THERMODYNAMIC FORMALISM AND ITS APPLICATIONS TO DEFORMATION SPACES

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

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May 2017
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Abstract

by

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In this dissertation, we discuss applications of Thermodynamic Formalism to various deformation spaces, namely, deformation spaces of metric graphs and immersed surfaces in hyperbolic 3-manifolds.

For spaces of metric graphs, we define two conformal but different pressure metrics and study Riemannian geometry induced by these two metrics. Moreover, we compare the Riemannian geometry features, such as curvature and completeness, of these two pressure metrics with the Weil-Petersson metric on Teichmüller spaces.

For immersed surfaces in hyperbolic 3-manifolds, we find relations between the critical exponent of the surface group (with respect to the ambient hyperbolic distance) and the topological entropy of the geodesics flow on the unit tangent bundle of the immersed surface (with respect to the induced Riemannian metric). These relations lead us to recover the remarkable Bowen rigidity for quasi-Fuchsian manifolds as well as give a Riemannian metric on the Fuchsian space which is bounded below by the classical Weil-Petersson metric.
To DPR, my family, and especially Chen-Hsuan
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SYMBOLS

$M$ A Manifold

$X$ A Metric space

$C(X)$ The set of continuous functions on $X$

$C^\alpha(X)$ The set of $\alpha$-Hölder continuous functions on $X$

$(X, \mathcal{A}, \mu)$ A measure space

$\mathcal{G}$ A undirected finite graph of valence no less than 3

$\mathbb{H}$ The hyperbolic plane

$\mathbb{H}^3$ The hyperbolic 3-space

$S$ A compact surface of genus no less than 2

$\mathcal{T}(S)$ The Teichmüller space of $S$

$\mathcal{M}(S)$ The Moduli space of $S$

$(\Sigma_A, \sigma)$ The subshift of finite type with adjacency matrix $A$

$\sigma$ The left shift map
ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Professor François Ledrappier, for his invaluable and boundless support, encouragement, guidance, and sharing. I am truly grateful for his mentorship. François teaches me how to learn, to think, and, most importantly, to enjoy mathematics. His words—never say no to somebody who wants to learn—reflects how open minded and how kind he is. I am truly fortunate to have him as my advisor. I also would like to thank Professor Mark Pollicott for many inspiring discussions and supports for my career. Without François and Mark, I would certainly not be able to see the beauty of ergodic theory, dynamical systems, and, especially, thermodynamic formalism.

Lastly, I would like to thank my family, especially to my mom, my sisters, my brother and my wife, for their continuously and endless love. Their understanding and support is the foundation of my life. Without their support, no matter how high I could be, it would be shaky.
CHAPTER 1

INTRODUCTION

1.1 Overview

Using methods developed in dynamical systems to study geometry has a very glorious history in mathematics. One can trace back to remarkable works of Poincaré, Hopf, and Birkhoff. This approach has been carried on and extensively studied by many prominent mathematicians such as Ruelle, Sinai, Margulis, Thurston, Bowen, McMullen et al. In this thesis, we aim to follow and continue this storyline of glimpsing geometry through a dynamics point of view. The dynamics approach that we undertake here is—*Thermodynamic Formalism*.

With the same spirit of statistical mechanics and thermodynamics, thermodynamic formalism in mathematics was introduced by Sinai, Ruelle, and Bowen in the 1970s as a tool for studying long-term behavior of flow (or diffeomorphism) and its corresponding entropy. Entropy is an important dynamics quantity meaning the growth rate of dynamical complexity. More interestingly, entropy can also be thought of as a geometric quantity which represents coarse and global geometric features of dynamical systems. Thermodynamic formalism intertwines these two different perspectives of entropy; hence, it becomes a convenient tool in understanding dynamics and geometry.

In this dissertation, we draw techniques and ideas from works of Bowen, Ledrappier, McMullen, Pollicott, Thurston and others, as well as thermodynamic formalism to study problems under a variety of geometric settings, namely, metric graphs and
negatively curved manifolds. Results here are extracted from two of my works: “Pressure type metrics on spaces of metric graphs” [19] and “Entropy, critical exponent and immersed surfaces in hyperbolic 3-manifolds” [18].

This thesis is organized as follows. In the rest of this chapter, we summarize results in the above two papers. Details are presented in Chapter 3 and Chapter 4, respectively. In Chapter 2 we discuss related background knowledge of dynamics and geometry.

1.2 Metric Graphs

Chapter 3 is a further study of dynamical-system-theoretically defined Riemannian metrics on deformation spaces—pressure metrics. The study of pressure metrics is ignited by McMullen’s study of the Weil-Petersson metric on Teichmüller space. In [28], McMullen proved that one can realize the Weil-Petersson metric on Teichmüller space by a pressure metric (on a certain functional space), which we call the McMullen pressure metric. This dynamics approach has been a great tool for defining and studying the Weil-Petersson metrics in a variety of contexts: Teichmüller spaces [28], quasi-Fuchsian spaces [6], Anosov representations [7], and Blaschke products [28]. However, at this point, most of our understanding of the geometry of pressure metrics is coming from their relation with the “classical” Weil-Petersson metric. In this work, carrying over ideas from Pollicott-Sharp’s work [32], we focus on investigating the pressure metric geometry from a more dynamical approach.

More explicitly, we follow Pollicott-Sharp’s construction of pressure metrics on the moduli space of metric graphs, and we consider two “natural” pressure metrics on the moduli space of graphs. We correct a formula in [32], and using the revised formulas we examine geometric features of these two natural pressure metrics on a moduli space of “typical” graphs. These examples show that geometric behaviors of these two natural pressure metrics are very far from what we know about the classical
Weil-Petersson metric (i.e., the McMullen pressure metric) on Teichmüller space.

To state our result more precisely and to put it in context, we first review basic setups in Teichmüller theory. Let $S$ be a compact topological surface with negative Euler characteristic. The Teichmüller space $\mathcal{T}(S)$ of $S$ can be thought of as the set of isotopy classes of Riemannian metrics with constant curvature $-1$, and the moduli space $\mathcal{M}(S)$ of $S$ can be described as the set of isometry classes of Riemannian metrics with constant curvature $-1$. Moreover, the moduli space $\mathcal{M}(S)$ is obtained by quotienting Teichmüller space $\mathcal{T}(S)$ by the mapping class group $\text{MCG}(S)$. The Weil-Petersson metric is a naturally defined and well-studied MCG-invariant metric on Teichmüller space (thus on the moduli space) with several striking features:

- the Weil-Petersson metric is negatively curved;
- the sectional curvature is neither bounded away from $0$ nor $-\infty$; and
- the Weil-Petersson metric is incomplete.

McMullen’s result in [28] shows that on Teichmüller space we can define a Riemannian metric, the *McMullen pressure metric*, via thermodynamic formalism, and which is exactly the Weil-Petersson metric. In other words, the McMullen pressure metric shares these notable geometric features with the Weil-Petersson metric on Teichmüller space.

**Definition.** Given an undirected finite graph $\mathcal{G}$ of valence no less than three with edge set $\mathcal{E}$. The edge weighting function $l : \mathcal{E} \rightarrow \mathbb{R}_{>0}$ assigns to each edge a length. We call the pair $(\mathcal{G}, l)$ a metric graph.

From a dynamical point of view, the metric graphs possess properties very similar to Riemann surfaces. Dynamics of paths on metric graphs is analogous to dynamics of the geodesic flow for Riemann surfaces. It is because the length weighting function $l$ on $\mathcal{G}$ plays the same role as a Riemannian metric on surfaces. Hence, it is natural
to begin the study of pressure metric geometry from deformation spaces of metric graphs. Here, our deformation space corresponding to the graph $\mathcal{G}$ is the space $\mathcal{M}_G$ of all edge weighting functions.

**Definition.** For a graph $\mathcal{G}$ and an edge weighting function $l$ the *entropy* $h(l)$ is defined by

$$h(l) = \lim_{T \to \infty} \frac{1}{T} \log \# \{ \gamma : l(\gamma) < T \}$$

where $\gamma = (e_0, e_1, ..., e_n = e_0)$ is a closed cycle of edges in $\mathcal{G}$ (without backtracking) and $l(\gamma) = \sum_{i=0}^{n-1} l(e_i)$.

From the dynamical perspective, the entropy $h(l)$ for metric graphs, as we have seen in the surface case, is an important and informative quantity. We recall that the moduli space $\mathcal{M}(S)$ is the collection (up to isometry) of Riemannian metrics on $S$ with constant curvature $-1$. We notice that the constant negative curvature condition of $\mathcal{M}(S)$ could be interpreted dynamically by using the constant (topological) entropy (of the geodesic flow on $S$) condition. More precisely, because when $S$ has a constant negative curvature, say $K(S)$, the topological entropy of the geodesic flow for $S$ is equal to $\sqrt{|K(S)|}$. Thus, to derive a close analogy to the moduli space $\mathcal{M}(S)$, it is natural and dynamics meaningful to consider the condition that entropy $h(l)$ is equal to 1. Moreover, there is one more reason for us to consider this entropy 1 normalization; however, this reason is more technical, and it’s from the nature of pressure metrics. We will explain this reason in Remark 16. In what follows, we will focus on the deformation space $\mathcal{M}^1_G$ of metric graphs with entropy 1, i.e.,

$$\mathcal{M}^1_G = \{ l : \mathcal{E} \to \mathbb{R}_{>0} : h(l) = 1 \}.$$
1.2.1 Main Results

Inspired by McMullen [28], Pollicott and Sharp constructed a pressure metric (they call it a Weil-Petersson type metric) for metric graphs [32]. We follow their work and construct another pressure metric for $\mathcal{M}_G^1$. We call these two pressure metrics the *Pollicott-Sharp pressure metric* and the *McMullen pressure metric* for $\mathcal{M}_G^1$, and denote them by $|| \cdot ||_{PS}$ and $|| \cdot ||_M$, respectively. We want to stress that we name the second pressure metric after McMullen instead of Weil-Petersson (unlike [32]) because our construction of the second pressure metric is closer to McMullen’s point of view of the Weil-Petersson metric on Teichmüller space. Although we know that there is no difference between these two Riemannian metrics (the Pollicott-Sharp and McMullen pressure metrics) on Teichmüller space, however, for moduli spaces of metric graphs our results here point out that this equivalence is no longer true. Therefore, to emphasize this assertion, we call the second pressure metric the McMullen pressure metric.

The McMullen pressure metric $|| \cdot ||_M$ for $\mathcal{M}_G^1$ is indeed conformal to the Pollicott-Sharp pressure metric $|| \cdot ||_{PS}$ and the scaling function could be thought of as a volume term of $\mathcal{G}$. Through normalizing the pressure metric $|| \cdot ||_{PS}$ by the volume term of $\mathcal{G}$, the resulting metric $|| \cdot ||_M$ is a closer analogy between McMullen’s definition of the Weil-Petersson metric on Teichmüller space and the definition of the pressure metric on Hitchin components (cf. Theorem 14 and Section 3.2 for more details). Moreover, in Section 3.2.2, we propose several formulas and properties of pressure metrics, which help to illustrate the usefulness of the definition.

Using formulas that we get in Section 3.2.1, we discuss four typical metric graphs: a figure 8 graph, a belt buckle, a dumbbell, and a three-petal rose. The first three graphs share the same fundamental group $F_2$ (the free group of rank 2). This makes an interesting connection with the *outer space* [12] in rank 2. For brevity, throughout this thesis, we denote the figure 8 graph, belt buckle, dumbbell, and the three-petal
TABLE 1.1

TYPICAL GRAPHS AND PRESSURE METRICS

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<th>Figure 8</th>
<th>Belt buckle</th>
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where $K$ denotes the Gaussian curvature.

rose by $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$, respectively. We summarize several results in Section 3.3 in the following.

**Proposition** (Proposition 6, 7, 10, 13, and Observation 3.3.1, 3.3.2, 3.3.3, 3.3.4).

1. **There exist examples of graphs for which the metric $|| \cdot ||_{PS}$ is not complete.**

2. **There exist examples of graphs for which the curvature of the metric $|| \cdot ||_{PS}$ is positive.**

3. **There exist examples of graphs for which the metric $|| \cdot ||_M$ is complete.**

4. **There exist examples of graphs for which the curvature of the metric $|| \cdot ||_M$ takes positive and negative values.**

**Remark 1.**

1. In this thesis, we rename the Weil-Petersson type metric defined in [32] by the Pollicott-Sharp pressure metric $|| \cdot ||_{PS}$. It is because that the McMullen pressure which is also a Weil-Petersson type metric (in the sense of Pollicott-Sharp [32]). To eliminate this ambiguity we endow them different names.
2. One should compare these results with results in Pollicott-Sharp’s paper [32]. Although Pollicott-Sharp and we do work on the same examples, the results do not quite match. It is because Lemma 3.3 in [32] (the main formula for calculating the pressure metric used in [32]) is true only if we input an additional condition that $f$ is normalized with respect to the transfer operator (i.e., $\mathcal{L}_f 1 = 1$). For more details, one can check Remark 19.

3. The difference between Propositions and Observations is that for Propositions we give proofs and for Observations we give computer computation evidences. The reason that we skip proofs for Observations is because computations of the curvature with respect to $|| \cdot ||_PS$ and $|| \cdot ||_M$ are not hard but tedious and complicated, which make no sense to analyze them in great detail rather than using the help from computers.

4. It is surprising that for all of our examples the sectional curvature under the McMullen pressure metric $|| \cdot ||_M$ is positive.

From the above proposition, we can see that both the Pollicott-Sharp pressure metric and the McMullen pressure metric for metric graphs shares some similarities with the Weil-Petersson metric on Teichmüller space. However, through working on these explicit examples, we can conclude that, from a pure dynamical setting, pressure metrics for metric graphs do not behave like the Weil-Petersson metric on Teichmüller space. In other words, the topology and the geometry of closed surfaces force the pressure metric to be the Weil-Petersson metric on Teichmüller space, but for graphs, without these special structures, the pressure metrics cannot reflect these well-known features in the Weil-Petersson geometry.
1.3 Immersed Surfaces in Hyperbolic 3-Manifolds

In Chapter 4, we consider a $\pi_1$–injective immersion $f : S \to M$ from a compact surface $S$ to a hyperbolic 3–manifold $M$. Let $\Gamma$ denote the copy of $\pi_1 S$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion $f$, and we endow $S$ with the induced Riemannian metric $g$ from the given hyperbolic metric $h$ on $M$ (i.e., $g = f^* h$). The topological entropy $h_{\text{top}}(S)$ of the geodesic flow on $T^1 \Sigma$, and the critical exponent $\delta_{\Gamma}$ of $\Gamma$ on $\mathbb{H}^3$ are two natural geometric quantities associated with this setting. Recall that when $(S, g)$ is a negatively curved manifold, each closed geodesic on $S$ corresponds to a unique conjugacy class $[\gamma] \in [\pi_1 S]$, and vice versa. We can write the topological entropy of the geodesic flow on $T^1 S$ as

$$h_{\text{top}}(S) = \lim_{T \to \infty} \frac{1}{T} \log \{ [\gamma] \in [\pi_1 S] : l_g[\gamma] \leq T \},$$

where $l_g[\gamma]$ is the length of the closed geodesic $[\gamma]$ with respect to the metric $g$. Moreover, the critical exponent $\delta_{\Gamma}$ can be understood by lengths of closed geodesics as well. By Sullivan’s theorem [36], when $\Gamma$ is convex cocompact, we have

$$\delta_{\Gamma} = \lim_{T \to \infty} \frac{1}{T} \log \{ [\gamma] \in [\pi_1 S] : l_h[\gamma] \leq T \},$$

where $l_h[\gamma]$ is the length of $[\gamma]$ using the hyperbolic metric $h$.

The primary tool used in Chapter 4 is, again, Thermodynamic Formalism. Specifically, the reparametrization method introduced by Ledrappier [24] and Sambarino [34]. Through the reparametrization method, we can link two different Anosov flows on $\mathbb{H}^3$ by a Hölder continuous function. Thus, we can compare periods of closed orbits associated with these two different Anosov flows, therefore, their topological entropies.
1.3.1 Main Results

Throughout this section, \( f : S \to M \) denotes a \( \pi_1 \)-injective immersion from a compact surface \( S \) to a hyperbolic 3-manifold \( (M, h) \), \( \Gamma \) denotes the copy of \( \pi_1 S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion \( f \), and \( g = f^* h \) denotes the induced Riemannian metric on \( S \). Moreover, we assume that \( \Gamma \) is convex cocompact and \( (S, g) \) is negatively curved.

We first introduce two geometric constants \( C_h(f) \) and \( C_g(f) \) which can be thought of as asymptotic geodesic distortions associated with the immersion \( f : S \to M \) with respect to \( h \) and \( h \), respectively.

**Definition** (Asymptotic geodesic distortions). We define two geometric constants associated to the immersion \( f : S \to M \) with respect to \( h \) and \( g \), respectively, as

\[
C_g(f) = \limsup_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(g)} l_g[\gamma]},
\]

\[
C_h(f) = \limsup_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(h)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(h)} l_g[\gamma]},
\]

where

\[
R_T(g) := \{[\gamma] \in [\pi_1 S] : l_g[\gamma] \leq T\} \text{ and } R_T(h) := \{[\gamma] \in [\pi_1 S] : l_h[\gamma] \leq T\}.
\]

**Remark** 2. This type of asymptotic geodesic distortions dates back to Thurston’s work [41]. More precisely, let \( g_1 \) and \( g_2 \) be two hyperbolic metrics on \( S \). Thurston
proves that the following limit exists

\[ I(g_1, g_2) := \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g_2)} l_2[\gamma]}{\sum_{[\gamma] \in R_T(g_1)} l_1[\gamma]} \]

Moreover, \( I(g_1, g_2) \) has a remarkable rigidity property that \( I(g_1, g_2) \geq 1 \) and \( I(g_1, g_2) = 1 \) iff \( g_1 \) and \( g_2 \) are isometric. Nowadays, this number \( I(g_1, g_2) \) is known as Thurston’s intersection number.

Our main theorem below shares similar flavor as the above theorem of Thurston. We relate the two geometric/dynamics quantities \( h_{\text{top}}(S) \) and \( \delta_\Gamma \) and show that they are governed by \( C_h(f) \) and \( C_g(f) \) through inequalities. Moreover, we also point out that the equality cases exhibit rigidity features.

**Theorem** (Theorem 26). Let \( f : S \to M \) be a \( \pi_1 \)-injective immersion from a compact surface \( S \) to a hyperbolic 3-manifold \( M \). Let \( \Gamma \) be the copy of \( \pi_1S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion \( f \). Suppose \( \Gamma \) is convex cocompact and \( (S, g) \) is negatively curved. Then

1. The limit-sups in \( C_h(f) \) and \( C_g(f) \) are limits.

2. \( 0 < C_h(f) \leq C_g(f) \leq 1 \).

3. Let \( h_{\text{top}}(S) \) be the topological entropy of the geodesic flow on \( T^1S \) and \( \delta_\Gamma \) be the critical exponent, then

\[ C_h(f) \cdot \delta_\Gamma \leq h_{\text{top}}(S) \leq C_g(f) \cdot \delta_\Gamma. \quad (1.3.1) \]

4. The first (resp. second) equality in (1.3.1) holds if and only if the marked length spectrum of \( S \) is proportional to the marked length spectrum of \( M \), and the proportion is the ratio \( \frac{\delta_\Gamma}{h_{\text{top}}(S)} \) (resp. \( \frac{h_{\text{top}}(S)}{\delta_\Gamma} \)).
5. If $C_h(f) = 1$ or $C_g(f) = 1$, then $S$ is a totally geodesic submanifold in $M$.

Remark 3.

1. By Sullivan’s theorem [37], one can replace the critical exponent $\delta_\Gamma$ in the above theorem by the Hausdorff dimension $\dim_H \Lambda(\Gamma)$ of the limit set $\Lambda(\Gamma)$.

2. We shall mention that our approach, from a dynamics point of view, is similar to Sambarino’s work [33], since we both use thermodynamic formalism and reparametrization method. However, our the geometry setup and applications are different from his work, especially the interpretations of $C_g(f)$ and $C_h(f)$ and their relations with Knieper’s geodesic stretches.

3. This result should also be compared with Theorem 1 [9] and Theorem 1.2 [23]. All these results possess a similar flavor of comparing entropies.

4. In Glorieux’s thesis [15], he follows Knieper’s method and deduces an upper bound on $h_{\text{top}}(S)$ in the case that $\Sigma$ is embedded in a quasi-Fuchsian manifold $M$ (cf. Section 4.4.2.2 for definitions). We prove that the upper bound on Glorieux’s thesis is exactly the same as the one in Theorem 26.

Moreover, when $f : S \to M$ is an embedding, we have another geometric interpretation of $C_h(f)$ and $C_g(f)$. The following theorem shows that $C_h(f)$ and $C_g(f)$ are equal to the geodesic stretches of $S$ relative to $M$ with respect to certain Gibbs measures. The precise definition of geodesic stretch will be given in Section 4.3.

Theorem (Theorem 27). Under the same assumptions in Theorem 26 plus $f : S \to M$ is an embedding. Then, the asymptotic geodesic distortions $C_h(f)$, $C_g(f)$ are, precisely, geodesic stretches relative to Gibbs measures. More precisely,

$$C_h(f) = I_\mu(S, M),$$

$$C_g(f) = I_{\mu_{BM}}(S, M),$$
where $\mu$ is a $\phi$–invariant Gibbs measure and $\mu_{BM}$ is the Bowen-Margulis measure of the geodesic flow $\phi_t$ on $T^1S$.

Remark 4.

1. Details of the Gibbs measure $\mu$ in $C_h(f) = I_\mu(S, M)$ will be presented in the proof of Theorem 26 (cf. Remark 31).

2. Our definition of the geometric stretch $I_\mu(S, M)$ in Section 4.3 is inspired by Knieper [23].

1.3.2 Applications

By the Gauss equation, immersed minimal surfaces in hyperbolic 3–manifolds are negatively curved. Therefore, we shall see how Theorem 26 can be used to study immersed minimal surfaces. Minimal surfaces in hyperbolic 3–manifolds is a very rich subject that has drawn a lot of attention, with significant contributions by Uhlenbeck [42] and Taubes [38]. In the Chapter 4, we take a glance at this rich subject from a dynamical system point of view.

The following corollary is a consequence of the main theorem.

**Corollary** (Corollary 2). Let $f : S \to M$ be a $\pi_1$–injective minimal immersion from a compact surface $S$ to a hyperbolic 3–manifold $M$, and $\Gamma$ be the copy of $\pi_1S$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion. Suppose $\Gamma$ is convex cocompact. Then assertions (1) – (5) in Theorem 26 are true.

Remark 5. From [42], we learn that the $\pi_1$–injectivity is guaranteed if we put some curvature conditions on $S$. Namely, all principal curvatures are between $-1$ and 1. Furthermore, in such cases, immersed minimal surfaces are indeed embedded. Therefore, we can interpret the constants $C_h(f)$ and $C_g(f)$ of such pairs $(S, M)$ as geodesic stretches.
In the last part of Chapter 4, we change gear to the Taubes’ moduli space of $S$. Taubes [38] constructs the space of minimal hyperbolic germs $\mathcal{H}$ which is a deformation space for the set whose archetypal element is a pair that consists of a Riemannian metric $g$ and the second fundamental form $B$ from a closed, oriented, negative Euler characteristic minimal surfaces $S$ in some hyperbolic 3–manifold $M$.

Uhlenbeck [42] proved that there exists a representation $\rho : \pi_1(S) \to \text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ leaving this minimal immersion invariant. In other words, there is a map

$$\Phi : \mathcal{H} \to \mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C})),$$

where $\mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ is the space of conjugacy classes of representations of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$.

The following corollary gives an upper and a lower bound on the topological entropy $h_{\text{top}}(g, B)$ of the geodesic flow on $T^1S$ for the data $(g, B) \in \mathcal{H}$. Moreover, using the same argument, we can also construct two geometric constants $C_1(g, B)$, $C_2(g, B)$ as $C_h(f)$, $C_g(f)$ in Theorem 26.

**Corollary (Corollary 3).** Let $\rho \in \mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ be a discrete, faithful and convex cocompact representation. Suppose $(g, B) \in \Phi^{-1}(\rho)$, then there are explicit geometric constants $C_1(g, B)$, $C_2(g, B)$ with $0 \leq C_1(g, B) \leq C_2(g, B) \leq 1$ such that

$$C_1(g, B) \cdot \delta_{\rho(\pi_1S)} \leq h_{\text{top}}(g, B) \leq C_2(g, B) \cdot \delta_{\rho(\pi_1S)} \leq \delta_{\rho(\pi_1S)}$$

with the last equality if and only if $B$ is identically zero which holds if and only if $\rho$ is Fuchsian.

**Remark 6.**

1. When $\rho$ is quasi-Fuchsian, the above upper bound on $h_{\text{top}}(g, B)$ is a special case treated in Sanders’ paper [35]. However, the inequality in Sanders’ work has no
information about the constant $C_2(g, B)$.

2. The lower bound given in Sanders’ paper [35] is derived from Manning’s formula [26], which gives a lower bound on the topological entropy $h_{\text{top}}(g, B)$ in terms of the curvature of $(S, g)$. Whereas, our method doesn’t see the curvature directly. It will be interesting to compare our lower bound with Sanders’ lower bound on $h_{\text{top}}(g, B)$.

From above corollary, we recover Bowen’s rigidity theorem [5].

**Corollary** (Bowen’s rigidity [5]). A quasi-Fuchsian representation $\rho \in \mathcal{QF}$ is Fuchsian if and only if $\dim_H \Lambda(\Gamma) = 1$.

Lastly, we focus on two special subsets of the minimal hyperbolic germs: the Fuchsian space $\mathcal{F}$ and the almost-Fuchsian space $\mathcal{AF}$. The Fuchsian space is the space of all hyperbolic metrics on $S$, i.e.,

$$\mathcal{F} = \{(m, 0) \in \mathcal{H};\, m \text{ is a hyperbolic metric on } S\}$$

and the almost-Fuchsian space $\mathcal{AF}$ is defined by the condition that $\|B\|_g^2 < 2$. By Uhlenbeck’s theorem in [42] we know that if $(g, B) \in \mathcal{AF}$ then there exists a unique quasi-Fuchsian 3–manifold $M$, up to isometry, such that $S$ is an embedded minimal surface in $M$ with the induced metric $g$ and the second fundamental form $B$.

The following theorem was first proved in [35], and we recover this theorem by the reparametrization method.

**Corollary** (Theorem 28, Theorem 3.5 [35]). Consider the entropy function restricted to the almost-Fuchsian space $h_{\text{top}} : \mathcal{AF} \to \mathbb{R}$, then

1. The entropy function $h_{\text{top}}$ realizes its minimum at the Fuchsian space $\mathcal{F}$; and
2. For \((m, 0) \in \mathcal{F}\), \(h_{\text{top}}\) is monotone increasing along the ray \(r(t) = (g_t, t B)\) provided \(\|t B\|_{g_t} < 2\), i.e., \(r(t) \subset A\mathcal{F}\), where \(g_t = e^{2u} m\).

In the end of Chapter 4, we give another proof of the following theorem given in [35]. Similar to Bridgeman’s method in [6], the pressure form is used to show that the Hessian of the entropy defines a metric on \(\mathcal{F}\), which is bounded below by the Weil-Petersson metric.

**Theorem** (Theorem 29, Theorem 3.8 [35]). *One can define a Riemannian metric on the Fuchsian space \(\mathcal{F}\) by using the Hessian of \(h_{\text{top}}\). Moreover, this metric is bounded below by \(2\pi\) times the Weil-Petersson metric on \(\mathcal{F}\).*

It is natural to ask if this metric is different from the Weil-Petersson metric. It will be interesting to learn more relations between this metric coming from the Hessian of the entropy and the Weil-Petersson metric or pressure metric on the quasi-Fuchsian spaces.
2.1 Thermodynamic Formalism for Symbolic Dynamics

2.1.1 Measure Preserving Dynamical Systems

Before we jump into definitions of thermodynamics, we shall introduce and recall some basic notions from ergodic theory.

Let $T$ be a measure-preserving transformation defined on a probability space $(X, A, \mu)$, that is, $T^{-1}(A) \subset A$ and $\mu(T^{-1}(A)) = \mu(A)$ for $A \in A$. If $\mathcal{P}$ is a finite measurable partition of $X$ and $\mathcal{C} \subset A$ where $\mathcal{C}$ is a sub-$\sigma$-algebra, we define the condition information $I_\mu$ of $\mathcal{P}$ given $\mathcal{C}$ as

$$I_\mu(\mathcal{P}|\mathcal{C}) := - \sum_{A \in \mathcal{P}} \log (\mu(A|\mathcal{C})) \cdot \chi_A$$

and the condition entropy $H_\mu$ of $\mathcal{P}$ given $\mathcal{C}$ as

$$H_\mu(\mathcal{P}|\mathcal{C}) := \int_X I_\mu(\mathcal{P}|\mathcal{C}) d\mu = \int_X - \sum_{A \in \mathcal{P}} \log (\mu(A|\mathcal{C})) \mu(A|\mathcal{C})$$

where $\mu(A|\mathcal{C}) = \mathbb{E}_\mu(\chi_A|\mathcal{C})$ and we use the convention $x \log x$ is zero at $x = 0$.

The information and entropy of $T$ with respect to $\mathcal{P}$ are defined, respectively, as $I_\mu(T, \mathcal{P}) := I_\mu(\mathcal{P}|T^{-1}\mathcal{C})$ and $H_\mu(T, \mathcal{P}) := H_\mu(\mathcal{P}|T^{-1}\mathcal{C})$ where $\mathcal{C} = \bigvee_{i=0}^{\infty} T^{-i}\mathcal{P}$ is the smallest $\sigma$-algebra containing $\bigcup_{i=1}^{\infty} T^{-i}\mathcal{P}$. 

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Definition 1. The measure-theoretic entropy of $T$ is defined as

$$h_\mu(T) := \sup_{\mathcal{P}} h(T, \mathcal{P})$$

where the supremum is taken over all finite measurable partitions $\mathcal{P}$.

Remark 7. A well-known theorem of Kolmogorov and Sinai asserts $h_\mu(T) = h(T, \mathcal{P})$ when $\mathcal{A}$ is the smallest $T$-invariant $\sigma$-algebra containing $\mathcal{P}$ (i.e., if $T$ is invertible and $\mathcal{A} = \bigvee_{i=-\infty}^{\infty} T^{-1}\mathcal{P}$, or more generally if $\mathcal{A} = \bigvee_{i=0}^{\infty} T^{-1}\mathcal{P}$).

2.1.2 Subshifts of Finite Type

A $k \times k$ matrix $A$ is called irreducible, if for each pair $(i, j)$, $1 \leq i, j \leq k$, there exists $n \geq 1$ such that $A^n(i, j) > 0$, where $A^n$ is an $n$-fold product of $A$ with itself. Suppose $A$ is an irreducible matrix consisting of 0 and 1, we define

$$\Sigma_A^+ = \{ \mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \{1, 2, ..., k\}^\mathbb{N}; A(x_n, x_{n+1}) = 1 \}.$$  

We consider the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $(\sigma(\mathbf{x}))_n = x_{n+1}$, and then call $(\Sigma_A^+, \sigma)$ a (one-sided) subshift of finite type. We notice that $\Sigma_A^+$ is a compact zero-dimensional space with respect to the Tychonoff product topology, and we can endow a metric $d$ on $\Sigma_A^+$. More precisely, this space is compact with respect to the metric

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^n},$$

where $\delta(i, j)$ is the standard Kronecker delta.

Remark 8. A is irreducible implies that $(\Sigma_A^+, \sigma)$ is (topological) transitive (i.e., $\sigma : \Sigma_A^+ \to \Sigma_A^+$ has a dense orbit).

We denote the set of continuous functions on $\Sigma_A^+$ by $C(\Sigma_A^+)$. A function $f : \Sigma_A^+ \to \mathbb{R}$ is called $\alpha$–Hölder continuous if there exists $C > 0$ and $\alpha \in (0, 1]$ such that for all
We have \( f(x) - f(y) \leq Cd(x, y)\alpha \), and we denote by \( C^\alpha(\Sigma^+_A) \) the space of \( \alpha \)-Hölder continuous functions on \( \Sigma^+_A \). We call a function \( f : \Sigma^+_A \to \mathbb{R} \) is Hölder continuous if it is \( \alpha \)-Hölder continuous for some \( \alpha \). Two functions \( f, g \in C(\Sigma^+_A) \) are called Livšic cohomologous, denoted by \( f \sim g \), if there exists \( h \in C(\Sigma^+_A) \) such that

\[
f - g = h \circ \sigma - h.
\]

### 2.1.3 Transfer Operator, Pressure, and Variance

Most of the following definitions could be generalized to continuous functions \( C(\Sigma^+_A) \) on \( \Sigma^+_A \). However, we don’t need such a generality, so we will only focus on Hölder continuous functions on \( \Sigma^+_A \).

**Definition 2.** The transfer operator (or Ruelle operator) \( \mathcal{L}_f : C(\Sigma^+_A) \to \mathbb{R} \) of a Hölder continuous function \( f \) is defined as

\[
(\mathcal{L}_fw)(x) = \sum_{y \in \sigma^{-1}(x)} e^{f(y)}w(y).
\]

**Theorem 1** (Ruelle-Perron-Frobenius Theorem, Theorem 2.2 [30]). Let \( f \) be a Hölder continuous function on \( \Sigma^+_A \) and suppose \( A \) is irreducible.

1. There is a simple maximal positive eigenvalue \( \beta_f \) of \( \mathcal{L}_f : C(\Sigma^+_A) \to C(\Sigma^+_A) \) with a corresponding strictly positive eigenfunction \( v_f \in C^\alpha(\Sigma^+_A) \), and \( \beta_f \) realizes the spectral radius of \( \mathcal{L}_f : C(\Sigma^+_A) \to C(\Sigma^+_A) \); and

2. There exists a unique probability measure \( \mu_f \) such that \( \mathcal{L}^*_f\mu_f = \beta_f\mu_f \), i.e.,

\[
\int \mathcal{L}_f \phi d\mu_f = \beta_f \int \phi d\mu_f \text{ for all } \phi \in C(\Sigma^+_A).
\]

By the Ruelle-Perron-Frobenius theorem, we are now ready to define the pressure.
Definition 3. **Pressure** is the map $P : C^0(\Sigma_A^+) \to \mathbb{R}$ given by

$$P(f) = \log \beta_f,$$

where $\beta_f$ is a simple maximal positive eigenvalue of $\mathcal{L}_f$ given in Theorem 1 (the Ruelle-Perron-Frobenious theorem).

**Theorem 2** (Variational Principle, Theorem 3.5 [30]). For $f \in C^0(\Sigma_A^+)$, we have

$$P(f) = \sup \{ h(\mu) + \int_{\Sigma_A^+} f d\mu : \mu \text{ is a } \sigma\text{-invariant probability measure} \},$$

where $h(\mu)$ is the measure-theoretic entropy of $\sigma$ w.r.t $\mu$. Moreover, there exists a unique $\sigma-$invariant probability measure $m_f$ realizing the equality, i.e.,

$$P(f) = h(m_f) + \int_{\Sigma_A^+} f dm_f.$$

**Definition 4.** The **equilibrium state (or equilibrium measure)** $m_f$ for $f \in C^0(\Sigma_A^+)$ the unique probability measure realizing the equality in the variance principle, i.e.,

$$P(f) = h(m_f) + \int_{\Sigma_A^+} f dm_f.$$

**Remark 9.**

1. One can prove that we can characterize the pressure via periodic points of $(\Sigma_A^+, \sigma)$. More precisely, for every $f \in C^0(\Sigma_A^+),$

$$P(f) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{\sigma^n x = x} e^{f^n(x)} \right),$$

where $f^n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1} x)$; and

2. Suppose $f \in C^0(\Sigma_A^+)$ such that $P(f) = 0$, then one can alternatively define the
equilibrium state of \( f \) as the measure

\[
m_f = \frac{v_f \cdot \mu_f}{\int v_f d\mu_f},
\]

where \( \mu_f \) is the unique probability measure associated with \( f \) given in Theorem 1.

We notice that the equilibrium state is an ergodic, \( \sigma \)-invariant probability measure with positive entropy. Moreover, it is unique and is the same for two Livšic cohomologous functions.

**Definition 5.** Let \( \mu \) be a \( \sigma \)-invariant measure. For any \( u, v \in C(\Sigma_A^+) \) with mean zero with respect to \( \mu \) (i.e., \( \int ud\mu = \int vd\mu = 0 \)), we define the covariance of \( u \) and \( v \) with respect to \( \mu \) by

\[
\text{Cov}(u, v, \mu) := \lim_{n \to \infty} \frac{1}{n} \int (u^n(x)) (v^n(x)) d\mu
\]

where \( u^n(x) = \sum_{i=0}^{n-1} u \circ \sigma^i(x) \). Similarly, for \( w \in C(\Sigma_A^+) \) with mean zero with respect to \( \mu \), the variance for \( w \) with respect to \( \mu \) is defined as

\[
\text{Var}(w, \mu) := \text{Cov}(w, w, \mu).
\]

In this thesis, we only consider variances with respect to equilibrium states, which simplifies the discussion. In the sequel, we first recall some facts and useful formulas in calculating variances with respect to equilibrium states. The theorem below is given by Jordan and Pollicott in [17].

**Theorem 3** (Lemma 3.4 [17]). Suppose \( f \in C^\alpha(\Sigma_A^+) \), \( \mathcal{L}_f 1 = 1 \) and \( \int wdm_f = 0 \), then

\[
\text{Var}(w, m_f) = \int_{\Sigma_A^+} w^2 dm_f,
\]

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where \( m_f \) is the equilibrium state with respect to \( f \).

We also recall several handy properties of the pressure.

**Theorem 4** (Analyticity of the Pressure, Prop. 4.7, 4.12 [30]). Let \( f, g \in C^\alpha(\Sigma_A^+) \) and \( m_f \) be the equilibrium state of \( f \). Then

1. The function \( t \to P(f + tg) \) is analytic; and

2. \( \text{Var}(g, m_f) = 0 \) if and only if \( g \) is Livšic cohomologous to a constant.

By analyticity of the pressure, we can differentiate the pressure, and, more importantly, derivatives of the pressure give us some handy formulas for computing variance.

**Theorem 5** (Derivatives of the pressure, Prop. 4.10, 4.11 [30], Theorem 2.2 [28]).

Let \( \psi_t \) be an analytic path in \( C^\alpha(\Sigma_A^+) \), \( m_0 = m_{\psi_0} \) be the equilibrium state of \( \psi_0 \), and \( \dot{\psi}_0 := \frac{d\psi_t}{dt} \bigg|_{t=0} \). Then

\[
\frac{dP(\psi_t)}{dt} \bigg|_{t=0} = \int_{\Sigma_A^+} \dot{\psi}_0 dm_0. \tag{2.1.1}
\]

Moreover, if the first derivative is zero (i.e., \( \int_{\Sigma_A^+} \dot{\psi}_0 m_0 = 0 \)) then

\[
\frac{d^2 P(\psi_t)}{dt^2} \bigg|_{t=0} = \text{Var}(\dot{\psi}_0, m_0) + \int_{\Sigma_A^+} \ddot{\psi}_0 dm_0. \tag{2.1.2}
\]

We now state a remarkable property found by Bowen, so-called Bowen’s formula, which relates the growth rate of weighted periodic orbits (or topological entropy of a certain system) with the pressure.

**Theorem 6** (Bowen's Formula, Prop. 6.1 [30]). If \( f : \Sigma_A^+ \to \mathbb{R} \) is a positive continuous function, then

\[
P(-s \cdot f) = 0
\]
if and only if \( s = h_f \) where

\[
h_f = \lim_{T \to \infty} \frac{1}{T} \log \# \{ \bar{x} \in \Sigma^+_A; \sigma^n \bar{x} = \bar{x} \text{ and } f^n(\bar{x}) < T \text{ for some } n \in \mathbb{N} \}.
\]

2.1.4 Locally Constant Functions

Now we change gear and focus on a particular type of Hölder continuous functions: locally constant functions, which are functions only depending on first two coordinates (i.e., \( f : \Sigma^+_A \to \mathbb{R} \) and \( f(\bar{x}) = f(x_0, x_1) \), where \( \bar{x} = x_0x_1x_2... \)). For every such function \( f \), we have explicit formulas of the eigenvalue \( \beta_f \), the eigenfunction \( v_f \), and the measure \( \mu_f \) derived in Theorem 1.

**Proposition 1** (Remark 1, p.27 [30]). Let \( f : \Sigma^+_A \to \mathbb{R} \) be a function depending only on first two coordinates and \( \mathcal{L}_f v_f = \beta_f v_f \). Suppose \( A_f \) is the matrix defined as

\[
A_f(i, j) = A(i, j)e^{f(i,j)}, \quad 1 \leq i, j \leq k.
\]

Then

1. \( \beta_f \) is the maximal eigenvalue of \( A_f \).

2. \( v_f \) is the (left) eigenvector \( v_I \) of \( A_f \) w.r.t. \( \beta_f \), i.e.,

\[
\sum_i v_f(i)A(i, j)e^{f(i,j)} = \beta_f \cdot v_f(j).
\]

3. If we define \( g(i, j) = \log v_f(i) - \log v_f(j) - \log \beta_f + f(i, j) \), then \( \mathcal{L}_g 1 = 1 \) and the matrix \( P_f \) corresponding to \( \mathcal{L}_g \) is

\[
P_f(i, j) = A(i, j)e^{g(i,j)} = \frac{A(i, j)v_f(i)e^{f(i,j)}}{\beta_f \cdot v_f(j)}.
\]

Moreover, \( P_f \) is column stochastic, i.e., \( \sum_i A(i, j)e^{g(i,j)} = 1 \), and the equilibrium
state \( m \) w.r.t. \( g \) is given by

\[
m_g[i_0, \ldots, i_n] = P_f(i_0, i_1) \cdot \ldots \cdot P_f(i_{n-1}, i_n) p_f(i_n),
\]

where \( P_f p_f = p_f \) and \( \sum_i p_f(i) = 1 \) and we use the notation \([i_0, i_1, \ldots, i_n] = \{ x \in \Sigma_A^+; x_j = i_j, j = 0, 1, \ldots, n \}\).

Remark 10. We call \([i_0, i_1, \ldots, i_n]\) a cylinder set, and in fact cylinder sets form a basis for the topology on \( \Sigma_A^+ \).

2.1.5 Pressure Metrics for Subshifts of Finite Type

In this section, we introduce a construction of Riemannian metrics on the space of Hölder continuous functions on \( \Sigma_A^+ \). The main idea of this construction is using variance which is indeed the second derivative of the pressure (cf. Theorem 5), therefore, we call Riemannian metrics constructed in this manner by pressure metrics.

Here we keep the same setting as in the previous section that \((\Sigma_A^+, \sigma)\) is a subshift of finite type and \( A \) is irreducible. We consider the space \( \mathcal{P}(\Sigma_A^+) \) of Livšic cohomology classes of pressure zero Hölder continuous on \( \Sigma_A^+ \), i.e.,

\[
\mathcal{P}(\Sigma_A^+) := \{ f; f \in C^\alpha(\Sigma_A^+) \text{ for some } \alpha \text{ and } P(f) = 0 \} / \sim,
\]

where \( \sim \) denotes the Livšic cohomology relation.

The tangent space of \( \mathcal{P}(\Sigma_A^+) \) at \( f \) is defined as

\[
T_f \mathcal{P}(\Sigma_A^+) = \ker \textbf{D} P(f) = \{ \phi; \phi \in C^\alpha(\Sigma_A^+) \text{ for some } \alpha \text{ and } \int \phi dm_f = 0 \} / \sim,
\]

where \( \textbf{D} P(f) \) is derivative of \( P \) at \( f \) and \( m_f \) is the equilibrium state of \( f \).

We notice that \( \text{Cov}(\cdot, \cdot, m_f) \) is bilinear, symmetric, and positive semi-definite (i.e., \( \text{Cov}(u, u, m_f) \geq 0 \) and \( \text{Cov}(u, u, m_f) = 0 \) iff \( u \) cohomologous to a constant).
Therefore, Cov(·, ·, mf) defines an inner product over $T_f\mathcal{P}(\Sigma^+_A)$ and thus a norm on $T_f\mathcal{P}(\Sigma^+_A)$.

**Definition 6** (Pressure Metrics for Subshifts of Finite Type). Let $f \in \mathcal{P}(\Sigma^+_A)$ and $\phi \in T_f\mathcal{P}(\Sigma^+_A)$, then we define two pressure metrics:

$$\|\phi\|_{PS}^2 := \text{Var}(\phi, mf)$$

and

$$\|\phi\|_M^2 := \frac{\text{Var}(\phi, mf)}{-\int f dm_f}.$$ 

The former one is called the Pollicott-Sharp (PS) pressure metric on $\mathcal{P}(\Sigma^+_A)$ and denoted by $\|\cdot\|_{PS}$, and the latter one is called the McMullen (M) pressure metric on $\mathcal{P}(\Sigma^+_A)$ and denoted by $\|\cdot\|_M$.

One can immediately see that $\|\cdot\|_{PS}$ and $\|\cdot\|_M$ are conformal and they only differ by a normalization. We call $\|\cdot\|_M$ the McMullen pressure metric because McMullen points out that after normalizing by the area, one can recover the Weil-Petersson metric on Teichmüller space (cf. Theorem 14).

The following proposition is our most important formula for computing the Pollicott-Sharp pressure metric and the McMullen pressure metric.

**Proposition 2.** If $\{\phi_t\}_{t \in (-1,1)}$ is an analytic one parameter family contained in $\mathcal{P}(\Sigma^+_A)$, then

$$\|\dot{\phi}_0\|_M^2 = \frac{\int \ddot{\phi}_0 dm_{\phi_0}}{\int \phi_0 dm_{\phi_0}} \quad \text{and} \quad \|\dot{\phi}_0\|_{PS}^2 = -\int \dddot{\phi}_0 dm_{\phi_0}$$

where $\dot{\phi}_0 = \frac{d}{dt}\phi_t|_{t=0}$ and $\ddot{\phi}_0 = \frac{d^2}{dt^2}\phi_t|_{t=0}$.

**Proof.** This follows from the direct computation of the (Gâteaux) second derivative.
of $P(\phi_t)$:

$$\left. \frac{d^2}{dt^2} P(\phi_t) \right|_{t=0} = (D^2 P)(\phi_0)(\dot{\phi}_0, \dot{\phi}_0) + (DP)(\phi_0)(\ddot{\phi}_0)$$

$$= \text{Var}(\dot{\phi}_0, m_{\phi_0}) + \int \ddot{\phi}_0 dm_{\phi_0}.$$ 

Since $P(\phi_t) = 0$, we have

$$\|\dot{\phi}_0\|^2_M := \frac{\text{Var}(\dot{\phi}_0, m_{\phi_0})}{-\int \ddot{\phi}_0 dm_{\phi_0}} = \frac{\int \ddot{\phi}_0 dm_{\phi_0}}{-\int \ddot{\phi}_0 dm_{\phi_0}}$$

and

$$\|\dot{\phi}_0\|^2_{PS} := \text{Var}(\dot{\phi}_0, m_{\phi_0}) = -\int \ddot{\phi}_0 dm_{\phi_0}.$$

\[\square\]

2.2 Thermodynamic Formalism for Anosov Flows

2.2.1 Flows and Reparametrization

Let $X$ be a compact metric space with a continuous flow $\phi = \{\phi_t\}_{t \in \mathbb{R}}$ on $X$ without any fixed point and $\mu$ be a $\phi$–invariant probability measure on $X$. Consider a positive continuous function $F : X \to \mathbb{R}_{>0}$ and define, for $t > 0$

$$\kappa(x, t) := \int_0^t F(\phi_s(x)) ds,$$

and $\kappa(x, t) := -\kappa(\phi_t(x), -t)$ for $t < 0$. The function $\kappa$ satisfies the cocycle property $\kappa(x, t + s) = \kappa(x, t) + \kappa(\phi_t x, s)$ for all $x, t \in \mathbb{R}$ and $x \in X$.

Since $F > 0$ and $X$ is compact, $F$ has a positive minimum and $\kappa(x, \cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have a map $\alpha : X \times \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(x, \kappa(x, t)) = \kappa(x, \alpha(x, t)) = t.$$

for all $(x, t) \in X \times \mathbb{R}.$
Definition 7. Let $F : X \to \mathbb{R}$ be a positive continuous function. The \textit{reparametrization} of the flow $\phi$ by $F$ is the flow $\phi^F = \{\phi^F_t\}_{t \in \mathbb{R}}$ defined as $\phi^F_t(x) = \phi_{\alpha(x,t)}(x)$.

Definition 8. Two continuous functions $F, G : X \to \mathbb{R}$ are \textit{Livšic cohomologous} if there exists a continuous function $V : X \to \mathbb{R}$ which is $C^1$ in the flow direction such that
\[
F(x) - G(x) = \frac{\partial}{\partial t} \bigg|_{t=0} V(\phi_t(x)),
\]
and we denote this relation by $F \sim G$.

2.2.2 Periods and Measures

Let $O$ be the set of closed orbits of $\phi$. For $\tau \in O$, let $l(\tau)$ be the period of $\tau$ with respect to $\phi$, then the period of $\tau$ with respect to the reparametrized flow $\phi^F$ is
\[
\int_0^{l(\tau)} F(\phi_s(x))ds,
\]
where $x$ is any point on $\tau$. Let $\delta_\tau$ be the Lebesgue measure supported by the orbit $\tau$, and we denote
\[
\langle \delta_\tau, F \rangle = \int_0^{l(\tau)} F(\phi_s(x))ds.
\]

Let $\mu$ be a $\phi$–invariant probability measure on $X$, $F : X \to \mathbb{R}$ be a continuous function, and $\phi^F$ be the reparametrization of $\phi$ by $F$. We define $\widehat{F \cdot \mu}$ to be the probability measure: for any continuous function $G$ on $X$
\[
\widehat{F \cdot \mu}(G) = \frac{\int_X G \cdot F d\mu}{\int_X F d\mu}.
\]
Then $\widehat{F \cdot \mu}$ is a $\phi^F$–invariant probability measure.
2.2.3 Entropy, Pressure, and Equilibrium States

We denote by $h_\phi(\mu)$ the measure-theoretic entropy of $\phi$ with respect to a $\phi$–invariant probability measure $\mu$. Let $\mathcal{M}^\phi$ denote the set of $\phi$–invariant probability measures, and $C(X)$ denote the set of continuous functions on $X$.

**Definition 9.** The pressure of a function $F : X \to \mathbb{R}$ is defined as

$$P_\phi(F) := \sup_{m \in \mathcal{M}^\phi} \left( h_\phi(m) + \int_X F \, dm \right).$$

We define the topological entropy of the flow $\phi$ by

$$h_\phi = P_\phi(0).$$

If there is no ambiguity on which flow we refer to, such as $\phi$, then we might drop the subscript $\phi$ and use $h$ to denote the topological entropy, and $h(\mu)$ to denote the measure theoretic entropy of $\phi$ with respect to $\mu$.

For a continuous function $F$, if there exists a measure $m \in \mathcal{M}^\phi$ on $X$ such that

$$P_\phi(F) = h_\phi(m) + \int_X F \, dm,$$

then $m$ is called an equilibrium state of $F$, and denoted it by $m = m_F$. An equilibrium state of the function $F \equiv 0$ is called a measure of maximum entropy.

**Remark 11.** From the definition of the pressure, we list two immediate properties:

1. $h_\phi = \sup_{m \in \mathcal{M}^\phi} h(m)$.
2. $P_\phi$ is monotone, in the sense that if $F \geq G$ then $P_\phi(F) \geq P_\phi(G)$.

The following Abramov formula relates the measure theoretic entropies of the flow $\phi$ and its reparametrization $\phi^F$. 

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**Theorem** (Abramov formula, [1]). Suppose $\phi$ is a continuous flow on $X$ and $\phi^F$ is the reparametrization of $\phi$ by a positive continuous function $F$, then for all $\mu \in \mathcal{M}^0$

$$h_{\phi^F}(\overline{F_*\mu}) = \frac{h_\phi(\mu)}{\int_X Fd\mu}.$$ 

The following Bowen’s formula links the topological entropy of the reparametrized flow $\phi^F$ and the reparametrization function $F$.

**Theorem 7** (Bowen’s formula, Sambarino [34]). If $\phi$ is a continuous flow on a compact metric space $X$ and $F : X \to \mathbb{R}$ is a positive continuous function, then

$$P_\phi(-hF) = 0$$

if and only if $h = h_{\phi^F}$. Moreover, if $h = h_{\phi^F}$ and $m$ is an equilibrium state of $-hF$, then $\overline{F_*m}$ is a measure of maximal entropy of the reparametrized flow $\phi^F$.

### 2.2.4 Anosov Flows

A $C^{1+\alpha}$ flow $\phi_t : X \to X$ on a compact manifold $X$ is called **Anosov** if there is a continuous splitting of the unit tangent bundle $T^1X = E^0 \oplus E^s \oplus E^u$, where $E^0$ is the one-dimensional bundle tangent to the flow direction, and there exists $C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$ for $t \geq 0$. We say that the flow is **transitive** if there is a dense orbit.

**Example.** Let $M$ be a compact Riemannian manifold with negative sectional curvature and $\phi : T^1M \to T^1M$ be the geodesic flow on the unit tangent bundle of $M$. Then $\phi_t : T^1M \to T^1M$ is a transitive Anosov flow.

Recall that a function $F : X \to \mathbb{R}$ is called **$\alpha$–Hölder continuous** if there exists $C > 0$ and $\alpha \in (0, 1]$ such that for all $x, y \in X$ we have $|F(x) - F(y)| \leq C \cdot d_X(x, y)^\alpha$. In most cases, we will abbreviate **$\alpha$–Hölder continuous** to **Hölder continuous**.
2.2.4.1 Suspension Flows

Although we will not use the following results, we still want to state them and to point out that dynamics of Anosov flows over compact metric spaces can be understood through symbolic dynamics, namely, suspension flows.

Let \((\Sigma^+_\mathcal{A}, \sigma)\) be a subshift of finite type and \(r \in C^\alpha(\Sigma^+_\mathcal{A})\) a positive Hölder continuous function. The \textit{suspension space} (relative to \(r\)) is defined as

\[ \Sigma_r := \{ (x, t) : x \in \Sigma^+_\mathcal{A}, 0 \leq t \leq r(x) \} \]

with the identification \((x, r(x)) = (0, \sigma(x))\), and the function \(r\) is called the \textit{roof function} of \(\Sigma_r\). We further define the suspension flow \(\sigma^r\) on \(\Sigma_r\) by

\[ \sigma^r_t(x, s) = (x, t + s) \]

for \(-s \leq t \leq r(x) - s\).

The following theorem describes how one can understand Anosov flows through suspension flow. Hence we can easily transform problems for Anosov flows into symbolic dynamics problems.

\textbf{Theorem 8} (Lemma 9.1, [30]). \textit{We can associate to a topologically transitive Anosov flow} \(\phi_t : X \to X\) \textit{a suspension flow} \(\sigma^r_t : \Sigma_r \to \Sigma_r\) \textit{and a continuous map} \(\pi : \Sigma_r \to X\) \textit{such that}

1. \(\pi\) \textit{is surjective and} \(\phi_t \pi = \pi \sigma^r_t\);

2. \(\pi\) \textit{is bounded-to-one, and one-to-one on a residual set, that is, a full measure set for every ergodic invariant measure of total support for} \(\sigma^r_t\); \textit{and}

3. \(\pi\) \textit{is an isomorphism (with respect to the unique measures of maximal entropy) and in particular, they have the same topological entropy.}
2.2.5 Variance and Derivatives of the Pressure

In this section, we shall recall the definition and basic properties of the variance. Let \( \phi_t : X \to X \) be a transitive Anosov flow on a compact metric space \( X \), and \( C^\alpha(X) \) be the set of \( \alpha \)-Hölder continuous function on \( X \). One should compare these definitions and result with their symbolic dynamic versions we stated in Section 2.1.

**Definition 10.** Suppose \( F \in C^\alpha(X) \) and \( m_F \) is the equilibrium state of \( F \). For any \( H, G \in C^\alpha(X) \) with mean zero (i.e., \( \int G \, dm_F = \int H \, dm_F \)), we define the covariance of \( G \) and \( H \) with respect to \( m_F \) by

\[
\text{Cov}(G, H, m_F) := \lim_{T \to \infty} \frac{1}{T} \int_X \left( \int_0^T H(\phi_t(x)) \, dt \right) \left( \int_0^T G(\phi_t(x)) \, dt \right) \, dm_F(x).
\]

Similarly, for \( G \in C^\alpha(X) \) with mean zero we define the variance of \( G \) with respect to \( m_F \) as

\[
\text{Var}(G, m_F) := \text{Cov}(G, G, m_F).
\]

The following properties give us some handy formulas of the derivatives of the pressure.

**Proposition 3** (Parry-Pollicott, Prop 4.10, 4.11 [30]). Suppose that \( \phi_t : X \to X \) is a transitive Anosov flow on a compact metric space \( X \), and \( F, G \in C^\alpha(X) \). If \( m_F \) is the equilibrium state of \( F \), then

1. The function \( t \mapsto P_\phi(F + tG) \) is analytic;

2. The first derivative is given by

\[
\left. \frac{dP_\phi(F + tG)}{dt} \right|_{t=0} = \int_X G \, dm_F;
\]
3. If $\int G \text{d}m_F = 0$, then the second derivative can be formulated as

$$\left. \frac{d^2 P_\phi(F + tG)}{dt^2} \right|_{t=0} = \text{Var}(G, m_F); \text{ and}$$

4. If $\text{Var}(G, m_F) = 0$, then $G$ is Livšic cohomologous to zero.

Remark 12. The above theorem should be compared with Theorem 5. In fact, this theorem can be proved by conjugating the Anosov flow to a suspension flow (cf. Theorem 2.1) and then the above result is a direct consequence of Theorem 5.

2.2.6 Pressure Metric

Here we keep the same setting as in the previous section that $\phi_t : X \to X$ is a transitive Anosov flow on a compact metric space $X$. We consider the space $\mathcal{P}(X)$ of Livšic cohomology classes of pressure zero Hölder continuous functions on $X$, i.e.,

$$\mathcal{P}(X) := \{F ; F \in C^\alpha(X) \text{ for some } \alpha \text{ and } P_\phi(F) = 0 \} / \sim .$$

The tangent space of $\mathcal{P}(X)$ at $F$ is

$$T_F \mathcal{P}(X) = \ker ((\mathcal{D}P_\phi)(F)) = \{G ; G \in C^\alpha(X) \text{ for some } \alpha \text{ and } \int G \text{d}m_F = 0 \} / \sim ,$$

where $m_F$ is the equilibrium state of $F$. For $G \in T_F \mathcal{P}(X)$, we define the (McMullen) pressure metric as

$$\|G\|_P^2 := \frac{\text{Var}(G, m_F)}{-\int F \text{d}m_F}.$$ 

Proposition 4. If $\{c_t\}_{t \in (-1,1)}$ is an analytic one parameter family contained in $\mathcal{P}(X)$, then

$$\|\dot{c}_0\|_P^2 = \frac{\int \dot{c}_0 \text{d}m_{\phi_0}}{\int c_0 \text{d}m_{\phi_0}}.$$
where $\dot{c}_0 = \left. \frac{d}{dt} c_t \right|_{t=0}$ and $\ddot{c}_0 = \left. \frac{d^2}{dt^2} c_t \right|_{t=0}$.

**Proof.** The proof is identical to the proof of Proposition 2 which is the symbolic dynamics version of this result.

\[\square\]

### 2.3 Negatively Curved Manifolds and the Group of Isometries

In this section, we survey several facts of $\delta$–hyperbolic spaces and their group of isometries. A good reference is the book [14] edited by Ghys and de la Harpe.

#### 2.3.1 $\delta$–Hyperbolic Spaces

A metric space $(X, d)$ is said to be *geodesic* if any two points $x, y \in X$ can be joined by a *geodesic segment* $[x, y]$ that is a naturally parametrized path from $x$ to $y$ whose length is equal to $d(x, y)$, and is called *proper* if all closed balls are compact.

**Definition 11.** A geodesic metric space $(X, d)$ is called $\delta$–*hyperbolic* (where $\delta \geq 0$ is some real number) if for any geodesic triangle in $X$ each side of the triangle is contained in the $\delta$–neighborhood of the union of two other sides. A metric space $(X, d)$ is called *hyperbolic* if it is $\delta$–*hyperbolic* for some $\delta \geq 0$.

In the following, we list two types of hyperbolic spaces appearing in this thesis.

**Example.**

1. **Pinched Hadamard manifold** $(\widetilde{M}, d_{\tilde{g}})$: a complete and simply connected Riemannian manifold $(\widetilde{M}, \tilde{g})$ whose sectional curvature is bounded by two negative numbers. The metric on $\widetilde{M}$ is the distance function $d_{\tilde{g}}$ induced by the Riemannian metric $\tilde{g}$.

2. The **Cayley graph** $C(G, S)$ and its *word metric* $w$: given a finitely generated group $G$ and a finite generating set $S$ of $G$, $C(G, S)$ is a graph whose vertices are elements of $G$. Two vertices $g, h \in G$ are connected by an edge if and only
if there is a generator $s \in S$ such that $h = gs$. The word metric $w$ on $C(G, S)$ is defined as assuming that each edge has unit length, and $w(g, h)$ is the minimum of the length of all paths connecting $g$ and $h$.

**Remark 13.** A group $G$ is called hyperbolic if for one (and for all) finite generating set the Cayley graph is hyperbolic. For example, finitely generated free groups and surface groups for surfaces with genus $> 1$.

We say that two geodesic rays $\tau_1 : [0, \infty) \to X$ and $\tau_2 : [0, \infty) \to X$ are equivalent and write $\tau_1 \sim \tau_2$ if there is a $K > 0$ such that for all $t > 0$

$$d(\tau_1(t), \tau_2(t)) < K.$$  

It is easy to see that $\sim$ is indeed an equivalence relation on the set of geodesic rays. We then define the geometric boundary $\partial_\infty X$ of $X$ by

$$\partial_\infty X := \{ [\tau] : \tau \text{ is a geodesic ray in } X \}.$$ 

Moreover, we know that when $X$ is proper, $\partial_\infty X$ is metrizable. More precisely, the following visual metric, given by Gromov, defines a metric on $\partial_\infty X$.

**Definition 12.** Let $(X, d)$ be a $\delta$–hyperbolic proper metric space. Let $a > 1$ and let $o \in X$ be a basepoint. We say that a metric $d_a$ on $\partial_\infty X$ is a visual metric with respect to the basepoint $o$ and the visual parameter $a$ if there exists $C > 0$ such that for any two distinct point $\xi, \eta \in \partial_\infty X$, and for any biinfinite geodesic $\tau$ connecting $\xi$ to $\eta$ in $X$ we have

$$\frac{1}{C} a^{-d(o, \tau)} \leq d_a(\xi, \eta) \leq C a^{-d(o, \tau)}.$$ 

**Theorem** (Gromov, cf. Theorem 1.5.2 [3]). There is $a_0 > 1$ such that for any basepoint $o \in X$ and any $a \in (1, a_0)$ the boundary $\partial_\infty X$ admits a visual metric $d_a$ with respect to $o$. 

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Remark 14 (cf. Ch.11 [10], Remark 1.5.3 [3]). :

1. For a pinched Hadamard manifold $M$ whose sectional curvature is not larger than $-b^2$, the boundary $\partial_\infty M$ admits a visual metric $d_a$ for all $a \in (1, e^b]$.

2. Let $d_a$ and $d_{a'}$ be two different visual metrics with respect to a fixed basepoint $o \in X$ and different visual parameters $a$ and $a'$, then $d_a$ and $d_{a'}$ are Hölder equivalent, i.e., there exists $C \geq 1$ and $\alpha = \frac{\log a'}{\log a}$ such that for $\xi, \eta \in \partial_\infty X$

$$\frac{1}{C} \cdot (d_a(\xi, \eta))^\alpha \leq d_{a'}(\xi, \eta) \leq C \cdot (d_a(\xi, \eta))^\alpha.$$

3. Let $d_a$ and $d'_a$ be two different visual metrics with respect to a fixed visual parameter $a$ and different basepoints $o, o' \in X$, then the metric $d_a$ and $d'_a$ are Lipchitz, i.e., there exists $C \geq 1$ such that for all $\xi, \eta \in \partial_\infty X$

$$\frac{1}{C} \cdot d_a(\xi, \eta) \leq d'_a(\xi, \eta) \leq C \cdot d_a(\xi, \eta).$$

2.3.2 Quasi-Isometries

Definition 13. A function $q : X \to Y$ from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$ is called a $(C, L)$-quasi-isometry embedding if there is $C, L > 0$ such that:

For any $x, x' \in X$, we have

$$\frac{1}{C} d_X(x, x') - L \leq d_Y(q(x), q(x')) \leq C \cdot d_X(x, x') + L.$$

If, in addition, there exists an approximate inverse map $\tilde{q} : Y \to X$ that is a $(C, L)$-quasi-isometric embedding such that for all $x \in X$ and $y \in Y$

$$d_X(\tilde{q}q(x), x) \leq L, \quad d_Y(q\tilde{q}(y), y) \leq L,$$
then we call $q$ a $(C, L)$-quasi-isometry. $(X, d_X)$ and $(Y, d_Y)$ are called quasi-isometric.

In most cases, the quasi-isometry constants $C$ and $L$ do not matter, so we shall use the words quasi-isometry and quasi-isometry embedding without specifying constants.

**Theorem 9** (Bourdon, Theorem 1.6.4 [3]). Let $(X, d_X)$ and $(Y, d_Y)$ be hyperbolic spaces. Suppose the boundaries are equipped with visual metrics. Then

1. Any quasi-isometry embedding $q : X \to X'$ extends to a bi-Hölder embedding $q : \partial_X \to \partial_Y$ with respect to the corresponding visual metrics.

2. Any quasi-isometry $q : X \to X'$ extends to a bi-Hölder homeomorphism $q : \partial_X \to \partial_Y$ with respect to the corresponding visual metrics.

**Definition 14.** A $(C, L)$-quasi-geodesic is a $(C, L)$-quasi-isometry embedding $q : \mathbb{R} \to X$.

**Theorem 10** (Morse Lemma, cf. Ch.5, Theorem 6 [14]). Suppose $X$ and $Y$ are hyperbolic spaces, and $q : X \to Y$ is a $(C, L)$-quasi-isometry. Then, for every geodesic $\gamma \subset X$, its image $q(\gamma)$ is a quasi-geodesic on $Y$ and is within a bounded distance $R$ from a geodesic on $Y$. Moreover, this constant $R$ is only depending on $X$, $Y$, and the quasi-isometry constants $C$ and $L$.

**Remark 15.** When the space $Y$ is a pinched Hadamard manifold, we have a stronger result of the above theorem. Namely, every geodesic $\gamma \subset X$ its image $q(\gamma)$ is a quasi-geodesic on $Y$ and is within a bounded distance $R$ from a unique geodesic on $Y$.

Let $X$ be a hyperbolic space. We denote its group of isometries by $\text{Isom}(X)$. The following lemma connects some subgroups of $\text{Isom}(X)$ and the hyperbolic space $X$.

**Theorem 11** (Svarc-Milnor lemma, cf. Lemma 3.37 [20]). Let $X$ be a proper geodesic metric space. Let $G$ be a subgroup of $\text{Isom}(X)$ acting properly discontinuously and
cocompactly on $X$. Pick a point $o \in X$. Then the group $G$ is finitely generated; for any choice of finitely generating set $S$ of $G$, the map $q: G \to X$, given by $q(\gamma) = \gamma(o)$, is a quasi-isometry. Here $G$ is given the word metric induced from $C(G, S)$.

2.3.3 Negatively Curved Manifolds and the Group of Isometries

Let $(X, g)$ be a negatively curved compact Riemannian manifold. The universal covering $(\tilde{X}, \tilde{g})$ of $(X, g)$ is a pinched Hadamard manifold, and $\pi_1 X$ is finitely generated and acting canonically on $\tilde{X}$. Let $\Gamma$ denote the group of deck transformations of the covering $\tilde{X}$. We know that $\Gamma \subset \text{Isom}(\tilde{X})$, $\Gamma \cong \pi_1 X$, and $X$ is isometric to $\Gamma \backslash \tilde{X}$. More precisely, using generators there exists a natural isomorphism $i_X : \pi_1 X \to \Gamma$, given by $\gamma_X := i_X(\gamma)$, $\forall \gamma \in \pi_1 X$. Thus, using this isomorphism, we can define a $\pi_1 X$-action on $\tilde{X}$ by $\gamma \cdot x = (\gamma_X)(x)$. It’s clear that this $\pi_1 X$-action is nothing different from the $\Gamma$-action on $\tilde{X}$.

Because $(X, g)$ is negatively curved, every $\gamma \in \Gamma$ corresponds to a unique geodesic $\tau^X_\gamma$ on $X$. Besides, each conjugacy class $[\gamma] \in [\Gamma]$ corresponds to a unique closed geodesic $\tau^X_\gamma$ on $X$ and vice versa. Moreover, the length of the closed geodesic $\tau^X_\gamma$ is exactly the translation distance of $\gamma \in \pi_1 X$ (i.e., $l_g(\tau^X_\gamma) = l_g[\gamma] := \inf_{x \in X} d_g(x, \gamma \cdot x)$).

The following remarkable theorem of Margulis gives another characterization of the topological entropy of geodesic flow through the (exponential) growth rate of closed geodesics. This perspective of the topological entropy is crucial in the rest of this thesis.

**Theorem 12** (Margulis [27]). Let $(X, g)$ be a compact negatively curved Riemannian manifold and $\Gamma$ be the group of deck transformations of $\tilde{X}$, then the topological entropy $h(X)$ of the geodesic flow on $T^1 X$ is given by

$$h(X) = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1 X] : l_g[\gamma] \leq T \}.$$
Now let’s consider a compact 3–manifold $M$ equipped with a hyperbolic metric $h$. Then there exists a discrete and faithful representation $\rho : \pi_1 M \to \text{Isom}(\mathbb{H}^3)$ such that $M \cong \rho(\pi_1 M) \backslash \mathbb{H}^3$ where $(\mathbb{H}^3, \tilde{h})$ is the universal covering of $(M, h)$. For the sake of brevity, in what follows we will denote the lifted metric of $\tilde{h}$ on $\mathbb{H}^3$ by $h$.

**Definition 15.** Let $\Gamma$ be a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$. The limit set $\Lambda(\Gamma)$ is the set of limit points $\Gamma x$ for any $x \in \mathbb{H}^3$.

**Definition 16.** The critical exponent $\delta_\Gamma$ is defined as

$$\delta_\Gamma := \inf \{ s : \sum_{\gamma \in \Gamma} e^{-sd_h(x, \gamma x)} < \infty \},$$

for any point $x \in \mathbb{H}^3$ and $d_h$ is the hyperbolic distance on $\mathbb{H}^3$.

**Definition 17.** A discrete subgroup $\Gamma$ of $\text{Isom}(\mathbb{H}^3)$ is called convex cocompact if $\Gamma$ acts cocompactly on the convex hull of the limit set of $\Gamma$, i.e., $\Gamma \backslash \text{Conv}(\Lambda(\Gamma))$ is compact.

The following theorem of Sullivan gives another point of view of the critical exponent which is similar to Margulis’ theorem for entropy stated above.

**Theorem 13 (Sullivan [36]).** Suppose $\Gamma$ is a non-elementary, convex cocompact, and discrete subgroup of $\text{Isom}(\mathbb{H}^3)$, then

$$\delta_\Gamma = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\Gamma] : l_h(\gamma) \leq T \},$$

where $l_h(\gamma) = d_h(o, \gamma o)$, $o$ is the origin of $\mathbb{H}^3$. 

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CHAPTER 3

SPACES OF METRIC GRAPHS

3.1 Overview and Background

This chapter is organized as follows. We start from recalling relevant background on the Weil-Petersson metric in Section 3.1.1. In Section 3.2, the main body of this chapter, we relate the deformation space of metric graphs with a subshift of finite type and construct pressure metrics for metric graph via this relation. Furthermore, we give several useful formulas for them. In the last section of this chapter, using formulas derived in Section 3.2, we discuss Riemannian geometry induced by these two pressure metrics through working on specific examples. Explicit computations carried out from these examples show us some interesting Riemannian geometry features which are different from “classical” Weil-Petersson geometry. We summarize these results in Proposition 16.

3.1.1 The Weil-Petersson-(McMullen) Metric on Moduli Spaces

In this section, we recall several facts of the Weil-Petersson metric on moduli spaces. These geometric features of the Weil-Petersson metric are critical clues for us to investigate the pressure metric geometry. The following approach to the Weil-Petersson metric is not traditional. The standard construction of the Weil-Petersson metric is built on the complex structure of Teichmüller space. Whereas, without employing the complex structure, McMullen in [28] gave a new characterization of the Weil-Petersson metric via the thermodynamic formalism. This point of view inspires
our study in this thesis. Moreover, we shall remark here that McMullen’s approach

to the “classical” Weil-Petersson metric is close to Thurston’s Riemannian metric on
Teichmüller space (cf. an elegant expository paper [8] by Bridgeman, Canary, and
Sambarino).

We summarize McMullen’s theorem in Theorem 14. Before we jump into the
statement, we recall some definitions. Given a compact topological surface $S$ with
negative Euler characteristic, the Teichmüller space $\mathcal{T}(S)$ describes the marked Rie-
mannian metrics on $S$, i.e., the conformal classes of Riemannian metric on $S$ with
constant Gaussian curvature $-1$. The moduli space $\mathcal{M}(S)$ describes the unmarked
Riemannian metrics on $S$ and is obtained by quotienting $\mathcal{T}(S)$ by the Mapping Class
Group of $S$.

We consider a $C^1$ family of metric $g_{\lambda} \in \mathcal{M}(S)$, $0 \leq \lambda \leq 1$. Let $T^1S$ be the
unit tangent bundle of the surface $S$ with respect to the metric $g_{\lambda_0}$. Let $\mu_{\lambda_0}$ be
the corresponding Liouville measure on $T^1S$. We denote by $\phi^{(\lambda_0)}_t : T^1S \to T^1S$ the
godesic flow. Since $g_{\lambda}$, for $0 \leq \lambda \leq 1$, is a volume preserving deformation we have

$$
\int \dot{g}_{\lambda_0}(v,v)d\mu_{\lambda_0}(v) = 0,
$$

where $\dot{g}_{\lambda_0}$ is defined via the expansion

$$
g_{\lambda} = g_{\lambda_0} + \dot{g}_{\lambda_0}(\lambda - \lambda_0) + O((\lambda - \lambda_0)^2).
$$

(cf. Lemma 7. (a) and (c) [31].)

**Definition 18.** The variance for $\dot{g}_{\lambda_0}(v,v)$ is defined as

$$
\text{Var}(\dot{g}_{\lambda_0}, \mu_{\lambda_0}) := \lim_{t \to \infty} \frac{1}{t} \int \left( \int_0^t \dot{g}_{\lambda_0}(\phi^{(\lambda_0)}_s(v), \phi^{(\lambda_0)}_s(v))ds \right)^2 d\mu_{\lambda_0}.
$$

**Theorem 14** (McMullen, Theorem 1.12 [28]). The Weil-Petersson metric is propor-
tional to the variance. More precisely,

$$\|\dot{g}_\lambda\|^2_{\text{Pressure}} := \frac{\text{Var}(\dot{g}_\lambda, \mu_\lambda)}{\int_{T^1S} g_\lambda(v, v) d\mu_\lambda} = \frac{4}{3} \cdot \frac{\|\dot{g}_\lambda\|^2_{WP}}{\text{area}(S, g_\lambda)}.$$ 

3.2 Pressure Metrics on the Space of Metric Graphs

In what follows, $\mathcal{G}$ denotes a finite, connected, nontrivial (i.e., which contain at least two distinct closed paths), and undirected graph with edge set $\mathcal{E}$ with valence no less than three. The length of each edge is given by the edge weighting function $l : \mathcal{E} \to \mathbb{R}_{>0}$.

**Definition 19.** Let $\mathcal{M}_\mathcal{G}$ denote the space of all edge weightings $l : \mathcal{E} \to \mathbb{R}_{>0}$ on $\mathcal{G}$.

**Definition 20.** The entropy $h(l)$ of the metric graph $(\mathcal{G}, l)$ is defined by

$$h(l) = \lim_{T \to \infty} \frac{1}{T} \log \# \{ \gamma : l(\gamma) < T \},$$

where $\gamma = (e_0, e_1, ..., e_n = e_0)$ is a closed cycle of edges in $\mathcal{G}$ (without backtracking) and $l(\gamma) = \sum_{i=0}^{n-1} l(e_i)$.

**Definition 21.** $\mathcal{M}_\mathcal{G}^1 = \{ l : \mathcal{E} \to \mathbb{R}_{>0} : h(l) = 1 \}$ is the space of all edge weightings with entropy $h(l) = 1$.

**Remark 16.** One might consider taking other normalizations on the space $\mathcal{M}_\mathcal{G}$, for example one can normalize $\mathcal{M}_\mathcal{G}$ by the volume of the graph, i.e., consider the moduli space $\mathcal{M}_\mathcal{G}^v := \{ l : \mathcal{E} \to \mathbb{R}_{>0} : \text{Vol}(\mathcal{G}) = \sum l(e_i) = 1 \}$. Notice that in the construction of pressure metrics for subshifts of finite type we only consider functions of pressure zero. In general, for any positive Hölder functions $f$, there exists a positive number $h_f$ such that $-h_f f$ is pressure zero (Bowen’s formula, cf. Theorem 6). Therefore, for each edge weighting function $l$, we can always scale $l$ to be the pressure zero function $-h_l l$. So, for example, pressure metrics on $\mathcal{M}_\mathcal{G}^v$ are the pullback of pressure metrics.
on $\mathcal{M}_G^1$ by the map $S : \mathcal{M}_G^0 \to \mathcal{M}_G^1$ where $S(l) = -hl$. Hence, it is enough to study pressure metrics on $\mathcal{M}_G^1$.

3.2.1 From Undirected Graphs to Directed Graphs

In this section, we closely follow the construction of Pollicott and Sharp in [32] associating a symbolic model with metric graphs. It is well-known that we can associate each directed graph with an adjacency matrix which records the directed edges connecting vertices to vertices.

To put directions on the undirected graph, we do the following. Given an undirected graph $G$, for each edge $e \in \mathcal{E}$ we associate $e$ with two directed edges which, abusing notation, we shall denote by $e$ and $\bar{e}$, i.e., two opposite directions $\rightarrow e \leftarrow$. We denote by $\mathcal{E}^0$ the set of all directed edges. We say $e' \in \mathcal{E}^0$ follows $e \in \mathcal{E}$ if $e'$ begins at the terminal endpoint of $e$, i.e., $\Rightarrow e'. We then define a $|\mathcal{E}^0| \times |\mathcal{E}^0|$ matrix $A$, with rows and columns indexed by $\mathcal{E}^0$, by

$$A(e, e') = \begin{cases} 1 & \text{if } e' \text{ follows } e \text{ and } e' \neq \bar{e} \\ 0 & \text{otherwise.} \end{cases}$$

Then the shift space

$$\Sigma_A = \{ \underline{e} = (e_n)_{n \in \mathbb{Z}} \in (\mathcal{E}^0)^\mathbb{Z} : A(e_n, e_{n+1}) = 1 \ \forall n \in \mathbb{Z} \}$$

can be naturally identified with the space of all two-sided infinite path (with a distinguished zeroth edge) in the graph $G$. We call $A$ the adjacency matrix of the undirected graph $G$.

Remark 17. It is not hard to see $A$ is irreducible. Recall that the graph $G$ is connected and with valence no less than three, so after we associate two (opposite) directions
to each edge of $\mathcal{G}$, we know that for each pair of vertices of $\mathcal{G}$ there exists a directed path connecting them.

Besides, Bowen showed that each two-sided subshift of finite type $(\Sigma_A, \sigma)$ can be characterized by a one-sided subshift of finite type $(\Sigma_A^+, \sigma)$. More precisely, each Hölder function on $\Sigma_A$ (two-sided sequences) is cohomologous to a Hölder function only depending on the future coordinates (one-sided sequences). Thus, it is enough to study the one-sided subshift of finite type $(\Sigma_A^+, \sigma)$, which is what we will do in the following.

The edge weighting function $l : \mathcal{E} \to \mathbb{R}_{>0}$, our primary player, is clearly well-defined under the directed graph setting. Moreover, each edge weighting can be identified as a locally constant function on $\Sigma_A^+$ by, abusing the notation, $l(\varepsilon) := l(e_0)$ where $\varepsilon = e_0e_1... \in \Sigma_A^+$. Therefore, $\mathcal{M}_G$ can be naturally regarded as a set of positive locally constant functions on $\Sigma_A^+$. By definition, we can rewrite the entropy of the metric graph $(\mathcal{G}, l)$ by

$$h(l) := \lim_{T \to \infty} \frac{1}{T} \log \# \{ \gamma : l(\gamma) < T \}$$

$$= \lim_{T \to \infty} \frac{1}{T} \log \# \left\{ \varepsilon \in \Sigma_A^+ : \sigma^n(\varepsilon) = \varepsilon \text{ and } \sum_{i=0}^{n-1} l(e_i) < T \text{ for some } n \in \mathbb{N} \right\}$$

$$= h_l.$$

Hence, we have $\mathcal{M}_G^1 = \{ l \in C(\Sigma_A^+); l > 0, l(\varepsilon) = l(e_0) \text{ and } h_l = 1 \}$.

**Lemma 1** (Lemma 3.1 [32]).

1. The pressure function is analytic on the space of locally constant functions.

2. The entropy $h(l)$ of $(\mathcal{G}, l)$ is characterized by $P(-h(l) \cdot l) = 0$.

3. The entropy function $\mathcal{M}_G \ni l \mapsto h(l) \in \mathbb{R}_{>0}$ varies analytically for $l > 0$.  

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Proof. The first two assertions come from Theorem 4 and Theorem 6, respectively. The last one is a consequence of the implicit function theorem.

3.2.2 Two Pressure Metrics

Following the discussion in Section 2.1.5, we can similarly define pressure metrics on $\mathcal{M}_G^1$. Specifically, because $l \in \mathcal{M}_G^1$, we know $h(l) = 1$ and $P(-h(l)l) = P(-l) = 0$. Hence, for each $l \in \mathcal{M}_G^1$ there exists a unique equilibrium state $m_{-l}$ with respect to the function $-l$.

**Definition 22.** We define the tangent space to $\mathcal{M}_G^1$ at $l$ by

$$T_l \mathcal{M}_G^1 = \left\{ \phi : \Sigma_A^+ \to \mathbb{R} : \phi(e) = \phi(e_0) \text{ for all } e \in \mathcal{E}^0, \text{ and } \int \phi dm_{-l} = 0 \right\}.$$

**Definition 23.** For $\phi \in T_l \mathcal{M}_G^1$, we define two pressure metrics:

$$\|\phi\|^2_{PS} := \text{Var}(\phi, m_{-l})$$

and

$$\|\phi\|^2_M := \frac{\text{Var}(\phi, m_{-l})}{\int ldm_{-l}}.$$

The former one is called the *Pollicott-Sharp pressure metric* on $\mathcal{M}_G^1$ and denoted by $\| \cdot \|_{PS}$, and the latter one is called the *McMullen pressure metric* on $\mathcal{M}_G^1$ and denoted by $\| \cdot \|_M$. We define the length of every continuously differentiable curve $\gamma : [0, 1] \to \mathcal{M}_G^1$ by

$$L_{PS}(\gamma) = \int_0^1 \|\dot{\gamma}\|_{PS} dt \text{ and } L_{M}(\gamma) = \int_0^1 \|\dot{\gamma}\|_{M} dt$$

and two path space metrics on $\mathcal{M}_G^1$ by $d_{PS}(l_1, l_2) = \inf_{\gamma}\{L_{PS}(\gamma)\}$ and $d_{M}(l_1, l_2) = \inf_{\gamma}\{L_{M}(\gamma)\}$, where the infimum is taken over all continuously differentiable curves.
with $\gamma(0) = l_1$ and $\gamma(1) = l_2$.

Remark 18.

1. Recall (cf. Section 2.1.5) that the covariance defines an inner product over $T_{l} \mathcal{M}_G$, and thus the variance defines a norm over $T_{l} \mathcal{M}_G$. Therefore, $| \cdot |_{PS}$ and $| \cdot |_{M}$ are Riemannian metrics.

2. Notice that our McMullen pressure metric for graphs is different from the Weil-Petersson type metric for graphs mentioned in Pollicott and Sharp’s paper [32]. In fact, their Weil-Petersson type metric is our Pollicott-Sharp pressure metric. Readers should be careful about this difference.

3. Another reason that we consider the McMullen pressure metric, $| \cdot |_{M}$, is because it also coincides with the definition of the pressure metric on the Hitchin component introduced by Bridgeman, Canary, Labourie and Sambarino in [7]. Moreover, their approach can trace back to Thurston (see Bridgeman-Canary-Sambarino [8] for more details.)

Because $l$ is a locally constant function, most of the above formulas for pressure metrics, variances, and equilibrium states can be simplified quite a bit. The remainder of this section is dedicated to expressing those quantities in simpler forms.

By Proposition 1, we consider the matrix $A_{-l}$ associating to $-l$ defined by $A_{-l}(i, j) = A(i, j)e^{-l(i)}$. Notice that because $h(l) = 1$, $A_{-l}$ has a simple maximum eigenvalue 1. Let $v_{-l}$ be the left eigenvector of $A_{-l}$ w.r.t. 1 (i.e., $v_{-l} \cdot A_{-l} = v_{-l}$). Then we construct the column stochastic matrix $P_{-l}$ associating with $-l$:

$$P_{-l}(i, j) = \frac{A(i, j)v_{-l}(i)e^{-l(i)}}{v_{-l}(j)} ,$$

and let $p_{-l}$ be the unit right eigenvector of $P_{-l}$ w.r.t. 1 (i.e., $P_{-l}p_{-l} = p_{-l}$) and
Thus we know that the equilibrium state \( m_{-l} \) is given by

\[
m_{-l}[i_0, i_1, \ldots, i_n] = P_{-l}(i_0, i_1) \cdot \ldots \cdot P_{-l}(i_{n-1}, i_n)p_{-l}(i_n).
\]

When there is no ambiguity, for shortness, we will drop the subscript \( l \) from \( v_{-l}, P_{-l} \) and \( p_{-l} \) and denote them by \( v, P \) and \( p \), respectively.

**Proposition 5.** We have

1. For each \( l \in M_G^1 \), the tangent space to \( M_G^1 \) at \( l \) is

\[
T_l M_G^1 = \left\{ \phi \in C(\Sigma_A^+) : \phi(\varepsilon) = \phi(\varepsilon_0) \text{ and } \sum_{e \in \mathcal{E}^0} \phi(e)p_e = 0 \right\},
\]

where \( p_e = p(e) \).

2. Let \( l_t \) be an analytic path in \( M_G^1 \), and \( g_t \) be the (up to cohomology) normalized function cohomologous to \( -l_t \) (i.e., \( \mathcal{L}_{g_t}1 = 1 \) and \( -l_t = g_t + h_t \circ \sigma - h_t \)) and let

\[
P(i, j) = P_{-l}(i, j) = A(i, j)e^{\theta_0(i, j)}, \text{ and } \text{p be the right eigenvector of } P \text{ w.r.t. } l_t (i.e., \text{Pp = p}) \text{ then}
\]

\[
\text{Var}(-\dot{\theta}_0, m_{-l_0}) = \int (\ddot{\theta}_0)dm_{-l_0} = \sum_i \ddot{\theta}_0(i)p_i = \int (\dot{g}_0)^2dm_{g_0} = \sum_{i,j} (\dot{g}_0(i, j))^2 P_{ij}p_j. \tag{3.2.3}
\]

**Proof.** (1): It is obvious.

(2): By Proposition 1, we know that \( m_{-l_0}[e_i] = p_{e_i} \) for each \( i \) and \( m_{-l_0}[e_i, e_j] = P(e_i, e_j)p(e_j) \) for all \( e_i, e_j \in \mathcal{E}^0 \).
For equation (3.2.2), by Theorem 5, we know

\[ \text{Var}(-\dot{l}_0, m_{-l_0}) = \int (\ddot{l}_0) dm_{-l_0} = \sum_i \int (\ddot{l}_0) dm_{-l_0} = \sum_i \ddot{l}_0(i) p_i. \]

For equation (3.2.3), since \( P(-h(l))l = 0 \) and by Theorem 5, we have \( \int \dot{l}_0 dm_{-l_0} = 0 \). Moreover, because for each Hölder continuous function the equilibrium state is unique in each Livšic cohomology class we know that \( m_{-l_0} = m_{g_0} \). Since \( -\dot{l}_0 \sim \dot{g}_0 \) and \( -\ddot{l}_0 \sim \ddot{g}_0 \), we have \( \int \dot{l}_0 dm_{-l_0} = \int \dot{g}_0 dm_{g_0} = 0 \) and \( \int -\ddot{l}_0 dm_{-l_0} = \int \ddot{g}_0 dm_{g_0} \).

Because \( L_{g_0} 1 = 1 \) and \( \int \dot{g}_0 dm_{g_0} = 0 \), by Theorem 3 we have

\[ \text{Var}(-\dot{l}_0, m_{-l_0}) = \text{Var}(\dot{g}_0, m_{g_0}) = \int (\dot{g}_0)^2 dm_{g_0} = \sum_{i,j} \int_{i,j} (\dot{g}_0)^2 dm_{g_0} = \sum_{i,j} (\dot{g}_0(i, j))^2 P_{ij} P_j. \]

\[ \square \]

Remark 19. Equation (3.2.3) is very close to the formula given in Lemma 3.3 [32]. However, Lemma 3.3 in [32] is NOT true for all functions in the tangent space \( T_lM^1_G \). It is only true for normalized functions, i.e., \( f \in T_lM^1_G \) and \( L_f 1 = 1 \).

Notice that it is convenient to interpret \( M^1_G \) as a hypersurface in \( M_G \). Because each edge weighting function \( l \) is depending on the weighting (i.e., length) of each edge, it can be regarded as a \( k \)-variable function \( l = l(e_1, e_2, ..., e_k) \) where \( e_1, e_2, ..., e_k \) are edges of \( G \). Thus, from this perspective \( M_G \) is \( \mathbb{R}^k_{\geq 0} \). Moreover, the condition \( h(l) = 1 \) gives us an equation of \( e_1, e_2, ..., e_k \), so \( M^1_G \) is a co-dimension one submanifold in \( M_G \).
3.3 Examples

In this section, we follow the recipe below to calculate pressure metrics. Let $\mathcal{G}$ be an undirected finite graph and $l$ be an edge weighting function then

- First, associate two opposite directions to each edge as we defined in Section 3.2.1, then write down the adjacency matrix $A$ and the weighted adjacency matrix $A_{-h(l)}$ associated with $l$.

- Second, solve the equation $h(l) = 1$. Explicitly, notice that $h(l) = 1$ means that $1$ is an eigenvalue of the matrix $A_{-l}$. Therefore, the characteristic polynomial of $A_{-l}$ is a function of edges $e_1,...,e_k$ of $\mathcal{G}$, i.e., $\det(A_{-l} - \text{Id}) = 0$. By solving the characteristic polynomial, we can write $l = (l(e_1), l(e_2), ..., l(e_k))$ where $l(e_k) = w(l(e_1), ..., l(e_{k-1}))$ is an analytic function depending $l(e_1), ..., l(e_{k-1})$.

- Third, compute the right eigenvector $v_{-l}$ of $A_{-l}$ w.r.t the eigenvalue $1$.

- Fourth, write down the normalized weighted matrix $P_{-l}$ in (3.2.1), and compute the equilibrium state $p$ of $-l$, i.e., the unit left eigenvector $p$ of $P_{-l}$ w.r.t. $1$.

- Fifth, consider $l = (l(e_1), l(e_2), ..., w(l(e_1), l(e_2), ..., l(e_k)))$ as a parametrization of $\mathcal{M}_{\mathcal{G}}^1$, and then compute the tangent vectors $\frac{\partial}{\partial l(e_i)}$ for $1 \leq i \leq k$.

- Last, compute pressure metrics $\frac{\partial}{\partial l(e_i)}$ for $1 \leq i \leq k$.

We notice that graphs of a belt buckle $\mathcal{G}_2$, a dumbbell $\mathcal{G}_3$, and a three-petal rose $\mathcal{G}_3$ are graphs with 3 edges. Therefore, the moduli spaces $\mathcal{M}_{\mathcal{G}_2}^1$, $\mathcal{M}_{\mathcal{G}_3}^1$, and $\mathcal{M}_{\mathcal{G}_3}^1$ are two-dimensional manifolds sitting in $\mathbb{R}^3$. Moreover, in the Riemannian geometry setting, it is natural to study geometric invariants such as curvatures. In dimension two, we shall focus on the Gaussian curvature.

Before we start working on examples, we recall two useful asymptotic notations that $A(x) \sim B(x)$ means $\lim A(x)/B(x) = 1$ and $A(x) \asymp B(x)$ means there exists positive constants $c$ and $c'$ such that $c < A(x)/B(x) < c'$. 47
3.3.1 Figure 8 Graphs

The first example is a figure 8 graph which we denote by $G_1$. The picture below is a figure 8. In the sake of brevity, we denote $l(e_1)$ and $l(e_2)$ by $x$ and $y$, respectively.

![Figure 8](image)

Figure 3.1. Figure 8

Following the recipe, we have

- $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ and $A_{-l} = \begin{pmatrix} e^{-x} & e^{-x} & 0 & e^{-x} \\ e^{-y} & e^{-y} & e^{-y} & 0 \\ 0 & e^{-x} & e^{-x} & e^{-x} \\ e^{-y} & 0 & e^{-y} & e^{-y} \end{pmatrix}$.

- $h(l) = 1 \implies \det(\text{Id} - A_{-l}) = 0 \implies e^{-y} = \frac{1 - e^{-x}}{1 + 3e^{-x}}$.

- $v = \left( \frac{2}{1 + 3e^{-x}}, 1, \frac{2}{1 + 3e^{-x}}, 1 \right)$.

- $P = \begin{pmatrix} e^{-x} & 2e^{-x} & 0 & 2e^{-x} \\ \frac{1 - e^{-x}}{2} & \frac{1 - e^{-x}}{2} & \frac{1 - e^{-x}}{2} & 0 \\ 0 & \frac{2e^{-x}}{1 + 3e^{-x}} & e^{-x} & \frac{2e^{-x}}{1 + 3e^{-x}} \\ \frac{1 - e^{-x}}{2} & 0 & \frac{1 - e^{-x}}{2} & \frac{1 - e^{-x}}{2} \end{pmatrix}$ and
\[ p = \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \end{pmatrix} = \begin{pmatrix} \frac{2e^x}{6e^x + e^{2x} - 3} \\ \frac{2e^x + e^{2x} - 3}{2(6e^x + e^{2x} - 3)} \\ p_1(x) \\ p_2(x) \end{pmatrix}. \]

- Consider the path \( M_{\mathcal{G}_1} \ni l = (l_1, l_2) = (x, -\log \frac{1-e^{-x}}{1+3e^{-x}}) =: c(x). \)

- \( \dot{c}(x) = (1, -\frac{4e^x}{(e^x-1)(e^x+3)}) \) and \( \ddot{c}(x) = (0, \frac{4e^x(e^{2x}+3)}{(e^x-1)^2(e^x+3)^2}). \)

**Proposition 6.** The moduli space of figure 8 graphs is **incomplete** under the Pollicott-Sharp pressure metric \( \| \cdot \|_{|P_S|} \), i.e., \( (M_{\mathcal{G}_1}^1, \| \cdot \|_{P_S}) \) is incomplete.

**Proof.** By Proposition 5, we have

\[
\| \dot{c}(x) \|_{P_S}^2 = \text{Var}(\dot{c}(x), m_c(x)) \\
= 2 \cdot \frac{4e^x(e^{2x} + 3)}{(e^x - 1)^2(e^x + 3)^2} \cdot P_2(x) \\
= \frac{4e^x(e^{2x} + 3)}{-24e^x + 6e^{2x} + 8e^{3x} + e^{4x} + 9}.
\]

When \( x \) is close to zero, we have the expansion

\[
\| \dot{c}(x) \|_{P_S}^2 = \frac{1}{x} - \frac{5}{4} + \frac{85x}{48} + o(x^2).
\]

This estimate shows that when \( x \to 0 \)

\[
\sqrt{\int \| \dot{c}(x) \|_{P_S}^2 dm_c(x)} \asymp \frac{1}{\sqrt{x}}.
\]

Finally, since \( \int_0^1 \frac{1}{\sqrt{x}} dx \) is convergent we see that the metric is incomplete, i.e., the curve arrives at \( l_1 = x = 0 \) in finite time with respect to this metric.

**Remark 20.** There is a hidden natural condition \( l_2 > 0 \) that we have to take into account. This condition was missing in Section 6 [32]. For \( \mathcal{G}_1 \) the condition is \( 1 - e^{-x} > \)
0 and \( \frac{e^{-x}}{1+e^{-x}} < 1 \), which is equivalent to \( x > 0 \).

**Proposition 7.** The moduli space of figure 8 graphs is complete under the McMullen pressure metric \( \| \cdot \|_M \), i.e., \( (\mathcal{M}^1_{\mathcal{G}_1}, \| \cdot \|_M) \) is complete.

**Proof.** Continue with the computation in the previous proposition, we know that

\[
\| \dot{c}(x) \|_M^2 = \frac{\text{Var}(\dot{c}(x), m_c(x))}{2x p_1(x) + 2c(x) p_2(x)} = -\frac{4e^x (e^{2x} + 3)}{(e^x - 1) (e^x + 3) ((2e^x + e^{2x} - 3) \log \left( \frac{e^x - 1}{e^x + 3} \right) - 4e^x x)}.
\]

When \( x \) is close to zero, we know that

\[
\| \dot{c}(x) \|_M^2 \asymp \frac{-1}{x^2 \log x},
\]

and when \( x \) tends to infinity we have

\[
\| \dot{c}(x) \|_M^2 \asymp \frac{1}{x}.
\]

This shows that the curve arrives the boundary of \( \mathcal{M}^1_{\mathcal{G}_1} \), i.e., \( x = 0 \) and \( x = \infty \), in infinite time. Because \( \mathcal{M}^1_{\mathcal{G}_1} = \{(x, c(x)); x \in (0, \infty)\} \) is a one dimensional smooth manifold, we can conclude it is complete.

3.3.2 Belt Buckles

The second example is a graph with two vertices and connected to each other by three edges, which we denote by \( \mathcal{G}_2 \) and call it a belt buckle. The picture below is a picture of a belt buckle. For brevity, we denote \( l(e_1), l(e_2) \) and \( l(e_3) \) by \( x, y \) and \( z \), respectively.

Following the recipe, we have
Figure 3.2. Belt Buckles

- \( A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \) and 
- \( A_{-l} = \begin{pmatrix} 0 & 0 & 0 & 0 & e^{-x} & e^{-x} \\ 0 & 0 & 0 & e^{-y} & 0 & e^{-y} \\ 0 & 0 & e^{-z} & e^{-z} & 0 \\ 0 & e^{-x} & e^{-x} & 0 & 0 & 0 \\ e^{-y} & 0 & e^{-y} & 0 & 0 & 0 \\ e^{-z} & e^{-z} & 0 & 0 & 0 & 0 \end{pmatrix} \).

- \( h(l) = 1 \Rightarrow \det(A_{-l} - \text{Id}) = 0 \Rightarrow e^{-z} = \frac{1-e^{-x-y}}{2e^{-x-y}+e^{-x}+e^{-y}}. \)

- \( v = \left( \frac{e^{-y+1}}{2e^{-x-y}+e^{-x}+e^{-y}}, \frac{e^{-x+1}}{2e^{-x-y}+e^{-x}+e^{-y}}, 1, \frac{e^{-y+1}}{2e^{-x-y}+e^{-x}+e^{-y}}, \frac{e^{-x+1}}{2e^{-x-y}+e^{-x}+e^{-y}}, 1 \right). \)

- \( P = \begin{pmatrix} 0_{3 \times 3} & U \\ U & 0_{3 \times 3} \end{pmatrix} \) where \( 0_{3 \times 3} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \) and
where we need to derive the corresponding first fundamental forms on the Gaussian curvature. In order to calculate the Gaussian curvature, we first remark that in this example, we are interested in curvatures of \( U \) and \( S \).

Now we consider the surface \( U \) and \( S \) with respect to di \( \partial \), we focus on the Gaussian curvature. In order to calculate the Gaussian curvature, we first need to derive the corresponding first fundamental forms

\[
U = \begin{pmatrix}
0 & \frac{e^{-x}(e^{-y}+1) - e^{-x-y}}{e^{-x}-1} & \frac{e^{-x}(e^{-y}+1) - e^{-x-y}}{2e^{-x-y} + e^{-x} - e^{-y}} \\
\frac{(e^{-x}+1) e^{-y}}{e^{-y}+1} & 0 & \frac{(e^{-x}+1) e^{-y}}{2e^{-x-y} + e^{-x} - e^{-y}} \\
\frac{1-e^{-x-y}}{e^{-y}+1} & \frac{1-e^{-x-y}}{e^{-x}-1} & 0
\end{pmatrix}.
\]

\[
\mathbf{p} = \begin{pmatrix}
p_1(x, y) \\
p_2(x, y) \\
p_3(x, y) \\
p_4(x, y) \\
p_5(x, y) \\
p_6(x, y)
\end{pmatrix} = \begin{pmatrix}
\frac{e^{x}(e^{y}+1)^2}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)} \\
\frac{(e^{x}+1)^2 e^{y}}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)} \\
\frac{(e^{x}+1)^2 e^{y}}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)} \\
\frac{e^{x}(e^{y}+1)^2}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)} \\
\frac{e^{x}(e^{y}+1)^2}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)} \\
\frac{e^{x}(e^{y}+1)^2}{4(3e^{x+y} + e^{2x+y} + e^{x+y} - 1)}
\end{pmatrix}.
\]

Now we consider the surface \( M_{G_2}^1 \) that \( l = (l_1, l_2, l_3) = (x, y, - \log \left( \frac{e^{x+y} - 1}{e^{x}+e^{y}+2} \right)) = (x, y, S(x, y)) = l(x, y) \).

\[
\frac{\partial}{\partial x} l = (1, 0, -\frac{e^{x}(e^{y}+1)^2}{(e^{x}+e^{y}+2)(e^{x+y} - 1)}), \quad \frac{\partial}{\partial y} l = (0, 1, -\frac{(e^{x}+1)^2 e^{y}}{(e^{x}+e^{y}+2)(e^{x+y} - 1)}),
\]

\[
\frac{\partial^2}{\partial x^2} l = (0, 0, \frac{e^{x}(e^{y}+1)^2(e^{2x+y}+e^{x}+2)}{(e^{x}+e^{y}+2)^2(e^{x+y} - 1)^2}), \quad \frac{\partial^2}{\partial x \partial y} l = (0, 0, -\frac{(e^{x}+1)(e^{y}+1)e^{x+y}(e^{x+y} - e^{y} - 3)}{(e^{x}+e^{y}+2)^2(e^{x+y} - 1)^2}),
\]

and \( \frac{\partial^2}{\partial y^2} l = (0, 0, \frac{(e^{x}+1)^2 e^{y}(e^{x}+e^{y}+e^{x}+2)}{(e^{x}+e^{y}+2)^2(e^{x+y} - 1)^2}) \).

Remark 21. For this graph, the hidden natural condition \( l_3 > 0 \) is equivalent to

\[ e^{x+y} < 3 + e^{x} + e^{y}. \]

In this example, we are interested in curvatures of \( M_{G_2}^1 \) with respect to different metrics \( || \cdot ||_{PS} \) and \( || \cdot ||_{M} \). Since \( M_{G_2}^1 \) is a two-dimensional manifold in \( \mathbb{R}^3 \), we focus on the Gaussian curvature. In order to calculate the Gaussian curvature, we first need to derive the corresponding first fundamental forms

\[
ds^2 = E(x, y)dx^2 + F(x, y)dxdy + G(x, y)dy^2
\]

where \( x = l_1 \) and \( y = l_2 \).
Lemma 2 (Brioschi formula). If a metric has local coordinates
\[ ds^2 = E(x, y)dx^2 + F(x, y)dxdy + G(x, y)dy^2, \]
then the Gaussian curvature is given by
\[
K(x, y) = \frac{\begin{vmatrix}
-\frac{E_{yy}}{2} + F_{xy} - \frac{G_{xx}}{2} & E_x & F_x - \frac{E_y}{2} \\
F_y - \frac{G_x}{2} & E & F \\
\frac{G_y}{2} & F & G
\end{vmatrix}}{(E G - F^2)^2}.
\]

By direct computations, we have the following propositions.

Proposition 8. The first fundamental form of \((\mathcal{M}^1_{\mathcal{G}_2}, \| \cdot \|_{PS})\)
is
\[
E_{PS}(x, y) = \text{Var} \left( \frac{\partial l}{\partial x}, m_{-l} \right) = 2 \cdot \frac{\partial^2 S}{\partial x^2} \cdot \mathbf{p}_3(x, y)
= \frac{e^x (e^y + 1)^2 (e^{2x+y} + e^y + 2)}{2 (e^x + e^y + 2) (e^{x+y} - 1) (3e^{x+y} + e^{2x+y} + e^{x+2y} - 1)},
\]
\[
F_{PS}(x, y) = \frac{1}{2} \left( \text{Var} \left( \frac{\partial l}{\partial x} + \frac{\partial l}{\partial y}, m_{-l} \right) - \text{Var} \left( \frac{\partial l}{\partial x}, m_{-l} \right) - \text{Var} \left( \frac{\partial l}{\partial y}, m_{-l} \right) \right)
= 2 \cdot \frac{\partial^2 S}{\partial x \partial y} \cdot \mathbf{p}_3(x, y).
= \frac{(e^x + 1) (e^y + 1) e^{x+y} (e^{x+y} + e^x + e^y + 3)}{2 (e^x + e^y + 2) (e^{x+y} - 1) (3e^{x+y} + e^{2x+y} + e^{x+2y} - 1)},
\]
\[
G_{PS}(x, y) = \text{Var} \left( \frac{\partial l}{\partial y}, m_{-l} \right) = 2 \cdot \frac{\partial S}{\partial y} \cdot \mathbf{p}_3(x, y)
= \frac{(e^x + 1)^2 e^y (e^{x+2y} + e^x + 2)}{2 (e^x + e^y + 2) (e^{x+y} - 1) (3e^{x+y} + e^{2x+y} + e^{x+2y} - 1)},
\]
where \(S(x, y) = -\log \left( \frac{e^{x+y} - 1}{e^{x+y} + 2} \right)\) and \(l(x, y) = (x, y, S(x, y)) \in \mathcal{M}^1_{\mathcal{G}_2}\).

Proof. Only \(F_{PS}(x, y)\) needs some elaborations. By the parallelogram formula and
Proposition 5, we have

$$F_{PS}(x, y) = \frac{1}{2} \left( \|\frac{\partial l}{\partial x} + \frac{\partial l}{\partial y}\|_P^2 - \|\frac{\partial l}{\partial x}\|_P^2 - \|\frac{\partial l}{\partial y}\|_P^2 \right)$$

$$= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 S(x, y) \cdot p_3(x, y) - \left( \frac{\partial}{\partial x} \right)^2 S(x, y) \cdot p_3(x, y)$$

$$- \left( \frac{\partial}{\partial y} \right)^2 S(x, y) \cdot p_3(x, y)$$

$$= 2 \frac{\partial^2}{\partial x \partial y} S(x, y) \cdot p_3(x, y).$$

We notice that the same computation holds for McMullen pressure metric ($\|\cdot\|_M$).

Therefore, we have the following result.

**Proposition 9.** The first fundamental form of $(M_{G_2}, \|\cdot\|_M)$ is

$$E_M(x, y) = \frac{\text{Var}( \frac{\partial}{\partial x}, m_{-l})}{V(l)}$$

$$= e^x (e^y + 1)^2 (e^{2x+y} + e^y + 2)$$

$$\cdot \frac{(e^x + e^y + 2) (e^{x+y} - 1) f(x, y)}{(e^x + e^y + 2) (e^{x+y} - 1) f(x, y)},$$

$$F_M(x, y) = \frac{1}{2} \left( \text{Var}( \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, m_{-l}) - \text{Var}( \frac{\partial}{\partial x}, m_{-l}) - \text{Var}( \frac{\partial}{\partial y}, m_{-l}) \right)$$

$$- \frac{\text{Var}( \frac{\partial}{\partial y}, m_{-l})}{V(l)}$$

$$= - \frac{(e^x + 1) (e^y + 1) e^{x+y} (e^{x+y} - e^x - e^y - 3)}{(e^x + e^y + 2) (e^{x+y} - 1) f(x, y)},$$

$$G_M(x, y) = \frac{\text{Var}( \frac{\partial}{\partial y}, m_{-l})}{V(l)}$$

$$= \frac{(e^x + 1)^2 e^y (e^{x+2y} + e^x + 2)}{(e^x + e^y + 2) (e^{x+y} - 1) f(x, y)},$$

where $V(l) = \int l dm_{-l} = 2(l_1 \cdot p_1 + l_2 \cdot p_2 + l_3 \cdot p_3)$ and
\[ f(x, y) = (xe^{x+2y} + ye^{2x+y} + 2e^{x+y}(x + y) + \\
(-2e^{x+y} - e^{2x+y} - e^{x+2y} + e^x + e^y + 2) \log \left( \frac{e^{x+y} - 1}{e^x + e^y + 2} \right) + e^x + e^y) \]

**Proposition 10.** The moduli space of belt buckles is bounded positively curved under the Pollicott-Sharp pressure metric \( || \cdot ||_{PS} \), i.e., \((\mathcal{M}^1_{G_2}, || \cdot ||_{PS})\) is positively curved and the Gaussian curvature is bounded. Moreover, \((\mathcal{M}^1_{G_2}, || \cdot ||_{PS})\) is incomplete.

**Proof.** By the Brioschi formula, we can write down the curvature explicitly as the following

\[
K_{PS}(x, y) = \frac{1}{4 (e^x + 1)^2 (e^y + 1)^2 (3e^{x+y} + e^{2x+y} + e^{x+2y} - 1) \\
(5 + 6e^x + 3e^{2x} + 6e^y + 3e^{2y} + 3e^{x+y} + 45e^{2(x+y)} + 19e^{3(x+y)} + \\
11e^{2x+y} + 9e^{3x+y} + 3e^{4x+y} + 11e^{x+2y} + 33e^{3x+2y} + 8e^{4x+2y} + 9e^{x+3y} + \\
+33e^{2x+3y} + 3e^{4x+3y} + 3e^{x+4y} + 8e^{2x+4y} + 3e^{3x+4y}) \cdot 
\]

We observe that the numerator and the denominator of \( K(x, y) \) have the same highest exponents \( e^{3x+4y} \) and \( e^{4x+3y} \). Consider the polar coordinate \( e^x = r \cos \theta \) and \( e^y = r \sin \theta \) where \( 1 < r < \infty \) and \( \theta \in (0, \frac{\pi}{2}) \)

\[
K_{PS}(x, y) = \frac{6r^7(\cos^3 \theta \sin^4 \theta + \cos^4 \theta \sin^3 \theta) + HOT(r^6)}{8r^7(\cos^3 \theta \sin^4 \theta + \cos^4 \theta \sin^3 \theta) + HOT(r^6)}. 
\]

Therefore, we know \( K_{PS}(x, y) \) is close to \( \frac{3}{4} \) when \( r \) is large, and it is easy see that \( K_{PS}(x, y) > 0 \) for \( x, y \geq 0 \). Because \( B := \{(x, y); x \geq 0, y \geq 0, e^{2x} + e^{2y} \leq r^2\} \) is a compact set, we know \( c_1 \geq K_p(x, y) \geq c_0 > 0 \).
To prove the second assertion, we consider the path 

\[ c(x) = l(x, x) = (x, x, -\log \left( \frac{1 - e^{-2x}}{2e^{-2x} + 2e^{-x}} \right)). \]

We repeat the argument that we used in Proposition 6. Because when \( x \) goes to 0, \( c(x) \) goes to the boundary of \( \mathcal{M}_{G_2}^1 \). To prove the incompleteness we only need to show that \( c(x) \) goes to boundary in finite time. Since

\[
||\dot{c}(x)||_{PS}^2 \begin{pmatrix} P_3(x, x) & \frac{d^2}{dx^2} \left( -\log \left( \frac{1 - e^{-2x}}{2e^{-2x} + 2e^{-x}} \right) \right) \\ \frac{d^2}{dx^2} \left( -\log \left( \frac{1 - e^{-2x}}{2e^{-2x} + 2e^{-x}} \right) \right) & e^x \end{pmatrix} \begin{pmatrix} P_0(x, x) \\ \frac{d^2}{dx^2} \left( -\log \left( \frac{1 - e^{-2x}}{2e^{-2x} + 2e^{-x}} \right) \right) \end{pmatrix} \begin{pmatrix} e^x \\ -3e^x + 2e^{2x} + 1 \end{pmatrix},
\]

it is clear that \( \int_0^1 ||\dot{c}(x)||_{PS}dx \) is convergent. Hence the geodesic distance from \( c(1) \) to the boundary point \( c(0) \) is finite. \( \square \)

**Observation 3.3.1.** The Gaussian curvature of \( (\mathcal{M}_{G_2}^1, |||\cdot|||_M) \) is negative and bounded.

**Evidence.** By the Brioschi formula, we can explicitly write down the curvature. However, it is too long and unnecessary to state it in here. We put the explicit expression of \( K_M(x, y) \) in the appendix for whom is interested in that.

The following is the graph of \( K_M(x, y) \) for \( 0 < x, y < 5 \) and the hidden condition \( e^{x+y} < 3 + e^x + e^y \). With the explicit formula of \( K_M \), we can easily plot the picture and estimate the maximum and the minimal of \( K_M \) through software, and here we use Mathematica. By Mathematica, we know the maximum of \( K_M(x, y) \) for \( 0 < x, y \) and \( e^{x+y} < 3 + e^x + e^y \) happens around \( (x_0, y_0) \approx (1.09861, 1.09861) \) where \( K_M(x_0, y_0) \approx -0.485025 \), and the lower bound of \( K_M \) is around \(-0.564958 \). \( \square \)
3.3.3 Dumbbells

In this section, we discuss the graph of a dumbbell. We denote this graph by $G_3$ pictured below. Likewise, for short, we denote $l(e_1)$, $l(e_2)$ and $l(e_3)$ by $x$, $y$ and $z$, respectively.

Following the same recipe, we have
\[ A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_{-l} = \begin{pmatrix} e^{-x} & 0 & e^{-x} & 0 & 0 & 0 \\ 0 & e^{-y} & 0 & 0 & 0 & e^{-y} \\ 0 & e^{-z} & 0 & 0 & 0 & e^{-z} \\ 0 & 0 & e^{-x} & e^{-x} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-y} & e^{-y} \\ e^{-z} & 0 & 0 & e^{-z} & 0 & 0 \end{pmatrix}. \]

- \( h(l) = 1 \implies \det(A_{-l} - \text{Id}) = 0 \implies e^{-z} = \sqrt{\frac{e^{-x-y}-e^{-x-y}+1}{4e^{-x-y}}}. \)

- \( v = \begin{pmatrix} \sqrt{\frac{(e^{-x-1})(e^{-y-1})}{2-2e^{-x}}} \\ 1 \end{pmatrix}, \quad e^{-z} \sqrt{\frac{(e^{-x-1})(e^{-y-1})}{2-2e^{-x}}}, \quad \frac{1}{2e^{-y}}, 1 \).

- \( p = \begin{pmatrix} e^{-x} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & e^{-y} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1-e^{-y} & 0 & 0 & 1-e^{-y} & 0 \\ 0 & 0 & \frac{1}{2} & a & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-y} & \frac{1}{2} \\ 1-e^{-x} & 0 & 0 & 1-e^{-x} & 0 & 0 \end{pmatrix} \)

Now we consider the surface

\[ \mathcal{M}_{G_3} \ni l = (l_1, l_2, l_3) = (x, y, \log(2) - \frac{1}{2} \log((e^x - 1)(e^y - 1))) = l(x, y). \]

- \( \frac{\partial}{\partial x} l = (1, 0, \frac{e^x}{2-e^{2x}}), \quad \frac{\partial}{\partial y} l = (0, 1, \frac{e^y}{2-e^{2y}}), \quad \frac{\partial^2}{\partial x^2} l = (0, 0, \frac{e^x}{2(e^x-1)^2}), \frac{\partial^2}{\partial x \partial y} l = (0, 0, 0), \) and

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\[ \frac{\partial^2 l}{\partial y^2} = (0, 0, \frac{e^y}{2(e^y-1)^2}). \]

**Remark 22.** There is a hidden condition \( l_3 > 0 \). In this case, the condition is equivalent to

\[ 4 > (e^x - 1)(e^y - 1). \]

Set \( x = l_1 \) and \( y = l_2 \). Since \( \mathcal{M}_{y_3} \) is also a two-dimensional manifold in \( \mathbb{R}^3 \), we repeat the same argument as for \( \mathcal{M}_{y_2} \). The following propositions come from direct computations.

**Proposition 11.** The first fundamental form of \( (\mathcal{M}_{y_3}, \| \cdot \|_{PS}) \) is

\[
E_{PS}(x, y) = \text{Var}(\frac{\partial l}{\partial x}, m_{-l}) = 2 \cdot \frac{\partial l}{\partial x} \cdot p_3(x, y) = \frac{e^x (e^y - 1)}{(e^x - 1)(4e^{x+y} - 3e^x - 3e^y + 2)},
\]

\[
F_{PS}(x, y) = \frac{1}{2} \left( \text{Var}(\frac{\partial l}{\partial x} + \frac{\partial l}{\partial y}, m_{-l}) - \text{Var}(\frac{\partial l}{\partial x}, m_{-l}) - \text{Var}(\frac{\partial l}{\partial y}, m_{-l}) \right) = 0,
\]

\[
G_{PS}(x, y) = \text{Var}(\frac{\partial l}{\partial y}, m_{-l}) = 2 \cdot \frac{\partial l}{\partial y} \cdot p_3(x, y) = \frac{(e^x - 1)e^y}{(e^y - 1)(4e^{x+y} - 3e^x - 3e^y + 2)}.
\]

**Proposition 12.** The first fundamental form of \( (\mathcal{M}_{y_3}, \| \cdot \|_M) \) is

\[
E_M(x, y) = \frac{\text{Var}(\frac{\partial l}{\partial x}, m_{-l})}{V(l)} = \frac{e^x (e^y - 1)}{(e^x - 1)f(x, y)},
\]
\[ F_M(x, y) = \frac{1}{2} \left( \text{Var}(\frac{\partial}{\partial x}, m_{-1}) - \text{Var}(\frac{\partial}{\partial x}, m_{-1}) - \text{Var}(\frac{\partial}{\partial y}, m_{-1}) \right) \]
\[ = 0, \]
\[ G_M(x, y) = \frac{\text{Var}(\frac{\partial}{\partial y}, m_{-1})}{V(l)} \]
\[ = \frac{(e^x - 1) e^y}{(e^y - 1) f(x, y)}, \]

where \( V(l) = \int l \, dm_{-l} = 2(l_1 \cdot p_1 + l_2 \cdot p_2 + l_3 \cdot p_3) \) and

\[ f(x, y) = (x e^{x+y} + y e^{x+y} - \log((e^x - 1)(e^y - 1)) + \]
\[ 2(-e^{x+y} + e^x + e^y) \log\left(\frac{1}{2} \sqrt{(e^x - 1)(e^y - 1)}\right) - e^x x - e^y y + \log(4) \]

**Proposition 13.** The moduli space of dumbbells is positively curved under the Pollicott-Sharp pressure metric \( || \cdot ||_{PS} \), i.e., \((\mathcal{M}_1^{G_3}, || \cdot ||_{PS})\) is positively curved. More precisely, the Gaussian curvature is strictly bigger than zero but has no upper bound. Moreover, \((\mathcal{M}_1^{G_3}, || \cdot ||_{PS})\) is incomplete.

**Proof.** Applying the Brioschi formula, we can write down the curvature explicitly:

\[ K_{PS}(x, y) = \frac{2e^{x+y} - 1}{4e^{x+y} - 3e^x - 3ey + 2} \]

where \( x = l_1 \) and \( y = l_2 \). It is clear that \( K_{PS}(x, y) > 0 \) for all \( x, y > 0 \), and

\[ \lim_{(x,y) \to (0,0)} K_{PS}(x, y) = \infty. \]

Moreover, set \( a = e^x \) and \( b = e^y \),

\[ K_{PS}(a, b) = \frac{2ab - 1}{4ab - 3a - 3b + 2} \]

where \( a > 1, b > 1 \) and \((a-1)(b-1) < 4\). Since \( \partial_a K_{PS}(a, b) = 0 \) and \( \partial_b K_{PS}(a, b) = 0 \) have no real solution, we know the extreme values of \( K_{PS}(a, b) \) can only happen at boundaries \( a = 1, b = 1 \) or \((a-1)(b-1) < 4\). Easy computation shows that \( K_{PS}(a, b) \)
is strictly bigger than 0 at these boundaries points. Therefore, we can conclude that $K_{PS}(x, y) > c_0 > 0$ for some $c_0$. And Mathematica computation indicates that $c_0$ is about 0.85.

To prove the last assertion, we consider the path $c(x) = l(x, x) = (x, x, \log(2) - \frac{1}{2} \log((e^x - 1) (e^y - 1)))$. We repeat the argument that we used in Proposition 6. Notice that $c(x)$ goes to the boundary of $M_{G_2}^1$ when $x$ goes to 0. To prove the incompleteness we only need to show that $c(x)$ goes to boundary in finite time. Since

$$\|\dot{c}(x)\|_{PS}^2 = P_3(x, x) \cdot \frac{d^2}{dx^2} (\log(2) - \log(e^x - 1)) + P_6(x, x) \cdot \frac{d^2}{dx^2} (\log(2) - \log(e^x - 1)) = \frac{e^x}{-3e^x + 2e^{2x} + 1}$$

it is clear that $\int_0^1 \|\dot{c}(x)\|_{PS} dx$ is convergent. Hence the geodesic distance from $c(1)$ to the boundary point $c(0)$ is finite.

**Observation 3.3.2.** The Gaussian curvature of $(M_{G_2}^1, \| \cdot \|_M)$ takes positive and negative values.

**Evidence.** It is the same as in the belt buckle case, using the Brioschi formula, we can have the explicit expression of $K_M(x, y)$ which, for the sake of brevity, is stated in the appendix. The following figure is the graph of $K_M(x, y)$ produced by Mathematica for $0.05 < x, y < 3$ and, moreover, Mathematica computation shows that $K_M$ takes positive (e.g., when $x$ is close to 0.05 and $y$ is close to 3) and negative values (e.g., when $x$ and $y$ are close to 0.05).

3.3.4 Three-Petal Roses

The fourth example is a three-petal rose $G_4$, which can be thought of as a generalization of figure 8. The picture below is a picture of a three-petal rose. For brevity, we denote $l(e_1), l(e_2)$ and $l(e_3)$ by $x, y$ and $z$, respectively.
Figure 3.5. $K_M$ for Dumbbells

Figure 3.6. Three-Petal Roses
Following the recipe, we have:

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix},
A_{-l} = \begin{pmatrix}
e^{-x} & e^{-x} & 0 & e^{-x} \\
e^{-y} & e^{-y} & 0 & e^{-y} \\
e^{-z} & e^{-z} & 0 \\
0 & e^{-x} & e^{-x} \\
e^{-y} & 0 & e^{-y} \\
e^{-z} & 0
\end{pmatrix}.
\]

\[h(l) = 1 \implies \det(A_{-l} - Id) = 0 \implies e^{-z} = \frac{2(1+e^{-y})}{e^{-x}(3+5e^{-y})+3e^{-y}+5},\]

\[v = (v_i)_{i=1}^{6} \text{ where } v_1 = v_3 = \frac{2(e^{-y}+1)}{e^{-x}(5e^{-y}+3)+3e^{-y}+1},\]
\[v_2 = v_4 = \frac{2(e^{-x}+1)}{e^{-x}(5e^{-y}+3)+3e^{-y}+1} \text{ and } v_3 = v_6 = 1.\]

\[P = (P_{ij})_{6 \times 6} \text{ where } P_{ij} = 0 \text{ for } i \neq j \text{ and } P_{11} = P_{44} = \frac{2(1+e^{-y})}{e^{-x}(3+5e^{-y})+3e^{-y}+5},\]
\[P_{22} = P_{55} = \frac{2(1+e^{-y})}{e^{-x}(3+5e^{-y})+3e^{-y}+1}, \text{ and } P_{33} = P_{66} = 1.\]

\[
\mathbf{p} = \begin{pmatrix}
p_1(x,y) \\
p_2(x,y) \\
p_3(x,y) \\
p_4(x,y) \\
p_5(x,y) \\
p_6(x,y)
\end{pmatrix} = \begin{pmatrix}
\frac{2e^x(e^{y}+1)^2}{12e^x+y+e^{2(x+y)}+6e^{x+y}+6e^{x+y}+10e^{x+y}-3e^{x+y}-10e^x-y-3e^{x+y}+15} \\
\frac{2(e^{x+y}+1)^2e^y}{12e^x+y+e^{2(x+y)}+6e^{x+y}+6e^{x+y}+10e^{x+y}-3e^{x+y}-10e^x-y-3e^{x+y}+15} \\
\frac{(e^{x+y}-e^{-y}-3)(e^{x+y}+3e^x+3e^y+5)}{2(12e^{x+y}+e^{2(x+y)}+6e^{x+y}+6e^{x+y}+10e^{x+y}-3e^{x+y}-10e^x-y-3e^{x+y}+15)} \\
p_1(x,y) \\
p_2(x,y) \\
p_3(x,y)
\end{pmatrix}.
\]

Now we consider the surface

\[M_{\mathcal{G}_4}^1 \ni l = (l_1, l_2, l_3) = (x, y, -\log(\frac{-e^{x+y}+e^x+e^y+3}{e^{x+y}+3e^x+3e^y+5})) = l(x, y).\]

\[
\frac{\partial l}{\partial x} = (1, 0, \frac{4e^x(e^{y}+1)^2}{(-e^{x+y}+e^{x+y}+3)(e^{x+y}+3e^x+3e^y+5)}),
\]
\[
\frac{\partial l}{\partial y} = (0, 1, \frac{4(e^{x+y}+1)^2e^y}{(-e^{x+y}+e^{x+y}+3)(e^{x+y}+3e^x+3e^y+5)}),
\]
\[
\frac{\partial^2 l}{\partial x^2} = (0, 0, \frac{4e^x(e^{y}+1)^2(e^{x+y}+3)(e^{x+y}+3e^x+3e^y+5)}{(-e^{x+y}+e^{x+y}+3)(e^{x+y}+3e^x+3e^y+5)^2}),
\]

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Proposition 15.

\[
\frac{\partial^2}{\partial x \partial y} l = (0, 0, \frac{32(e^x+1)(e^y+1)e^{x+y}(e^x+e^y+2)}{(-e^{x+y}+e^x+e^y+3)^2(e^{x+y}+3e^x+3e^y+5)^2}),
\]

and \( \frac{\partial^2}{\partial y^2} l = (0, 0, \frac{4(e^x+1)^2(e^y+3)e^{x+y+2y}+3e^x-e^{2y}+5}{(-e^{x+y}+e^x+e^y+3)^2(e^{x+y}+3e^x+3e^y+5)^2}) \).

Remark 23. There is a hidden condition \( l_3 > 0 \). In this case, the condition is equivalent to

\[-e^{x+y} + e^x + e^y + 3 < 0.\]

Since \( M_{G_4} \) is still a two-dimensional manifold in \( \mathbb{R}^3 \), we repeat the same argument as for \( M_{G_2} \). Set \( x = l_1, y = l_2 \) we have the following results of the first fundamental form with respect to \( || \cdot ||_{PS} \) and \( || \cdot ||_M \).

Proposition 14. The first fundamental form of \( (M_{G_4}, || \cdot ||_{PS}) \) is

\[
E_{PS}(x, y) = \text{Var}(\frac{\partial l}{\partial x}, m_{-l}) = 2 \cdot \frac{\partial l}{\partial x} \cdot p_3(x, y)
\]
\[
= -\frac{4(e^x+1)^2(e^y+3)(e^{2x+y} - e^{2x} + 3e^y + 5)}{(-e^{x+y} + e^x + e^y + 3)(e^{x+y} + 3e^x + 3e^y + 5)w(x, y)},
\]
\[
F_{PS}(x, y) = \frac{1}{2} \left( \text{Var}(\frac{\partial l}{\partial x} + \frac{\partial l}{\partial y}, m_{-l}) - \text{Var}(\frac{\partial l}{\partial x}, m_{-l}) - \text{Var}(\frac{\partial l}{\partial y}, m_{-l}) \right)
\]
\[
= -\frac{32(e^x+1)(e^y+1)(e^{x+y} + 3e^x + 3e^y + 5)w(x, y)}{(-e^{x+y} + e^x + e^y + 3)(e^{x+y} + 3e^x + 3e^y + 5)w(x, y)},
\]
\[
G_{PS}(x, y) = \text{Var}(\frac{\partial l}{\partial y}, m_{-l}) = 2 \cdot \frac{\partial l}{\partial y} \cdot p_3(x, y)
\]
\[
= -\frac{4(e^x+1)^2(e^x+3)e^{x+y+2y} + 3e^x - e^{2y} + 5}{(-e^{x+y} + e^x + e^y + 3)(e^{x+y} + 3e^x + 3e^y + 5)w(x, y)}.
\]

where \( w(x, y) = (12e^{x+y} + e^{2(x+y)} + 6e^{2x+y} + 6e^{x+2y} - 10e^x - 3e^{2x} - 10e^y - 3e^{2y} - 15) \)

Proposition 15. The first fundamental form of \( (M_{G_4}, || \cdot ||_M) \) is

\[
E_M(x, y) = \frac{\text{Var}(\frac{\partial l}{\partial x}, m_{-l})}{V(l)}
\]
\[
= \frac{4(e^x+1)^2(e^y+3)(e^{2x+y} - e^{2x} + 3e^y + 5)}{f_1(x, y)f_2(x, y)f_3(x, y)}.
\]

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\[ F_M(x, y) = \frac{1}{2} \left( \text{Var}(\frac{\partial l}{\partial x}, m_{-1}) - \text{Var}(\frac{\partial l}{\partial y}, m_{-1}) - \text{Var}(\frac{\partial l}{\partial y}, m_{-1}) \right) \]
\[ G_M(x, y) = \frac{1}{2} \left( \text{Var}(\frac{\partial l}{\partial y}, m_{-1}) \right) \]
\[ = \frac{32 (e^x + 1) (e^y + 1) e^{x+y} (e^x + e^y + 2)}{f_1(x, y) f_2(x, y) f_3(x, y)}, \]
\[ = \frac{4 (e^x + 1)^2 (e^x + 3) e^y (e^{x+2y} + 3e^x - e^{2y} + 5)}{f_1(x, y) f_2(x, y) f_3(x, y)}, \]

where \( V(l) = \int l \, dm_{-l} = 2(l_1 \cdot p_1 + l_2 \cdot p_2 + l_3 \cdot p_3) \) and

\[ f_1(x, y) = -e^{x+y} + e^x + e^y + 3, \]
\[ f_2(x, y) = e^{x+y} + 3e^x + 3e^y + 5 \]
\[ f_3(x, y) = \left[ -4 (xe^{x+2y} + ye^{2x+y} + 2e^{x+y}(x + y) + e^x x + e^y y) + (-4e^{x+y} + e^{2(x+y)} + 2e^{2x+y} + 2e^{x+2y} - 14e^x - 3e^{2x} - 14e^y - 3e^{2y} - 15) \right] \]
\[ \log \left( \frac{-e^{x+y} + e^x + e^y + 3}{e^{x+y} + 3e^x + 3e^y + 5} \right) \]

**Observation 3.3.3.** The moduli space of three-petal roses is positively curved under the Pollicott-Sharp pressure metric \( || \cdot ||_{PS} \), i.e., \((M_{\mathcal{G}_3}, || \cdot ||_{PS})\) is positively curved and the Gaussian curvature is bounded. Moreover, \((M_{\mathcal{G}_3}, || \cdot ||_{PS})\) is incomplete.

**Evidence.** Applying the Brioschi formula, we can write down the curvature explicitly; however, in this thesis, we only give the figure of the curvature and avoid stating lengthy results.

Moreover, numerical results indicate that \(0.2 < K_{PS}(x, y) < 1\).

The second assertion is because each three-petal rose contains a figure 8, and we know figure 8 is incomplete with respect to the Pollicott-Sharp pressure metric.

\[ \square \]

**Observation 3.3.4.** The sectional curvature of \((M_{\mathcal{G}_3}, || \cdot ||_M)\) takes positive and negative values.
Figure 3.7. $K_{PS}$ for Three-Petal Roses

Evidence. The explicit formula of $K_M$ is complicated, and it makes no sense to state in here. The following figure of $K_M$ for $0.5 < x, y < 20$ is produced by Mathematica, which indicates that $K_M$ is positive when $(x, y) = (5, 15)$ and negative when $(x, y) = (19, 19)$.

3.3.5 Summary

**Proposition 16** (Proposition 6, 7, 10, 13, and Observation 3.3.1, 3.3.2, 3.3.3, 3.3.4).

1. *There exist examples of graphs for which the metric $|| \cdot ||_{PS}$ is not complete.*

2. *There exist examples of graphs for which the curvature of the metric $|| \cdot ||_{PS}$ is positive.*

3. *There exist examples of graphs for which the metric $|| \cdot ||_M$ is complete.*
4. There exist examples of graphs for which the curvature of the metric $||·||_M$ takes positive and negative values.
TABLE 3.1

TYPICAL GRAPHS AND PRESSURE METRICS

<table>
<thead>
<tr>
<th></th>
<th>Figure 8</th>
<th>Belt buckle</th>
<th>Dumbbell</th>
<th>Three-petal rose</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\cdot|_{PS}$</td>
<td>incomplete</td>
<td>incomplete</td>
<td>incomplete</td>
<td>incomplete</td>
</tr>
<tr>
<td></td>
<td>$0 &lt; C_2 \leq K \leq C_1$</td>
<td>$0 &lt; K \leq \infty$,</td>
<td>$0 &lt; C_2 &lt; K &lt; C_1$</td>
<td></td>
</tr>
<tr>
<td>$|\cdot|_M$</td>
<td>complete</td>
<td>$C_2 \leq K \leq C_1 &lt; 0$</td>
<td>$K$ takes + and −</td>
<td>$K$ takes + and −</td>
</tr>
</tbody>
</table>

where $K$ denotes the Gaussian curvature.
CHAPTER 4

IMMERSED SURFACES IN HYPERBOLIC 3-MANIFOLDS

4.1 Overview and Background

This chapter is organized as follows. We begin by recalling necessary background knowledge on Anosov flows and immersed surfaces in hyperbolic 3-manifolds. In Section 4.2 we give the proof of our main result, and in Section 4.4 we discuss applications of our results to the Taubes’ moduli spaces.

Throughout this chapter, $\phi_t$ is a transitive Anosov flow on a compact manifold $X$ (see Section 2.2.4 for definitions). We first recall some properties of the pressure and equilibrium states.

**Theorem 15** (Bowen-Ruelle, Theorem 1, Theorem 8). If $\phi_t$ is a transitive Anosov flow on a compact manifold $X$, then for each $F : X \to \mathbb{R}$ Hölder continuous function, there exists a unique equilibrium state $m_F$ of $F$, which is also known as the Gibbs measure of $F$. Moreover, if $F$ and $G$ are Hölder continuous functions such that $m_F = m_G$, then $F - G$ is Livšic cohomologous to a constant.

**Remark.** Because the equilibrium state $m_F$ of $F$ is unique, we know that $m_F$ is ergodic, i.e., the Gibbs measure of $F$ is ergodic.

Recall that $O$ is the set of period orbits of $\phi$. For a continuous function $F : X \to \mathbb{R}_{>0}$ and $T \in \mathbb{R}$, we define

$$R_T(F) = \{ \tau \in O : \langle \delta_\tau, F \rangle \leq T \}.$$
**Proposition 17** (Bowen [4]). The topological entropy $h_\phi$ of a transitive Anosov flow $\phi$ is finite and positive. Moreover,

$$h_\phi = \lim_{T \to \infty} \frac{1}{T} \log \# \{ \tau \in O : l(\tau) \leq T \}.$$

If $F : X \to \mathbb{R}$ is a positive H"older continuous function, then

$$h_F := h_{\phi^F} = \lim_{T \to \infty} \frac{1}{T} \log \# R_T(F),$$

is finite and positive.

**Theorem 16** (Equidistribution, Bowen [4], Parry-Pollicott [30]). Suppose $\phi$ is a transitive Anosov flow on a compact manifold $X$. Then there exists a unique probability measure of maximum entropy $\mu_\phi$. Moreover, for all continuous function $G$ on $X$, we know

$$\int_X G d\mu_\phi = \mu_\phi(G) = \lim_{T \to \infty} \frac{1}{\# R_T(1)} \sum_{\tau \in R_T(1)} \langle \delta_\tau, G \rangle = \lim_{T \to \infty} \frac{\sum_{\tau \in R_T(1)} \langle \delta_\tau, G \rangle}{\sum_{\tau \in R_T(1)} \langle \delta_\tau, 1 \rangle}.$$

The probability measure $\mu_\phi$ is called the **Bowen-Margulis measure** of the flow $\phi$.

4.1.1 Livšic Type Theorems

Recall that two continuous functions $F, G : X \to \mathbb{R}$ are Livšic cohomologous if there exists a continuous function $V : X \to \mathbb{R}$ which is $C^1$ in the flow direction such that

$$F(x) - G(x) = \frac{\partial}{\partial t} \bigg|_{t=0} V(\phi_t(x)).$$

**Remark 24.** By definition, the following properties are immediate:

1. If $F$ and $G$ are Livšic cohomologous then they have the same integral over any
The pressure $P_\phi(F)$ only depends on the Livšić cohomology class of $F$.

3. $R_T(F)$ only depends on the Livšić cohomology class of $F$.

In the rest of this chapter, we will only discuss the Livšić cohomology of Hölder continuous functions on $X$. Specifically, two Hölder continuous $F,G : X \to \mathbb{R}$ are called Livšić cohomologous if there exists a Hölder continuous $V : X \to \mathbb{R}$ which is $C^1$ in the flow direction such that

$$F(x) - G(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\phi_t(x)).$$

**Theorem 17** (Livšić Theorem [25]). Let $\phi_t : X \to X$ be a transitive Anosov flow. Let $F : X \to \mathbb{R}$ be a Hölder continuous function such that $\langle \delta_{\tau}, F \rangle = \int_0^{l(\tau)} F \circ \phi_t(x) dt = 0$ for each $\phi-$closed orbit $\tau$ for any $x_\tau \in \tau$. Then $F$ is cohomologous to $0$.

**Theorem 18** (Positive Livšić Theorem [34]). Let $\phi_t : X \to X$ be a transitive Anosov flow. Let $F : X \to \mathbb{R}$ be a Hölder continuous function such that $\langle \delta_{\tau}, F \rangle > 0$ for each $\phi-$closed orbit $\tau$. Then $F$ is cohomologous to a Hölder continuous function $G(x)$ such that $G(x) > 0$, $\forall x \in X$.

For the geodesic flow on the unit tangent bundle of negatively curved manifolds, we have a similar theorem as the following:

**Theorem 19** (Nonnegative Livšić Theorem [39]). Suppose $X$ is a compact Riemannian manifold with negative sectional curvature. Let $\phi_t : T^1X \to T^1X$ be the geodesic flow on $T^1X$. Let $F : T^1X \to \mathbb{R}$ be a Hölder continuous function such that $\langle \delta_{\tau}, F \rangle \geq 0$ for each $\phi-$closed orbit $\tau$. Then $F$ is cohomologous to a Hölder continuous function $G(x)$ such that $G(x) \geq 0$, $\forall x \in T^1X$.

Here we recall the Anosov closing lemma [2]:
Theorem 20 (Anosov Closing Lemma). Let $\phi_t : X \to X$ be a transitive Anosov flow. Then there exists a $t_0 > 0$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, d_X) > 0$ such that if for $v \in X$, $t > t_0$ satisfying

$$d_X(v, \phi_t v) < \delta,$$

then there exists a closed orbit $\tau_w = \{\phi_{s}w\}_{s=0}^{t'}$ of period $t'$, where $|t - t'| < \varepsilon$, such that

$$d_X(\phi_sv, \phi_sw) < \varepsilon \quad \text{for} \ 0 \leq s \leq t.$$

4.1.2 Hölder Cocycles

Let $(X, g)$ be a compact negatively curved manifold, $\tilde{X}$ be its universal covering, and $\Gamma$ be the group of deck transformations of the covering $\tilde{X}$. Recall that the $\pi_1X$-action on $\tilde{X}$ is defined by $\gamma \cdot x = i_X(\gamma)(x)$, where $i$ is the isomorphism $i_X : \pi_1\Sigma \to \Gamma$.

Definition 24. A Hölder cocycle is a function $c : \pi_1X \times \partial_\infty \tilde{X} \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 \cdot x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \pi_1X$ and $x \in \partial_\infty \tilde{X}$, and $c(\gamma, \cdot)$ is Hölder continuous for every $\gamma \in \pi_1X$.

Given a Hölder cocycle $c$ we define the periods of $c$ to be the number

$$l_c[\gamma] := c(\gamma, \gamma_X^+),$$

where $\gamma_X^+$ is the attracting fixed point of $\gamma \in \pi_1X \setminus \{e\}$ on $\partial_\infty \tilde{X}$.

Remark 25. The cocycle property implies that the period of an element $\gamma$ only depends on its conjugacy class $[\gamma] \in [\pi_1X]$.

Two cocycles $c$ and $c'$ are said to be cohomologous if there exists a Hölder contin-
uous function $U : \partial_\infty \tilde{X} \to \mathbb{R}$ such that, for all $\gamma \in \pi_1 X$, one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma \cdot x) - U(x).$$

One easily deduces from the definition that the set of periods of a cocycle is a cohomological invariant.

**Definition 25.** The *exponential growth rate* for a Hölder cocycle $c$ is defined as

$$h_c := \limsup_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1 X] : l_c[\gamma] \leq T \}.$$

### 4.1.3 From Cocycle Cohomology to Livšic Cohomology

**Theorem 21** (Ledrappier [24]). For each Hölder cocycle $c : \pi_1 X \times \partial_\infty \tilde{X} \to \mathbb{R}$, there exists a Hölder continuous function $F_c : T^1 X \to \mathbb{R}$, such that, for all $\gamma \in \pi_1 X - \{e\}$, one has

$$l_c[\gamma] = \int_{[\gamma]} F_c.$$

The map $c \mapsto F_c$ induces a bijection between the set of cohomology classes of $\mathbb{R}$-valued Hölder cocycles, and the set of Livšic cohomology classes of Hölder continuous functions from $T^1 X$ to $\mathbb{R}$.

Using the above Theorem 21, Sambarino give the following reparametrization theorem in [34].

**Theorem 22** (Sambarino [34]). Let $c$ be a Hölder cocycle with positive periods such that $h_c$ is finite. Then the action of $\Gamma$ on $(\partial_\infty \tilde{X} \times \partial_\infty \tilde{X} - \text{diagonal}) \times \mathbb{R}$ via $c$, that is,

$$\gamma(x, y, s) = (\gamma x, \gamma y, s - c(\gamma, y)),$$

is proper and compact. Moreover, the flow $\psi$ on $\pi_1 X \setminus (\partial_\infty \tilde{X} \times \partial_\infty \tilde{X} - \text{diagonal}) \times \mathbb{R}$
\( \mathbb{R} \), defined as
\[
\psi_t \Gamma(x, y, s) = \Gamma(x, y, s - t),
\]
is conjugated to \( \phi^{F_c} : T^1X \to T^1X \) which is the reparametrization of the geodesic flow \( \phi \) on \( T^1X \) by a H"older continuous function \( F_c \) provided \( l_c[\gamma] = \int_{[\gamma]} F_c \) for all \( [\gamma] \in [\pi_1X] \). Furthermore, the conjugating map is also H"older continuous, and the topological entropy of \( \psi \) is \( h_c \).

4.1.4 Immersed Surfaces in Hyperbolic 3–Manifolds

In this section, we review some well-known facts about immersed surfaces in hyperbolic 3–manifolds. Let \( S \) be a differentiable 2-manifold and \( M \) be a 3-manifold, we say a differentiable mapping \( f : S \to M \) is an immersion if \( df_p : T_pS \to T_{f(p)}M \) is injective for all \( p \in S \). If, in addition, \( f \) is a homeomorphism onto \( f(S) \subset M \), where \( f(S) \) has the subspace topology induced from \( M \), we say that \( f \) is an embedding. Moreover, if the induced homomorphism \( f_* : \pi_1S \to \pi_1M \) is injective, then we say \( f \) is \( \pi_1 \)-injective.

Throughout, we consider that \( M \) is a hyperbolic 3–manifold equipped with a hyperbolic metric \( h \) and \( S \) is a compact, 2–dimension manifolds with negative Euler characteristic. Before moving further, we recall several terminologies in differential geometry. Given an immersion \( f : S \to M \), let \( g = f^*g \) be the induced Riemannian metric on \( S \), \( \nabla \) be the Levi-Civita connection on \( (M, h) \), \( N \) be the unit outward normal vector field to the surface \( f(S) \subset M \), and \( \partial_1 \) and \( \partial_2 \) be the coordinate fields of \( TS \).

The second fundamental form \( B : TS \times TS \to \mathbb{R} \) of \( f \) is the symmetric 2-tensor on \( S \) defined by, locally,
\[
B(\partial_i, \partial_j) = \langle \partial_i, -\nabla_{\partial_j}N \rangle_h,
\]
where \( \langle \cdot, \cdot \rangle_h \) is the hyperbolic metric \( h \) on \( M \).

The \textit{shape operator} \( S_g : TS \to TS \) is the symmetric self-adjoint endomorphism defined by raising one index of the second fundamental form \( B \) with respect to the metric \( g \).

The \textit{mean curvature} \( H \) of the immersion \( f : S \to M \) (or, of the immersed surface \( (S, g) \)) is the trace of the shape operator. We call an immersion \( f : S \to M \) \textit{minimal} if the mean curvature \( H \) vanishes identically.

Moreover, we can relate the induced Riemannian metric \( g \) and shape operator \( S_g \) by Gauss-Codazzi equations:

\[
K_g = -1 + \det S_g, \quad \text{(Gauss eq.) (4.1.1)}
\]
\[
\nabla_{df(X)}(S_g(Y)) - \nabla_{df(Y)}S_g(X) = S_g([X, Y]). \quad \text{(Codazzi eq.) (4.1.2)}
\]

where \( X, Y \in TS \) and \([\cdot, \cdot]\) is the Lie bracket on \( TS \).

\textit{Remark 26.}

1. We call real eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( S_g \) the \textit{principal curvatures} of the immersion \( f : S \to M \).

2. The shape operator \( S_g \) and the second fundamental form \( B \) are linked by

\[
B(X, Y) = \langle X, S_g(Y) \rangle_g, \quad \forall X, Y \in TS.
\]

3. If \( f \) is a minimal immersion, then Gauss-Codazzi equations can be expressed in terms of \( B \) by

\[
K_g = -1 - \frac{1}{2} \| B \|^2_g,
\]
\[
(\nabla_{\partial_i} B)_{jk} = (\nabla_{\partial_j} B)_{ik},
\]

where \( \langle \cdot, \cdot \rangle_h \) is the hyperbolic metric \( h \) on \( M \).
where \( \| \cdot \|_g \) is the tensor norm w.r.t. metric \( g \) and \( \partial_1 \) and \( \partial_2 \) are coordinate fields of \( TM \). Moreover, in this case the Gauss equation implies \( K_g \leq -1 \), i.e., \((S, g)\) is a negatively curved surface.

### 4.1.5 Minimal Hyperbolic Germs

In the context of minimal hyperbolic germs, the surface \( S \) is always assumed to be closed. Let \((g, B)\) be a pair consisting of a Riemannian metric \( g \) and a symmetric 2-tensor \( B \) on \( S \).

**Definition 26.** A pair \((g, B)\) is called a **minimal hyperbolic germ** if it satisfies the following equations

\[
\begin{align*}
K_g &= -1 - \frac{1}{2} \| B \|_g^2, \\
(\nabla_{\partial_i} B)_{jk} &= (\nabla_{\partial_j} B)_{ik}, \\
B &\text{ is traceless w.r.t. } g.
\end{align*}
\]

Recall that \( \text{Diff}_0(S) \) is the space of orientation preserving diffeomorphisms of \( S \) isotopic to the identity. There is a natural \( \text{Diff}_0(S) \) action (i.e., by pullback) on the space of minimal hyperbolic germs, and we are mostly interested in the following quotient space.

**Definition 27.** The space \( \mathcal{H} \) of minimal hyperbolic germs is the quotient:

\[
\mathcal{H} = \{ \text{minimal hyperbolic germs} \} / \text{Diff}_0(S).
\]

Taubes shows that \( \mathcal{H} \) is a smooth manifold of dimension \( 12g - 12 \) where \( g \) is the genus of \( S \). The fundamental theorem of surface theory ensures that each \((g, B)\in\mathcal{H}\) can be integrated into an immersed minimal surface in a hyperbolic 3-manifold with the Riemannian metric \( g \) and the second fundamental form \( B \).

Moreover, \( \mathcal{H} \) is closely related with Teichmüller space. To be more precise, we first recall facts in Teichmüller theory. The **Teichmüller space** \( \mathcal{T} \) of \( S \) is the space
of isotopic classes of complex structures on $S$. Alternatively, by the uniformization theorem, $\mathcal{T}$ can also be identified with the space of isotopic classes of conformal structures on $S$, i.e., conformal classes of Riemannian metrics with curvature $-1$. It is clear that we can identify $\mathcal{T}$ with a subspace $\mathcal{F}$ of $\mathcal{H}$. Namely, the Fuchsian space $\mathcal{F}$ is the set

$$\mathcal{F} = \{(m, 0) \in \mathcal{H} : m \text{ is a Reimannian metric of constant curvature } -1\}.$$ 

Let $[g]$ be the conformal class of a Riemannian metric $g$ on $S$ and $X = (S, [g])$ be the Riemann surface associated with $g$. It is well-known that $T^*_X$ the fiber of the holomorphic cotangent bundle over $X$ can be identified with $Q(X)$ the space of holomorphic quadratic differentials on $X$.

The following theorem of Hopf [16] helps us see the relation between $\mathcal{H}$ and $Q(X)$.

**Theorem 23** (Hopf [16]). If $(g, B) \in \mathcal{H}$, then $B$ is the real part of a (unique) holomorphic quadratic differential $\alpha \in Q(X)$. More precisely, if $(x_1, x_2) = x_1 + ix_2 = z$ is a local isothermal coordinate of $X$ and $B = B_{11}dx_1^2 + B_{22}dx_2^2 + 2B_{12}dx_1dx_2$, then

$$\alpha(g, B) = (B_{11} - iB_{12})(x_1, x_2)dz^2.$$ 

**Remark.** In fact, $B_{11} = -B_{22}$ because $(S, g)$ is minimal, and it is not hard to see $\|\alpha\|_g = \|B\|_g$.

Moreover, the space $\mathcal{H}$ admits a smooth map to $T^*\mathcal{T}$ given by

$$\Psi : \mathcal{H} \to T^*\mathcal{T}$$

$$(g, B) \mapsto ([g], \alpha(g, B)).$$

For any two holomorphic quadratic differentials $\alpha$ and $\beta$ in $Q(X)$, the Weil-
Petersson pairing is given by

\[ \langle \alpha, \beta \rangle_{WP} = \int_S \frac{\alpha \beta}{m}, \]

where \( m \) is the hyperbolic metric on \( S \) conformal to \( g \). It’s also well-known that this pairing defines a Kähler metric, the Weil-Petersson metric, on the Teichmüller space whose geometry has been intensely studied. In the last section, we will discuss several applications of our results related with the Weil-Petersson metric on \( \mathcal{F} \).

We now change gear to the Gauss equation. Since every Riemannian metric \( g \) on \( S \) is conformal to a unique hyperbolic metric \( m \), we can write \( g = e^{2u}m \) where \( e^{2u} \) is the conformal factor. Therefore, we can rewrite the Gauss equation as the following.

**Theorem 24** (Gauss equation, Theorem 4.2 [42]). The Gauss equation can be written, in terms of \( m \),

\[ -1 - \frac{1}{2} \| B \|^2_m = K_g = e^{-2u}(\Delta_m u - 1), \]

where \( K_g \) is the Gaussian curvature of \((S, g)\).

From another point of view, using Uhlenbeck’s result in [42] we can relate the space of minimal hyperbolic germs \( \mathcal{H} \) with the character variety \( \mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C})) \), where \( \mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C})) \) is the space of conjugacy classes of representations of \( \pi_1(S) \) into \( \text{PSL}(2, \mathbb{C}) \). More precisely, Uhlenbeck [42] proves that for each data \((g, B) \in \mathcal{H}\) there exists a representation \( \rho : \pi_1(S) \to \text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}) \) leaving this minimal immersion invariant, i.e., there is a map

\[ \Phi : \mathcal{H} \to \mathcal{R}(\pi_1(S), \text{PSL}(2, \mathbb{C})). \]
4.1.6 Almost-Fuchsian Germs

**Definition 28.** The space of almost-Fuchsian germs $\mathcal{AF}$ is defined as

$$\mathcal{AF} = \{(g, B) \in H : \|B\|_g^2 < 2\}.$$

**Remark 27.** $\|B\|_g^2 < 2$ is equivalent to that the principal curvatures $\lambda_1$ and $\lambda_2$ are between $-1$ and $1$.

This definition is motivated by the observation of Uhlenbeck [42] that when $(g, B) \in \mathcal{AF}$ there exists a unique quasi-Fuchsian manifold $M$, up to isometry, such that $S$ is an embedded minimal surface in $M$ with the induced metric $g$ and the second fundamental form $B$. In other words, almost-Fuchsian germs are more than quasi-Fuchsian, but not Fuchsian.

In the following, we discuss a ray in $\mathcal{AF}$, which could be considered as a path in the Teichmüller space $\mathcal{T}$. Specifically, for a hyperbolic metric $m \in \mathcal{F}$, and a holomorphic quadratic differential $\alpha \in Q((S, [m]))$, we consider the ray

$$r(t) = (g_t, tB) \subset \mathcal{AF},$$

where $g_t$ and $B = \text{Re}(\alpha)$ satisfying $\|tB\|_{g_t}^2 < 2$. Notice that $g_t$ is conformal to the hyperbolic metric $m$, so we can write $g_t = e^{2u_t}m$ where the conformal factor $e^{2u_t}$ is a $C^2$ function on $S$. By studying the Gauss equation, Uhlenbeck proved $u_t$ is smooth on $t$; hence, $r(t)$ is smooth when $t$ is small. We state that result in the below.

**Theorem 25** (Uhlenbeck, Theroem 4.4 [42]). Consider the maps $F : W^{2,2}(S) \times [0, \infty) \to L^2(S)$,

$$F(u, t) = \Delta_m u + 1 - e^{2u} - \frac{1}{2} \|tB\|_m^2 e^{-2u},$$

where $W^{2,2}(S)$ is the classical Sobolev space. Then there exists $\tau_0 > 0$ and a smooth
solution curve

\[ c : [0, \tau_0] \to W^{2,2}(S) \times [0, \infty) \]

\[ t \mapsto (u(t), t) \]

such that \( c(0) = (0, 0) \) and \( F(c(t)) = 0 \) for all \( t \in [0, \tau_0] \).

4.1.7 Asymptotic Geodesic Distortions

Let \( S \) be a compact 2–dimensional manifold with negative Euler characteristic, \( f : S \to M \) be a \( \pi_1 \)–injective immersion from \( S \) to a hyperbolic 3-manifold \((M, h)\), and \( \Gamma \) be the copy of \( \pi_1 S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion \( f \). Throughout this subsection, we assume that \( \Gamma \) is a convex cocompact and \((S, g)\) is negatively curved where \( g = f^* h \) and \( h \) is the given hyperbolic metric on \( M \).

We introduce two geometric constants which can be interpreted as asymptotic geodesic distortions with respect to \( g \) and \( h \), respectively. As we mentioned in the introduction, this type of geodesic distortions was first introduced in Thurston’s work [40] (a.k.a. Thurston’s intersection number). One can also check Burger [9], Knieper [23], and Croke-Fathi [11] for more discussions on geometric constants constructed in similar manners.

**Definition 29** (Asymptotic geodesic distortions). We define two geometric constants associated to the immersion \( f : S \to M \) as

\[
C_g(f) = \limsup_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g)} l_{h}[\gamma]}{\sum_{[\gamma] \in R_T(g)} l_g[\gamma]},
\]
\[ C_h(f) = \limsup_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(h)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(h)} l_g[\gamma]}, \]

where

\[ R_T(g) := \{ [\gamma] \in [\pi_1 S] : l_g[\gamma] \leq T \} \quad \text{and} \quad R_T(h) := \{ [\gamma] \in [\pi_1 S] : l_h[\gamma] \leq T \}. \]

4.2 Main Results

Throughout this section, \( S \) denotes a compact 2–dimensional manifold with negative Euler characteristic. Let \( f : S \to M \) be a \( \pi_1 \)–injective immersion from \( S \) to a hyperbolic 3-manifold \( M \) and \( \Gamma \) be the copy of \( \pi_1 S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion \( f \). More precisely, let \( \rho : \pi_1 M \to \text{Isom}(\mathbb{H}^3) \) be the discrete and faithful representation, up to conjugacy, corresponding to \( M \), i.e., \( M = \rho(\pi_1 M) \backslash \mathbb{H}^3 \). Then \( \Gamma = \rho(f_*(\pi_1 S)) \) where \( f_* \) is the induced homomorphism of \( f : S \to M \).

The hypotheses throughout here are: \( \Gamma \) is a convex cocompact and \( (S, g) \) is negatively curved where \( g = f^*h \) and \( h \) is the given hyperbolic metric on \( M \).

Notice that because \( (S, g) \) is a compact negatively curved surface, its universal covering \( (\tilde{S}, \tilde{g}) \) is a pinched Hadamard manifold. Let \( \Gamma_\Sigma \) denote the group of deck transformations of the covering \( \tilde{S} \). Then we know \( \Gamma_\Sigma \cong \pi_1 S \) and \( \Gamma_\Sigma \subset \text{Isom}(\tilde{S}) \).

**Lemma 3.** There exists a quasi-isometry \( q : \tilde{S} \to \text{Conv}(\Lambda(\Gamma)) \), where \( \text{Conv}(\Lambda(\Gamma)) \) is the convex hull of \( \Lambda(\Gamma) \) in \( \mathbb{H}^3 \). Moreover, \( q \) extends to a bi-Hölder and \( \Gamma \)-equivariant map between boundaries.

**Proof.** Because \( (\tilde{S}, \tilde{g}) \) and \( (\text{Conv}(\Lambda(\Gamma)), h) \) are both pinched Hadamard manifold, \( \Gamma_\Sigma \in \text{Isom}(\tilde{S}) \) acts cocompactly on \( \tilde{S} \), and \( \Gamma \) acts convex cocompactly on \( \mathbb{H}^3 \), by Theorem 11 (Švarc-Milnor lemma), we know that there are quasi-isometries \( q_1 : \tilde{S} \to C(\Gamma_\Sigma, S'') \) and \( q_2 : C(\Gamma, S') \to \text{Conv}(\Lambda(\Gamma)) \), where \( C(\Gamma, S') \) is the Cayley graph of \( \Gamma = \langle S' \rangle \). Because \( \Gamma \) and \( \Gamma_\Sigma \) are both isomorphic to \( \pi_1 S \), by Švarc-Milnor lemma
the identity map \( i : C(\Gamma, S') \rightarrow C(\Gamma, S) \) is a quasi-isometry. So, \( q_2 i q_1 : \tilde{S} \rightarrow \text{Conv}(\Lambda(\Gamma)) \) is the desired quasi-isometry. The second assertion is a consequence of Proposition 9, because the quasi-isometry \( q = q_2 i q_1 \) extends to a bi-Hölder map \( q : \partial_\infty \tilde{S} \rightarrow \Lambda(\Gamma) = \partial_\infty \text{Conv}(\Lambda(\Gamma)) \). Lastly, by the construction of \( q \), it is easy to see that \( q \) is \( \Gamma \)-equivariant.

Now, we have two different objects \( \Gamma_S \subset \text{Isom}(\tilde{S}) \) and \( \Gamma \subset \text{Isom}(\mathbb{H}^3) \). Nevertheless, they are the same as a group. Because \( \pi_1 S \) is finitely generated, there are canonical isomorphisms between \( \pi_1 S, \Gamma_S \) and \( \Gamma \), by sending generators to generators. Namely, \( i_S : \pi_1 S \rightarrow \Gamma_S \) and \( i_M : \pi_1 S \rightarrow \Gamma \). For brevity, we denote elements in \( \pi_1 S, \Gamma_S \) and \( \Gamma \) by \( \gamma, \gamma_S \) and \( \gamma_M \), respectively, where \( i_S(\gamma) = \gamma_S \) and \( i_M(\gamma) = \gamma_M \).

**Lemma 4.** The above quasi-isometry \( q : \partial_\infty \tilde{S} \rightarrow \Lambda(\Gamma) \) sends the attracting (resp. repelling) limit point \( \gamma_S^+ \) (resp. \( \gamma_S^- \)) of the hyperbolic element \( \gamma_S \in \Gamma_S \subset \text{Isom}(\tilde{S}) \) to the attracting (resp. repelling) limit point \( \gamma_M^+ \) (resp. \( \gamma_M^- \)) of \( \gamma_M \in \Gamma \subset \text{Isom}(\mathbb{H}^3) \).

**Proof.** Notice that the boundary map \( q : \partial_\infty \tilde{S} \rightarrow \Lambda(\Gamma) \) is an equivariant homeomorphism, and having an attracting (repelling) point is a topological feature. Therefore, \( q \) maps the attracting point of \( \gamma_S \in \Gamma_S \) to the attracting point of \( \gamma_M \in \Gamma \).

Now we are ready to state and prove the main theorem. However, we chop the proof of the following theorem into several lemmas, so the complete proof will be presented in the end of this section.

**Theorem 26** (Main theorem). Let \( f : S \rightarrow M \) be a \( \pi_1 \)-injective immersion from a compact surface \( S \) to a hyperbolic 3-manifold \( M \). Let \( \Gamma \) be the copy of \( \pi_1 S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion \( f \). Suppose \( \Gamma \) is convex cocompact and \((S, g)\) is negatively curved. Then

1. The limit-sups in \( C_h(f) \) and \( C_g(f) \) are limits.

2. \( 0 < C_h(f) \leq C_g(f) \leq 1 \).
3. Let $h_{\text{top}}(S)$ be the topological entropy of the geodesic flow on $T^1S$ and $\delta_\Gamma$ be the critical exponent, then

$$C_h(f) \cdot \delta_\Gamma \leq h_{\text{top}}(S) \leq C_g(f) \cdot \delta_\Gamma \quad (4.2.1)$$

4. The first (resp. second) equality in (4.2.1) holds if and only if the marked length spectrum of $S$ is proportional to the marked length spectrum of $M$, and the proportion is the ratio $\frac{\delta_\Gamma}{h_{\text{top}}(S)}$ (resp. $\frac{h_{\text{top}}(S)}{\delta_\Gamma}$).

5. If $C_h(f) = 1$ or $C_g(f) = 1$, then $S$ is a totally geodesic submanifold in $M$.

We first introduce the key lemma.

**Lemma 5.** Under the same assumptions as Theorem 26. Then

1. There exists a Hölder continuous function $F : T^1S \to \mathbb{R}$ such that $0 < F \leq 1$ and $\int_{\tau} F = l_h[\tau]$ for all closed orbits $\tau$ on $T^1S$ where $l_h[\tau]$ is the length of the closed geodesic in the free homotopy class containing $f(\tau) \subset T^1M$ with respect to the hyperbolic metric $h$.

2. Let $\mu_{-h_F}$ be the equilibrium for $-h_F F$ and $\mu_{BM}$ be the Bowen-Margulis measure for the geodesic flow on $T^1S$ where

$$h_F := \lim_{T \to \infty} \frac{1}{T} \log \# \{ \tau \text{ is a closed orbit on } T^1S : \int_{\tau} F < T \}.$$  

We have $C_1 := \int F d\mu_{-h_F}$ and $C_2 := \int F d\mu_{BM}$ satisfy

$$C_1 \delta_\Gamma \leq h_{\text{top}}(S) \leq C_2 \delta_\Gamma. \quad (4.2.2)$$

3. Each equality in (4.2.2) holds iff the marked length spectrum of $S$ is proportional to the marked length spectrum of $M$.  

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Proof. Let \( \phi \) denote the geodesic flow on the unit tangent bundle of \((S, g)\).

The first step is to construct a Hölder reparametrization function \( F : T^1 S \to \mathbb{R}_{>0} \) such that the topological entropy \( h_F \) of the reparametrized flow \( \phi^F \) is the critical exponent \( \delta_\Gamma \) of \( \Gamma \) in \( \mathbb{H}^3 \).

Recall the Busemann function \( B^\eta_h(\gamma, \xi) : \partial_\infty \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R} \), for \( \eta \in \partial_\infty \mathbb{H}^3 \) and \( x, y \in \mathbb{H}^3 \) is given by

\[
B^\eta_h(x, y) := \lim_{z \to \eta} d_h(x, z) - d_h(y, z).
\]

Using the quasi-isometry \( q \) given in Lemma 3, we define a map \( c : \pi_1 S \times \partial_\infty \tilde{S} \to \mathbb{R} \) by

\[
c : \pi_1 S \times \partial_\infty \tilde{S} \to \mathbb{R}
\]

\[
(\gamma, \xi) \mapsto B^\eta_h(f(o), \gamma^{-1} \cdot f(o)),
\]

for \( o \in \tilde{S} \).

Claim: \( c \) is a Hölder cocycle.

pf.

\[
c(\gamma_1 \gamma_2, \xi) = B^\eta_h(f(o), (\gamma_1 \gamma_2)^{-1} \cdot f(o))
\]

\[
= B^\eta_h(f(o), (\gamma_2^{-1} \gamma_1^{-1}) \cdot f(o))
\]

\[
= B^\eta_h(f(o), \gamma_2^{-1} \cdot f(o)) + B^\eta_h(f(o), (\gamma_2^{-1} \gamma_1^{-1}) \cdot f(o))
\]

\[
= c(\gamma_2, \xi) + B^\eta_{\gamma^2 q(\xi)}(f(o), f(o))
\]

\[
= c(\gamma_2, \xi) + B^\eta_{\gamma q(\xi)}(f(o), \gamma_1^{-1} \cdot f(o))
\]

\[
= c(\gamma_2, \xi) + B^\eta_{\gamma q(\xi)}(f(o), \gamma_1^{-1} \cdot f(o)) \quad \text{by Lemma 4}
\]

\[
= c(\gamma_2, \xi) + c(\gamma_1, \gamma_2 \xi).
\]
Therefore, \( c \) is a cocycle. To see \( c \) is Hölder, we first notice that the boundary map \( q : \partial_{\infty}\tilde{S} \to \Lambda(\Gamma) \subset \partial_{\infty}\mathbb{H}^3 \) is bi-Hölder as we have discussed in the beginning of this section. Moreover, we know that \( \Lambda(\Gamma) \) embeds in \( \partial_{\infty}\mathbb{H}^3 \) and \( B_{\eta}(x, y) \) is smooth on \( \partial_{\infty}\mathbb{H}^3 \). Therefore, \( c(\gamma, \cdot) \) is Hölder continuous on \( \partial_{\infty}\tilde{S} \), and we finish the proof of this claim.

Notice that the period \( c(\gamma, \gamma_S^+) = B_{q(\gamma_S^+)}^h(f(o), \gamma^{-1}f(o)) = l_h[\gamma] \) > 0 for all \([\gamma] \in [\pi_1 S]\). Thus, \( l_c[\gamma] = l_h[\gamma] \) for all \([\gamma] \in [\pi_1 S]\), and we can easily see that

\[
h_c = \delta_T = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1 S] : l_h[\gamma] \leq T \} < \infty.
\]

Thus, by Theorem 22, there exists a positive Hölder continuous maps \( F_c \) on \( T^1 S \) such that the translation flow defined by the Hölder cocycle \( c \) is conjugated to the reparametrization \( \phi^{F_c} \) of the geodesic flow \( \phi_t : T^1 S \to T^1 S \) by \( F_c \). In particular, for all \([\gamma] \in [\pi_1 S]\)

\[
c(\gamma, \gamma_S^+) = \int_{[\gamma]} F_c = l_h[\gamma],
\]

and the topological entropy of the flow \( \phi^{F_c} \) is exactly the exponential growth rate of \( c \), i.e., \( h_{F_c} = h_c \).

Notice that for the constant function \( 1 \) on \( T^1 S \), we have \( l_g[\gamma] = \int_{[\gamma]} 1 \) for all \([\gamma] \in [\pi_1 S]\). Therefore, we have the pressure of the function \(-h_1 \cdot 1\) is zero, i.e., \( P(-h_1 \cdot 1) = 0 \), where

\[
h_1 = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1 S] : l_g[\gamma] \leq T \}
\]

is the topological entropy of the geodesic flow \( \phi \) on \( T^1 S \).

From now on we denote \( F_c \) by \( F \).

The second step is to show that
\[ h_{\text{top}}(S) \leq \int_{C_{2}} F d\mu_{BM} \cdot h_{F}, \]

where \( \mu_{BM} \) the Bowen-Margulis measure of the geodesic flow \( \phi : T^{1}S \rightarrow T^{1}S. \)

Since

\[ P(-h_{F} \cdot F) = 0 = h(\mu_{-h_{F}F}) - h_{F} \int F d\mu_{-h_{F}F} \]
\[ P(-h_{\text{top}}(S) \cdot 1) = 0 = h(\mu_{BM}) - h_{\text{top}}(S) \cdot \int 1d\mu_{BM} = h(\mu_{BM}) - h_{\text{top}}(S). \]

where \( \mu_{-h_{F}F} \) is the equilibrium state of \(-h_{F}F. \) Since \( \mu_{BM} \in \mathcal{M}^{\phi}, \) by the variational principle we have

\[ P(-h_{F} \cdot F) = 0 \geq h(\mu_{BM}) - h_{F} \int F d\mu_{BM}. \]

Furthermore,

\[ h_{F} \int F d\mu_{BM} \geq h(\mu_{BM}) = h_{\text{top}}(S). \]

**The third step** is to show the inequality

\[ \int_{C_{1}} F d\mu_{-F_{h_{F}}} \cdot h_{F} \leq h_{\text{top}}(S). \]

By Remark 11 , we have

\[ h_{\text{top}}(S) \geq h(\mu_{-F_{h_{F}}}) \]
\[ \iff h_{\text{top}}(S) - h_{F} \int F d\mu_{-F_{h_{F}}} \geq h(\mu_{-F_{h_{F}}}) - h_{F} \int F d\mu_{-F_{h_{F}}} = 0 \]
\[ \iff h_{\text{top}}(S) \geq h_{F} \cdot \int F d\mu_{-F_{h_{F}}}. \]
The fourth step is to show that $0 \leq C_1 \leq 1$ and $0 \leq C_2 \leq 1$.

Because $C_1 = \int F d\mu_{-Fh_F}$, $C_2 = \int F d\mu_{BM}$ and $F$ is positive, it is enough to show that $F$ can be chosen to be smaller or equal than 1.

Claim: $F \leq 1$.

This is a consequence of Theorem 19. For each conjugacy class $[\gamma] \in [\pi_1 S]$ there exists a unique closed geodesic $\tau^S_\gamma$ on $S$ such that $l_g[\gamma] = l_g(\tau^S_\gamma)$. Because $f$ is $\pi_1$-injective, $f$ maps $\tau^S_\gamma$ to a closed curve $f(\tau^S_\gamma)$ on $M$ which is in the same free homotopy class generated by $[\gamma]$. More precisely, let $\tau^M_\gamma$ denote the closed geodesic on $M$ in the conjugacy class $[\gamma]$, then we know that $f(\tau^S_\gamma)$ and $\tau^M_\gamma$ are in the same free homotopy class. Moreover, because $g$ is the induced metric $f^*h$, we know that $(S, g)$ is Riemannian isometric to $(f(S), h)$. Thus, $l_g(\tau^S_\gamma) = l_h(f(\tau^S_\gamma))$. Therefore, for all $[\gamma] \in [\pi_1 S]$,

$$l_g[\gamma] = l_g(\tau^S_\gamma) = l_h(f(\tau^S_\gamma)) \geq l_h(\tau^M_\gamma) = l_h[\gamma].$$

Therefore, for all $[\gamma] \in [\pi_1 S]$

$$\int_{[\gamma]} 1 = l_g[\gamma] \geq l_h[\gamma] = \int_{[\gamma]} F.$$

By Theorem 19, we have $1 - F$ is cohomologous to a nonnegative Hölder continuous function $H$, and $H$ is unique up to cohomology. Thus, we have that $F \sim 1 - H$ and $1 - H \leq 1$. By choosing $F$ to be $1 - H$, we now finish the proof of this claim.

The fifth step is to examine the equality cases.

If $h_{top}(S) = h_F \int F d\mu_{-Fh_F}$, then $h_{top}(S) = h(\mu_{-Fh_F})$, i.e., $\mu_{-Fh_F}$ is the equilibrium state of the constant function $-h_{top}(S) \cdot 1$. By the uniqueness part of Theorem 15, we have that $Fh_F$ is cohomologous to the constant $h_{top}(S)$, i.e., $F \sim \frac{h_{top}(S)}{h_F}$. Similarly, if $h_{top}(S) = h_F \int F d\mu_{BM}$, then $\mu_{BM} = \mu_{-h_F F}$. Hence, again, $h_{top}(S) \sim F \cdot h_F$, i.e., $F \sim \frac{h_{top}(S)}{h_F}$. 

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Lemma 6. If $h_{\text{top}}(S) = \delta_T$, then $S$ is a totally geodesic submanifold in $M$.

Proof. Notice $h_{\text{top}}(S) = \delta_T$ implies $F = 1$. This means that the length of each closed geodesic on $S$ has the same length with the corresponding closed geodesic on $M$. Furthermore, we know that the closed geodesics in $S$ are dense, that is, for any point $p \in S$, the set of tangent vectors $v \in T_p S$ such that the exponential map $\exp_p tv$ gives a closed geodesic is dense in $T_p S$. Therefore, the shape operator $S_g$ is zero when evaluating on this dense subset of vectors on $T_p S$. By the continuity of the shape operator $S_g$, we have $S_g \equiv 0$. Therefore $S$ is totally geodesic in $M$. □

Lemma 7. Let $\mu_{BM}$ be the Bowen-Margulis measure of the geodesic flow $\phi : T^1 S \to T^1 S$ and $\mu_{-h_F F}$ be the Gibbs measure for $-h_F F$ defined in Theorem 26. Then

$$C_2 := \int F d\mu_{BM} = \lim_{T \to \infty} \frac{1}{\# R_T(g)} \sum_{[\gamma] \in R_T(g)} \frac{l_h[\gamma]}{l_g[\gamma]} = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(g)} l_g[\gamma]} = C_g(f)$$

and

$$C_1 := \int F d\mu_{-h_F F} = \left( \lim_{T \to \infty} \frac{1}{\# R_T(h)} \sum_{[\gamma] \in R_T(h)} \frac{l_g[\gamma]}{l_h[\gamma]} \right)^{-1} = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(h)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(h)} l_g[\gamma]} = C_h(f)$$

where

$$R_T(g) := \{[\gamma] \in \pi_1 S : l_g[\gamma] \leq T\} \text{ and } R_T(h) := \{[\gamma] \in \pi_1 S : l_h[\gamma] \leq T\}.$$ 

Proof. This is a consequence of the equidistribution theorem (Theorem 16).
By Theorem 16, we have

\[ C_2 := \int F \, d\mu_{BM} \]

\[ = \lim_{T \to \infty} \frac{1}{\# R_T(1)} \sum_{\tau \in R_T(1)} \frac{\langle \delta_{\tau}, F \rangle}{\langle \delta_{\tau}, 1 \rangle} \]

\[ = \lim_{T \to \infty} \sum_{\tau \in R_T(1)} \langle \delta_{\tau}, F \rangle \]

\[ = \lim_{T \to \infty} \sum_{\tau \in R_T(1)} \langle \delta_{\tau}, 1 \rangle. \]

Notice that every closed orbit \( \tau \) of the geodesic flow \( \phi \) on \( T^1 S \) corresponds to a unique conjugacy class \([\gamma^\tau]\) of \( \pi_1 \Sigma \), and vice versa. Moreover, the period of \( \tau \) is the length of \( \gamma^\tau \) on \( S \), i.e.,

\[ l_h(\gamma^\tau) = \langle \delta_{\tau}, F \rangle, \ l_g(\gamma^\tau) = \langle \delta_{\tau}, 1 \rangle. \]

Since there is a one-to-one correspondence between \( R_T(1) \) and \( R_T(g) \), we can rewrite the equation above by

\[ C_2 := \int F \, d\mu_{BM} \]

\[ = \lim_{T \to \infty} \frac{1}{\# R_T(g)} \sum_{[\gamma] \in R_T(g)} \frac{l_h[\gamma]}{l_g[\gamma]} \]

\[ = \lim_{T \to \infty} \sum_{[\gamma] \in R_T(g)} \frac{l_h[\gamma]}{l_g[\gamma]} \]

\[ =: C_f(g). \]
For the other equation, by Theorem 7, we know that $\mu_{\phi^F} = \mu_{-h_FF}$. Therefore

$$
\mu_{\phi^F}(\frac{1}{F}) = \int F \mu_{-h_FF}(\frac{1}{F}) = \frac{\int (\frac{1}{F}) \cdot F d\mu_{-h_FF}}{\int F d\mu_{-h_FF}} = \frac{1}{\int F d\mu_{-h_FF}}.
$$

By Theorem 16, we have

$$
\mu_{\phi^F}(\frac{1}{F}) = \lim_{T \to \infty} \frac{1}{\# R_T(F)} \sum_{\tau' \in R_T(F)} \langle \delta^F_{\tau'}, \frac{1}{F} \rangle = \lim_{T \to \infty} \frac{\sum_{\tau' \in R_T(F)} \langle \delta^F_{\tau'}, \frac{1}{F} \rangle}{\sum_{\tau' \in R_T(F)} \langle \delta^F_{\tau'}, 1 \rangle}.
$$

Notice that for a closed geodesic $\tau'$ of the geodesic flow $\phi : T^1 S \to T^1 S$, $\langle \delta^F_{\tau'}, \frac{1}{F} \rangle = \int_0^{l_h(\tau')} \frac{1}{F(\phi_t)} \cdot F(\phi_t) dt = l_g(\tau')$ and similarly $\langle \delta^F_{\tau'}, F \rangle = \int_0^{l_h(\tau')} F(\phi_t) dt = l_h(\tau')$. By the one-to-one correspondence between closed orbit $\tau'$ and conjugacy class $[\gamma']$, we have a one-to-one correspondence between $R_T(F)$ and $R_T(h)$.

Hence, we have the following equation

$$
C_1 = \int F d\mu_{-h_FF} = \left( \mu_{\phi^F}(\frac{1}{F}) \right)^{-1} = \left( \lim_{T \to \infty} \frac{1}{\# R_T(h)} \sum_{[\gamma] \in R_T(h)} \frac{l_g[\gamma]}{l_h[\gamma]} \right)^{-1} = \frac{\sum_{[\gamma] \in R_T(h)} l_h[\gamma]}{\sum_{[\gamma] \in R_T(h)} l_g[\gamma]} =: C_h(f)
$$

\[ \square \]

**Remark 28.**

1. Conversely, suppose we reparametrize the geodesic flow $\psi$ on $T^1 M$ to conjugate the geodesic flow $\phi_t$ on $T^1 \Sigma$. Let $G$ be the reparametrization function. Then,
\[ \int F d\mu_{-FhF} \text{ is exactly } \int G d\mu'_{BM} \text{ where } \mu'_{BM} \text{ is the Bowen-Margulis measure of } \psi. \]

2. From the above expression of \( C_h(f) \) and \( C_g(f) \), we can also see \( C_h(f) \leq 1 \) and \( C_g(\Sigma, M) \leq 1 \). It is because for each \([\gamma] \in [\pi_1 S]\), we know \( l_g[\gamma] \geq l_h[\gamma] \) (cf. step 4 in the proof of Theorem 26).

Now, we are ready to prove the main theorem.

**Proof of the main theorem (Theorem 26).** Assertion 1: it is Lemma 7.

Assertion 2, 3, 4: are consequences of Lemma 5 and Lemma 7.

Assertion 5: it is Lemma 6.

\[ \square \]

4.3 Embedding

In this section, we assume that \( f : S \to M \) is an embedding. To state our results more precisely and to put it in context, we first introduce the geodesic stretch and discuss the relation between the geodesic stretch and \( C_h(f), C_g(f) \).

Notice that we can lift \( f : S \to M \) to an embedding between their universal coverings, i.e., \( \widetilde{f} : \widetilde{S} \to \widetilde{M} = \mathbb{H}^3 \). Moreover, one can easily check this lifting is \( \pi_1 S \)-equivariant. Specifically, for each \( \gamma \in \pi_1 S \), let \( \gamma_S \in \Gamma_S \) and \( \gamma_M \in \Gamma \) be the corresponding element of \( \gamma \) in the deck transformation groups \( \Gamma_S \subset \text{Isom}\widetilde{S} \) and \( \Gamma \subset \text{Isom}(\mathbb{H}^3) \), respectively. Then for each \( \tilde{x} \in \widetilde{S} \) we have

\[ \tilde{f}(\gamma \cdot \tilde{x}) := \tilde{f}(\gamma S(\tilde{x})) = \gamma_M(f(\tilde{x})) =: \gamma \cdot \tilde{f}(\tilde{x}). \]

Using this embedding \( \tilde{f} : \widetilde{S} \to \mathbb{H}^3 \) we can define a tangent map \( f : T^1 \widetilde{S} \to T^1 \mathbb{H}^3 \) by

\[ f : (\tilde{x}_0, w) \mapsto (\tilde{f}(\tilde{x}_0), d\tilde{f}_{\tilde{x}_0}(w)) \]
where $\tilde{x}_0 \in \tilde{S}$ and $w$ is a unit vector on the tangent plane $T_{\tilde{x}_0}\tilde{S}$. Notice that $\pi_1 \Sigma$ acts on $T^1\tilde{S}$ and $T^1\mathbb{H}^3$ in an obvious way. Thus $f$ is also $\pi_1 S$–equivariant. More precisely, 
\[
\gamma \cdot f(\tilde{x}_0, w) = (\gamma \cdot \tilde{f}(\tilde{x}_0), d\tilde{f}_{\tilde{x}_0}(w)) = (\tilde{f}(\gamma \cdot x_0), d\tilde{f}_{\tilde{x}_0}(w)) = f(\gamma \cdot (\tilde{x}_0, w)).
\]

The following lemma depicts a key feature of the embedding $f : S \to M$.

**Lemma 8.** $(\tilde{S}, d_g)$ is quasi-isometric to $(\tilde{f}(\tilde{S}), d_h) \subset (\mathbb{H}^3, d_h)$ where $d_g$ is the distance on $\tilde{S}$ induced by $g$ and $d_h$ is the hyperbolic distance on $\mathbb{H}^3$.

**Proof.** Because $\tilde{f}$ is an embedding and $\pi_1 S$–equivariant, we know that $(\tilde{f}(\tilde{S}), d_h)$ is a proper geodesic space and $\Gamma \in \text{Isom}(\tilde{f}(\tilde{S})) \subset \text{Isom}(\mathbb{H}^3)$ acts properly discontinuously and compactly on $\tilde{f}(\tilde{S})$. Hence, by Theorem 11 (Švarc-Milnor lemma), $(\tilde{S}, d_g)$ and $(\tilde{f}(\tilde{S}), d_h)$ are quasi-isometric. (Because $(\tilde{S}, d_g)$ and $(\tilde{f}(\tilde{S}), d_h)$ are both quasi-isometric to the Cayley graph of $\pi_1 S$ with a word metric.)

**Definition 30.** For all $v \in T^1\tilde{S}$ and $t > 0$, we define

\[
a(v, t) := d_h(\pi \circ f(v), \pi \circ f \circ \tilde{\phi}(v))
\]

where $\pi : T^1\tilde{S} \to \tilde{S}$ is the natural projection and $\tilde{\phi}$ is the lift of $\phi$.

**Remark 29.** $a(v, t)$ is $\pi_1 S$–invariant, because $f$ is $\pi_1 S$–equivariant and $\pi_1 S$ is acting on $\tilde{S}$ via $\Gamma_S \subset \text{Isom}(\tilde{S})$.

**Lemma 9.** For all $v \in T^1\tilde{S}$ and $t_1, t_2 > 0$,

\[
a(v, t_1 + t_2) \leq a(v, t_1) + a(\tilde{\phi}_{t_1}(v), t_2).
\]

**Proof.** It’s easy consequence of the triangle inequality of $d_h$. 

The following corollary is a consequence of Kingman’s sub-additive ergodic theorem [22].
Corollary 1. Let $\mu$ be a $\phi_t$-invariant probability measure on $T^1 S$. Then for $\mu$-a.e. $v \in T^1 S$

$$I_\mu(S, M, v) := \lim_{t \to \infty} \frac{a(v, t)}{t},$$

exists and defines a $\mu$-integrable function on $T^1 \Sigma$, invariant under the geodesic flow $\phi_t$.

Proof. To adapt Kingman’s sub-additive ergodic theorem [22] to flow case, it is sufficient to check:

$$\sup \{a(v, t); v \in T^1 S, \ 0 \leq t \leq 1 \} \in L^1(\mu).$$

We notice that $(T^1 S, d_g)$ and $(f(T^1 S), d_h)$ are quasi-isometric because $(S, d_g)$ and $(f(S), d_h)$ are. Therefore we have

$$a(v, 1) = d_h(\pi \circ f(v), \pi \circ f \circ \phi_1(v)) \leq C d_g(v, \phi_1(v)) + L < C + L$$

where $C, L$ are the quasi-isometry constants. Hence, $a(v, 1)$ is bounded. $\square$

From the above corollary, we can define the geodesic stretch as the following.

Definition 31. Let $\mathcal{M}^{\phi}$ be the set of $\phi_t$-invariant probability measures. The geodesic stretch $I_\mu(S, M)$ of $S$ relative to $M$ and $\mu \in \mathcal{M}^{\phi}$ is defined as

$$I_\mu(S, M) := \int_{T^1 \Sigma} I_\mu(S, M, v) d\mu.$$

Remark 30. If $\mu \in \mathcal{M}^{\phi}$ is ergodic, then $I_\mu(S, M) = \lim_{t \to \infty} \frac{a(v, t)}{t}$ for $\mu$-a.e. $v \in T^1 S$.

Since $f : (S, d_g) \to (f(S), d_h)$ is a quasi-isometry, by Theorem 9 we know that $f$ extends to a bi-Hölder map between $\partial S$ and $\partial f(S) = \Lambda(\Gamma)$. By the same discussion as in Lemma 4, we know that $f$ maps the attracting (resp. repelling) fixed point $\gamma_S^+$ (resp. $\gamma_S^-$) of $\gamma_S \in \Gamma_S$ to the corresponding attracting (resp. repelling) fixed point $\gamma_M^+$ (resp. $\gamma_M^-$) of $\gamma_M \in \Gamma$.  

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Moreover, each conjugacy class \([\gamma] \in [\pi_1 S]\) corresponds to a unique closed geodesic \(\tau^S_\gamma\) on \(S\) and \(\tau^M_\gamma\) on \(M\), and \(\tau^S_\gamma\) also corresponds to the unique geodesic \(\tilde{\tau}^S_\gamma\) connecting \(\gamma^-_S\) and \(\gamma^+_S\) on \(\partial_\infty S\). Notice that \(\tilde{f}(\gamma^-_S) = \gamma^-_M\) and \(\tilde{f}(\gamma^+_S) = \gamma^+_\Sigma\) on \(\partial_\infty \tilde{f}(S) = \Lambda(\Gamma) \subset \partial_\infty \mathbb{H}^3\), so \(f(\tau^S_\gamma)\) is a quasi-geodesic on \(\mathbb{H}^3\) within a bounded Hausdorff distance from the geodesic \(\tau^M_\gamma\) on \(\mathbb{H}^3\), where \(\tau^M_\gamma\) is the geodesic on \(\text{Conv}(\Lambda(\Gamma)) \subset \mathbb{H}^3\) connecting \(\gamma^-_M\) and \(\gamma^+_M\) on \(\Lambda(\Gamma)\).

Now we are ready to state the main result of this section. However, because the proof of the main theorem consists of several lemmas, we postpone the proof until the end of this section.

**Theorem 27.** Suppose that \(S\) is a compact and negatively curved surface embedded in a hyperbolic 3-manifold \(M\) and \(\Gamma\) is convex cocompact as in Theorem 26. Then the geometric constants \(C_h(f)\) and \(C_g(f)\) in Theorem 26 are geodesic stretches relative to Gibbs measures. More precisely,

\[
C_h(f) = I_\mu(S, M), \\
C_g(f) = I_{\mu_{BM}}(S, M),
\]

where \(\mu\) is a \(\phi\)-invariant Gibbs measure and \(\mu_{BM}\) is the Bowen-Margulies measure of the geodesic flow \(\phi_t\) on \(T^1 S\).

**Remark 31.**

1. The Gibbs measure \(\mu\) in \(C_h(f) = I_\mu(S, M)\) is indeed the measure \(\mu_{-hF}\) derived in the proof of Theorem 26.

2. Theorem 27 also indicates that \(C_h(f), C_g(f) \leq 1\), because \(a(v, t) \leq t\) for all \(t > 0\) and \(v \in T^1 S\).

Before we start proving Theorem 27, we shall introduce two useful lemmas.
Lemma 10. Suppose $\mu \in \mathcal{M}^\phi$ and ergodic. Then there exists a sequence of conjugacy classes $\{[\gamma_n]\} \subset [\pi_1 S]$, i.e., closed geodesics, such that

$$\int F d\mu = \lim_{n \to \infty} \frac{h[\gamma_n]}{l_g[\gamma_n]},$$

where $F$ is the reparametrization function defined in Theorem 26.

Proof. First, by the sub-additive ergodic theorem we know that for $\mu$-a.e. $v \in T^1 S$

$$\lim_{t \to \infty} \frac{a(v, t)}{t} = I_\mu(S, M). \tag{4.3.1}$$

By the Birkhoff ergodic theorem we have for $\mu$-a.e. $v \in T^1 S$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t F(\phi_s v) ds = \int F d\mu. \tag{4.3.2}$$

We define two sets

$$A := \{v \in T^1 S : v \text{ satisfies (4.3.1)}\}$$

$$B := \{v \in T^1 S : v \text{ satisfies (4.3.2)}\}.$$

Since $A$ and $B$ are both full $\mu$-measure, we have $A \cap B \neq \emptyset$.

Pick $v \in A \cap B$, and $\varepsilon_n \searrow 0$ as $n \to \infty$. By the Anosov Closing Lemma (Theorem 20), for each $\varepsilon_n$, there exists $\delta_n = \delta_n(\varepsilon_n)$ such that for $v \in T^1 S$ and $T_n = T_n(\delta_n) > 0$ satisfying $D_g(\phi_{T_n}(v), v) < \varepsilon_n$, then there exists $w_n \in T^1 S$ generates a periodic orbit $\tau_n^S$ on $S$ of period $l_g(\tau_n^S) = T_n'$ such that $|T_n - T_n'| < \varepsilon_n$ and $D_g(\phi_{s}(v), \phi_{s}(w_n)) < \varepsilon_n$ for all $s \in [0, T_n]$.

Furthermore, because the geodesic flow $\phi_t$ on $T^1 S$ is a transitive Anosov flow and $T^1 S$ is compact, by the Poincaré recurrent theorem, for each $\delta_n$ given as above, we can pick $T_n$ to be the $n$-th return time of the flow $\phi_t$ to the set $B_{\delta_n}(v)$, i.e.,
$D_g(\phi_{\tau_n}(v), v) < \delta_n$ for each $n$.

Suppose $\tau_n^S$ corresponds to $[\gamma_n] \in [\pi_1 S]$, then since $\mu$ is ergodic, by the Birkhoff ergodic theorem we have

$$\int_{T^1 S} F d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(\phi_t v) dt.$$ 

**Claim:** $\int F d\mu = \lim_{n \to \infty} \int_{[\gamma_n]} F.$

pf. Notice that

$$\frac{1}{l_g[\gamma_n]} \int_0^{l_g[\gamma_n] - \varepsilon_n} F(\phi_t v) \leq \frac{1}{l_n} \int_0^{l_n} F(\phi_t v) \leq \frac{1}{l_g[\gamma_n] - \varepsilon_n} \int_0^{l_g[\gamma_n] + \varepsilon_n} F(\phi_t v).$$

Because $F$ is Hölder, we know that $|F(\phi_t v) - F(\phi_t w_n)| \leq C \cdot D_g(\phi_t v, \phi_t w_n)^\alpha \leq C \cdot \varepsilon_n^\alpha$.

When $n$ is big enough such that $l_g[\gamma_n] > 2\varepsilon_n$ (notice that $\varepsilon_n \searrow 0$ and $l_g[\gamma_n] \nearrow \infty$), we have

$$\left| \frac{1}{l_n} \int_0^{l_n} F(\phi_t v) - \frac{1}{l_g[\gamma_n]} \int_0^{l_g[\gamma_n]} F(\phi_t w_n) \right|$$

$$\leq \frac{l_g[\gamma_n]}{l_g[\gamma_n]} \int_0^{l_g[\gamma_n]} |F(\phi_t v) - F(\phi_t w_n)| dt$$

$$\leq \frac{1}{l_g[\gamma_n] - \varepsilon_n} (l_g[\gamma_n] \cdot C \cdot \varepsilon_n^\alpha + 2\varepsilon_n \cdot \|F\|_{\infty})$$

$$\leq 2C \cdot \varepsilon_n^\alpha + \frac{2\varepsilon_n}{l_g[\gamma_n] - \varepsilon_n} \cdot \|F\|_{\infty}.$$ 

So, we finish the proof of this claim.

Moreover, from the construction of $F$, $\forall [\gamma_n] \in [\pi_1 S]$ we have

$$\int_{[\gamma_n]} F = l_h[\gamma_n].$$
Therefore,
\[
\int F \, d\mu = \lim_{n \to \infty} \frac{\int_{\gamma_n} F}{l_g[\gamma_n]} = \lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]}
\]

Lemma 11. Let \(\{[\gamma_n]\} \subset [\pi_1 S]\) be the sequence constructed in the proof of Lemma 10. Then
\[
\lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]} = I_\mu(S, M).
\]

Proof. Claim:
\[
\lim_{n \to \infty} \frac{a(w_n, l_g[\gamma_n])}{l_g[\gamma_n]} = \lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]}
\]

pf. By definition,
\[
a(w_n, l_g[\gamma_n]) := d_h(\pi \circ f \circ w_n, \pi \circ f \circ \tilde{\phi}_{l_g[\gamma_n]} w_n).
\]

For such \([\gamma_n] \in [\pi_1 S]\), let \(\tau_n^S, \tau_n^M\) denote the corresponding closed geodesics on \(S\) and \(M\), and \(\tilde{\tau}_n^S\) and \(\tilde{\tau}_n^M\) denote their lifting on \(\tilde{S}\) and \(\text{Conv}(\Lambda(\Gamma))\), respectively. Then we know that \(\tilde{f}(\tilde{\tau}_n^S)\) and \(\tilde{\tau}_n^M\) are at most Hausdorff distance \(R\) from each other. Therefore we can choose \(x_n \in \tilde{\tau}_n^M\) such that \(d_h(\pi w_n, x_n) < R\). Because \(d_h\) is \(\Gamma\)-invariant, \(\tilde{f} : \tilde{S} \to \mathbb{H}^3\) is an embedding, and \(\pi \circ f \circ w_n\) and \(\pi \circ f \circ \tilde{\phi}_{l_g[\gamma_n]} w_n\) project to the same point on \(S\), we have \(d_h(\gamma_n \cdot x_n, \pi \circ f \circ \tilde{\phi}_{l_g[\gamma_n]} w_n) = d_h(\pi \circ f \circ w_n, x_n) < R\). Hence, by the triangle inequality

\[
\left| d_h(\pi \circ f \circ w_n, x_n) - d_h(x_n, \gamma_n \cdot x_n) \right| \leq d_h(\pi \circ f \circ w_n, x_n) \leq R\]
\[
+ d_h(\gamma_n \cdot x_n, \pi \circ f \circ \tilde{\phi}_{l_g[\gamma_n]} w_n) \leq R\]
\[
= 2R.
\]
Therefore,
\[
\lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]} = \lim_{n \to \infty} \frac{l_h[\gamma_n] - 2R}{l_g[\gamma_n]} \leq \lim_{n \to \infty} \frac{a(w_n, l_g[\gamma_n])}{l_g[\gamma_n]} \leq \lim_{n \to \infty} \frac{l_h[\gamma_n] + 2R}{l_g[\gamma_n]} = \lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]},
\]
and we finish the proof of this claim.

**Claim:**
\[
I_\mu(S, M) = \lim_{t \to \infty} \frac{a(v, t)}{t} = \lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]}.
\]

pf. Pick the \( t_n \) as we mentioned in the first paragraph. Then
\[
|a(v, t_n) - a(w_n, l_g[\gamma_n])| \leq |d_h(\pi \circ f \circ v, \pi \circ f \circ \tilde{\phi}_{t_n} v) - d_h(\pi \circ f \circ w_n, \pi \circ f \circ \tilde{\phi}_{t_n} w_n)|
\]
(triangle inequality)
\[
\leq d_h(\pi \circ f \circ v, \pi \circ f \circ w_n) + d_h(\pi \circ f \circ \tilde{\phi}_{t_n} w_n, \pi \circ f \circ \tilde{\phi}_{t_n} v) + 2L
\]
(quas-isometry Lemma 8)
\[
\leq C \cdot (\delta_2 + \varepsilon) + 2L
\]
(Anosov closing lemma),

where \( C \) and \( L \) are the quasi-isometry constants only depending on the embedding \( f : S \to M \).

Therefore,
\[
\lim_{t_n \to \infty} \frac{a(v, t_n)}{t_n} = \lim_{n \to \infty} \frac{a(w_n, l_g[\gamma_n])}{l_g[\gamma_n]} = \lim_{n \to \infty} \frac{l_h[\gamma_n]}{l_g[\gamma_n]}.
\]

\[\square\]

**Proof of Theorem 27.** To see \( C_h(f) = I_\mu(S, M) \), as mentioned in Remark 31 that \( \mu = \mu_{-h_F F} \) is the equilibrium state for \( -h_F F \) and thus an ergodic measure. By Lemma 10 and Lemma 11, there exists a sequence \( \{[\gamma_n]\} \subset [\pi_1 S] \) (which depends on
\( \mu \) such that

\[
C_h(f) = \int F \, d\mu
\]

(Lemma 10) = \( \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)} \)

(Lemma 11) = \( I_\mu(S, M) \).

Since \( \mu_{BM} \) is also a Gibbs measure, the same argument works for \( \mu_{BM} \); therefore, we have \( C_g(f) = I_{\mu_{BM}}(S, M) \).

\[
\square
\]

4.4 Applications to Immersed Minimal Surfaces in Hyperbolic 3–manifolds

4.4.1 Immersed Minimal Surfaces in Hyperbolic 3–Manifolds

In what follows, \( M \) denotes a hyperbolic 3–manifold equipped with a hyperbolic metric \( h \) and \( S \) is a compact, 2–dimension manifolds with negative Euler characteristic. Recall that \( f : S \to M \) is called a minimal immersion if \( f \) is an immersion and the mean curvature \( H \) vanishes identically.

Let \( g = f^*h \) denote the induced metric on \( S \) via the immersion \( f \). By the Gauss equation, when \( f : S \to M \) is a minimal immersion, the Gaussian curvature \( K_g \leq -1 \).

So, applying the Theorem 26 to this case, we have the following corollary.

**Corollary 2.** Let \( f : S \to M \) be a \( \pi_1 \)–injective minimal immersion from a compact surface \( S \) to a hyperbolic 3–manifold \( M \), and \( \Gamma \) be the copy of \( \pi_1S \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion. Suppose \( \Gamma \) is convex cocompact. Then assertions (1) – (5) in Theorem 26 are true.
4.4.2 Minimal Hyperbolic Germs

4.4.2.1 Minimal Hyperbolic Germs

In the next three sections, following Uhlenbeck’s assumptions in [42], we shall assume $S$ is a closed surface.

Recall that $H$ is the set of the isotopy classes of pairs consisting of a Riemann metric $g$ and a symmetric 2-tensor $B$ on $S$ such that the trace of $B$ w.r.t. $g$ is zero and $(g, B)$ satisfies the Gauss-Codazzi equations (cf. Remark 26). Such a pair $(g, B) \in H$ can be integrated into an immersed minimal surface of a hyperbolic 3-manifold with the induced metric $g$ and second fundamental form $B$. Moreover, we know that for each data $(g, B) \in H$ there exists a representation $\rho : \pi_1(S) \to \text{Isom} (\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ leaving this minimal immersion invariant. Thus, we have a map

$$\Phi : H \to R(\pi_1(S), \text{PSL}(2, \mathbb{C})), \quad (4.4.1)$$

where $R(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ is the space of conjugacy classes of representations of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$.

The following corollary is an obvious consequence of Theorem 26. Recall that $h_{\text{top}}(g, B)$ denotes the topological entropy of the geodesic flow for the immersed surface $(S, g)$ with second fundamental form $B$.

**Corollary 3.** Let $\rho \in R(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ be a discrete, convex cocompact representation and suppose $(g, B) \in \Phi^{-1}(\rho) \neq \emptyset$. Then there are explicit geometric constants $C_1(g, B)$ and $C_2(g, B)$ with $0 \leq C_1(g, B) \leq C_2(g, B) \leq 1$ such that

$$C_1(g, B) \cdot \delta_{\rho(\pi_1 \Sigma)} \leq h_{\text{top}}(g, B) \leq C_2(g, B) \cdot \delta_{\rho(\pi_1 \Sigma)} \leq \delta_{\rho(\pi_1 \Sigma)}$$

with the last equality if and only if $B$ is identically zero which holds if and only if $\rho$ is Fuchsian.

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Proof. Notice that \((g, B) \in \Phi^{-1}(\rho)\) means that there exists a \(\pi_1\)-injective immersion \(f : \Sigma \to \rho(\pi_1S)\setminus \mathbb{H}^3 = M\) such that the induced metric is \(g\) and the second fundamental form is \(B\), where \((M, h)\) is a convex cocompact hyperbolic 3–manifold. Therefore, by Theorem 26 we have

\[
C_h(f) \cdot \delta_{\rho(\pi_1\Sigma)} \leq h_{top}(S) \leq C_g(f) \cdot \delta_{\rho(\pi_1\Sigma)}.
\]

Then we pick \(C_1(g, B) = C_h(f)\) and \(C_2(g, B) = C_g(f)\). The rightmost inequality is because \(C_2(g, B) = C_g(f) \leq 1\), and the rigidity is the consequence of Corollary 6.

\[\square\]

Remark 32. By Sullivan’s theorem, we know that \(\delta_{\rho(\pi_1S)} = \dim H \Lambda(\rho(\pi_1S))\). Thus, we can replace the critical exponent by the Hausdorff dimension in the above corollary.

4.4.2.2 Quasi-Fuchsian Spaces

We call a discrete faithful representation \(\rho : \pi_1(S) \to \text{Isom} (\mathbb{H}^3)\) quasi-Fuchsian if and only if the limit set \(\Lambda(\rho(\pi_1S))\) of \(\rho(\pi_1S)\) is a Jordan curve and the domain of discontinuity \(\partial_\infty \mathbb{H}^3 \backslash \Lambda(\rho(\pi_1S))\) is composed by two invariant, connected, simply-connected components. \(Q\mathcal{F}\) denotes the set of quasi-Fuchsian representations.

We notice that if \(\rho \in Q\mathcal{F}\), then elements in \(\Phi^{-1}(\rho)\) are \(\pi_1(S)\)–injective minimal immersions from \(S\) to \(\rho(\pi_1(S))\setminus \mathbb{H}^3\). Moreover, Uhlenbeck in [42] points out that for \(\rho \in Q\mathcal{F}\), \(\Phi^{-1}(\rho)\) is always a non-empty set.

**Corollary 4.** Let \(\rho \in Q\mathcal{F}\) be a quasi-Fuchsian representation and \((g, B) \in \Phi^{-1}(\rho)\).

Then there are explicit geometric constants \(C_1(g, B)\) and \(C_2(g, B)\) with \(0 \leq C_1(g, B) \leq C_2(g, B)\) such that

\[
C_1(g, B) \cdot \delta_{\rho(\pi_1S)} \leq h_{top}(g, B) \leq C_2(g, B) \cdot \delta_{\rho(\pi_1S)} \leq \delta_{\rho(\pi_1S)}
\]

with the last equality if and only if \(B\) is identically zero which holds if and only if \(\rho\)
is Fuchsian.

Using the above corollary, we can give another proof of the famous Bowen’s rigidity theorem.

**Corollary 5** (Bowen’s rigidity [5]). A quasi-Fuchsian representation \( \rho \in QF \) is Fuchsian if and only if \( \dim_H \Lambda_\Gamma = 1 \).

**Proof.** For any \((g,B) \in \Phi^{-1}(\rho)\), we have \( S \) is an immersed minimal surface in a quasi-Fuchsian manifold \( M = \rho(\pi_1 S) \setminus \mathbb{H}^3 \) with the induced metric \( g \) and the second fundamental form \( B \). Let \( K(S) \) denote the Gaussian curvature of \( S \) in \( M \), then by the Gauss-Codazzi equation \( K(S) \leq -1 \). Therefore using the Theorem B in [21], we have

\[
\text{htop}(g, B) \geq \left( \frac{-\int K(S) dA}{\text{Area}(S)} \right)^{\frac{1}{2}} \geq 1.
\]

Hence the result is derived by the above lower bound of \( \text{htop}(S) \) plus the above corollary. \( \square \)

4.4.2.3 Almost-Fuchsian Spaces

Recall that the *almost-Fuchsian* space \( AF \) is the space of minimal hyperbolic germs \((g,B) \in H \) such that \( \|B\|_g < 2 \). Given a hyperbolic metric \( m \in F \) and a holomorphic quadratic differential \( \alpha \in Q(|m|) \), we discuss an informative smooth path

\[
r(t) = (g_t, tB) \subset AF,
\]

where \( g_t = e^{2u_t}m \) and \( B = \text{Re}(\alpha) \) satisfying \( \|tB\|_{g_t}^2 < 2 \). Notice that \( u_t : \Sigma \rightarrow \mathbb{R} \) is well-defined and smooth on \( t \) (cf. Theorem 25).

Through studying this path, we can learn many geometric features of the Fuchsian space \( F \). For example, we will see how the entropy behaves while we change the data along the ray \( r(t) \) in \( AF \). In the following, we denote the unit tangent bundle of \( S \) associated with the data \( r(t) \) by \( T^1 S_{r(t)} \).
We recover the theorem below by employing the reparametrization method.

**Theorem 28** (Sanders, Theorem 3.5 [35]). Consider the entropy function restricted to the almost-Fuchsian space $h_{\text{top}} : \mathcal{AF} \to \mathbb{R}$, then

1. The entropy function $h_{\text{top}}$ realizes its minimum at the Fuchsian space $\mathcal{F}$; and

2. For $(m, 0) \in \mathcal{F}$, $h_{\text{top}}$ is monotone increasing along the ray $r(t) = (g_t, tB)$ provided $\|tB\|_{g_t} < 2$, i.e., $r(t) \subset \mathcal{AF}$, where $g_t = e^{2u_t}m$.

Fixed $t_0 > 0$, $r(t_0) = (e^{2u_{t_0}}m, t_0B)$ defines the geodesic flow $\phi^{t_0} : T^1S_{r(t_0)} \to T^1S_{r(t_0)}$. For any $t > t_0$, we want to show $h_{\text{top}}(t_0, g_0) \leq h(t, g)$ and equality holds if and only if $B = 0$. Since $(S, g_t)$ is negatively curved, using the distance function $d_{g_t}$, we can construct the Busemann cocycle $B^{\phi^t}_{\xi}(x, y)$ like we did in the proof of Theorem 26. Then by Theorem 22, we can reparametrize the geodesic flow $\phi^{t_0}$ induced by the data $r(t_0) = (e^{2u_{t_0}}m, t_0B)$ on $T^1S_{r(t_0)}$ by a Hölder function $F_t$ on $T^1S_{r(t_0)}$ such that $\phi^t : T^1S_{r(t)} \to T^1S_{r(t)}$ is conjugated to $(\phi^{t_0})^{F_t} : T^1S_{r(t_0)} \to T^1S_{r(t_0)}$. We consider the pressure $P_{\phi^{t_0}} : C(T^1S_{r(t_0)}) \to \mathbb{R}$, and we have

$$P_{\phi^{t_0}}(-h_{F_t} \cdot F_t) = 0 = P_{\phi^{t_0}}(-h(g_t, tB) \cdot F_t)$$

$$P_{\phi^{t_0}}(-h_{F_{t_0}} \cdot 1) = 0 = P_{\phi^{t_0}}(-h(g_{t_0}, t_0B) \cdot 1).$$

**Remark 33.** Without using Theorem 22, when $t_0$ and $t$ are small, the structure stability of Anosov flows also gives us the reparametrizing function $F_t$. We will see more details about this perspective in the next section.

Before we start proving Theorem 28, we first prove a key observation.

**Lemma 12.** The Riemannian metric $g_t = e^{2u_t}m$ is decreasing, i.e., $\frac{d}{dt}u_t < 0$ for all $t > 0$. 

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Proof. By Theorem 24 (the Gauss equation), we have

\[ K_{gt} = -1 - \frac{1}{2} t^2 e^{-4u_t} \|B\|^2_m = e^{-2u_t} (\Delta_m u_t - 1). \]  

(4.4.2)

Taking the derivative of equation 4.4.2 w.r.t. \( t \) and evaluating at \( t_0 \) provided \( \|t_0 B\|_{g_0}^2 < 2 \), then

\[ -\Delta_m \dot{u}_{t_0} = e^{2u_{t_0}} \cdot \dot{u}_{t_0} (\|t_0 B\|_{g_0}^2 - 2) - t_0 e^{-2u_{t_0}} \|B\|^2_m. \]

Because for the fixed \( t_0 \), at a maximum of \( u_{t_0} \) we have \( -\Delta_m \dot{u}_{t_0} \geq 0 \). Thus

\[ e^{2u_{t_0}} \cdot \dot{u}_{t_0} (\|t_0 B\|_{g_0}^2 - 2) - t_0 e^{-2u_{t_0}} \|B\|^2_m \geq 0. \]

The hypothesis \( \|t_0 B\|_{g_0}^2 < 2 \) implies that \( \dot{u}_{t_0} \leq 0 \) at each maximum; hence \( \dot{u}_{t_0} \leq 0 \) for all points on \( S \). Moreover, if \( \dot{u}_{t_0} = 0 \) for some \( t_0 > 0 \), then we have \( B = 0 \). This means \( u_t \equiv 0 \) for all \( t \).

Proof of Theorem 28. Because almost-Fuchsian is quasi-Fuchsian, the first assertion is a consequence of Corollary 4 and the fact that \( h_{top}(g, B) \geq 1 \) (cf. the proof of Corollary 5). The remaining part of this proof is devoted to the second assertion.

It’s enough to show \( l_{g_t} [\gamma] \leq l_{g_{t_0}} [\gamma] \), for \( t > t_0 \) and \( \forall [\gamma] \in [\pi_1 S] \). Because if \( l_{g_t} [\gamma] \leq l_{g_{t_0}} [\gamma] \) for all \( [\gamma] \in [\pi_1 S] \), then we have \( F_t \) is cohomologous to a function which is bigger then \( F_{t_0} (= 1) \), abusing the notation, we denote this function by \( F_t \) again, i.e., \( l_{g_t} [\gamma] \leq l_{g_{t_0}} [\gamma] \implies F_t \leq F_{t_0} = 1. \)

Claim: \( F_t \leq F_{t_0} = 1 \implies h_t \geq h_{t_0} \) for \( t \leq t_0 \) and the equality holds if and only if \( F_t \sim 1. \)

pf. we repeat the same trick used in the proof of Theorem 26. Since the pressure is monotone (see Remark 11), we have
Thus,

$$0 = P_{\phi_{t_0}}(-h_{t_0}) = P_{\phi_{t_0}}(-h_{t_0}F_{t_0}) = P_{\phi_{t_0}}(-h_{t}F_{t}) \geq P_{\phi_{t_0}}(-h_{t}F_{t_0}) = P_{\phi_{t_0}}(-h_{t}).$$

By definition,

$$0 = P_{\phi_{t_0}}(-h_{t_0}) = \sup_{\nu \in \mathcal{M}^{\phi_{t_0}}} h(\nu) + \int (-h_{t_0})d\nu \implies \sup_{\nu \in \mathcal{M}^{\phi_{t_0}}} h(\nu) = h(\mu_{h_{t_0}}) = h_{t_0}$$

where $\mu_{h_{t_0}}$ is the equilibrium state of the function $-h_{t_0} \cdot 1$ and

$$0 \geq P_{\phi_{t_0}}(-h_{t}) = \sup_{\nu \in \mathcal{M}^{\phi_{t_0}}} h(\nu) + \int (-h_{t})d\nu = h_{t_0} - h_{t}.$$  

To see the equality case, we notice that $h_{t_0} = h_{t}$ implies that $h(\mu_{h,F_{t}}) = h_{t} = h_{t_0}$, i.e.,

$$0 = P_{\phi_{t_0}}(-h_{t_0} \cdot 1) = h(\mu_{h,F_{t}}) - \int h_{t_0} \cdot 1d\mu_{h,F_{t}}.$$  

In other words, $\mu_{F_{t}h_{t}}$ is the equilibrium state of the constant function $-h_{t_0} \cdot 1$. Hence, by the uniqueness of equilibrium state (cf. Theorem 15) we have $\mu_{h,F_{t}} = \mu_{h_{0} \cdot 1}$ and which implies $h_{t}F_{t} \sim h_{0} \cdot 1$, i.e., $F_{t} \sim 1$. We now complete the proof of this claim.

By the above claim, we know the equality holds if and only if $F_{t} \sim 1$, i.e., $l_{g_{0}}[\gamma] = l_{g}[\gamma]$ for all $[\gamma] \in [\pi_{1}S]$. By the marked length spectrum theorem [29], this means that $g_{0} = g_{1}$. In other words, $h(g_{t}, tB)$ is strictly increasing when $u_{t} \neq 0$.

To prove $l_{g_{t}}(\gamma) \leq l_{g_{0}}(\gamma)$ for $t > t_{0}$, it is enough to show that $g_{t} = e^{2u_{t}m}$ is decreasing, i.e., $\frac{d}{dt}u_{t} < 0$ for all $t > 0$ (which is proved in the above Lemma).

To see this, fixing a free homology class of a closed curve $\tau$ and let $c_{t}$ denote the
closed geodesic in this class under the metric $g_t$, assuming that $g_t$ is decreasing, then we have $|| \cdot ||_{g_t} \leq || \cdot ||_{g_t}$ for $t > t_0$. Thus,

$$l_{g_t}(c_t) \leq l_{g_t}(c_{t_0}) = \int_{c_{t_0}} ||v||_{g_t} \leq \int_{c_{t_0}} ||v||_{g_{t_0}} = l_{g_{t_0}}(c_{t_0}),$$

where $v$ is the unit tangent vector of $c_{t_0}$ for the metric $g_{t_0}$, i.e.,

$$v(s) := \frac{d}{ds}(c_{t_0}(s))/||\frac{d}{ds}(c_{t_0}(s))||_{g_{t_0}}.$$

Remark 34. The main difference between our proof and Sanders’ proof in [35] is that in [35] he used a sophisticated formula of the derivative of the topological entropy whereas in our reasoning we examine the length changing along the path directly.

4.4.2.4 Another Metric on $\mathcal{F}$

Following the previous theorem, Sanders proves that one can define a metric on the Fuchsian space $\mathcal{F} \subset \mathcal{H}$ by taking the Hessian of the topological entropy along the path $r(s) = (e^{2u_s}m, sB)$. The striking point is that this metric is bounded below by the Weil-Petersson metric on $\mathcal{F}$.

Recall that the fiber of the cotangent bundle of $m \in \mathcal{F}$ is identified with the space of holomorphic quadratic differentials on the Riemann surface $(S, m)$. Thus, in order to connect the Hessian of the entropy with the Weil-Petersson metric, we will prove that the Hessian of the topological entropy along the given path $r(s)$ gives us a way to measure holomorphic quadratic differentials on $(S, m)$. It is because $r(s)$ is defined by the data $(m, B)$, where $B$ is given by a holomorphic quadratic differential $\alpha$ such that $B = \text{Re}(\alpha)$.

In the following, we give another proof of Sanders’ theorem by using the pressure
Before we start proving this result, we recall several important concepts of Anosov flows. We first notice that by the structure stability of the Anosov flow (cf. Prop. 1 in [31] or [13]), when $s$ is small, the geodesic flows $\phi^s : T^1 S_{r(s)} \to T^1 S_{r(s)}$ conjugates to the reparametrized flow $\phi^{F_s} : T^1 S_{r(0)} \to T^1 S_{r(0)}$ where $\phi : T^1 S_{r(0)} \to T^1 S_{r(0)}$ is the geodesic flow on $T^1 S_{r(0)}$ and $F_s$ the is the reparametrizing Hölder continuous function. Since the path $r(s)$ is a smooth one parameter family in $\mathcal{AF}$, the structure stability theorem also indicates that $\{F_s\}$ form a smooth one parameter family of Hölder continuous functions on $T^1 S_{r(0)}$.

Since we shall only be interested in metrics $g_s$ close to $g_0(= m)$, it suffices to consider one parameter families given by perturbation series: for $s$ small,

$$g_s = g_0 + s \cdot \dot{g}_0 + \frac{s^2}{2} \ddot{g}_0 + ..., \text{ and } F_s = F_0 + s \cdot \dot{F}_0 + \frac{s^2}{2} \ddot{F}_0 + ..., $$

where $\dot{g}_0, \ddot{g}_0, ...$ are symmetric 2-tensors on $T^1 S_{r(0)}$ and $\dot{F}_0, \ddot{F}_0, ...$ are Hölder continuous functions on $T^1 S_m$.

The following lemma connects the derivatives of $g_s$ with $F_s$.

**Lemma 13** (Pollicott, Lemma 5 [31]).

$$\int_{T^1 S} \dot{F}_0 d\mu_0 = \int_{T^1 S} \dot{g}_0(v, v) d\mu_0, \quad (4.4.3)$$

and

$$\int_{T^1 S} \dddot{F}_0 d\mu_0 \leq \int_{T^1 S} \dddot{g}_0(v, v) \frac{1}{2} d\mu_0. \quad (4.4.4)$$

**Remark.** The proof of above lemma is a straightforward computation. However, in the sake of brevity we omit the proof.

The following lemma reveals the relation between the Weil-Petersson metric and the second derivative of the metric $g_s$. 

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Lemma 14.

\[ \int_{T^1S} \frac{\ddot{g}_0(v,v)}{2} d\mu_0 = \int_{T^1S} \ddot{u}_0 d\mu_0 = -2\pi \int_S \|\alpha\|^2_m dV_m. \]

**Proof. Claim:**

\[ \int \ddot{u}_0 d\mu_0 = -2\pi \int \|\alpha\|^2_m dV_m. \]

pf. For \( m \in \mathcal{F} \) and \( \alpha \in Q([m]) \), we notice that the Gauss equation (Theorem 24) for this given data \( r(s) = (e^{2u_s}, s \cdot \text{Re} \alpha) \) is that

\[ \Delta_m u_s + 1 - e^{-2u_s} - se^{-2u_s} \|\alpha\|^2_m = 0, \]

where \( \|\alpha\|_m \) is the norm of \( \alpha \) with respect to the hyperbolic metric \( m \).

Taking the derivative with respect to \( s \), we have

\[ -\Delta_m \dot{u}_s = e^{2u_s} \cdot \dot{u}_s (\|s\alpha\|^2_m - 2) - se^{-2u_s} \|\alpha\|^2_m. \]

The maximum principle implies that \( \dot{u}_0 = 0 \). We differentiate the above equation again and evaluate at \( s = 0 \), then we get

\[ -\Delta_m \ddot{u}_0 = -2\ddot{u}_0 - 2\|\alpha\|^2_m. \quad (4.4.5) \]

By integrating the equation (4.4.5) with respect to the volume form of \( m \), and because \( S \) is has no boundary we have

\[ 0 = -2\int_S \ddot{u}_0 dV_m - 2\int_S \|\alpha\|^2_m dV_m. \]

Because in this case the Bowen-Margulis measure \( \mu_0 \) of the geodesic flow \( \phi : T^1S_m \to T^1S_m \) is the Liouville measure, we have
\[ 2\pi \int_S \dddot{u}_0 dV_m = \int_{T^1 S} \dddot{u}_0 d\mu_0. \]

**Claim:**
\[ \int_{T^1 S} \frac{\dddot{g}_0(v, v)}{2} d\mu_0 = \int_{T^1 S} \dddot{u}_0 d\mu_0. \]

pf. It is straightforward, because \( \dddot{u}_0 = 0 \) and \( \dddot{g}_0 = 2\dddot{u}_0 m. \)

Now we are ready to state and prove the main result of this section.

**Theorem 29** (Sanders, Theorem 3.8 [35]). *One can define a Riemannian metric on the Fuchsian space \( F \) by using the Hessian of \( h_{\text{top}} \). Moreover, this metric is bounded below by \( 2\pi \) times the Weil-Petersson metric on \( F \).*

**Proof of Theorem 29.** Here we consider \( c_t := -h_{\text{top}}(g_t, tB) \cdot F_t = -h_t F_t \). Since \( P_\phi(c_t) = 0 \), we know that \( \int \dot{c}_0 d\mu_0 = 0 \) where \( \mu_0 \) is the Bowen-Margulis measure of the flow \( \phi : T^1 S_{r(0)} \to T^1 S_{r(0)} \), i.e., \( m_{c_0} = \mu_0 \) and \( \dot{c}_0 \in T_{c_0} \mathcal{P}(T^1 S_m) \). Therefore, by Proposition 4, the pressure metric of \( \dot{c}_0 \) is
\[
\| \dot{c}_0 \|^2_P = -\frac{\text{Var}(\dot{c}_0, \mu_0)}{\int \dot{c}_0 d\mu_0} = \int \dddot{c}_0 d\mu_0. 
\]

Notice that \( h_0 = 1, F_0 = 1, \) and \( \dddot{u}_0 = 0 \), so by Lemma 13
\[
\int \dddot{F}_0 d\mu_0 = \int \dddot{g}_0 d\mu_0 = \int 2\dddot{u}_0 m(v, v) d\mu_0 = 0,
\]
and hence
\[
0 \leq \| \dot{c}_0 \|^2_P = \dddot{h}_0 + 2\dddot{h}_0 \int \dddot{F}_0 d\mu_0 + \dddot{h}_0 \int \dddot{F}_0 d\mu_0 = \dddot{h}_0 + \int \dddot{F}_0 d\mu_0.
\]
Therefore,

\[
\ddot{h}_0 \geq - \int_{T^1S} \dot{F}_0 \, d\mu_0 \\
\geq - \int_{T^1S} \frac{\ddot{g}_0(v,v)}{2} \, d\mu_0 \quad \text{Lemma 13} \\
= - \int_{T^1S} \ddot{u}_0 m(v,v) \, d\mu_0 \quad \text{Lemma 14} \\
= - \int_{T^1S} \ddot{u}_0 \, d\mu_0 \\
= 2\pi \int_S \|\alpha\|_m^2 \, dV_m \quad \text{Lemma 14} \\
= 2\pi \|\alpha\|_{WP}^2.
\]

Remark 35. Comparing with Bridgeman's results in [6], we suspect:

1. \( \int_{T^1S} \dot{F}_0 \, d\mu_0 = \int_{T^1S} \frac{\ddot{g}_0(v,v)}{2} \, d\mu_0 \), and

2. \( \|\dot{c}_0\|_P = 0 \).
APPENDIX A

CURVATURE FORMULA FOR \((\mathcal{M}_G^1, \| \cdot \|_M)\)

**Proposition.** The curvature of \((\mathcal{M}_G^1, \| \cdot \|_M)\) can be written explicitly as the following.

For \(0 < x, y \) and \( e^{x+y} < 3 + e^x + e^y \), we have

\[
K_M(x, y) = \frac{1}{4 (e^x + 1)^2 (e^y + 1)^2 f(x, y)} \cdot \left( e^x x(x - y - 4) + 2e^{2x} x(x - y - 4) + 2x(y - 2)e^{3x+4y} + 2(x - 2)ye^{4x+3y} - (x + 4)e^{3y} y - e^{2x} x(y + 4) + e^y y(-x + y - 4) + 2e^{2y} y(-x + y - 4) + ye^{4x+4y}(2x + y - 4)2ye^{4x+2y}(2x + y - 4) + xe^{x+4y}(x + 2y - 4) + 2xe^{2x+4y}(x + 2y - 4) + 4e^{x+y} \left(x^2 - 4x + (y - 4)y\right) + 2e^{x+3y} \left(2x^2 + x(3y - 8) - 8y\right) + 2e^{3x+y}(x(3y - 8) + 2(y - 4)y) + 4e^{3(x+y)}(x(3y - 4) - 4y) + e^{2x+y} \left(8x^2 + x(5y - 32) + 6(y - 4)y\right) + e^{x+2y} \left(6x^2 + x(5y - 24) + 8(y - 4)y\right) + 4e^{2(x+y)} \left(3x^2 + x(5y - 12) + 3(y - 4)y\right) + e^{2x+3y} \left(8x^2 + x(17y - 32) - 24y\right) + e^{3x+2y}(x(17y - 24) + 8(y - 4)y) - (5(x + 4)e^y + 4(x + 4)e^{2y} + (x + 4)e^{3y} + 4e^{2x+4y}(x + y - 2) + 5e^x(y + 4) + 4e^{2x}(y + 4) + e^{3x}(y + 4) \right) conti. (next page)
\[
\text{CONTI. = }
4e^{4x+2y}(x + y - 2) + 2(x + y + 4) + 2e^{x+y}(x + y + 20) + e^{4x+y}(2x + y - 4) + \\
e^{4x+3y}(2x + y - 4) + 4e^{3x+y}(2x + y - 2) + \\
e^{x+4y}(x + 2y - 4) + e^{3x+4y}(x + 2y - 4) + 4e^{x+3y}(x + 2y - 2) + \\
2e^{3(x+y)}(5x + 5y - 12) + e^{2x+y}(11x + 5y + 16) + \\
e^{x+2y}(5x + 11y + 16) + e^{2x+3y}(19x + 17y - 32) \\
+ e^{3x+2y}(17x + 19y - 32) + e^{2(x+y)}(26x + 26y - 24) \\
(\cdot (x+y - 1)^2 (6e^{x+y} + e^{2x+y} + e^{x+2y} + 7e^x + 2e^{2x} + 7e^y + 2e^{2y} + 6) \\
\cdot \log^2 \left( \frac{e^{x+y} - 1}{e^x + e^y + 2} \right) + \log \left( \frac{e^{x+y} - 1}{e^x + e^y + 2} \right) \right)
\]

where

\[
f(x, y) = \left\{ xe^{x+2y} + ye^{2x+y} + 2e^{x+y}(x + y) + \\
(-2e^{x+y} - e^{2x+y} - e^{x+2y} + e^x + e^y + 2) \log \left( \frac{e^{x+y} - 1}{e^x + e^y + 2} \right) + e^x x + e^y y \right\}
\]

**Proposition.** The curvature of \((M^1_{G}, \|\cdot\|_M)\) can be written explicitly as the following. For \(0 < x, y \) and \(4 > (e^x - 1)(e^y - 1)\), we have

\[
K_M(x, y) = -\{ 4(e^x - 1)(e^y - 1) \cdot \\
[xe^{x+y} + ye^{x+y} - \log ((e^x - 1)(e^y - 1)) + \\
2(-e^{x+y} + e^x + e^y) \log \left( \frac{1}{2} \sqrt{(e^x - 1)(e^y - 1)} \right) - e^x x - e^y y + \log(4) \}\}^{-1} \\
\cdot \text{CONTI (NEXT PAGE)}
\[ \left\{ -4e^x x + 4e^{2x} x + 8xe^{x+y} + 4xe^{2(x+y)} - 8xe^{2x+y} - 4xe^{x+2y} + \\
e^x x^2 - e^{2x} x^2 - 2xe^{x+y} - x^2e^{2(x+y)} + 2xe^{2x+y} \\
x^2e^{x+2y} - 4e^y y + 4e^{2y} y + 8ye^{x+y} + 4ye^{2(x+y)} - 4ye^{2x+y} \\
- 8ye^{x+2y} - e^x xy + e^{2x} xy - xe^y y + xe^{2y} y - \\
- 2ye^{x+y} - y^2e^{2(x+y)} + y^2e^{2x+y} + 2y^2e^{x+2y} + \\
2xye^{2(x+y)} + xye^{2x+y} + xye^{x+2y} + e^y y^2 - e^2y y^2 - \\
x \log(4) + xe^y \log(4) - y \log(4) + e^x y \log(4) - \\
(x(e^y - 1) + (e^x - 1)y) \log ((e^x - 1)(e^y - 1)) + \\
2 \left[ -2(x - 4)e^{2x+y} - 16e^{x+y} + (3x + 8)e^y + \\
e^{2x}(x - y - 4) - 2(y - 4)e^{x+2y} + \\
e^{2y}(-x + y - 4) \cdot (e^{2(x+y)}(x + y - 4) - 2(x + y + 2) + e^x(3y + 8)) \right] \cdot \\
\log \left( \frac{1}{2} \sqrt{(e^x - 1)(e^y - 1)} \right) \right\} \]


