms from the constraint equations,
and \( F \) is a vector of Lagrange multipliers.
When the contact constraints are integrable, the equations of motion can be reformulated to remove the $E^TF$ term and also reduce the number of generalized coordinates, resulting in Eq. 2.1, which is used to simulate the system during the single support phase. This model simplification, however, prevents determination of contact forces and thus cannot be used in the computation of the impact map. Consequently, the extended formulation (Eq. 2.3) requires retaining the Lagrange multipliers, and two additional generalized coordinates are needed to detach the biped model from the ground. For this work, the x- and y-components of the location of the hip relative to an arbitrary origin on the ground are chosen as the additional generalized coordinates, although other choices are possible. The vector $q_e = [q^T, p_x, p_y]^T$ is the extended generalized coordinates vector, where $p_x$ and $p_y$ are the x- and y-coordinates of the hip.

Rewriting Eq. 2.3 in the form of Eq. 2.1 with the Lagrange multipliers term on the right-hand side gives

$$D_e(q_e)\ddot{q}_e + C_e(\dot{q}_e, q_e)\dot{q}_e + G_e(q_e) = B_eu + E(q_e)^TF, \quad (2.4)$$

where $D_e$ is the $(N + 2) \times (N + 2)$ inertia matrix in extended coordinates, $C_e$ is an $(N + 2) \times (N + 2)$ matrix containing the centripetal and Coriolis terms in extended coordinates, $G_e$ is an $(N + 2) \times 1$ vector containing the effects of the link weights, $E$ is a $2 \times (N + 2)$ matrix of constraint equations, and $F$ is a $2 \times 1$ vector of Lagrange multipliers. Eq. 2.4 can be integrated over the duration of impact. By assumption 3, $q_e^+ = q_e^- = q_e$. As a result, $D_e$, $G_e$ and $E$ can come out of the integrals. Recall that Lagrange multipliers are the generalized constraint forces (Howland, 2006), which means that $F$ is the impulsive force at the impacting foot (assumptions 1 and 4). Using assumptions 2, 3 and 6, the changes in the integrals of the second, third and fourth terms over the duration of the impact are much smaller than the changes in
the first and last terms. Making these simplifications yields

\[ D_e(q_e)(\dot{q}_e^+ - \dot{q}_e^-) = E(q_e)^T f, \quad (2.5) \]

where \( f \) is the integrated impulsive force, which has units of N·s. The velocities of the biped at the instant before impact, \( \dot{q}_e^- \), are known from the results of the single support equations. The velocities of the biped at the instant after impact, \( \dot{q}_e^+ \), and the impulsive force, \( f \), are not known. Thus, Eq. 2.5 represents \( N + 2 \) linear equations in \( N + 4 \) unknowns.

The final two linear equations are obtained using the constraint equations

\[ E\dot{q}_e^+ = 0. \quad (2.6) \]

These constraint equations are also used to mathematically reattach the biped to the ground after impact. Equation 2.6 represents a purely kinematic relationship between the generalized coordinates and the velocity of the point of contact. Note that while the form of Eq. 2.6 used here is identical to the form for a point-foot biped, the terms in \( E \) are different for a curved foot biped because \( E \) captures the rolling motion of the impacting foot.

The matrix \( E \) in Eq. 2.6 is found by first applying assumption 5 that the impacting foot rolls without slip immediately after impact. The velocity of the hip, \([\dot{p}_x, \dot{p}_y]^T\), and the velocity of the point of contact can be related using standard rigid body kinematic relationships (Hibbeler, 2006). If needed, the equation that defines the relationship can be rearranged so that the left (or right) side is equal to zero. Then, to find \( E \), take the Jacobian of the equation with respect to \( \dot{q}_e \).

As an example, the two-link biped shown in Fig. 2.3 consists of two identical rigid legs of length \( L \) pinned at the hip. The circular feet each have a radius of \( R_0 \). The trailing leg makes an angle of \( q_2 \) with the ground, measured clockwise. The impacting
Figure 2.3. McGeer’s two-link passive biped at impact. The legs are symmetric and have equal mass.

A relationship between the generalized coordinates and the velocity of the point of contact can be found using

$$v_{F/0} = v_{H/0} + \omega_{I/0} \times r_{F\rightarrow H},$$

where $v_{F/0}, v_{H/0}$ and $\omega_{I/0}$ are the velocity of the point of contact on the impacting foot, the velocity of the hip, and the angular velocity of the impacting leg, all relative to the ground. The position vector from the point of impact to the hip is $r_{F\rightarrow H}$.

Substituting the vectors into Eq. 2.7 yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} + \begin{bmatrix} (\dot{q}_1 - \dot{q}_2)r_{F\rightarrow H,y} \\ -(\dot{q}_1 - \dot{q}_2)r_{F\rightarrow H,x} \end{bmatrix},$$

where

$$r_{F\rightarrow H,x} = -(L - R_0) \sin(q_1 - q_2) - X_F \cos(q_1 - q_2)$$

$$r_{F\rightarrow H,y} = (L - R_0) \cos(q_1 - q_2) - X_F \sin(q_1 - q_2) + R_0.$$
Note that $v_{F/0} = 0$ because of the no-slip condition (assumption 5). The left-hand side of Eq. 2.8 is already zero, so the Jacobian with respect to $\dot{\mathbf{q}}_e = [\dot{q}_1, \dot{q}_2, \dot{p}_x, \dot{p}_y]^T$ can be taken directly to find $\mathbf{E}$.

$$
\mathbf{E} = \begin{bmatrix}
    r_{F\rightarrow H,y} & -r_{F\rightarrow H,y} & 1 & 0 \\
    -r_{F\rightarrow H,x} & r_{F\rightarrow H,x} & 0 & 1
\end{bmatrix}.
$$

(2.9)

Clearly, plugging Eq. 2.9 into the constraint equation (Eq. 2.6) recovers Eq. 2.8. The complete mathematical model, including all of the single support matrices, can be found in Appendix A.

An alternative method to find the matrix $\mathbf{E}$ is to use a vector loop approach. This is the method described in Westervelt et al. (2007) for point-foot bipeds. Define the vector loop so that it passes through an arbitrary origin on the ground, the hip and the point of contact of the impacting foot.

$$
0 = \mathbf{r}_{O\rightarrow H} + \mathbf{r}_{H\rightarrow F} + \mathbf{r}_{F\rightarrow O},
$$

(2.10)

where $\mathbf{r}_{O\rightarrow H}$ is the vector from the origin to the hip, $\mathbf{r}_{H\rightarrow F}$ is the vector from the hip to the point of contact of the impacting foot, and $\mathbf{r}_{F\rightarrow O}$ is the vector from the point of contact of the impacting foot to the origin. To obtain the constraint equation (Eq. 2.6), Eq. 2.10 is differentiated with respect to time using the chain rule. This gives $\mathbf{E}$ as the Jacobian of Eq. 2.10 with respect to the extended generalized coordinates $\mathbf{q}_e$.

One can see the effect of curved feet in the constraint equation by observing how the x-component of the vector from the impacting foot to the origin differs for non-zero foot radii. For a point-foot biped, $\dot{r}_{F\rightarrow O,x} = 0$ because the foot hits the ground and sticks. For a curved foot biped, $|\dot{r}_{F\rightarrow O,x}| = R|\dot{\phi}|$, where $R$ is the radius of the foot and $\dot{\phi}$ is the absolute angular velocity of the impacting foot. Clearly, for a non-zero
foot radius, \( r_{F \rightarrow O, x} \neq 0 \) in general, whereas for a point-foot biped, it is identically zero because of the zero foot radius. Thus, as expected, the impact map for curved foot bipeds reduces to the impact map for point-foot bipeds when the foot radius is zero.

In the first part of the impact map, the goal is to find a relationship between the pre- and post-impact biped states. By assumption 3, the pre- and post-impact biped configurations are the same, so the position relationship is trivial. The velocity relationship can be found by solving Eqs. 2.5 and 2.6 for \( \dot{q}^+ \). The second part of the impact map simply requires switching the coordinate definitions. After a significant amount of algebra, the impact map can be written as

\[
q^+ = Sq^- 
\]
(2.2a repeated)

\[
\dot{q}^+ = A(q^-)q^-, 
\]
(2.2b repeated)

where

\[
A = S(\Lambda_{11} - \Lambda_{12}E_1S), 
\]
(2.11)

\[
\Lambda = I_{(N+2) \times (N+2)} - D_e^{-1}E^T(ED_e^{-1}E^T)^{-1}E 
\]
(2.12)

\[
= \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{bmatrix},
\]

\( I \) is the \((N + 2) \times (N + 2)\) identity matrix, \( \Lambda_{11} \) is an \( N \times N \) matrix, and \( \Lambda_{12} \) is an \( N \times 2 \) matrix. The term \( E_1 \) transforms the pre-impact single support joint velocities into the pre-impact hip velocities and can be found by taking the first \( N \) columns of \( E(Sq^-) \).
2.2 Model Validation

To validate the impact map, simulation results are compared with prior experimental data of McGeer’s two-link, passive dynamic walker traversing a decline and the five-link, actuated biped ERNIE walking on a treadmill with a supporting boom.

2.2.1 Validation using a Passive Biped

McGeer’s two-link planar robot (McGeer, 1990) can walk down a shallow slope without any input power except for the small amount required to lift the swing foot up to avoid foot scuffing, making it an ideal system for validation because there are no inputs driving the system to a desired trajectory. A model of the biped is shown in Fig. 2.3 with the relevant parameters indicated and values given in Table 2.1. The leg length is $L$, and the foot radius is $R_f$. The foot center of curvature is located a distance of $X_F$ away from the leg. Each leg has mass $m$ located at $L_{COM}$ down the leg and a radius of gyration $r_{Gyr}$ about the COM. The ramp angle $\gamma$ is 1.4°. The counterclockwise leg splay from the stance leg to the swing leg is $q_1$. The stance leg angle $q_2$ is measured clockwise from the perpendicular of the ramp.

The mathematical model of a step, consisting of the dynamic equations (Eq. 2.1) followed by the impact map (Eq. 2.2), can be viewed as a nonlinear function of the initial conditions. Given a reasonably accurate initial guess, a fixed point can be found using numerical techniques. When experimental data is available, the initial angle can be estimated from the measured leg splay at impact, and the initial angular velocity can be estimated by dividing the leg splay by the step duration.

McGeer developed a mathematical model of his biped using linearized equations for both the single support phase and the impact. Instead of using Lagrange multipliers to determine the impact map, McGeer used conservation of angular momentum of the entire biped about the impact point and conservation of angular momentum.
TABLE 2.1
PARAMETER VALUES FOR McGEER’S PASSIVE DYNAMIC BIPED
WITH CURVED FEET AND NO KNEES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>0.5 m</td>
<td>Leg length</td>
</tr>
<tr>
<td>$R_0$</td>
<td>0.41$L$</td>
<td>Radius of foot</td>
</tr>
<tr>
<td>$L_{COM}$</td>
<td>0.355$L$</td>
<td>Distance along leg to COM</td>
</tr>
<tr>
<td>$X_F$</td>
<td>0.002$L$</td>
<td>Perpendicular distance from leg to foot center of curvature</td>
</tr>
<tr>
<td>$r_{Gyr}$</td>
<td>0.348$L$</td>
<td>Radius of gyration of leg about leg COM</td>
</tr>
<tr>
<td>$m$</td>
<td>1.75 kg</td>
<td>Mass of leg</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.4°</td>
<td>Slope of ramp</td>
</tr>
</tbody>
</table>

From McGeer (1990)
of the trailing leg about the hip. In the present analysis, angular momentum about
the point of impact of the entire biped is conserved, but no simplifying assumptions
about the conservation of angular momentum of the trailing leg are used. McGeer’s
mathematical model predicted a slightly shorter step duration and a significantly
larger leg splay at impact than was observed experimentally. To increase the accu-
racy of his model, McGeer added a rolling friction term to the single support phase in
his model so that the predicted leg splay at impact matched the experimental value.

A comparison of the reported experimental values, McGeer’s original linear model,
and the current curved foot model is shown in Fig. 2.4. With appropriate initial
conditions, the physical system is reported to have a step duration of 0.56 to 0.67
seconds, a leg splay angle of 0.54 to 0.66 radians (30° to 38°) at impact and an average
speed of 0.38 m/s to 0.50 m/s (McGeer, 1990). The variation in the step duration
and other parameters is likely because the physical system was close to the periodic
orbit, but not exactly on it. McGeer’s original linear model predicted a step duration
of 0.55 seconds, a leg splay angle of 0.82 radians (47°) at impact and an average speed
of 0.72 m/s. The simulation using the impact map developed in this work has a step duration of 0.67 seconds, a leg splay angle of 0.58 radians (34°) and an average speed of 0.44 m/s, which are all within the ranges of the experimental results, and a much better match than the predictions of McGeer’s linearized model.

2.2.2 Validation using an Actuated Biped

To further validate the model, simulation results using a five-link model were compared with prior experimental results (McCandless, 2008) from the biped ERNIE. Because these hardware experiments occurred before the transition to overground walking, ERNIE walked on a treadmill and had slightly different parameters from those given in Section 1.2 (Table 2.2, Fig. 2.5).

To determine the impact map, an equation expressing the velocity of the hip in terms of the generalized coordinates is required.

\[
\mathbf{v}_{H/0} = \mathbf{v}_{IP/0} + \omega_{IL/0} \times \mathbf{r}_{IP\rightarrow IK} + \omega_{IU/0} \times \mathbf{r}_{IK\rightarrow H},
\]

(2.13)

where \(\mathbf{v}_{H/0}\) and \(\mathbf{v}_{IP/0}\) are the velocities of the hip and the point of impact relative to the ground, \(\omega_{IL/0}\) and \(\omega_{IU/0}\) are the angular velocities of the impacting lower leg and impacting upper leg relative to the ground, and \(\mathbf{r}_{IP\rightarrow IK}\) and \(\mathbf{r}_{IK\rightarrow H}\) are the position vectors from the point of impact to the impacting knee and from the impacting knee to the hip. Using the no-slip condition, \(\mathbf{v}_{IP/0} = \mathbf{0}\). Rearranging so that the right-hand side is zero and taking the Jacobian with respect to the extended coordinates yields

\[
\mathbf{E} = \begin{bmatrix}
0 & e_2 & 0 & e_1 & e_2 & 1 & 0 \\
0 & e_4 & 0 & e_3 & e_4 & 0 & 1
\end{bmatrix},
\]

(2.14)
TABLE 2.2

GEOMETRIC AND MASS PARAMETERS OF THE BIPED ERNIE
WHEN CONFIGURED FOR TREADMILL WALKING

<table>
<thead>
<tr>
<th>Link</th>
<th>Length (m)</th>
<th>COM (m)</th>
<th>Mass (kg)</th>
<th>Inertia (kg·m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body</td>
<td>0.2</td>
<td>0.135</td>
<td>13.6</td>
<td>0.0905</td>
</tr>
<tr>
<td>Upper Leg</td>
<td>0.356</td>
<td>0.125</td>
<td>1.47</td>
<td>0.0238</td>
</tr>
<tr>
<td>Lower Leg</td>
<td>0.378</td>
<td>0.119</td>
<td>0.97</td>
<td>0.03</td>
</tr>
<tr>
<td>Foot Radius</td>
<td>0.205</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Boom</td>
<td>2.20</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Reflected Rotor Inertia</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.173</td>
</tr>
</tbody>
</table>

From McCandless (2008)
where

\[ e_1 = R - (L_L - R) \cos(q_2 + q_4 + q_5) + X_F \sin(q_2 + q_4 + q_5) \]
\[ e_2 = e_1 - L_U \cos(q_2 + q_5) \]
\[ e_3 = -(L_L - R) \sin(q_2 + q_4 + q_5) - X_F \cos(q_2 + q_4 + q_5) \]
\[ e_4 = e_3 - L_U \sin(q_2 + q_5). \]

Note that when the equation is rearranged, care should be taken so that the \( v_{H/0} \) term is positive, or the sign of the impulsive forces will be opposite of what is expected.

In each hardware experiment, ERNIE’s ankle offset was 0, 5 or 10 mm, and it walked on a treadmill running at a speed of 0.40, 0.45, 0.50, 0.55 or 0.60 m/s (McCandless, 2008). Because ERNIE walked on a treadmill, its average speed had to match the treadmill speed, or it walked/fell off of the front/back of the treadmill. Ev-
ery combination of speed and ankle offset was attempted experimentally. To control ERNIE, a gait file consisting of the control law for the actuators and a set of initial conditions for the instant before impact was loaded into the controller. Since the experimental gaits were found using an incorrect model for the impact map, it was not always possible for ERNIE to achieve steady-state walking. However, for every combination of average speed and ankle offset, except for 0.60 m/s with an ankle offset of 5 mm, a gait was eventually found such that ERNIE was able to walk at the desired speed for at least 20 seconds without stumbling or requiring intervention. When ERNIE was able to achieve steady-state walking, ERNIE’s state at the instant before impact was different than predicted by the gait file.

Similar to the hardware experiments, each simulation using the dynamic model presented herein requires a control law and ERNIE’s state at the instant before impact. For validation, the control law for simulation matched the experimental control law used for the particular combination of average speed and ankle offset being simulated. The initial conditions were extracted from the recorded experimental data. The instant before impact can be determined using the recorded stance foot data. Once the impact times are known, it is straightforward to extract the angles at each impact from the recorded joint angles and determine the mean and range for each angle. Finding the angular velocities at impact is somewhat more challenging. Attempts to numerically differentiate the joint angles did not produce accurate estimates of the joint angular velocities. Instead, it was observed that shortly before impact, all of the angles vary approximately linearly in time. To ascertain the approximate angular velocity at the instant before impact, a line was fit through the linear portion by hand, with the slope of the line giving the approximate angular velocity. For each gait and joint, the angular velocity was estimated using one step from the right leg and one step from the left leg.

Because ERNIE was attached to a boom, it was constrained to walk in a large
circular arc. This creates differences between the strictly planar mathematical model and the physical system because the boom adds additional unmodeled inertia to the system, forces stance foot slip in the radial direction and causes ERNIE to deviate from strictly planar motion. Because ERNIE walked in a circular arc, its outside foot had to travel farther than its inside foot, leading to asymmetries in its gait. In most cases, the angular position asymmetries were small, less than $4^\circ$. The only exception was the trailing knee angle whose range of angles was about $5^\circ$. On the other hand, the angular velocity asymmetries were large, particularly for the impacting knee and the torso. Table 2.3 shows representative data. Simulations using initial conditions from just the left leg or just the right leg predicted falls after a few steps. Thus, the initial conditions for simulation had to be determined from within the range of experimental initial conditions. To find the initial conditions for simulation, the following optimization was employed.

$$\begin{align*}
\min_{q_1, q_2, q_4, \dot{q}} & \quad f(q, \dot{q}) \\
\text{s. t.} & \quad \begin{bmatrix} q_1 \\ q_2 \\ q_4 \end{bmatrix} \leq \begin{bmatrix} a_{\text{min}} \\ a_{\text{max}} \end{bmatrix} \\
& \quad b_{\text{min}} \leq \dot{q} \leq b_{\text{max}}, \quad (2.15)
\end{align*}$$
where

\[ f(q, \dot{q}) = \left( \frac{v - v_{des}}{v_{des}} \right)^2 + \sum_{k=1}^{5} \left( \frac{q_{k,0} - q_{k,1}}{q_{k,0}} \right)^2 + \sum_{k=1}^{5} \left( \frac{\dot{q}_{k,0} - \dot{q}_{k,1}}{\dot{q}_{k,0}} \right)^2, \] (2.16)

\( a_{\text{min}} \) and \( a_{\text{max}} \) are vectors of the smallest and largest angles at impact for \( q_1 \), \( q_2 \) and \( q_4 \), \( b_{\text{min}} \) and \( b_{\text{max}} \) are vectors of the smallest and largest angular velocities at impact, \( v \) is the actual average speed, \( v_{des} \) is the desired average speed, \( q_{k,0}^- \) is the initial condition for the angle \( q_k \), \( q_{k,1}^- \) is the angle \( q_k \) at the instant before the impact at the end of the first step, \( \dot{q}_{k,0}^- \) is the initial condition for the angular velocity \( \dot{q}_k \), and \( \dot{q}_{k,1}^- \) is the angular velocity \( \dot{q}_k \) at the instant before the impact at the end of the first step. The average speed is defined as the step length divided by the step duration. The first term in Eq. 2.16 ensures that ERNIE walks at the desired speed. The two summations ensure that ERNIE’s state is the same at the end of every step, which is required for a one-step periodic gait. Equation 2.16 is always greater than or equal to zero. For a periodic gait at the desired speed, Eq. 2.16 must equal zero. For about half of the gaits, the bounds had to be increased slightly to achieve this condition in the simulations. The torso angle \( q_5 \) is held constant at its mean experimental value because it was found that the optimization performed better when the torso angle was a constant. Because both feet must be on the ground at impact, the initial condition for the trailing knee angle was determined as a function of the other joint angles.

In all cases, a set of initial conditions for simulation that produced periodic walking at the desired speed was found. Because both the simulation and the physical
### TABLE 2.3

MAXIMUM, MINIMUM AND SIMULATED ANGULAR POSITION AND VELOCITY AT THE INSTANT BEFORE IMPACT FOR $v = 0.5 \text{ m/s}$ AND $X_F = 5 \text{ mm}$ FOR ERNIE WALKING ON A TREADMILL

<table>
<thead>
<tr>
<th>N</th>
<th>$q_{\text{max}}$ (rad)</th>
<th>$q_{\text{min}}$ (rad)</th>
<th>$q_{\text{sim}}$ (rad)</th>
<th>$\dot{q}_{\text{max}}$ (rad/sec)</th>
<th>$\dot{q}_{\text{min}}$ (rad/sec)</th>
<th>$\dot{q}_{\text{sim}}$ (rad/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.22</td>
<td>3.19</td>
<td>3.19</td>
<td>-0.0456</td>
<td>-0.129</td>
<td>-0.0124</td>
</tr>
<tr>
<td>2</td>
<td>3.69</td>
<td>3.66</td>
<td>3.63</td>
<td>-0.340</td>
<td>-0.900</td>
<td>-0.421</td>
</tr>
<tr>
<td>3</td>
<td>-0.449</td>
<td>-0.522</td>
<td>-0.506</td>
<td>-2.27</td>
<td>-2.52</td>
<td>-2.28</td>
</tr>
<tr>
<td>4</td>
<td>-0.0623</td>
<td>-0.115</td>
<td>-0.119</td>
<td>1.11</td>
<td>-2.29</td>
<td>1.11</td>
</tr>
<tr>
<td>5</td>
<td>-0.122</td>
<td>-0.144</td>
<td>-0.134</td>
<td>0.204</td>
<td>-1.07</td>
<td>0.0249</td>
</tr>
</tbody>
</table>

The simulation values were found using Eq. 2.15.
Figure 2.6. A comparison of the experimental and simulated step durations when the ankle offset is (a) 0 mm, (b) 5 mm, and (c) 10 mm for ERNIE walking on a treadmill. For the experimental data, the error bars show ± one standard deviation.
system were forced to walk at a specified average speed, if the simulated step duration is close to the experimental step duration, then the simulated step length will also be close to the experimental step length. In all but two of the cases, the simulated step period was within one standard deviation of the average experimental step duration (Fig. 2.6). When $x_f = 0$ mm and $v = 0.5$ m/s, the simulated period is slightly longer than the average experimental period plus one standard deviation. However, in this particular experiment, ERNIE was only able to walk for 26 sec. This indicates, together with the large standard deviation, that the actual, experimental gait was not very stable. Most likely the simulation would agree better with the experiment if similar, but slightly less optimal, initial conditions were used. When $x_f = 0$ mm and $v = 0.6$ m/s, the simulated step period is significantly longer than the experimental step period. In both simulation and experiment, the hip angles at impact are much larger than in all of the other cases, while the knee angles are about the same. As a result, the distance between the points of contact at impact is much larger than in the other cases. In experiment, ERNIE almost certainly rolled to the end of its foot and then pivoted about the end, effectively switching to a point foot at the end of each step. Because this transition is not modeled, the experimental and simulated results do not agree.

Even without system identification, the step duration in simulation and experiment were successfully correlated in twelve of fourteen cases. This indicates that the mathematical model is robust enough to handle small discrepancies between the model and the physical system. These results provide additional validation for the model, particularly for the impact phase. Combined with the results of the previous section, this indicates that the curved foot impact map derived herein, in combination with the standard continuous equations of motion describing single support, accurately models the dynamics of planar curved foot bipeds with instantaneous transfer of support. Further, this validated model is general enough to handle any planar
curved foot biped as long as the foot/ground interface is unactuated and the impact can be modeled as instantaneous. This validated impact map, then, enables examination of the zero dynamics.

2.3 Zero Dynamics

The development in this section follows the work in Westervelt et al. (2007) using the concepts of zero dynamics and feedback linearization (Isidori, 1995) and is presented here for completeness. It is more convenient to model the swing phase of the biped (Eq. 2.1) using first order differential equations by defining $x = [q^T, \dot{q}^T]^T$. The system can then be written as

$$\dot{x} = f(x) + g(x)u,$$  \hspace{2cm} (2.17)

where

$$f(x) = \begin{bmatrix} \dot{q} \\ -D^{-1}(Cq + G) \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix}.$$

The desired configuration of the biped is defined using the output function

$$y = h(q),$$  \hspace{2cm} (2.18)

where $h$ is a vector function of length $N - 1$ indicating the tracking error for the desired trajectory. The approach requires that the output is a smooth function of position only.

Feedback linearization is applied to obtain a new system that is input/output
linear. Differentiating Eq. 2.18 once gives

\[ \dot{y} = \frac{\partial h}{\partial x} \dot{x} \]  

\[ = \begin{bmatrix} \frac{\partial h}{\partial q} & \frac{\partial h}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \dot{q} \\ -D^{-1}(C\dot{q} + G) \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} \cdot u \]  

\[ = \frac{\partial h}{\partial q} \dot{q} \]  

\[ = L_f h, \]  

where \( L_f h \) is the Lie derivative of \( h \) (Isidori, 1995). The derivative of \( h \) with respect to \( \dot{q} \) is 0 because \( h \) is a function of configuration only. Differentiating a second time gives

\[ \ddot{y} = \left[ \frac{\partial}{\partial q} (L_f h) \right] \begin{bmatrix} \dot{q} \\ -D^{-1}(C\dot{q} + G) \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} \cdot u \]  

\[ = \frac{\partial h}{\partial q} \ddot{q} \]  

\[ = L_f h, \]
\[
\begin{align*}
\dot{\eta}_1 &= h(q), \\
\dot{\eta}_2 &= L_f h(q, \dot{q}).
\end{align*}
\] (2.21)  
\[\eta_1 = h(q), \quad \eta_2 = L_f h(q, \dot{q}).\] (2.22)

The input is chosen as
\[
u = (L_g L_f h)^{-1} (v - L^2_f h),
\] (2.23)

which creates the double integrator \(\ddot{y} = v\). Choosing \(v\) appropriately, for example as high gain PD feedback, collapses the dynamics to the zero dynamics manifold (i.e. it is tracking the desired path perfectly). As a result, \(\eta_1\) and \(\eta_2\) are always zero. The final two coordinates are chosen to be the same as in Westervelt et al. (2007),
\[
\begin{align*}
\xi_1 &= \theta(q) \\
\xi_2 &= D_N(q) \dot{q}.
\end{align*}
\] (2.24)  
\[\xi_1 = \theta(q), \quad \xi_2 = D_N(q, \dot{q}).\] (2.25)

where \(D_N\) is the \(N^{th}\) row of \(D\) and \(\theta\) is a smooth function. Note that although this choice for \(\xi_2\) is not typical when dealing with zero dynamics systems in general, it simplifies the later analysis of this system significantly. The system of differential
equations in the transformed system is

\[ \dot{\eta}_1 = \eta_2 \]
\[ = L_f h(q, \dot{q}), \quad (2.26) \]
\[ \dot{\eta}_2 = L_f^2 h + L_g L_f h \cdot u \]
\[ = v, \quad (2.27) \]
\[ \dot{\xi}_1 = \frac{\partial \theta}{\partial q} \dot{q}, \quad (2.28) \]
\[ \dot{\xi}_2 = q^T \frac{\partial D_N^T}{\partial q} \dot{q} + D_N \ddot{q} \]
\[ = q^T \frac{\partial D_N^T}{\partial q} \dot{q} - C_N \dot{q} - G_N, \quad (2.29) \]

where the subscript \( N \) refers to the \( N^{th} \) row of the matrix.

Because \( \eta_1 \) and \( \xi_1 \) are functions of position only, it is possible to find the coordinate transformation between \( [\eta_1; \xi_1] \) and position \( q \). Then, this transformation is used in the coordinate transformation between \( [\eta_2; \xi_2] \) and velocity \( \dot{q} \). Let

\[
\begin{bmatrix}
\eta_1 \\
\xi_1
\end{bmatrix} = \begin{bmatrix} h(q) \\
\theta(q)
\end{bmatrix} = \Phi(q). \quad (2.30)
\]

Note that \( \Phi \) must be a diffeomorphism between \( x \) and the new coordinate system (i.e. \( \Phi \) must be invertible). Then,

\[ q = \Phi^{-1}(\xi_1), \quad (2.31) \]

where Eq. 2.31 holds because the biped is on the zero dynamics manifold \( (\eta_1 = 0) \).
Then, $\eta_2$ and $\xi_2$ can be related to $\dot{q}$ by

$$
\begin{bmatrix}
\eta_2 \\
\xi_2
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial h(q)}{\partial q} \\
D_N(q)
\end{bmatrix} \dot{q}.
$$

(2.32)

Inverting the relationship gives

$$
\dot{q} = \bar{\lambda}_q(\xi_1)\xi_2,
$$

(2.33)

where

$$
\begin{align*}
\bar{\lambda}_q &= 
\begin{bmatrix}
\frac{\partial h(\Phi^{-1}(\xi_1))}{\partial q} \\
D_N(\Phi^{-1}(\xi_1))
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
1
\end{bmatrix}.
\end{align*}
$$

(2.34)

Again, recall that $\eta_2 = 0$ on the zero dynamics manifold.

For curved foot bipeds, the nonlinear transformations (Eqs. 2.31 and 2.33) to take the dynamical system into its zero dynamics form remain the same as for point-foot bipeds. Similarly, the dynamical system written in zero dynamics form (Eqs. 2.26-2.29) does not change with the addition of curved feet to the model. Thus, the next logical step is to perform a stability analysis for curved foot bipeds similar to the one performed for point-foot bipeds.

2.4 Defining a Stable Gait

As discussed above, the mathematical model of a step, consisting of the impact map (Eq. 2.2) followed by the integration of the dynamic equation (Eq. 2.17), can be viewed as a discrete, nonlinear function of the pre-impact biped state. Because a one-step periodic gait is desired, the system must have a fixed point, which is equivalent to requiring a constant pre-impact state. To determine the stability of the step, the fixed point can be analyzed using the method of Poincaré.

In the transformed zero dynamics system, only the zero dynamics coordinates
\(\xi_1\) and \(\xi_2\) are non-zero, so their equations (Eqs. 2.28 and 2.29) are the only two equations of interest. For point feet (Westervelt et al., 2007), by cleverly combining Eqs. 2.28 and 2.29, it is possible to remove time from the system and make \(\xi_2\) a function of \(\xi_1\). Integrating this combined equation over the single support phase and applying the impact map yields an analytic equation for both the fixed point and the Poincaré map. However, the proof relies on the fact that the Jacobian of the inertia matrix \(D\) is not a function of \(q_N\), the absolute angle. For curved feet, because there is a rolling constraint between the curved foot and the ground, the Jacobian of the inertia matrix depends on \(q_N\). Nevertheless, an analytical equation can still be found to identify the fixed point and evaluate stability, but the proof from Westervelt et al. (2007) must be modified to account for the changes due to curved feet.

The initial portion of the modified proof is given by Srinivasan (2007) and presented below. \(C_N\) is related to the inertia matrix (Spong and Vidyasagar, 1989) by

\[
C_N = \dot{q}^T \frac{\partial D_T}{\partial q} - \frac{1}{2} \dot{q}^T \frac{\partial D}{\partial q_N}.
\]  

(2.35)

Substituting Eq. 2.35 into Eq. 2.29 and simplifying gives

\[
\dot{\xi}_2 = \frac{1}{2} \dot{q}^T \frac{\partial D(q)}{\partial q_N} \dot{q} - G_N(q).
\]  

(2.36)

At this point, it is clear that \(\dot{\xi}_2\) is a function of both \(q\) and \(\dot{q}\). For point-foot bipeds, because the inertia matrix does not depend on \(q_N\), the first term in Eq. 2.36 is zero. Thus, Eq. 2.36 for point feet reduces to the equation for \(\dot{\xi}_2\) given in Westervelt et al. (2007). The proof in Srinivasan (2007) stops here.
Using the coordinate transformations in Eqs. 2.31 and 2.33, Eq. 2.36 becomes

\[
\dot{\xi}_2 = \frac{1}{2} (\tilde{\lambda}_q(\xi_1)\xi_2)^T \frac{\partial \mathbf{D}(\Phi^{-1}(\xi_1))}{\partial q_N} \tilde{\lambda}_q(\xi_1)\xi_2
\]
\[
- \mathbf{G}_N(\Phi^{-1}(\xi_1)).
\] (2.37)

Recalling that \(\xi_2\) is a scalar, Eq. 2.37 becomes

\[
\dot{\xi}_2 = \xi_2^2 \left( \frac{1}{2} \tilde{\lambda}_q^T(\xi_1) \frac{\partial \mathbf{D}(\xi_1)}{\partial q_N} \tilde{\lambda}_q(\xi_1) \right) - \mathbf{G}_N(\xi_1).
\] (2.38)

Thus, Eq. 2.29 can be written as

\[
\dot{\xi}_2 = \kappa_{21}(\xi_1)\xi_2^2 - \kappa_{22}(\xi_1),
\] (2.39)

where

\[
\kappa_{21} = \frac{1}{2} \tilde{\lambda}_q^T(\xi_1) \frac{\partial \mathbf{D}(\xi_1)}{\partial q_N} \tilde{\lambda}_q(\xi_1)
\]

\[
\kappa_{22} = \mathbf{G}_N(\xi_1).
\]

Note that when the foot radius is zero, \(\kappa_{21} = 0\) because the inertia matrix does not depend on \(q_N\).

Eq. 2.28 can also be written as a function of only \(\xi_1\) and \(\xi_2\). Substituting in the coordinate transformations gives

\[
\dot{\xi}_1 = \kappa_1(\xi_1)\xi_2,
\] (2.40)

where

\[
\kappa_1 = \frac{\partial \theta(\mathbf{q})}{\partial \mathbf{q}} \tilde{\lambda}_q(\xi_1).
\]

By the same proof given in Westervelt et al. (2007), \(\theta\) must be monotonically
increasing over a step. Thus, $\dot{\theta} > 0$ over a step, which means that $\dot{\xi}_1$ is never equal to zero over a step. Dividing Eq. 2.39 by Eq. 2.40 gives

$$\frac{\dot{\xi}_2}{\xi_1} = \frac{\kappa_{21}(\xi_1)\xi_2^2 - \kappa_{22}(\xi_1)}{\kappa_1(\xi_1)\xi_2}. \quad (2.41)$$

To turn Eq. 2.41 into a linear differential equation, define a new coordinate (Westervelt et al., 2007)

$$\zeta = \frac{1}{2}\xi_2^2. \quad (2.42)$$

Its derivative is given by

$$\dot{\zeta} = \dot{\xi}_2 \xi_2. \quad (2.43)$$

Substituting Eq. 2.42 and a manipulation of Eq. 2.43 into Eq. 2.41 gives

$$\frac{\dot{\xi}_2}{\xi_1} = \frac{2\kappa_{21}(\xi_1)\zeta - \kappa_{22}(\xi_1)}{\kappa_1(\xi_1)} \dot{\xi}_2. \quad (2.44)$$

Rearranging and simplifying yields

$$\frac{\dot{\zeta}}{\xi_1} = \frac{d\zeta}{d\xi_1} \quad (2.45a)$$

$$= \frac{2\kappa_{21}(\xi_1)}{\kappa_1(\xi_1)} \zeta - \frac{\kappa_{22}(\xi_1)}{\kappa_1(\xi_1)}. \quad (2.45b)$$

This is a first order linear differential equation of the form

$$\frac{d\zeta}{d\xi_1} + P(\xi_1)\zeta = Q(\xi_1). \quad (2.46)$$
where

\[ P(\xi_1) = -\frac{2\kappa_{21}(\xi_1)}{\kappa_1(\xi_1)} \]
\[ Q(\xi_1) = -\frac{\kappa_{22}(\xi_1)}{\kappa_1(\xi_1)}. \]

The particular solution (Greenberg, 1998) to Eq. 2.46 is

\[ \zeta(\xi_1) = e^{-c(\xi_1)} \left( \int_{\xi_1^-}^{\xi_1^+} e^{c(t)} Q(t) dt + \zeta(\xi_1^+) \right), \tag{2.47} \]

where

\[ c(\xi_1) = \int_{\xi_1^-}^{\xi_1^+} P(\tau) d\tau. \]

From Eq. 2.47, it is clear that \( \zeta \) is a function of \( \xi_1 \). Considering the definition of \( \zeta \) in Eq. 2.42, \( \xi_2 \) can also be expressed as a function of \( \xi_1 \). Note that for point feet, \( P = 0 \). As a result, \( e^{c(\xi_1)} = 1 \), and Eq. 2.47 reduces to the equation given in Westervelt et al. (2007).

The next step is to determine the fixed point \([\xi_1^-, \xi_2^-]^T\), which is needed to fully specify the gait and to define the Poincaré map. Recall that the superscript ‘-’ indicates the instant before impact. The configuration of the biped at the instant before impact, \( \xi_1^- \), can be found most easily by looking at the original system and then changing to the zero dynamics coordinates. At impact, both feet are on the ground, and the actuated joint angles are known (i.e. Eq. 2.18 equals zero). This gives a system of \( N \) equations in \( N \) unknowns which can be solved for the joint angles at the instant before impact. The choice of the output function \( h(q) \) will determine how difficult this task is. Once the configuration at the instant before impact is known, \( \xi_1^- \) can be found using Eq. 2.30. At this point, it is also convenient to calculate \( \xi_1^+ \) using the position component of the impact map (Eq. 2.2a) because \( \xi_1^+ \) appears in Eq. 2.47. Now, \( \xi_2^- \) needs to be found, and finding it requires looking
at the entire step.

By definition (Eq. 2.25), at the instant after impact,

\[ \xi_2^+ = D_N(q^+) \dot{q}^+. \]  

(2.48)

Substituting the impact map (Eq. 2.2b) and velocity coordinate transformation (Eq. 2.33) into Eq. 2.48 gives

\[ \xi_2^+ = \delta_{\text{zero}} \xi_2^-, \]  

(2.49)

where

\[ \delta_{\text{zero}} = D_N(q^+) A(q^-) \bar{\lambda}_q(\xi^-). \]

Substituting the square of Eq. 2.49 into Eq. 2.42 constructs a relationship between \( \zeta^+ \) and \( \zeta^- \).

\[ \zeta(\xi^+) = \frac{1}{2} \delta_{\text{zero}}^2 [\xi^-]^2. \]  

(2.50)

Integrating Eq. 2.47 over the step from \( \xi_1^+ \) to \( \xi_1^- \) and substituting in Eq. 2.50 gives

\[ \zeta(\xi^-) = \frac{1}{2} [\xi^-]^2 = b_1 \left( b_2 + \frac{1}{2} \delta_{\text{zero}}^2 [\xi^-]^2 \right), \]  

(2.51)

where

\[ b_1 = e^{-c(\xi^-)}, \]

\[ b_2 = \int_{\xi_1^+}^{\xi_1^-} e^{c(t)} Q(t) dt. \]

Solving Eq. 2.51 for \( \xi^- \) gives

\[ \xi^- = \pm \sqrt{\frac{2b_1b_2}{1 - b_1 \delta_{\text{zero}}^2}}, \]  

(2.52)

which is a function of configuration only. The sign of Eq. 2.52 is chosen so that \( \dot{\xi}^- > 0, \)
which can be easily calculated. If \( b_1 \delta_{zero}^2 < 1 \), a solution always exists because \( P \) and \( Q \) are both continuous, bounded functions when \( \xi_1^+ \leq \xi_1 \leq \xi_1^- \).

To prevent imaginary trajectories, \( \zeta \geq 0 \). Note that because \( e^{-c(\xi_1)} > 0 \), the term inside the parentheses in Eq. 2.47 determines the sign of \( \zeta \). Thus,

\[
\int_{\xi_1^+}^{\xi_1^-} e^{c(t)} Q(t) dt + \frac{1}{2} \delta_{zero}^2 (\xi_2^-)^2 \geq 0
\]  

(2.53)

for all \( \xi_1^+ \leq \xi_1 \leq \xi_1^- \).

Step-to-step stability is evaluated using the method of Poincaré. Choosing the instant before impact as the Poincaré section, the Poincaré return map is

\[
\rho(\zeta^-) = b_1 \left( b_2 + \delta_{zero}^2 \zeta^- \right).
\]  

(2.54)

For a gait to be stable, the eigenvalue of Eq. 2.54 must be between -1 and +1. Taking the Jacobian to get the eigenvalue gives

\[
\frac{\partial \rho}{\partial \zeta^-} = b_1 \delta_{zero}^2.
\]  

(2.55)

Because \( \delta_{zero} \) is a real number, \( \delta_{zero}^2 \geq 0 \). Further, \( b_1 > 0 \), so \( b_1 \delta_{zero} \geq 0 \). If \( \delta_{zero} = 0 \), \( \xi_2^+ = 0 \), which means that \( \dot{q} \) is also zero, i.e. the biped does not move. Thus, an exponentially stable, non-trivial fixed point given by Eq. 2.52 must satisfy

\[
0 < b_1 \delta_{zero}^2 < 1.
\]  

(2.56)

Note that as mentioned above, for a point-foot biped, \( b_1 = 1 \) and is not a function of configuration. Thus, determining stability for a curved foot biped requires the calculation of an additional term.

Using the results in Morris and Grizzle (2005), an exponentially stable fixed point
of the zero dynamics defined by Eq. 2.52 can be shown to result in an exponentially stable periodic orbit of the full system if several conditions are met. Briefly,

- the feedback $v$ must be a function of the state of the biped $x$ but not of time,
- $L_g L_f h$ must be invertible,
- the closed-loop equations for $[\eta_1^T, \eta_2^T]^T$ must be able to be written as $M(\epsilon)[\eta_1^T, \eta_2^T]^T$, where $\epsilon$ is a tuning parameter, and
- $\lim_{\epsilon \to 0} e^{M(\epsilon)} = 0$.

Then, for $\epsilon$ sufficiently small, a stable fixed point of the zero dynamics results in a stable fixed point for the full system. The reader is referred to Morris and Grizzle (2005) for details. Because the proof of the stability for the full system does not depend on the form of the zero dynamics equations (Eqs. 2.28 and 2.29), it does not change from the point-foot case.

While Eq. 2.52 is critical for determining the stability criterion for curved foot bipeds, it is not particularly useful for actually calculating the $\xi_2^-$ value of the fixed point that is needed to both fully define the gait and to specify the initial conditions for simulation. Due to the nonlinear nature of the system, the integrals have to be computed numerically, which can be slow. In particular, computing the nested integral in Eq. 2.51 is very slow. Instead, the problem can be reformulated as a scalar, first-order boundary value problem with Eq. 2.46 as the differential equation and Eq. 2.49 as the boundary condition. For a stable, periodic gait, a solution is guaranteed to exist because Eq. 2.51 always has a solution. Thus, a numerically tractable solution exists for determining periodic gaits for a curved foot biped.

Determining the fixed point for a periodic walking gait and checking stability for a curved foot biped requires knowledge of the desired joint trajectories, just as with point feet. However, the rolling contact at the foot-ground interface for a curved foot
biped introduces an additional term into the zero dynamics equations. Because of this additional term, it is no longer practical to use the direct method of calculating the fixed point (Eq. 2.52) for curved foot bipeds. Instead, as discussed above, the fixed point can be found by solving a boundary value problem. When evaluating step-to-step stability, the extra term in the zero dynamics equations results in an additional term in the stability equation (Eq. 2.56). However, just as with point feet, the stability of curved foot walking remains a function of configuration only.

2.5 Gait Design

The ability to model and control a curved foot biped enables the design of gaits in simulation. Because there are many possible gaits at a given speed and step length, it is necessary to perform an optimization to obtain the best gait. The optimizations were performed using MATLAB’s \texttt{fmincon} function with the “active-set” algorithm. The termination tolerance on the coefficients was 1e-3, while the termination tolerances on the function values and constraints were both 1e-5. The finite-difference gradient was bounded between [1e-6, 1e-2].

2.5.1 Choice of Optimization Variables

Different gaits are obtained by changing the output function of the controller (Eq. 2.18), which is encoded using a Bézier polynomial of degree $Q$. The general form of the output is

$$\mathbf{y} = \mathbf{H}_0 \cdot \mathbf{q} - \sum_{i=0}^{Q} \frac{\alpha_i Q!}{i!(Q-i)!} s^i (1-s)^{Q-i} = 0, \quad (2.57)$$

where

$$\mathbf{H}_0 = \begin{bmatrix} \mathbf{I}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times1} \end{bmatrix},$$
I is the \((N - 1) \times (N - 1)\) identity matrix, \(0\) is a \((N - 1) \times 1\) vector of zeros, \(0 \leq s \leq 1\) is a monotonically increasing measure of step progression, and \(\alpha_i\) is a polynomial coefficient. For convenience, the polynomial coefficients are contained in an \((N - 1) \times (Q + 1)\) matrix \(\alpha\). Because step progression is measured using \(\xi_1 = \theta\),

\[
s = \frac{\theta - \theta^+}{\theta^- - \theta^+}. \tag{2.58}
\]

For this work, just as in Westervelt et al. (2007),

\[
\theta = c \cdot q, \tag{2.59}
\]

where \(c\) is a \(1 \times N\) vector. To ensure that the position coordinate transformation is a diffeomorphism, the entry in the \(N^{th}\) column of \(c \neq 0\).

Note that at the start of a step, \(s = 0\), so

\[
H_0 \cdot q^+ = \alpha_0 \tag{2.60}
\]

\[
H_0 \cdot \dot{q}^+ = (\alpha_1 - \alpha_0) \frac{Q(c \cdot \dot{q}^+)}{\theta^- - \theta^+}. \tag{2.61}
\]

Similarly, at the end of a step, \(s = 1\), so

\[
H_0 \cdot q^- = \alpha_Q \tag{2.62}
\]

\[
H_0 \cdot \dot{q}^- = (\alpha_Q - \alpha_{Q-1}) \frac{Q(c \cdot \dot{q}^-)}{\theta^- - \theta^+}. \tag{2.63}
\]

Further, both feet must be on the ground at the end of the step, so there is an additional constraint of the form

\[
f(q^-) = 0. \tag{2.64}
\]
In order to simulate the gait, the state at the instant after impact, $q^+$ and $\dot{q}^+$, must be known. To ensure a periodic gait, the impact map (Eq. 2.2) must be satisfied. Note that in general, the $A$ matrix in the velocity component of the impact map is not full rank, so the inverse equation to Eq. 2.2b is not well defined. In addition, the equation of motion for the single support phase (Eq. 2.51) must be satisfied.

There are $(Q + 1) \times (N - 1) + 4N$ quantities that need to be determined to define the controller and ensure a periodic gait. These variables are given by $\alpha$ ($(Q + 1) \times (N - 1)$ variables), $q^-$ ($N$ variables), $q^+$ ($N$ variables), $\dot{q}^-$ ($N$ variables), and $\dot{q}^+$ ($N$ variables). There are $4(N - 1) + 2N + 2$ constraints defined by the position impact map (Eq. 2.2a, $N$ constraints), the velocity impact map (Eq. 2.2b, $N$ constraints), the equation of motion (Eq. 2.51, 1 constraint), the start of step configuration (Eq. 2.60, $N - 1$ constraints), the start of step velocity (Eq. 2.61, $N - 1$ constraints), the end of step configuration (Eq. 2.62, $N - 1$ constraints), the end of step velocity (Eq. 2.63, $N - 1$ constraints), and the ground contact requirement (Eq. 2.64, 1 constraint). As a result, there are $(N - 1) \times (Q - 1)$ independent variables.

As an example, suppose the biped has three DOF ($N = 3$). The Bézier polynomial will be assumed to have a degree of five ($Q = 5$). Thus, the following 24 quantities (shown in blue) must be determined:

$$\alpha = \begin{bmatrix} \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 & \alpha_5^1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \end{bmatrix} \quad (2.65)$$

$$q^- = \begin{bmatrix} q_{1^-} & q_{2^-} & q_{3^-} \end{bmatrix}^T \quad (2.66)$$

$$q^+ = \begin{bmatrix} q_{1^+} & q_{2^+} & q_{3^+} \end{bmatrix}^T \quad (2.67)$$

$$\dot{q}^- = \begin{bmatrix} \dot{q}_{1^-} & \dot{q}_{2^-} & \dot{q}_{3^-} \end{bmatrix}^T \quad (2.68)$$

$$\dot{q}^+ = \begin{bmatrix} \dot{q}_{1^+} & \dot{q}_{2^+} & \dot{q}_{3^+} \end{bmatrix}^T \quad (2.69)$$
There are 16 constraints, which means there are 8 independent variables.

It must be decided which variables the optimization can change directly (called the optimization variables) and which variables will be calculated within each iteration (called the calculated variables). The middle \( Q - 3 \) Bézier coefficients \( \alpha_{2..Q - 2} \) must be in the optimization coefficients. In theory, the remaining optimization variables can be chosen to be any set of the remaining variables as long as there are at least \( (N - 1) \times (Q - 1) \) total optimization variables. Eqs. 2.2, 2.51, and 2.60-2.64 can either be used to calculate the calculated variables or they can be used as constraints in the optimization. The choice of optimization variables will determine how each equation will be used. Returning to the example, it is possible to choose all 24 variables to be optimization variables and use all 16 scalar constraint equations as constraints in the optimization. Another option is to choose 14 of the variables to be the optimization variables, calculate the remaining 10 variables using 10 of the scalar equations each iteration, and use the remaining 6 equations as constraints in the optimization. In practice, having close to the minimal set of optimization variables is convenient because it reduces the dimension of the optimization problem. Further, it is difficult or impossible to invert some of the equations, so some sets of optimization variables work better than others. For example, it is easy to find \( \dot{q}^+ \) given \( \dot{q}^- \) but impossible to find a unique \( \dot{q}^- \) given \( \dot{q}^+ \). Thus, the choice of optimization variables can influence both the speed of convergence and how long each iteration takes.

It was first attempted to use the minimal set of optimization parameters. If the optimization variables are chosen to be \( \alpha_{2..Q} \), all of the other variables can be calculated. Returning to the example, the 8 optimization variables are shown in red,
and the 16 computed variables are shown in purple.

\[
\alpha = \begin{bmatrix}
\alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 & \alpha_5^1 \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2
\end{bmatrix}
\] (2.70)

\[
q^- = \begin{bmatrix}
q_1^- \\
q_2^- \\
q_3^-
\end{bmatrix}^T 
\] (2.71)

\[
q^+ = \begin{bmatrix}
q_1^+ \\
q_2^+ \\
q_3^+
\end{bmatrix}^T 
\] (2.72)

\[
\dot{q}^- = \begin{bmatrix}
\dot{q}_1^- \\
\dot{q}_2^- \\
\dot{q}_3^-
\end{bmatrix}^T 
\] (2.73)

\[
\dot{q}^+ = \begin{bmatrix}
\dot{q}_1^+ \\
\dot{q}_2^+ \\
\dot{q}_3^+
\end{bmatrix}^T 
\] (2.74)

The following procedure can be used to find the calculated variables.

1. Calculate \(q^-\) using Eqs. 2.62 and 2.64 and the variable \(\alpha_Q\).

2. Calculate \(q^+\) using Eq. 2.2a and the variable \(q^-\).

3. Calculate \(\alpha_0\) using Eq. 2.60 and the variable \(q^+\).

4. Calculate \(\alpha_1\) using Eqs. 2.61, 2.63, and 2.2b and the variables \(q^-\), \(q^+\) and \(\alpha_{Q-1,Q}\).

5. Calculate \(\dot{q}^-\) using Eqs. 2.63, 2.2b, and 2.51 and the variable \(\alpha\).

6. Calculate \(\dot{q}^+\) using Eq. 2.2b and the variable \(\dot{q}^-\).

This method ensures that all the constraints are satisfied at each iteration in the optimization. Unfortunately, this choice results in very slow optimizations that can have trouble converging.

A better choice of optimization variables is to use slightly more than the minimal set of optimization variables and use the equation of motion (Eq. 2.51) as a constraint. If the numerical methods do not have any error and if Eq. 2.51 is enforced at every
iteration step, the two methods are identical. However, there is numerical error, and Eq. 2.51 is not enforced at every step, so the two methods are similar, but not exactly identical. For this strategy, the optimization variables are $\alpha_{2\ldots Q-2}$, $q_{1\ldots N-1}$, and $\dot{q}^-$. Returning once again to the example, the 9 optimization variables are shown in red, and the 15 computed variables are shown in purple.

\[
\alpha = \begin{bmatrix}
\alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 & \alpha_5^1 \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2
\end{bmatrix}
\] (2.75)

\[
q^- = \begin{bmatrix}
q_{-1}^- & q_{-2}^- & q_{-3}^-
\end{bmatrix}^T
\] (2.76)

\[
q^+ = \begin{bmatrix}
q_{+1}^+ & q_{+2}^+ & q_{+3}^+
\end{bmatrix}^T
\] (2.77)

\[
\dot{q}^- = \begin{bmatrix}
\dot{q}_{-1}^- & \dot{q}_{-2}^- & \dot{q}_{-3}^-
\end{bmatrix}^T
\] (2.78)

\[
\dot{q}^+ = \begin{bmatrix}
\dot{q}_{+1}^+ & \dot{q}_{+2}^+ & \dot{q}_{+3}^+
\end{bmatrix}^T
\] (2.79)

The computed variables can be found using the following procedure.

1. Calculate $q_{-N}$ using Eq. 2.64 and the variable $q_{1\ldots N-1}$.

2. Calculate $q^+$ using Eq. 2.2a and the variable $q^-$.  

3. Calculate $\dot{q}^+$ using Eq. 2.2b and the variable $\dot{q}^-$.  

4. Calculate $\alpha_0$ and $\alpha_Q$ using Eqs. 2.60 and 2.62 and the variables $q^-$ and $q^+$.  

5. Calculate $\alpha_1$ and $\alpha_{Q-1}$ using Eqs. 2.61 and 2.63 and the variables $\alpha_0$, $\alpha_Q$, $q^+$, $q^-$, $\dot{q}^+$, and $\dot{q}^-$.  

This method does not enforce a periodic gait at every iteration, although the final gait will be periodic as long as the optimization converges. Unfortunately, requiring the equation of motion (Eq. 2.51) to be exactly satisfied to within numerical tolerance.
results in fairly slow convergence for the optimization. By allowing a gait that is almost but not quite periodic, the optimization can be significantly sped up.

2.5.2 Constraints

Beyond the constraints needed to ensure a periodic gait, constraints are needed to prescribe the desired walking speed, ensure stability (Eq. 2.56) and achieve a physically possible gait.

- Average walking speed – this is an equality constraint;
- No imaginary motion – Eq. 2.53 is satisfied;
- Velocity transformation exists – \( \bar{\lambda}_q(\xi_1) \) from Eq. 2.33 does not become singular;
- Joint limits – the maximum and minimum permitted angles for each joint are listed in Table 2.4;
- Step length – used to constrain the step length (Table 2.5);
- Positive vertical ground reaction forces – this was enforced for both the single support and impact phases;
- Ground reaction forces in the friction cone – for both the single support and impact phases, the coefficient of friction was 0.38 (ERNIE), 0.37 (human) or 0.6 (two-link);
- Foot clearance – used to ensure that the swing foot only contacted the ground at impact;
- No roll back – used for the human models to ensure that the stance foot does not roll backward.
TABLE 2.4

JOINT LIMIT CONSTRAINTS IN DEGREES FOR THE
OPTIMIZATION USED TO FIND MODEL GAITS

<table>
<thead>
<tr>
<th>Joint</th>
<th>ERNIE</th>
<th>Human</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max</td>
<td>Min</td>
</tr>
<tr>
<td>Stance Hip</td>
<td>270</td>
<td>120</td>
</tr>
<tr>
<td>Swing Hip</td>
<td>270</td>
<td>120</td>
</tr>
<tr>
<td>Stance Knee</td>
<td>0</td>
<td>-25</td>
</tr>
<tr>
<td>Swing Knee</td>
<td>0</td>
<td>-90</td>
</tr>
<tr>
<td>Stance Ankle</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Swing Ankle</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Torso</td>
<td>10</td>
<td>-10</td>
</tr>
</tbody>
</table>

Except for the walking speed constraint, all constraints were implemented as inequality constraints. The foot clearance constraint was not used for the two-link model because it is impossible for a two-link biped to walk without scuffing the swing foot.

2.5.3 Additional Constraints for Hardware Gaits

Beyond the “standard” constraints for biped walking listed above, several additional constraints were needed to successfully transfer gaits from simulation to the ERNIE hardware. This section provides a brief overview; details can be found in Post (2013) and Martin et al. (2014a).

Due to joint errors and backlash in the drivetrain, the swing foot often touches the
### Table 2.5

**STEP LENGTH CONSTRAINTS FOR THE OPTIMIZATION USED TO FIND MODEL GAITS**

<table>
<thead>
<tr>
<th>Biped</th>
<th>Max</th>
<th>Min</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-Link</td>
<td>–</td>
<td>–</td>
<td>No step length constraint(^1)</td>
</tr>
<tr>
<td>ERNIE</td>
<td>0.55 m</td>
<td>0.35 m</td>
<td>Section 2.6 experiments(^2)</td>
</tr>
<tr>
<td></td>
<td>0.65 m</td>
<td>0.30 m</td>
<td>Section 3.1.2 experiments(^2)</td>
</tr>
<tr>
<td></td>
<td>0.70 m</td>
<td>0.40 m</td>
<td>Section 3.1.3 experiments(^2)</td>
</tr>
<tr>
<td>Healthy Human</td>
<td>0.16-0.45 m</td>
<td>0.96-1.25 m</td>
<td>Constraint was never active(^3)</td>
</tr>
<tr>
<td>Amputee</td>
<td>1.09 m</td>
<td>0.29 m</td>
<td></td>
</tr>
</tbody>
</table>

\(^1\) Rather than using a step length constraint, the angle between legs at impact was held constant.

\(^2\) A series of optimizations swept across this range of step lengths in 0.01 m increments.

\(^3\) For the matching gaits (Section 4.2), step length was constrained to be between ± 0.4 m of the human experimental step length. Since the step length constraint was never active, it has no effect on the optimization, so the original step length constraints were not modified for the prediction portion of the study.
ground earlier than expected, which in turn causes the impact to occur earlier than expected. When this occurs, ERNIE is not in the expected transition configuration, which results in large joint errors at the start of the new step. To reduce the effect of this issue, the absolute angle of the lower swing leg is not permitted to change more than $3^\circ$ during the last 20% of the gait cycle so that even if the impact occurs early, ERNIE is close to the expected configuration. Since the swing hip typically does not move very much during the end of the gait cycle, a similar constraint is not needed for the hip.

Due to unmodeled backlash in the drivetrain and relatively small motor inertias, ERNIE’s stance leg compresses at the start of each step regardless of the desired motion. As a result, constraints were added to the optimization to enforce the natural dynamics of the system in simulation. Specifically, gaits were required to have at least $5^\circ$ of stance hip flexion and forward torso pitch during the first 20% of the gait cycle. For point-foot gaits, the stance knee was also required to have at least $5^\circ$ of flexion during the first 20% of the gait cycle. Finally, to ensure that ERNIE does not require a foot that is longer than the physical foot, a constraint on the absolute shank angle was implemented.

Step progression in experiment is estimated using both the position of the hip and the measured joint angles.

\[
s_{\text{exp}} = \frac{\theta + x_H/L_i - \theta^+}{\theta^- + x_H^-/L_i - \theta^+},
\]

where $x_H$ is the forward progression of the hip as measured by the encoder on the boom, $L_i$ is the average length of the leg over the step, and the other terms are the same as in Eq. 2.58. The angles at the joints can be measured using the motor encoders or potentiometers at the joints, but the potentiometers are too noisy to be used in a feedback loop. Because there is backlash in the drivetrain, the motor
encoders only provide an estimate of the joint angles, and as a result, directly using Eq. 2.58 caused frequent gait failure. Simply using the position of the hip as the measure of gait progression also led to frequent gait failure because it does not capture any tracking errors. Using a combination of both methods as in Eq. 2.80 mitigates the problems with each method and was reliably successful in experiment.

2.6 Control Validation

To validate the control method, three sets of hardware experiments were conducted using ERNIE (Fig. 2.7). Gaits were designed in simulation for both the point- and curved foot ERNIE configurations, and the gaits were executed using matched ERNIE hardware (Post, 2013). In addition, experiments were conducted with curved feet on the ERNIE hardware but using the gaits designed in simulation for point feet (mixed gaits). The mixed gaits are needed to determine if stable experimental walking for the curved foot gaits occurs simply due to the inherent robustness of the control method or if it occurs due to the extension of the modeling and control methods. For the experiments in this section, the curved foot radius was 0.201 m, and the ankle offset was zero. Because the point feet are constructed from racquetballs, they have a radius of 0.029 m. However, this is small enough to be neglected. In addition, the lower leg is slightly smaller in the point-foot configuration than in the curved foot configuration. The parameters for the point-foot configuration are given in Table 2.2.

Gaits were optimized using specific energetic cost of transport (SECT) (Collins et al., 2005) as the cost function.

\[
SECT = \frac{W_{\text{total}}}{x^c_H m_{\text{tot}} g},
\]  

(2.81)
where

\[ W_{total} = \sum_{i=1}^{4} \int_{0}^{t_f} |u_i \dot{q}_i| dt \]  

(2.82)

is the sum of the absolute value of the work done by each actuator over a step, \( x_H \) is the step length, \( m_{tot} \) is the total mass of the biped, and \( g \) is the gravitational constant. Note that although SECT typically suggests perfect regeneration of negative work, the absolute value in Eq. 2.82 was included to better suit the hardware. (Because ERNIE is not capable of regenerating much, if any negative work, and because it is backdrivable, it consumes energy in performing negative work.) Similar forms of the total work calculation have previously been used to analyze human (Bianchi et al., 1998), underactuated biped robot (Pekarek et al., 2007; Silva and Tenreiro Machado, 1999), and humanoid robot (Liu et al., 2012) walking. Because the optimization is known to be sensitive to the step length constraints, the optimization swept across
TABLE 2.6

AVERAGE WALKING SPEED FOR EACH EXPERIMENTAL CONDITION FOR ERNIE WALKING OVERGROUND

<table>
<thead>
<tr>
<th></th>
<th>Point</th>
<th>Curved</th>
<th>Mixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min Speed (m/s)</td>
<td>0.47</td>
<td>0.48</td>
<td>0.52</td>
</tr>
<tr>
<td>Max Speed (m/s)</td>
<td>0.76</td>
<td>0.75</td>
<td>0.76</td>
</tr>
<tr>
<td>Speed Err (m/s)</td>
<td>0.09</td>
<td>0.06</td>
<td>0.13</td>
</tr>
</tbody>
</table>

step lengths ranging from 0.35 m to 0.55 m in 0.01 m increments, with the gait from the previous optimization serving as the seed for the next optimization. This resulted in hundreds of possible gaits for each speed, and in most cases, an optimal step length emerged. Experimental gaits were chosen from gaits near the optimal step length.

For each experiment, ERNIE was started from rest, and data were collected for one lap of the room. Due to small differences between the model and physical system, the torso pitch often had to be shifted forward slightly, as in Yang et al. (2008, 2009), until stable walking was achieved. A total of 11 point-foot, 18 curved foot, and 11 mixed gaits were tested.

Point-foot experimental trials were run with simulation gait profiles optimized for speeds of 0.40 to 0.60 m/s in 0.05 m/s increments, with the actual speeds ranging from 0.47 to 0.76 m/s in experiment (Table 2.6). The mean error in speed was 0.09 m/s. Given the variability in resultant speed of the point-foot gaits, curved foot gaits were designed for speeds of 0.40 to 0.70 m/s in 0.10 m/s increments, and the actual speed range in experiment for these curved foot gaits was 0.48 to 0.75 m/s (Table 2.6). The mean error in speed was 0.06 m/s. The most likely reason for the reduction in error between point- and curved foot gaits is the probable reduction in the impact
at the step-to-step transition for curved feet. A reduced impact should reduce the amount the motors are backdriven, which in turn should reduce the tracking errors.

For the mixed gaits, stable walking was achieved with curved feet on ERNIE using all of the point-foot gaits. The experimental speed ranged from 0.52 to 0.76 m/s, with a mean difference from the point-foot design speed of 0.13 m/s (Table 2.6). As expected, the mixed gaits consistently resulted in a faster walking speed than the point-foot gaits due to the rolling of the foot and the potentially smaller impacts. A predicted walking speed for the mixed gaits was generated in simulation by changing the foot radius and simulating the point-foot gait until steady-state was reached. In all cases, the difference between the experimental and simulated speed was smaller when the predicted mixed gait speed was used, rather than the point-foot speed. The reduced errors for the curved foot gaits as opposed to the mixed gaits indicates that the curved foot formulation is necessary to accurately control a curved-foot biped with HZD-based control.

Beyond the error in walking speed, two metrics for joint tracking error, maximum motor error and root mean square (RMS) error, were utilized. RMS error is defined as

\[ E_{RMS} = \frac{1}{M} \sum_{j=1}^{M} \sum_{k=1}^{n_j} \sqrt{\frac{\sum e_k^2}{n_j}} \]  

(2.83)

where \( e_k \) is the motor or torso error at sample \( k \) of step \( j \), with \( M \) steps of \( n_j \) samples each (Post, 2013). \( E_{RMS} \) is calculated for each joint.

In general, curved foot gaits had smaller maximum errors than either the point-foot or mixed gaits at equivalent speeds (Fig. 2.8). As expected, the error increased with increasing speed for all experimental conditions, likely due to the greater motor power required and harder impacts at faster speeds. Similarly, the maximum RMS error for curved foot gaits is less than the maximum RMS error for point-foot or mixed gaits at equivalent speeds (Fig. 2.9). The reduced errors for the curved foot
Figure 2.8. Maximum error for motor or torso tracking during experiment when ERNIE walked overground with point feet and gaits designed for point feet (Point), or with curved feet and gaits designed for curved feet (Curved) or point feet (Mixed).

gaits as compared to the mixed gaits indicates that correctly accounting for the curved feet in the control method improves the ability of the experimental system to track the desired gait. The minimum RMS error for all experimental conditions was approximately equal for a given speed. While it appears that point-foot HZD-based control is robust enough to achieve stable walking even with very large errors in foot radius present, using the curved foot model and control method results in better performance. Further, validating the control method, the curved foot gaits consistently had errors that were less than or equal to the errors in the point-foot gaits, indicating that the new control formulation has similar performance for curved foot gaits as the original formulation has for point-foot gaits. As discussed in Section 1.1.4, curved feet offer several benefits over point-feet, so the ability to systematically design curved-foot gaits should result in improved robotic gait.
Figure 2.9. Maximum and minimum RMS error for motor and torso tracking during experiment when ERNIE walked overground with point feet and gaits designed for point feet (Point), or with curved feet and gaits designed for curved feet (Curved) or point feet (Mixed).

2.7 Summary

When the foot radius is properly accounted for, the mathematical model of a curved foot biped is no more difficult to formulate than the mathematical model of a point-foot biped. Further, the mathematical model of a curved foot biped reduces to that of a point-foot biped when the foot radius is zero. This mathematical model has been verified by comparing the results of simulation to previous experimental results for two very different bipedal robots – McGeer’s two-link, passive dynamic walker traversing a decline and the five-link, actuated biped ERNIE walking on a treadmill with a supporting boom. In both cases, the step duration and average velocity from simulation were within the ranges of experimental results. This indicates that the mathematical model captures the dominate dynamics of bipedal systems with curved feet.

Beyond just modeling, it is also possible to extend HZD-based control to planar,
curved foot bipeds. Due to the rolling motion of the foot, the equation that captures the zero dynamics has an additional term as compared to a point-foot biped. This additional term carries through to the equations that define the fixed point (Eq. 2.51) and the stability condition (Eq. 2.56). Despite the additional term, both equations remain analytic, although it is no longer practical to use the analytical solution to find the fixed point for a specific gait. Instead, a scalar boundary value problem must be solved. The control method was validated using the five-link planar biped ERNIE walking overground with a supporting boom. While gaits designed for point feet did produce stable walking when the robot was equipped with curved feet, errors in desired walking speed and joint tracking were significantly higher than when gaits designed for curved feet were run. Therefore, the rigorous gait design approach described herein is needed to reliably extend the provably stable HZD-based control formulation to curved foot bipeds.
CHAPTER 3

THE EFFECTS OF FOOT GEOMETRIC PROPERTIES ON GAIT

This chapter discusses how foot design, specifically foot radius and ankle offset, affects gait. Using both two- and five-link bipeds, the effects on joint kinematics and energy efficiency are explored. Section 3.1 discusses the biped models and methods used. Section 3.2 discusses how foot radius affects gait in simulation, and Section 3.3 discusses how ankle offset affects gait in simulation. Section 3.4 discusses the combined effect of foot radius, ankle offset and speed on gait using both simulation and hardware experiments with the five-link planar biped robot ERNIE.

3.1 Methods

The effects of foot radius and ankle offset on gait both individually and in combination were investigated. In all cases, gait were designed by minimizing the specific energetic cost of transport (SECT, Eq. 2.81). Three planar models were utilized – a two-link biped, a five-link biped without robustness constraints, and a five-link biped with robustness constraints.

3.1.1 Two-Link Biped

The two-link biped consists of two symmetric legs pinned at the hip (Fig. 3.1, Table 3.1) with an actuator to control the angle between the legs. Because it is impossible to avoid foot scuffing in the middle of the step for a two-link biped, foot clearance issues were ignored. When designing gaits, the angle between the legs ($q_1$)
at impact was held constant, which effectively holds the step length constant. This reduces the dimension of the controller design space to two, making it feasible to completely explore the remaining design space. The step length with point feet was chosen to be 45% $LL$ (0.45 m). When investigating the effects of foot radius, the ankle offset was held constant at zero, and the foot radius was varied from 0 to 60% $LL$ (0 to 0.6 m). When investigating the effects of ankle offset, the foot radius was held constant at 30% $LL$ (0.3 m), and the ankle offset was varied from -1.4% to 0.6% $LL$ (-0.014 to 0.006 m).

3.1.2 Five-Link Biped without Robustness Constraints

The five-link model was based on the physical ERNIE robot prior to the transition to overground walking (Fig. 3.2(a), Table 2.2). Gaits were designed using the same methodology as discussed in Section 2.6 except the additional robustness constraints discussed in Section 2.5.3 were not utilized. In addition, the minimum permitted stance knee angle was -90°, not -25° as given by Table 2.4. The step length was systematically varied from 41% to 89% $LL$ (0.30 to 0.65 m). The gait with the
lowest SECT for each condition was included in this study. When investigating the effect of foot radius, the ankle offset was held constant at zero, the speed was held constant at 0.55 m/s and the foot radius was varied from 0 to 45% $LL$ (0 to 0.33 m). When investigating the effect of ankle offset, the foot radius was held constant at 28% (0.21 m), the speed was held constant at 0.50 m/s and the ankle offset was varied from 0% to 16% $LL$ (0 to 0.12 m).

3.1.3 Five-Link Biped with Robustness Constraints

Because the gaits designed in Section 3.1.2 do not have the additional robustness constraints as described in Section 2.5.3, they are unlikely to work in hardware experiments. Additionally, it was desired to investigate the effect of speed on gait, so new gaits were designed. A full factorial study was performed in both simulation and hardware to investigate the effects of foot radius, ankle offset and speed. For the hardware experiments, the ERNIE robot was configured for overground walking (Fig. 3.2, Section 1.2).

Three foot radii ($R$) were used (Fig. 3.3) – 17% $LL$ (0.125 m), 27%$LL$ (0.201 m), and 37%$LL$ (0.272 m). Regardless of the foot radius, the distance from the hip to
Figure 3.2. (a) Schematic of the biped ERNIE. (b) ERNIE configured for overground walking with curved feet.
the floor remained constant when the leg was upright (i.e. \( LL \) is constant). Because the difference in mass between the smallest foot and largest foot is only 0.05 kg, the change in mass was neglected in the model. Because it was expected that gaits for larger foot radii would have more of a rolling motion, the feet were designed to have differing arc lengths, and thus, the constraint that was used to ensure that ERNIE did not roll past the end of its foot varied with foot radius. The maximum angle the stance foot was allowed to roll forward was 40° for the smaller two radii and 48° for the largest foot radius. The values were chosen based on both the results from the study described in Section 3.1.2 and the desire to have feet of reasonable length. Because the maximum distance the foot can be shifted increases as the foot radius increases, the ankle offsets were chosen as a function of foot radius. Three ankle offsets were used – 0\( \%R \), 12.5\( \%R \), and 25\( \%R \), rounded to the nearest 0.5 cm. In terms of leg length, the nonzero ankle offsets were 2\( \%LL \), 3\( \%LL \), 4\( \%LL \), 5\( \%LL \), 7\( \%LL \), and 9\( \%LL \). An ankle offset of 25\( \%R \) was the largest distance the foot could be shifted without increasing the length of the lower leg. Three different speeds were also considered – 0.40 m/s, 0.55 m/s, and 0.70 m/s. The slowest speed at which ERNIE can reliably walk is approximately 0.40 m/s. Because ERNIE does not have any designed compliance within the drivetrain, the large impacts during the step-to-step transitions at speeds in excess of 0.70 m/s may pose a risk to the drivetrain.

Gaits were designed using the same methodology discussed in Sections 2.5 and 2.6. The optimization swept across step lengths ranging from 0.4 m to 0.7 m (54 to 95\( \% \) \( LL \)) in 0.01 m increments, with the gait from the previous optimization serving as the seed for the next optimization. The initial seeds for the string of optimizations were three gaits that had previously worked well in experiment for a foot radius of 0.201 m and no ankle offset. A total of 270 gaits (a series of 90 gaits from each initial seed) for each set of foot radius, ankle offset, and speed were found and then reduced in number based on step length constraints. Since the optimal step length is expected
to increase with increasing foot radius due to the rolling of the foot, step length was constrained based on the radius of the foot. The step length for each foot radius was chosen as the mean step length of the three gaits (one for each ankle offset) that had the lowest SECT at 0.40 m/s. For $R = 17\% \ LL$, the step length was 60\% $LL$ (0.44 m), for $R = 27\% \ LL$, the step length was 64\% $LL$ (0.47 m), and for $R = 37\% \ LL$, the step length was 70\% $LL$ (0.52 m). For each foot radius, ankle offset and speed combination, the three gaits with the lowest SECT at the correct step length (plus or minus 0.01 m) were included in the study. This led to a total of 81 gaits that were further examined in both simulation and hardware.

For each hardware experiment, ERNIE was started from rest at the same room location ($0^{\circ}$ of boom rotation) with the right (inside) leg as the initial stance leg. Data from one lap of the room, starting with the first impact after $180^{\circ}$ of boom rotation, was evaluated. The initial half lap was not evaluated to remove the effects of the manual start. Even with an initial push that was much too hard or soft, ERNIE’s speed consistently stabilized within a quarter lap, indicating that the initial push
had no effect on the evaluated data. Due to small differences between the model and physical system, the torso angle often had to be shifted forward slightly as in Yang et al. (2008, 2009) and Post (2013). This angle was adjusted from $0^\circ$ to $12^\circ$ in $2^\circ$ increments as needed to obtain the complete 1.5 laps of the room at approximately the design speed. If data from multiple trials with different torso offsets were collected, only the trial with the average speed closest to the design speed was analyzed.

3.2 Varying Only Foot Radius

The two- and five-link bipeds described in Sections 3.1.1 and 3.1.2 were used to investigate the effect of varying only foot radius in simulation.

3.2.1 Effect on Kinematics

For the two-link biped, the unactuated stance leg angle ($q_2$ in Fig. 3.1) exhibits only slight changes with increasing foot radius, while the hip angle changes significantly (Fig. 3.4). Because the step length was held constant, the angles at the beginning and end of the step must be identical for all foot radii. However, the trajectory between the initial and final angles can and does change as the foot radius changes. For all foot radii, the hip angle increases for most of a step. Near the end of the step, the hip angle peaks, and then the leg swings back slightly before the impact. As the foot radius increases, the peak becomes smaller and occurs later in the step. The timing of the peak appears to be heavily correlated with the SECT of the gait (Fig. 3.8(a)). This is because later peaks tend to make the pre-impact leg angular velocities more similar to the post-impact leg angular velocities, reducing impact losses. For many foot radii/speed combinations, it is impossible to select control parameters so that the peak occurs at the end of the step. The optimal set of control parameters are those that place the peak as close to the end of the step as possible.
As the foot radius increases for the five-link biped, there is no consistent trend for the torso angles. For the hip angles, both the maximum and minimum angles decrease slightly as foot radius increases (Fig. 3.5(a)). However, the range remains at about 26° regardless of the foot radius.

On the other hand, increasing foot radius has a significant effect on the knee kinematics (Fig. 3.5(b)). The range of angles occurring over a step increases significantly. For a point-foot gait, the stance knee angle remains at a relatively constant 12° of flexion throughout the step. As the foot radius increases up to about the biomimetic foot radius (28% of leg length), the stance knee straightens during the first 20% of the step, remains relatively constant for the next 60% of the step, and then rebends during the last 20% of the step. With a biomimetic foot radius, the range of the stance knee flexion is 21°. As the foot radius increases further, the stance knee straightens earlier in the step, and then immediately begins to bend again. For the largest foot radius considered, the range increases to 48° of flexion. The deeply bent stance knee shortens the distance between contact points at impact, which may be advantageous from an energy standpoint (Donelan et al., 2002a). For all foot radii, the swing knee begins flexed, flexes more as the step progresses, then straightens toward the end of...
Figure 3.5. Joint angles over two steps, one stride, for the (a) hip and (b) knee of the five-link biped model. The solid lines denote the stance side, and the dashed lines the swing side. As the lines get darker, the foot radius increases.
the step. The larger the foot radius, the earlier the swing knee reaches peak flexion. For the largest foot radius, the peak knee flexion occurs at approximately 30% of the step, while for point feet, the peak knee flexion occurs at approximately 60% of the step. In addition, the change in knee angle from the beginning of the swing phase to peak flexion decreases while the change in angle from peak flexion to the end of the swing phase increases as foot radius increases. By shifting the time of peak knee flexion earlier, the knee velocity at the start and end of the swing phase can remain similar, and presumably close to the optimal value, regardless of foot radius.

For the larger foot radii, the feet become unreasonable large (Fig. 3.6). In a real world situation, the large radii feet could present serious problems for obstacle avoidance. While the feet were assumed to be massless in the model, it is not possible to build massless feet, although the change in mass is relatively small compared to the total mass of the biped. In a worst-case scenario, assuming that the foot is made of a solid 1/4 inch thick piece of aluminum just large enough to allow the foot to roll without pivoting about the ends, for a foot radius of 45% of leg length, the additional foot mass is 0.253 kg, or 26% of the lower leg mass and 1% of the total mass. For the

Figure 3.6. The required arc length of the foot as a function of foot radius for the five-link biped model. The least-squares parabolic fit has an $R^2$ value of 0.997.
Figure 3.7. The optimal step length as a function of foot radius for the five-link biped model. The least-squares linear fit has an $R^2$ value of 0.956.

For the five-link biped, the optimal step length increased as foot radius increased (Fig. 3.7). This is expected because as the stance leg rotates from leaning back to leaning forward, the stance foot rolls along the ground without requiring any additional actuator work. As the foot radius increases, the distance the foot rolls for a given change in stance leg angle increases, leading to longer steps. The step length results are slightly noisy because SECT is only weakly correlated with step length.
near the optimal step length. Further, because gaits were optimized using MATLAB’s \texttt{fmincon} function, only local minima for SECT can be guaranteed. However, it is likely that gaits near the global minima were found because SECT forms a convex cost function for the two-link biped, which is likely true for the five-link biped as well.

Regardless of whether the gaits found were globally optimal, there is a clear trend toward longer step lengths. The percentage of the step length that is due to the rolling of the foot also increases with increasing foot radius. With point feet, the foot does not roll, and all of the step length comes from moving the swing leg from behind to in front of the stance leg. For the biomimetic foot radius of 28% of leg length, 35% of the step length is due to the rolling of the foot. With a foot radius slightly shorter than the lower leg (45% LL), 67% of the step length is due to the rolling of the foot. Because of the distance the stance foot rolls, the feet are closer together at impact, which helps reduce impact losses (Donelan et al., 2002a), thereby lowering the SECT.

3.2.2 Effect on Efficiency

For both bipeds, the SECT decreased as the foot radius increased (Fig. 3.8). The decrease in SECT is significant, over 60% of the point-foot value in all cases. This is to be expected because as the foot radius tends toward the leg length, the biped more closely approximates a wheel, reducing impact losses.

For the two-link biped, as the foot radius increases, the controller design space that produces stable, one-step periodic gaits without slip shrinks. An important consequence of this is that for large foot radii, the more efficient gaits are unstable. As a result, feet with large radii are less energetically efficient than feet with moderate radii. If the foot radius is held constant, the SECT generally increases with increasing speed. However, if the optimal foot radius is used for each speed, the SECT only increases slightly with increasing speed. Because a five-link biped has a significantly
Figure 3.8. SECT (Eq. 2.81) vs. foot radius for the (a) two-link and (b) five-link bipeds models. For the five-link biped model, the speed is 0.55 m/s.

larger controller design space than a two-link biped, the SECT strictly decreases with increasing foot radius for the five-link biped.

Because of the increase in combined foot/lower leg mass, the decrease in SECT in a physical system would be slightly smaller than predicted by the model. Both the energy lost at impact and the work required to move the swing leg could increase when the foot mass is included in the model (Fig. 3.9). However, due to the small change in overall mass, the added mass only has a small effect on the energy requirements of the system (Fig. 3.8(b)).

It appears that gaits designed by minimizing SECT with the five-link ERNIE model do not capture some key aspects of human gait because the model does not predict a moderately sized optimal foot radius, whereas human subject experiments do (Adamczyk et al., 2006; Hansen and Wang, 2011). In one study that manipulated the effective rollover shape (Hansen and Wang, 2011; Wang and Hansen, 2010), subjects did not change their hip and knee joint kinematics regardless of the effective foot-ankle radius. (Adamczyk et al. (2006) did not report joint kinematics.) In contrast, the knee kinematics of the five-link model changed substantially. Because of
the way gaits were generated for this study, there are several hundred potential gaits for each foot radius. If the pool of potential gaits is limited to the generated gaits that have a human-like maximum knee flexion, the optimal radius has a moderate value, agreeing with the experimental studies. This indicates that if the goal is to model human gait, SECT is not an appropriate objective function.

3.3 Varying Only Ankle Offset

The two- and five-link bipeds described in Sections 3.1.1 and 3.1.2 were used also to investigate the effect of varying only the ankle offset in simulation.

3.3.1 Effect on Kinematics

For the two-link biped, the joint kinematics remain essentially unchanged with shifts in ankle offset (Fig. 3.10). While there are some differences in the magnitudes of the angles (on the order of $2^\circ$), the shape and timing of the joint trajectories remain substantially the same. The shifts in the angles are necessary so that the biped can have both feet on the ground during the instantaneous double support.
phase. However, as the ankle offset shifts back, the hip angular velocity becomes slightly more constant over a step. The small changes in joint trajectories are not surprising because shifts in the ankle offset only change the distribution of the mass relative to the foot. It does not change the inverted pendulum-like motion of the stance leg. Thus, the optimal motion does not substantially change with changing ankle offset. This means that for a two-link biped, it is possible to adjust the ankle offset without significantly altering its motion.

For the five-link biped, as the ankle offset shifts forward, the range of hip angles remains relatively constant (Fig. 3.11(a)). The maximum flexion of both the stance and swing knee, though, depend strongly on the ankle offset (Fig. 3.11(b)). The large swing knee flexion is a secondary effect because when the stance knee is significantly flexed, the swing knee must bend more in order to satisfy the ground clearance constraints. Unlike the effect of foot radius, shifts in the ankle offset do not change the shape of the knee angle kinematics. The only real change is a shift of the joint angles to greater knee flexion.
Figure 3.11. Maximum and minimum joint angles for the (a) hip and (b) knee joints of the five-link biped model.

A simple model similar to the one in Adamczyk et al. (2006) can help explain why the maximum stance knee flexion increases when the ankle offset is increased. Assume that the velocity of the hip is perpendicular to the line from the point of contact to the hip. Further assume that the impact losses scale with the angle between the pre- and post-impact hip velocity directions. Define the angle between the pre- and post-impact hip velocity directions as the velocity angle. The velocity angle scales strongly with step length, as expected. For a given step length, the velocity angle does not increase much between relatively straight legs and slightly more flexed legs. If the legs are significantly bent, the velocity angle increases significantly for slightly more knee flexion. Increasing the ankle offset effectively reduces the knee angle because the important angle is not $q_3$, the angle between the stance upper leg and the line between the knee and the stance lower leg center of mass, but rather $\phi$, the angle between the stance upper leg and the line between the knee and the point of contact (see Fig. 3.12(a) for a schematic). As the center of curvature shifts farther forward, the effective knee angle $\phi$ decreases for a given $q_3$. Thus, the knees can be more flexed before large changes in the velocity angle occur.
Figure 3.12. (a) Schematic showing the stance knee angle ($q_3$) and the effective stance knee angle ($\phi$) of the five-link biped model. (b) The effective impacting knee angle as a function of ankle offset.

While having a smaller stance knee angle at impact can help reduce impact losses, having the knee too straight can also increase energy consumption. Because the hip is moving faster than the point of contact at impact, the passive dynamics tend to straighten the stance knee. If the new stance knee is too straight at impact, the hip momentum will cause the stance knee to hyperextend, which was not permitted in the optimization. Because the model does not have mechanical knee stops, it requires energy to prevent hyperextension. As a result, having the stance knee sufficiently bent at impact can reduce the energy required. Again, it appears that the effective knee angle $\phi$ is a much better measure of whether the knee will hyperextend than the actual knee angle $q_3$.

For a biped without mechanical knee stops, there is a trade-off between reducing impact losses and reducing stance knee work, both of which are dependent on the effective knee angle $\phi$. As a result, the effective knee angle remains relatively constant for the optimal gaits (Fig. 3.12(b)). If the model’s knees had mechanical stops to prevent hyperextension, it is likely that the stance knee would fully extend. With
mechanical knee stops, preventing hyperextension is entirely passive, which eliminates the trade-off.

3.3.2 Effect on Efficiency

Compared to the effect of foot radius, the effect of ankle offset on energy efficiency is small (Fig. 3.13). For the two-link biped, the SECT decreases slightly as ankle offset decreases from positive to negative. This finding agrees with earlier experimental findings (McGeer, 1990). The reason for this decrease is a reduction in the work required to move the swing leg through. As discussed above, as the ankle offset shifts backward, the hip angular velocity becomes more constant, reducing the work required to accelerate and decelerate the leg. Because the two-link walker does not have knees, moving the ankle offset back does not cause the stance leg to buckle. Further, for the two-link biped, the velocity angle does not depend on the center of curvature location, so the impact losses remain constant. Thus, the small changes in the hip angular velocity are sufficient to reduce the SECT of the two-link biped as the ankle offset decreases.

For the five-link biped, the SECT remains fairly constant regardless of the ankle offset. This is to be expected because of the trade-off between preventing hyperextension and reducing impact losses.

3.4 Combined Effects of Foot Radius, Ankle Offset and Speed

The five-link ERNIE biped with robustness constraints (described in Section 3.1.3) was used to investigate the combined effects of varying foot radius, ankle offset, and speed in both simulation and hardware experiments. A discussion of how the results in this section differ from the results in the previous two sections is provided in Appendix B.
Figure 3.13. SECT vs. ankle offset for the (a) two-link and (b) five-link biped models. For the five-link biped, the speed is 0.50 m/s.

3.4.1 Metrics

The effects of foot radius, ankle offset and speed on tracking error, joint kinematics, and energy efficiency were evaluated. Since tracking error measures the deviation between simulation and experiment, only experimental data exists. For energy efficiency and joint kinematics, the trends for both simulation and experiment are presented. For the experimental results, means over one lap of the room are presented. The experimental tracking errors are reported first to demonstrate that ERNIE successfully walked with gaits adequately approximating the optimized gaits. Then, the joint kinematics results are presented to examine how the motions of the stance and swing hip and knee were influenced by foot radius, ankle offset, and speed. These changes in kinematics are important for interpreting the energetic efficiency results presented last.

To verify that the hardware was capable of generating the planned gaits with acceptable accuracy, tracking error was evaluated using root mean square (RMS) error (Eq. 2.83) and correlations between the desired and actual angles. To quantify the correlation, Pearson’s correlation coefficient was used (Ferrari et al., 2008). The
correlation coefficient measures how accurately a linear transformation would transform the commanded angle into the actual angle. Informally, it provides information about how well the shapes of the joint angle plots match between the commanded and executed cases. A value of 1 indicates a perfect match, while 0 indicates no correlation.

To help identify whether the changes in SECT occur due to changes in joint motion, changes in joint torque, or a combination of the two, the effects of foot radius, ankle offset and speed on joint kinematics were quantified. The ranges of motion of the hip, knee, and torso over a stride were analyzed, along with the timing of the peak joint angles.

Energy efficiency was measured using SECT and nondimensional torque squared ($\tau^2$). To calculate SECT for simulation data, Eqs. 2.81 and 2.82 were used directly. For experimental data, Eq. 2.81 was also used, but

$$W_{total,exp} = \sum_{i=1}^{4} \sum_{k=1}^{n_j-1} |\bar{u}_{i,k} \cdot \Delta m_{i,k}|,$$

where $n_j$ is the number of samples for a particular step. $\Delta m_{i,k}$ is the change in motor angle, and $\bar{u}_{i,k}$ is average motor torque at joint $i$ for the time period between the $k^{th}$ and $(k + 1)^{th}$ sample. The motor torque is calculated from the measured current, and the motor angle is measured using the motor encoder. Nondimensional torque-squared is defined as

$$\tau^2 = \frac{\sum_{i=1}^{4} \int_0^T (u_i(t))^2 dt}{(m_{tot} gx_H)^2},$$

where $T$ is the duration of the step, $u_i$ is the torque at the $i^{th}$ joint, $m_{tot}$ is ERNIE’s total mass (19.6 kg), $g$ is gravitational acceleration, and $x_H$ is step length. The simulated/experimental torque and step length were used to evaluate simulation/experimental results. SECT provides a measure of how much work is required, while $\tau^2$ provides
a measure of how much motor torque is required.

For each metric, regression equations were found using the statistical package R’s (R Core Team, 2013) stepwise section function `step`. In all cases, the initial fit was given by

\[
f(R, X_F, v) = \beta_0 + \beta_v \cdot v + \beta_R \cdot R + \beta_F \cdot X_F + \beta_{vR} \cdot v \cdot R + \beta_{vF} \cdot v \cdot X_F + \beta_{RF} \cdot R \cdot X_F, \quad (3.3)
\]

where \( f \) is the metric of interest, \( \beta_j \) are the coefficients, \( R \) is the foot radius in meters, \( X_F \) is the ankle offset in meters, and \( v \) is the speed in meters per second. For simulation fits, the simulated speed was used, and for experimental fits, the experimental speed was used. The `step` function chooses which terms to remove from (or add back in to) the equation using the AIC metric in an iterative fashion. The adjusted coefficient of multiple determination \( (R^2_a) \) was used to quantify how well the regression equation fit the data (Montgomery et al., 2004). Eq. 3.3 finds a single regression fit using all of the experimental data. For each coefficient, a p-value is calculated that gives the probability that the actual coefficient value is zero even though the estimated coefficient is not zero. For small p-values (< 0.05), there is a less than a 5% chance that the sign of the coefficient is incorrect, which provides confidence that at least the signs of the terms are correct. For large p-values, the noise-to-signal ratio is too high, and as a result, it is not possible to make claims about the sign of that coefficient.

The presented regression equations are likely only valid for interpolation and not for extrapolation. The range of foot parameters studied covers a significant portion of the practical design space, so this is not a burdensome constraint. The trends are certainly not valid at speeds below 0.40 m/s because ERNIE cannot sustain walking at slower speeds. If the foot radius were larger than the length of the lower leg, the center of curvature would shift forward, rather than backward, when the knee is
flexed, which could affect the ideal motion, so it is unknown if the results are valid for foot radii larger than 0.378 m. Obviously, a foot radius of less than zero is not physically possible. The ankle offset was assumed to be much less than the foot radius due to the challenges of building feet with very large ankle offsets. While legs could certainly be designed such that large ankle offsets are possible, they would likely not look like legs found in nature due to how far the center of curvature is shifted forward. Further, in this study, only non-negative ankle offsets were examined, and it seems likely that the trends for negative ankle offsets would be different than for positive ankle offsets.

3.4.2 Effect on Tracking Error

As expected, the RMS error at all joints increases with increasing speed for the range of parameters studied (Table 3.2). (Because of the interaction effects between speed and the foot properties, a negative $\beta_v$ does not necessarily mean that the error decreases with increasing speed.) For the set of design parameters studied, the strongest speed effects are seen at the stance hip, stance knee, and torso. Foot radius does not have a consistent effect on RMS error. For some joints (stance hip and torso), larger foot radii have smaller errors. For other joints (swing hip and knee), larger foot radii have larger errors. For the stance knee, larger foot radii can have larger or smaller errors depending on the ankle offset. Although $\beta_F$ is consistently negative, ankle offset also does not have a consistent effect due to the interaction terms. Ankle offset has very little effect on the stance knee or torso RMS error. Increasing ankle offset increases RMS error at the swing hip and knee, while increasing ankle offset decreases RMS error at the stance hip. The magnitudes of the RMS errors for each joint are approximately the same. The RMS errors range from $3^\circ$ to $6^\circ$ at the stance hip, $3^\circ$ to $5^\circ$ at the swing hip, $2^\circ$ to $5^\circ$ at the stance knee, $2^\circ$ to $7^\circ$ at the swing knee,
# TABLE 3.2

**COEFFICIENT VALUES FOR THE REGRESSION EQUATION GIVEN BY EQ. 3.3 FOR MEAN RMS ERROR IN THE ERNIE HARDWARE EXPERIMENTS**

<table>
<thead>
<tr>
<th>Metric</th>
<th>$R^2_a$</th>
<th>$\beta_0$</th>
<th>$\beta_v$</th>
<th>$\beta_R$</th>
<th>$\beta_F$</th>
<th>$\beta_{vR}$</th>
<th>$\beta_{vF}$</th>
<th>$\beta_{RF}$</th>
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</thead>
<tbody>
<tr>
<td>Stance</td>
<td>0.81</td>
<td>1.92</td>
<td>6.21</td>
<td>-5.42</td>
<td>-16.4</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Hip</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Swing</td>
<td>0.68</td>
<td>1.72</td>
<td>1.53</td>
<td>2.06</td>
<td>-7.76</td>
<td>X</td>
<td>X</td>
<td>98.6</td>
</tr>
<tr>
<td>Hip</td>
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<td>0.050</td>
<td>0.392</td>
<td></td>
<td></td>
<td></td>
<td>0.011</td>
</tr>
<tr>
<td>Stance</td>
<td>0.84</td>
<td>-3.69</td>
<td>10.6</td>
<td>7.63</td>
<td>-30.8</td>
<td>-15.6</td>
<td>X</td>
<td>182</td>
</tr>
<tr>
<td>Knee</td>
<td>0.006</td>
<td>&lt; 0.001</td>
<td>0.193</td>
<td>&lt; 0.001</td>
<td>0.096</td>
<td></td>
<td></td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>Swing</td>
<td>0.83</td>
<td>4.51</td>
<td>-4.40</td>
<td>-12.0</td>
<td>-94.5</td>
<td>26.3</td>
<td>113</td>
<td>196</td>
</tr>
<tr>
<td>Knee</td>
<td>0.033</td>
<td>0.187</td>
<td>0.223</td>
<td>&lt; 0.001</td>
<td>0.095</td>
<td>0.003</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
<tr>
<td>Torso</td>
<td>0.52</td>
<td>-7.56</td>
<td>22.6</td>
<td>29.7</td>
<td>-11.2</td>
<td>-60.1</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>0.027</td>
<td>&lt; 0.001</td>
<td>0.051</td>
<td>0.007</td>
<td>0.015</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*NOTE: For each case, the top row gives the value of the coefficient, and the bottom row gives the p-value. If a coefficient is not in the regression equation, it is indicated with an “X”.*
and 3° to 8° at the torso.

For all but the stance knee, the executed motor angles track the commanded motor angles well (Fig. 3.14). None of the correlations depend on speed, foot radius or ankle offset. For both hips and the swing knee, the mean correlation is 0.99, indicating almost perfect tracking. For the torso, the mean correlation is 0.85, which is a reasonable value considering that the torso angle measures the unactuated motion of ERNIE. The mean correlation of the stance knee is 0.27. However, the stance knee has a mean range of motion of only 3°, which means that small tracking errors have a much larger effect on the stance knee correlation than they have on any other joint.

Because foot radius and ankle offset do not have a large effect on tracking error, and because the tracking errors were generally low, the effects of foot design on joint kinematics and energy efficiency can be examined with confidence.
3.4.3 Effect on Joint Kinematics

The range of motion of the hip increases with increasing foot radius and ankle offset and decreases with increasing speed (Fig. 3.15, Table 3.3), with most of the change occurring due to increased stance hip extension. The swing hip flexion, particularly over the middle part of the step, is remarkably similar regardless of foot radius, ankle offset or speed. Because the constrained step length is larger for larger radius feet, the stance hip extension likewise increases. On the other hand, as speed increases, the distance the foot rolls increases, which reduces the required stance hip extension.

In terms of timing, the stance hip is the only joint whose motion is dependent on any of the varying parameters, in this case speed. For slower speeds, the stance hip quickly moves from front to back and then remains at an approximately constant position for the remainder of the step. At faster speeds, the stance hip reaches the point of maximum extension at impact. Thus, at slow speeds, the biped simply falls into the impact, whereas at faster speeds, the stance hip is used to help push into the impact. This might be a similar, although likely less efficient, mechanism to human toe-off, which becomes more significant at faster speeds (Section 4.6; Donelan et al., 2002a).

The range of motion of the knee increases strongly with increasing foot radius (Fig. 3.16, Table 3.3), largely due to the foot clearance requirements in swing. As foot radius increases, the distance from the knee to the lowest point on the swing foot increases, while the distance from the ground to the hip remains relatively constant. Thus, to prevent foot scuffing, the swing knee must flex more. A large ankle offset exacerbates the issue, which explains the positive interaction term $\beta_{RF}$ between foot radius and ankle offset in Table 3.3. Previous experiments (Section 2.6; Post, 2013) with point feet exhibited a knee range of motion of approximately $25^\circ$. Thus, the range of knee flexion appears to plateau for foot radii less than about 0.1 m. The
Figure 3.15. Left column contains the mean hip joint angles over all steps in hardware; right column contains the hip joint angles in simulation. From 0 to 1, the stance angles are shown, and from 1 to 2, the swing angles are shown. The top row shows the slowest speed, the middle row shows the moderate speed, and the bottom row shows the fast speed. Line width indicates foot radius, and line style indicates ankle offset. The general shapes of the hip angle traces are similar in simulation and hardware. At the slow speed, the hip angle is held approximately constant for a period near the end of stance. As speed increases, the duration of this hold decreases. The maximum hip angle is approximately constant in all cases, while the minimum hip angle varies with foot design.
Figure 3.16. Left column contains the mean knee joint angles over all steps in hardware; right column contains the knee joint angles in simulation. From 0 to 1, the stance angles are shown, and from 1 to 2, the swing angles are shown. The top row shows the slowest speed, the middle row shows the moderate speed, and the bottom row shows the fast speed. Line width indicates foot radius, and line style indicates ankle offset. The knee angle remains relatively constant at about -30° over stance regardless of the foot geometry or speed, but flexes significantly during the swing phase, with the amount of flexion depending on the foot geometry.
Figure 3.17. Left column contains the mean torso joint angle over all steps in hardware; right column contains the torso joint angles in simulation. The top row shows the slowest speed, the middle row shows the moderate speed, and the bottom row shows the fast speed. Line width indicates foot radius, and line style indicates ankle offset. The motion of the torso is qualitatively similar for both simulation and hardware.
<table>
<thead>
<tr>
<th>Metric</th>
<th>$R^2_a$</th>
<th>$\beta_0$</th>
<th>$\beta_v$</th>
<th>$\beta_R$</th>
<th>$\beta_F$</th>
<th>$\beta_{vR}$</th>
<th>$\beta_{vF}$</th>
<th>$\beta_{RF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hip Hardware</td>
<td>0.70</td>
<td>19.9</td>
<td>25.3</td>
<td>228</td>
<td>-152</td>
<td>-284</td>
<td>X</td>
<td>1280</td>
</tr>
<tr>
<td></td>
<td>0.302</td>
<td>0.411</td>
<td>0.009</td>
<td>0.153</td>
<td>0.039</td>
<td>0.005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hip Simulation</td>
<td>0.82</td>
<td>23.1</td>
<td>22.0</td>
<td>143</td>
<td>9.59</td>
<td>-162</td>
<td>X</td>
<td>609</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>0.021</td>
<td>&lt; 0.001</td>
<td>0.895</td>
<td>0.001</td>
<td>0.046</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Knee Hardware</td>
<td>0.96</td>
<td>16.7</td>
<td>9.08</td>
<td>98.6</td>
<td>-284</td>
<td>X</td>
<td>X</td>
<td>2840</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>0.133</td>
<td>&lt; 0.001</td>
<td>0.001</td>
<td>&lt; 0.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Knee Simulation</td>
<td>0.97</td>
<td>14.4</td>
<td>X</td>
<td>88.8</td>
<td>-274</td>
<td>X</td>
<td>X</td>
<td>2640</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Torso Hardware</td>
<td>0.53</td>
<td>-8.12</td>
<td>36.5</td>
<td>81.5</td>
<td>29.9</td>
<td>-94.0</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.363</td>
<td>0.014</td>
<td>0.044</td>
<td>0.007</td>
<td>0.146</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Torso Simulation</td>
<td>0.86</td>
<td>18.6</td>
<td>-6.23</td>
<td>10.2</td>
<td>-34.5</td>
<td>63.0</td>
<td>174</td>
<td></td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>0.092</td>
<td>0.015</td>
<td>0.006</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

NOTE: For each case, the top row gives the value of the coefficient, and the bottom row gives the p-value. If a coefficient is not in the regression equation, it is indicated with an “X”.
Figure 3.18. The distance the foot rolls (and hence the required length of the foot) as a percentage of step length for both hardware and simulation. Because the hardware does a good job of matching the planned step length, the data points overlap. As foot radius increases, the required arc length of the foot increases.

timings of the stance and swing knee peaks were not affected by foot radius, ankle offset or speed.

Because the torso motion is strongly affected by the heuristic controller, it is not expected that the motion in simulation and hardware will be quantitatively similar. As expected, the simulation does not do a good job of predicting trends in the hardware torso range of motion (Table 3.3). Rather, qualitatively comparing the torso motion provides additional validation that the hardware gaits track the simulation gaits with acceptable accuracy. In both simulation and hardware, the minimum torso angle occurs during the first half of the step, and the maximum torso angle occurs shortly before impact (Fig. 3.17), providing the expected validation.

The distance the foot rolls as both an absolute value and as a percentage of step length also increases with increasing foot radii (Fig. 3.18). Therefore, larger radii feet need to be substantially longer to prevent rolling onto the ends and transitioning to a point contact in late stance.

In total, the dominant change in joint kinematics is an increase in swing knee
flexion with both foot radius and ankle offset. In addition, the required foot length increases with increasing foot radius. The former has implications for energetic efficiency.

3.4.4 Effect on Energy Efficiency

Both the simulation and hardware results show that both SECT (given by Eq. 2.81) and $\tau^2$ (given by Eq. 3.2) depend on walking speed, foot radius, ankle offset, and the interaction terms (Figs. 3.19-3.20, Tables 3.4-3.5). In addition, mean absolute power (Winter, 1979; Caldwell and Forrester, 1992; Bianchi et al., 1998) was also used to quantify energy requirements, although it is not reported here. As expected, the magnitudes of all three metrics were much lower in simulation than in hardware due to the ideal actuators, absence of backlash, and lack of error in the simulation. The signs of the coefficients, however, were consistent for all three metrics and between simulation and experiment. Because of the consistency, it appears that the actual torque required varies systematically with foot geometry and speed. Combined with the fact that most coefficients have p-values of less than 0.001, it is possible to confidently state that the observed changes occur due to the changing speed and foot geometry and do not simply arise from noise.

From a design perspective, the effects of varying foot radius and ankle offset at a constant speed are the most interesting. Looking at SECT, both the foot radius ($\beta_R$) and ankle offset ($\beta_F$) terms are negative. When the ankle offset is zero, increasing foot radius decreases the SECT. Similarly, when the foot radius is small, increasing ankle offset decreases the SECT. Due to the positive interaction term ($\beta_{RF}$), however, for large ankle offsets, increasing foot radius increases SECT. Similarly, for large foot radii, increasing ankle offset increases SECT. Thus, it is better to have a large foot radius with a small ankle offset or a small foot radius with a large ankle offset. Unless the robot will primarily be walking at very slow speeds, the most energetically efficient
TABLE 3.4

COEFFICIENT VALUES FOR THE REGRESSION EQUATION GIVEN BY EQ. 3.3 FOR SECT FOR BOTH SIMULATION AND EXPERIMENT

<table>
<thead>
<tr>
<th>Metric</th>
<th>$R^2_a$</th>
<th>$\beta_0$</th>
<th>$\beta_v$</th>
<th>$\beta_R$</th>
<th>$\beta_F$</th>
<th>$\beta_{vR}$</th>
<th>$\beta_{vF}$</th>
<th>$\beta_{RF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp</td>
<td>0.48</td>
<td>0.535</td>
<td>-0.135</td>
<td>-0.693</td>
<td>-13.4</td>
<td>X</td>
<td>14.9</td>
<td>19.3</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>0.236</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
<tr>
<td>Sim</td>
<td>0.89</td>
<td>0.036</td>
<td>0.145</td>
<td>-0.194</td>
<td>-2.63</td>
<td>X</td>
<td>2.52</td>
<td>8.53</td>
</tr>
<tr>
<td></td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: For each case, the top row gives the value of the coefficient, and the bottom row gives the p-value. If a coefficient is not in the regression equation, it is indicated with an “X”.

foot design is to have a foot radius equal to the length of the lower leg and no ankle offset. The arc length of the foot must be sufficiently long (Fig. 3.18), however, or it is difficult for the optimization to find gaits.

If it is not desirable to have a very long foot, a moderate foot radius can be used in conjunction with a large ankle offset to achieve almost the same energy efficiency. The difference in energy efficiency between the two foot design strategies depends on walking speed because of the positive interaction term between ankle offset and speed ($\beta_{vF}$). The faster the speed, the greater the loss in efficiency when the small foot radius/large ankle offset strategy is used compared to the large foot radius/no ankle offset strategy. When the small foot radius/large ankle offset strategy is used, there is interplay between the radius of the foot and the ankle offset. From an efficiency standpoint, the unnormalized value of the ankle offset is what matters, but the regression equations are likely only valid if the ankle offset is less than the foot...
Figure 3.19. Left column contains hardware SECT vs. hardware speed; right column contains simulation SECT vs. simulation speed. The top row shows the smallest foot radius, the middle row shows the moderate foot radius, and the bottom row shows largest foot radius. In order to plot the regression equation, constant values of foot radius and ankle offset were used in Eq. 3.3, which is not equivalent to splitting the data into nine parts and finding nine separate regression equations in which the only independent variable is speed. As foot radius increases, the transition speed at which the optimal ankle offset becomes zero is slower.
Figure 3.20. Left column is hardware $\tau^2$ vs. hardware speed; right column is simulation $\tau^2$ vs. simulation speed. The top row shows the smallest foot radius, the middle row shows the moderate foot radius, and the bottom row shows largest foot radius. In order to plot the regression equation, constant values of foot radius and ankle offset were used in Eq. 3.3, which is not equivalent to splitting the data into nine parts and finding nine separate regression equations in which the only independent variable is speed. As foot radius increases, the transition speed at which the optimal ankle offset becomes zero is slower.
### TABLE 3.5

**COEFFICIENT VALUES FOR THE REGRESSION EQUATION GIVEN BY EQ. 3.3 FOR TORQUE-SQUARED FOR BOTH SIMULATION AND EXPERIMENT**

<table>
<thead>
<tr>
<th>Metric</th>
<th>$R^2_a$</th>
<th>$\beta_0$</th>
<th>$\beta_v$</th>
<th>$\beta_R$</th>
<th>$\beta_F$</th>
<th>$\beta_{vR}$</th>
<th>$\beta_{vF}$</th>
<th>$\beta_{RF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp</td>
<td>0.87</td>
<td>2.57e-4</td>
<td>0.309</td>
<td>0.072</td>
<td>-3.28</td>
<td>-0.952</td>
<td>1.95</td>
<td>6.80</td>
</tr>
<tr>
<td></td>
<td>0.996</td>
<td>&lt; 0.001</td>
<td>0.778</td>
<td>&lt; 0.001</td>
<td>0.021</td>
<td>0.048</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
<tr>
<td>Sim</td>
<td>0.95</td>
<td>6.68e-4</td>
<td>0.114</td>
<td>-0.047</td>
<td>-0.650</td>
<td>-0.259</td>
<td>0.397</td>
<td>1.86</td>
</tr>
<tr>
<td></td>
<td>0.897</td>
<td>&lt; 0.001</td>
<td>0.086</td>
<td>&lt; 0.001</td>
<td>&lt; 0.001</td>
<td>0.003</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: For each case, the top row gives the value of the coefficient, and the bottom row gives the p-value.

radius. (As discussed below, it is likely that very large ankle offsets will result in less energetically efficient gaits than moderate ankle offsets.) As a result, the foot radius needs to be large enough that it is possible to have a significant ankle offset, but not so large that the interaction term dominates. For ERNIE, this foot radius is about 0.2 m (27%LL). Referring back to the foot radius and ankle offset values reported in Table 1.1, note that the small foot radius/large ankle offset strategy is employed by many passive and minimally actuated bipeds.

If the metric is $\tau^2$, however, the optimal ankle offset is different for some foot radii. In experiment, for the range of conditions studied, having a larger foot radius and larger ankle offset always produces a more efficient gait, which differs from the findings for SECT. However, as foot radius increases, the effect of ankle offset becomes less significant. At a foot radius of about 0.3 m, which is slightly less than the length of the lower leg, the optimal ankle offset switches to zero. The simulation predicts
that a larger foot radius is more efficient, consistent with the experimental results and the SECT optimal design. It also predicts that the optimal ankle offset is large for small radii feet, and zero for large radii feet, which is consistent with the experimental results and the SECT optimal design. However, the simulation predicts the transition from a large optimal ankle offset to a small optimal ankle offset at a smaller foot radius than is observed experimentally. For both experiment and simulation, the radius at which the optimal ankle offset switches is a function of speed, with the transition occurring at smaller foot radii for faster speeds.

For both metrics, the signs of the terms in the regression equation can be explained by considering energy flows. The negative term for foot radius ($\beta_R$) is not surprising because as foot radius increases, the losses at impact decrease (Fig. 3.21) and the motion more closely approximates the rolling of a wheel (McGeer, 1990). Similarly, the positive speed term ($\beta_v$) is not surprising because as speed increases, the energy lost at impact increases. The effect of speed decreases as foot radius increases ($\beta_{vR} < 0$) because the motion more closely approximates a rolling wheel, which requires no input energy in the ideal case regardless of speed.

Figure 3.21. The energy lost at impact in simulation for all foot radii, ankle offset and speed conditions. Differing ankle offsets are not identified because the effect is small. As foot radius increases, the energy lost at impact decreases, which in turn, lowers the SECT.
For SECT, one of the primary factors in the positive $\beta_{RF}$ is the increase in swing knee work that occurs due to the increased range of motion needed to avoid foot scuffing (Fig. 3.16). Because $\tau^2$ does not consider motion, and since the torque remains relatively small regardless of foot radius or ankle offset (Fig. 3.22), the increased swing knee motion does not affect the value of $\tau^2$. This difference likely explains why the transition foot radius between large ankle offset and no ankle offset occurs at larger foot radii when considering $\tau^2$ rather than SECT.

For both metrics, the increase in torque at the stance knee (Fig. 3.22) is a significant factor in the positive interaction term between foot radius and ankle offset ($\beta_{RF}$). This increase in torque can be explained by considering the moment the ground reaction force (GRF) creates about the stance knee. A rough approximation of the direction of the GRF is the line through the point of contact and the hip. At the beginning of the step, this line is well behind the knee joint, which causes a negative moment at the knee. As the step progresses, the line moves closer to the knee joint and in some cases moves in front of the knee joint. As ankle offset increases, the line shifts forward. When the line passes near or through the knee joint over much of the step, the moment generated by the GRF is small. This explains why larger ankle offsets are more efficient independent of the other parameters (i.e. the negative $\beta_F$ in the regression model). Because increasing the foot radius has a similar effect to increasing the ankle offset, it is detrimental to have both a large foot radius and a large ankle offset. When both foot radius and ankle offset are large, the line quickly passes in front of the knee, which then requires a larger knee torque. (Note that in Fig. 3.22, the stance knee torque is positive for small foot radii and ankle offsets, while it is mostly negative for large foot radii and ankle offsets.) This effect is captured in the positive interaction term between foot radius and ankle offset ($\beta_{RF}$) in the regression model. If ERNIE had mechanical knee stops to prevent hyperextension, the interaction term in the SECT regression equation would likely be smaller.
but still positive because the stance knee motion could be largely passive, while the swing knee would still have to provide adequate foot clearance. The interaction term would likely drop out of the $\tau^2$ regression equation because the stance knee motion could be passive.

As speed increases, the vertical GRF decreases over most of the step, consistent with the decrease in vertical GRF with increasing speed in human gait during the single support phase (Stansfield et al., 2006). This results in a smaller stance knee torque for faster speeds (Fig. 3.22). As a result, the effect of ankle offset becomes less important as speed increases, as indicated by the opposite signs of the ankle offset term ($\beta_F$) and the interaction term for speed and ankle offset ($\beta_{vF}$).

In summary, the most energetically efficient walking for the ERNIE robot under HZD-based control with gaits designed using a work-based objective function can be achieved by using a foot with a large radius (equal to shank length) and a relatively long arc length attached to the shank with no ankle offset. When large feet are not desirable, similar performance can be achieved by using a moderate radius foot and shifting the ankle offset forward of the shank.

3.5 Summary

For the most part, foot radius and ankle offset have very little effect on the timing of the joint kinematics. The range of motion and the maximum knee flexion are notable exceptions that do depend on foot radius and in some cases, ankle offset, with larger foot radii and ankle offsets leading to larger ranges of motion. For the planar biped ERNIE, walking under HZD-based control with gaits designed using the robustness constraints and a work-based objective function, increasing foot radius and increasing ankle offset have much the same effect on joint kinematics.

Regardless of the morphology of the biped, increasing foot radius tends to increase energy efficiency. The effect of ankle offset on efficiency depends on the morphology
Figure 3.22. Left column contains the mean knee joint torque over all steps in hardware; right column contains the knee joint torque in simulation. From 0 to 1, the stance torques are shown, and from 1 to 2, the swing torques are shown. The top row shows the slowest speed, the middle row shows the moderate speed, and the bottom row shows the fast speed. Line width indicates foot radius, and line style indicates ankle offset. As foot radius and ankle offset increase, the stance torque shifts from primarily positive to primarily negative. Swing knee torque is very low.
of the biped and the constraints used. For a two-link biped, shifting the foot center of curvature backward increases the efficiency of walking at a particular speed. For a five-link biped walking with gaits designed without robustness constraints, changes in the ankle offset did not significantly affect the efficiency of the gait. For a five-link biped walking with gaits designed with robustness constraints, increasing ankle offset can either increase or decrease the efficiency due to interaction effects with the foot radius. For the range of foot parameters studied on the planar biped ERNIE walking under HZD-based control with gaits designed using a work-based objective function and the robustness constraints, there are two foot design strategies that result in energetically efficient gaits – a small foot radius/large ankle offset strategy and a large foot radius/no ankle offset strategy. For ERNIE, the large foot radius/no ankle offset strategy produces the most efficient gaits.
This chapter discusses how the model developed in Chapter 2 can be further extended to model human gait. It also presents a validated objective function that can be used with one version of the model to predict normal human walking at speeds ranging from very slow to very fast. Section 4.1 presents the three models considered. Section 4.2 discusses how the existing experimental data used for validation was processed. Sections 4.3 and 4.4 discuss how gaits were found. Section 4.5 presents the metrics used to compare gaits. Section 4.6 reports how well each of the considered models can match human walking, and Section 4.7 reports how well the final model can predict human walking.

4.1 Proposed Human Models

4.1.1 Overview of Models

Three distinct models were considered for use in predicting human gait. All three are left-right symmetric planar models with instantaneous transfer of support at impact. The base model is a four-link model without any form of toe-off (Fig. 4.1) that consists of revolute hip and knee joints connecting two thighs and two shanks, both with mass and inertia, and a point mass at the hip representing the mass of the head, arms and trunk. Constant radius circular feet are rigidly attached to the shanks with their centers of curvature located anterior to the shanks (Hansen et al., 2004a). The upper body is modeled as a point mass because the available experimental data
used for validation mostly did not provide torso kinematics. The impulsive model is identical to the base model except that at impact, an impulsive force is applied to the hip to capture the effects of toe-off. The integral of this instantaneous impulsive force should be approximately equal to the integral of the finite-time toe-off force applied by the trailing leg. The ankles model is identical to the base model except the feet, which have mass but no inertia, are pinned to the shanks with revolute joints (Fig. 4.2). The primary purpose of including ankle joints is to allow positive work at the stance ankle that represents the effects of both ankle plantar flexion and toe extension in humans. In addition to capturing toe extension, the model ankle differs from the human ankle in that the circular foot shape itself achieves some of the human ankle’s functionality by capturing much of the center of pressure (CoP) movement relative to the shank. As a result, the model ankle motion is expected to differ somewhat from the physiological ankle motion.

To model slope walking, the ground can be rotated by a counter-clockwise (CCW) angle of $\gamma$ from horizontal, although it was kept level for this work. The coordinate
Figure 4.2. Schematic of the six-link biped used in the ankles model. The joint angles used in the model are indicated with the $q_i$'s. The unactuated angle is $q_1$; the actuated angles are $q_2$ through $q_6$. The plots, however, transform the model angles into a more typical biomechanics convention as indicated with the $\theta_i$'s.

The system is aligned so that $\hat{x}$ is parallel to the direction of travel, $\hat{z}$ points out of the page, and $\hat{y}$ is mutually perpendicular to $\hat{x}$ and $\hat{z}$. As a result, there is an angle of $\gamma$ between $-\hat{y}$ and the gravity vector. All joint angles are measured CCW positive. The absolute angle is $q_1$ and is defined as the angle between $-\hat{y}$ and the stance thigh. The leg splay is measured from the stance thigh to the swing thigh and is denoted by $q_2$. The angle of the stance knee is $q_3$, and the angle of the swing knee is $q_4$. For the ankles model, the angle of the stance ankle is $q_5$, and the angle of the swing ankle is $q_6$. For consistency with the biomechanics convention, the angles for the reported results are transformed (Fig. 4.2). The hip angle is measured CCW positive from $-\hat{y}$ in both stance and swing. Flexion angles are positive, and extension angles are negative. The knee angles are measured clockwise positive such that flexion is positive and extension is negative. The ankle angles are measured CCW positive from the perpendicular of the shank, so plantarflexion is negative, and dorsiflexion is positive.
Although the models have differing numbers of links, the geometric parameters are the same. Fig. 4.1 shows most of the geometric properties. The length of the thigh is $L_t$, with the center of mass located a distance of $c_t$ from the hip. The length of the shank from the knee to the ankle is $L_s$, and the center of mass is located a distance of $c_s$ from the knee. The radius of the foot is $R$. For the base and impulsive models, the center of curvature is located a distance of $X_f$ in front of the ankle, and for the ankles model, the distance from the ankle joint to the foot center of curvature is $X_f$. The inertia of the foot is neglected because it is an order of magnitude smaller than the other, already small inertias. The center of mass of the foot is located at $(c_{f,x}, c_{f,y})$, where $c_{f,y}$ is the distance from the model ankle to the physiological ankle and $c_{f,x}$ is the distance from the physiological ankle to the physiological foot center of mass. The mass of the body is $m_b$.

The mass and geometric properties were scaled from the given height and mass of the subjects using the anthropomorphic tables in Winter (2009) (Table 4.1). The effective radius of the foot was set at $0.145H$, where $H$ is the height of the subject (Hansen et al., 2004a). The ankle offset was adjusted for speed by fitting lines to the data in Hansen et al. (2004a). Specifically, the ankle offset was scaled for speed using

$$X_f = (0.0093v + 0.0040)H, \quad (4.1)$$

where

$$v = \frac{\nu_{\text{dim}}}{\sqrt{0.530gH}} \quad (4.2)$$

is the nondimensional speed and $\nu_{\text{dim}}$ is the dimensioned speed.

The point of contact between the foot and ground is unactuated, and the unactuated motion is captured in the stance hip absolute angle ($q_1$ in Figs. 4.1 and 4.2). The leg joints are actuated using ideal torque generators, with a single hip actuator controlling the angle between the thighs ($q_2$ in Figs. 4.1 and 4.2). For the base and
TABLE 4.1

ANTHROPOMORPHIC PARAMETERS FOR HUMAN MODELS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Model</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_t$</td>
<td>B, I &amp; A</td>
<td>0.245$H$</td>
</tr>
<tr>
<td>$L_s$</td>
<td>B, I &amp; A</td>
<td>0.246$H$</td>
</tr>
<tr>
<td>$R$</td>
<td>B, I &amp; A</td>
<td>0.145$H$</td>
</tr>
<tr>
<td>$X_f$</td>
<td>B, I &amp; A</td>
<td>Eq. 4.1</td>
</tr>
<tr>
<td>$c_t$</td>
<td>B, I &amp; A</td>
<td>0.106$H$</td>
</tr>
<tr>
<td>$c_s$</td>
<td>B &amp; I</td>
<td>0.173$H$</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>0.107$H$</td>
</tr>
<tr>
<td>$c_{f,x}$</td>
<td>A</td>
<td>0.076$H$</td>
</tr>
<tr>
<td>$c_{f,y}$</td>
<td>A</td>
<td>0.106$H$</td>
</tr>
<tr>
<td>$m_b$</td>
<td>B, I &amp; A</td>
<td>0.678$M$</td>
</tr>
<tr>
<td>$m_t$</td>
<td>B, I &amp; A</td>
<td>0.100$M$</td>
</tr>
<tr>
<td>$m_s$</td>
<td>B &amp; I</td>
<td>0.061$M$</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>0.0465$M$</td>
</tr>
<tr>
<td>$m_f$</td>
<td>A</td>
<td>0.0145$M$</td>
</tr>
<tr>
<td>$I_t$</td>
<td>B, I &amp; A</td>
<td>0.000626$MH^2$</td>
</tr>
<tr>
<td>$I_s$</td>
<td>B, I &amp; A</td>
<td>0.000857$MH^2$</td>
</tr>
</tbody>
</table>

NOTE: $H$ is subject height and $M$ is subject mass. “B” stands for the base model, “I” stands for the impulsive model and “A” stands for the ankles model.
impulsive models, the actuated angles are $q_2$ through $q_4$, and for the ankles model, the actuated angles are $q_2$ through $q_6$.

The models were controlled using HZD-based control (Chapter 2). The phase variable $\theta$ (Eq. 2.59) was chosen as the linearized horizontal position of the hip.

\[
\theta = \begin{bmatrix} -(R + L_t + L_s) & 0 & -(R + L_s) & 0 \end{bmatrix} q \tag{4.3a}
\]

for the base and impulsive models, and

\[
\theta = \begin{bmatrix} -(R + L_t + L_s) & 0 & -(R + L_s) & 0 & -R & 0 \end{bmatrix} q \tag{4.3b}
\]

for the ankles model. The Bézier polynomials that parameterize the output function (Eq. 2.57) were fifth order because this typically results in coefficient of determination ($R^2$) values of 1.0000 at all joints except the ankle when fitting curves to human experimental data without regard to HZD constraints. Polynomial functions do not fit the ankle motion as well, with $R^2$ values of about 0.9997 (stance) or 0.9747 (swing) for fifth order polynomials and 1.0000 (stance) or 0.9986 (swing) for ninth order polynomials. Since increasing the order of the ankle polynomial only slightly improves the fit and since the motion of the model ankle is not expected to match the motion of the physiological ankle particularly well, the improvement when using a higher order polynomial is not worth the additional optimization effort.

4.1.2 Extension of HZD-Based Control for the Impulsive Model

Because the impulsive model has an impulsive force at the hip during impact, the $\Lambda$ matrix used to find the $A$ matrix in the velocity impact map (Eq. 2.2b) changes. The impulsive force at the hip can have two components – one that scales with
pre-impact velocity and one that is constant. This gives

\[ f_h = \beta \dot{q}_e + \beta_0 \]  

(4.4)
as the impulsive force at the hip, where \( \beta \) is a \( 2 \times 6 \) matrix and \( \beta_0 \) is a \( 2 \times 1 \) vector. This yields the following dynamic equation (Eq. 2.5) for the impact map.

\[ D_e \dot{q}_e^+ = (D_e + B_h \beta) \dot{q}_e^- + E^T f + B_h \beta_0, \]  

(4.5)
where \( B_h \) relates the impulsive force at the hip to the extended generalized coordinates. For this work,

\[ \beta = \begin{bmatrix} \beta_v & 0_{2 \times 2} \end{bmatrix}, \]  

(4.6)
\[ B_h = \begin{bmatrix} 0_{4 \times 2} \\ I_{2 \times 2} \end{bmatrix}, \]  

(4.7)
where \( I \) is the identity matrix, \( 0 \) is a matrix of zeros, and \( \beta_v \) is a \( 2 \times 4 \) matrix. The coefficients in \( \beta_v \) were added to the optimization variables and optimized along with the rest of the gait. After a considerable amount of manipulation, it can be shown that if \( \beta_0 \neq 0 \), there does not exist a closed-form solution for the model of the step. Thus, for this study, it was decided to set \( \beta_0 = 0 \). With \( \beta_0 = 0 \), the \( \Lambda \) matrix (Eq. 2.12) becomes

\[ \Lambda = \left( I_{6 \times 6} - D_e^{-1} E^T (E D_e^{-1} E^T)^{-1} E \right) \left( I_{6 \times 1} + D_e^{-1} B_h \beta \right). \]  

(4.8)

4.1.3 Definition of a Step

A human stride is assumed to begin with heel contact (HC). This is followed by a double support phase ending with opposite toe-off (OTO). The single support stance
phase goes from OTO to opposite heel contact (OHC). A second double support phase occurs after OHC until toe-off (TO). After TO, there is a swing phase lasting until the next HC. Because the models assume instantaneous impacts, it is not possible to exactly match the human phases. Clearly, at least part of the model’s stance phase will consist of the human single support stance phase, and at least part of the model’s swing phase will consist of the human swing phase. Determining what to do with the double support phases is less clear.

Fig. 4.3 shows the four options considered. In all cases, the model stance and swing phases are the same length. Step definitions 1-3 use the whole gait cycle, while step definition 4 ignores the first double support phase. For step definition 1, the stance phase goes from HC to OHC, and the swing phase goes from OHC to HC. For step definition 2, the stance phase goes from OTO to TO, and the swing phase goes from TO to OTO. For step definition 3, each of the double support phases is split in

\[
\begin{array}{cccc}
  \text{HC} & \text{IE} & \text{OTO} & \text{OHC} & \text{OIE} & \text{TO} & \text{HC} \\
  \text{Human} & & & & & & \\
  \text{Option 1} & & & & & & \\
  \text{Option 2} & & & & & & \\
  \text{Option 3} & & & & & & \\
  \text{Option 4} & & & & & & \\
\end{array}
\]
half at what will be called the impact event (IE) and opposite impact event (OIE). The stance phase goes from the IE to the OIE, and the swing phase goes from the OIE to the IE. For step definition 4, the stance phase goes from OTO to the OIE, and the swing phase goes from the OIE to HC. This definition of the stance and swing phases provided the best match to human experimental data when performing optimizations using SECT and the ERNIE model. Step definition 3 was ultimately chosen, as discussed in Section 4.3.

4.2 Human Experimental Data

4.2.1 Overview of Data

In order to validate both the models and the prediction function (Section 4.4), the motion of the models was compared to the measured motion of humans across a range of speeds. For this study, published experimental walking data from a variety of sources were used. The data were divided into two groups – a training group that was used to both evaluate the models and perform initial validation of the prediction function, and a validation group that was used for final validation of the prediction function. In addition, a single paper was used for pilot studies.

In most papers, means for a group of subjects were reported. Papers that did not provide subject height and mass were excluded. All included papers provided the average speed, and most provided step length. When step length was not provided, it was estimated from the joint angles. In order to identify the transitions between the phases of a stride, papers had to provide at least the time of toe-off, or equivalently, the duration of the stance phase. Papers that did not were excluded.

Only studies with healthy subjects were included. Child studies were included because it appears that gait is mature by age 7 (Ganley and Powers, 2005; Sutherland, 1997). While the dimensional values of self-selected step length and frequency change
as children age, it appears to be purely a function of skeletal growth (Lythgo et al., 2011; Todd et al., 1989). It also appears that by age three, children use their lower-limbs to generate an effective foot radius similar to adults (Hansen and Meier, 2010). Data from elderly subjects were excluded because their gait is different than that of younger adults (Oberg et al., 1993).

There are several ways that subject speed can be controlled during overground walking. Studies that controlled subject speed by enforcing a specified step cadence or step length were excluded because these can alter gait (Bertram, 2005). Most of the studies gave general verbal directions such as “Walk slowly” or “Walk at a comfortable speed.” Only nondimensional speeds greater than 0.2 were included because the double support phase is too long at slower speeds for the model to accurately capture the gait dynamics.

Two treadmill studies (Stoquart et al., 2008; van Hedel et al., 2006) were originally included but were later removed due to much larger differences with the model than found for the overground studies. There are well documented differences between overground and treadmill walking (Alton et al., 1998; Watt et al., 2010), so the difference in model performance is not surprising. Additionally, the quality of the data from the treadmill studies was generally lower than that from the overground walking studies, which could have influenced the results as well.

All of the studies provided hip, knee and ankle angles in the sagittal plane. A few provided torso kinematics as well, although these data were not used. In all cases, it was assumed that the gait was left-right symmetric. Because human joints have a fairly complex motion that is approximated using a spherical or pivot joint, the experimental protocol influences the joint angles. Further complicating matters is the fact that the clinically relevant terms of joint flexion/extension, internal/external rotation, and ab/adduction do not refer to mutually perpendicular axes. In all common current protocols, one of the joint axes is fixed to the proximal bone, one of the
joint axes is fixed to the distal bone and the third axis is the line that is mutually perpendicular to the other axes (Grood and Suntay, 1983). This more closely matches the clinical terms than a set of three mutually perpendicular axes would. Different protocols define the axes using slightly different marker locations. Different protocols also use slightly different methods to locate the center of the hip joint. As a result, the absolute magnitudes of the angles vary somewhat between protocols (Ferrari et al., 2008). For most flexion/extension angles, scaling and shifting the joint angles from one protocol to another would allow an almost perfect match (Ferrari et al., 2008). Unfortunately, there are no papers that attempt this. Since the differences are generally small, they were ignored for this work.

Most studies provided the joint kinematics in the form of plots, although a few provided electronic tables of data. The quality of the plots had significant variation. The overall quality of each paper’s data, which was primarily influenced by the quality of the plots but also considered the reported experimental protocol and how much information had to be estimated, was given a subjective rating of 1 to 4, with higher numbers indicating better data.

Tables 4.2, 4.3, and 4.4 give details on the papers, subjects and speeds used. The papers were split so that the overall quality of the training and validation groups were comparable, each experimental protocol was distributed between groups, each group had exactly one child study, the subject footwear between the groups were comparable, the ratios of male to female subjects were similar, and the training group had slightly more papers and speeds. Two of the papers had two sets of data; each group received one set. In general, having the two groups be as similar as possible is desirable because it equally distributes the effect of uncontrolled variables. However, it is also desirable to have more data in the training group than in the validation group because the training group was used to determine several key features of the model, such as the number of links and the step definition. In the final set of gaits
considered, the training group had 5 unique sets of data for a total of 16 unique gaits, and the validation group had 3 unique sets of data for a total of 10 unique gaits.

4.2.2 Extracting Data from Graphs

As mentioned above, most of the data available are in the form of black-and-white graphs. In order to use the data, it must be digitized, and then the digitized joint angle data must be converted into the model coordinate system as given in Figs. 4.1 and 4.2. The general method is as follows:

1. Digitize the data by collecting a large number of points from each line on each plot. To improve the accuracy of the results, the x-axis ranged from 0 to 100% of the gait cycle. Because the plot axes were not always parallel to the edges of the paper, particularly for older studies, trigonometry had to be used to convert an x-y point on the screen to the plot coordinate system. Since angles were provided in degrees, having the x-axis range from 0 to 100 generally resulted x and y values of approximately equal magnitude, which in turn prevented either value from having an unduly large effect on the other value.

2. Determine when each of the step events occur. Depending on the paper, they are either given in a table or included in a plot. The step events are heel contact (HC), opposite toe-off (OTO), opposite heel contact (OHC), and toe-off (TO). Also determine when the impact event (IE) and the opposite impact event (OIE) occur. These are defined as the midpoint between HC and OTO, and the midpoint between OHC and TO, respectively.

3. Split the digitized data into stance and swing components.

4. Because it was highly unlikely that the number of points collected for each line were the same and that the collected points were evenly spaced, the collected
# TABLE 4.2

## PAPERS FOR HUMAN SUBJECT DATA

<table>
<thead>
<tr>
<th>Reference</th>
<th>Use</th>
<th>Directions</th>
<th>Protocol</th>
<th>Footwear</th>
<th>Subjects</th>
<th>Speeds</th>
<th>Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwartz et al. (2008)</td>
<td>P</td>
<td>General verbal</td>
<td>PiG</td>
<td>NS</td>
<td>83</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>T</td>
<td>General verbal</td>
<td>ELITE</td>
<td>Barefoot</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Borghese et al. (1996)</td>
<td>T</td>
<td>General verbal</td>
<td>ELITE</td>
<td>Barefoot</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>T</td>
<td>General verbal</td>
<td>LAMB</td>
<td>NS</td>
<td>20</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Han and Wang (2011)</td>
<td>T</td>
<td>Precise speed, no details</td>
<td>Han</td>
<td>Sneakers</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Murray et al. (1984)</td>
<td>T</td>
<td>General verbal</td>
<td>NS</td>
<td>Sneakers</td>
<td>7</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>V</td>
<td>General verbal</td>
<td>ELITE</td>
<td>Barefoot</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>V</td>
<td>General verbal</td>
<td>LAMB</td>
<td>NS</td>
<td>20</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Liu et al. (2008)</td>
<td>V</td>
<td>General verbal</td>
<td>PiG</td>
<td>NS</td>
<td>8</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

NOTE: “P” stands for pilot, “T” stands for training, “V” stands for validation, “NS” stands for not specified, and “EG” stands for electrogoniometers. Quality is a numerical ranking of 1-4, with higher numbers indicating better data. The subjects column gives the number of subjects in the study, and the speeds column gives the number of speeds in the study.
## TABLE 4.3
### SUBJECT INFORMATION

<table>
<thead>
<tr>
<th>Ref.</th>
<th>Age (Years)</th>
<th>Sex</th>
<th>Height (m)</th>
<th>Mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwartz et al. (2008)</td>
<td>10.5 ± 3.5</td>
<td>48M, 35F</td>
<td>1.56 ± 0.19</td>
<td>34.1 ± 9.6</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>30</td>
<td>1M</td>
<td>1.81</td>
<td>73</td>
</tr>
<tr>
<td>Borghese et al. (1996)</td>
<td>Adult</td>
<td>1M</td>
<td>1.72</td>
<td>67</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>10.8 ± 3.2</td>
<td>9M, 11F</td>
<td>1.47 ± 0.20</td>
<td>41.4 ± 15.5</td>
</tr>
<tr>
<td>Han and Wang (2011)</td>
<td>25 ± 2</td>
<td>NS</td>
<td>1.7 ± 0.03</td>
<td>56 ± 6.8</td>
</tr>
<tr>
<td>Murray et al. (1984)</td>
<td>30 (20-36)</td>
<td>7F</td>
<td>1.65 (1.61-1.70)</td>
<td>57 (45-64)</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>27</td>
<td>1F</td>
<td>1.68</td>
<td>52</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>42.1 ± 15.4</td>
<td>9M, 11F</td>
<td>1.71 ± 0.10</td>
<td>68.5 ± 15.8</td>
</tr>
<tr>
<td>Liu et al. (2008)</td>
<td>12.9 ± 3.3</td>
<td>2M, 6F</td>
<td>1.53 ± 0.17</td>
<td>51.8 ± 19.2</td>
</tr>
</tbody>
</table>

NOTE: “NS” stands for not specified, “M” stands for male and “F” stands for female.
<table>
<thead>
<tr>
<th>Ref.</th>
<th>Very Slow</th>
<th>Slow</th>
<th>Normal</th>
<th>Fast</th>
<th>Very Fast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwartz et al. (2008)</td>
<td>0.488, 0.171</td>
<td>0.823, 0.289</td>
<td>1.221, 0.429</td>
<td>1.589, 0.558</td>
<td>1.931, 0.678</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>–</td>
<td>0.98, 0.320</td>
<td>–</td>
<td>–</td>
<td>2.00, 0.652</td>
</tr>
<tr>
<td>Borghese et al. (1996)</td>
<td>–</td>
<td>1.279, 0.428</td>
<td>1.602, 0.535</td>
<td>2.102, 0.703</td>
<td>–</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>0.79, 0.286</td>
<td>1.02, 0.369</td>
<td>1.29, 0.467</td>
<td>1.32, 0.478</td>
<td>1.73, 0.626</td>
</tr>
<tr>
<td>Han and Wang (2011)</td>
<td>–</td>
<td>0.8, 0.269</td>
<td>1.3, 0.437</td>
<td>1.7, 0.572</td>
<td>–</td>
</tr>
<tr>
<td>Murray et al. (1984)</td>
<td>–</td>
<td>0.782, 0.284</td>
<td>1.417, 0.485</td>
<td>–</td>
<td>1.917, 0.656</td>
</tr>
<tr>
<td>Bianchi et al. (1998)</td>
<td>–</td>
<td>0.93, 0.315</td>
<td>–</td>
<td>–</td>
<td>2.06, 0.697</td>
</tr>
<tr>
<td>Bovi et al. (2011)</td>
<td>0.84, 0.282</td>
<td>1.19, 0.399</td>
<td>1.22, 0.409</td>
<td>1.49, 0.500</td>
<td>1.97, 0.661</td>
</tr>
<tr>
<td>Liu et al. (2008)</td>
<td>0.54, 0.192*</td>
<td>0.75, 0.266</td>
<td>1.15, 0.408</td>
<td>1.56, 0.553</td>
<td>–</td>
</tr>
</tbody>
</table>

NOTE: The first value is the speed in m/s, and the second is the nondimensional speed.

* Not used in final study.
points were interpolated so that each line from each plot had one hundred equally spaced points. This is needed for step 5.

5. Transform the data from the experimental coordinate system to the model coordinate system.

Two methods of digitizing the data were considered – a custom MATLAB program and a free program called Engauge Digitizer (Mitchell, 2002). The MATLAB program requires the user to manually click on each point. Engauge Digitizer can automatically place the points if the line is reasonably solid. In addition, it allows the user to manually add or remove points. Because Engauge Digitizer is partially automatic, digitizing plots was significantly faster than with the MATLAB code. Regardless of the method used, once the plot was digitized, the data points were evenly spaced using the piecewise cubic Hermite interpolation that is built into MATLAB. This method was chosen because it allows extrapolation as well as interpolation, which is needed because the range of the digitized data was typically less than 0 to 100 due to the difficulty of placing a digitization point at exactly 0 and exactly 100.
TABLE 4.5

COMPARISON OF DIGITIZATION RESULTS USING MATLAB AND ENGAUGE DIGITIZER

| Line | Method | |||max|mean|||max|mean|
|------|--------|--------|--------|--------|--------|--------|
| 1 | MATLAB | 0.979 | 2.056 | 0.891 |
| 1 | Engauge | 2.158 | 3.402 | 2.088 |
| 2 | MATLAB | 1.413 | 1.986 | 1.223 |
| 2 | Engauge | 1.414 | 2.822 | 1.319 |
| 3 | MATLAB | 1.429 | 2.964 | 1.263 |
| 3 | Engauge | 1.566 | 2.528 | 1.450 |

To ensure that the chosen method of digitization provides acceptably accurate results, a sample plot with quality comparable to the lower quality graphs was generated (Fig. 4.4) and then digitized using both methods. The x-axis corresponds to normalized time, and the y-axis corresponds to a joint angle in degrees. The lines were chosen to roughly approximate the shape of the human experimental data. Further, the lines were chosen such that there were several difficult-to-digitize regions. Namely, there were several points where the lines crossed and a region where two of the lines were almost on top of each other. The chosen lines are given by

\[
y_1 = 19 \sin(12\pi x/500) \cos(\pi x/150 - \pi/6) - 2
\]

(4.9)

\[
y_2 = 30 \sin(11\pi x/500) \cos(\pi x/150 + \pi/18) - 7
\]

(4.10)

\[
y_3 = 25 \sin(11\pi x/500) \cos(\pi x/150) - 4,
\]

(4.11)

where \(0 \leq x \leq 100\).
Figure 4.5. Comparison of the digitized results to the original lines for (a) $y_1$, (b) $y_2$, and (c) $y_3$. 
From a visual inspection (Fig. 4.5), both methods appear to work well. The maximum error for digitization using MATLAB is 0.547, while the maximum error for digitization using Engauge Digitizer is 0.576 using a pointwise comparison over 100 equally spaced points. This indicates that the expected error for digitizing the actual joint angle plots should be under one degree. This is an acceptable level of accuracy because the standard deviation of the measured joint angles is approximately 5° (Bovi et al., 2011; Schwartz et al., 2008; Winter, 1984). To better quantify the error between the digitized data and the original data, three metrics were used.

\[
\|e\| = \frac{\|y_d - y_o\|}{\|y_o\|} \times 100 \quad (4.12)
\]

\[
\max |e| = \frac{\max (y_d - y_o)}{\max |y_o|} \times 100 \quad (4.13)
\]

\[
\text{mean } |e| = \frac{\sum_{i=0}^{100} |y_d(x_i) - y_o(x_i)|}{\sum_{i=0}^{100} |y_o(x_i)|} \times 100, \quad (4.14)
\]

where \(y_d\) is the digitized data and \(y_o\) is the original data. In almost all cases, digitization using MATLAB performed better (Table 4.5). However, digitization using Engauge Digitizer performed acceptably well, and because it was significantly faster, it was used.

4.3 Choosing the Matching Function

To determine which of the three proposed models were appropriate to model human gait, gaits that mimicked the training gaits (from Section 4.2) were found. In order to do this, model gaits, specifically the Bézier polynomials that define the output function, must be determined. The most straightforward way would be to simply fit a polynomial to the experimental data and then find the pre-impact state as described in Section 2.4. Unfortunately, this does not typically result in a physically
possible, periodic gait due to the differences between the model and the human. Instead, the gaits must be found using an optimization as described in Section 2.5.

4.3.1 Proposed Objective Functions

Clearly, the choice of objective function has a strong influence on how well different gait features are matched since it is impossible to perfectly replicate the human gait. Ideally, the model gait should match the experimental step length and average speed, as well as the joint angles. The overall objective function for the optimization was a weighted linear combination of terms for matching step length, speed, and joint angles.

\[
f = c_{SL} \cdot f_{SL} + c_v \cdot f_v + c_c \cdot f_c + \sum_{j=1}^{N} c_{p,j} \cdot f_{p,j} + \sum_{j=1}^{N} c_{PC,j} \cdot f_{PC,j} + c_{\Delta} \cdot f_{\Delta} + \sum_{j=1}^{N} c_{AA,j} \cdot f_{AA,j},
\]

(4.15)

where \( c_i \) is the weighting factor and \( f_i \) is the function.

The objective function for step length is

\[
f_{SL} = (S_{sim} - S_{exp})^2,
\]

(4.16)

where \( S_{sim} \) is the simulated dimensioned step length and \( S_{exp} \) is the experimental dimensioned step length. The objective function for speed is

\[
f_v = (v_{sim} - v_{exp})^2,
\]

(4.17)

where \( v_{sim} \) is the simulated dimensioned average speed and \( v_{exp} \) is the experimental dimensioned average speed.

To match the joint angles, three options for the objective function were considered – a point-to-point error calculation, a comparison of the least-squares experimental polynomial coefficients to the model controller coefficients and Pearson’s correlation
The point-to-point objective function value can be calculated as follows:

1. Define $0 \leq t_n(i) \leq 1$ as a vector of 100 equally spaced points. $t_n$ is time normalized by step duration and $i = \{1, 2, \ldots 100\}$.

2. Interpolate the simulation data to find $q_{j,sim}(i)$, which is the joint angle of joint $j$ at time $t_n(i)$ in degrees. The experimental data $q_{j,exp}(i)$ has already been interpolated as part of the digitization process.

3. For each time point, calculate an error value as

$$e_j(i) = 0.0006889(q_{j,sim}(i) - q_{j,exp}(i))^4 + 0.02278(q_{j,sim}(i) - q_{j,exp}(i))^2.$$  \hspace{1cm} (4.18)

This choice of polynomial gives a value of 1 when the error is $5^\circ$ (± one standard deviation of the experimental data) and a value of 40 when the error is $15^\circ$ (± three standard deviations of the experimental data).

4. Sum the error values over each time point to obtain the value of the objective function for each joint,

$$f_{p,j} = \frac{\sum_{i=1}^{100} e_j(i)}{100}.$$  \hspace{1cm} (4.19)

Calculating the objective function using polynomial coefficients is simpler. To do so,

1. Find the least squares polynomial of degree 5 for each experimental joint angle as a function of $s$, where $s$ is calculated as for HZD-based control (Eq. 4.3).

This gives $q_{j,exp} \approx \sum_{k=0}^{5} p_{j,k} s^k$.

2. Convert the polynomial coefficients into coefficients for a Bézier polynomial
using

\[ \alpha_{j,0} = p_{j,5} \]  
\[ \alpha_{j,k} = p_{j,5-k} \left( \frac{(5-k)!}{5!} \right) j! - j! \sum_{v=0}^{k-1} \frac{(-1)^{k+v} \alpha_{j,v}}{k!(k-v)!}. \]  

(4.20)  
(4.21)

Store the coefficients for the swing hip \((q_2)\) and both knees \((q_3\) and \(q_4)\) in a 3 × 6 matrix called \(\alpha_{\text{exp}}\). Store the coefficients for each ankle in its own matrix (also called \(\alpha_{\text{exp}}\)). The coefficients that define the model’s output function are stored in a matrix called \(\alpha_{\text{sim}}\).

3. Calculate the value of the objective function as

\[ f_{c,5} = \| \alpha_{\text{sim}} - \alpha_{\text{exp}} \|_2. \]  

(4.22)

Because there is no controller for the unactuated angle, it was chosen to use a more direct method to compare the model and experimental stance hip angles, so \(f_c\) is not defined for \(q_1\). In addition, the definition of \(f_c\) was modified for the stance ankle so that only the middle two control parameters \((\alpha_2\) and \(\alpha_3)\) were utilized because they specify the ankle motion in the middle of the step where the model and experimental stance ankle motions should be similar. Thus, Eq. 4.22 becomes

\[ f_c = \| \alpha_{\text{sim},2\&3} - \alpha_{\text{exp},2\&3} \|_2 \]  

(4.23)

for the stance ankle \((q_5)\).

The third objective function used to match the joint kinematics is

\[ f_{PC,j} = (1 - \rho(q_{\text{exp},j}, q_{\text{sim},j})), \]  

(4.24)
where $-1 \leq \rho(\cdot) \leq 1$ is the Pearson linear correlation coefficient.

In human experiments in which subjects wore rocker-bottomed shoes, the range of motion of the stance ankle was approximately $5.7^\circ$ (Wang and Hansen, 2010). In the objective function, this behavior is captured by

$$f_{\Delta} = (q_{5,\text{sim}}(t_{\text{OHC}}) - q_{5,\text{sim}}(t_{\text{OTO}}) - 5.7)^2,$$

where $t_{\text{OTO}}$ and $t_{\text{OHC}}$ are the times of OHC and OTO.

If Pearson’s correlation coefficient is used to match joint kinematics at more than one joint, it is possible that the angles could become unreasonable because it does not penalize shifts between $q_{\text{exp}}$ and $q_{\text{sim}}$. To avoid this issue, the difference between the simulated and experimental angle at OHC was used.

$$f_{AA,j} = (q_{j,\text{sim}}(t_{\text{OHC}}) - q_{j,\text{exp}}(t_{\text{OHC}}))^2.$$  

To determine which objective function and step definition (Section 4.1.3) to use, a pilot study was conducted using child gait data from Schwartz et al. (2008). Because it was possible that different objective functions would be needed for different models, the pilot study was conducted first for the base and impulsive models and then for the ankles model.

4.3.2 Base and Impulsive Models Pilot Study

Using the data for the natural speed, four sets of experimental data were created, one for each of the step definitions. A subject-specific model was developed using the reported experimental height and mass. Each set of data was optimized using objective functions 1 through 4 as given in Table 4.6. The weighting factors ensure terms of approximately equal magnitude when each represents a marginal fit. For
TABLE 4.6

WEIGHTING FACTORS FOR EQ. 4.15 FOR THE BASE AND IMPULSE MODELS PILOT STUDY

<table>
<thead>
<tr>
<th>Name</th>
<th>c_{SL}</th>
<th>c_v</th>
<th>c_{p,1}</th>
<th>c_{p,2...4}</th>
<th>c_{c,1}</th>
<th>c_{c,2...4}</th>
<th>c_{PC,1}</th>
<th>c_{PC,2...4}</th>
<th>c_{\Delta}</th>
<th>c_{AA}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>10^5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>10^4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>10^4</td>
<td>0.1</td>
<td>0</td>
<td>1</td>
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<td>0</td>
<td>0</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

NOTE: All coefficients not listed are zero for all objective functions.

step length and speed, the marginal fit is about one experimental standard deviation. For the angle measures, the marginal fit was found by qualitatively examining plots of joint angles with known errors added and calculating the unscaled value of the appropriate term.

A few additional objective functions were tried, but they did not perform well in initial tests. Objective function 3 originally did not have the point-to-point error term for the unactuated angle, but the optimization got stuck in unfeasible regions very frequently. Adding the point-to-point error term reduced this problem. An objective function consisting of just the step length and speed term was tried, and although the gait matched the experimental speed and step length very well, it did not match the joint kinematics well. This is not surprising because there are many gaits for a particular speed and step length, so the objective function needs to have a term that penalizes non-human-like joint motions. Objective function 5 was added later in the study to replace objective function 4. Objective function 4 uses point-to-point error matching for the absolute angle, which means that the value is highly sensitive to
shifts in the experimental angle. Since the magnitude of the experimental absolute angle depends more strongly on the experimental protocol than the waveform does (Ferrari et al., 2008), it is preferable to match the waveform of the angle rather than the magnitude. This can be accomplished by using Pearson’s correlation coefficient. A potential issue with using Pearson’s coefficient is that it does not penalize differences in magnitudes at all, which means the angle could be shifted to an unreasonable value without any penalty. However, because objective function 5 tries to match both the absolute values of the relative angles \((q_2 - q_4)\) and step length, the absolute angle should remain reasonable.

Using MATLAB’s \texttt{fmincon} function with the “active-set” algorithm, repeated optimizations were performed with the result of the current optimization serving as the seed for the next optimization. All strings of optimizations for the base model started from the same initial guess, and all strings of optimizations for the impulsive started from the same initial guess. The initial guesses were similar, but the impulsive model initial guess had a small amount of toe-off included because preliminary studies showed that the no toe-off gait is a local minimum. For the first several optimizations in a string, fairly tight (10% in most cases) bounds on the amount the variables could change (i.e. the linear constraints) were used to prevent the optimization from taking too large of a step and getting stuck in an unfeasible region. Once the current solution was not against any of the linear bounds, the magnitude of the bound was increased, usually to 100%. Additional optimizations were performed until a gait was found in which additional optimizations did not improve the value of the objective function. As needed, the bound on the linear constraints was tightened or relaxed. At this point, the string of optimizations was stopped, and the final gait was taken to be the optimal gait. Using this final gait, the value of each of the five proposed objective functions was calculated.

The results of the optimization are shown in Table 4.7 for the base model and
in Table 4.8 for the impulsive model. A representative set of gaits for the impulsive model is shown in Fig. 4.6. (Plots of the remaining gaits for the impulsive model are given in Appendix C.1.1.) To determine the best definition for the step phases and which objective function to use, four series of comparisons were made between gaits with differing step definitions and objective functions. Details are available in Appendix C.1.2.

As a first comparison, it was checked that when the step definition was held constant and the objective function was varied, for a particular objective function, the optimization using that objective function had the lowest value. Unfortunately, for objective functions 2 and 3 this was not the case. For example, for the impulsive model with step definition 1, a gait optimized using objective function 3 yields a higher value of objective function 3 than performing the optimization with objective function 4. As a result, objective functions 2 and 3 consistently had higher values for all of the objective functions than objective functions 1 and 4. Thus, it was decided that objective function 1 or 4 would be used in the full study. Considering just objective functions 1 and 4, the three best choices are objective function 1 with step definition 3, objective function 4 with step definition 2 and objective function 4 with step definition 3. These choices perform significantly better than the alternatives, but almost the same with respect to each other.

To help determine which option to use, optimizations were performed using just the top three options to match the slow speed gait from Schwartz et al. (2008) (Table 4.10). Matching the gross motion of the gait (i.e. speed and step length) is more important than matching the joint kinematics because those are the more clinically relevant parameters. The performance of the impulsive model is more important than the performance of the base model because the impulsive model performs better in general. Thus, it was initially decided to use objective function 4 with step definition...
## Table 4.7

Comparison of Different Objective Functions and Step Definitions for the Base Model

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Step Length (m)</th>
<th>Speed (m/s)</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seed 1</td>
<td>0.392</td>
<td>1.219</td>
<td>40.375</td>
<td>2.229</td>
<td>2.431</td>
<td>561.426</td>
<td></td>
</tr>
<tr>
<td>Seed 2</td>
<td>0.392</td>
<td>1.219</td>
<td>89.805</td>
<td>1.118</td>
<td>1.155</td>
<td>558.818</td>
<td></td>
</tr>
<tr>
<td>Seed 3</td>
<td>0.392</td>
<td>1.219</td>
<td>13.990</td>
<td>0.634</td>
<td>0.649</td>
<td>558.105</td>
<td></td>
</tr>
<tr>
<td>Seed 4</td>
<td>0.392</td>
<td>1.219</td>
<td>13.788</td>
<td>1.357</td>
<td>1.373</td>
<td>558.700</td>
<td></td>
</tr>
</tbody>
</table>

| 1     | 1 | 0.383 | 1.222 | 14.779 | 1.122 | 1.256 | 606.434 |
| 1     | 2 | 0.406 | 1.222 | 9.383  | 1.115 | 1.207 | 497.908 |
| 1     | 3 | 0.404 | 1.222 | 5.864  | 0.767 | 0.779 | 502.990 |
| 1     | 4 | 0.440 | 1.222 | 5.377  | 1.327 | 1.339 | 357.228 |

| 2     | 1 | 0.290 | 1.222 | 60.966 | 1.459 | 1.807 | 1147.603 |
| 2     | 2 | 0.342 | 1.222 | 41.012 | 0.575 | 0.693 | 821.189 |
| 2     | 3 | 0.389 | 1.222 | 12.687 | 0.551 | 0.567 | 572.762 |
| 2     | 4 | 0.353 | 1.223 | 11.023 | 0.540 | 0.571 | 757.231 |

| 3     | 1 | 0.432 | 1.221 | 45.392 | 1.362 | 1.397 | 386.104 |
| 3     | 2 | 0.376 | 1.222 | 45.927 | 0.825 | 0.896 | 640.656 |
| 3     | 3 | 0.399 | 1.222 | 8.267  | 0.509 | 0.517 | 526.963 |
| 3     | 4 | 0.366 | 1.222 | 10.348 | 0.805 | 0.832 | 690.850 |

| 4     | 1 | 0.566 | 1.202 | 771.304 | 7.055 | 7.367 | 49.629  |
| 4     | 2 | 0.621 | 1.221 | 426.114 | 2.733 | 3.848 | 14.453  |
| 4     | 3 | 0.549 | 1.206 | 494.323 | 5.601 | 5.629 | 68.730  |
| 4     | 4 | 0.579 | 1.206 | 552.430 | 5.577 | 5.616 | 30.163  |

NOTE: The human experimental gait had a step length of 0.628 m and a speed of 1.222 m/s. The Obj. Fun. column indicates which objective function was used in the optimization, while columns $f_1$ to $f_4$ give the value of each objective function. The Set column indicates which step definition was used.
Figure 4.6. Comparison of the impulsive model gaits using step definition 3. Note that the plots use the model angles and not the transformed biomechanics angles. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using $f_1$, the green line shows the gait found using $f_2$, the black line shows the gait found using $f_3$, the pink line shows the gait found using $f_4$, and the yellow line shows the gait found using $f_5$. 
TABLE 4.8
COMPARISON OF DIFFERENT OBJECTIVE FUNCTIONS AND
STEP DEFINITIONS FOR THE IMPULSIVE MODEL

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Step Length (m)</th>
<th>Speed (m/s)</th>
<th>f₁</th>
<th>f₂</th>
<th>f₃</th>
<th>f₄</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seed 1</td>
<td>1</td>
<td>0.392</td>
<td>1.219</td>
<td>40.369</td>
<td>2.229</td>
<td>2.430</td>
<td>561.443</td>
<td>559.726</td>
</tr>
<tr>
<td>Seed 2</td>
<td>2</td>
<td>0.392</td>
<td>1.219</td>
<td>89.809</td>
<td>1.117</td>
<td>1.155</td>
<td>558.836</td>
<td>558.494</td>
</tr>
<tr>
<td>Seed 3</td>
<td>3</td>
<td>0.392</td>
<td>1.219</td>
<td>13.988</td>
<td>0.634</td>
<td>0.649</td>
<td>558.123</td>
<td>558.056</td>
</tr>
<tr>
<td>Seed 4</td>
<td>4</td>
<td>0.392</td>
<td>1.219</td>
<td>13.784</td>
<td>0.634</td>
<td>0.649</td>
<td>558.718</td>
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<td>1.222</td>
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<td>1.952</td>
<td>1.973</td>
<td>218.089</td>
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<td>0.959</td>
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<th>Obj. Fun.</th>
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<th>Speed</th>
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<th>$f_2$</th>
<th>$f_3$</th>
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<th>$f_5$</th>
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<td>1.222</td>
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<td>61.588</td>
<td>0.842</td>
<td>0.930</td>
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<td>1.220</td>
<td>0.262</td>
<td>0.264</td>
<td>0.276</td>
<td>0.267</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.627</td>
<td>1.222</td>
<td>16.398</td>
<td>1.196</td>
<td>1.282</td>
<td>2.072</td>
<td>1.277</td>
</tr>
</tbody>
</table>

Continued on next page
### TABLE 4.8 – continued from previous page

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Step Length</th>
<th>Speed</th>
<th>f₁</th>
<th>f₂</th>
<th>f₃</th>
<th>f₄</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0.629</td>
<td>1.223</td>
<td>19.262</td>
<td>0.560</td>
<td>0.574</td>
<td>0.695</td>
<td>0.576</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.628</td>
<td>1.222</td>
<td>2.911</td>
<td>0.177</td>
<td>0.308</td>
<td>1.486</td>
<td>0.195</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.628</td>
<td>1.222</td>
<td>0.677</td>
<td>0.239</td>
<td>0.254</td>
<td>0.384</td>
<td>0.243</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.628</td>
<td>1.222</td>
<td>2.487</td>
<td>0.339</td>
<td>0.422</td>
<td>1.162</td>
<td>0.347</td>
</tr>
</tbody>
</table>

NOTE: The human experimental gait had a step length of 0.628 m and a speed of 1.222 m/s. The Obj. Fun. column indicates which objective function was used in the optimization, while columns $f_1$ to $f_5$ give the value of each objective function. The Set column indicates which step definition was used.
TABLE 4.9
COMPARISON OF THE MOST PROMISING OBJECTIVE FUNCTIONS AND STEP DEFINITIONS FOR THE IMPULSIVE MODEL

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Exp. Speed (m/s)</th>
<th>Exp. Step Length (m)</th>
<th>Sim. Speed (m/s)</th>
<th>Sim. Step Length (m)</th>
<th>f₁</th>
<th>f₄</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.513</td>
<td>0.321</td>
<td>124.153</td>
<td>124.043</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.623</td>
<td>315.048</td>
<td>4.133</td>
<td>1.726</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>2.454</td>
<td>0.463</td>
<td>0.387</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>244.585</td>
<td>10.663</td>
<td>1.385</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>12.121</td>
<td>1.888</td>
<td>0.670</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.574</td>
<td>0.500</td>
<td>30.45</td>
<td>30.352</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.627</td>
<td>44.91</td>
<td>2.277</td>
<td>0.933</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>1.220</td>
<td>0.276</td>
<td>0.267</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>2.911</td>
<td>1.486</td>
<td>0.195</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>0.677</td>
<td>0.384</td>
<td>0.243</td>
</tr>
</tbody>
</table>

Continued on next page
<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Exp. Speed (m/s)</th>
<th>Exp. Step Length (m)</th>
<th>Sim. Speed (m/s)</th>
<th>Sim. Step Length (m)</th>
<th>f₁</th>
<th>f₄</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>1.589</td>
<td>0.698</td>
<td>1.589</td>
<td>0.698</td>
<td>17.663</td>
<td>–</td>
<td>0.856</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1.589</td>
<td>0.698</td>
<td>1.589</td>
<td>0.698</td>
<td>0.794</td>
<td>–</td>
<td>0.145</td>
</tr>
</tbody>
</table>
3 because objective function 4 does a much better job of matching step length than objective function 1 and step definition 3 performs better than step definition 2 at both speeds for the impulsive model. Further, the combination of objective function 4 with step definition 3 appears to be the most robust optimization in that it is the least likely to get stuck in an unfeasible region and has fewer problems with local minima than the other options.

After it was determined that multiple experimental protocols would have to be used, objective function 5 was added. The procedure described in the previous paragraphs was repeated. Because it had already been determined that including toe-off by using the impulsive model improved the accuracy of the results, only optimizations with the impulsive model were performed for objective function 5. As expected, objective function 5 provides a good substitution for objective function 4. Thus, the final choice was to use objective function 5 with step definition 3.

Unfortunately, objective function 5 appears to have some issues with local minima (Table 4.9). However, it appears that if \( f_5 < 0.35 \), the model gait matches the experimental gait well enough for this study.

4.3.3 Ankles Model Pilot Study

A similar study to the one described in Section 4.3.2 was conducted for the ankles model. Several objective functions were tested (Table 4.11). Similar to the previous study, the weights were chosen so that if each component had a reasonably acceptable error, the magnitude of each term would be approximately the same except for the ankle terms, which were smaller. As discussed in Section 4.1.1, during normal human walking, the stance ankle is used to both help form an effective rocker with the ground (Hansen et al., 2004a; Wang and Hansen, 2010) and provide an additional propulsive moment. Because the model’s foot is curved, the model ankle motion is expected
## TABLE 4.10

COMPARISON OF DIFFERENT OPTIMIZATION SEEDS FOR THE IMPULSIVE MODEL

<table>
<thead>
<tr>
<th>Seed</th>
<th>Speed</th>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Step Length</th>
<th>Speed</th>
<th>f₁</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial</td>
<td>0.628</td>
<td>1.222</td>
<td>0.677</td>
<td>0.243</td>
<td>0.677</td>
<td>0.243</td>
</tr>
<tr>
<td></td>
<td>0.823</td>
<td>5</td>
<td>3</td>
<td>0.628</td>
<td>1.221</td>
<td>1.130</td>
<td>0.317</td>
</tr>
<tr>
<td></td>
<td>1.222</td>
<td>5</td>
<td>2</td>
<td>0.628</td>
<td>1.222</td>
<td>5.688</td>
<td>0.959</td>
</tr>
<tr>
<td></td>
<td>1.222</td>
<td>1</td>
<td>3</td>
<td>0.628</td>
<td>1.222</td>
<td>2.288</td>
<td>0.357</td>
</tr>
<tr>
<td></td>
<td>1.589</td>
<td>5</td>
<td>3</td>
<td>0.628</td>
<td>1.222</td>
<td>10.762</td>
<td>0.582</td>
</tr>
</tbody>
</table>

NOTE: Step definition 3 and objective function 5 were used. The human experimental gait has a step length of 0.628 m and a speed of 1.222 m/s. The step length is given in meters, and the speed is given in m/s.
TABLE 4.11
WEIGHTING FACTORS FOR EQ. 4.15 FOR THE ANKLES MODEL
PILOT STUDY

<table>
<thead>
<tr>
<th>Name</th>
<th>$c_{SL}$</th>
<th>$c_v$</th>
<th>$c_{p,1..4}$</th>
<th>$c_{c,2..4}$</th>
<th>$c_{c,5..6}$</th>
<th>$c_{PC,1}$</th>
<th>$c_{PC,5}$</th>
<th>$c_{PC,6}$</th>
<th>$c_{\Delta}$</th>
<th>$c_{AA,5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$10^5$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$10^4$</td>
<td>$10^4$</td>
<td>0</td>
<td>1</td>
<td>0.1</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$10^4$</td>
<td>$10^4$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>$10^4$</td>
<td>$10^4$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0.001</td>
<td>0.001</td>
</tr>
</tbody>
</table>

NOTE: All coefficients not listed are zero for all objective functions.

to differ from the physiological ankle motion. In addition, the best possible fit of the physiological ankle motion, particularly during the swing phase, is not quite as good as at the other joints. Taken together, this means that it is less important to match the ankle motion than the motion of the other joints. To provide continuity from the previous study, objective functions that are similar have the same number. Objective function 1 was exactly the same as before so that the results for all three models could be easily compared. However, it was not be used to generate gaits.

As before, strings of optimizations were performed. The bounds for the first several optimizations were increased to 20% because initial tests indicated that the optimization performed better with looser bounds. Because toe-off primarily comes from the trailing leg during double support, at least some of the double support phase must occur at the end of the step. As a result, step definition 1 was omitted from the pilot study because it places the entire double support phase at the beginning of the step. Step definition 4 was also excluded because it does not include the entire human step.
Unlike the previous pilot study, the initial seed was determined using the experimental data, which was not possible for the base and impulsive models. As long as the average speed was fast enough, fitting polynomials to the experimental angles and then finding the appropriate Bézier polynomial yielded a gait that could take at least one step. While this gait typically violated one or more constraints, the optimization was consistently able to find a valid gait within a single optimization. If the average speed of the gait was too slow, however, the gait found by fitting polynomials to the experimental data fell within one step, and the optimization rarely converged. For those cases, the initial condition was changed to a predetermined set of polynomial coefficients that produced a valid gait. When possible, it is desirable to use the experimental data as the initial guess because it is should be close to the global minimum, which means there should be fewer issues with local minima. In addition, it is possible that the string of optimizations will converge to a minimum more quickly. In practice, it does not seem to matter very much.

After finding gaits for the natural speed, it was determined that step definition 3 produced the best results (Table 4.12). It was not clear, however, which objective function produced the most human-like gait. Thus, further optimizations were performed using a slower and a faster speed. In all cases, all three objective functions performed well (Table 4.13, Fig. 4.7, Appendix C.2), although objective function 5 performed best. Thus, it was decided that objective function 5 should be used for the full study. A further advantage of using objective function 5 is that matching gaits will be found for all three models using essentially the same objective function.

4.3.4 Full Study Design

As a quick check to ensure that adding toe-off to the model makes the motion more human-like, the values of objective function 1 can be compared. The value of $f_1$ is 5.9 for the base model, 0.68 for the impulsive model, and 0.79 for the ankles model.
Figure 4.7. Comparison of the model gaits for the ankles model walking at a natural speed (1.22 m/s) using step definition 3. Note that the plots use the model angles, not the transformed biomechanics angles. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using $f_5$, the green line shows the gait found using $f_6$, and the black line shows the gait found using $f_7$. 
TABLE 4.12

COMPARISON OF DIFFERENT OBJECTIVE FUNCTIONS AND STEP DEFINITIONS FOR THE ANKLES MODEL

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Set</th>
<th>Step Length (m)</th>
<th>Ave. Speed (m/s)</th>
<th>$f_1$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seed 2</td>
<td>0.631</td>
<td>1.526</td>
<td>9301.173</td>
<td>927.397</td>
<td>927.986</td>
<td>927.990</td>
<td></td>
</tr>
<tr>
<td>Seed 3</td>
<td>0.598</td>
<td>0.915</td>
<td>9455.775</td>
<td>955.036</td>
<td>956.208</td>
<td>955.660</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.628</td>
<td>1.222</td>
<td>23.616</td>
<td>0.665</td>
<td>0.728</td>
<td>2.291</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.629</td>
<td>1.222</td>
<td>33.528</td>
<td>0.619</td>
<td>0.651</td>
<td>1.349</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.629</td>
<td>1.223</td>
<td>18.105</td>
<td>0.995</td>
<td>1.196</td>
<td>0.843</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.628</td>
<td>1.222</td>
<td>0.788</td>
<td>0.258</td>
<td>1.498</td>
<td>1.722</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.628</td>
<td>1.222</td>
<td>2.321</td>
<td>0.423</td>
<td>0.322</td>
<td>0.375</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.628</td>
<td>1.222</td>
<td>1.359</td>
<td>0.439</td>
<td>1.903</td>
<td>0.273</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: The human experimental gait has a step length of 0.628 m and a speed of 1.222 m/s.
### TABLE 4.13

**COMPARISON OF OBJECTIVE FUNCTIONS AT DIFFERENT SPEEDS FOR THE ANKLES MODEL**

<table>
<thead>
<tr>
<th>Obj. Fun.</th>
<th>Exp. Speed</th>
<th>Exp. Step</th>
<th>Sim. Speed</th>
<th>Sim. Step</th>
<th>$f_1$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(m/s)</td>
<td>Length (m)</td>
<td>(m/s)</td>
<td>Length (m)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>1.055</td>
<td>0.362</td>
<td>2.541</td>
<td>2.288</td>
</tr>
<tr>
<td>6</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>20.403</td>
<td>0.658</td>
<td>0.630</td>
<td>0.824</td>
</tr>
<tr>
<td>7</td>
<td>0.823</td>
<td>0.624</td>
<td>0.823</td>
<td>0.624</td>
<td>6.422</td>
<td>0.607</td>
<td>1.994</td>
<td>0.489</td>
</tr>
<tr>
<td>5</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>0.788</td>
<td>0.258</td>
<td>1.498</td>
<td>1.722</td>
</tr>
<tr>
<td>6</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>2.321</td>
<td>0.423</td>
<td>0.322</td>
<td>0.375</td>
</tr>
<tr>
<td>7</td>
<td>1.222</td>
<td>0.628</td>
<td>1.222</td>
<td>0.628</td>
<td>1.359</td>
<td>0.439</td>
<td>1.903</td>
<td>0.273</td>
</tr>
<tr>
<td>5</td>
<td>1.589</td>
<td>0.689</td>
<td>1.589</td>
<td>0.698</td>
<td>3.189</td>
<td>0.254</td>
<td>0.767</td>
<td>0.561</td>
</tr>
<tr>
<td>6</td>
<td>1.589</td>
<td>0.689</td>
<td>1.589</td>
<td>0.698</td>
<td>3.940</td>
<td>0.420</td>
<td>0.299</td>
<td>0.800</td>
</tr>
<tr>
<td>7</td>
<td>1.589</td>
<td>0.689</td>
<td>1.589</td>
<td>0.698</td>
<td>4.584</td>
<td>0.423</td>
<td>1.290</td>
<td>0.246</td>
</tr>
</tbody>
</table>

**NOTE:** Step definition 3 was used.
when using step definition 3. This indicates that some method of including toe-off in the model significantly improves the model. It also suggests that both the impulsive toe-off method and the articulated ankle method perform comparably. These claims were evaluated more thoroughly in the full study.

For all models, step definition 3 and objective function 5 were used in the full study. The gaits to be matched were the training gaits given in Section 4.2. Note that no explicit effort was made to match human kinetics. For reference, the objective function was

$$f = 10^4 \left( S_{\text{sim}} - S_{\text{exp}} \right)^2 + 10^4 \left( v_{\text{sim}} - v_{\text{exp}} \right)^2 + 0.1 \left\| \alpha_{\text{sim}, \text{Hip and Knee}} - \alpha_{\text{exp}, \text{Hip and Knee}} \right\|^2 + 0.1 \left\| \alpha_{\text{sim}, \text{Stance Ankle}} - \alpha_{\text{exp}, \text{Stance Ankle}} \right\|^2 + 0.1 \left\| \alpha_{\text{sim}, \text{Swing Ankle}} - \alpha_{\text{exp}, \text{Swing Ankle}} \right\|^2 + 10 \left( 1 - \rho(q_{1,\text{exp}}, q_{1,\text{sim}}) \right)$$

where \( S \) is the dimensioned step length, \( v \) is the dimensioned speed, \( \alpha_{\text{sim}} \) is a matrix of the control parameters from Eq. 2.57, \( \alpha_{\text{exp}} \) is a similar matrix found by fitting the experimental joint trajectories with fifth order polynomials, and \( \rho \) is the Pearson linear correlation coefficient. The subscripts ‘exp’ and ‘sim’ indicate values from experiment and the model, respectively. For the base and impulsive models, the two ankle terms were omitted.

The optimizations were performed as in the pilot studies. To reduce the effects of local minima, up to four strings of optimizations were performed for each experimental gait. The seed for the first optimization was the same initial gait that was
used for the pilot study (base or impulsive models) or the initial gait found by fitting polynomials to the experimental data (ankles model). The seeds for the remaining optimizations were chosen from gaits found for other studies and/or speeds. If, after a string of optimizations completed, the value of objective function 5 was less than 0.35 (base or impulsive models) or 0.40 (ankles model), that model gait was used for the study, and no further optimizations were performed. If it was clear after three strings of optimization that no gait with $f_5 < 0.35$ (base or impulsive models) or 0.40 (ankles model) could be found, the gait with the smallest value of objective function 5 was used. Similarly, after a total of four strings of optimization were completed, the gait with the smallest value of objective function 5 was used.

4.4 Choosing the Prediction Function

In order to predict normal human gait without a priori knowledge of the gait, an objective function needs to be determined. Many objective functions were considered initially. They were

- Work-related functions,
- Power-related functions,
- Torque magnitude functions,
- Torque-squared functions,
- Torque-cubed functions,
- Maximum torque functions,
- Minimum jerk functions, and
- Ground reaction force (GRF) functions,
all with multiple normalization schemes. The values of all of the possible objective functions were computed for each matching gait of the base and ankles models. (As discussed in Section 4.6, the ankles model is the best model for predicting human gait, so the other models were dropped. The base model is included here only for comparison purposes.) The value of each objective function was plotted against nondimensional speed and visually inspected for trends. The original hypothesis was that for an objective function that could predict human gait, the base model would have larger values than the ankles model, and the ankles model would have parabolic trends. The values of the work-based, GRF, torque magnitude and maximum torque objective functions were similar for both models, indicating that they are poor choices. Of the remaining options, only hip jerk had a strong quadratic relationship with speed. However, initial optimizations with hip jerk as the objective function resulted in gaits with extremely long steps, and combining hip jerk with torque-squared or power-based functions did not result in accurate gait predictions.

Since it seems likely that torque squared is correlated to the square of muscle activations, which in turn, are generally believed to be correlated with metabolic cost (Silder et al., 2012), and humans tend to minimize metabolic cost when walking (Ralston, 1958), it was decided to proceed with a torque-squared-based objective function. At each actuated joint \( k \), the nondimensionalized torque squared is

\[
\tau_k^2 = \frac{\int_0^T u_k^2(t)dt}{T(Mg\ell_0)^2}, \tag{4.28}
\]

where \( T \) is the step duration, \( u \) is the joint moment, \( M \) is subject mass, \( g \) is gravitational acceleration and \( \ell_0 \) is subject leg length. Simply summing up \( \tau^2 \) over all the joints did not result in gaits that predicted step length or knee kinematics well. Comparing the torque at each joint for a gait predicted using \( \tau^2 \) and its corresponding matching gait, it became apparent that the predicted gaits utilized far less ankle
torque than the matching gaits. To combat this issue, each joint was given its own weighting factor, which resulted in the following objective function.

\[ g = \sum_{k=1}^{N} c_k \cdot \tau_k^2 + g_p, \]  

(4.29)

where \( c_k \) is a weighting factor and \( g_p \) is a penalty function.

The model foot length is defined as the distance the CoP travels from touch-down to toe-off. To prevent unreasonably long feet, and hence unreasonably long step lengths, a penalty \( g_p \) was added when the required foot length exceeded the physiological foot length.

\[ g_p = \begin{cases} 
0 & \text{if } L_f/\ell_0 \leq 0.287 \\
c_p (L_f - 0.287\ell_0)^2/(L_f^* - 0.287\ell_0)^2 & \text{otherwise}
\end{cases}, \]  

(4.30)

where \( c_p \) is a weighting factor, \( L_f \) is the required model foot length, \( L_f^* \) is the speed-dependent optimal foot length from a linear fit of data from the initial matching gaits, \( \ell_0 \) is leg length, and the physiological foot length is 0.287\( \ell_0 \). Note that \( g_p \) is zero when the model foot length is less than or equal to the physiological foot length and positive otherwise.

A total of 53 different combinations of weighting factors were tested, and the predicted gait was compared to the experimental gait. The weighting factor for the swing hip was kept fixed at 1. The weighting factor for the knees were 0.5, 0.4, 0.3, 0.2, and 0.1. The weighting factors for the ankles were 0.3, 0.2, 0.1, 0.15, and 0.05. The weighting factors for the penalty function were 1, 0.5, 0.2, 0.08, and 0.02. Not all possible combinations of weighting factors were tested. Using the speed and corresponding subject-specific model from five matching gaits, chosen so that each of the matching studies were included and representing the full range of speeds, gaits were found for each of the trial objective functions. The kinematics of these predicted
gaits were then compared to the kinematics of the experimental gaits. It was found that weighting factors of 0.2 for the knees and ankles and 0.08 for the penalty function matched the gait kinematics best, although the predictive capability is not highly sensitive to the choice of weighting factors. (Recall that the stance hip motion is unactuated, so $\tau_1^2 = 0$.) Hip work is motion dominated, while stance knee and ankle work are torque dominated, so these weighting factors result in gaits that perform approximately equal magnitudes of work at the hip, stance knee, and stance ankle, consistent with normal human gait (DeVita et al., 2007).

To fully evaluate Eq. 4.29’s ability to predict normal human walking, a study utilizing all of the experimental data was conducted. Optimizations were conducted as described in Section 2.5. Speed was enforced as a constraint. “Prediction gaits” consisted of the 16 training gaits (Section 4.2), and “validation gaits” consisted of the 10 validation gaits (Section 4.2). Subject-specific six-link models based on subject height and mass were developed (Section 4.1.1). For each speed, the control parameters $\alpha_{sim}$ were determined to minimize the objective function (Eq. 4.29) subject to the constraint on walking speed. Because the optimization is not convex, the initial guess can affect the results due to local minima. For the prediction gaits, the initial guesses were the next slowest and fastest gaits from the matching study. For the validation gaits, the initial guesses were the three prediction gaits at about the self-selected speed.

4.5 Evaluating the Results

Because of the range of subject sizes, all reported data were nondimensionalized using the factors proposed in Hof (1996). Optimizations and simulations were conducted using a fully dimensioned model, and results were nondimensionalized as needed. The factors are given in Table 4.14. For reference, the nondimensional self-selected speed is about 0.42, and the nondimensional walk-to-run transition speed is
### Table 4.14

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step Length</td>
<td>$0.53H$</td>
</tr>
<tr>
<td>Speed</td>
<td>$\sqrt{0.53gH}$</td>
</tr>
<tr>
<td>Torque</td>
<td>$0.53MgH$</td>
</tr>
<tr>
<td>Power</td>
<td>$Mg^{1.5}(0.53H)^{0.5}$</td>
</tr>
</tbody>
</table>

**Source:** Hof (1996)

**Note:** $H$ is subject height, $0.53H$ is leg length, $M$ is subject mass, and $g$ is gravitational acceleration.

Both kinematic and kinetic metrics were used to evaluate the model gaits' agreement with normal human walking data. Instead of reporting p-values, 95% confidence intervals are reported. Kinematic metrics included step length, average speed, and the individual hip, knee and ankle joint kinematics. For the experimental values, means were utilized. Model and experimental step lengths and speeds were directly compared, while joint angle trajectories were compared via Pearson’s correlation coefficient. Peak hip flexion and extension were compared directly. For the knee, peak stance flexion during weight acceptance (Worthen-Chaudhari et al., 2014), peak stance extension, and peak swing flexion were examined. For the ankle, using peak angles directly is useless because inter-protocol mean ankle angles vary by over 20° (Ferrari et al., 2008). Instead, the range of motion was directly compared.

For kinetics, the energetic cost of walking was compared using mean absolute power (Winter, 1979; Caldwell and Forrester, 1992; Bianchi et al., 1998). While it is about 0.71.
likely that torque-squared (as computed in this study) is correlated with metabolic cost, there have been no studies demonstrating this, so there is no clear benefit to using the torque-squared metric as opposed to the more common metric of mean absolute power.

\[
P = \sum_{i=1}^{N} \int_{0}^{T} \left| P_i(t) \right| dt, \quad (4.31)
\]

where \( P_i \) is the instantaneous power at joint \( i \), \( T \) is the step duration, and \( t \) is time. Due to the limited kinetic data, all studies with reported joint moments were used when reporting experimental mean power for the matching portion of the study, not just those from the training group. One study (Han and Wang, 2011) was omitted because it used an unpublished method that computed unusually high joint torques.

In the prediction study, differences between the prediction and validation groups were evaluated by computing the differences in the linear regression equations.

\[
f = \beta_0 + \beta_v v + \beta_{0V} x_V + \beta_V v_V, \quad (4.32)
\]

where

\[
x_V = \begin{cases} 
0 & \text{if prediction gait} \\
1 & \text{if validation gait} 
\end{cases}, \quad (4.33)
\]

\[
v_V = \begin{cases} 
0 & \text{if prediction gait} \\
v & \text{if validation gait} 
\end{cases}, \quad (4.34)
\]

\( f \) is the metric of interest, the \( \beta_i \)'s are the regression coefficients, and \( v \) is nondimensional speed. \( \beta_{0V} \) gives the difference between the two groups in the intercept and \( \beta_V \) gives the difference between the two groups in the slope with respect to speed. When there is no relationship with speed, the \( \beta_v \) and \( \beta_V \) terms are omitted from the equation.

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Ankles
Impulsive
Base
Experimental
Normalized Step Length
Normalized Speed
Matching Gaits

Figure 4.8. Normalized step length vs. normalized average speed for the human experimental data and all three models when matching experimental data. The points from experiment and the base model overlap at slow speeds. The points from experiment and the impulsive and ankles models overlap at all speeds.

4.6 Results of Matching Study

4.6.1 Results

The base model matches both human step length and speed only at slow speeds (Fig. 4.8). At faster speeds, the model almost matches speed but with much shorter steps than observed in the human data. In contrast, both the impulsive model and the ankles model match experimental step length and speed almost exactly at all speeds.

All three models accurately match the shape of human hip motion during both stance and swing across speeds (Fig. 4.9). For the base model, the correlation coefficients are all greater than 0.9, with a mean of 0.97. For the impulsive model, the correlation coefficients are all greater than 0.98, with a mean of 1.0. Similarly, the correlation coefficients for the ankles model are all greater than 0.96, with a mean of
The correlations for each joint are grouped. Within each group, the order is the base model gaits (B), the impulsive model gaits (I), and then the ankles model gaits (A). The circular points indicate outliers.

0.99. The base model, however, does not accurately match knee motion shape during either stance or swing across speeds (Fig. 4.9), with mean correlations of 0.30 for stance and 0.76 for swing. The impulsive model and the ankles model both perform much better, with mean correlations greater than 0.9 for both stance and swing. The joint angles for all three models over a full stride are compared with those from a representative experimental gait in Fig. 4.10. Plots of all of the matching gaits are given in Appendix C.3

The base model predicts a hip range of motion $9.7^\circ \pm 4.6^\circ$ larger than observed experimentally (Fig. 4.11), mostly because the leg swings out too far prior to heel strike, as in Fig. 4.10(a). The impulsive model shifts the hip range of motion forward $3.2^\circ \pm 1.3^\circ$, although the range itself differs little from that observed experimentally.
Figure 4.10. Joint kinematics at the (a) hip, (b) knee, and (c) ankle for a nondimensional speed of 0.46. From left to right, the vertical lines indicate the transfer of support, human toe-off, and human heel contact. The results are qualitatively similar for all gaits.
Figure 4.11. Difference between the model and experimental peak angles at the hip and knee. Each metric is grouped. From left to right, the groups are hip flexion, hip extension, early stance knee flexion, stance knee extension, and swing knee flexion. Within each group, the order is base model (B), impulsive model (I), and ankles model (A).

(0.8° ± 2.0° larger in simulation). The ankles model underestimates the hip range of motion by 6.4° ± 4.6°. The base model has a larger (11.3° ± 8.3°) range of stance knee motion than observed experimentally and much less swing knee flexion (18.8° ± 11.7°), particularly at faster speeds. Because the error in the base model’s swing knee flexion increases with increasing speed, the range of errors shown in Fig. 4.11 is particularly large. The impulsive model matches the knee motion very well (less than 2° error). The ankles model underestimates the range of stance knee motion (8.6° ± 3.8°) and the swing knee flexion (2.8° ± 3.2).

The mean absolute power increases linearly with walking speed (Fig. 4.12) at a nondimensional rate of 0.26 ± 0.04 for humans ($R^2 = 0.93$). The base model has
a slope that is about four times larger (difference of $0.81 \pm 0.15$). The other two models slightly overestimate the increase in power with walking speed, although the difference is not statistically significant (difference of $0.05 \pm 0.15$ for the impulsive model and $0.13 \pm 0.15$ for the ankles model). The smaller difference for the impulsive model is somewhat misleading because it does not include the impulsive force that also inputs energy into the gait cycle. Although the base model requires more power, it also consumes less energy per step, largely due to the shorter steps. Thus, the base model overestimates some and underestimates other kinetic measures of human walking, while both the impulsive model and the ankles model capture the overall human kinetic response even without explicitly attempting to match human energetic data.
4.6.2 Discussion

The four-link model’s inability to accurately match human experimental data, particularly at high speeds, can be explained by considering energy flows. The energy loss at the foot-ground impact increases with increasing step length, and both the legs inject energy to compensate. In the ankles model, the hip, knees, and stance ankle all perform work, as in normal human gait. Notably, the model’s stance ankle provides a large push-off torque just prior to impact, similar to the human ankle (Fig. 4.13). The impulsive model replaces this ankle torque with an impulse force at the hip during impact. Lacking both ankles and the additional impulsive force, the base model compensates with shorter steps to reduce impact losses and a pumping of the stance hip and knee to inject additional energy (Fig. 4.10). Still, the base model requires more power than a human, with both the kinematic and kinetic differences more pronounced at higher speeds when more energy is required.
By introducing the effects of toe-off into the model, the accuracy in matching normal human walking data improves substantially for all speeds. Further, both the impulsive and ankles models capture the overall effect of speed on mean absolute power (Burdett et al., 1983) without explicitly attempting to match human energetic data. Thus, the impulsive and ankles models simultaneously capture the joint kinematics and the overall energetic effort of normal human walking. In general, the impulsive model mimics human gait slightly better than the ankles model, likely because the impulsive model has an extra four independent optimization parameters. There is not an obvious way to map the toe-off impulse of the impulsive model into prosthesis design characteristics because the impulse is likely affected by both the foot properties and the configuration of the stance leg. To model a prosthesis for the ankles model, it should be possible to split the function of the prosthetic foot into a rollover shape portion and an ankle stiffness portion. Because the accuracy of the two models is similar and because the ankles model has a more natural connection to prosthesis design, the ankles model is the preferred model. As the preferred model, the ankles model is the only model used in the prediction portion of the study.

4.7 Results of Prediction Study

4.7.1 Results

The results for the prediction and validation groups were not statistically different (Table 4.15), so the discussion reports combined values. Step length increases linearly with speed at approximately the same rate for humans (0.58±0.09, $R^2 = 0.87$) and the ankles model (0.59 ± 0.12, $R^2 = 0.81$), indicating that the optimization successfully predicts the increase in step length with speed (Fig. 4.14).

The model gaits well match hip kinematics, with mean correlations greater than 0.98 (Fig. 4.15). The error in peak hip angles is not statistically different from zero.
<table>
<thead>
<tr>
<th>Metric</th>
<th>( \beta_V )</th>
<th>( \beta_{0V} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step Length</td>
<td>-0.208 ± 0.245</td>
<td>0.090 ± 0.118</td>
</tr>
<tr>
<td>Correlation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stance Hip</td>
<td>-</td>
<td>-0.002 ± 0.008</td>
</tr>
<tr>
<td>Swing Hip</td>
<td>-</td>
<td>0.001 ± 0.004</td>
</tr>
<tr>
<td>Stance Knee</td>
<td>-</td>
<td>-0.210 ± 0.360</td>
</tr>
<tr>
<td>Swing Knee</td>
<td>-</td>
<td>-0.073 ± 0.107</td>
</tr>
<tr>
<td>Stance Ankle</td>
<td>-</td>
<td>0.091 ± 0.518</td>
</tr>
<tr>
<td>Swing Ankle</td>
<td>-</td>
<td>0.306 ± 0.394</td>
</tr>
<tr>
<td>Err. Peak Angles</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hip Flexion</td>
<td>-</td>
<td>2.108 ± 4.563</td>
</tr>
<tr>
<td>Hip Extension</td>
<td>-</td>
<td>2.360 ± 4.390</td>
</tr>
<tr>
<td>Stance Knee Flexion</td>
<td>-</td>
<td>3.538 ± 8.413</td>
</tr>
<tr>
<td>Stance Knee Extension</td>
<td>-</td>
<td>0.285 ± 6.482</td>
</tr>
<tr>
<td>Swing Knee Flexion</td>
<td>-</td>
<td>4.999 ± 6.001</td>
</tr>
<tr>
<td>Ankle Range</td>
<td>-</td>
<td>6.106 ± 12.596</td>
</tr>
<tr>
<td>Mean Abs. Power</td>
<td>0.032 ± 0.066</td>
<td>-0.012 ± 0.032</td>
</tr>
</tbody>
</table>

NOTE: Coefficients are for Eq. 4.32.
Figure 4.14. Normalized step length vs. normalized average speed for the human experimental data and the predicted model gaits. The linear least-squares lines for the experimental (solid), prediction (dashed), and validation (broken) gaits are shown.

(Fig. 4.16), with a mean error of 1.5° ± 1.5°. Knee motion is not predicted quite as accurately, particularly in stance, where the mean correlation is 0.61 ± 0.18 (Fig. 4.15) and the mean error in peak flexion is 2.6° ± 4.1° (Fig. 4.16). Swing knee motion is better predicted, with a mean correlation of 0.87 ± 0.05 (Fig. 4.15) and a mean error in peak flexion of 1.9° ± 3.0° (Fig. 4.16). The model does a poor job of capturing the physiological ankle motion, with a mean correlation of 0.36 ± 0.25 for the stance ankle and 0.53 ± 0.20 for the swing ankle (Fig. 4.15), although the difference in range of motion is not statistically different from zero (4.1° ± 6.1° larger in simulation, Fig. 4.16). Plots of all of the prediction gaits are available in Appendix C.4.

The optimization well predicts mean absolute power (Eq. 4.31, Fig. 4.17). As reported in Section 4.6, the mean absolute power increases at a rate of 0.27 ± 0.05 for humans. The model predicts a rate of increase of 0.32 ± 0.03 ($R^2 = 0.95$). While the model slightly underestimates the power required at lower speeds and overestimates
Figure 4.15. Correlation between the model and experimental joint angles when predicting normal human walking. “St” stands for stance and “Sw” stands for swing. The correlations for each joint are grouped. Within each group, the order is the prediction gaits (P) followed by the validation gaits (V). The circular points indicate outliers.
Figure 4.16. Difference between the experimental and model peak joint angles when predicting normal human walking. Each metric is grouped. From left to right, the groups are hip flexion, hip extension, early stance knee flexion, stance knee extension, swing knee flexion, and ankle range of motion. Within each group, the order is the prediction gaits (P) followed by the validation gaits (V). The circular points indicate outliers.
the increase with speed (although not by a statistically significant amount), the error at a particular speed is small compared to the total increase in power.

4.7.2 Discussion

Using a torque-squared objective function, the ankles model can accurately predict normal human walking gaits regardless of walking speed. The hip kinematics are well predicted, with the model capturing both the time-based trajectories and the peak magnitudes. The stance knee kinematics are acceptably captured, particularly at slower speeds. The main source of error is insufficient knee flexion during the model’s weight acceptance phase. In both experiment and simulation, the peak stance knee flexion increases with increasing speed, but not as fast in simulation as in experiment, leading to larger errors at faster speeds. Likewise, the swing knee kinematics are well captured. As anticipated, the model ankle motion correlates
poorly with the physiological ankle motion, but the model does predict the range of motion well, perhaps because the anticipated increase in model ankle flexion to achieve toe-off effects roughly cancels the anticipated decrease in flexion associated with the curved feet taking on some of the human ankle functionality. The model slightly underestimates the mean absolute power. Minimizing torque squared, the model penalizes positive and negative work equally, although human muscles and tendons store and return some energy in a spring-like manner (Ishikawa et al., 2005). Still, even without springs, the model can well approximate normal human walking as evidenced by the small errors in both the joint angles and mean absolute power.

Because the results for the prediction and validation gaits are not statistically different, the model is capable of predicting human walking independent of the initial seed. Thus, it is possible to obtain accurate results without motion capture data. As a result, this simple six-link planar model can be used to accurately predict both the kinematics and energetic effort of normal human walking at speeds ranging from very slow to very fast.

4.8 Summary

Three planar models with instantaneous transfer of support were developed. All models used HZD-based control. The base model consists of four links and has no ability to generate toe-off. The impulsive model also has four links, but it models toe-off via an impulsive force at the hip that helps to redirect the hip velocity during impact. Due to the ankle joints, the ankles model adds two links to the base model, which allows the ankles to generate a toe-off effect. The ability of all three models to mimic normal human walking at speeds ranging from very slow to very fast was evaluated. The base model performed poorly, particularly at faster speeds. Both the impulsive model and the ankles model matched human kinematics and mechanical energetic cost well, indicating that including toe-off in a model of this nature is...
necessary for accurate modeling of human walking. This can be accomplished either by including ankles in the model or with an impulsive force at impact that redirects the hip velocity.

With the ankles model, it is possible to predict normal human walking at speeds ranging from very slow to very fast using a torque-squared-based objective function. The model is able to predict the human kinematics and mechanical energetic cost accurately even without prior knowledge of the experimental gait. Thus, a six-link planar model with instantaneous transfer of support and curved feet, controlled using HZD-based control, can predict normal human walking across a range of speeds using a torque-squared-based objective function.
CHAPTER 5

QUANTIFYING THE TRADE-OFFS BETWEEN SYMMETRY AND EFFICIENCY FOR AN ASYMMETRIC AMPUTEE MODEL

This chapter discusses how the human model developed in Chapter 4 can be further extended to investigate asymmetric amputee gait. Section 5.1 presents the asymmetric model. Section 5.2 quantifies how the asymmetrical model’s gait changes as the importance of kinematic symmetry and energetic efficiency are varied in the objective function used to design the gait.

5.1 Asymmetric Amputee Model

Because prostheses do not completely replicate the lost function of amputated joints and because prostheses typically have different mass properties than the intact leg, a model of an amputee must be asymmetric. For below-knee amputees wearing a passive prosthesis, a five-link model is appropriate (Fig. 5.1). This model is similar to the symmetric six-link model used for healthy human gait, but one of the ankle joints has been removed. This results in an amputated leg with a hip and knee joint, but no ankle joint, and a contralateral leg with a hip, knee and ankle joint. The amputated foot is rigidly attached to the amputated shank. The point of contact between the foot and ground is unactuated, and the unactuated motion is captured in the stance hip absolute angle ($q_1$ in Fig. 5.1). The leg joints are actuated using ideal torque generators, with a single hip actuator controlling the angle between the thighs ($q_2$ in Fig. 5.1). The actuated angles are $q_2$ through $q_5$. 

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Figure 5.1. Schematic of the five-link, asymmetric amputee model. (a) shows the amputated leg in stance and (b) shows the contralateral leg in stance. The amputated leg has hip and knee joints, but no ankle joint. The contralateral leg has hip, knee and ankle joints. The joint angles used in the model are indicated with the $q_i$’s. The unactuated angle is $q_1$; the actuated angles are $q_2$ through $q_5$. The plots, however, transform the model angles into a more typical biomechanics convention as indicated with the $\theta_i$’s.
The mass and length parameters were chosen for a 50\textsuperscript{th} percentile male (Table 5.1; NHTSA, 2014). Because prostheses are typically much lighter than the physiological leg (Lehmann et al., 1998; Mattes et al., 2000), the mass and inertia of the amputated shank were chosen to be half of the physiological values.

Since the model is no longer left-right symmetric, a one-step periodic gait is not appropriate. Instead, a two-step periodic gait is used. As a result, there are two output functions (Eq. 2.57), one for when the amputated leg is in stance and one for when the contralateral leg is in stance. Both output functions are parameterized using fifth order Bézier polynomials. To ensure continuity across impact events, $\alpha_0$ and $\alpha_1$ are chosen as functions of the other polynomial coefficients. Specifically, for the amputated polynomial,

$$
\begin{bmatrix}
\alpha_{A0} \\
\theta_A^+
\end{bmatrix} = H_A S H C^{-1} \begin{bmatrix}
\alpha_{C5} \\
\theta_C^-
\end{bmatrix},
$$

(5.1)

$$
\alpha_{A1} = \frac{(\theta_A^- - \theta_A^+)}{5(c_A A C \omega C)} H_0 A C \omega C + \alpha_{A0},
$$

(5.2)

where

$$
H_A = \begin{bmatrix} c_A \\ H_0 \end{bmatrix},
$$

$$
H_C = \begin{bmatrix} c_C \\ H_0 \end{bmatrix},
$$

$$
c_A = \begin{bmatrix} - (R_A + L_A t + L_A s) & 0 & - (R_A + L_A s) & 0 & 0 \\ - (R_C + L_C t + L_C s) & 0 & - (R_C + L_C s) & 0 & - R_C \end{bmatrix},
$$

$$
c_C = \begin{bmatrix} - (R_A + L_A t + L_A s) & 0 & - (R_A + L_A s) & 0 & 0 \\ - (R_C + L_C t + L_C s) & 0 & - (R_C + L_C s) & 0 & - R_C \end{bmatrix},
$$

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TABLE 5.1

MASS AND LENGTH PARAMETERS FOR THE AMPUTEE MODEL

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Amputated</th>
<th>Contralateral</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of Thigh</td>
<td>$L_t$</td>
<td>0.429 m</td>
<td>0.429 m</td>
</tr>
<tr>
<td>Length of Shank</td>
<td>$L_s$</td>
<td>0.245 m</td>
<td>0.245 m</td>
</tr>
<tr>
<td>Radius of Foot</td>
<td>$R$</td>
<td>0.254 m</td>
<td>0.254 m</td>
</tr>
<tr>
<td>Ankle Offset</td>
<td>$L_f$</td>
<td>0.010 m</td>
<td>0.010 m</td>
</tr>
<tr>
<td>Thigh COM</td>
<td>$C_t$</td>
<td>0.186 m</td>
<td>0.186 m</td>
</tr>
<tr>
<td>Shank COM</td>
<td>$C_s$</td>
<td>0.303 m</td>
<td>0.187 m</td>
</tr>
<tr>
<td>Foot COM</td>
<td>$(C_{xf}, C_{yf})$</td>
<td>–</td>
<td>(0.133, 0.186) m</td>
</tr>
<tr>
<td>Mass of Hip</td>
<td>$M_h$</td>
<td>52.9 kg</td>
<td></td>
</tr>
<tr>
<td>Mass of Thigh</td>
<td>$M_t$</td>
<td>7.80 kg</td>
<td>7.80 kg</td>
</tr>
<tr>
<td>Mass of Shank</td>
<td>$M_s$</td>
<td>2.38 kg</td>
<td>3.63 kg</td>
</tr>
<tr>
<td>Mass of Foot</td>
<td>$M_f$</td>
<td>–</td>
<td>1.13 kg</td>
</tr>
<tr>
<td>Inertia of Thigh</td>
<td>$J_t$</td>
<td>0.150 kg·m²</td>
<td>0.150 kg·m²</td>
</tr>
<tr>
<td>Inertia of Shank</td>
<td>$J_s$</td>
<td>0.102 kg·m²</td>
<td>0.205 kg·m²</td>
</tr>
</tbody>
</table>

NOTE: The COM location is measured from the proximal joint. The inertia is measured about the COM.
\[ \theta_A = c_A \cdot q, \]
\[ \theta_C = c_C \cdot q, \]
\[ H_0 = \begin{bmatrix} I_{(N-1)\times(N-1)} & 0_{(N-1)\times1} \end{bmatrix}, \]
\[ S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \]
\[ \omega_A = H_A^{-1} \begin{bmatrix} \frac{5(\alpha_{A5} - \alpha_{A4})}{\theta_A^+ - \theta_A^-} \\ 1 \end{bmatrix}, \]
\[ \omega_C = H_C^{-1} \begin{bmatrix} \frac{5(\alpha_{C5} - \alpha_{C4})}{\theta_C^+ - \theta_C^-} \\ 1 \end{bmatrix}, \]

\( A_C \) is the velocity impact map for the transition from the contralateral to the amputated side (based on Eq. 2.2b), \( A_A \) is the velocity impact map for the transition from the amputated to the contralateral side (based on Eq. 2.2b), \( I \) is the \((N-1)\times(N-1)\) identity matrix, \( 0 \) is a \((N-1)\times1\) vector of zeros, and \( \alpha_i \) is a polynomial coefficient. The subscripts “A” and “C” stand for amputated and contralateral, respectively. The superscripts “-” and “+” refer to the instants before and after impact, respectively. \( \theta_A \) and \( \theta_C \) represent the linearized horizontal position of the hip and serve as metrics of step progression. The matrix \( S \) switches the angle definitions. Similarly, for the contralateral polynomial,

\[
\begin{bmatrix} \alpha_{C0} \\ \theta_C^+ \end{bmatrix} = H_C SH_A^{-1} \begin{bmatrix} \alpha_{A5} \\ \theta_A^- \end{bmatrix},
\]

\( \alpha_{C1} = \frac{(\theta_C^- - \theta_C^+)}{5(c_C A_A \omega_A)} H_0 A_A \omega_A + \alpha_{C0}. \)
Stability was not checked because the equations used to check stability for symmetric bipeds are no longer valid. This omission is acceptable only because for a kneed biped it appears that all gaits that satisfy the other conditions for a valid gait are also stable.

Gaits were designed using the same methodology as in Section 2.5. To quantify the changes in amputee gait as symmetry and efficiency become more or less important, a total of five objective functions were used. At one extreme was the predictive function used for normal human gait (Eq. 4.29). This is referred to as the efficiency function. At the other extreme was the symmetry function given by

\[ g_{sym} = 10(\Delta_v^2 + \Delta_S^2) + \sum_{k=1}^{4} (1 - \rho(q_{k,amp}, q_{k,con})), \] (5.5)

where \( \Delta_v \) is the difference in average speed and \( \Delta_S \) is the difference in step length between the amputated and contralateral sides. \( \rho(q_{k,amp}, q_{k,con}) \) is the correlation coefficient used to evaluate similarity between the amputated and contralateral joint angle traces. Specifically, \( \rho \) measures how accurately a linear transformation would transform the amputated joint angle \( q_{k,amp} \) into the contralateral joint angle \( q_{k,con} \). Recall that a value of 1 indicates a perfect match, while 0 indicates no correlation. In general, in the amputee literature, symmetry refers to kinematic symmetry rather than kinetic symmetry (Boes et al., 2013; Robinson et al., 1977; Roerdink et al., 2012). In addition, amputees retain essentially the same gait kinematics, but not necessarily kinetics, when the masses of their prostheses are perturbed (Selles et al., 2004). The most common measures of amputee gait symmetry are the differences in amputated/contralateral step length and step period (or equivalently step length and step speed) (Bamberg et al., 2010; Isakov et al., 1996, 2000; Mattes et al., 2000; Rusaw and Ramstrand, 2011; Sagawa et al., 2011). These metrics alone are insufficient for an objective function because simply constraining step length and speed does not
yield a unique gait. As a result, a measure of similarity between the amputated and contralateral joint angles must also be included in the objective function. There is no commonly used metric of joint symmetry (Sagawa et al., 2011), so two measures were tested – joint correlations and the similarity of the control output functions (similar to the matching function used in Chapter 4). Both metrics yielded similar gaits in test cases when both optimizations converged, but the correlation metric converged more reliably and typically required fewer iterations, so it was selected. Since differences in step length and speed are the most common measures of symmetry, they received the largest weighting in Eq. 5.5. The weighting factor of 10 was chosen so that the step length and speed terms were the most important quantities but did not completely dominate the equation.

The overall objective function is a weighted combination of the two extremes.

\[ g_{amp} = 10w g_{eff} + (1 - w) g_{sym}, \]  

(5.6)

where \( g_{eff} \) is Eq. 4.29 and \( w \) is a weighting factor. In preliminary work, the efficiency term (Eq. 4.29) had values of about 0.005, while the symmetry term (Eq. 5.5) had values of about 0.05, prompting the extra factor of 10 before the efficiency term. A total of five weighting factors were examined – 0, 0.5, 0.75, 0.95, and 1. When the weighting factor is 0, the objective function only contains the symmetry term. When the weighting factor is 1, the objective function only contains the efficiency term. A total of three average stride speeds were examined – 0.8, 1.0, and 1.2 m/s. The amputee self-selected walking speed is about 1 m/s.

To quantify the results, metrics similar to the ones used to evaluate normal human gait were utilized (Section 4.5), although the agreement between the amputated and contralateral sides was evaluated instead of the agreement between simulation and experiment.
5.2 Effects of Varying the Objective Function on Asymmetric Gait

Figure 5.2 shows the trade-off curve between symmetry and efficiency. The range of efficiency values spans one order of magnitude, from $10^{-3}$ to $10^{-2}$. The range of symmetry values spans four orders of magnitude, from $10^{-3}$ to 10. From a clinical standpoint, once the symmetry metric becomes less than about $10^{-2}$, further reductions are unlikely to be noticeable. In general, achieving a symmetry metric of less than $10^{-2}$ requires less than a 0.03 m difference in step length, 0.03 m/s difference in step velocity, and correlations greater than 0.99. While there is definitely a trade-off between symmetric and efficient gaits, the trade-off curve is such that it is possible to have a fairly efficient and fairly symmetric gait.

5.2.1 Kinematic Results

For all except the most efficient gaits, the average speed was the same for the amputated and contralateral steps (Fig. 5.3). For the most efficient gaits (weighting
factor of 1), the average speed on the amputated side was slower than the average speed on the contralateral side, which is consistent with experimental data (Isakov et al., 2000).

As efficiency becomes more important, the difference in step length between the amputated and contralateral sides increases (Fig. 5.4). For the most symmetric gaits, the step length is the same between the amputated and contralateral sides. For the rest of the gaits, the amputated step length is shorter than the contralateral step length, which is consistent with experimental amputee gait (Robinson et al., 1977). This difference becomes more pronounced as efficiency becomes more important and as speed increases.

As expected, the correlation between the amputated and contralateral joint angles generally increases as the importance of symmetry increases (Table 5.2, Fig. 5.5, Appendix D). However, for all but the most efficient gaits, the correlation is almost perfect between the amputated and contralateral sides at all joints. Excluding the
Figure 5.4. The step length for each weighting factor used to find gaits for the asymmetric biped model. The step length when the amputated leg is in stance (“A” in the legend) is indicated with the open symbols, and the step length when the contralateral leg is in stance (“C” in the legend) is indicated with the filled symbols. For the weighting factor of 0, the amputated and contralateral values completely overlap. As the weighting factor increases, efficiency becomes more important.
most efficient gaits (weighting factor of 1), the mean correlation over all joints and weighting factors is 0.995, and the minimum correlation is 0.969. This means the shape of the joint trajectories as a function of step percentage is almost identical between the amputated and contralateral sides. Gaits designed with only the efficiency objective function have much lower correlations at all joints. The correlations at the hip are still quite high (> 0.95), but the correlations at the knee are very low. As a result, for the most efficient gaits, the motion of the amputated knee differs drastically from the motion of the contralateral knee. Regardless of objective function, the amputated/contralateral hip motion is more symmetric than the amputated/contralateral knee motion. This is not surprising because the range of feasible hip motions is much smaller than the range of feasible knee motions due to the fact that the hip must begin the stance phase flexed and end the stance phase extended.

In general, as the weighting factor increases, the differences between the peak amputated and contralateral joint angles increase (Fig. 5.5, Appendix D). For all joints, the contralateral angles tend to be more extended, which is not surprising considering the longer contralateral steps. The difference in peak hip angles were generally less than 10°, with means of about 4° regardless of speed or weighting factor, although the fastest speed tended to have the largest differences. At the knee, stance knee extension had very small differences between the amputated and contralateral sides, with means of 0.3° for the slow speed, 1.7° for the moderate speed, and 6.5° for the fast speed. There was no consistent effect of weighting factor. The maximum stance knee flexion was highly variable and also had no clear trend with weighting factor. The differences ranged from 0° to 18° with means of 1.7° for the slow speed, 4.1° for the moderate speed, and 10.3° for the fast speed. The largest effect of the weighting factor was seen in the difference between the amputated and contralateral peak swing knee flexion, with an increase of about 10° for a given speed.
### TABLE 5.2

**CORRELATION BETWEEN THE AMPUTATED AND CONTRALATERAL JOINT KINEMATICS FOR THE ASYMMETRIC BIPED MODEL**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Weighting Factor</th>
<th>Slow Speed</th>
<th>Moderate Speed</th>
<th>Fast Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stance Hip</strong></td>
<td>0.00</td>
<td>0.9999</td>
<td>0.9998</td>
<td>0.9992</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.9994</td>
<td>0.9994</td>
<td>0.9996</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.9992</td>
<td>0.9985</td>
<td>0.9987</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.9981</td>
<td>0.9982</td>
<td>0.9959</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.9908</td>
<td>0.9707</td>
<td>0.9800</td>
</tr>
<tr>
<td><strong>Swing Hip</strong></td>
<td>0.00</td>
<td>0.9998</td>
<td>0.9998</td>
<td>0.9807</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1.0000</td>
<td>0.9985</td>
<td>0.9881</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.9997</td>
<td>0.9967</td>
<td>0.9961</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.9933</td>
<td>0.9962</td>
<td>0.9704</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.9658</td>
<td>0.9540</td>
<td>0.9659</td>
</tr>
<tr>
<td><strong>Stance Knee</strong></td>
<td>0.00</td>
<td>0.9996</td>
<td>0.9992</td>
<td>0.9981</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.9982</td>
<td>0.9975</td>
<td>0.9996</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.9946</td>
<td>0.9917</td>
<td>0.9955</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.9881</td>
<td>0.9686</td>
<td>0.9935</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.8410</td>
<td>-0.0698</td>
<td>-0.0477</td>
</tr>
<tr>
<td><strong>Swing Knee</strong></td>
<td>0.00</td>
<td>0.9996</td>
<td>0.9993</td>
<td>0.9984</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.9979</td>
<td>0.9976</td>
<td>0.9996</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.9950</td>
<td>0.9956</td>
<td>0.9955</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.9890</td>
<td>0.9750</td>
<td>0.9934</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.7530</td>
<td>-0.9156</td>
<td>-0.2685</td>
</tr>
</tbody>
</table>

**NOTE:** As the weighting factor increases, efficiency becomes more important.
Figure 5.5. Comparison of asymmetric biped model gaits designed with different objective functions. The left column contains the hip angles; the right column contains the knee angles. The amputated joint angles are indicated with the solid line; the contralateral angles are indicated with the dashed line. The stance phase occurs from 0 to 1; the swing phase occurs from 1 to 2. As the weighting factor \( w \) increases, efficiency becomes more important.
as the weighting factor increased from 0 to 0.95. Interestingly, the difference drops about 10° as the weighting factor increased from 0.95 to 1. Swing knee flexion also had the largest differences in general, with differences ranging from 1° to 23°. Once again, the fastest speed tended to have larger differences than the slower speeds. This is not surprising because the effect of toe-off is more significant at faster speeds, so one would expect greater differences in symmetry for faster speeds. In predicting normal human gait, the hardest features to capture were the peak knee flexions, so it is not surprising that the largest differences between sides in asymmetric gait occur for peak knee flexion.

For the most efficient gaits, the model’s intact ankle has an excessive range of motion during stance, particularly for faster speeds (Fig. 5.6). At the fastest speed, the model’s ankle range of motion is 50°. For normal gait, the ankle range of motion during stance is approximately 20° even for fast walking speeds (Schwartz et al., 2008). The excessive ankle motion likely occurs because the ankle joint is very effective at driving the hip forward (Anderson and Pandy, 2003) and increasing step length. For normal gait, the weighting factors in the objective function (Eq. 4.29) balance the work among joints, and as a result, the model’s ankle range of motion is very similar to the physiological range of motion. For an amputee, the optimal contralateral step length is fairly long, and as a result, the optimal ankle work increases. Since the optimization minimizes torque squared, increasing the range of motion of the ankle increases the ankle work without necessarily increasing the torque. In a human, the ankle joint itself prevents excessive motion, but the model has no such built-in limitations. Instead, the excessive motion must be prevented using constraints, which was not done for these gaits.

Taken together, if the symmetry term is included in the objective function even with a very small weighting factor, kinematic symmetry between the amputated and contralateral sides is generally very high. As the importance of symmetry decreases,
the gait does become less symmetric, but the changes are small enough that they have no practical significance. On the other hand, if the symmetry term is completely omitted from the objective function, kinematic symmetry is significantly reduced.

5.2.2 Efficiency Results

Mean absolute power (Eq. 4.31) significantly decreases as the importance of efficiency increases (Fig. 5.7). Unlike the symmetry metrics, mean absolute power decreases in an approximately quadratic manner as the weighting factor increases. From the least efficient to the most efficient gaits, the required power is approximately halved for a given speed. Consistent with both normal (Burdett et al., 1983) and amputee (Genin et al., 2008) experimental gait, the mechanical power required to walk increases as speed increases.

In general, the power required per step depends on which leg is in stance (Fig. 5.8). For all but the most symmetric gaits, the contralateral step requires more power. This
is not surprising because when the contralateral leg is in stance, the intact ankle can inject significant energy into the gait cycle. In contrast, for the most symmetric gaits, the amputated step requires more power. This is also not surprising considering the matching results for normal human gait (Section 4.6). Much like the base model required more energy to match normal human gait than the ankles model did, the amputated step also requires more energy than the contralateral step for the most symmetric gaits.

5.3 Summary

For a planar asymmetric model controlled using HZD-based control walking at a given speed, energetic efficiency significantly decreases as the importance of kinematic symmetry increases. Likewise, kinematic symmetry decreases as the importance of energetic efficient increases for a given speed. The trade-off between the two strategies is such that small decreases in kinematic symmetry result in large increases in
energetic efficiency. As a result, it is possible to design a gait that is both fairly efficient and fairly symmetric. This indicates that amputees or other similar asymmetric bipeds should be physically capable of walking in an approximately symmetric manner without unreasonably increasing the energy required for gait. In order to obtain such a gait, the objective function must contain terms for both symmetry and efficiency, with the efficiency term having a much larger weight.
This work has extended the HZD-based modeling and control technique to bipeds with curved feet (Chapter 2). The results were successfully validated using both existing and new biped hardware experiments. The extended control technique was then utilized to systematically investigate the effects of foot design on robot gait (Chapter 3). The major finding was that similar energetic benefits across a range of speeds can be achieved with either a larger radius foot or a smaller radius foot whose center of curvature is located forward of the shank. The model can also be used to predict normal human walking at speeds ranging from very slow to very fast using a torque-squared-based objective function (Chapter 4). Extending the model further allows the investigation of asymmetric amputee gait. Specifically, it was shown that if the objective function used to generate gait contains a heavily-weighted efficiency term and a kinematic symmetry term, the asymmetric model can walk in a manner that is both fairly symmetric and fairly efficient (Chapter 5). Omitting the symmetry term results in a drastic decrease in symmetry. Each aspect of the work can be further extended.

6.1 Future Work Relating to HZD-Based Control

The ability to model and control bipeds with variable radii feet would increase the possible design space. If it is assumed that the foot radius is a known function of configuration, the changes to both the controller and the model are expected to be modest. The control method should remain essentially the same because the
derivation does not need to assume constant foot radius – it merely requires the form of the equations to remain the same. Similarly, the model should not require significant changes. The complexity of the equations, however, is likely to increase significantly, particularly in the $C$ matrix of Eq. 2.1.

The assumption that foot radius is a known function of configuration effectively limits the feet to be rigid bodies. While having compliant feet could further increase the design space and could possibly be more relevant to prosthesis design, it is likely to produce significant complications in both the model and the control. Because the foot radius would not be known, an iterative approach would be required in which the foot radius is estimated, the forces at the foot are calculated and then the foot radius is updated. This is feasible for simulation; however, it is not clear how the zero dynamics could be calculated without performing a full simulation first. It is also possible that the impact map would no longer be linear in joint angular velocity. If this is the case, it would require significant changes to the HZD stability analysis. It is the opinion of the author that the increased difficulty and complexity of modeling bipeds with compliant feet will not produce a corresponding increase in performance, and thus, it is not worth the effort to model compliant feet. Instead, it is likely that many of the effects of compliance can be modeled using non-constant radii feet in combination with an ankle joint. However, the extension of HZD to allow compliant feet could be an interesting exercise.

6.2 Future Work Relating to Foot Design

While the experiments presented in Chapter 3 covered a large range of the design space, additional experiments in both hardware and simulation could give a more complete characterization of the effect of foot design on robotic biped gait. Specifically, the effects of step length and negative ankle offset should be investigated. Because only three distinct foot radii, ankle offset and speeds were tested, the
fits were limited to linear fits. There was some evidence that quadratic fits would be more appropriate. Performing additional experiments both within the currently studied range and beyond it would allow additional, potentially better, fits to be considered. It would also help to determine how extreme the foot design can become before the fits are no longer accurate. Due to the number of potential combinations of foot radius, ankle offset, walking speed, and step length, a full-factorial study will not be feasible. Instead, an optimal study design (Chaloner and Verdinelli, 1995) will need to be used.

6.3 Future Work Relating to Modeling Healthy Human Gait

To further improve the model’s ability to predict normal human gait, two enhancements to the method can be made. By developing a computationally feasible method to quantify the robustness of a gait to perturbations, the accuracy of the stance knee motion might be improved. By augmenting the objective function, the ability to predict gait at the (unspecified) self-selected walking speed as a function of age could be achieved.

As seen in Chapter 2, HZD-based control is a very robust control method in the sense that it is able to generate stable walking even in the presence of large modeling errors. However, just because the system can achieve stable walking does not mean that the actual efficiency is similar to the designed efficiency. For example, in a pilot study using a work-based objective function (Appendix E), small changes in the ankle offset led to extremely large changes in the required knee work. Since human gait is known to be variable (Kang and Dingwell, 2008) and since there is noise in the feedback loop (Harris and Wolpert, 1998), it seems likely that human gait patterns are insensitive to both geometric parameter variation and signal noise. Based on the results of the pilot study and on the fact that the model matches stance knee motion less well than the other joints, it is hypothesized that including a robustness
term in the objective function could improve the model’s predictive performance. In addition, these results could be very useful in the design of robot gaits.

Because walking speed is a common measure of walking ability (Andriacchi et al., 1977; Hermodsson et al., 1994; Steffen et al., 2002), being able to predict the self-selected walking speed would increase the usefulness of the model and make it easier to compare the results to clinical studies. As people age, they typically walk more slowly (Alexander, 1996). However, the rate of metabolic energy expenditure at the self-selected speed does not increase compared to young adults (Mian et al., 2006). Similarly, for amputees, the rate of metabolic energy expenditure at the self-selected speed typically does not increase compared to healthy, young adults (Waters and Mulroy, 1999). This indicates that there is a preferred rate of metabolic energy expenditure and people walk at the speed that requires this cost. There are two potential ways to capture this preference in the model. The first method is to determine appropriate weighting factors for an objective function containing an energetic cost term based on the current objective function (Eq. 4.29), a robustness term based on the work in the previous paragraph, and a new speed term. The second method is to change the objective function to a weighted combination of robustness and speed and use energetic cost as a constraint. Presumably, for both methods, the weighting factors would change as subjects age, which could provide insights into why elderly adults walk differently than young adults (DeVita and Hortobagyi, 2000).

6.4 Future Work Relating to Modeling Amputee Gait

Currently, stability is not formally checked for the asymmetric model. This should be added. To do so, the methods developed for aperiodic gait on symmetric bipeds can likely be used (Yang et al., 2009).

This work established that it is physically possible for an asymmetric biped to walk in a manner that is both reasonably efficient and reasonably symmetric. How-
ever, no effort was made to determine if such a gait is similar to experimental amputee
gait or if an objective function in the form of Eq. 5.6 is capable of predicting amputee
gait. This is an obvious extension to the current work. Before the predictive objec-
tive function is found, however, the choice of components in the objective function
should receive further attention. The efficiency objective function is likely adequate
because it is identical (except for the omitted the amputated ankle term) to the ob-
jective function for healthy human gait. The symmetry objective function should
be modified. Considering the results in Section 5.2.1, the weighting factor for speed
and possibly step length should be reduced. In addition, separate weighting factors
for each correlation term might need to be used. All of these weighting factors can
likely be estimated using amputee gait. By determining the approximate size of each
symmetry term for experimental amputee gait, the weighing factors can be chosen
so that each term in the objective function is approximately equal. A third objective
function might also need to be added. The symmetry objective function does not
penalize non-human-like gaits; it only penalizes decreases in amputated/contralateral
symmetry. As a result, it is possible to have a very symmetric gait that looks nothing
like a human gait, which is obviously not an acceptable amputee gait. An objective
function that compares the amputee gait to a size- and speed-matched healthy hu-
man gait could solve this issue. It is expected that this objective function will be
a function of step length, average speed, and joint angles, evaluated either using
Pearson’s correlation coefficient or by taking the norm of the difference in the poly-
nomial coefficients that define the actuated joint motion. This objective function
should prevent unreasonable intact ankle motions, but if it does not, the joint limit
constraints on the ankles will be reduced. To predict amputee gait, it is expected
that a linear combination of the three objective function will be used as the overall
predictive function.

\[ g = c_{eff}g_{eff} + c_{sym}g_{sym} + c_{HH}g_{H}, \]  

(6.1)
where $c_k$ are weighting factors to be determined, $g_{eff}$ is the efficiency metric given by the objective function used to predict normal human gait (Eq. 4.29), $f_{sym}$ is the modified symmetry objective function that measures the similarity of the amputated and contralateral steps, and $f_H$ is the new measure of similarity between the contralateral step and a healthy human step.

To validate the objective function, experimental amputee gaits at a variety of speeds will be needed. These gaits should be divided into a training pool used to determine the weighting factors and a validation pool.

To determine how well the model is able to predict amputee gait, the following metrics can be calculated for both the amputated and contralateral sides:

- The error in the linear fit of step length as a function of average walking speed between the model and experiment;

- The error in the linear fit of one-step average walking speed as a function of one-stride average walking speed between the model and experiment;

- The error in peak joint angles between the model and experiment for hip flexion, hip extension, early stance knee flexion, stance knee extension, and swing knee flexion;

- Pearson’s correlation coefficient between the model and experimental joint angles for the hip and knee joints; and

- The error in the linear fit of mean absolute power as a function of one-stride average walking speed between the model and experiment.

It is expected that the error in the linear fits will not be significantly different from zero, the correlation will be greater than 0.95 for the hip, and the mean error in peak joint angles will be less than 5°. In addition, the differences between the training and
validation groups can be calculated. In all cases, it is expected that the differences will not be statistically different than zero.

Once a predictive model of amputee gait has been developed, it can be used to investigate the effects of foot design and alignment on amputee gait. Previous work on the design and alignment of prostheses has focused on the effect of changing the form of the foot, which results in simultaneously varying all of the functions of the foot and ankle and could contribute to the lack of statistically significant results (Neumann, 2009) even though amputees have consistent preferences (Hafner et al., 2002). For the sagittal plane, the anterior-posterior alignment of the prosthetic foot and the compliance of the foot and ankle have typically been varied independently. These parameters affect three model parameters controlling the function of the foot and ankle - the radius of the foot, the ankle offset and the stiffness of the ankle (Hansen, 2008). To determine how foot properties affect amputee gait, the foot parameters could be varied in simulation. The gains in efficiency and amputated-contralateral symmetry with increasing work at the amputated ankle could also be quantified. To validate the findings, experiments could be conducted using healthy subjects wearing a tethered robotic prosthetic foot similar to the one in Caputo and Collins (2014).

Investigating the effect of ankle stiffness will not be trivial because the passive ankle adds an additional degree of underactuation. The HZD-based control method used for the model assumes a single degree of underactuation, although it has been extended to more than one degree of underactuation for certain cases (Chevallereau et al., 2005, 2009). Unfortunately, there is not a straightforward translation from the prior work to the amputee model. To capture the second degree of underactuation, either the control method could be extended or the ankle could be treated as if it were actuated and constraints used to remove non-spring-like torques (Erez and Todorov, 2012). The second method worked well in a pilot study using a very simple model.
APPENDIX A

MODEL OF TWO-LINK BIPED

The matrices required to model the two-link, curved foot biped shown in Fig. 2.3.

A.1 Single Support Equations

The vectors and matrices in Eq. 2.1 are given below for the two-link walker.

*Generalized Coordinates:*

\[ \mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T \quad \text{(A.1)} \]

*Inertia Matrix:*

\[ \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix} \quad \text{(A.2)} \]

where

\[
\begin{aligned}
d_{11} &= m \left\{ L_{COM}^2 + r_{Gyr}^2 \right\} \\
d_{12} &= m \left\{ LL_{COM} \cos q_1 + L_{COM} X_F \sin q_1 - (L_{COM}^2 - r_{Gyr}^2) \\
&\quad + R_0 L_{COM} (\cos(q_1 - q_2) - \cos q_1) \right\} \\
d_{22} &= m \left\{ ((L - R_0) \cos q_2 + X_F \sin q_2 - L_{COM} \cos(q_1 - q_2) + R_0)^2 \\
&\quad + ((L - R_0) \sin q_2 - X_F \cos q_2 + L_{COM} \sin(q_1 - q_2))^2 \\
&\quad + ((R - L + L_{COM}) \sin q_2 + X_F \cos q_2)^2 \\
&\quad + (R_0 + X_F \sin q_2 - (R_0 - L + L_{COM}) \cos q_2)^2 + 2r_{Gyr}^2 \right\}
\end{aligned}
\]
Coriolis and Centripital Matrix:

\[
\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]  

(A.3)

where

\[
\begin{align*}
c_{11} &= 0 \\
c_{12} &= mL_{COM}(R_0 \sin q_1 - L \sin q_1 + X_F \cos q_1) \dot{q}_2 \\
c_{21} &= -mL_{COM}(L \sin q_1 - R_0 \sin q_1 - X_F \cos q_1 + R_0 \sin(q_1 - q_2)) \cdot (\dot{q}_1 - \dot{q}_2) \\
c_{22} &= mX_F(2R_0 \dot{q}_2 \cos q_2 - L_{COM} \dot{q}_1 \cos q_1) + 2m(R_0^2 - LR_0) \dot{q}_2 \sin q_2 \\
&\quad + mR_0 L_{COM}(\dot{q}_1 - \dot{q}_2) \sin(q_1 - q_2) - mR_0 L_{COM}(\dot{q}_1 \sin q_1 - \dot{q}_2 \sin q_2) \\
&\quad + mL \dot{L}_{COM} \dot{q}_1 \sin q_1
\end{align*}
\]

Gravity Vector:

\[
\mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T
\]

(A.4)

where

\[
\begin{align*}
g_1 &= -mgL_{COM} \sin(\gamma - q_1 + q_2) \\
g_2 &= mg \{L_{COM} \sin(\gamma - q_1 + q_2) + 2(R_0 - L + L_{COM}) \sin(\gamma + q_2) \\
&\quad + 2X_F \cos(\gamma + q_2)\}
\end{align*}
\]

Actuator Mapping: For a passive biped,

\[
\mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T.
\]

(A.5)
For a biped with actuation at the hip,

\[ B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T. \quad (A.6) \]

A.2 Impact Map

The complete impact model (Eq. 2.2) for a two-link walker is given below.

**Extended Coordinates:**

\[ q_e = \begin{bmatrix} q_1 & q_2 & p_x & p_y \end{bmatrix}^T \quad (A.7) \]

where

\[ \dot{p}_x = \dot{q}_2 (R_0 + (L - R_0) \cos q_2 + X_F \sin q_2) \]
\[ \dot{p}_y = -\dot{q}_2 ((L - R_0) \sin q_2 - X_F \cos q_2) \]

**Switching Matrix:**

\[ S = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \quad (A.8) \]
Velocity Map:

\[ A = S(\Lambda_{11} - \Lambda_{12}E_1S), \]  
\[ \Lambda = I_{(N+2) \times (N+2)} - D_e^{-1}E^T(ED_e^{-1}E^T)^{-1}E \]  
\[ = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \]  
\[ E_1 = \begin{bmatrix} R_0 + (L - R_0) \cos q_2 + X_F \sin q_2 & -R_0 - (L - R) \cos q_2 - X_F \sin q_2 \\ X_F \cos q_2 - (L - R_0) \sin q_2 & (L - R_0) \sin q_2 - X_F \cos q_2 \end{bmatrix} \]  

where \( I \) is the \((N + 2) \times (N + 2)\) identity matrix, \( \Lambda_{11} \) is an \( N \times N \) matrix, and \( \Lambda_{12} \) is an \( N \times 2 \) matrix.

System of Equations: The velocity portion of the impact map can also be computed by solving Eq. 2.5 and 2.6.

\[ D_e \dot{q}_e^+ - E^Tf = D_e \dot{q}_e^- \]  
\[ Eq_e^+ = 0 \]  

(2.5 & 2.6 repeated)

After solving for \( \dot{q}_e^+ \) and \( f \), switch the coordinates

\[ q_f = Sq^+ \]  
\[ \dot{q}_f = S\dot{q}_e^+, \]  

where \( q^+ \) is the vector of joint angles at the instant after impact but before the coordinates have been switched and \( q_f \) is the vector of joint angles at the instant after impact after the coordinates have been switched.
Impulsive Forces:
\[
f = \begin{bmatrix} f_T & f_N \end{bmatrix}^T
\]  \hspace{1cm} (A.14)

Inertia Matrix:
\[
D_e = \begin{bmatrix}
  d_{11} & d_{12} & d_{13} & d_{14} \\
  d_{12} & d_{22} & d_{23} & d_{24} \\
  d_{13} & d_{23} & d_{33} & d_{34} \\
  d_{14} & d_{24} & d_{34} & d_{44}
\end{bmatrix}
\]  \hspace{1cm} (A.15)

where
\[
\begin{align*}
d_{11} &= m(L_{COM}^2 + r_{Gyr}^2) \\
d_{12} &= -m(L_{COM}^2 + r_{Gyr}^2) \\
d_{13} &= mL_{COM} \cos(q_1 - q_2) \\
d_{14} &= mL_{COM} \sin(q_1 - q_2) \\
d_{22} &= 2m(L_{COM}^2 + r_{Gyr}^2) \\
d_{23} &= -mL_{COM}(\cos(q_1 - q_2) + \cos q_2) \\
d_{24} &= mL_{COM}(\sin q_2 - \sin(q_1 - q_2)) \\
d_{33} &= 2m \\
d_{34} &= 0 \\
d_{44} &= 2m
\end{align*}
\]

Constraint:
\[
E = \begin{bmatrix}
r_{F \rightarrow H,y} & -r_{F \rightarrow H,y} & 1 & 0 \\
-r_{F \rightarrow H,x} & r_{F \rightarrow H,x} & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (2.9 repeated)
where

\[ r_{F \rightarrow H, x} = -(L - R_0) \sin(q_1 - q_2) - X_F \cos(q_1 - q_2) \]

\[ r_{F \rightarrow H, y} = (L - R_0) \cos(q_1 - q_2) - X_F \sin(q_1 - q_2) + R_0. \]
APPENDIX B

EFFECT OF ROBUSTNESS CONSTRAINTS ON ERNIE’S GAIT

There are several differences between the optimal gaits for ERNIE when the robustness constraints are or are not used. One of the differences is the stance hip motion. Because of the robustness constraint requiring stance hip flexion at the start of the step, the stance hip first flexes and then extends (Fig. 3.15), whereas for the gaits without the robustness constraints, the stance hip just extends (Fig. 3.5(a)). The hip motion for the rest of the stride is similar for both sets of gaits.

The motion of the knee is significantly different between the two sets of gaits. For gaits without the robustness constraints, the swing knee motion is very smooth, with approximately constant velocity during the last half of the swing phase (Fig. 3.5(b)). For gaits with the robustness constraints, because of the constraint requiring minimal swing lower leg motion at the end of the step, the swing knee extends quickly near the middle of the step before slowing down near the end of the step (Fig. 3.16). The stance knee motion is also drastically different between the two sets of gaits. Because the objective function is work-based, there are two possible strategies for minimizing cost – either use minimal torque or have minimal motion. The gaits without the robustness constraints use the minimal torque method, which results in significant stance knee motion, while the gaits with the robustness constraints use the minimal motion method. For gaits with the robustness constraints, since the upper leg must move in the direction opposite that required for passive motion, it is likely that the minimal stance knee torque strategy will not yield acceptable gaits. As a result, the optimal gait with robustness constraints cannot use the minimal
torque strategy and so must use the minimal motion strategy. As a result, without the robustness constraints, the stance knee has significant motion, while with the robustness constraints, the stance knee has very little motion.

Not surprisingly, the gaits without the robustness constraints have a lower SECT than the gaits with the robustness constraints. Both sets of gaits predict that more efficient gaits can be designed with a larger radius foot. The effect of ankle offset on energy efficiency differs between the two sets of gaits. Due to the fact that each set utilizes a different strategy for minimizing stance knee work and given that stance knee work is a major factor in how ankle offset influences the SECT, it is not surprising that the effect of ankle offset on gait differs between the two sets of gaits.
APPENDIX C

SUPPLEMENTARY MATERIAL FOR CHAPTER 4

C.1 Supplementary Figures for the Pilot Study with the Base and Impulsive Models

C.1.1 Plots of Gaits for Pilot Study

Fig. C.1 shows the gaits found using step definition 1 and the impulsive model. Fig. C.2 shows the gaits found using step definition 2 and the impulsive model. Fig. 4.6 shows the gaits found using step definition 3 and the impulsive model. Fig. C.3 shows the gaits found using step definition 4 and the impulsive model. Note that the plots use the model angles, not the transformed biomechanics angles.
Figure C.1. Comparison of the impulsive model gaits using step definition 1. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using $f_1$, the green line shows the gait found using $f_2$, the black line shows the gait found using $f_3$, the pink line shows the gait found using $f_4$, and the yellow line shows the gait found using $f_5$. 
Figure C.2. Comparison of the impulsive model gaits using step definition 2. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using \( f_1 \), the green line shows the gait found using \( f_2 \), the black line shows the gait found using \( f_3 \), the pink line shows the gait found using \( f_4 \), and the yellow line shows the gait found using \( f_5 \).
Figure C.3. Comparison of the impulsive model gaits using step definition 4. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the free knee angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using $f_1$, the green line shows the gait found using $f_2$, the black line shows the gait found using $f_3$, the pink line shows the gait found using $f_4$, and the yellow line shows the gait found using $f_5$. 
C.1.2 Details on the Comparison of Different Matching Functions

To evaluate which step definition and objective function performed best, a series of four comparisons was made. A graphical representation of the results is shown in Fig. C.4. In the first comparison, the results for the base model were compared by holding the objective function used in the optimization constant and varying the step definition. The step definition that produced a gait with the step length closest to the experimental value was determined. In Fig. C.4, this step definition is indicated with the thin-lined, filled orange circles. Similarly, the step definition(s) that produced a gait with the average speed closest to the experimental average speed was determined and marked on Fig. C.4 with thin-lined, filled orange triangles. The step definition that had the lowest value of objective function one \( f_1 \) was determined and marked on Fig. C.4 with thin-lined, filled orange squares. In the same manner, the step definitions that had the lowest values of objective functions two \( f_2 \), three \( f_3 \), and four \( f_4 \) were also determined and marked on Fig. C.4 with thin-lined, filled orange stars \( f_2 \), diamonds \( f_3 \), and crosses \( f_4 \). A second, similar comparison was performed for the base model except that this time, the definition of the step was held constant, and the objective function used in the optimization was varied. In Fig. C.4, these results are shown with the thin-lined, open teal shapes. The final two comparisons used the impulsive model. In the first comparison, the objective function used in the optimization was held constant, and the definition of the step was varied. In Fig. C.4, these results are shown with the thick-lined, filled red shapes. In the second comparison, the definition of the step was held constant, and the objective function used in the optimization was varied. In Fig. C.4, these results are shown with the thick-lined, open blue shapes.

Additional details on Fig. C.4 are provided here. Each cell represents one combination of objective function and step definition. If that combination won a comparison, a symbol was placed in the cell. The more symbols a cell contains, the more closely
Figure C.4. Graphical representation of the data in Tables 4.7 and 4.8. Note that optimizations using objective function five were not performed on the base model.
it matched the experimental data. In each cell, the top line shows the results of the comparison between the step definition sets with the objective function held constant for the base model. The second line shows the results of the comparison between the objective functions with the step definition set held constant for the base model. The third line shows the results of the comparison between the step definition sets with the objective function held constant for the impulsive model. The bottom line shows the results of the comparison between objective functions with the step definition set held constant for the impulsive model.

The shape and color of the symbol indicates what is being compared. Filled symbols mean the objective function used in the optimization was held constant and the step definition was varied. Empty symbols mean the step definition was held constant and the objective function used in the optimization was varied. Considering the shapes, a circle means the model gait matched the experimental step length, and a triangle means the model gait matched the experimental speed. A square means the gait had the lowest value of objective function 1, a star means the gait had the lowest value of objective function 2, a diamond means the gait had the lowest value of objective function 3, a cross means the gait had the lowest value of objective function 4, and a heart means the gait had the lowest value of objective function 5.

For example, for the base model, the gait using step definition 4 and optimized with objective function 1 matched the experimental step length more closely than any of the other base models also optimized using objective function 1. This is indicated by the filled orange circle in the top right cell. As a second example, for the impulsive model, the gait using step definition 3 and optimized with objective function 5 had the lowest value of objective function 5 as compared to the other impulsive models also using step definition 3, as indicated with the empty blue heart in that cell.
C.2 Supplementary Figures for the Pilot Study with the Ankles Model

Fig. C.5 shows the gaits found using step definition 3 and the ankles model at a slow speed. Fig. 4.7 shows the gaits found using step definition 3 and the ankles model at the natural speed. Fig. C.6 shows the gaits found using step definition 3 and the ankles model at a fast speed. Note that the plots use the model angles, not the transformed biomechanics angles.
Figure C.5. Comparison of the model gaits for the ankles model walking at a slow (0.83 m/s) speed using step definition 3. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using \( f_5 \), the green line shows the gait found using \( f_6 \), and the black line shows the gait found using \( f_7 \).
Figure C.6. Comparison of the model gaits for the ankles model walking at a fast (1.59 m/s) speed using step definition 3. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The blue line shows the experimental data, the blue dashed lines show plus or minus one standard deviation of the experimental data, the red line shows the gait found using $f_5$, the green line shows the gait found using $f_6$, and the black line shows the gait found using $f_7$. 
C.3 Plots of All Matching Gaits

Figs. C.7-C.11 show all of the matching gaits for the base model grouped by speed. Figs. C.12-C.16 show all of the matching gaits for the impulsive model grouped by speed. Figs. C.17-C.21 show all of the matching gaits for the ankles model grouped by speed. Note that the plots use the model angles, not the transformed biomechanics angles.
Figure C.7. Comparison of the experimental and simulated gaits for the base model walking very slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.8. Comparison of the experimental and simulated gaits for the base model walking slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.9. Comparison of the experimental and simulated gaits for the base model walking at a moderate speed. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.10. Comparison of the experimental and simulated gaits for the base model walking fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.11. Comparison of the experimental and simulated gaits for the base model walking very fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.12. Comparison of the experimental and simulated gaits for the impulsive model walking very slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.13. Comparison of the experimental and simulated gaits for the impulsive model walking slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.14. Comparison of the experimental and simulated gaits for the impulsive model walking at a moderate speed. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.15. Comparison of the experimental and simulated gaits for the impulsive model walking fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.16. Comparison of the experimental and simulated gaits for the impulsive model walking very fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, and (d) shows the swing knee angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.17. Comparison of the experimental and simulated gaits for the ankles model walking very slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.18. Comparison of the experimental and simulated gaits for the ankles model walking slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.19. Comparison of the experimental and simulated gaits for the ankles model walking at a moderate speed. (a) shows the absolute angle of the stance leg, (b) shows the hip splay; (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.20. Comparison of the experimental and simulated gaits for the ankles model walking fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
Figure C.21. Comparison of the experimental and simulated gaits for the ankles model walking very fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the simulated gait and the dashed lines show the experimental gait.
C.4 Plots of All Prediction Gaits

Figs. C.22-C.26 show all of the matching gaits for the ankles model grouped by speed. Note that the plots use the model angles, not the transformed biomechanics angles.
Figure C.22. Comparison of the experimental and predicted gaits for the ankles model walking very slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the predicted gait and the dashed lines show the experimental gait.
Figure C.23. Comparison of the experimental and predicted gaits for the ankles model walking slowly. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the predicted gait and the dashed lines show the experimental gait.
Figure C.24. Comparison of the experimental and predicted gaits for the ankles model walking at a moderate speed. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the predicted gait and the dashed lines show the experimental gait.
Figure C.25. Comparison of the experimental and predicted gaits for the ankles model walking fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the predicted gait and the dashed lines show the experimental gait.
Figure C.26. Comparison of the experimental and predicted gaits for the ankles model walking very fast. (a) shows the absolute angle of the stance leg, (b) shows the hip splay, (c) shows the stance knee angle, (d) shows the swing knee angle, (e) shows the stance ankle angle, and (f) shows the swing ankle angle. The solid lines show the predicted gait and the dashed lines show the experimental gait.
APPENDIX D

SUPPLEMENTARY MATERIAL FOR CHAPTER 5

Figs. D.1-D.3 show all of the asymmetric gaits grouped by speed.
Figure D.1. Comparison of asymmetric biped model gaits at 0.8 m/s designed with different objective functions. The amputated joint angles are indicated with the solid line; the contralateral angles are indicated with the dashed line. The stance phase occurs from 0 to 1; the swing phase occurs from 1 to 2. As the weighting factor $w$ increases, efficiency becomes more important.
Figure D.2. Comparison of asymmetric biped model gaits at 1 m/s designed with different objective functions. The amputated joint angles are indicated with the solid line; the contralateral angles are indicated with the dashed line. The stance phase occurs from 0 to 1; the swing phase occurs from 1 to 2. As the weighting factor $w$ increases, efficiency becomes more important.
Figure D.3. Comparison of asymmetric biped model gaits at 1.2 m/s designed with different objective functions. The amputated joint angles are indicated with the solid line; the contralateral angles are indicated with the dashed line. The stance phase occurs from 0 to 1; the swing phase occurs from 1 to 2. As the weighting factor $w$ increases, efficiency becomes more important.
APPENDIX E

PILOT STUDY ON THE EFFECT OF ALIGNMENT PERTURBATIONS WHEN GAIT PATTERNS ARE CONSTANT

It has been observed that amputees do not change their self-selected speed or gait cadence when their prostheses are incorrectly aligned (Neumann, 2009). Further, the peak ground reaction forces (GRF) do not appear to depend on alignment. It has also been observed that incorrectly aligned prostheses increase stump loading (Sanders et al., 1998) and likely increase the arthritis risk in the contralateral limb (Hurley et al., 1990). Of these experimental findings, the most convincing result is for self-selected speed, and the least convincing result is for arthritis risk.

To explore if simple planar models can capture these results, a pilot study of the effects of ankle offset ($X_f$) perturbations on gait when the gait kinematics were held constant was conducted. Because this pilot study occurred before the development of the asymmetric amputee model, symmetric models were used. Further, the objective function for normal human gait had not yet been developed, so the objective function was specific energetic cost of transport (Eq. 2.81).

E.1 Methods

Both two- and five-link models were utilized (Fig. E.1). For the two-link model, the mass and geometric properties (Table 3.1) were chosen to match the two-link biped in Westervelt et al. (2007). The foot radius was set to 0.30 m. For the five-link model, the mass and geometric properties (Table 1.2) were chosen to match ERNIE’s. For both models, the nominal ankle offset was 0 mm, and the nominal
speed was 0.5 m/s. To perturb the system, the ankle offset was changed from the nominal value in small increments until steady-state gait could not be achieved. In the most relevant experimental studies (Sanders et al., 1998; Schmalz et al., 2002), the foot anterior/posterior alignment was shifted by about 20 mm.

It is not possible to exactly match the kinematics of the perturbed system to the kinematics of the nominal system. At the very least, the configuration at impact must be different. There are several potential methods that can be used to keep the gait kinematics the same. In this pilot study, three different methods were used.

1. Keep the output function exactly the same regardless of what the ankle offset is and simulate until the biped either falls over or reaches steady-state.

2. Keep the control parameters ($\alpha_2...Q$) the same, but change the initial state and the dependent polynomial coefficients such that the controller guarantees a steady-state gait. This keeps the joint angles the same as a function of $\theta$ but not necessarily time.
### TABLE E.1

ANKLE OFFSETS USED IN PERTURBATION PILOT STUDY

<table>
<thead>
<tr>
<th>Method</th>
<th>Two-Link</th>
<th>Five-Link</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td>1</td>
<td>-30</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>-6</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>-30</td>
<td>30</td>
</tr>
</tbody>
</table>

All values are given in mm. In all cases, when the maximum and minimum values do not have the same magnitude, it is because the biped was unable to achieve a steady-state gait at the missing ankle offset.

3. Determine a new set of control parameters such that the error between the actual and nominal joint angles and angular velocities as a function of time is minimized. This keeps the joint angles the same as a function of time but not necessarily $\theta$.

The ranges of ankle offsets used for each biped and method are given in Table E.1. Because some of the gait matching methods were more capable of producing steady-state gait in the perturbed model than others, the range of perturbed ankle offsets vary significantly. The maximum differences in joint angles as a function of normalized time (current time divided by step period) are given in Table E.2. For a particular method, the error increases as ankle offset is shifted farther from the nominal alignment. For the five-link biped, the differences are fairly small. The differences are larger for the two-link biped, but still within acceptable limits.
Because increased stump loading and arthritis risk are not quantities that can be
directly calculated in the model, they must be correlated with quantities that can be. Increased amputee stump loading is likely indicated by increased

- maximum stance knee torque and/or
- mean stance knee torque.

Increased arthritis risk is likely correlated with an increase in the

- impact impulse norm,
- maximum stance hip torque,
- maximum swing hip torque,
- maximum swing knee torque,
- mean stance hip torque,
- mean swing hip torque, and/or
- mean swing knee torque.
For the two-link model, there is only a single hip torque and no knee torque. Thus, an increase in the two-link biped’s hip torque could indicate increased stump loading and/or increased arthritis risk. In this study, the mean joint torque was approximated by the work at the joint. This is a reasonable approximation because the joint kinematics remained similar, so changes in work correspond to changes in joint torque.

E.2 Results

The effect of ankle offset perturbation on walking speed and cadence depends on the method used. Because the configurations of the bipeds at impact were almost the same regardless of ankle offset, step length remained essentially constant, which means step cadence and walking speed vary together. For both methods 1 and 2, the speed strongly depends on the ankle offset (Fig. E.2). For method 3, the speed does not depend on the ankle offset. However, it should be noted that in the optimization used to solve for the new control parameters in method 3, the average speed was constrained. Only method 3 matches the human experimental results because it is the only method that does not predict a change in speed.

For the two-link biped, method 1 produces large changes in the peak vertical GRF when the ankle offset is shifted forward (Fig. E.3(a)) but almost no change in peak vertical GRF when the ankle offset is shifted backward. This occurs because ankle offset primarily affects the GRF just after impact. For positive ankle offsets, the vertical GRF just after impact increases, while for negative ankle offsets it decreases relative to the nominal gait. For all ankle offsets, there is a local maximum near the middle of the gait cycle. With positive ankle offsets, the global maximum is just after impact instead of near the middle of the gait as for negative ankle offsets. Method 2 produces a small ($\approx 3\%$) increase in the peak vertical GRF as the ankle offset is shifted from negative to positive. The peak vertical GRF does not have a consistent trend for method 3. For the five-link biped, there are definite trends in the peak
Figure E.2. Change in average speed vs. ankle offset for the (a) two- and (b) five-link biped models in the perturbation study. Note the dramatic range in speeds exhibited.

Figure E.3. Change in peak vertical GRF vs. ankle offset for the (a) two- and (b) five-link biped models in the perturbation study. Blue circles indicate gaits found using method 1, green squares indicate gaits found using method 2, and red triangles indicate gaits found using method 3.
vertical GRF (Fig. E.3(b)). However, the magnitude of the change is so small that it is clinically irrelevant. This finding agrees with the amputee experimental results.

For both the two- and five-link bipeds, the magnitude of the impact impulse depends on the ankle offset (Fig. E.4). For methods 1 and 2, the change is likely due the change in average speed. For method 3, the impulse generally decreases as the ankle offset is shifted from behind the leg to in front of the leg. The magnitude of the change is fairly small ($\approx \pm 5\%$). This trend could explain an increased arthritis risk if the foot center of curvature is too far back.

For the two-link biped, the hip work varies quadratically with ankle offset (Fig. E.5). Although the variation is significant, the minimum does not occur when the ankle offset is zero, as would be expected if alignment perturbations affect either the arthritis risk and/or the stump loading. For method 1, the minimum occurs at $X_f = -7.8$ mm, for method 2, the minimum occurs at $X_f = -3.2$ mm, and for method 3, the minimum occurs at $X_f = -17$ mm. The maximum magnitude hip torque also does not provide convincing evidence for alignment perturbations affecting either the arthritis
Figure E.5. Change in hip work vs. ankle offset for the two-link biped model in the perturbation study. Blue circles indicate gaits found using method 1, green squares indicate gaits found using method 2, and red triangles indicate gaits found using method 3. The blue solid line is the least squares quadratic fit for method 1, the green dashed line is the least squares quadratic fit for method 2, and the red broken line is the least squares quadratic fit for method 3. In all cases, $R^2 > 0.99$.

There are huge changes in the stance knee torque for all three methods (Fig. E.7). Both the stance knee work and the maximum magnitude stance torque depend quadratically on ankle offset. In all cases, the minimum occurs near the nominal
Figure E.6. Change in (a) stance and (b) swing hip work vs. ankle offset for the five-link biped model in the perturbation study. Blue circles indicate gaits found using method 1, green squares indicate gaits found using method 2, and red triangles indicate gaits found using method 3. The blue solid line is the least squares quadratic fit for method 1, the green dashed line is the least squares quadratic fit for method 2, and the red broken line is the least squares linear fit for method 3. For methods 1 and 2, $R^2 > 0.98$ for all cases. For method 3, for the stance hip $R^2 = 0.91$, and for the swing hip $R^2 = 0.33$. 
Figure E.7. Change in (a) stance knee work and (b) stance knee maximum magnitude torque vs. ankle offset for the five-link biped model in the perturbation study. Blue circles indicate gaits found using method 1, green squares indicate gaits found using method 2, and red triangles indicate gaits found using method 3. The blue solid line is the least squares quadratic fit for method 1, the green dashed line is the least squares quadratic fit for method 2, and the red broken line is the least squares quadratic fit for method 3. Except for the maximum torque for method 3, $R^2 > 0.9$. For the maximum torque for method 3, $R^2 = 0.78$. 
alignment. The stance knee work increases by more than a factor of ten for even small changes in ankle offset. The maximum magnitude stance knee torque increases by about 500% for method 3. The large changes in stance knee kinetics likely capture the increased stump loading in amputees with misaligned prostheses.

E.3 Conclusions

For the most part, the GRF results match the results from human subject experiments when using methods 1 and 2. In both methods, the torque at the stance knee is increased with misalignment, which likely indicates increased stump loading. Torques at the swing hip and knee are also increased, indicating an increased arthritis risk. Unfortunately, neither method 1 nor 2 match the experimental results for walking speed. Further, only a small range of ankle offsets can be achieved without the biped falling over, particularly for the five-link model. Because amputees are able to walk with a much larger range of alignments, it appears that modeling the human response to alignment perturbations must change the control parameters. As a result, neither method has much predictive power.

The results for method 3 match the human subject experimental results for walking speed, step duration, and ground reaction forces. Further, for the five-link model, it predicts the increase in stump loading that is observed in amputees with misaligned prostheses. Method 3 only partially captures the increased arthritis risk. It predicts that having the foot center of curvature behind the nominal alignment increases the arthritis risk while having the foot center of curvature in front of the nominal alignment reduces the arthritis risk. It is possible that if an asymmetric model were used, the intact side hip work would depend quadratically on the prosthetic ankle alignment, thus capturing the increased arthritis risk. It appears that method 3 could provide an adequate method for exploring the effect of alignment perturbations once the optimal alignment is known.
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