SOME MODEL THEORY OF FIELDS AND DIFFERENTIAL FIELDS

Abstract

by

Gregory David Cousins

In this dissertation, we attempt to study connections between various properties of fields, namely boundedness, largeness, and strong form of model completeness called “almost quantifier elimination”. The first chapter consists of background information, and the second chapter is more or less devoted to the study of fields in the ring language, possible expanded by constants. We show that large, perfect fields with almost quantifier elimination are geometric and fail to have a strong notion of unboundedness.

The third chapter is dedicated to the study, and characterization, of differential fields with a notion of largeness for differential-algebraic sets. We also make some brief comments on notions of largeness for fields equipped with an automorphism.

The fourth chapter is based on joint work with Quentin Brunette, Anand Pillay, and Françoise Point in which we prove that if \( T \) is a theory of large, bounded fields of characteristic 0 with almost quantifier elimination, and \( T_D \) is the model companion of \( T \cup \{ \text{“}\partial \text{ is a derivation} \} \), then for any model \( (U, \partial) \) of \( T_D \), differential subfield \( K \) of \( U \) such that \( C(K) \models T \), and logarithmic differential equation \( d\log_G(z) = a \) (where \( G \) is an algebraic group defined over \( C(K) \), then there is a strongly normal extension \( L \) of \( K \) for the equation with \( K \subseteq L \subseteq U \).
To my parents, John and Josie, and my sister, Helena.
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Much of the work that went into this dissertation was motivated possible connections between various properties of fields, and the research was very much exploratory. In particular, we have been interested in three notions:

- almost quantifier elimination,
- largeness,
- and boundedness.

It seems to be the case that all of the most beloved fields have these three properties in common, and yet how they relate to each other in general has proven to be mysterious.

The notion of almost quantifier elimination (almost QE) is a strengthening of model completeness that is enjoyed by algebraically closed fields, real closed fields, p-adically closed fields, pseudo-finite fields, and, more generally, bounded PAC fields. While all of these fields (and their theories) are well-studied, it seems that the notion of almost QE as a property of a given theory of fields has not been explored.

Large fields were introduced by Pop in [67] as a class of fields over which one can prove strong inverse-galois results. For example, if $K$ is a large field, then every finite group $G$ can be realized as $G \cong \text{Gal}(L/K(t))$ for some finite galois extension $L/K(t)$. If $K$ is large and hilbertian, then the same thing is true for $K$ itself. Though largeness is an diophantine property of fields, the class of large fields is elementary and, furthermore, one can show that a field $K$ is large iff $K$ is existentially closed in the field of formal Laurent series $K((t))$. The PAC fields form an elementary subclass of the class of large fields and they have been studied intensely starting with the characterization of their elementary invariants in [15]. Disappointingly, there has
not been a concerted effort to analyse the theory of large fields model theoretically in a similar fashion. Of course, the class of large fields is much broader than the class of PAC fields, and so hoping for such an analysis may be unreasonable. Despite this, it would be interesting to know if largeness is implied by any model theoretic properties, analogous to the way that superstable fields are algebraically closed, \cite{14}. More generally, it is interesting to ask what diophantine properties of fields are implied by the well-known properties/dividing lines of theories (the stable fields conjecture is an example of this). However, the case appears to be that most of the results in this program conclude things about the absolute galois group of a field $K$. It is also this sort of result that leads us to consider bounded fields.

A field $K$ is called bounded if, for every $n$, $K$ has only finitely many algebraic extensions of degree $n$. Perfect bounded fields were studied by Serre in \cite{71} (where he called them “fields of type (F)” as a condition that implies the finiteness of certain cohomology groups (a related condition is $[K^\times : (K^\times)^n]$ being finite for every $n$, which is equivalent to $K$ having only finitely many solvable extension of any finite degree). As any field $K$ interprets its finite extensions, bounded fields have the property that for any $L \models \text{Th}(K)$, $\text{Gal}(L^{\text{alg}}/L) \cong \text{Gal}(K^{\text{alg}}/K)$, and so the theory “knows” the absolute galois group of $K$ (in fact, is is an if and only if, by compactness). In \cite{10} and \cite{28}, Chatzidakis and Hrushovski show that if $K$ is a PAC field, then Th($K$) is simple iff $K$ is bounded. Montenegro in \cite{52} showed that if $K$ is an unbounded PAC field, then Th($K$) has TP$_2$. One may expect (or hope) that a similar result may hold for large fields, but this conjecture is spectacularly false; in \cite{18}, the authors show that $K((\mathbb{Z}^\omega))$ (where $\mathbb{Z}^\omega$ is equipped with component-wise addition and the lexicographic ordering) is unbounded, large, and NIP whenever $K$ is a characteristic 0 field such that Th($K$) is NIP. Another result in a similar vein can be found in \cite{35}, where the authors show that NIP fields of characteristic $p$ are Artin-Schreier closed, and which in turn implies that model complete NIP of characteristic $p$ are
algebraically closed. There are two tantalizing conjectures that would fit nicely within the study of large fields: 1) infinite super-simple fields are PAC (and hence bounded) and 2) bounded fields are large. The origin of the first conjecture is unclear, while the second conjecture was suggested by Koenigsmann in [33]. While there has been some results towards the first conjecture (see [59], where the authors show that if \( K \) is supersimple then every rational variety over \( K \) has a \( K \)-point by proving that \( K \) has trivial Brauer group), the second conjecture seems to be wide open. Modulo the results in [57], where it is shown that every supersimple field is perfect and bounded, the second conjecture would imply that any supersimple field is large. We believe furthermore that fields whose theory has almost QE should be bounded, but were only able to show a weak version of this, namely that fields with almost QE are not unbounded in a strong way.

In later chapters, we study notions of largeness for differential fields and, briefly, fields with an automorphism. The motivation for this is the work by Hrushovski in [28], where he introduces the notion of a PAC substructure of a strongly minimal structure, and proves that bounded PAC substructures of a strongly minimal structure are simple (with their induced structure). This notion of PAC was later generalized in Polkowska’s thesis and the resulting papers [64, 58], and then studied further in a galois theoretic light by Hoffmann, [27, 26]. While PAC fields are typically defined in terms of the existence of \( K \)-points for absolutely irreducible \( K \)-varieties, large fields rely on a notion of smoothness for \( K \)-points of varieties that does not clearly generalize to other, less algebro-geometric, structures. PAC fields are, equivalently, those fields which are existentially closed in every regular extension, and so Poizat’s galois theory [62] gives another way to conveniently describe PAC substructures by generalizing the notion of regular extension. This remains troublesome for generalizing largeness, since large fields \( K \) are precisely those which are existentially closed in every regular extension that admits a (discrete, not necessarily definable)
$K$-rational $K$-place, [44], and so one would need a notion of discrete specialization of structures. There are, of course, many obstacles to this sort of generalization. Even in the case of fields, given a pair of bounded PAC fields $K \subseteq L$, there is no henselian, $K$-rational $K$-place on $L$, definable or otherwise. Despite these difficulties, the work in this dissertation on differential and difference fields will serve as motivation for future work on notions of largeness for more general structures.

In the last chapter, we are motivated by the theory of strongly normal extension and the theory of logarithmic differential equations introduced by Kolchin. In joint work with Brouette, Pillay, and Point, we prove that under the assumptions of largeness, almost QE, and boundedness, one can give strong results on the embeddability of strongly normal extensions for certain differential fields. Our hope is that this work can be extended further to the setting of multiple (say, two) commuting derivations, with the assumption that the field of constants with respect to one derivation is differentially large with respect to the other.

The first chapter of this dissertation lays out the background, terminology, and notation for the remaining sections. For the most part, the results in this section are standard and well-known.

In the second chapter, we initiate a study of fields with “almost quantifier elimination”. We say that a theory $T$ of fields in the language of rings (possibly with extra constants) has almost QE if for every $K \models T$ and every relatively algebraically closed substructure $A \subseteq K$, the theory $T \cup \text{qftp}(A)$ is complete. We go on to show that a theory $T$ has almost QE iff it is the model completion of the class of relatively algebraically closed substructures of a models of $T$ \ref{2.0.2}, and give a syntactic characterization of almost QE \ref{2.0.13} and provide several examples (in particular, we show that $\mathbb{C}((t_1)) \ldots ((t_n))$ has almost QE in the language of rings expanded by constants for $t_1, \ldots, t_n$, see \ref{2.0.28}). With the basics of fields with almost QE in hand, we show \ref{2.0.42} that large fields with almost QE are geometric in the sense of [55].
We conjecture that fields with almost QE are bounded. Though we were not able to confirm this conjecture, we introduce a notion that we call “strong unboundedness” (more generally for PAC substructures of stable structures) and observe that fields with almost QE are not strongly unbounded, see (2.1).

Chapter three begins with a comparison of two notions of largeness for differential fields. Focusing on the one of these notions, (3.1.2), which is known to be elementary (by work of Tressl [76]), we show that differential largeness of a field \((K, \partial)\) can be characterized in terms of particular embeddings into \((K((t)), d/dt)\) (analogous to how a field \(K\) is large iff \(K \preceq \exists K((t)),\) see (3.1.15). In Example 3.1.18, we construct a field that is differentially large. By results in [70], the algebraic closure of this field gives an example of a differentially closed field. In the last section (3.2) of the chapter, we introduce a notion of largeness for fields equipped with an automorphism, and make some minor observations analogous to those shown to hold for differentially large fields.

The final chapter, chapter four, is based on joint work with Quentin Brouette, Anand Pillay, and Françoise Point, [7]. We show that if \(T\) is the model companion of a theory of large, bounded fields with almost QE then, for \((U, \partial) \models T\) and certain “nice” subfields \((K, \partial) \subset (U, \partial)\), strongly normal extensions for logarithmic differential equations over \(K\) can be found embedded inside \((U, \partial)\) (4.0.7).
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O sweet spontaneous
earth how often have
the
doting

fingers of
prurient philosophers pinched
and
poked
thee
,has the naughty thumb
of science prodded
thy

beauty how
often have religions taken
thee upon their scraggy knees
squeezing and

buffeting thee that thou mightest conceive
gods

(but
true

to the incomparable
couch of death thy
rhythmic
lover

thou answerest

them only with

spring)

- e.e. cummings

CHAPTER 1
PRELIMINARIES

This chapter is dedicated to the background material needed for the rest of the dissertation. We cover the most of the basics of model theory, field theory, algebraic geometry, and differential algebra.

1.1 Model Theory

1.1.1 Notation and Basics

For this dissertation, we will suppose that the reader is familiar with the basic notions of model theory, see [50, 75, 25, 61, 9]. Let $\mathcal{L}$ be a language. We assume the convention of identifying an $\mathcal{L}$-structure $M$ with its underlying universe. In some cases we will indicate a distinguished symbol, such as in the case of a differential field $(K, \partial)$ or a field with an automorphism $(K, \sigma)$.

An $\mathcal{L}$-theory $T$ will be a consistent set of $\mathcal{L}$-sentences (everything will be first-order). In general, we will not assume that a given theory $T$ is complete. Given an $\mathcal{L}$-structure $M$, $\text{Th}(M)$ will be the complete theory of $\mathcal{L}$-sentences $\varphi$ such that $M \models \varphi$. For two $\mathcal{L}$-structures $M$ and $N$, $M \equiv N$ means that $\text{Th}(M) = \text{Th}(N)$. If $\Sigma$ and $\Delta$ are sets of $\mathcal{L}$-sentences, we write $\Sigma \vdash \Delta$ to mean $M \models \Sigma \Rightarrow M \models \Delta$ for any $\mathcal{L}$-structure $M$. Given an $\mathcal{L}$-structure $M$ and a symbol $R \in \mathcal{L}$, it is customary to write $R^M$ to mean the interpretation of $R$ in $M$. We will follow this convention for some basic definitions, but will ignore it for the most part when the context is clear.

In general $\bar{x}, \bar{y}, \bar{z}...$ will indicate finite (unless otherwise indicated) tuples of variables, and $\bar{a}, \bar{b}, \bar{c}...$ will indicate finite tuples of elements from some structure.
By an abuse of notation, we will write $\bar{a} \in M$ to mean that $\bar{a}$ is an element of $M^n$ for some $n \geq 1$. For an $\mathcal{L}$-structure $M$ and a formula $\varphi(\bar{x})$, we will often write $\varphi(M)$ for the subset of $M^n$ defined by $\varphi(\bar{x})$, i.e. $\varphi(M) := \{ \bar{a} : M \models \varphi(\bar{a}) \}$ and say that a set $X \subseteq M^n$ is $A$-definable (or definable over $A$) if there is some formula $\varphi(\bar{x})$ with parameters in $A$ such that $X = \varphi(M)$. We will write $\varphi(\bar{x}) \in \mathcal{L}(A)$ to mean that $\varphi(\bar{x})$ is an $\mathcal{L}$-formula with parameters from $A$. $\mathcal{L}(A)$ denotes the language $\mathcal{L}$ expanded by constants for $A$, and we consider $M$ as an $\mathcal{L}(A)$-structure.

Given an $\mathcal{L}$-structure $M$, an $\mathcal{L}$-structure $A \subseteq M$ will be called a ($\mathcal{L}$-) substructure of $M$ if $c^A = c^M$ for all constant symbols $c$, $f^A(\bar{a}) = f^M(\bar{a})$ for all $\bar{a} \in A$ and all functions symbols $f \in \mathcal{L}$, and $R^A(\bar{a})$ if $R^M(\bar{a})$ for all $\bar{a} \in A$ and all relation symbols $R \in \mathcal{L}$. If $A \subseteq M$ is an $\mathcal{L}$-structure such that $M \models \varphi(\bar{c}) \iff A \models \varphi(\bar{c})$ for every $\bar{c} \in A$ and every formula $\varphi(\bar{x}) \in \mathcal{L}$, then we say that $A$ is an elementary substructure of $M$, and write $A \preceq M$. Note that substructures always interpret constants in $\mathcal{L}$, so for example, the field of rational functions over the complex numbers, $\mathbb{C}(t)$, may be considered an $\mathcal{L}$-structure where $\mathcal{L}$ is the language of rings with an extra constant symbol "$t$", in which case $\mathbb{Q}(t) \subset \mathbb{C}(t)$ is an $\mathcal{L}$-substructure, but $\mathbb{Q} \subset \mathbb{C}(t)$ is not.

Given an $\mathcal{L}$-structure $M$ and a subset $A \subseteq M$, the definable closure of $A$ in $M$, $\text{dcl}_M(A)$, will be the set of elements of $M$ which are definable over $A$. The algebraic closure of $A$ in $M$, $\text{acl}_M(A)$ will be the set of elements in $c$ in $M$ such that there is a formula $\varphi(x) \in \mathcal{L}(A)$ with $M \models \varphi(c)$ and $\varphi(M)$ is finite. Given a tuple $\bar{a}$ in a structure $M$, we write $\text{dcl}_M(\bar{a})$ and $\text{acl}_M(\bar{a})$ the be the definable and, respectively, the algebraic closure of $\bar{a}$ considered as an unordered set. We not that if $A \subseteq M \models T$, then $\text{dcl}_M(A)$ and $\text{acl}_M(A)$ are both substructures of $A$. Furthermore, after fixing a theory $T$, we shall often feel free to write $\text{dcl}(A)$ and $\text{acl}(A)$ without indicating a superstructure $M$ by assuming that all of our computations take place in some big ambient model of $T$.

Let $T$ be some $\mathcal{L}$-theory, $M \models T$, and $A \subseteq M$ some set. A (partial) $\mathcal{L}$-type over
\( A, \Sigma(\bar{x}), \) is a consistent (with \( T \)) set of formulas \( \varphi(\bar{x}) \in \mathcal{L}(A) \). A complete type over \( A \) is a maximally consistent (with \( T \)) set of formulas with parameters from \( A \). For the rest of this dissertation, we will assume that all types are complete, unless otherwise indicated. Arbitrary types will most often be written as \( p, q, \ldots \), or \( p(\bar{x}), q(\bar{x}), \ldots \) if we have a particular tuple of variables in mind. Given a tuple \( \bar{c} \in M \), the type of \( \bar{c} \) over \( A \) is the complete type

\[
\text{tp}_M(\bar{c}/A) := \{ \varphi(\bar{x}) \in \mathcal{L}(A) : M \models \varphi(\bar{c}) \},
\]

and for two tuples \( \bar{b} \) and \( \bar{c} \) of the same length, we will write \( \bar{a} \equiv_A \bar{c} \) to mean \( \text{tp}_M(\bar{a}/A) = \text{tp}_M(\bar{c}/A) \). Similarly, the quantifier-free type of \( \bar{c} \) over \( A \) will be the set

\[
\text{qftp}_M(\bar{c}/A) := \{ \varphi(\bar{x}) \in \mathcal{L}(A) : M \models \varphi(\bar{c}) \text{ and } \varphi(\bar{x}) \text{ is quantifier-free} \},
\]

and write \( \bar{a} \equiv_{A}^q \bar{b} \) to mean that \( \bar{a} \) and \( \bar{b} \) have the same quantifier-free type over \( A \). Though \( \text{qftp}_M(\bar{c}/A) \) is an incomplete type, it is complete with respect to quantifier-free formulas in the sense that, for every quantifier-free \( \varphi(\bar{x}) \in \mathcal{L}(A) \), either \( \varphi(\bar{x}) \) or \( \neg \varphi(\bar{x}) \) is in \( \text{qftp}_M(\bar{c}/A) \). If \( T \) is a complete theory we will sometimes write \( \text{tp}_T(\bar{c}/A) \) or even \( \text{tp}(\bar{c}/A) \) if the context is clear (and similarly for \( \text{qftp} \)). For a tuple of variables of finite length \( \bar{x} \), \( S_{\bar{x}}(A) \) (or \( S^T_{\bar{x}}(A) \) if the ambient theory is unclear) will be the compact, hausdorff, totally disconnected space whose elements are complete types \( p(\bar{x}) \) over \( A \), with topology generated by clopen sets \([\varphi(\bar{x})] := \{ p \in S_{\bar{x}}(A) : \varphi(\bar{x}) \in p \}\), where \( \varphi(\bar{x}) \in \mathcal{L}(A) \). Informally, \( S(A) \) will be the set of all complete types over \( A \). By another abuse of notation, given a substructure \( A \subseteq M \), we write \( \text{tp}_M(A) \) (and \( \text{qftp}(A) \)) to mean the type (respectively, quantifier-free type) of \( A \) considered as an infinite tuple ordered with respect to some (any) ordering of \( A \). Some authors call \( \text{tp}_M(A) \) the “elementary diagram of \( A \)” (respectively, the “diagram of \( A \)”)

Given a
theory $T$ and a set $A \subseteq M \models T$, $T_A$ will be the $\mathcal{L}(A)$-theory defined by $T \cup \text{qftp}(A)$. Given a type $p(\vec{x})$ and a model $M \models T$, we will say that a tuple $\vec{c} \in M$ is a realization for $p(\vec{x})$ (and less specifically, $M$ realizes $p(\vec{x})$) if $M \models \varphi(\vec{c})$ for every $\varphi(\vec{x}) \in p(\vec{x})$. A type $p(\vec{x})$ over $A$ is called isolated (by a formula) if there is $\varphi(\vec{x}) \in \mathcal{L}(A)$ such that $T_A \vdash \forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$ for all $\psi(\vec{x}) \in p(\vec{x})$. Equivalently, $p(\vec{x})$ is isolated if it is an isolated point of the space $S_{\vec{x}}(A)$. Any type that is algebraic (in the sense that it only has finitely many realizations in any model) is isolated. It is an easy exercise to check that every isolated type in $S_{\vec{x}}(A)$ is realized in every model of $T_A$.

Fix a complete theory $T$. For an infinite cardinal $\kappa \geq \aleph_1$, we will say that a model $M \models T$ is $\kappa$-saturated if, for every $A \subseteq M$ such that $|A| < \kappa$ and every type $p \in S(A)$, $M$ realizes $p$. A model $M$ is saturated if it is saturated in its own cardinality. In general, one cannot expect saturated models to exist without making some set theoretic assumptions, but by compactness and taking unions of elementary chains, one can always find a model that is $\kappa$-saturated for arbitrarily large $\kappa$ (a bigly saturated model). Many proofs rely on having a model that is saturated to some unspecified degree and so we will often use the phrase “$M$ is sufficiently saturated” to be interpreted by the reader as “$M$ is saturated enough to make the following proof work”. In such a situation, a “small” subset of $M$ refers to a set $A \subseteq M$ such that the cardinality $|A|$ is strictly less than the degree of saturation of $M$.

For an $\mathcal{L}$-theory $T$ and and a set of formulas $\Delta \subseteq \mathcal{L}$, we will say that $T$ has relative quantifier-elimination (with respect to $\Delta$) if, for every formula $\varphi(\vec{x}) \in \mathcal{L}$, there is a formula $\psi(\vec{x}) \in \Delta$ such that $T \vdash \forall \vec{x}(\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}))$. If $\Delta$ is the set of quantifier-free formulas, we say that $T$ has quantifier-elimination. It is a basic exercise to show that a theory $T$ has quantifier-elimination if and only if it is “substructure complete”; that is, for any $M \models T$ and any substructure $A \subseteq M$, the $\mathcal{L}(A)$ $T_A = T \cup \text{qftp}(A)$ is complete. A weaker notion is model completeness. A theory $T$ is model complete if and only if, for any $M \models T$, $T \cup \text{qftp}(M)$ is complete. Given an $\mathcal{L}$-theory $T$, we
say that an \( \mathcal{L} \)-theory \( T' \) is the model companion of \( T \) if \( T' \) is model complete, and every model of \( T \) embeds into a model of \( T' \) and vice versa. If, in addition, for any model \( M \) of \( T \), we have that \( T' \cup \text{qftp}(M) \) is complete, then we say that \( T' \) is the model completion of \( T \). For example, ACF is the model completion of the theory of fields; specifying \textit{any} field \( F \), \( \text{ACF} \cup \text{qftp}(F) \) determines a completion of ACF since \( \text{qftp}(F) \) fixes a characteristic.

Given a class of \( \mathcal{L} \)-structures \( \Sigma \), we say that \( M \in \Sigma \) is existentially closed in \( \Sigma \) if, for any \( M \subseteq N \in \Sigma \) and any \( \mathcal{L} \)-formula \( \varphi(\bar{x}) \in \mathcal{L}(M) \), if \( N \models \exists \bar{x}\varphi(\bar{x}) \), then \( M \models \exists \bar{x}\varphi(\bar{x}) \) (in much model theoretic literature, “\( M \preceq_1 N \)” is used to denote that \( M \) is existentially closed in \( N \), though we will typically write \( M \preceq_\exists N \)). When \( \Sigma \) is the class of models of a theory \( T \), we say that the existentially closed structures of \( \Sigma \) are existentially closed models of \( T \). If a theory \( T \) is inductive (meaning that any union of a chain of models is again a model), then \( T \) has a model companion if and only if the existentially closed models of \( T \) form an elementary class (and the model companion is precisely their theory).

Though we will only make passing remarks about imaginaries and their elimination, we recall their basic theory here anyway. Let \( T \) be an \( \mathcal{L} \)-theory and \( \varphi(\bar{x}, \bar{y}) \) defined over \( \emptyset \). Then the formula \( E(\bar{y}, \bar{z}) \) defined by \( \forall \bar{x}(\varphi(\bar{x}, \bar{y}) \iff \varphi(\bar{x}, \bar{z})) \) defines an equivalence relation on \( M^n \) for some \( n \) for any model \( M \models T \) (\( M \models E(\bar{a}, \bar{b}) \) iff \( \varphi(M, \bar{a}) = \varphi(M, \bar{b}) \)). In fact, every \( \emptyset \)-definable equivalence relation arises in this way. From \( T \), we may construct a canonical, multi-sorted expansion \( T^{eq} \) as follows. For each \( n \geq 1 \) and each \( \emptyset \)-definable equivalence relation \( E \) on \( \mathcal{n-}\)many variables, we have a sort \( S_E \) and a function symbol \( f_E : S_{\preceq}^n \to S_E \) such that \( T^{eq} \vdash \forall \bar{x}, \bar{y}(E(\bar{x}, \bar{y}) \iff f_E(\bar{x}) = f_E(\bar{y})) \). Every model \( M \models T \) can be canonically expanded to a model \( M^{eq} \) of \( T^{eq} \) by identifying the universe \( M \) with \( S_{\preceq} \); for any model \( N \models T^{eq} \), the reduct to the “home sort” \( S_{\preceq}(N) \) is a model of \( T \) and so every model of \( T^{eq} \) is of the form \( M^{eq} \) for some model \( M \) of \( T \). The construction \( T \mapsto T^{eq} \)
essentially preserves all “nice” properties of a theory, so in many situations it is convenient to assume $T = T^{eq}$ (though we shall not). An element of a sort $S_E$ (with $E \neq \{=\}$) is called an imaginary.

Given a theory $T$, and a set $A \subseteq M$, $dcl^{eq}(A)$ and $acl^{eq}(A)$ are the definable and algebraic closures of $A$ considered as a subset of $M^{eq}$. Let $S$ be a subset of the collection of sorts of $T^{eq}$ and write $\bar{a} \in S$ to mean that $\bar{a}$ is an elements from some product of sorts in $S$. We will say that $T$ has:

1. elimination of imaginaries relative to $S$ if, for every imaginary $e$, there is a finite tuple $\bar{a} \in S$ such that $e \in dcl^{eq}(\bar{a})$ and $\bar{a} \in dcl^{eq}(e)$,
2. weak elimination of imaginaries relative to $S$ if, for every imaginary $e$, there is a finite tuple $\bar{a} \in S$ such that $e \in dcl^{eq}(\bar{a})$ and $\bar{a} \in acl^{eq}(e)$,
3. geometric elimination of imaginaries relative to $S$ if, for every imaginary $e$, there is a finite tuple $\bar{a} \in S$ such that $e \in acl^{eq}(\bar{a})$ and $\bar{a} \in acl^{eq}(e)$.

If, in the above definitions, $S = \{S_{\bar{a}}\}$, we just say that $T$ has elimination of imaginaries, weak elimination of imaginaries, or geometric elimination of imaginaries. If $T$ is a theory that eliminates imaginaries, we make no distinction between $dcl$ and $dcl^{eq}$, or between $acl$ and $acl^{eq}$.

1.1.2 Stability, Simplicity, and Related Concepts

In this subsection, we recall some basics of stability theory and related properties of first order theories. Most claims will be stated without proof and we refer the reader to the many excellent references on the material, in particular [56, 56, 8, 75].

We fix a language $\mathcal{L}$, a complete $\mathcal{L}$-theory $T$, and a sufficiently saturated model $M$.

**Definition 1.1.1.** Let $\varphi(\bar{x}, \bar{b})$ be a $\mathcal{L}$ formula, and let $A \subseteq M$ be some set. We say that $\varphi(\bar{x}, \bar{b})$ divides over $A$ if there is a sequence $\bar{b}_i \equiv_A \bar{b}$ and $k \geq 2$ such that $\{\varphi(\bar{x}, \bar{b}_i) : i < \omega\}$ is $k$-inconsistent. A (possibly incomplete) type $\Sigma(\bar{x})$ divides over $A$ if there is $\varphi(\bar{x}, \bar{b})$ that divides over $A$ and such that $\Sigma(\bar{x}) \vdash \varphi(\bar{x}, \bar{b})$. 
Dividing formulas are in some sense “small”, akin to null sets with respect to some measure, or sets of small dimension. Just as the finite union of null sets is again null, we want finite disjunctions of dividing formulas (and their subsets) to be small as well. The notion of forking provides what we need:

**Definition 1.1.2.** A partial type \( \Sigma(\bar{x}) \) forks over \( A \) if \( \Sigma(\bar{x}) \vdash \bigvee^n_i \varphi_i(\bar{x}, \bar{b}_i) \), where each \( \varphi_i(\bar{x}, \bar{b}_i) \) divides over \( A \).

It is clear that if \( p \) is a complete type, then \( p \) divides over \( A \) iff \( p \) forks over \( A \). If \( p \in S(A) \) is a complete type, \( A \subseteq B \) and \( q \in S(B) \) is an extension of \( p \), we say that \( q \) is a non-forking extension of \( p \) if \( q \) does not fork over \( A \).

**Definition 1.1.3.** A complete type \( p \in S(A) \) is called stationary if \( p \) has a unique non-forking extension to any set \( B \supseteq A \).

The notion of forking introduces a notion of independence to model theory: for any theory \( T \), \( M \models T \), and sets \( A, B, C \subseteq M \), we say \( A \) is forking independent from \( B \) over \( C \), written \( A \downarrow_C B \) if, for every finite tuple \( \bar{a} \in A \), the type \( tp(\bar{a}/BC) \) does not fork over \( C \). The class of simple theories are those for which forking independence is particularly nice.

**Definition 1.1.4.** We will say that a first order theory \( T \) is simple if for every \( B \subseteq M \models T \), and for every complete type \( p \in S(B) \), there is \( A \subseteq B \) with \( |A| < |T| \) such that \( p \) does not fork over \( A \).

This is not the typical way that one defines simple theories, but it is equivalent, see [75] Chapter 7. Forking independence in simple theories satisfies the following theorem, which is a statement about amalgamating types:

**Fact 1.1.5** (Independence Theorem over Models). Let \( T \) be simple, \( M \models T \), and suppose there are tuples \( \bar{c}_0, \bar{c}_1 \) and sets \( A_0, A_1 \) such that

- \( \bar{c}_0 \equiv_M \bar{c}_1 \),
• $A_0 \downarrow_M A_1$,
• $\bar{c}_i \downarrow_M A_i$, for $i = 0, 1$.

Then there is $\bar{d}$ (in some sufficiently saturated model extending $M$) such that $\bar{c}_0 \equiv_{A_0} \bar{d} \equiv_{A_1} \bar{a}_1$ and such that $\bar{d} \downarrow_M A_0 A_1$.

In fact, Kim and Pillay famously showed that the existence of a “well behaved” abstract independence relation on sets characterizes simple theories:

**Fact 1.1.6 ([39]).** Let $T$ be a theory, $N \models T$ a sufficiently saturated model, and suppose that $\downarrow^*$ is a ternary relation between subsets of $N$ satisfying the following properties ($\bar{a}, \bar{b}, \ldots$ will be finite unordered tuples):

1. (automorphism invariance) for all $\sigma \in \text{Aut}(M)$, $A \downarrow^*_B C$ implies $\sigma(A) \downarrow^*_\sigma(B)$,
2. (monotonicity and transitivity) $\bar{a} \downarrow^*_A BC$ iff $\bar{a} \downarrow^*_A B$ and $\bar{a} \downarrow^*_{AB} C$,
3. (symmetry) $\bar{a} \downarrow^*_A \bar{b}$ iff $\bar{b} \downarrow^*_A \bar{a}$,
4. (finite character) $\bar{a} \downarrow^*_A B$ iff $\bar{a} \downarrow^*_A \bar{b}$ for every finite $\bar{b} \subseteq B$,
5. (local character) there is $\kappa$ such that for all $\bar{a}$ and all sets $B$, there is $A \subseteq B$, with $|A| < \kappa$ such that $\bar{a} \downarrow^*_A B$,
6. (existence) for all $\bar{a}$ and sets $A, B$, there is $\bar{a}' \equiv_A \bar{a}$ such that $\bar{a}' \downarrow^*_A B$, and
7. (independence over models) for all (small) $M \preceq N$, all tuples $\bar{c}_0, \bar{c}_1$ and sets $A_0, A_1$ such that

   • $\bar{c}_0 \equiv_M \bar{c}_1$,
   • $A_0 \downarrow_M A_1$,
   • $\bar{c}_i \downarrow_M A_i$, for $i = 0, 1$,

   there is $\bar{d}$ such that $\bar{c}_0 \equiv_{A_0} \bar{d} \equiv_{A_1} \bar{a}_1$ and such that $\bar{d} \downarrow_M A_0 A_1$.

Then $T$ is simple and $\downarrow^*$ is precisely forking independence, $\downarrow$.

A subclass of the simple theories that tend to be particularly well behaved are the stable theories:
Definition 1.1.7. Let $\kappa \geq \aleph_1$ be a cardinal. A theory $T$ is said to be $\kappa$-stable if, whenever $A \subseteq M \models T$ is a set such that $|A| = \kappa$, then $|S_\bar{x}(A)| = \kappa$ (for any finite tuple of variable $\bar{x}$). We say that $T$ is stable if it is $\kappa$-stable for some $\kappa$.

Historically, $\aleph_1$-stable theories have been referred to as $\omega$-stable, and we continue this tradition in this dissertation.

Fact 1.1.8. Let $T$ be stable with $T = T^{eq}, M \models T$. For any $\bar{c} \in M$ and $A \subseteq M$, $\text{tp}(\bar{c} / \text{dcl}(A\bar{c}) \cap \text{acl}(A))$ is stationary.

Definition 1.1.9 ([50]). Let $M \models T$ and $A \subseteq M$. We say that $M$ is prime over $A$ if, for any $N \models T$, if $f : A \rightarrow N$ is a partial elementary map, then $f$ extends to an elementary embedding $f' : M \rightarrow N$.

Fact 1.1.10 (Theorem 4.2.20, [50]). Let $T$ be $\omega$-stable, with $A \subseteq M \models T$. Then there is $M_0 \preceq M$, prime over $A$, such that every element of $M_0$ realizes an isolated type over $A$.

Finally, we would like to at least define what it means for a theory $T$ to have the negation of the independence property (NIP), since we will refer to such theories occasionally throughout this dissertation:

Definition 1.1.11. Let $T$ be a theory. A formula $\varphi(\bar{x}, \bar{y})$ is said to have the independence property (IP) if there is $M \models T$ and sequences $(\bar{a}_i \in M : i < \omega)$, and $(\bar{b}_\eta \in M : \eta \in 2^\omega)$ such that $M \models \varphi(\bar{a}_i, \bar{b}_\eta)$ iff $\eta(i) = 0$. (Here, $2^\omega$ is the set of functions $\eta : \omega \rightarrow 2 = \{0, 1\}$).

A theory $T$ is said to have the negation of the independence property ($T$ is NIP) if there is no formula with the independence property. The class of NIP theories is “independent” from the simple theories in the sense that a theory $T$ is stable iff $T$ is simple and NIP. Note that NIP theories are sometimes referred to by various other names, such as “dependent theories”, or “theories with IP”, or NNIP, etc.
1.1.3 PAC Substructures of Stable Structures

The references for this section are mostly Polkowska’s thesis, [63], and the papers by Polkowska, [64], and Pillay and Polkowska, [58], which generalize the notion of PAC substructure of a strongly minimal structure as introduced by Hrushovski, [28]. We will not make any attempt to compare these notions to PAC structures as defined by Hoffmann, [27] [26].

Fix a stable $\mathcal{L}$-theory $T$ and let $P \subseteq M \models T$ be a substructure, with $\text{dcl}_M(P) = P$. We will assume that $T$ has quantifier-elimination and elimination of imaginaries. The “induced structure” on $P$ is $P$ together with subsets of $P^n$ of the form $\varphi(M^n) \cap P^n$, for $\varphi(\bar{x})$ a formula with parameters in $M$. Let $\mathcal{L}_P$ be the language $\mathcal{L}$ expanded by a predicate $P(x)$ for the substructure.

**Definition 1.1.12.** For a cardinal $\kappa > |T|$, we will say that $P$ is $\kappa$-PAC if, for every $A \subseteq P$ with $|A| < \kappa$, every complete, stationary type $p \in S^T(A)$ is realized in $P$. Say that $P$ is PAC if it is $|T|^+-\text{PAC}$.

**Definition 1.1.13.** Say that the PAC property is first-order of $T$ iff there is a set of $\mathcal{L}_P$ sentences $\Sigma$ such that:

1. when $P$ is a PAC substructure of $M \models T$, then $(M,P) \models \Sigma$, and
2. if $(M,P)$ is a $\kappa$-saturated model of $T \cup \Sigma$, where $\kappa > |T|$, then $P$ is a $\kappa$-PAC substructure of $M$.

**Definition 1.1.14.** A substructure $P \subseteq M$ will be called bounded iff there is some cardinal $\kappa$ such that, for any model $(M',P') \models (M,P)$, $|\text{Aut}(\text{acl}_{M'}(P')/P)| \leq \kappa$.

1.2 Field Theory and Algebraic Geometry

Much of this dissertation will be dedicated to the study of various properties of fields, both with and without extra structure, and so in this section outline some results and terminology concerning fields and algebro-geometric objects associated
to them. In the spirit of Weil's approach to algebraic geometry, we fix a "universal
domain" $\Omega$, which one may take to be a very saturated algebraically closed field
(eventually, we will stop talking about $\Omega$ and just assume that every field we consider
are subfields of it).

1.2.1 Fields

We assume that the reader is familiar with the basic terminology and algebra of
fields. Given a field $K$, an extension $L/K$ (all subfields of $\Omega$) will be called normal if
$L$ is the splitting field of some collection of irreducible polynomials over $K$ and will
be called separable if, for any $a \in L$, the minimal polynomial of $a$ over $K$ has distinct
roots in any extension of $L$. In characteristic 0, every field extension is separable. If
$L/K$ is normal and separable, we say that $L$ is a galois extension of $K$, and we write
$\text{Gal}(L/K)$ to be the group of automorphisms of $L$ that fix $K$ point-wise. If $L/K$ is
an algebraic extension, the the degree, $[L : K]$, of $L$ over $K$ will be the size of a basis
of $L$ considered as a $K$-vector space. For a field $K$ and two extensions $L \supseteq K \subseteq F$,
the compositum $LF := K(L \cup F)$ is defined to be the smallest subfield of $\Omega$ that
contains both $L$ and $F$.

Recall that an element $c \in \Omega$ is transcendental over a field $K$ if $P(c) \neq 0$ for
every polynomial $P(x) \in K[x]$. More generally, a finite set $\{c_1, \ldots, c_n\} \subseteq \Omega$ is
called algebraically independent over $K$ if $P(c_1, \ldots, c_n) \neq 0$ for any $P(x_1, \ldots, x_n) \in
K[x_1, \ldots, x_n]$ and an infinite set $C \subseteq \Omega$ is algebraically independent over $K$ if every
finite subset of $C$ is algebraically independent over $K$. A field $L/K$ is a transcendental
extension of $K$ if $L$ is an algebraic extension of a field of the form $K(C)$, for $C$ a set of
elements algebraically independent over $K$. For a transcendental extension $L/K$, the
transcendence degree of $L$ over $K$, $\text{tr. deg}(L/K)$ is the supremum of $|C|$, where $C \subseteq L$
is an algebraically independent set over $K$. For a tuple $\bar{c} \in \Omega$, we will sometimes
write $\text{tr. deg}(\bar{c}/K)$ as an abbreviation of $\text{tr. deg}(K(\bar{c})/K)$. It is straightforward to
check that for any tuple \( \bar{c} \) and any field \( K \), \( \text{tr}. \text{deg}(\bar{c}/K) = \text{tr}. \text{deg}(\bar{c}/K^{\text{alg}}) \).

Recall that in the theory of algebraically closed fields, ACF, algebraic independence gives an independence relation, \( \perp^{\text{ACF}} \), that characterizes forking. For a tuple \( \bar{c} \in \Omega \), and fields \( K \subseteq L \subseteq \Omega \) (not necessarily algebraically closed), \( \text{tp}_{\text{ACF}}(\bar{c}/K) \) forks over \( L \) iff \( \text{tr}. \text{deg}(\bar{c}/K) < \text{tr}. \text{deg}(\bar{c}/L) \). Given a field \( K \) and two field extensions \( F \supseteq K \subseteq L \subseteq \Omega \), \( F \perp^{\text{ACF}} L \) iff \( \bar{c} \perp^{\text{ACF}} L \) for every finite tuple \( \bar{c} \in F \).

Another useful notion of independence for fields is linear disjointness: given field extensions \( F \supseteq K \subseteq L \), we say that \( L \) is linearly disjoint from \( F \) over \( K \) iff every finite set \( \{c_1, \ldots, c_n\} \subseteq L \) that is linearly independent over \( K \) remains linearly independent over \( F \). Linear disjointness is a symmetric notion and is stronger than algebraic independence.

**Definition 1.2.1.** Let \( L/K \) be a extension of fields. We will say that \( L/K \) is regular iff \( L \) is linearly disjoint from \( K^{\text{alg}} \) over \( K \).

Given an extension of fields \( L/K \), we say that \( K \) is relatively algebraically closed in \( L \) iff for any \( a \in L \), if there is \( P(x) \in K[x] \) such that \( P(a) = 0 \), then \( a \in L \). If \( A \subseteq K \) is any subfield, then \( A^{\text{alg}} \cap K \) is the relative algebraic closure of \( A \) in \( K \).

**Fact 1.2.2.** An extension \( L/K \) is regular iff \( L/K \) is separable, and \( K \) is relatively algebraically closed in \( L \) (so in characteristic 0, \( L/K \) is regular iff \( K \) is relatively algebraically closed in \( L \)).

**Proposition 1.2.3.** Suppose \( K \) is any field, and \( E/K \) and \( F/K \) are regular extension. If \( E \) and \( F \) are algebraically independent over \( K \), then \( EF/K \) is a regular extension.

### 1.2.2 Valued Fields

Here, we make a brief foray into the theory of valued fields, since it will be convenient to have the terminology of valued fields on hand when we introduce large
Definition 1.2.4. An integral domain $\mathcal{O}$ is a valuation ring iff, for every $c \in \text{Frac}(\mathcal{O})$, either $c \in \text{Frac}(\mathcal{O})$ or $c^{-1} \in \text{Frac}(\mathcal{O})$.

Given a valuation ring $\mathcal{O}$, one may show that the ideals of $\mathcal{O}$ are linearly ordered by inclusion, and that there is a unique maximal ideal $m$. The quotient map $\pi: \mathcal{O} \to \mathcal{O}/m$ is a ring homomorphism called the residue map, and the $\mathcal{O}/m$ is the residue field.

Definition 1.2.5. A pair $(K, \mathcal{O})$ is called a valued field if $\mathcal{O}$ is a valuation ring and $K = \text{Frac}(\mathcal{O})$.

The reason for this terminology is that, given a valued field $(K, \mathcal{O})$, there is a surjective group homomorphism (a “valuation map”) $v: K^\times \to \Gamma$, where $\Gamma := K^\times / \mathcal{O}$ is always a totally ordered abelian group (we may extend $v$ to all of $K$ by setting $v(0) = \infty$ and so, abusing notation, we will usually just write $v: K \to \Gamma$) called the value group. The valuation map has the property that $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ and $m = \{x \in K : v(x) > 0\}$. Though $\mathcal{O}$ does not uniquely determine the map $v: K^\times \to \Gamma$, $\mathcal{O}$ determines an equivalence class of valuations on $K$, and so it enough to define a valued field as a field $K$ together with a surjective group homomorphism $v: K^\times \to \Gamma$. Given a valued field $(K, v)$ it is customary to write $vK$ for the value group and $Kv$ for the residue field. A valued field $(K, v)$ is called discrete if $vK \cong (\mathbb{Z}, +)$.

Definition 1.2.6. For fields $K$ and $k$, a place of $K$ is a homomorphism $\pi: \mathcal{O} \to k$, where $\mathcal{O}$ is a subring of $K$, such that:

1. if $c \in K \setminus \mathcal{O}$, then $c^{-1} \in \mathcal{O}$ and $\pi(c^{-1}) = 0$,
2. $\pi(c) \neq 0$ for some $c \in K$. 

fields. The main references for this material are [81, 77].
By construction, given a $K$-place $\pi$, the subring $O = \text{dom}(\pi)$ is always a valuation ring, with corresponding residue field $\pi(O)$ and so specifying a $K$-place also determines a valued field. Model theoretically, there are many languages that are convenient for the study of valued fields. For the most part, it suffices to assume that the language of valued fields is the language of rings expanded by a unary predicate for the valuation ring. In this situation, the value group, valuation, residue map, and residue field are all interpretable.

**Definition 1.2.7.** Let $K/k$ be a field extension and let $\varphi : K \to k^{\text{alg}} \cup \{\infty\}$ a place for $K$ with domain $O$. We say that:

1. $\varphi$ is a $k$-place if $\varphi |_k = \text{id}_k$,
2. $\varphi$ is $k$-rational if $\varphi(O) = k$.

We will say that a finite field extension $F/K$ is a $K$-rational extension if there is some $K$-rational $K$-place $\varphi : F \to K \cup \{\infty\}$.

For the most part, we will usually write $K$-rational place rather than $K$-rational $K$-place, as all of the places in question will be $K$-places.

**Fact 1.2.8.** Every $K$-rational extension $F/K$ is a regular extension of $K$.

**Proof.** See [20], Lemma 2.6.9. 

Many natural examples of valued fields are examples of Hahn fields. Given an ordered abelian group $\Gamma$, and a field $K$, the Hahn field $K((\Gamma))$ is defined to be the collection of formal sums of the form

$$
\sum c_\gamma t^\gamma,
$$

where $\{\gamma : c_\gamma \neq 0\}$ is a well-ordered subset of $\Gamma$, and $c_\gamma \in K$. The field $K((\Gamma))$ comes
equipped with a natural “$t$-adic” valuation, defined by

$$v \left( \sum_{\gamma \in \Delta} c_\gamma t^\gamma \right) := \min \{ \gamma : c_\gamma \neq 0 \}. $$

In the case where $\Gamma = \mathbb{Z}$, $K((\mathbb{Z})) = K((t))$ is the familiar field of formal power-series over $K$. Observe that we can also specify the valuation on $K((t))$ by defining a place; for example, the “evaluation at 0” map $\pi : K((t)) \to K \cup \{\infty\}$, which sends $t$ to 0, gives $\pi(\sum c_i t^i) = c_0$ so long as $v(\sum c_i t^i) \geq 0$.

Let $\Gamma$ be an ordered abelian group. A subgroup $\Delta \leq \Gamma$ is called convex if, $-\delta \leq \gamma \leq \delta$ implies that $\gamma \in \Delta$ for every $\gamma \in \Gamma$ and $\delta \in \Delta$. Given a convex subgroup $\Delta \leq \Gamma$, the quotient $\Gamma / \Delta$ can be equipped with a canonical ordering that makes the quotient map $\Gamma \to \Gamma / \Delta$ a homomorphism of ordered abelian groups. Given a valued field $v : K \to \Gamma$ and a convex subgroup $\Delta \subseteq \Gamma$ the coarsening of $v$ induced by $\Gamma$ is the valuation $\dot{v} : K \to \Gamma / \Delta$ defined by the composition of $v$ with the quotient map $\Gamma \to \Gamma / \Delta$.

**Fact 1.2.9.** Let $v : K \to \Gamma$ be a valued field, and let $F \subseteq K$ be a subfield such that $v |_F$ is a non-trivial valuation on $F$ with value group $vF$. Then for any convex subgroup $\Delta \subseteq \Gamma$, $\Delta \cap vF$ is a convex subgroup of $vF$ (which may be trivial or all of $vF$), and so the coarsening, $\dot{v}$, of $v$ on $K$ induced by $\Delta$ restricts to a coarsening of $v$ on $F$ with value group $\dot{v}F \cong vF/(\Delta \cap vF)$.

If $v : K \to \Gamma$ is a valuation with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$, and $\dot{v} : K \to \Gamma / \Delta$ is a coarsening of $v$, then $v$ induces a valuation on the residue field $K\dot{v}$ given by $\dot{v}(c + \dot{\mathfrak{m}}) := v(c) \in \Delta$, for $c \in \hat{\mathcal{O}}$ and $\hat{\mathfrak{m}}$ the maximal ideal of the valuation ring $\hat{\mathcal{O}}$ of $\dot{v}$. The residue field $(K\dot{v})v$ is isomorphic to $Kv$ and so if $\pi : \mathcal{O} \to K\dot{v}$ is the place associated to $v$, then a coarsening of $v$ give a decomposition $\pi = \pi_2 \circ \pi_1$, where $\pi_1 : \hat{\mathcal{O}} \to K\dot{v}$ is the place associated to $\dot{v}$, and $\pi_2 : \hat{\mathcal{O}} \to Kv$ is the place associated to the induced valuation $\dot{v} : K \to \Delta$. 
Finally, we would like to include a few words about henselianity and Ax-Kochen-Ershov (AKE) type theorems, at least in the residue characteristic 0 case. Henselianity was introduced as a lifting property first observed to be true for \( p \)-adic fields (or more generally, for complete, discrete valued fields) in the form of Hensel’s Lemma. In the modern study of valued fields, we say that a valued field \((K, \mathcal{O})\) is henselian if it satisfies the statement of a more general version of Hensel’s Lemma. Let \((K, v)\) be a valued field with residue map \( \pi : \mathcal{O} \to Kv \). For a polynomial \( P(x) \in \mathcal{O}[x] \), denote by \( \overline{P}(x) \in Kv[x] \) the polynomial obtained by applying \( \pi \) to the coefficients of \( P(x) \).

**Lemma 1.2.10** (Hensel’s Lemma). Let \((K, v)\) be a valued field with residue map \( \pi : \mathcal{O} \to Kv \). Then \((K, v)\) is henselian iff, for every polynomial \( P(x) \in \mathcal{O}[x] \), if there is \( c \in Kv \) such that \( \overline{P}(c) = 0 \) and \( \overline{P}'(c) \neq 0 \) (where \( \overline{P}'(x) \) is the formal derivative of \( \overline{P}(x) \)), then there is a unique \( a \in K \) such that \( P(a) = 0 \) and \( \pi(a) = c \).

In general, if \((K, v)\) is a valued field, \((K, v)\) can be embedded into a unique, minimal, henselian, local ring, called the henselization of \( K \) (usually denoted by \( K^h \)). The henselization of a henselian valued field is itself. Though we do not wish to go into detail about henselizations here, as it is quite complicated, we mention one example where the henselization can be easily described: for any field \( K \), if we equip \( K(t) \) with the \( t \)-adic valuation (induced by the place \( K(t) \to K \) sending \( t \) to 0), then the henselization of \( K(t) \) is precisely the relative algebraic closure of \( K(t) \) in the field of formal power series \( K((t)) \), so \( K(t)^h = K(t)^{alg} \cap K((t)) \). Hahn series fields in general provide a wealth of examples of henselian valued fields.

An underlying philosophy in the study of valued fields is that one should be able to reduce the study of a valued field \((K, v)\) to the study of its residue field \( Kv \) and its value group \( vK \). This is not always possible but, in certain nice situations, this principle holds in various strong ways. These results are typically referred to as Ax-Kochen-Ershov (AKE) theorems in reference to the first results of this type by Ax and Kochen, and (independently) Ershov. We mention one of these results here,
since we will need it for an example in the next chapter:

**Theorem 1.2.11 (AKE3,15).** Let \((K, v) \subseteq (L, v)\) be an extension of henselian valued fields of residue characteristic 0. If \(Kv \preceq_\exists Lv\) (as fields) and \(vK \preceq_\exists vL\), then \((K, v) \preceq_\exists (L, v)\) as valued fields. The statement of the theorem also holds when \(\preceq_\exists\) is replaced by \(\preceq\).

### 1.2.3 Varieties, Function Fields, and Generic Points

For any \(n \geq 1\), we write \(A^n := \Omega^n\) to be affine \(n\)-space. In this dissertation, we define an **affine variety** \(V\) (as a subset of affine \(n\)-space \(A^n\)) to be the zero-set of set of polynomials \(A \subseteq \Omega[x_1, \ldots, x_n]\) (we will write \(V = V(A)\) to mean that \(V\) is the affine variety cut out by the set of polynomials \(A\)). If \(K \subseteq \Omega\) is some field, we will say that \(V \subseteq A^n\) is defined over \(K\) (or that \(V\) is an affine \(K\)-variety) if the radical ideal \(I(V)\) is generated by polynomials over \(K\). This terminology implies that if \(V\) is an affine \(K\)-variety, and \(L \supseteq K\) is a field extension, then \(V\) is also an \(L\)-variety.

An affine \(K\)-variety \(V\) is said to be \(K\)-irreducible iff \(V\) cannot be written as the union \(V = V_1 \cup \ldots \cup V_m\) of proper affine subvarieties. Equivalently, if \(V = V(A)\) and \(I = \sqrt{I(A)} \subseteq \Omega[\bar{x}]\) is the radical of the ideal generated by \(A\), then the ideal \(I_K := I \cap K[\bar{x}]\) is prime. An affine \(K\)-variety \(V\) is absolutely irreducible if \(I_{K^{alg}} \subseteq K^{alg}[\bar{x}]\) is prime. We will use the following important fact regularly, though implicitly, throughout this dissertation:

**Fact 1.2.12 (32).** If \(K \models ACF\), then for any definable family \(\{X_\bar{a} : \bar{a} \in K\}\) of affine varieties, the set of \(\bar{a}\) such that \(X_\bar{a}\) is \(K\)-irreducible (i.e. absolutely irreducible) is definable.

A consequence of this fact (and quantifier-elimination in ACF) is that for any field \(K\), one can express in a first order way that a \(K\)-variety is absolutely irreducible. This will be important for axiomatizing many classes of structures we will be discussing.
If $V$ is reducible over $K$, the varieties $V_i$ are called the components of $V$. Since it is rather tedious to always write “$K$-irreducible $K$-variety,” we will assume that an irreducible $K$-variety is a $K$-variety which is irreducible over $K$. We note that our terminology differs from that of most authors, who call solution-sets of systems of polynomials “algebraic sets” and reserve the term “affine $K$-variety” for $K$-irreducible algebraic sets. Given a $K$-variety $V$, we will write $V(K) := V \cap K^n$ to be the “$K$-points” of $V$.

Affine varieties come equipped with a natural topology called the Zariski topology. For any $n \geq 1$, the Zariski topology on $A^n$ is generated by the closed sets $\mathbb{V}(I)$ for $I \subseteq \Omega[x_1, \ldots, x_n]$, and so affine varieties themselves are Zariski-closed. For an affine variety $V \subseteq A^n$, the Zariski topology on $V$ is the subspace topology induced by the Zariski topology on $A^n$. The basic open sets of the Zariski topology are complements of affine varieties. If $V$ is an affine variety, then a basic open subset $U \subseteq V$ is called quasi-affine. Quasi-affine varieties are also equipped with the Zariski topology induced by the Zariski topology on affine space and all topological notions, such as irreducibility, equally make sense. A quasi-affine variety $W \subseteq V$, where $V$ is an affine $K$-variety and $W$ is a Zariski open subset of $V$, is defined over $K$ if the complement of $W$ in $V$ is an affine $K$-variety.

Let $V \subseteq A^n$ be an affine, irreducible $K$-variety and suppose that $\mathbb{I}(V)$ is generated by a basis $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Naïvely, we define the Zariski tangent bundle of $V$ to be the affine $K$-variety $T(V) \subseteq A^{2n}$ cut out by the equations

$$
\begin{cases}
  f_j(x_1, \ldots, x_n) = 0, & \text{for } 1 \leq j \leq m \\
  \sum_{i=1}^{n} \frac{df_j(x_1, \ldots, x_n)}{dx_i} y_i = 0, & \text{for } 1 \leq j \leq m.
\end{cases}
$$

For a point $\bar{a} \in V$, the Zariski tangent space of $V$ at $\bar{a}$, $T_{\bar{a}}(V)$, is the linear affine
space defined by the equations

\[ \sum_{i=1}^{n} \frac{df_j(\bar{a})}{dx_i} y_i = 0, \text{ for } 1 \leq j \leq m. \]

Equivalently, if \( \bar{a} \in V \), then \( T_{\bar{a}}(V) \) can be defined as \( \ker(J_V(\bar{a})) \), were \( J_V(\bar{a}) \) is the jacobian matrix of \( V \) at \( \bar{a} \),

\[
J_V(\bar{a}) := \begin{pmatrix}
\frac{df_1(\bar{a})}{dx_1} & \cdots & \frac{df_1(\bar{a})}{dx_n} \\
\vdots & \ddots & \vdots \\
\frac{df_m(\bar{a})}{dx_1} & \cdots & \frac{df_m(\bar{a})}{dx_n}
\end{pmatrix},
\]

considered as a linear map \( J_V(\bar{a}) : K^n \to K^m \). By general linear algebra,

\[
\dim(T_{\bar{a}}(V)) = n - \text{Rank}(J_V(\bar{a})),
\]

where \( \text{Rank}(J_V(\bar{a})) \) is the row/column rank or, equivalently, the dimension of the image. It is a fact from algebraic geometry (or even just commutative algebra in terms of Krull dimension of noetherian local rings) that \( \dim(V) \leq \dim(T_{\bar{a}}(V)) \) for any point \( \bar{a} \in V \) and \( \dim(V) = \dim(T_{\bar{a}}(V)) \) just when \( \text{Rank}(J_V(\bar{a})) \) is maximized. As in undergraduate calculus, the tangent space \( T_{\bar{a}}(V) \) is the “best linear approximation” of \( V \) at the point \( \bar{a} \), and so, roughly, we expect that the tangent space has dimension equal to the dimension of the variety at a given point. In general this will be true, but there will be some points where this fails. We say that \( \bar{a} \) is smooth (sometimes regular, or non-singular) if \( \dim(V) = \dim(T_{\bar{a}}(V)) \). The points of \( V \) that are not smooth are called singular, and they can be detected by the jacobian matrix. Recall that for any \( n \times m \) matrix \( M \), \( \text{Rank}(M) \) is always equal to the largest \( k \) such that there is a \( k \times k \) submatrix \( B \) with \( \det(B) \neq 0 \). Hence for any point \( \bar{a} \in V \), \( \bar{a} \) is smooth iff \( \text{Rank}(J_V(\bar{a})) = n - \dim(V) \). Equivalently, \( \bar{a} \in V \) is singular iff every
\((n-\dim(V)) \times (n-\dim(V))\) submatrix of \(J_V(\bar{a})\) has zero determinant. This statement does not actually depend on the point \(\bar{a}\) and so the set of singular points, \(V^{\text{sing}}\), is a (proper) affine subvariety of \(V\) (which implies that the set of non-singular points is dense in \(V\)). In particular, if \(C\) is an affine curve, then \(C^{\text{sing}}\) is finite (and so \(C(K)\) is dense in \(C\) iff \(C(K)\) is infinite). A smooth variety is one for which every point is smooth. If \(V\) is a quasi-affine variety, \(\bar{a}\) is smooth for \(V\) iff \(\bar{a}\) is smooth for the Zariski closure of \(V\).

**Example 1.2.13.** Consider the affine curve (defined over \(\mathbb{Q}\)) given by

\[
C := \{(x, y) \in A^2 : y^2 - x^2(x + 1) = 0\}.
\]

Letting \(f(x, y) := y^2 - x^2(x + 1)\), we can compute the general jacobian matrix \(J_C(x, y)\) (without specifying any point) as

\[
J_C(x, y) = \begin{pmatrix}
-x(3x + 2) & 2y
\end{pmatrix}.
\]

The matrix \(J_C(x, y)\) has rank at most 1 for any choice of \((x, y)\) and so a point \((a, b) \in C\) is singular iff \(J_C(a, b)\) has rank 0, i.e.

\[
\begin{aligned}
b^2 - a^2(a + 1) &= 0 \\
-a(3a + 2) &= 0 \\
2b &= 0.
\end{aligned}
\]

The last equation implies that \(b = 0\). The second equation implies that \(a = 0\) or \(a = -2/3\), but the point \((-2/3, 0)\) is not on the curve \(C\) and so \(C\) has one singular point, \((0, 0)\). This singularity is clearly visible in a plot of the real points of \(C\):
The tangent space $T_{(0,0)}(C)$ is isomorphic to all of $\mathbb{A}^2$, and is isomorphic to $\mathbb{A}^1$ at any other point.

Though most sections of this dissertation will not require us to work with the abstract notion of variety, we will define them here for completeness, and so that we may refer to algebraic groups that are not necessarily affine. The definition of abstract variety that we discuss here was introduced by Weil in his book *Foundations of Algebraic Geometry* [80], which defines them as a sort of algebraic manifold, and we avoid any reference to the more popular scheme-theoretic approach to algebraic geometry. Before we continue to the definition of abstract varieties, we recall the definition of morphism of quasi-affine varieties, since it will be necessary to refer to them when we talk about “glueing” affine spaces and elsewhere in the dissertation (all of these definitions can be found in [24]).

Let $V \subseteq \mathbb{A}^n$ be a quasi-affine $K$-variety and let $\mathbb{I}(V) \subseteq K[\bar{x}]$ be the radical ideal of polynomials over $K$ that vanish on $V$. The ring $K[V] := K[\bar{x}]/\mathbb{I}(V)$ is called the coordinate ring of $V$. A polynomial $P(\bar{x}) \in K[\bar{x}]$ may be viewed as a function $P : \mathbb{A}^n \to \mathbb{A}^1$, and so we may interpret the coordinate ring $K[V]$ as the set of equivalence classes of such polynomial functions, where $P_1$ is equivalent to $P_2$ iff $P_1 \mid_V = P_2 \mid_V$. Observe that $K[V]$ is an integral domain iff $V$ is $K$-irreducible iff $\mathbb{I}(V)$ is prime (and $V$ is absolutely irreducible iff $K[V] \otimes_K K^{alg}$ is an integral domain).
Suppose that $V$ is a $K$-irreducible, quasi-projective $K$-variety. Then the fraction field $K(V) := \text{Frac}(K[V])$ is called the field of rational functions on $V$, or the function field of $V$. The field $K(V)$ may also be defined as the set of equivalence classes of “$K$-regular” functions on $V$: a function $f : V \to \mathbb{A}^1$ is $K$-regular at a point $c \in V$ if there is an open set $U \subseteq V$ containing $c$, and $P_1, P_2 \in K[x]$ such that $P_2$ is non-zero on $U$, and such that $f \upharpoonright U = P_1/P_2$ (and $f$ is just $K$-regular if it is $K$-regular at all points in $V$).

**Definition 1.2.14** ([24], 1.3). Let $V$ and $W$ be two $K$-irreducible, quasi-affine $K$-varieties. A morphism (defined over $K$) $\Phi : V \to W$ is a continuous map such that, for every open $U \subseteq W$ and every $K$-regular function $f : U \to \mathbb{A}^1$, the function $f \circ \Phi : \Phi^{-1}(U) \to \mathbb{A}^1$ is $K$-regular.

For certain algebro-geometric reasons (see [24], 1.4), morphisms of quasi-affine varieties as defined above are somewhat restrictive, and so one introduces rational maps to allow for a little bit more flexibility.

**Definition 1.2.15** ([24]). Let $V$ and $W$ be $K$-irreducible, quasi-affine $K$-varieties. A $K$-rational map $f : V \to U$ is an equivalence class $\langle U, f_U \rangle$, where $U \subseteq V$ is a non-empty open subset defined over $K$ and $f_U : U \to W$ is a $K$-regular map. Two pairs $\langle U_1, f_{U_1} \rangle$ and $\langle U_2, f_{U_2} \rangle$ are equivalent iff $f_{U_1} \upharpoonright_{U_1 \cap U_2} = f_{U_2} \upharpoonright_{U_1 \cap U_2}$.

More succinctly, a $K$-rational map $\Phi : V \to W$ is a morphism from some open subset of $V$ to $W$, and so the class of rational maps properly contains the class of morphisms. A rational map $\Phi : V \to W$ is called dominant if the image $\Phi(V)$ is Zariski dense in $W$. Dominant rational maps are composable, and so we may consider the category of quasi-affine varieties with dominant rational maps. Two varieties are called birationally equivalent if they are isomorphic in this category. The category of absolutely irreducible $K$-varieties with dominant rational maps is dual to the category of finite regular extensions of the field $K$; a $K$-embedding of
function fields $K(V) \hookrightarrow K(W)$ induces a dominant rational map $W \rightarrow V$. As a matter of terminology, we will often say that a projection $\pi : W \rightarrow V$ is generic (or just that $W$ projects generically onty $V$) if the image $\pi(W) \subseteq V$ contains a Zariski open subset of $V$ (this is a particular case of a dominant rational map).

We now define what it means to be an abstract $K$-variety (see the section by Pillay in [6], or Definition 7.4.1 in [50]):

**Definition 1.2.16.** An abstract $K$-variety is a topological space $V$ with an open covering $V = V_1 \cup \ldots \cup V_m$ such that for each $1 \leq i \leq m$, there is a quasi-affine $K$-variety $U_i$ and a homeomorphism $f_i : V_i \rightarrow U_i$ such that $U_{i,j} := f_i(U_i \cap U_j)$ is quasi-affine, and for each $1 \leq i \leq j \leq m$, $f_{i,j} := f_j \circ f_i^{-1} : U_{i,j} \rightarrow U_{j,i}$ is an isomorphism of quasi-affine varieties defined over $K$.

Abstract varieties are endowed with the Zariski topology induced by declaring the “charts”, $f_i$, to be homeomorphisms. The notions of morphism, rational map, and function field all extend naturally to the setting of abstract varieties (see [80]) and an abstract variety is irreducible if it is irreducible as a topological space. A point of an abstract variety is smooth if it is smooth for some affine neighbourhood. The fact that varieties in general are locally quasi-affine means that, for nearly every application in this dissertation, one may take “$K$-variety” to mean “quasi-affine $K$-variety”.

Given a variety $V$ and a field $K$, we may talk about the $K$-Zariski topology on $V$, which is the coarsening of the full Zariski topology generated by closed sets defined over $K$. For a quasi-affine $K$-variety $V$, we say that a point $\bar{c} \in V$ is generic for $V$ over $K$ if $\bar{c} \in U$ for every basic Zariski-open subset $U$ of $V$ defined over $K$. It is an easy exercise to show that if $V$ is a $K$-irreducible, quasi-affine $K$-variety, and $\bar{c}$ is generic for $V$ over $K$, then $K(V) \cong K(\bar{c})$ as $K$-algebras. If $V$ is affine, then this is the same as saying that $V$ is the $K$-Zariski closure of $\{\bar{c}\}$ in the $K$-Zariski topology. Note that if $\bar{c}$ is generic for $V$, then it is necessarily the case that $\bar{c} \notin V(K)$, since otherwise the $K$-Zariski closure of $\{\bar{c}\}$ would be $\{\bar{c}\}$. With this idea in hand, we may introduce
some model theory. As stated earlier, \( \text{Th}(\Omega) \) is a completion of ACF (depending only on the characteristic of \( \Omega \)). Thus, \( \text{Th}(\Omega) = \text{ACF}_p \) \((p = 0 \text{ or prime})\) is \( \omega \)-stable, with elimination of both quantifiers and imaginaries. By quantifier-elimination, every ACF\(_p\)-type is, modulo ACF\(_p\), implied by its quantifier-free part.

Let \( K \) be a field. To each \( K \)-irreducible quasi-affine \( K \)-variety \( V \), we may associate a complete type \( p_V(\bar{x}) \) \( \in S^\text{ACF}_{\bar{x}}(K) \), which we will call the generic type of \( V \) over \( K \). The type \( p_V(\bar{x}) \) is axiomatized by the formulas of the form “\( \bar{x} \in V \land \bar{x} \notin W \)”, where \( W \subset V \) is a proper \( K \)-subvariety. Note that if \( V \) is an affine or quasi-affine variety, and \( U \subset V \) is a \( K \)-Zariski open subvariety, then \( p_V = p_U \). The concept of a generic type of a variety gives us a somewhat non-standard way of defining the dimension of a variety:

**Definition 1.2.17.** Let \( K \) be a field and let \( V \) be an irreducible (quasi-affine) \( K \)-variety. Then the dimension of \( V \) is defined to be

\[
\dim(V) = \max \{ \text{tr. deg}(K(\bar{c})/K) : \bar{c} \in \Omega \text{ realizes } p_V(\bar{x}) \}
\]

If \( V \) is an abstract variety, then the dimension is defined to be the maximum of \( \dim(U) \) for some open quasi-affine \( U \subset V \). A \( K \)-variety \( C \) of dimension 1 will be called a \( K \)-curve. It is a fact of algebraic geometry that if \( V \) is an \( K \)-irreducible quasi-affine \( K \)-variety which fails to be reducible over \( K^{\text{alg}} \), then all the components of \( V \) as a \( K^{\text{alg}} \)-variety have the same dimension, and the absolute galois group \( \text{Gal}(K^{\text{alg}}/K) \) permutes them transitively.

**Remark 1.2.18.** Let \( V \) be a quasi-affine \( K \)-variety and \( \bar{c} \in V \) generic for \( V \) over \( K \). If \( L \supseteq K \) is a field extension, then it is not necessarily the case that \( \bar{c} \) is generic for \( V \) with respect to the \( L \)-Zariski topology, for example, if \( L = K(\bar{c}) \). For any \( L \supseteq K \), \( \bar{c} \in V \) is generic for \( V \) over \( L \) iff \( \dim(V) = \text{tr. deg}(\bar{c}/L) \).

**Fact 1.2.19.** Let \( V_1, V_2 \subseteq \mathbb{A}^n \) be \( K \)-irreducible affine \( K \)-varieties with \( \dim(V_1) = \)
dim(V_2). Then either \( V_1 = (V_1 \cap V_2) = V_2 \) or \( \dim(V_1 \cap V_2) \leq \dim(V_1) \).

**Remark 1.2.20.** To prove this fact, recall that the dimension of an irreducible affine variety \( V \) may equivalently be defined as the maximum length of a chain \( V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V \), where each \( V_i \) is a proper, irreducible affine subvariety.

**Proposition 1.2.21.** Let \( K \) be a field and \( V \) an irreducible quasi-affine \( K \)-variety. Then the generic type \( p_V(\bar{x}) \) is stationary iff \( V \) is absolutely irreducible.

**Proof.** If \( V \) is not absolutely irreducible, then \( V = V_1 \cup \ldots \cup V_n \) where each \( V_i \) is defined over \( K^{alg} \) and \( p_{V_i} \) is a non-forking extension of \( p_V \) or each \( i = 1, \ldots n \). On the other-hand, suppose that \( p_1 \) and \( p_2 \) are two distinct non-forking extension of \( p_V \) to \( K^{alg} \). Let \( \bar{c}_i \) be a realization of \( p_i \), and let \( X_i \) be the \( K^{alg} \)-Zariski closure of \( \{ \bar{c}_i \} \), for \( i = 1, 2 \). By Hilbert’s basis theorem, each \( X_i \) is definable by a formula \( \varphi_i(\bar{x}) \) with parameters in \( K^{alg} \). By assumption, \( p_V \cup \{ \varphi_i(\bar{x}) \} \) does not fork over \( K \) for \( i = 1, 2 \) so \( \varphi_i(\bar{x}) \) defines a closed subvariety of \( V \) of dimension \( \dim(V) \). However \( X_1 \) is absolutely irreducible by construction, so \( X_1 \cap X_2 \) is either all of \( X_1 \) or a proper subvariety of dimension less than \( \dim(V) \). Since \( p_2 \) is a non-forking extension of \( p_V \), and since \( \bar{c}_2 \) realizes \( p_2 \), \( \bar{c}_2 \not\in X_1 \), and hence \( X_1 \neq X_2 \). This gives \( V = X_1 \cup X_2 \), a union of proper Zariski closed subvarieties, and so \( V \) is not absolutely irreducible. \( \square \)

**Proposition 1.2.22.** Let \( K \) be any field and let \( \bar{c} \) be such that \( K(\bar{c})/K \) is a regular extension. Then \( K(\bar{c}) \cong K(V) \) for some absolutely irreducible affine \( K \)-variety \( V \).

**Proof.** For any \( \bar{c} \in M \) and any set \( A \subseteq M \), \( \text{tp}(\bar{c}/\text{acl}(A)) \cap \text{dcl}(A) \) is stationary. Let \( T = \text{ACF} \), let \( K \) be any field, and let \( M \models T \). If \( \bar{c} \in M \) is such that \( K(\bar{c})/K \) is a regular extension, then \( K(\bar{c}) \cap K^{alg} = K \), and so \( \text{tp}_{\text{ACF}}(\bar{c}/K) \) is stationary. In other words, if \( V \) is the \( K \)-Zariski closure of \( \{ \bar{c} \} \), then \( V \) is an absolutely irreducible affine \( K \)-variety. \( \square \)

**Proposition 1.2.23.** Let \( K \) be any field and let \( V \) be a \( K \)-irreducible \( K \)-variety. If \( V(K) \) contains a smooth point, then \( V \) is absolutely irreducible.
Proof. Suppose that \( \bar{c} \in V(K) \) is smooth. For a contradiction, suppose that \( V = W_1 \cup \ldots \cup W_n \), where each \( W_i \) is defined over \( K^{alg} \). Since \( V \) is \( K \)-irreducible, the \( W_i \)'s are all of the same dimension and are permuted by elements of \( \text{Aut}(K^{alg}/K) \). However, since \( \bar{c} \) is fixed by \( \text{Aut}(K^{alg}/K) \), it must be the case that \( \bar{c} \in \cap_i W_i \), contradicting the fact that \( \bar{c} \) is smooth.

Proposition 1.2.24. Let \( K \) be any field and let \( F/K \) be a regular extension. Then \( \text{tp}_{\text{ACF}}(F/K) \) is stationary.

Proof. It suffices to consider the case when \( F = K(\bar{c}) \) is finitely generated and to show that \( \text{tp}_{\text{ACF}}(\bar{c}/K) \) is stationary. Then \( \bar{c} \) is generic for some absolutely irreducible \( K \)-variety \( V \) by Proposition 1.2.22 and hence \( \text{tp}_{\text{ACF}}(\bar{c}/K) \) is stationary.

Finally, we mention two useful facts that often allow one to reduce statements about varieties in general to statements about plane curves.

Fact 1.2.25 (Lemma 5.1.2, [31]). Let \( K \) be an infinite field and \( C \) an absolutely irreducible \( K \)-curve with a smooth \( K \)-point \( \bar{c} \in C(K) \). Then there is an absolutely irreducible plane curve \( C' = \{(x,y) : f(x,y) = 0\} \) defined over \( K \) and a birational correspondence \( \Phi : C \to C' \) such that \( \Phi(\bar{c}) = (0,0) \) and \( \partial f/\partial y(0,0) \neq 0 \) (i.e. \( (0,0) \) is smooth for \( C' \)).

Fact 1.2.26 (Lemma 5.1.3, [31]). Let \( K \) be an infinite field and \( V \) an absolutely irreducible \( K \)-variety with \( \dim(V) > 0 \). For any finite set of points \( P \subseteq V(K^{alg}) \) there is an absolutely irreducible \( K \)-curve \( C \subseteq V \) such that \( P \subseteq C(K^{alg}) \), and such that if \( \bar{c} \in P \) is a smooth point of \( V \), then \( \bar{c} \) is smooth for \( C \) as well.

1.2.4 Large Fields

The following definition can be taken from Pop’s paper [65] where it was originally defined, though Pop’s whimsically titled *Little Survey on Large Fields* [66] gives a
A thorough and pleasant exposition of large fields in general. For a more extensive study of large fields, including the construction of fields that are \textit{not} large, see Jarden’s book \cite{31} (note that Jarden and some others use the term “ample” in place of “large”).

\textbf{Definition 1.2.27.} Let $K$ be a field. We say that $K$ is large if, for every absolutely irreducible $K$-variety $V$, if $V(K)$ contains a smooth point, then $V(K)$ is dense in $V$.

\textbf{Fact 1.2.28.} Let $K$ be a field. The following are equivalent:

1. $K$ is large,
2. for every absolutely irreducible $K$-curve $C$, if $C(K)$ contains a smooth point, then $C(K)$ is infinite,
3. $K$ is existentially closed in $K((t))$,
4. $K$ is existentially closed in every rational extension $F/K$ of transcendence degree 1.

The notion of largeness was introduced by Pop as a class of fields over which one can prove strong inverse galois results. Among other things, Pop shows that all henselian valued fields are large, and so large fields form a huge subset of all commonly studied “nice” fields. Of course, many fields commonly studied are not large; as a spectacular example of using a sledgehammer to crack a nut, observe that Faltings’ theorem implies that number fields are not large. With some work, one may also show that function fields are not large, see \cite{31}. Of course, there are many large fields that are also “not nice”: in \cite{2}, Ax shows that for any field $F$ (in the language of rings), $F((t))$ interprets (again, in the language of rings without an extra symbol for “$t$”) the constant field $F$, and so $F((t))$ is a large field that is as model-theoretically poorly behaved as $F$ is (so for example, $\mathbb{Q}((t))$ interprets Peano arithmetic). Despite this, many of model theorists’ favourite fields are large. Besides the obvious examples of algebraically closed, separably closed, and real closed fields, we also have that $p$-adically closed fields are large, and, more generally, Pop showed in \cite{67} that every henselian valued field is large. Fields with \textit{pro}-$p$ galois group are
large as well, see [31]. Another class of large fields that model theorists love are the pseudo-algebraically closed fields:

**Definition 1.2.29.** Let $K$ be a field. We say that $K$ is pseudo-algebraically closed (PAC) if, for every absolutely irreducible $K$-variety $V$, $V(K)$ is non-empty.

It is clear that PAC fields are large. Unsurprisingly, one has an analogue of Fact 1.2.28 for PAC fields:

**Fact 1.2.30.** Let $K$ be a field. The following are equivalent:

1. $K$ is PAC,
2. for every absolutely irreducible $K$-curve $C$, $C(K)$ is infinite,
3. $K$ is existentially closed in every regular extension $F/K$.

A non-trivial example of a PAC field is a pseudo-finite field (an infinite model of the theory of finite fields). Sufficiently saturated PAC fields are precisely PAC substructures of algebraically closed fields in the sense of subsection 1.1.3.

**Remark 1.2.31.** A field $K$ is said to be virtually large if there is a finite extension $L/K$ that is large. Similarly, one may say that a field $K$ is said to be virtually PAC if there is a finite extension of $K$ which is PAC. The existence of virtually PAC fields is well know: if $K$ is a PRC field, then $K(\sqrt{-1})$ is PAC. It is natural to ask if there are virtually PAC fields that are not PRC. In [3], the authors ask whether every virtually large field is large. This question was answered negatively in [73] where the author provides an extreme counterexample; their construction gives a field $K$ that has a PAC quadratic extension and such that $C(K)$ is finite for every absolutely irreducible $K$-curve $C$ with genus at least 2. The author constructs this field inductively, starting with $\mathbb{Q}$ however the construction works identically if one starts the construction with any field $F$ for which Faltings’ Theorem applies and so one can construct a similar counter example which is not PRC (for example, start with $\mathbb{Q}(i)$). Note that from
and [10], we get that every unbounded, virtually PAC field has TP$_2$. Question: is there an example of a bounded, virtually PAC field which is neither PAC nor PRC?

Question 1.2.32. Let $T$ be a theory of fields in the ring language, possible with extra constants. Suppose $\varphi(\bar{x}, \bar{y})$ is a formula in the language of rings, and suppose that for any $K \models T$ and any $\bar{a} \in K$, $\varphi(K, \bar{a})$ is a quasi-affine algebraic set. The set of $\bar{a}$ such that $\varphi(K, \bar{a})$ is an absolutely irreducible set is definable, see Johnson’s thesis [32], Section 10.2. Say that $T$ has the definable density property (DDP) if for every $K \models T$ and every definable family $\{\varphi(K, \bar{a}) : \bar{a} \in K\}$, the set of $\bar{a}$ such that $\varphi(K, \bar{a})$ is $V(K)$ for some absolutely irreducible $V$ and $V(K)$ is Zariski-dense in $V$, is definable.

Large fields have the DDP, since for any absolutely irreducible variety $V$ over a large field $K$, $V(K)$ is dense iff $V(K) \setminus V^{\text{sing}} \neq \emptyset$. Does $T$ have the DDP iff every model of $T$ is large?

Question 1.2.33. Let us say that a field $K$ is “$n$-large” for some $n \geq 1$ if, for every absolutely irreducible affine plane curve $C$, if $|C(K) \setminus C^{\text{sing}}| \geq n$, then $C(K)$ is infinite. Large fields are clearly 1-large. The class of $n$-large fields is elementary. Is there a field that is $n$-large, but not $n - 1$-large for some $n > 1$?

1.2.5 Algebraic Groups

In this section, we briefly outline some terminology and facts from the theory of algebraic groups, which will come up in the context of strongly normal extensions in Chapter 4. The information in this section is sourced from Borel, [5].

Definition 1.2.34. An algebraic group is a tuple $(G, e, \mu, \iota)$ where:

1. $G$ is an abstract variety,
2. $e$ is a distinguished element of $G$, and
3. $\mu : G \times G \to G$ and $\iota : G \to G$ are morphisms of abstract varieties.
such that $G$ is a group with group operation $gh := \mu(g,h)$, identity $e$, and inversion map $g^{-1} := \iota(g)$.

An algebraic group $G$ is said to be defined over a field $K$ if $G$ is a $K$-variety and $\mu$ and $\iota$ are defined over $K$. Typically, we will write $G$ to be an algebraic group without specifying the rest of the data. If $G$ is an algebraic group, then $G^0$ is the component (in the sense of varieties) of the identity $e$, i.e. $G$ is irreducible iff $G = G^0$ (and we say in this case that $G$ is connected).

**Fact 1.2.35** ([5], 1.2). Let $G$ be an algebraic group.

1. $G$ is smooth.
2. $G^0$ is a normal subgroup of finite index.
3. If $G$ is defined over $K$, then so is $G^0$.

**Example 1.2.36.** The basic example of an algebraic group is the group of invertible $n \times n$ matrices over a field $K$, with group operation given by matrix multiplication, $\text{GL}_n(K)$ (more precisely, $\text{GL}_n(K)$ is the group of $K$-points of the algebraic group $\text{GL}_n$, which is a quasi-affine subvariety of $\mathbb{A}^{n^2}$). A Zariski closed algebraic subgroup of $\text{GL}_n$ is called a linear algebraic group.

**Fact 1.2.37** ([5], Proposition 1.10). Every affine algebraic $K$-group is isomorphic over $K$ to a closed $K$-subgroup of $\text{GL}_n$ for some $n$.

**Example 1.2.38.** The affine line $\mathbb{A}^1$ equipped with addition induced by the field structure is an affine algebraic group, sometimes denoted $\mathbb{G}_a$. For any $a, b \in \mathbb{G}_a$, we have

\[
\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}
\]

and so the map $b \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ gives the embedding of $\mathbb{G}_a$ into $\text{GL}_2$. 

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1.2.6 Bounded Fields

Let $G$ be a profinite group. We will say that $G$ is small if, for each $n$, $G$ has only finitely many open normal subgroups of index $n$. This property was introduced by Serre in [71], where he called such groups “type (F)”. Serre also calls fields $K$ type (F) if $\text{Gal}(K) := \text{Gal}(K^\text{alg}/K)$ is a profinite group of type (F). However, in keeping with what seems to be the trend in model theory, we will say that a field $K$ is bounded if it has small absolute galois group.

**Fact 1.2.39.** Let $K$ be a field. Then $K$ is bounded iff $K$ has finitely many degree $n$ extensions for every $n$.

This is because for every open normal subgroup $H \leq G$ of index $n$, the subfield of $K^\text{alg}$ fixed by $H$ is a galois extension of $K$ of degree $n$ (and every non-galois extension has a galois closure). A corollary of this is that a field $K$ is bounded iff for every $L \equiv K$, $\text{Gal}(K) \cong \text{Gal}(L)$. This is because any field $K$ uniformly interprets finite extension of degree $n$: for any field $K$ and any irreducible polynomial $P(x, \bar{c}) = \sum_{i=0}^{n} c_i x^i$ over $K$ (with out loss of generality, $c_n = 1$), if $\alpha$ is a root of $P(x)$, then the field extension $K(\alpha)$ is an $n$-dimensional $K$-vector space with basis \{1, $\alpha, \ldots, \alpha^{n-1}$\}. It follows that $K(\alpha)$ is interpretable in $K$ as the vector space $K^n$ (with $(a_0, \ldots, a_{n-1})$ representing $\sum a_i \alpha^i$) equipped with multiplication defined by

$$
(a_0, \ldots, a_{n-1}) \times (b_0, \ldots, b_{n-1}) \equiv \left(a_0 I + a_1 L_\alpha + \ldots + a_{n-1} L_\alpha^{n-1}\right) \begin{pmatrix}
    b_0 \\
    b_1 \\
    \vdots \\
    b_{n-1}
\end{pmatrix},
$$
where $L_\alpha$ is the matrix

$$L_\alpha = \begin{pmatrix}
0 & 0 & \ldots & 0 & -c_0 \\
1 & 0 & \ldots & 0 & -c_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -c_{n-1}
\end{pmatrix}$$

(multiplication by $\alpha$ is a linear map on $K(\alpha)$, and $L_\alpha$ is the corresponding matrix on $K^n$). Since these extensions are interpretable uniformly in the parameter $\bar{c} = (c_0, \ldots, c_{n-1})$, there is a formula $\theta(\bar{x}, \bar{y})$ in the language of rings such that, for any field $K$, $K \models \theta(\bar{c}, \bar{d})$ iff $P(x, \bar{c})$ and $P(x, \bar{d})$ are both irreducible over $K$ and have the same splitting field. It follows that if $K$ is a field with $d$-many algebraic extensions of degree $n$, then there is a sentence expressing this, and so any field elementarily equivalent to $K$ has $d$-many extensions of degree $n$. On the other hand, if $K$ is a field with infinitely many galois extensions of degree $n$, an easy compactness argument gives a field $K' \succ K$ with a degree $n$ galois extension that is not the splitting field of any irreducible polynomial over $K$ (note that if a polynomial is irreducible over $K$, then it is irreducible over $K'$ as well).

There are several results are conjectures bounded fields and their theories. Poizat showed (see [60]) that if $K$ is a non-separably closed field such that $\text{Th}(K)$ is stable, then $K$ is unbounded. Hrushovski [28] showed that bounded PAC fields are simple, and Chatzidakis [10] conversely showed that simple PAC fields are bounded. In particular, the only stable PAC fields are separably closed or algebraically closed fields.

In [33], Koenigsmann conjectures that any bounded field is large. It is easily seen that bounded fields satisfy a weakening of largeness: if $K$ is bounded, then for every absolutely irreducible $K$-curve $C$, there is a finite extension $L/K$ such that $C(L)$ is infinite. To show this, take a plane curve $C'$ birationally equivalent to $C$ over $K$.
defined by a polynomial of degree $d$ and check that if $L$ is the compositum of the finitely many degree $d$ extensions of $K$, then $C'(L)$ is infinite (and hence $C(L)$ is too). In [33], the authors also ask (attributing the question to Macintyre) whether every model complete theory of fields has bounded models. In Chapter 2, we ask a similar question, but with the stronger assumption of almost QE.

Serre introduced bounded fields where one may prove finiteness results for galois cohomology groups with coefficients in linear algebraic groups. More generally, Pillay and Kamensky show the following:

**Fact 1.2.40** ([34], Theorem 5.2). Let $K$ be a bounded field of characteristic 0, and let $G$ be an arbitrary algebraic group defined over $K$. Then $H^1(K, G)$ is countable.

Here, $H^1(K, G)$ is the first galois cohomology group of $K$ with coefficients in $G$, see [71] Chapter 1, 5.2. This fact is essential for the results in Chapter 4.

**Remark 1.2.41.** We remark that if $K$ is a field, then $K$ is a bounded substructure of $K^{alg}$ (in the sense of 1.1.3) iff Gal($K$) is small. However, in the more general context of bounded substructures of stable structures, there are examples of bounded $P \subseteq M$ (even examples where $M$ is strongly minimal!) such that Aut(acl($P$)/$P$) is not small, see [63].

As with largeness, boundedness seems to be a property shared by many of the nicest fields:

**Example 1.2.42.**

1. It is a classical result that Gal($K$) is trivial iff $K$ is separably closed.

2. A theorem of Artin-Schreier gives that Gal($K$) is finite and non-trivial iff $K$ is real closed (in which case Gal($K$) $\cong \mathbb{Z}/2\mathbb{Z}$).

3. $\mathbb{C}((t))$ has a unique extension of degree $n$ for every $n \geq 2$, given by $\mathbb{C}((t))[t^{1/n}]$, so Gal($\mathbb{C}((t))$) $\cong \mathbb{Z}$. This follows from Puiseux’s theorem that the field given by $\bigcup_n \mathbb{C}((t^{1/n}))$ is algebraically closed. More generally, $\mathbb{C}((t_1))\ldots((t_n))$ has absolute galois group isomorphic to $\hat{\mathbb{Z}}^n$. 
4. Any finite field has a unique degree \( n \) extension for any \( n \geq 2 \), and hence every pseudo-finite field \( K \) is bounded with \( \text{Gal}(K) \cong \hat{\mathbb{Z}} \).

5. For every prime \( p \), \( \mathbb{Q}_p \) is bounded. Krasner in [13] calculates the number of degree \( n \) extensions of \( \mathbb{Q}_p \) for every \( n \) and every \( p \).

1.3 Differential Algebra

This section will be dedicated to the basic theory of differential fields and differential algebra. The language \( \mathcal{L}_\partial \) of differential rings is the language of rings expanded by a unary function symbol \( \partial \). Our reference for most of the differential algebra in this section is [23], especially Hardouin’s section, [22], which contains a very concise introduction to the differential analogue of algebraic geometry.

1.3.1 Differential Rings and Fields

**Definition 1.3.1.** A differential ring \((R, \partial)\) is an \( \mathcal{L}_\partial \)-structure such that \( R \) is a commutative, unital ring and \( \partial : R \to R \) is an additive group homomorphism satisfying the Leibniz identity: for all \( a, b \in R \), \( \partial(ab) = a\partial(b) + \partial(a)b \). Such a map \( \partial \) is called a derivation on \( R \).

If \( K \) is a field and \( \partial \) is a derivation on \( K \), then we will say that \((K, \partial)\) is a differential field. Note that if \((R, \partial)\) is a differential ring such that \( R \) is a domain, then \( \partial \) extends uniquely to \( K := \text{Frac}(R) \) by setting

\[
\partial \left( \frac{a}{b} \right) = \frac{a\partial(b) - \partial(a)b}{b^2}.
\]

A morphism of differential rings will simply be a homomorphism of \( \mathcal{L}_\partial \)-structures.

**Definition 1.3.2.** If \((R, \partial)\) is a differential ring, we call the subring \( \mathcal{C}_\partial(R) := \{a \in R : \partial(a) = 0\} \) the ring of constants of \((R, \partial)\). If \((K, \partial)\) is a differential field, \( \mathcal{C}_\partial(K) \) is also a field.
If the context is clear, we will usually write \( \mathcal{C}(R) \) for the ring of constants rather than \( \mathcal{C}_\partial(R) \).

**Example 1.3.3.**

- For any commutative, unital ring \( R \), we can view \( R \) as a differential ring by equipping it with the trivial derivation \( \partial(a) = 0 \) for all \( a \in R \). Here \( \mathcal{C}(R) = R \).

- Let \( \mathbb{C}(t) \) be the field of rational functions over \( \mathbb{C} \). Then \( (\mathbb{C}(t), d/ dt) \) is a differential field with \( \mathcal{C}(\mathbb{C}(t)) = \mathbb{C} \).

- Let \( R \) be a commutative, unital ring such that \( (R, +) \) is divisible. The ring of formal power series over \( R \), \( R[[t]] \), can be equipped with the derivation \( D = d/ dt \) so that \( \mathcal{C}_D(R[[t]]) = R \). There is a canonical embedding of differential rings
  
  \[
  \rho : (R, \partial) \hookrightarrow (R[[t]], D)
  \]
  
  \[
  a \mapsto \sum_{n=0}^{\infty} \frac{\partial^n(a)}{n!} t^n.
  \]

  The map \( \rho \) is analogous to the map taking a function, infinitely differentiable at 0, to its Taylor series around 0.

- Let \((R, \partial)\) be any differential ring. For a set of variables \(x_1, \ldots, x_n\), we write \( R\{x_0, \ldots, x_n\} \) for the ring generated over \( R \) by the symbols \( \partial^i x_j \), for \( i, j \geq 0 \). We extend \( \partial \) to \( R\{x_0, \ldots, x_n\} \) by \( \partial(\partial^i(x_j)) = \partial^{i+1}(x_j) \) for any \( i, j \geq 0 \). This \( (R\{x_0, \ldots, x_n\}, \partial) \) is called the differential polynomial ring in \( n + 1 \)-variables over \( R \). If \( f \in K\{x_1, \ldots, x_n\} \), define the order of \( f \) to be the maximum integer \( m \) such that, for any \( i \), \( \partial^m(x_i) \) appears in \( f \).

**Question 1.3.4.** Let \( K \) be a field and let \( K((t)) \) be the field of formal power series over \( K \). The field \( K(t)^h := K(t)_{\text{alg}} \cap K((t)) \) is the henselization of \( K(t) \) with respect to the \( t \)-adic valuation. Suppose now that \( (K, \partial) \) is a differential field. Let \( (K', \partial) \subseteq (K, \partial) \) be the differential field with underlying pure field given by \( K' := \rho^{-1}(K(t)^h \cap \rho(K)) \). Observe that \( K' \) is the relative field-theoretic algebraic closure in \( K \) of the fraction field of the \( \forall \)-definable set \( \bigcup_n \{ a \in K : \partial^n(a) = 0 \} \) (since if \( \partial^{n+1}(a) = 0 \) and \( \partial^n(a) \neq 0 \), then \( \rho(a) \) is a polynomial of degree \( n \)). Are there conditions on \( K \) that imply \( (K', \partial) \preceq (K, \partial) \) or \( (K', \partial) \preceq (K, \partial) \)?

**Definition 1.3.5.** Let \((R, \partial)\) be a differential ring. An ideal \( \mathcal{I} \subseteq R \) is called a differential ideal (or \( \partial \)-ideal) if \( \partial(I) \subseteq I \).
**Example 1.3.6.** 1. Let \((R, \partial) = (\mathbb{C}[x], \frac{d}{dx})\). Then \((R, \partial)\) has no non-trivial ideals: \(\langle 0 \rangle\) is clearly a \(\partial\)-ideal; suppose that \(I\) is a non-zero \(\partial\)-ideal. Then there is a polynomial \(P(x) = ax^n + R(x) \in I\) such that \(\deg(R(x)) < n\) and \(a \in \mathbb{C}^x\). Then \(\partial^n(P(x)) = P^{(n)}(x) = a \in I\) and hence \(1 \in I\).

2. Let \((R, \partial) = (\mathbb{C}[x, e^x], \frac{d}{dx})\). The ideal generated by \(e^x\) is a non-trivial \(\partial\)-ideal.

**Remark 1.3.7.** Unless otherwise stated, we will always assume that any differential rings \((R, \partial)\) are of characteristic 0 and are such that \(\mathbb{Q} \subseteq R\). The assumption that \(\mathbb{Q} \subseteq R\) guarantees that for any differential ideal \(I \subseteq R\), the radical \(\sqrt{I}\) is also a differential ideal. Suppose that \(a \in \sqrt{I}\) and so \(a^n \in I\) for some \(n > 0\) and \(a \in R\). Since \(I\) is a differential ideal, \(\partial(a^n) = na^{n-1}\partial(a) \in I\). Differentiating \(n - 1\) more times gives

\[
n!\partial(a)^n + \sum_{i=1}^{n} r_ia^i \in I
\]

for some \(r_i \in R\). Since each \(a^i \in I\), \(n!\partial(a)^n \in I\), and since \(\mathbb{Q} \subseteq R\), \(\partial(a)^n \in I\). Hence \(\partial(a) \in \sqrt{I}\). In general, the radical of a differential ideal need not be a differential ideal, but since we will mostly be concerned with differential fields and polynomial rings over differential fields or their quotients, we will not be concerned with such examples.

**Fact 1.3.8 ([22], Lemma 2.6).** Suppose that \((R, \partial)\) is a differential ring, and \(I \subseteq R\) is a \(\partial\)-ideal. Then there is a differential \(\partial'\) on \(R/I\) such that the quotient map \(R \to R/I\) is a homomorphism of differential rings.

**Definition 1.3.9.** Let \(I \subseteq (R, \partial)\) be a differential ideal. A subset \(B \subseteq I\) differentially generates \(I\) (or just generates, if the context is clear) if \(I\) as generated as an ideal by the set \(\{\partial^i(b) : b \in B, i \geq 0\}\).

Recall that a ring \(R\) is called noetherian iff every ideal \(I \subseteq R\) is finitely generated. Hilbert’s basis theorem then states that if \(R\) is noetherian, then \(R[x_1, \ldots, x_n]\) is
noetherian for any \( n \geq 1 \). There is an analogue of noetherianity in the case of differential rings, though it is not quite as strong:

**Definition 1.3.10.** A differential ring \((R, \partial)\) is called rittian if every radical differential ideal \( I \subseteq R \) is finitely generated as a differential ideal.

**Fact 1.3.11** (The Ritt-Raudenbush Basis Theorem). If \((R, \partial)\) is rittian, then so is the differential polynomial ring \( R\{x_1, \ldots, x_n\} \).

In particular, every radical, differential ideal of a differential polynomial ring (in finitely many variable) over a differential field is finitely generated, as differential fields are always rittian.

Notice that if \((R, \partial)\) is a differential ring where \( \partial \) is the trivial derivation, then \((R, \partial)\) is rittian if and only if \( R \) is noetherian.

**Remark 1.3.12.** For a field \( K \) and any finite set \( A \subseteq K[x_1, \ldots, x_n] \), there is an algorithm (with input \( A \)) which produces a basis for the ideal \( I(A) \) (for example, a Gröbner basis). On the other hand, there is no known general algorithm to produce a basis for a differential ideal, even if the given ideal is known to be prime. A famous problem of Ritt ask for an algorithm to determine containment for two given prime ideals. An algorithm that produced a basis for a given prime ideal would give such an algorithm.

### 1.3.2 Differentially Closed Fields

A differential field \((K, \partial)\) is called differentially closed if it contains all possible solutions of systems of differential equations defined over \( K \). More precisely, the differentially closed fields are the existentially closed models of the theory of differential fields. Differentially closed fields were introduced by Robinson, [68], where he gave an axiomatization of the theory of differentially closed fields of characteristic 0, \( \text{DCF}_0 \).
In her thesis, [4], Blum gave a particularly simple axiomatization of DCF₀ that refers only to differential polynomials in a single variable:

**Fact 1.3.13.** A differential field \((K, \partial)\) is a model of DCF₀ iff for all differential polynomials \(f(x), g(x) \in K\{x\}\) such that the order of \(g(x)\) is less than the order of \(f(x)\), there is \(a \in K\) such that \(f(a) = 0\) and \(g(a) \neq 0\).

Pierce and Pillay, [54], later gave an alternate, more “geometric” axiomatization of DCF₀. Let \(V \subseteq A^n\) be an affine variety defined over a field \(K\) and suppose that \(I(V)\) is generated by polynomials \(f_1, \ldots, f_m \in K[x_1, \ldots, x_n]\). The “twisted tangent bundle” of \(V\) (by which we mean, the tangent bundle “twisted” by the differential, and not a twist in the sense of arithmetic geometry), is the \(K\) variety \(\tau(V) \subseteq A^{2n}\) defined by \(\{f_1 = 0, \ldots, f_m = 0\}\) together with the equations of polynomials in \(K[x_1, \ldots, x_n, y_1, \ldots, y_n]\) given by

\[
f_j^\partial(\bar{x}) + \sum_{i=1}^n \frac{df_j(\bar{x})}{dx_i} y_i = 0,
\]

where \(1 \leq j \leq m\) and \(f_j^\partial(\bar{x})\) is the polynomial obtained by applying \(\partial\) to the coefficients of \(f_j(\bar{x})\). Observe that the polynomial appearing in equation (1.1) is just \(\partial(f_j(\bar{x}))\) taken in \(K\{\bar{x}\}\) with \(y_i\) substituted in for \(\partial(x_i)\). In particular, for any \(\bar{a} \in V(K)\), we have \((\bar{a}, \partial(\bar{a})) \in \tau(V)\). As a special case, if \(V\) is a variety defined over the constant field \(\mathcal{C}(K)\) by polynomial \(f_1, \ldots, f_m \in \mathcal{C}(K)[\bar{x}]\), then \(f_j^\partial \equiv 0\), and so \(\tau(V)\) is just the usual Zariski tangent bundle \(T(V)\). It is useful to note that for any point \(\bar{a} \in V\), the fibre \(\tau_{\bar{a}}(V)\) above \(\bar{a}\) in \(\tau(V)\) is a hyperplane, just as in the classical case.

**Fact 1.3.14 (54).** A differential field \((K, \partial)\) is a model of DCF₀ iff, for every absolutely irreducible quasi-affine \(K\)-variety \(V\) and every rational map \(s : V \to \tau(V)\) that is a section of the canonical projection \(\tau(V) \to V\), there is \(\bar{a} \in V(K)\) such that \(s(\bar{a}) = (\bar{a}, \partial(\bar{a}))\).
We emphasize this axiomatization of $\text{DCF}_0$ because we will use it as a prototype for axiomatizing certain “large” fields with extra structure.

**Fact 1.3.15** ([51]). *The theory $\text{DCF}_0$ is complete, with elimination of quantifiers and imaginaries, and is $\omega$-stable. Every differential field has a differential closure, which is unique up to isomorphism.*

1.3.3 Differential Algebraic Geometry

In this section, we give some background information on rudimentary differential algebraic geometry. We will only go so far as discussing affine, or quasi-affine differential algebraic varieties, and make no attempt to define any further generalizations. For the interested reader, Freitag’s thesis [19] contains a more modern study of differential algebraic geometry, or see the paper by Gillet [21] for a scheme theoretic approach. As in the classical case of algebraic geometry, we imagine that everything we do takes place in a very big, very saturated differentially closed field $(\Omega, \partial)$. As before, we identify $\mathbb{A}^n$ with $\Omega^n$. We will say that a subset $\mathcal{V}$ of $\mathbb{A}^n$ is an affine differential variety if $\mathcal{V}$ is the set of solutions of a set of differential polynomials $A \subseteq \Omega\{x_1, \ldots, x_n\}$. As in the classical setting, there is a one-to-one correspondence between radical differential ideals of $\Omega\{x_1, \ldots, x_n\}$ and affine differential varieties (if $\mathcal{V}$ is an affine differential variety, then the ideal $\mathcal{I}(\mathcal{V}) := \{ f(\bar{x}) \in \Omega\{\bar{x}\} : f(\bar{a}) = 0 \forall \bar{a} \in \mathcal{V} \}$ is radical). If $(K, \partial) \subseteq (\Omega, \partial)$ is some differential subfield, then an affine differential variety $\mathcal{V}$ is said to be defined over $K$ if $\mathcal{I}(\mathcal{V})$ is (differentially) generated by polynomials from $K\{\bar{x}\}$. An affine differential $K$-variety $\mathcal{V}$ is $K$-irreducible if $\mathcal{I}(\mathcal{V}) \cap K\{\bar{x}\}$ is a prime ideal, and $\mathcal{V}$ is absolutely irreducible if $\mathcal{I}(\mathcal{V})$ is prime. Again, for an affine differential variety $\mathcal{V}$, $\mathcal{V}(K)$ will be the set of $K$-points of $\mathcal{V}$.

Differential varieties can be equipped with a differential analogue of the Zariski topology called the Kolchin topology. For each $n$, the Kolchin topology on $\mathbb{A}^n$ is generated by closed sets given by affine differential varieties. The Kolchin topology
on an affine differential variety is the subspace topology induced from the Kolchin topology on $\mathbb{A}^n$. Quasi-affine differential varieties are basic open subsets of affine differential varieties, and they too have an induced Kolchin topology. For this dissertation, we will require nothing more general the notion of “quasi-affine differential variety”, which we take to be a basic Kolchin open subset of an affine differential variety. As in the Zariski setting, one can consider the $K$-Kolchin topology on a differential $K$-variety $V$, which is the restriction of the full Kolchin topology over $\Omega, \partial$ to the topology generated by those closed sets defined over $K$. To say that differential variety $\mathcal{V}$ defined over $K$ is $K$-irreducible is equivalent to saying that $\mathcal{V}$ is irreducible as a topological space in the $K$-Kolchin topology.

Note that every affine (or quasi-affine) algebraic variety $V$ is also an affine differential variety, with $\mathcal{I}(V) \subseteq K\{x\}$ generated differentially by $\mathbb{I}(V) \subseteq K[x]$. It follows that the Kolchin topology strictly refines the Zariski topology; for example, in the Zariski, the only proper closed subsets of $\mathbb{A}^1$ are finite sets, while in the Kolchin topology, $C := C(\Omega) = \{x : \partial(x) = 0\}$ is an infinite, Kolchin closed subset of $\mathbb{A}^1$. Despite this, Kolchin proved that his eponymous topology is not fine enough to “split up” irreducible Zariski closed sets:

**Fact 1.3.16** (Kolchin’s Irreducibility Theorem, [10]). *Let $V$ be an affine algebraic $K$-variety defined over a differential field $(K, \partial)$. If $V$ is $K$-irreducible in the $K$-Zariski topology, then $V$ is $K$-Kolchin irreducible as well.*

**Definition 1.3.17.** Let $(K, \partial)$ be a differential field and let $\mathcal{V} \subseteq \mathbb{A}^n$ be a $K$-irreducible affine differential variety over $K$. Let $\mathcal{I}(\mathcal{V}) \subseteq K\{x_1, \ldots, x_n\}$ be the differential ideal of differential polynomials that vanish on $\mathcal{V}(K)$. The differential ring $K\{\mathcal{V}\} := K\{x_1, \ldots, x_n\}/\mathcal{I}(\mathcal{V})$ is called the differential coordinate ring of $\mathcal{V}$. The differential field

$$K(\mathcal{V}) := \text{Frac}(K\{\mathcal{V}\})$$
\[
\{ \frac{f}{g} : f, g \in K\{x_1, \ldots, x_n\}, g \neq 0 \text{ on } \mathcal{V} \}
\]

is called the differential function field of \( \mathcal{V} \) over \( K \).

**Proposition 1.3.18.** Let \( \mathcal{V} \) be an absolutely irreducible affine differential variety defined over \( (K, \partial) \) and let \( (K\langle \mathcal{V} \rangle, \partial) \) be the differential function field of \( \mathcal{V} \) over \( K \). Then \( \mathcal{V}(K) \) is Kolchin-dense in \( \mathcal{V} \) iff \( (K, \partial) \preceq \exists (K\langle \mathcal{V} \rangle, \partial) \).

**Proof.** The proof is exactly the same as in the classical case. First, suppose that \( (K, \partial) \preceq \exists (K\langle \mathcal{V} \rangle, \partial) \). Let \( \mathcal{U} \) be a basic Kolchin-open subset of \( \mathcal{V} \). The complement in \( \mathcal{V} \), \( \mathcal{V} \cap \mathcal{U}^c \), is a Kolchin-closed subset of \( \mathcal{V} \). Let \( \mathcal{W} \subseteq \mathcal{V} \) be the \( K \)-Kolchin-closure of \( \mathcal{V} \cap \mathcal{U}^c \), i.e. the zero-set of all differential polynomials over \( K \) which vanish on \( \mathcal{V} \cap \mathcal{U}^c \). Let \( \mathcal{U}' \subseteq \mathcal{U} \) be the complement \( \mathcal{W}^c \cap \mathcal{V} \). Then \( \mathcal{U}' \) is a basic Kolchin-open subset of \( \mathcal{V} \) defined over \( K \) and so, in particular, \( \mathcal{U}'(K\langle \mathcal{V} \rangle) \) is non-empty. Since \( (K, \partial) \preceq \exists (K\langle \mathcal{V} \rangle, \partial) \), \( \mathcal{U}'(K) \) is non-empty, hence \( \mathcal{U}(K) \supseteq \mathcal{U}'(K) \) is non-empty. Therefore, \( \mathcal{V}(K) \) is dense in \( \mathcal{V} \).

On the other hand, suppose that \( \mathcal{V}(K) \) is Kolchin-dense in \( \mathcal{V} \). By model theory, \( K \preceq \exists K\langle \mathcal{V} \rangle \) (as differential fields) iff \( K\langle \mathcal{V} \rangle \) embeds over \( K \) into some elementary extension \( K' \succ K \). For every \( \bar{a} \in \mathcal{V}(K) \), let \( M_{\bar{a}} := (K, \partial) \). Let

\[
\mathcal{F} = \{ \mathcal{U}(K) : \mathcal{U} \subseteq \mathcal{V} \text{ is open, } \mathcal{U} \neq \emptyset \}.
\]

Since \( \mathcal{V}(K) \) is dense in \( \mathcal{V} \), \( \mathcal{U}(K) \) is non-empty for every open \( \mathcal{U} \subseteq \mathcal{V} \). Furthermore, \( \mathcal{U}_1(K) \cap \mathcal{U}_2(K) \in \mathcal{F} \) for any pair of open subsets \( \mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{V} \) and so \( \mathcal{F} \) is a filter base for some filter on \( \mathcal{V}(K) \). Let \( \mathcal{D} \) be any ultrafilter extending the filter generated by \( \mathcal{F} \) and let

\[
(K', \partial) := \prod_{\bar{a}, \mathcal{D}} M_{\bar{a}} \succ (K, \partial).
\]
It suffices to show that \((K\langle V \rangle, \partial)\) embeds into \((K', \partial)\) over \(K\). Suppose \(h \in K\langle V \rangle\). Then \(U_h := \text{dom}(h)\) is a Kolchin open subset of \(V\). Define a function

\[
h \mapsto \left( \tilde{h}(\bar{a}) : \bar{a} \in V(K) \right) \in \prod_{\bar{a} \in V(K)} M_{\bar{a}}
\]

where

\[
\tilde{h}(\bar{a}) = \begin{cases} h(\bar{a}) & \bar{a} \in U_h(K) \\ 0 & \bar{a} \notin U_h(K). \end{cases}
\]

Set

\[
\chi : K\langle V \rangle \to K'
\]

\[
h \mapsto \left( \tilde{h}(\bar{a}) : \bar{a} \in V(K) \right) / D.
\]

It is straightforward to check that \(\chi\) is a \(\partial\)-ring homomorphism over \(K\). Suppose \(g \neq h \in K\langle V \rangle\). Then there is some non-empty open \(\mathcal{U} \subseteq V\) such that for all \(\bar{a} \in \mathcal{U}(K)\), \(g(\bar{a}) - h(\bar{a}) \neq 0\), hence \(\chi(g - h) \neq 0\). Therefore, \(\chi\) is the required embedding. \(\square\)

**Definition 1.3.19.** For a \(K\)-irreducible affine differential variety \(V\) over \(K\), a point \(\bar{e} \in V(\mathcal{U})\) is called Kolchin generic for \(V\) over \(K\) if for every basic open \(\mathcal{U} \subseteq V\) which is defined over \(K\), \(\bar{e} \in \mathcal{U}\). Equivalently, \(\bar{e} \in V\) and \(\bar{e} \notin W\) for any proper Kolchin-closed \(W \subset V\).

Let the generic type of \(V\) over \(K\), \(p_V(\bar{x})\), be the quantifier-free type in the language of differential rings which says \(\bar{x} \in V\) and \(\bar{x} \notin W\) for any Kolchin-closed proper subset \(W \subset V\). Then \(\text{Th}(K, \partial) \cup p_V(\bar{x})\) is consistent iff \(V(K)\) is Kolchin-dense in \(V\). Observe that for any \(K\)-irreducible affine differential variety \(V\) (or even quasi-affine), \(K\langle V \rangle\) is isomorphic (as differential fields) over \(K\) to \(K(\bar{e})\), the differential field generated over \(K\) by a \(K\)-generic point \(\bar{e}\) of \(V\). Therefore we have:
Corollary 1.3.20. Let $(K, \partial)$ be a differential field, and $\mathcal{V}$ a $K$-irreducible differential variety defined over $K$. Then the following are equivalent:

1. $\mathcal{V}(K)$ is Kolchin-dense in $\mathcal{V}$;
2. $(K, \partial) \preceq (K(\mathcal{V}), \partial)$;
3. Th$(K, \partial) \cup p_\mathcal{V}(\bar{x})$ is consistent.

To each (quasi-)affine differential variety $\mathcal{V}$, say defined and irreducible over a differential field $(K, \partial)$, we can associate a series of affine algebraic varieties called jets. Let $\bar{c}$ be Kolchin-generic for $\mathcal{V}$ over $K$. The $n$-th jet space of $\mathcal{V}$, Jet$_n(\mathcal{V})$, is defined to be the $K$-Zariski closure of the point $\{(\bar{c}, \partial(\bar{c}), \ldots, \partial^n(\bar{c}))\}$. Jet spaces essentially higher-order “twisted tangent bundles” attached to a differential variety. If $V$ is an affine algebraic variety, then Jet$_0(V) = V$, Jet$_1(V)$ is the usual twisted tangent bundle $\tau(V)$ and, more generally, there is an isomorphism of affine varieties between Jet$_n(V)$ and $\tau^n(V) := \tau(\tau^{n-1}(V))$ for every $n \geq 0$. We may equivalently define jet spaces as follows: let $K\{x_1, \ldots, x_n\}$ be the ring of differential polynomials over $K$, and let $K[y_{i,j} : 1 \leq i \leq n, 0 \leq j < \omega]$ be a polynomial ring in infinitely many variables. We define an isomorphism of $K$-algebras $\gamma : K\{x_1, \ldots, x_n\} \to K[y_{i,j} : 1 \leq i \leq n, 0 \leq j < \omega]$ by $\gamma(\partial^j(x_i)) := y_{i,j}$. If $\mathcal{V} \subset \mathbb{A}^n$ is a $K$-irreducible, affine, differential $K$-variety, and $\mathcal{I} = \mathcal{I}(\mathcal{V}) \subseteq K\{\bar{x}\}$ is the radical, prime differential ideal of differential polynomials over $K$ vanishing on $\mathcal{V}$, then define $I_m$ to be the ideal

$$\gamma(I) \cap K[y_{i,j} : 1 \leq i \leq n, 0 \leq j \leq m].$$

Then Jet$_m(\mathcal{V}) = \mathbb{V}(I_m)$. The sequence of jet spaces determines $\mathcal{V}$, since $\bar{a} \in \mathcal{V}$ iff $(\bar{a}, \partial(\bar{a}), \ldots, \partial^n(\bar{a}))$ is in Jet$_n(\mathcal{V})$ for all $n \geq 0$. 

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1.3.4 Picard-Vessiot Theory

Let \((K, \partial)\) be a differential field and let \(G\) be a connected algebraic group defined over \(\mathcal{C}(K)\) with identity element \(e\) and Lie algebra \(LG \cong T_e(G)\). Suppose that \(\varphi : G \times G \to G\) is the \(\mathcal{C}(K)\)-regular map defining the group operation on \(G\). The tangent bundle, \(T(G)\), can be given the structure of an algebraic \(\mathcal{C}(K)\)-group with operation “\(\ast\)” defined as

\[
(g_1, h_1) \ast (g_2, h_2) := (g_1 g_2, d\varphi_{(g_1, g_2)}(h_1, h_2)),
\]

where \(d\varphi_{(g_1, g_2)} : T_{(g_1, g_2)}(G \times G) \to T_{g_1 g_2}(G)\) is the differential of \(\varphi\) (a linear map of vector spaces) and the identity element of \(T(G)\) is \((e, 0)\). With this, we may write the group operation on the Lie algebra \(LG\) as

\[
(e, a) \ast (e, b) = (e, d\varphi_{(e, e)}(a, b)).
\]

For every \(g \in G\), we have that \((g, 0) \in T(G)\) (a rational section of the canonical projection \(T(G) \to G\)). Observe that, since \(G\) is defined over \(\mathcal{C}(K)\), we have \((g, \partial(g)) \in T(G)\) for every \(g \in G\). With this, we define the “differential logarithm” map with respect to \(G\) (introduced by Kolchin) as

\[
d\log_G(g) := (g, \partial g) \ast (g^{-1}, 0)
\]

\[
= (e, d\varphi_{(g, g^{-1})}(\partial g, 0)) \in T_e(G) \cong LG.
\]

Note that, by the linearity of \(d\varphi_{(g, g^{-1})}\), \(d\log_G(g) = (e, 0)\) iff \(\partial(g) = 0\), and so \(\ker(d\log_G) = G(\mathcal{C}(K))\). A “logarithmic differential equation over \(G\)” will be an equation of the form

\[
d\log_G(z) = a
\]
for some \( a \in LG \) (we will sometimes write “\( a \)”, rather than “\((e, a)\)” for elements of \( LG \) to avoid clutter). Note that \( d\log_G(g) = (e, a) \) iff \( (g, \partial(g)) = (e, a) \ast (g, 0) \).

**Example 1.3.21.** Let \( G = GL_1 = \mathbb{G}_m \) be the multiplicative group (of a field). The group operation \( \varphi : G \times G \to G \) is simply multiplication \( \varphi(x, y) = xy \) and so for \( (a, b) \in G \times G \), the differential map is given by \( d\varphi_{(a,b)}(x,y) = bx + ay \). Thus, the logarithmic derivative on \( G \) is defined as

\[
d\log_{GL_1}(x) = (1, d\varphi_{(x,x^{-1})}(\partial x, 0)) = (1, x^{-1}\partial(x)).
\]

The tangent bundle \( T(GL_1) \) is isomorphic to the additive group \( \mathbb{G}_a \) and so a logarithmic differential equation for \( GL_1 \) takes the form

\[
\frac{\partial(x)}{x} = a,
\]

which is the classical “logarithmic derivative” (and which is where the name comes from). In particular, if \( a = 0 \), then the solution set of \( \partial x / x = 0 \) is precisely \( \mathcal{C}(K) \).

**Fact 1.3.22.** Let \((K, \partial)\) be a differential field, and let \( d\log_G(z) = a \) be a logarithmic differential equation over an algebraic group \( G \) defined over \( \mathcal{C}(K) \). For any \( g_1, g_2 \in G \) such that \( d\log_G(g_i) = a, i = 1, 2 \), we have that \( g_1g_2^{-1} \in \ker(d\log_G) \).

Given a logarithmic differential equation \( d\log_G(x) = a \), we aim to study fields that are akin to the spitting field of a polynomial, in the sense that they are minimal and generated by solutions. Such fields were originally studied by Kolchin, see [42].

**Definition 1.3.23.** Let \((K, \partial)\) be a differential field, \( G \) a connected algebraic group defined over \( \mathcal{C}(K) \), and \( d\log_G(z) = a \) a logarithmic differential equation. A strongly normal extension of \((K, \partial)\) for \( d\log_G(z) = a \) is a differential field \((L, \partial)\) such that:

1. there is a solution \( g \in G(L) \) of \( d\log_G(z) = a \) such that \( L = K(g) \), and
2. there are no new constants, i.e. $\mathcal{C}(L) = \mathcal{C}(K)$.

If $G = \text{GL}_n$, then a strongly normal extension for $\text{dlog}_G(z) = a$ is called a Picard-Vessiot (PV) extension of $(K, \partial)$. A logarithmic differential with $G = \text{GL}_n$ is equivalent to one of the form

$$\partial(Z) = AZ,$$  \hspace{1cm} (1.2)

where $Z$ is a $n \times n$ matrix of unknowns, and $A$ is some $n \times n$ matrix over $K$ (the tangent bundle $T(\text{GL}_n)$ is isomorphic to the space of $n \times n$ matrices over $K$; $\text{GL}_n$ is isomorphic to an open set of $\mathbb{A}^{n^2}$ and the tangent space of $\mathbb{A}^{2n}$ at any point is itself). A particular solution, $C \in \text{GL}_n$ is a matrix whose $n$ linearly independent columns span the space of solutions of a system of linear, first order differential equations of the form $\partial Y = AY$, where $A$ is a column vector of length $n$. This basis is also called a fundamental system of solutions for this equation. A vector equation such as this can be further transformed into a linear homogeneous differential equation in a single variable. Given a vector equation $\partial Y = AY$ over a differential field $(K, \partial)$, if $Y_1, \ldots, Y_n$ is a fundamental system of solutions such that $\mathcal{C}(K(Y_1, \ldots, Y_n)) = \mathcal{C}(K)$, then $(K(Y_1, \ldots, Y_n), \partial)$ is a PV extension of $(K, \partial)$ for $\partial Y = AY$ (this is a bit of abuse of notation, in that we should say that the PV extension should be generated by entries of the vectors $Y_i$, not the vectors themselves).

1.4 Fields With An Automorphism

A difference field is a field $K$ equipped with a ring endomorphism $\sigma$. If $\sigma$ is an automorphism, then $(K, \sigma)$ is called an inversive difference field. In \cite{Macintyre}, Macintyre showed that the theory of fields with an automorphism has a model companion, ACFA, the theory of algebraically closed fields with a (generic) automorphism. ACFA has the following geometric axiom scheme (see \cite{Hrubes}):
Fact 1.4.1. A difference field \((K,\sigma)\models ACFA\) iff

1. \(K\models ACF\),

2. \(\sigma\) is an automorphism,

3. for any absolutely irreducible, quasi-affine \(K\)-variety \(V\), and any absolutely irreducible quasi-affine subvariety \(W \subseteq V \times \sigma(V)\) such that \(W\) projects generically onto \(V\) and \(\sigma(V)\), there is \(\bar{a} \in V(K)\) such that \((\bar{a},\sigma(\bar{a})) \in W(K)\).

See the paper by Chatzidakis and Hrushovski, [11], where the give a thorough model theoretic analysis of ACFA.
In this chapter, we explore a strong notion of model completeness for theories of fields called “almost quantifier elimination”. Recall that a theory $T$ is model complete if, modulo $T$, every formula is equivalent to an existential formula. In other words, in a structure whose theory is model complete, every definable set is equivalent to a projection of a quantifier-free definable set. In a structure whose theory has quantifier elimination, every definable set is definable by a quantifier-free formula. Almost QE is a property of some theories of fields that sits comfortably in between these two situations: every definable set is a projection of a quantifier-free definable set (a constructible set), and the fibres of the projection are finite and field-theoretically algebraic (by field-theoretically algebraic, we mean the fibres are algebraic in the sense of $\text{acl}_{\text{ACF}}$, as opposed to $\text{acl}_T$; in a non-algebraically closed field $K$, $A \subset K$, $\text{acl}_{Th(K)}(A)$ may properly contain the set of elements of $K$ that are solutions to polynomials over $A$). The term “almost quantifier elimination” was first used in print (as far as the author is aware) in a blog post by Tao [74] describing the phenomenon in the context of pseudo-finite fields.

The intended purpose of this chapter is to explore the relationship between almost QE and two other properties of fields: largeness, and boundedness. Our motivation for our belief that there even exist general relationships between these properties comes from the (admittedly naïve) observation that all the model-theoretically “nicest” fields share these three properties. We accomplish this investigation with varying degrees of success; we show that large, perfect fields with almost QE are
geometric in the sense of [29]. This implies that we may endow definable sets with a notion of dimension. We conjecture that fields with almost QE are bounded (various authors have asked whether fields with model complete theories must be bounded, see [33]), but fall short in this endeavour. We are able to show that PAC fields with almost QE have a weak notion of boundedness, and we show this in the more general setting of PAC substructures of stable theories. Unless otherwise stated \( \downarrow \) will mean independence in the sense of ACF.

**Definition 2.0.1.** Let \( T \) be a (possibly incomplete) theory of fields in the language of rings possibly expanded by constants, say \( \mathcal{L} \). We will say that \( T \) has almost QE if, whenever \( K \models T \) and \( A \subseteq K \) is a relatively algebraically closed \( \mathcal{L} \)-substructure \( (A = A_{\text{alg}} \cap K) \), then \( T \cup \text{qftp}_\mathcal{L}(A) \) is complete.

Note that any theory of fields, \( T \), with almost QE is model complete, since if \( M \models T \) then \( M \) is a relatively algebraically closed substructure of itself, and so \( T \cup \text{qftp}(M) \) is complete.

Let \( T \) be any theory of fields in some language \( \mathcal{L} \) extending the language of rings. Let \( \mathcal{L}_R \) be the language obtained by expanding \( \mathcal{L} \) by an \( n \)-ary predicate \( P_n(x_0, \ldots, x_{n-1}) \) for every \( n \in \mathbb{N} \). Let \( T_R \) be the definitional expansion of \( T \) to \( \mathcal{L}_R \) obtained by expanding \( T \) by the axioms

\[
\Phi_n \equiv \forall x_0, \ldots, x_{n-1} (P_n(x_0, \ldots, x_{n-1}) \leftrightarrow \exists y (y^n + x_{n-1}y^{n-1} + x_1 y + x_0 = 0)).
\]

Let \( T^* \) be the theory given by \( T_R \cup \{\Phi_n : n \geq 2\} \). If \( A \models T^* \), then \( A \models T_R \) and so \( A \) is a substructure of a model of \( T \). Suppose that \( A \models T^* \) and \( A \subseteq M \models T_R \). Then for any \( \bar{a} \in A^n \), \( M \models P_n(\bar{a}) \iff A \models P_n(\bar{a}) \) and hence \( A \) is relatively algebraically closed in \( M \). Let \( T_a \) be the \( \mathcal{L} \)-theory obtained by replacing every occurrence of \( P_n(\bar{x}) \) by \( \exists y (y^n + x_{n-1}y^{n-1} + x_1y + x_0 = 0) \) in every sentence in \( T^* \). Then \( T_a \) is an \( \mathcal{L} \)-theory whose models are precisely the relatively algebraically closed substructures of
models of $T$. From this observation, we get the following equivalent characterization of almost QE:

**Proposition 2.0.2.** Let $T$ be a theory of fields in the language of rings, possible expanded by constants, say $\mathcal{L}$. The following are equivalent:

1. $T$ has almost QE,
2. $T$ is the model completion of $T_a$.

**Proof.** If $T$ is the model completion of $T_a$, then it is immediate that $T$ has almost QE, since if $A$ is a relatively algebraically closed substructure of a model of $T$, then $A \models T_a$ and hence $T \cup \text{qftp}(A)$ is complete by the definition of model completion.

Suppose that $T$ has almost QE. Then $T$ is the model companion of $T_a$, since almost QE implies that $T$ is model complete. The fact that $T$ is the model completion of $T_a$ follows by definition: if $A \models T_a$, then $A$ is relatively algebraically closed in a model of $T$, and so by almost QE, $T \cup \text{qftp}(A)$ is complete. 

**Remark 2.0.3.** In [20], Chapter 27, Section 2, the authors show if $T$ is the theory of fields, then $T_R$ has a model companion $\tilde{T}_R$, which is precisely the theory of $\omega$-free PAC fields with degree of imperfection 1 (in the characteristic $p > 0$ case). They also provide an example showing that that $T_R$ does not have a model completion.

**Example 2.0.4.** Let $G$ be a small, projective profinite group. Let $T$ be the theory of fields $K$ of characteristic 0 such that $\text{Gal}(K) \cong G$ in the language of rings. Let $A \models T_a$. Then for any model $K \models T$ such that $A \subseteq K$, $A$ is relatively algebraically closed in $K$. This implies that $\text{Gal}(A) \cong \text{Gal}(K)$ and hence $A \models T$ i.e. $T_a \models T$.

It is easy to see that $T$ is closed under unions of chains (i.e. $T$ is inductive), so $T$ has a model companion iff the existentially closed models of $T$ form an elementary class. We claim that the model companion $\tilde{T}$ is precisely the theory of PAC fields with absolute galois group isomorphic to $G$. To see this, we will use the following fact from [20]:
Fact 2.0.5 (Theorem 23.1.1, [20]). Let \( L/K \) be a Galois extension, \( G \) a projective group, and \( \alpha : G \to \text{Gal}(L/K) \) an epimorphism. Then \( K \) has an extension \( E \) that is perfect, PAC, and linearly disjoint from \( L \), and there exists an isomorphism \( \gamma : \text{Gal}(E) \to G \) such that \( \alpha \circ \gamma = \text{res}_L \), where \( \text{res}_L : \text{Gal}(E) \to \text{Gal}(L/K) \) is the restriction map \((\text{res}_L(\sigma) = \sigma|_L, \text{ for } \sigma \in \text{Gal}(E))\). 

Let \( K \) be an existentially closed model of \( T \), and let \( L = K^{alg} \). By Fact 2.0.5 there is a PAC field \( E \) extending \( K \) such that \( E \) is linearly disjoint from \( K^{alg} \) over \( K \) (equivalently, \( E \) is a regular extension of \( K \)) and such that \( \text{Gal}(E) \cong \text{Gal}(K) \). This implies that \( E \models T \) with \( K \) a substructure of \( E \). Let \( V \) be an absolutely irreducible \( K \)-variety (not necessarily affine). As \( V \) is also defined over \( E \), \( V(E) \) is non-empty, since \( E \) is PAC. Since \( K \) is existentially closed in \( E \), \( V(K) \) is non-empty. This shows that \( K \) is PAC, and hence \( K \models \tilde{T} \).

Conversely, suppose that \( K \models \tilde{T} \) and let \( E \models T \) be any model extending \( K \). As \( E \) and \( K \) are both models of \( T_0 \), \( K \) is relatively algebraically closed in \( E \) and hence \( E \) is a regular extension of \( K \). Since \( K \) is PAC, \( K \) is existentially closed in \( E \).

One can show further that \( \tilde{T} \) is the model completion of \( T \) using the characterization of elementary invariants of PAC fields (see the unpublished manuscript [15]).

Example 2.0.6. Let \( T \) be the theory of fields with absolute galois group isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). By Artin-Schreier, \( T = \text{RCF} \), the theory of real closed fields. The theory of real closed fields is model complete, and for any real closed field \( \mathcal{R} \), if \( A \subseteq \mathcal{R} \) is relatively algebraically closed, then \( A \models \text{RCF} \). It follows that \( T \) is its own model completion and so \( \text{RCF} \) has almost QE.

Example 2.0.7. By [1], if \( T \) is the theory of pseudofinite fields, then \( T \) is the model completion of the theory of pro-cyclic fields (fields with absolute galois group a projective limit of cyclic groups).
**Conjecture 1.** Let $T$ be a complete theory of fields in the language of rings. If there is a expansion $\mathcal{L}^*$ of $\mathcal{L}_{\text{rings}}$ by constants and an extension $T^*$ of $T$ to $\mathcal{L}^*$ such that $T^*$ has almost QE, then every model of $T$ is bounded.

**Remark 2.0.8.** In regards to Conjecture 1, let $T$ be a theory of fields in the ring language possibly expanded by constants. Suppose that the models of $T$ are unbounded. It suffices to show that $T_a$ has no model completion. To do this, we need only to find, $A, B, C \models T_a$ such that $A \subseteq B \cap C$ and such that $B$ and $C$ cannot be amalgamated over $A$.

Let us consider just the characteristic 0 case. Observe that, in the field language, the class of characteristic 0 fields has the amalgamation property: for fields $A, B, C$ such that $A \subseteq B \cap C$, the compositum $BC$ is the smallest field containing $B$ and $C$ (we can even write $BC = A(B \cup C)$, since both fields extend $A$). Suppose now that $A, B, C \models T_a$ and $A \subseteq B \cap C$. Any amalgam $D$ will necessarily contain the compositum $BC$, and so to prove the conjecture it suffices to show that if $T$ is a theory of unbounded fields, then one can always find $A, B, C \models T_a$ such that $A \subseteq B \cap C$, $A$ is relatively algebraically closed in $B$ and $C$, but $A$ is not relatively algebraically closed in $BC$. Recall that, for fields of characteristic 0, $A/B$ is regular iff $B$ is relatively algebraically closed in $A$. It is a fact (see [20] Chapter 2) that if $B$ and $C$ are regular extension of a field $A$ such that $B$ and $C$ are algebraically independent over $A$, then the compositum $BC$ is also a regular extension over $A$, and so any counter-example would require two regular extensions that fail to be algebraically independent. For example, $\mathbb{Q}(t)$ is a regular extension of $\mathbb{Q}$, as is $\mathbb{Q}(\sqrt{2}t)$, but their compositum is $\mathbb{Q}(t, \sqrt{2})$.

Consider the following elementary fact concerning large fields:

**Lemma 2.0.9.** Let $T$ be a theory of large fields. Let $M \preceq M' \models T$. Then, for any set $C$ such that $M \subseteq C \subseteq M'$ and $\bar{a} \in M'$ such that $\bar{a} \downarrow_M C$, we have that

\[\text{ }\]
$tp_{ACF}(\bar{a}/C^{alg}) = qftp(\bar{a}/C^{alg})$ is finitely satisfiable in $M$ (i.e. $qftp(\bar{a}/C^{alg})$ is a coheir of $qftp(\bar{a}/M)$).

Proof. Without loss of generality, we may assume that $C$ is relatively algebraically closed in $M'$ and that $\bar{a} \in M'$ is a finite tuple. Let $V$ be the absolutely irreducible $M$-variety having $\bar{a}$ as an $M$-generic point. Since $\bar{a}$ is generic for $V$, it is smooth. By largeness, $V(M)$ is Zariski dense in $V$.

Since $V$ is absolutely irreducible, $qftp(\bar{a}/M)$ has a unique non-forking (in the sense of ACF) extension to $C^{alg}$, say $p(\bar{x})$. Since $\bar{a} \mid C$, $p(\bar{a}) = qftp(\bar{a}/C^{alg})$. It follows that every formula $\varphi(\bar{x}) \in qftp(\bar{a}/C^{alg})$ defined a Zariski open subset of $V$. Since $V(M)$ is dense in $V$, $\varphi(M)$ is non-empty.

Lemma 2.0.9 is proven in [64] as the “Coheir Lemma” (Lemma 3.18) for PAC substructures of a stable structure, and is a key component of the proof that bounded PAC structures are simple. The following conjecture is equivalent to a positive answer to Conjecture 1:

Conjecture 2. Suppose $T$ is a theory of PAC fields in the language of rings, possibly expanded by constants. If $T$ has almost QE, then $\mid$ satisfies the Independence Theorem over models.

**Proposition 2.0.10.** Suppose $T$ is an $L$-theory of fields, where $L$ is the language of rings, possibly expanded by constants. If $T$ has almost QE, then for any $M \models T$ and any substructure $A \subseteq M$ such that $A$ is relatively algebraically closed in $M$, if $N \models T \cup qftp(A)$, then $A \subseteq N$ is relatively algebraically closed in $N$ as well.

Proof. Let $M \models T$ and let $A \subseteq M$ be a substructure which is relatively algebraically closed in $M$ as a subfield. Let $N \models T \cup qftp(A)$. Suppose that $A$ is not relatively algebraically closed in $N$. Then there is a polynomial $P(x)$ over $A$ such that for every $a \in A$, we have that $N \models \exists x(P(x) = 0 \land P(a) \neq 0)$. Let $\Sigma := \{\exists x(P(x) = \}$
0 ∧ P(a) ≠ 0 : a ∈ A}. Now \( T \cup \text{qftp}(A) \vdash \Sigma \) by almost QE. Since \( M \models T \cup \text{qftp}(A) \), we have that \( M \models \Sigma \), but then \( P(x) = 0 \) has a root in \( M \) and no root in \( A \), a contradiction.

**Remark 2.0.11.** If \( T \) has almost QE in the language of rings, then every model of \( T \) is perfect, since \( T \) is model complete. This is not necessarily true if the language has been expanded by constants. For example, separably closed fields of finite degree of imperfection have quantifier-elimination after naming a \( p \)-basis.

**Definition 2.0.12.** We will say that a formula \( \varphi(\bar{x}) \) is good iff, modulo \( T \), \( \varphi(\bar{x}) \) is equivalent to a formula of the form

\[
\exists \bar{y} \psi(\bar{x}, \bar{y}),
\]

where \( \psi(\bar{x}, \bar{y}) \) is quantifier-free and is such that, for any \( M \models T \), if \( M \models \psi(\bar{a}, \bar{b}) \), then \( \bar{b} \in \text{dcl}_M(\bar{a})^{alg} \cap M \) and, moreover, there are only finitely many \( \bar{b}' \in M \) such that \( M \models \psi(\bar{a}, \bar{b}') \).

Note that we may assume that the quantifier-free formula \( \psi(\bar{x}, \bar{y}) \) in Definition 2.0.12 is positive, since any occurrence of a subformula of the form \( f(\bar{x}, \bar{y}) \neq 0 \) may be replaced by \( \exists z (f(\bar{x}, \bar{y})z - 1 = 0) \). Furthermore, if our theory is not the theory of algebraically closed fields, we may assume that \( \psi(\bar{x}, \bar{y}) \) is of the form \( f(\bar{x}, \bar{y}) = 0 \); if \( K \) is a field that is not algebraically closed, then for every \( n < \omega \) there is a some polynomial \( P(x_1, \ldots, x_n) \) over \( K \) such that \( P(a_1, \ldots, a_n) = 0 \) iff \( a_i = 0 \) for all \( i \). It follows that any conjunction of the form \( \bigwedge_i f_i(\bar{x}) = 0 \) may be replaced by \( P_n(f_1(\bar{x}), \ldots, f_n(\bar{x})) = 0 \). It is clear that any disjunction of the form \( \bigvee_i f_i(\bar{x}) = 0 \) may be replaced by the product \( \prod_i f_i(\bar{x}) = 0 \).

**Proposition 2.0.13.** Let \( T \) be a theory of perfect fields in the language of rings, possibly expanded by constants. The following are equivalent:
1. if $M \models T$ and $A \subseteq M$ is a relatively algebraically closed substructure, then $T \cup \text{qftp}(A)$ is complete,

2. if $M$ and $N$ are models of $T$ extending a common substructure $A$ such that $A$ is relatively algebraically closed in $M$, then $\text{tp}_M(A) = \text{tp}_N(A)$,

3. every formula $\varphi(\bar{x})$ is equivalent modulo $T$ to a good formula.

Proof. 1 $\Rightarrow$ 2: Let $M$ and $N$ be models of $T$ extending a common substructure $A$ such that $A$ is relatively algebraically closed in $M$. By 1, $T \cup \text{qftp}(A)$ is complete and so $T \cup \text{qftp}(A) \models \text{tp}_M(A)$. As $A$ is a substructure of $N$, $N \models T \cup \text{qftp}(A)$ and so $N \models \text{tp}_M(A)$. Hence $\text{tp}_N(A) = \text{tp}_M(A)$.

2 $\Rightarrow$ 1: Immediate.

1 $\Rightarrow$ 3: Let $\varphi(\bar{x})$ be a formula and let $\bar{c}$ be a new tuple of constants. We may assume, both $T \cup \{\varphi(\bar{c})\}$ and $T \cup \{\neg \varphi(\bar{c})\}$ are consistent. Let $\Gamma(\bar{x})$ be the set of all formulas $\neg \theta(\bar{x})$ such that $\theta(\bar{x})$ is good and $T \models \forall \bar{x} (\theta(\bar{x}) \rightarrow \varphi(\bar{x}))$.

Claim 2.0.14. The set $T \cup \{\varphi(\bar{c})\} \cup \Gamma(\bar{c})$ is inconsistent.

Proof of Claim. Suppose not. Then there is $M \models T \cup \{\varphi(\bar{c})\} \cup \Gamma(\bar{c})$. Let $A \subseteq M$ be the relative algebraic closure of $\text{dcl}_M(\bar{c}) \subseteq M$. By 1., $T \cup \text{qftp}(A)$ is complete. Since $M \models T \cup \text{qftp}(A) \cup \{\varphi(\bar{c})\}$, it follows that $T \cup \text{qftp}(A) \cup \{\neg \varphi(\bar{c})\}$ is inconsistent. By compactness, there is quantifier-free $\psi(\bar{x}, \bar{y})$ and $\bar{b} = (b_1, \ldots, b_n) \in A$ such that $T \cup \{\psi(\bar{c}, \bar{b})\} \cup \{\neg \varphi(\bar{c})\}$ is inconsistent. Now, for each $b_i \in \bar{b}$, let $g_i(y_i, \bar{c})$ be the minimal polynomial of $b_i$ over the substructure generated by $\bar{c}$. Then

$$T \cup \{\neg \varphi(\bar{c})\} \cup \left\{ \psi(\bar{c}, \bar{b}) \land \bigwedge_{i=1}^{n} g_i(b_i, \bar{c}) = 0 \right\}$$
is inconsistent as well. Hence
\[ T \vdash \forall \bar{x}, \forall \bar{y} \left[ \left( \psi(\bar{x}, \bar{y}) \land \bigwedge_{i=1}^{n} g_i(y_i, \bar{x}) = 0 \right) \rightarrow \varphi(\bar{x}) \right], \]
or, equivalently,
\[ T \vdash \forall \bar{x} \left[ \exists \bar{y} \left( \psi(\bar{x}, \bar{y}) \land \bigwedge_{i=1}^{n} g_i(y_i, \bar{x}) = 0 \right) \rightarrow \varphi(\bar{x}) \right]. \]

Clearly \( \chi(\bar{x}) \) is a good formula and so \( \neg \chi(\bar{x}) \in \Gamma(\bar{x}) \). By construction \( M \models \chi(\bar{c}) \).

Since \( M \models \Gamma(\bar{c}) \), \( M \models \chi(\bar{c}) \) and \( M \models \neg \chi(\bar{c}) \), a contradiction. \( \square \)

By the claim and compactness, there are \( \neg \theta_1(\bar{x}), \ldots, \neg \theta_m(\bar{c}) \in \Gamma(\bar{c}) \) such that
\[ T \cup \{ \varphi(\bar{c}) \} \cup \{ \neg \theta_1(\bar{x}) \land \ldots \land \neg \theta_m(\bar{x}) \} \]
is inconsistent. Hence \( T \vdash \varphi(\bar{c}) \rightarrow \bigvee_{i=1}^{m} \theta_i(\bar{c}) \) and so
\[ T \vdash \forall \bar{x} \left( \varphi(\bar{x}) \rightarrow \bigvee_{i=1}^{m} \theta_m(\bar{x}) \right). \]

Since, for each \( 1 \leq i \leq m \), \( \theta_i(\bar{x}) \in \Gamma(\bar{x}) \),
\[ T \vdash \forall \bar{x} \left( \varphi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{m} \theta_m(\bar{x}) \right). \]

As a finite disjunction of good formulas is good, this completes the proof.

3. \( \Rightarrow 1. \) : Suppose that, modulo \( T \), every formula is equivalent to a good formula.

Let \( A \) be a relatively algebraically closed substructure of some model of \( M \models T \cup \text{qftp}(A) \). Let \( \varphi(\bar{a}) \) be arbitrary and suppose that \( M \models \varphi(\bar{a}) \). Let \( N \models T \) be arbitrary such that \( A \subseteq M \cap N \). By 3, there is a good formula \( \chi(\bar{a}) \) such that \( T \vdash \varphi(\bar{a}) \leftrightarrow \chi(\bar{a}) \).
Since $\chi(\bar{a})$ is a good formula, $M \models \chi(\bar{a})$ iff $A \models \chi(\bar{a})$. Since existential quantifiers go up, $N \models \chi(\bar{a})$. Finally, as $N \models T$, $N \models \varphi(\bar{a})$. Hence $T \cup \text{qftp}(A)$ is complete.

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**Lemma 2.0.15.** Let $T$ be a model complete theory of fields, $K \models T$, and $a \in K^{alg}$. Then for any $K' \succ K$, $\text{qftp}(a/K) \vdash \text{qftp}(a/K')$.

**Proof.** Let $a \in K^{alg}$. Then the type $\text{qftp}(a/K)$ is isolated by the formula $m_a(x) = 0$, where $m_a(x)$ is the minimal polynomial of $a$ over $K$. By model completeness, $m_a(x)$ is also the minimal polynomial of $a$ over $K'$, and hence $m_a(x) = 0$ isolates $\text{qftp}(a/K')$.

\begin{flushright}
\qed
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**Lemma 2.0.16.** Let $T$ be a theory of perfect fields with almost QE, $M \models T$, $A \subseteq M$ a relatively algebraically closed substructure and $p(\bar{x}) \in S(A)$ a non-algebraic type. For $\bar{e}$ a realization of $p(\bar{x})$, let $V_p$ be the absolutely irreducible $A$-variety for which $\bar{e}$ is generic. Then $p(\bar{x})$ is axiomatized by $\text{qftp}(\bar{e}/A)$ together with the set of formulas in $p(\bar{x})$ of the form

\[ \bar{x} \in V_p \land \exists y((\bar{x}, \bar{y}) \in W) \]

where $W$ is an absolutely irreducible $A$-variety projecting generically onto $V_p$ such that if $\bar{c} \in V_p$ and $(\bar{c}, \bar{d}) \in W$ then $\bar{d} \in A(\bar{c})^{alg}$.

**Proof.** We may assume $M$ is a big model and that $A$ is small and let $\bar{e} \in M$ be a realization of $p(\bar{x})$. Let $\varphi(\bar{x}) \in p(\bar{x})$. We may also assume that $V_p$ is a smooth, quasi-affine variety after removing the singular locus. By almost QE, we may assume that $\varphi(\bar{x})$ is equivalent to a formula of the form

\[ \bar{x} \in X \land \exists y_1, \ldots, y_n \left[ f(\bar{x}, \bar{y}) = 0 \land \bigwedge_{i=1}^{n} (g_i(\bar{x}, y_i) = 0) \right], \]

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where $X$ is an algebraic set defined over $A$ (as remarked earlier, the quantifier-free part in the formula above may be replaced by a single polynomial, but we have written it in this way to emphasize that realizations of the $y_i$’s are algebraic). Any realization $ar{c}$ of $p(\bar{x})$ is generic for $V_p$ and so $\dim(X) \geq \dim(V_p)$. After possibly replacing $\varphi(\bar{x})$ by the conjunction of $\varphi(\bar{x})$ and a finite number of formulas from qftp($\bar{\epsilon}/A$), we may assume that $X = U \subseteq V_p$, a Zariski-open subset of $V_p$. Therefore, any $\varphi(\bar{x}) \in p(\bar{x})$ is implied, modulo the quantifier-free part of $p(\bar{x})$, by a formula of the form

$$\bar{x} \in U \land \exists \bar{y}(g(\bar{x}, \bar{y}) = 0).$$

To finish proving the lemma, it suffices to show that we may replace “$g(\bar{x}, \bar{y}) = 0$” with an absolutely irreducible $A$-variety $W_i$ projecting finite-to-one onto $U$ (and so generically onto $V_p$). Let $X$ be the $A$-variety defined by $X = \{(\bar{x}, \bar{y}) : \bar{x} \in V_p \land g(\bar{x}, \bar{y}) = 0\}$. Note that the set $U' = \{(\bar{x}, \bar{y}) \in X : \bar{x} \in U\}$ is an open subset of $X$ that projects onto $U$. Let $X = Y_1 \cup \ldots \cup Y_m$ be the irreducible components of $X$ over $A^{alg}$ (note that the $Y_i$ are also irreducible over $M^{alg}$). For each $1 \leq i \leq m$, let

$$W_i = \bigcap_{\sigma \in \text{Aut}(M^{alg}/M)} \sigma(Y_i).$$

Observe that $X(M) = W_1(M) \cup \ldots \cup W_m(M)$ and that each $W_i$ is definable over $M$ (and since $M$ is perfect, each $W_i$ is defined over $M$). Discarding the $W_i$ such that $W_i(M) = \emptyset$, may assume that the $W_i(M)$ are all non-empty.

Claim 2.0.17. Each $W_i$ is irreducible over $M$.

Proof of Claim. Suppose otherwise. Then for some $i$, we may write $W_i = V \cup Z$, where $V$ and $Z$ are proper closed sub-varieties defined over $M$. Hence,

$$V = W_i \cap V$$
\[
= \left( \bigcap_{\sigma \in \text{Aut}(M^{\text{alg}}/M)} \sigma(Y_i) \right) \cap V \\
= \bigcap_{\sigma \in \text{Aut}(M^{\text{alg}}/M)} (\sigma(Y_i) \cap V) \\
= \bigcap_{\sigma \in \text{Aut}(M^{\text{alg}}/M)} (\sigma(Y_i) \cap \sigma(V)) \text{ (since } V \text{ is defined over } M.) \\
= \bigcap_{\sigma \in \text{Aut}(M^{\text{alg}}/M)} \sigma(Y_i \cap V).
\]

As \( V \neq \emptyset \), we have \( Y_i \cap V \neq \emptyset \). Since \( V \subsetneq W_i \), it must be the case that \( Y_i \cap V \subsetneq Y_i \), and so \( Y_i \cap V \) is a proper, closed subset of \( V \). Similarly, we have

\[
Z = \bigcap_{\sigma \in \text{Aut}(M^{\text{alg}}/M)} \sigma(Y_i \cap V),
\]

and so \( Y_i \cap Z \) is a non-empty, proper, closed subset of \( Y_i \). Finally,

\[
(Y_i \cap V) \cup (Y_i \cap Z) = (Y_i \cup (Y_i \cap Z)) \cap (V \cup (Y_i \cap Z)) \\
= Y_i \cap (V \cup Y_i) \cap (V \cup Z) = Y_i,
\]

which contradicts the fact that \( Y_i \) is absolutely irreducible. Hence \( W_i \) is irreducible over \( M \).

\[ \square \]

**Claim 2.0.18.** For each \( 1 \leq i \leq m \), \( W_i \) is definable over \( A \).

**Proof of Claim.** As \( X \) is defined over \( A \), each \( Y_i \) is definable over \( A^{\text{alg}} \). Let \( \sigma \in \text{Aut}(M^{\text{alg}}/M) \). Then \( \sigma \) permutes the \( Y_i \)'s. As \( \sigma \upharpoonright A^{\text{alg}} \in \text{Aut}(A^{\text{alg}}/A) \), we have that \( \sigma(Y_i) = \sigma \upharpoonright A^{\text{alg}} (Y_i) \) as varieties over \( A^{\text{alg}} \). On the other hand, if \( \tau \in \text{Aut}(A^{\text{alg}}/A) \), then, since \( A \) is relatively algebraically closed in \( M \), \( \tau \) extends to an automorphism
\[ \tau \in \Aut(M_{\text{alg}}/M) \text{ such that } \tau(Y_i) = \hat{\tau}(Y_i) \text{ as varieties over } A. \] This shows that

\[
W_i = \bigcap_{\sigma \in \Aut(M_{\text{alg}}/M)} \sigma(Y_i) = \bigcap_{\tau \in \Aut(A_{\text{alg}}/A)} \tau(Y_i)
\]

and so \( W_i \) is definable over \( A \). Since \( A \) is relatively algebraically closed in \( M \), and \( M \) is perfect, \( W_i \) is defined over \( A \).

At this point, we have that our original formula is implied by one of the form

\[
\bigvee_{i=1}^{m} \bar{x} \in U \land \exists \bar{y}(\bar{x}, \bar{y}) \in W_i,
\]

where each \( W_i \) is \( M \)-irreducible and projects generically finite-to-one onto \( U \subseteq V_p \). As \( p(\bar{x}) \) is complete, we may assume that \( m = 1 \), and so the original formula is implied by something of the form

\[
\bar{x} \in U \land (\bar{x}, \bar{y}) \in W.
\]

\textbf{Claim 2.0.19.} The variety \( W \) is absolutely irreducible.

\textit{Proof of Claim.} As \( W \) is \( M \)-irreducible, it suffices to show that \( W(M) \) contains a non-singular point. Since \( p(\bar{x}) \) is realized in \( M \) by some \( \bar{c} \) generic for \( V_p \) over \( A \), there is \( \bar{d} \in M \) such that \( (\bar{c}, \bar{d}) \in W \). By assumption, \( W \) projects finite-to-one onto \( U \) and so \( \dim(W) = \dim(U) = \text{tr. deg}(\bar{c}/A) \). It follows that \( (\bar{c}, \bar{d}) \) is generic for \( W \) and so must be non-singular. Thus, \( W \) is absolutely irreducible.

Note that for any open set \( O \subseteq V_p \), the set \( O' = \{(\bar{x}, \bar{y}) \in W : \bar{x} \in O\} \) is an open subset of \( W \). If \( (\bar{c}, \bar{d}) \in W \) is generic, \( (\bar{c}, \bar{d}) \in O' \), hence \( \bar{c} \in O \) and so \( \bar{c} \) is generic for
This completes the proof.

**Definition 2.0.20.** Say that a formula \( \varphi(\bar{x}) \) is very good if, modulo \( T \), \( \varphi(\bar{x}) \) is equivalent to quantifier-free formula or a formula of the form

\[
\bar{x} \in U \land \exists \bar{y}(\bar{x}, \bar{y}) \in W,
\]

where \( U \) is a Zariski-open subset of an absolutely irreducible variety \( V \) and \( W \) is an absolutely irreducible variety projecting finite-to-one onto \( U \).

**Remark 2.0.21.** Observe that the proof of Proposition \( \underline{2.0.16} \) gives, in particular, that for any \( p(\bar{x}) \in S_\lambda(A) \), where \( A \) is a relatively algebraically closed substructure of a model, if \( \varphi(\bar{x}) \in p(\bar{x}) \) is a finite conjunction of very good formulas, then \( \varphi(\bar{x}) \) is implied by a very good formula modulo \( T \) together with the quantifier-free part of \( p(\bar{x}) \).

This gives the following:

**Corollary 2.0.22.** Let \( T \) be a theory of perfect fields with almost QE in a language \( \mathcal{L} \). Then every \( \mathcal{L} \)-formula \( \varphi(\bar{x}) \) is equivalent to a finite disjunction of very good formulas.
and so $\neg \chi(\bar{c}) \in \Gamma(\bar{c})$. Since $M \models \Gamma(\bar{c})$, $M \models \neg \chi(\bar{c})$. However, we assumed that $M \models p(\bar{c})$ and so $M \models \chi(\bar{c})$, a contradiction.

By compactness, there are formulas $\neg \theta_1(\bar{x}), \ldots, \neg \theta_n(\bar{x}) \in \Gamma(\bar{x})$ such that

$$T \cup \text{qftp}(A) \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{n} \theta_i(\bar{x})),$$

as required. \[\square\]

**Example 2.0.23.** The following are examples of fields with almost QE:

1. Algebraically closed fields in the ring language.

2. Real closed fields in the ring language: every formula is equivalent modulo $RCF$ to a boolean combination of formulas of the form $f(\bar{x}) = 0$ and $\exists y(y^2 - f(\bar{x}) = 0)$, where $f(\bar{x})$ is a polynomial.

3. For $p$ prime, $\text{Th}(\mathbb{Q}_p)$ in the ring language: one gets quantifier elimination after adding a predicate for each $n$ expressing “there exists an $n$-th root”. Note that the relative algebraic closure of a discrete valued field in its completion is the henselization, so if $A \subseteq M \models \text{Th}(\mathbb{Q}_p)$ is relatively algebraically closed, then $A \preceq M$.

4. $T$ the theory of pseudofinite fields in the ring language expanded by constants $c_{n,i}$, for $n \in \mathbb{N}$ and $0 \leq i \leq n - 1$ interpreted such that for any $K \models T$, $P_n(x) := x^n + c_n, n - 1 x^{n-1} + \ldots + c_{n,1}x + c_{n,0}$ is irreducible over $K$ and a solution of $P_n(x) = 0$ generates the unique degree $n$ algebraic extension of $K$.

5. (Non-example) Let $T = \text{Th}((\mathbb{C}(t)))$ be the $L_{\text{rings}}$-theory of the field of formal Laurent series over $\mathbb{C}$. Then $T$ does not have almost QE.

The fact that $\mathbb{C}(t)$ does not have almost QE in the ring language follows from the following:

**Proposition 2.0.24.** Let $T$ be a complete theory of fields such that for every $M \models T$, there is some algebraically closed field $K \subseteq M$. If $T$ is model complete, then $T = \text{ACF}$.

**Proof.** This proposition follows immediately from the fact that ACF is the model companion to the theory of fields (in fact, the model completion, since choosing a
field fixes the characteristic). Alternatively, if $M \models T$, there is some $N \models T$ large enough such that there is an algebraically closed field $K \subseteq N$ with $|K| = |M|$. Hence $M$ embeds into $K$ as a field and, if $T$ is model complete, the embedding is elementary.

In [35], the authors prove that any field $K$ of characteristic $p > 0$ that is NIP contains a copy of $\mathbb{F}_p^{alg}$ as a subfield. From this, we get the following amusing corollary:

**Corollary 2.0.25.** Let $T$ be a complete $\mathcal{L}_{\text{rings}}$-theory of characteristic $p > 0$ fields that is NIP and model complete. Then $T = \text{ACF}_p$.

This is in contrast to the characteristic 0 case, where $\mathbb{R}$ and $\mathbb{Q}_p$ are examples fields whose theory is NIP and model complete in the ring language.

**Remark 2.0.26.** Note that, in particular, the above corollary applies to stable fields.

In [60], Poizat show that if a field $K$ has a stable theory, then either $K$ is separably closed or $K$ is unbounded, and so any characteristic $p$ counter-example to the stable fields conjecture (that all stable fields are separably closed) must be quite poorly behaved in some sense.

**Proposition 2.0.27.** Let $T$ be a theory of fields in the language of rings possibly expanded by constants. Suppose $T$ has the property that for any $M \models T$, if $A \leq M$ is a relatively algebraically closed substructure of $M$, then $A \preceq_\Sigma M$. Then $T$ has almost QE.

**Proof.** Let $M$ and $N$ be models of $T$ which are countable saturated. Suppose that $A \subseteq M$ and $B \subseteq N$ be relatively algebraically closed substructures and let $f : A \to B$ be an isomorphism of substructures. Let $c \in M \setminus A$, and let $\bar{c} \in A(c)^{alg} \cap M$. As $A$ is relatively algebraically closed in $M$, $\bar{c}$ is generic over $A$ for some absolutely irreducible $A$-variety $V$. Let $p(\bar{x}) = \text{qftp}(\bar{c}/A)$ and let $q(\bar{x})$ be the image of $p(\bar{x})$ under the map $f : A \to B$. By saturation, it suffices to show that $\text{Th}(N) \cup q(\bar{x})$ is
consistent. Let $\varphi(\overline{x}) \in q(\overline{x})$. Then there is $\psi(\overline{x}) \in p(\overline{x})$ such that $f(\psi(\overline{x})) = \varphi(\overline{x})$. We have that $M \models \psi(\overline{c})$. Since $A \preceq M$, there is $\bar{a} \in A$ such that $M \models \psi(\bar{a})$. Hence $N \models \varphi(f(\bar{a}))$. \hfill \Box

Example 2.0.28. Let $\mathcal{L}$ be the language of rings with a constant $t$. Then the $\mathcal{L}$-theory $\text{Th}(\mathbb{C}((t)))$ has almost QE. Thanks to Sylvy Anscombe for helping me with the following proof.

Proof. By a result of J. Robinson \cite{Robinson}, the valuation ring of $\mathbb{C}((t))$ is definable by the formula $\mathcal{O}(x) = \exists y[y^2 = 1 + tx^2]$. It follows that $\mathfrak{m}(x)$, the maximal ideal of $\mathcal{O}(x)$, is definable by the formula $\exists y(\mathcal{O}(y) \land x = ty)$. Let $(K, \mathcal{O}(x)) \succ (\mathbb{C}((t)), \mathcal{O}(x))$ and let $v_t(x)$ be the valuation corresponding to $\mathcal{O}(x)$. Choose $A \leq K$ be a relatively algebraically closed substructure.

Claim 2.0.29. $(A, \mathcal{O}(A))$ is henselian.

Proof of Claim. Let $f(x) \in \mathcal{O}(A)[x]$ be a polynomial and let $\bar{f}(x) \in (\mathcal{O}(A)/\mathfrak{m}(A))[x]$ be its reduction to the residue field. Suppose that $\bar{f}(x) = 0$ has a simple root $c$ in $\mathcal{O}(A)/\mathfrak{m}(A) \leq \mathcal{O}(K)/\mathfrak{m}(K)$. By Hensel’s Lemma \cite{Hensel}, there is a solution $\alpha$ of $f(x) = 0$ in $M$ which is unique with residue $c$. As $A$ is relatively algebraically closed in $K$, $\alpha \in A$. Hence $A$ is henselian. \hfill \Box

Claim 2.0.30. $\mathcal{O}(A) = \{x \in A : A \models \exists y(y^2 - tx^2 - 1 = 0)\}$ (and so $\mathcal{O}(A)$ is induced by the restriction of $v_t$ to $A$).

Proof of Claim. Since existential quantifiers go up,

$$\mathcal{O}(A) \supseteq \{x \in A : A \models \exists y(y^2 - tx^2 - 1 = 0)\}.$$ 

On the other hand, if $a \in \mathcal{O}(A)$, then $ta^2 \in \mathfrak{m}(A)$, so we have a polynomial $f(y) = y^2 - ta^2 - 1$ over $\mathcal{O}(A)$. The residue $\bar{f}(y) = y^2 - 1$ has a simple root $(y = 1, -1)$ in
\( O(A)/\mathfrak{m}(A) \) and so \( A \models \exists y(y^2 - ta^2 - 1 = 0) \) by Hensel’s Lemma (1.2.10). Therefore \( a \in \{ x \in A : A \models \exists y(y^2 - tx^2 - 1 = 0) \} \).

Claim 2.0.31. Let \( Kv_t := O(K)/\mathfrak{m}(K) \) be the residue field of \( K \) and let \( Av_t := O(A)/\mathfrak{m}(K) \) be the residue field of \( A \). Then \( Av_t^{\text{alg}} \cap Kv_t = Av_t \).

Proof of Claim. Let \( \alpha \in Kv_t \) be algebraic over \( Av_t \). Let \( P(x) \) be the minimal polynomial of \( \alpha \) over \( Av_t \). As \( Av_t \) is characteristic 0, \( P(x) \) is separable (i.e. \( \alpha \) is a simple root). Let \( f(x) \in O(A)[x] \) be an irreducible polynomial such that \( \tilde{f}(x) = P(x) \). By Hensel’s Lemma, there is unique \( c \in K \) such that \( f(c) = 0 \) and \( c \) has residue \( \alpha \in Kv_t \). Since \( A \) is relatively algebraically closed in \( K \), \( c \in A \). Hence \( \alpha \in Av_t \).

It follows that \( (A, O(x)) \) is a henselian valued field with algebraically closed residue field, and so \( Av_t \preceq Kv_t \). Let \( v_t(K) \models \text{Th}(\mathbb{Z}, +, 0, 1) \) be the value group of \( K \) and let \( v_t(A) \subseteq v_t(K) \) be the value group of \( A \). By AKE (1.2.11) and Proposition 2.0.27, it suffices to show that \( v_t(A) \preceq v_t(K) \). Note that since we have a symbol for \( t \) and \( v_t(t) = 1 \), the theory of the value group in the given language; \( v_t(K) \) is a model of Presburger arithmetic, and the divisibility predicates are universally definable by \( \neg P_n(x) \leftrightarrow \exists y \bigvee_{i=1}^{n-1} ny = x - i \).

Let \( \pi : Kv_t \to Av_t \) be a homomorphism onto \( Av_t \) such that \( \pi(a) = a \) for all \( a \in Av_t \). Such a homomorphism exists: define \( \pi \) on a transcendence basis of \( Kv_t \) over \( Av_t \) and extend. The place \( \pi \) corresponds to a valuation \( w \) on \( Kv_t \) that is trivial on \( Av_t \) with residue field isomorphic to \( Av_t \). Composition of places gives a new valuation \( u : K \to uK \) such that \( u \upharpoonright_A = v_t \upharpoonright_A \) and such that \( Ku = Av_t \) (so \( \pi \) is an \( Av_t \)-rational \( Av_t \)-place. By Lemma 3.7 of [45], \( u(K)/u(A) \cong u(K)/v_t(A) \) is torsion-free.

Claim 2.0.32. \( v_t(K)/v_t(A) \) is torsion-free.

Proof of Claim. Let \( \Delta = w(Kv_t) \), and so \( v_t \) is a coarsening of \( u \) with \( v_t(K) \cong u(K)/\Delta \). It follows that \( v_t(A) \cong u(A)/(u(A) \cap \Delta) \). Since \( Kv_t \) is algebraically closed,
\( \Delta \) is divisible (if \( \delta = w(a) \) for some \( a \in K v_t \), then any \( b \) such that \( b^n = a \) has \( w(b) = \delta/n \)). Suppose that \( a \in v_t(A) \) is such that \( a = mb \) for some \( m > 1 \) and some \( b \in v_t(K) \). Then \( a = \alpha + \Delta \) and \( b = \beta + \Delta \) for \( \alpha \in u(A) \) and \( \beta \in u(K) \) and so \( \alpha - m\beta \in \Delta \). Since \( \Delta \) is divisible, there is \( \delta \in \Delta \) such that \( \alpha - m\beta = m\delta \). Therefore \( \alpha = m(\delta + \beta) \) where \( \delta + \beta \in u(K) \). Since \( u(K)/u(A) \) is torsion-free, there is \( \gamma \in u(A) \) such that \( \alpha = m\gamma \). Thus \( \alpha + \Delta = m\gamma + \Delta \in v_t(A) \).

This implies that \( v_t(A) \models \text{Th}(\mathbb{Z},+,0,1) \), and so the result follows from model completeness of \( \text{Th}(\mathbb{Z},+,0,1) \).

**Corollary 2.0.33.** For any \( n \leq \omega \), the theory of \( C((t_1)) \ldots ((t_n)) \) has almost QE in the language of rings together with constants for \( t_1, \ldots, t_n \).

**Proof.** By Example 2.0.28 and induction, it suffices to prove the following claim:

**Claim 2.0.34.** If \( T_{n-1} := \text{Th}(C((t_1)) \ldots ((t_{n-1}))) \) has almost QE in the language \( L_{n-1} \) of rings expanded by constants for \( t_1, \ldots, t_{n-1} \), then \( T_n := \text{Th}(C((t_1)) \ldots ((t_n))) \) has almost QE in the language \( L_n := L_{n-1} \cup \{t_n\} \).

**Proof of Claim.** Let \( K \models T_n \). As in Example 2.0.28, \( K \) is a henselian valued field with respect to the valuation ring \( \mathcal{O}_n(x) = \exists y[y^2 = 1 + t_n x^2] \) with corresponding valuation \( v_n \) and residue field \( Kv_n \), a model of \( T_{n-1} \) (here we are being somewhat informal in that if \( \pi_n \) is the place corresponding to the \( t_n \)-adic valuation \( v_n \), then \( Kv_n \) is a model of \( T_{n-1} \) where the new constants are the definable elements \( \pi_n(t_i) \) for \( i = 1, \ldots, n \)). Let \( A \subseteq K \) be a relatively algebraically closed subfield containing the interpretations of \( t_1, \ldots, t_n \). Then \( \mathcal{O}_n(A) \) is a henselian valued subfield with residue field \( Av_n \) a relatively algebraically closed subfield of \( Kv_n \). By the induction hypothesis \( Av_n \not\preceq Kv_n \), and so by AKE, it is enough to show that \( v_n(A) \not\preceq v_n(K) \models \text{Th}(\mathbb{Z},+,0,1) \).
By induction, there is a valuation $w$ on $Kv_n$ such that $(Kv_n)w = (Av_n)w \models ACF_0$. Let $u$ be the valuation on $K$ obtained by composing (the corresponding places of) $v_n$ and $w$. By Lemma 3.7 of [45], $u(K)/u(A)$ is torsion-free.

Now, by the proof of Claim 2.0.32 with $v_n$ in place of $v_t$ and $\Delta := w(Kv_n)$, we have that $v_n(K)/v_n(A)$ is torsion free. 

This completes the proof. 

**Question 2.0.35.** For each $n$, set $K_n = \mathbb{C}((t_1)) \ldots ((t_n))$ and let $\iota : K_n \hookrightarrow K_{n+1}$ be the inclusion map. Let $K = \lim \rightarrow K_n$. Does $\text{Th}(K)$ have almost QE in some language extending the language of rings? A positive answer would give a counter-example to the conjecture that fields with almost QE are bounded, though we believe that the answer should be “no”.

**Remark 2.0.36.** The argument in Example 2.0.28 and Corollary 2.0.33 relies on the fact that for any pair of algebraically closed fields $A \leq K$, one can always find a non-trivial, henselian valuation on $K$ that is trivial on $A$ with residue field isomorphic to $A$. This limits the number of situations where one can extend the argument to fields of the form $K((t))$. For example, let $T$ be a theory of bounded, non-separably closed, PAC fields (which is therefore simple), and suppose that we have $A \preceq K \models T$. Suppose that there is a non-trivial, henselian place on $K$ that is trivial on $A$ and with residue field isomorphic to $A$. Since $A$ is not separably closed, a theorem in [30] implies that the corresponding valuation is $\emptyset$-definable, but this contradicts the fact that $T$ is simple. On the otherhand, it may be interesting to see if $\text{Th}(\mathbb{Q}_p((t)))$ has almost QE in the language of rings expanded by a constant for $t$.

**Proposition 2.0.37.** Let $\mathcal{L}$ be the language of rings possibly expanded by constants. If $T$ is an $\mathcal{L}$-theory of fields such that $T$ has almost QE and every model of $T$ is large, then for every $K \models T$ and every substructure $A \subseteq K$, we have $\text{acl}(A) = A^{\text{alg}} \cap K$. 

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Remark 2.0.38. It is important here that \( A \) is a substructure, and not just an arbitrary subset. For an arbitrary subset \( A \subseteq K \), we have \( \text{acl}(A) = \text{dcl}(A)^{\text{alg}} \cap K \).

Proof. Let \( K \models T \) and let \( A \subseteq K \) be a substructure. Without loss of generality, \( A^{\text{alg}} \cap K = A \). Clearly \( A \subseteq \text{acl}(A) \). Suppose \( c \in \text{acl}(A) \setminus A \). Let \( \varphi(x, \bar{y}) \) and \( \bar{a} \in A \) be such that \( \varphi(x, \bar{a}) \) isolates \( \text{tp}(c/A) \). By Proposition 2.0.13 there is a good formula \( \exists \bar{z} \psi(x, \bar{y}, \bar{z}) \) such that

\[ T \vdash \forall x, \bar{y} (\varphi(x, \bar{y}) \leftrightarrow \exists \bar{z} \psi(x, \bar{z}, \bar{y})) \]

Since \( \varphi(x, \bar{a}) \) isolates \( \text{tp}(c/A) \), it follows that for any \( c' \in K \), \( c' \equiv_A c \) iff \( K \models \exists \bar{z} \psi(c', \bar{z}, \bar{a}) \). Observe that the set \( C(K) \) defined in \( K \) by the quantifier-free formula \( \psi(x, \bar{z}, \bar{a}) \) is an affine algebraic set of dimension 1; that is, \( C(K) \) is the set of \( K \)-points of an open subset of an \( A \)-irreducible curve \( C \) over \( A \) (irreducibility follows from the fact that \( \text{tp}(c/A) \) is isolated by \( \varphi(x, \bar{a}) \)). As \( A \) is relatively algebraically closed, \( C \) is irreducible over \( K \). As \( C \) is a curve, \( C \) has only finitely many singular points, all of which are in \( C(A^{\text{alg}}) \). Let \( \bar{d} \in A(c)^{\text{alg}} \cap K \) be such that \( K \models \psi(c, \bar{d}, \bar{a}) \). Since \( (c, \bar{d}) \in C(\text{acl}(A)) \setminus C(A) \), tr.\( \deg(c, \bar{d})/A = 1 \) and hence \( c, \bar{d} \) is necessarily a non-singular point of \( C \). Recall that for any field \( K \), every \( K \)-irreducible \( K \)-variety with a non-singular \( K \)-point is absolutely irreducible. Since \( K \) is a large field, it follows that \( C(K) \) is dense in \( C \) (in particular, \( C(K) \) is infinite). Therefore, there are infinitely many \( c' \in K \) such that \( K \models \exists \bar{w} \psi(c', \bar{w}, \bar{a}) \), which contradicts the fact that \( \text{tp}(c/A) \) has only finitely many realizations.

In [33], the authors say a fields \( K \) is very slim if, for all \( L \equiv K \) and all \( A \subseteq L \), \( \text{acl}_L(A) = A^{\text{alg}} \cap L \), i.e. the model theoretic algebraic closure in \( L \) is the same as relative algebraic closure. In fact, they show that any large field that is model complete is very slim and so Proposition 2.0.37 is a special case of this result, though the assumption of almost QE allows for a somewhat simplified proof.
Remark 2.0.39. In [79], the authors define an $L$-theory $T$ to be finitely model complete iff for every $L$-formula $\varphi(\bar{x})$, there is a quantifier-free $L$-formula $\theta(\bar{x}, \bar{y})$ and integer $n \geq 0$ such that $T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \exists \bar{y}\theta(\bar{x}, \bar{y}))$ and $T \models \forall \bar{x} \exists^{\leq n}\bar{y}\theta(\bar{x}, \bar{y})$. By definition, if $M$ is a model of a finitely model complete theory $T$, and if $T \models \forall \bar{x} \exists^{\leq n}\bar{y}\theta(\bar{x}, \bar{y})$, then for any $\bar{a}, \bar{b} \in M$, if $M \models \theta(\bar{a}, \bar{b})$, then $\bar{b} \in \text{acl}(\bar{a})$. It is clear that a theory of fields with almost QE is finitely model complete, though it is not clear that the converse holds in general. However, if $T$ is a theory in of large fields in the language of rings (possibly expanded by constants) that is finitely model complete then the result of [33] implies that $T$ is very slim, and hence $T$ has almost QE.

Remark 2.0.40. Any pure field of characteristic 0 with a non-trivial henselian valuation is large and very slim [33] (so, for example, $C((t))$, even without a constant for $t$).

Recall that a theory $T$ is called pregeometric if acl satisfies the exchange principle, i.e. for all $M \models T$, $X \subset M$ and $a, b \in M$,

$$b \in \text{acl}(X, a) \setminus \text{acl}(X) \Rightarrow a \in \text{acl}(X, b).$$

The results in [33] imply that large, model complete fields are pregeometric. Assuming aQE, we can get a bit more.

Definition 2.0.41. A pregeometric theory $T$ is said to be geometric if $T$ eliminates $\exists^\infty$. That is, for every formula $\varphi(x, \bar{y})$ (where $x$ is a singleton), the set of $\bar{b} \in M \models T$ such that $\varphi(M, \bar{b})$ is infinite (equivalently, finite) is definable.

Proposition 2.0.42. Let $T$ be a theory of large, perfect fields with almost QE. Then $T$ eliminates $\exists^\infty$ (and so $T$ is geometric).

To prove this, we will use the following fact from [13]:
Fact 2.0.43 (The Intersection-Decomposition Procedure). Let $F$ be any field and let $V = \mathbb{V}(f_1, \ldots, f_r)$ be an algebraic set where $f_1, \ldots, f_r \in F[x_1, \ldots, x_n]$ are all of degree $\leq d$. Then there is an $F$-algebraic set $V^*$ and constants $D$, $M$, and $R$ depending only on $(d, n, r)$ (and not $F$ nor the $f_i$’s) such that:

1. $V^*(F) = V(F)$,
2. $V^*$ has at most $M$ absolutely irreducible components, and
3. every absolutely irreducible component of $V^*$ is of the form $\mathbb{V}(h_1, \ldots, h_R)$, where $h_1, \ldots, h_R \in F[x_1, \ldots, x_n]$ all have degree $\leq D$.

Proof of Proposition 2.0.42. Let $\varphi(x, \bar{y}, \bar{a})$ be some formula with parameters $\bar{a} \in M \models T$ and let $A = \text{dcl}_M(\bar{a})^{ag} \cap M$. Without loss of generality, $T \cup \text{qftp}(A) \cup \{\exists x, \bar{y}\varphi(x, \bar{y}, \bar{a})\}$ is consistent. By Corollary 2.0.22, $\varphi(x, \bar{y}, \bar{a})$ is equivalent to a formula of the form

$$\bigvee_{i=1}^n [(x, \bar{y}) \in U_i \land \exists \bar{z}(x, \bar{z}, \bar{y}) \in W_i],$$

a finite disjunction of very good formulas over $A$. By the proof of Corollary 2.0.22, each $U_i$ is an Zariski-open subset of some absolutely irreducible variety $V$, and so the union of the $U = \bigcup_i U_i$ is a Zariski-open subvariety of the same variety $V$.

For any tuple $\bar{b}$, $\varphi(x, \bar{b}, \bar{a})$ defines a finite set iff each of the disjuncts defines a finite set. Therefore, it suffices to show that for any very good formula $(x, \bar{y}) \in U \land \exists \bar{z}(x, \bar{z}, \bar{y}) \in W$, the set of $\bar{b}$ such that $(x, \bar{b}) \in U \land \exists \bar{z}(x, \bar{z}, \bar{b}) \in W$ is finite is definable. Since the projection $W \to U$ is finite-to-one, it suffices to show that the set $\{\bar{b} : W(x, \bar{z}, \bar{b}) \text{ is finite}\}$ is definable.

Let $X_\bar{b} \subseteq \mathbb{A}^{1+n}$ be the set defined by $W(x, \bar{z}, \bar{b})$. Then $X_\bar{b}(M)$ is infinite iff $X_\bar{b}^*$ has an absolutely irreducible component of dimension 1 that contains a smooth point (by largeness of $M$). By Fact 2.0.43, this is expressible by a formula depending only
on $n$, the number of polynomials defining $X_b$, and their degree.

As noted in [33], every very slim field that eliminates $\exists^\infty$ is algebraically bounded, in the sense of [78]. Thus we have:

**Corollary 2.0.44.** Every large, perfect field with almost QE is algebraically bounded.

**Remark 2.0.45.** In [78] the author shows that every henselian valued field is algebraically bounded without any QE assumptions.

**Example 2.0.46.** Some examples of geometric fields:

1. $\mathbb{R}$,
2. $\mathbb{Q}_p$,
3. any perfect PAC field ([12]).

### 2.1 Strongly Unbounded Substructures

In this section, we work in the generality of PAC substructures of models of a stable theory $T$ for which we assume the PAC property is first order, see Section 1.1.3 for the definitions and general terminology we will use here. A special case of PAC substructures of a stable structure are the PAC fields (considered as substructures of algebraically closed fields). The section represents work towards the conjecture that large fields with almost QE are bounded. In particular, after introducing a strong form of unboundedness, we show that, PAC fields with almost QE are not strongly unbounded as a special case of Proposition 2.1.2.

**Definition 2.1.1.** A substructure $P_0 \subseteq M$ is strongly unbounded if there is $M \supset P \supset P_0$ and $a \in acl(P)$ such that $tp_M(a/P)$ has a unique extension to $dcl(P, acl(P_0))$.

In the case of fields, $P_0$ is strongly unbounded (as a substructure of an algebraically closed field) if there is $P \supset P_0$ and $a \in P_{alg}$ such that the minimal polynomial of $a$ over $P$ is irreducible over $P_0_{alg}P$. Note that this implies that $P_0$ is unbounded.
For a field $P_0$ and a sufficiently saturated elementary extension $P$, if $P_0$ is unbounded, then $P_0^{alg}P/P$ galois implies that $P_0$ is strongly unbounded: let $Q(x)$ be an irreducible polynomial over $P$. If $P_0^{alg}P$ is galois over $P$, then either $Q(x)$ remains irreducible over $P_0^{alg}P$ or it splits into linear factors. Since $P_0$ is unbounded and $P$ is sufficiently saturated, $P_0^{alg}P \neq P^{alg}$ and hence there is some polynomial $Q(x)$ irreducible over $P$ that remains irreducible over $P_0^{alg}P$, and so $P_0$ is strongly unbounded.

We may rephrase this in terms of galois groups. Let $P_0$ be a field and $P$ a sufficiently saturated elementary extension. Suppose that $P_0^{alg}P/P$ is galois. Then the restriction map $\text{res} : \text{Gal}(P_0^{alg}P) \to \text{Gal}(P)$ is a continuous surjection of profinite groups. Suppose $H \leq \text{Gal}(P)$ is an open normal subgroup of index $n$. Then the preimage $\text{res}^{-1}(H)$ is an open normal subgroup with index $n$.

**Proposition 2.1.2.** Suppose that $K_0$ is unbounded field. Then $K_0$ has a finite extension $L/K_0$ that is strongly unbounded.

**Proof.** As $K_0$ is unbounded, there is an elementary extension $K$ and $a \in K^{alg}$ such that $a \notin \text{dcl}(K,K_0^{alg})$. Let $Q(x)$ be the minimal polynomial of $a$ over $\text{dcl}(K,K_0^{alg})$. There is some finite extension $L_0/K_0$ such that $Q(x)$ is defined over $\text{dcl}(K,L_0)$. Then $L_0$ is strongly unbounded: let $L = \text{dcl}(K,L_0)$ be a finite extension of $K$.

**Claim 2.1.3.** $L \succ L_0$.

**Proof of Claim.** Let $T = \text{Th}(K)$ and consider the canonical expansion to $T^{eq}$. Without loss of generality, assume that $L_0 = K_0(a)$ and that $\{1, a, \ldots, a^n\}$ is a basis for $L_0$ as a $K_0$ vector space. By assumption, $L = K[1, a, \ldots, a^n]$ as well. Let $\varphi(\bar{x})$ be a formula in the language of rings with parameters in $L_0$ and suppose that $\varphi(L)$ is non-empty. Let $S$ be the sort of $K^{eq}$ interpreting $L$ and let $f : K^n \to S$ be the canonical map. Then $f^{-1}(\varphi(\bar{x}))$ (with $\varphi(L)$ considered as a subset of $S$) is definable in $K^n$ with parameters from $K_0$. Since $K_0 \preceq K$, $f^{-1}(\varphi(\bar{x}))(K_0)$ is non-empty.
Since \( L_0 \) is interpreted in the same way as \( L \), \( \varphi(L_0) \) is non-empty. Hence \( L_0 \preccurlyeq L \) by Tarski-Vaught.

By the claim, \( L \succ L_0 \). We have that \( L^{alg} = K^{alg} \), and by construction the minimal polynomial of \( a \) over \( L \) is \( Q(x) \), which remains irreducible over \( dcl(K, K_0^{alg}) = dcl(L, L_0^{alg}) \).

**Proposition 2.1.4.** Suppose that \( P \) is PAC, \( P_0 \preccurlyeq P \preccurlyeq \overline{P} \) and suppose that for any \( P' \subseteq \overline{P} \), if \( \text{qftp}(P'/P_0) = \text{qftp}(P/P_0) \), then \( P' \) is relatively algebraically closed in \( P \). Then \( P \) is not strongly unbounded.

**Proof.** For a contradiction, suppose that \( P \) is strongly unbounded. We aim to find \( P' \subseteq \overline{P} \) such that \( \text{qftp}(P'/P_0) = \text{qftp}(P/P_0) \) and that is not relatively algebraically closed in \( \overline{P} \).

By definition, \( P \) being strongly unbounded means that there is \( a \in \text{acl}(P) \setminus P \) such that \( \text{qftp}(a/P) \) has a unique extension to \( \text{dcl}(P, \text{acl}(P_0)) \), say \( q(x) \) (and so \( q(x) = \text{qftp}(a/\text{dcl}(P, \text{acl}(P_0))) \)). It follows that \( q(x) \) is isolated by some formula \( \chi(x, \overline{c}) \) over \( P \).

**Claim 2.1.5.** The type \( \text{qftp}(a, P/P_0) \) is stationary (in the sense of forking in the ambient theory).

**Proof of Claim.** By Lemma 3.7 of [10], \( \text{qftp}(\overline{c}/P_0) \) is stationary, and so the unique non-forking extension \( \text{qftp}(\overline{c}/\text{acl}(P_0)) \) is definable over \( P_0 \).

Now, since \( \text{qftp}(\overline{c}/P_0) \) is stationary and definable over \( P_0 \), there is a \( P_0 \)-definable type \( \overline{\Sigma}(x, \overline{y}) \) over \( M \), given by the condition that, for any formula \( \varphi(x, \overline{y}, \overline{z}) \in L \), there is a formula \( d(\varphi)(\overline{z}) \in L(P_0) \) such that for any \( \overline{e} \in M \),

\[
\varphi(x, \overline{y}, \overline{e}) \in \overline{\Sigma}(x, \overline{y}) \iff M \models d(\varphi)(\overline{e}) \\
\iff \forall x(\chi(x, \overline{y}) \rightarrow \varphi(x, \overline{y}, \overline{e})) \in p(\overline{y}),
\]
where \( p(\bar{y}) \) is the unique non-forking extension of \( qftp(\bar{c}/P_0) \) to \( M \).

To see that \( qftp(a, \bar{c}/P_0) \) is stationary, it suffices to show that \( (a, \bar{c}) \models \Sigma(x, \bar{y}) | P_0 \).

Let \( \varphi(x, \bar{y}, \bar{z}) \in L \) be any formula, and let \( \bar{e} \in P_0 \) be such that \( \varphi(x, \bar{y}, \bar{e}) \in \Sigma(x, \bar{y}) | P_0 \).

Then, by definition,

\[
\forall x(\chi(x, \bar{y}) \rightarrow \varphi(x, \bar{y}, \bar{e})) \in q(\bar{y}) | P_0 = qftp(\bar{c}/P_0).
\]

Therefore,

\[
M \models \forall x(\chi(x, \bar{c}) \rightarrow \varphi(x, \bar{c}, \bar{e})).
\]

Since \( M \models \chi(a, \bar{c}) \), it follows that \( M \models \varphi(a, \bar{c}, \bar{e}) \). Hence \( qftp(a, \bar{c}/P_0) \) is stationary.

By the claim and the fact that \( \mathcal{P} \) is PAC, \( qftp(a, P/P_0) \) is realized in \( \mathcal{P} \) by some \( a', P' \). Now \( qftp(P'/P_0) = qftp(P/P_0) \), but \( a' \in \mathcal{P} \cap (P')^{alg} \setminus P' \).

Remark 2.1.6. Here is a proof of the claim in the special case of fields:

It suffices to show that for any finite tuple \( \bar{c} \in P \), \( qftp(a, \bar{c}/P_0) \) is stationary. Without loss of generality, we may assume that \( m_a(x) = m_a(x, \bar{c}) \), the minimal polynomial of \( a \) over \( P \), is defined over \( dcl(P_0, \bar{c}) \). Consider the set of formulas \( \Sigma(x, \bar{y}) \) over \( P_0^{alg} \) given by the property that for any \( \varphi(x, \bar{y}) \in L(P_0^{alg}), \)

\[
\varphi(x, \bar{y}) \in \Sigma(x, \bar{y}) \leftrightarrow \varphi(x, \bar{c}) \in qftp(a/ dcl(P, P_0^{alg})) = q(x).
\]

The set \( \Sigma(x, \bar{y}) \) is complete: If \( \varphi(x, \bar{y}) \in L(P_0^{alg}) \setminus \Sigma(x, \bar{y}) \), then by definition \( \varphi(x, \bar{c}) \notin q(x) \); since \( q(x) \) is complete, \( \neg \varphi(x, \bar{c}) \in q(x) \) and so \( \neg \varphi(x, \bar{y}) \in \Sigma(x, \bar{y}) \).

Therefore, \( \Sigma(x, \bar{y}) \) is a complete type over \( P_0^{alg} \) that extends \( qftp(a, \bar{c}/P_0) \). We
claim that $\Sigma(x, \bar{y})$, does not fork over $P_0$. Observe that $\text{qftp}(\bar{c}/P_0)$ is stationary with $\text{qftp}(\bar{c}/P_0^{alg})$ the unique non-forking extension to $P_0^{alg}$. Since for any polynomial $g(\bar{y}) \in P_0^{alg}[\bar{y}]$, $(g(\bar{y}) = 0) \in \Sigma(x, \bar{y})$ iff $g(\bar{c}) = 0$, we have that $\Sigma(x, \bar{y})$ contains $\text{qftp}(\bar{c}/P_0^{alg})$. If there were a formula $\varphi(x, \bar{y}) \in \Sigma(x, \bar{y})$ which forks with $\text{qftp}(a, \bar{c}/P_0)$, then the the ideal $I \subseteq P_0^{alg}[\bar{y}]$ corresponding to the $P_0^{alg}$-Zariski closure of $\exists x \varphi(x, \bar{y})$ has a basis $\{g_1(\bar{y}), \ldots, g_m(\bar{y})\}$ such that $[\bigwedge_i g_i(\bar{y}) = 0] \in \Sigma(x, \bar{y})$ forks with $\text{qftp}(\bar{c}/P_0)$, which is impossible.

All that’s left to show is that $\Sigma(x, \bar{y})$ is the unique non-forking extension of $\text{qftp}(a, \bar{c}/P_0)$. Suppose that $\varphi(x, \bar{y}) \in L(P_0^{alg}) \setminus \Sigma(x, \bar{y})$. Then, by definition, $\varphi(y, \bar{c}) \notin \text{qftp}(a/\text{dcl}(P, P_0^{alg}))$. Since $\text{qftp}(a/\text{dcl}(P, P_0^{alg}))$ is isolated by “$m_a(x, \bar{c}) = 0$”, we have that

$$P_0^{alg} \models \forall x(m_a(x, \bar{c}) = 0 \to \neg \varphi(x, \bar{c})),$$

or, equivalently,

$$P_0^{alg} \models \neg \exists x(m_a(x, \bar{c}) = 0 \land \varphi(x, \bar{c})).$$

Therefore, $\exists x(m_a(x, \bar{y}) = 0 \land \varphi(x, \bar{y})) \notin \text{qftp}(\bar{c}/P_0^{alg})$ and so

$$\text{qftp}(\bar{c}/P_0) \cup \{\exists x(m_a(x, \bar{y}) = 0 \land \varphi(x, \bar{y}))\}$$

forks over $P_0$. Since $\text{qftp}(\bar{c}/P_0) \subseteq \text{qftp}(a, \bar{c}/P_0)$,

$$\text{qftp}(a, \bar{c}/P_0) \cup \{\exists x(m_a(x, \bar{y}) = 0 \land \varphi(x, \bar{y}))\}$$

forks over $P_0$. Finally, since $(m_a(x, \bar{y}) = 0) \in \text{qftp}(a, \bar{c}/P_0)$,

$$\text{qftp}(a, \bar{c}/P_0) \cup \{\varphi(x, \bar{y})\}$$
forks over $P_0$. Therefore, if $p(x, \bar{y}) \neq \Sigma(x, \bar{y})$ is any other extension of $qftp(a, \bar{c}/P_0)$ to $P_0^{alg}$, then there is $\varphi(x, \bar{y}) \in p(x, \bar{y})$ such that $\varphi(x, \bar{y}) \notin \Sigma(x, \bar{y})$, and so $qftp(a, \bar{c}/P_0) \cup \{\varphi(x, \bar{y})\}$ forks over $P_0$.

Note that in the case of fields, if $K_0$ is a field such that $\text{Th}(K_0)$ has almost QE, then for any $K \succ K_0$ and any $K_0 \subseteq F \subseteq K$, if $F$ is relatively algebraically closed in $K$, then any $F'$ realizing $qftp(F/K_0)$ is relatively algebraically closed in $K$. Thus we have the following corollary:

**Corollary 2.1.7.** Let $K$ be a field that is strongly unbounded. Then the $\mathcal{L}_{\text{ring}}$-theory $\text{Th}(K)$ does not have almost QE.

**Question 2.1.8.** Let $K$ be a field such that $\text{Th}(K)$ has almost QE in a language $\mathcal{L}$. Let $L/K$ be a finite extension of $K$ such that $K$ is an $\mathcal{L}$-substructure of $L$. Does $L$ have almost QE?

**Remark 2.1.9.** It seems unlikely that the answer to this question is yes in general. Suppose the answer is yes. Then if $K$ has almost QE, then every finite extension of $K$ has almost QE. Hence there is no finite extension of $K$ that is strongly unbounded. It follows that $K$ is bounded, since every unbounded field interprets a strongly unbounded field.

This question may be true for particular theories of fields, for example, [17], where the authors prove model completeness for finite extensions of $p$-adic fields in the language of rings.
3.1 Notions of Differential Largeness

In this section, we wish to explore analogues of large in the differential context. In [70], the authors propose the following definition:

**Definition 3.1.1.** A differential field \((K, \partial)\) is called differentially large if, for any \(K\)-irreducible differential variety \(V\), if, for infinitely many \(n \geq 0\), the variety \(\text{Jet}_n(V)\) contains a Zariski-smooth \(K\)-point, then \(V(K)\) is Kolchin dense in \(V\).

It is easy to see that if \((K, \partial)\) is differentially large, then \(K\) is large as a field.

**Proposition 3.1.2.** For a differential field \((K, \partial)\), the following are equivalent:

1. \((K, \partial)\) is differentially large,

2. \((K, \partial)\) is large as a field, and for every absolutely irreducible variety \(V\) over \(K\) and every absolutely irreducible variety \(W \subseteq \tau(V)\) projecting generically onto \(V\), and any Zariski-open \(U \subseteq W\), if \(V(K)\) contains a smooth point, then there is \(\bar{a} \in V(K)\) such that \((\bar{a}, \partial(\bar{a})) \in W(K)\),

3. \(K\) is large as a field and for every absolutely irreducible variety \(V\) over \(K\) and every rational section \(s : V \to \tau(V)\) of the projection of \(\tau(V)\) onto \(V\), if \(V(K)\) contains a smooth point, then for every Zariski open \(U \subseteq V\) defined over \(K\) on which \(s(\bar{x})\) is defined, there is \(\bar{a} \in U(K)\) such that \(s(\bar{a}) = (\bar{a}, \partial(\bar{a}))\),

4. \(K\) is large as a field and for every absolutely irreducible curve \(C\) over \(K\) and every rational section \(s : C \to \tau(C)\) of the projection of \(\tau(C)\) onto \(C\), if \(C(K)\) contains a smooth point, then for every Zariski open \(U \subseteq C\) defined over \(K\) on which \(s(\bar{x})\) is defined, there is \(\bar{a} \in U(K)\) such that \(s(\bar{a}) = (\bar{a}, \partial(\bar{a}))\).
Proof. The equivalence of 1., 2., and 3. can be found in [70]. The fact that 3. implies 4. is trivial. The last implication follows from the fact that for any irreducible $K$-variety $V \subseteq \mathbb{P}^n$, and any finite set of points $P \subseteq V(K_{\text{alg}})$, there is an irreducible $K$-curve $C \subseteq V$ passing though every point in $P$ and such that if $\bar{x} \in P$ is smooth for $V$, then it is smooth for $C$ as well. Furthermore, if $C \subseteq V$ is such a curve, and $s : V \to \tau(V)$, then $s |_{C} : C \to \tau(C) \subseteq \tau(V)$. \hfill \box

Next, we show that it is sufficient to consider just plane curves.

**Proposition 3.1.3.** Let $(K, \partial)$ be a differential field. Then $K$ is differentially large iff $K$ is large as a field and, for every absolutely irreducible plane curve $C \subseteq \mathbb{A}^2$ and every rational section $s : C \to \tau(C)$ of the natural projection of $\tau(C)$ onto $C$, if $C(K)$ contains a smooth point, then for every Zariski-open $U \subseteq C$ defined over $K$, there is $\bar{a} \in U(K)$ such that $s(\bar{a}) = (\bar{a}, \partial(\bar{a}))$.

Proof. The forward direction is trivial. Let $C \subseteq \mathbb{A}^n$ be an arbitrary absolutely irreducible $K$-curve and suppose that $s : C \to \tau(C)$ be a $K$-rational section of the natural projection. By general algebraic geometry, we may assume that $C$ is birationally equivalent to an absolutely irreducible plane $K$-curve $C' \subseteq \mathbb{A}^2$ where the birational map $\pi : C \to C'$ is simply the projection onto the first two coordinates. For $\bar{x} = (x_1, \ldots, x_n)$, write

$$s(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_n, s_1(\bar{x}), \ldots, s_n(\bar{x})).$$

Recall that, by the definition of birational map, we have a map $f : C' \to C$, which we will write as

$$f(x_1, x_2) = (x_1, x_2, h_3(x_1, x_2), \ldots, h_n(x_1, x_2)).$$
Observe that the projection $\pi : C \to C'$ induces a map $d\pi : \tau(C) \to \tau(C')$ given by

$$d\pi(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, x_2, y_1, y_2).$$

The composition $s' := d\pi \circ s \circ f : C' \to \tau(C')$ is clearly a rational section of the projection $\tau(C') \to \tau(C)$:

$$s'(x_1, x_2) = (x_1, x_2, s_1(f(x_1, x_2)), s_2(f(x_1, x_2))).$$

Suppose that $C(K)$ contains a smooth point of $C$. By largeness, $C(K)$ is dense in $C$ and so the image $\pi(C(K))$ is dense in $C'$. Let $U \subseteq C$ be any Zariski-open subset defined over $K$. Take $U' \subseteq C'$ to be any Zariski-open subset, defined over $K$, such that $f(U')$ is a subset of $U$. By assumption, there is $(a_1, a_2) \in U'(K)$ such that $s'(a_1, a_2) = (a_1, a_2, \partial(a_1), \partial(a_2))$. Recall that for a rational map $h(\bar{x})$ we can define the differential $dh(\bar{x}, \bar{y})$ to be the map such that for all $\bar{a} \in K$, $\partial(h(\bar{a})) = dh(\bar{a}, \partial(\bar{a}))$. It is easy to see that if $h(\bar{x})$ is a polynomial map, then

$$dh(\bar{x}, \bar{y}) = h^0(\bar{x}) + \sum_{i=1}^{n} \frac{dh}{dx_i} y_i,$$  \hfill (3.1)

where $h^0(\bar{x})$ is the polynomial obtained by applying $\partial$ to the coefficients of $h(\bar{x})$. If $h(\bar{x})$ is rational, $dh(\bar{x})$ still exists, though may not have such a nice form.

Consider the differential $df : \tau(C') \to \tau(C)$ induced by $f : C' \to C$,

$$df(x_1, x_2, y_1, y_2) = (x_1, x_2, h_3(x_1, x_2), \ldots, h_n(x_1, x_2), \ldots, y_1, y_2, dh_3(x_1, x_2, y_1, y_2), \ldots, dh_n(x_1, x_2, y_1, y_2)).$$
By construction of the differential map,

\[ df(a_1, a_2, \partial(a_1), \partial(a_2)) = (a_1, a_2, h_3(a_1, a_2), \ldots, h_n(a_1, a_2), \ldots) \]

\[ \ldots \partial(a_1), \partial(a_2), \partial(h_3(a_1, a_2)), \ldots, \partial(h_n(a_1, a_2))) \]  

As \( f : C' \to C \) is a birational equivalence, we get that \( df : \tau(C') \to \tau(C) \) is a birational equivalence as well. Observe now that the map \( s' : C' \to \tau(C') \) was chosen so that \( s' = d\pi \circ s \circ f \) and so, on an open subset of \( C \), we have that \( s = df \circ s' \circ \pi \).

Letting \( \bar{b} := (a_1, a_2, h_3(a_1, a_2), \ldots, h_n(a_1, a_2)) \), it follows that

\[ s(\bar{b}) = (\bar{b}, \partial(\bar{b})) \]

as required.

We will write \( (K, \partial) \models \text{DiLF} \) (for “Differentially Large Fields”) if \( (K, \partial) \) satisfies any of the equivalent axiom schemas for differentially large fields. We wish to compare this definition to a stronger notion of differential largeness. First, we suggest a definition of smoothness for a point of a differential variety.

For a tuple \( \bar{x} = (x_0, \ldots, x_m) \) we define a \( K \)-algebra homomorphisms

\[ \rho : K \{ \bar{x} \} \to K[x_i^j : 0 \leq i \leq m, 0 \leq j \leq \omega] \]

\[ \partial^{j}(x_i) \mapsto x_i^j \]

**Definition 3.1.4.** Let \( (K, \partial) \) be a differential field, and let \( V \subseteq K^m \) be a \( K \)-irreducible affine differential variety and let \( \mathcal{I}(V) \) be the differential ideal vanishing on \( V \). We say a point \( \bar{a} \in V(K) \) is Kolchin-smooth if, for every \( n \geq 0 \), the prolongation \( \nabla_n(\bar{a}) \) is a Zariski smooth point of the algebraic variety \( \text{Jet}_n(V) := V(I_n) \), where \( I_n := \rho(\mathcal{I}(V)) \cap K[x_i^j : 0 \leq i \leq m, 0 \leq j \leq n] \).
Proposition 3.1.5. Let $\mathcal{V} \subseteq \mathbb{A}^m$ be a $K$-irreducible affine differential variety let $\mathcal{I}(\mathcal{V}) \subseteq K\{\bar{x}\}$ be the radical differential ideal vanishing on $\mathcal{V}$. Suppose that $\mathcal{I}(\mathcal{V})$ is generated as a differential ideal by a finite subset of $K[\bar{x},\partial(\bar{x}),\ldots,\partial^r(\bar{x})]$. If there is $\bar{a} \in \mathcal{V}(K)$ such that $\nabla_r(\bar{a}) = (\bar{a},\partial(\bar{a}),\ldots,\partial^r(\bar{a}))$ is Zariski smooth for $\text{Jet}_r(\mathcal{V})$, then $\nabla_n(\bar{a})$ is Zariski smooth for $\text{Jet}_n(\mathcal{V})$ for all $n \geq r$.

Proof. We proceed by induction on $n$. Observe that the map

$$(\bar{x}_0,\bar{x}_1,\ldots,\bar{x}_n,\bar{x}_{n+1}) \mapsto (\bar{x}_0,\bar{x}_1,\ldots,\bar{x}_n,\bar{x}_1,\ldots,\bar{x}_n,\bar{x}_{n+1})$$

gives an isomorphism (of varieties) from $\text{Jet}_{n+1}(\mathcal{V})$ to $\tau(\text{Jet}_n(\mathcal{V}))$. Without loss of generality, we may assume that $\text{Jet}_n(\mathcal{V})$ is smooth, since we may just remove the singular locus. It suffices to show that $\tau(\text{Jet}_n(\mathcal{V}))$ is smooth. The proof of this statement can be found more generally in (1.4) of [49], however we will reproduce the proof here for the sake of clarity: let $F_1,\ldots,F_s \in K[x_1,\ldots,x_t]$ (in this case, $t = (n+1)m$) generate the radical ideal of polynomials vanishing on $\text{Jet}_n(\mathcal{V})$. Then the radical ideal of polynomials vanishing on $\tau(\text{Jet}_n(\mathcal{V}))$ is generated by $F_1,\ldots,F_s$, together with the set

$$\left\{ F_j^\partial(x_1,\ldots,x_t) + \sum_{i=1}^t \frac{dF_j(x_1,\ldots,x_t)}{dx_i} y_i : 1 \leq j \leq s \right\} \subset K[x_1,\ldots,x_s,y_1,\ldots,y_s]$$

(recall that $F_i^\partial$ is the polynomial obtained by applying $\partial$ to the coefficients of $F_i$).

Let $(\bar{a},\bar{b}) = (a_1,\ldots,a_t,b_1,\ldots,b_t)$ be a point of $\tau(\text{Jet}_n(\mathcal{V}))$. By assumption, $\bar{a}$ is a
smooth point of $\text{Jet}_n(\mathcal{V})$. The Jacobian $J_{\tau(\text{Jet}_n(\mathcal{V}))}(\vec{a}, \vec{b})$ is given by

$$
\begin{pmatrix}
\frac{dF_1(a)}{dx_1} & \cdots & \frac{dF_1(a)}{dx_t} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{dF_s(a)}{dx_1} & \cdots & \frac{dF_s(a)}{dx_t} & 0 & \cdots & 0
\end{pmatrix}
$$

which has rank at most $2t - 2 \dim(\text{Jet}_n(\mathcal{V}))$ (since $\dim(\tau(\text{Jet}_n(\mathcal{V}))) = 2 \dim(\text{Jet}_n(\mathcal{V}))$).

Since $\vec{a}$ is smooth for $\text{Jet}_n(\mathcal{V})$, the submatrix

$$
\begin{pmatrix}
\frac{dF_1(a)}{dx_1} & \cdots & \frac{dF_1(a)}{dx_t} \\
\vdots & \ddots & \vdots \\
\frac{dF_s(a)}{dx_1} & \cdots & \frac{dF_s(a)}{dx_t}
\end{pmatrix}
$$

(which is the Jacobian of $\text{Jet}_n(\mathcal{V})$) has rank $t - \dim(\text{Jet}_n(\mathcal{V}))$ and so the Jacobian of $\tau(\text{Jet}_n(\mathcal{V}))$ has rank at least $2t - 2 \dim(\text{Jet}_n(\mathcal{V}))$ and hence $(\vec{a}, \vec{b})$ is smooth for $\tau(\text{Jet}_n(\mathcal{V}))$. \qed

Remark 3.1.6. For an irreducible differential variety $\mathcal{V}$ defined over a differential field $(K, \partial)$, an alternative (likely weaker) notion of smoothness could be an “eventual smoothness”; that is, say $\vec{a} \in \mathcal{V}(K)$ is eventually Kolchin-smooth if for some (rather than for all) $n \geq 0$, $\nabla(\vec{a})$ is Zariski-smooth for $\text{Jet}_n(\mathcal{V})$. One issue with this definition is that it is not clear that an irreducible differential variety $\mathcal{V}$ defined over $K$ with an eventually smooth $K$-point should be absolutely irreducible, which is a property one would expect for a reasonable notion of smoothness.

Definition 3.1.7. A differential field $(K, \partial)$ is called weakly differentially large if, for any $K$-irreducible differential variety $\mathcal{V}$, if $\mathcal{V}(K)$ contains a Kolchin smooth point,
then $V(K)$ is Kolchin dense in $V$.

Remark 3.1.8. Let $(K, \partial)$ be a differential field. If $V$ is an absolutely irreducible $K$-variety then, by Kolchin’s irreducibility theorem, $V$ is irreducible as a Kolchin-closed set as well. It is immediate from the definition of Kolchin-smooth that a Kolchin-smooth point of $V$ is Zariski-smooth, and Proposition 3.1.5 shows conversely that a Zariski-smooth point of $V$ is Kolchin-smooth for $V$ considered as a differential variety. It follows that if $(K, \partial)$ is a weakly large differential field, then $K$ is large in the usual sense.

We note that the existence of a Kolchin smooth $K$-point in a differential $(K, \partial)$-variety $V$ implies that $V$ is absolutely irreducible (as in the case of classical varieties):

**Proposition 3.1.9.** Let $V$ be a $K$-irreducible differential variety over an arbitrary differential field $(K, \partial)$. Suppose that $V(K)$ contains a Kolchin-smooth point. Then $V$ is irreducible over any differential closure of $(K, \partial)$.

**Proof.** Suppose $V$ is a differential variety defined over $(K, \partial)$ and we know that $V$ is $(K, \partial)$-irreducible. Suppose there is $\bar{c}$ in $V(K)$ that is Kolchin-smooth. Let $(U, \partial)$ be any differential closure of $(K, \partial)$. Then $V$ is $(U, \partial)$-irreducible iff the differential function field, $U\langle V \rangle$, is an integral domain. Let $\mathcal{J} = \mathcal{J}(V/U) \subseteq U\{\bar{x}\}$ be the radical differential ideal corresponding to $V$ considered as a differential variety over $U$.

Suppose $V$ is not $U$-irreducible. Write $V = V_1 \cup \ldots \cup V_n$ as a unique, irredundant union of Kolchin closed sets defined over $U$. We proceed as in the classical case: let $G := \text{Aut}_\partial(U/K)$ be the group of differential field automorphisms of $(U, \partial)$ that fix $(K, \partial)$. Then $G$ acts transitively on the set $\{V_1, \ldots, V_n\}$, since otherwise the union of the orbits of each component would give $V$ as a union of proper Kolchin-closed subsets defined over $(K, \partial)$, which is impossible since we assumed that $V$ was irreducible over $(K, \partial)$. Since $\bar{c}$ is in $V(K)$, $\bar{c}$ is fixed by $G$ and so $\bar{c}$ is in $V_i(K)$ for every $i$. 83
Let

\[ O_{\bar{c}} := \left\{ \frac{f}{g} : f, g \in U\langle V \rangle \text{ and } g(\bar{c}) \neq 0 \right\}, \]

be the localization of \( U\langle V \rangle \) at the prime ideal of elements vanishing on \( \bar{c} \). Observe that \( O_{\bar{c}} \) can be written as \( \lim \rightarrow O_{\bar{c},n} \), where

\[ O_{\bar{c},n} := \{ f/g : f, g \in U[\bar{x}, \ldots, \partial^n(\bar{x})]/(J \cap U[\bar{x}, \ldots, \partial^n(\bar{x})]) \text{ and } g(\bar{c}, \ldots, \partial^n(\bar{c})) \neq 0 \}, \]

(the connecting maps are those induced by the inclusion

\[ U[\bar{x}, \ldots, \partial^n(\bar{x})] \hookrightarrow U[\bar{x}, \ldots, \partial^n(\bar{x}), \ldots, \partial^m(\bar{x})] \]

for \( m \geq n \)).

By assumption, \( \bar{c} \) is Kolchin-smooth, and so (by algebraic geometry) \( O_{n,\bar{c}} \) is an integral domain. It is a fact that the direct limit of integral domains is again an integral domain, and so \( O_{\bar{c}} \) is an integral domain.

Now, consider the embedding of \( U \)-algebras \( U\langle V \rangle \hookrightarrow O_{\bar{c}} \) given by \( f \mapsto f/1 \). As \( O_{\bar{c}} \) is an integral domain, the proposition will follow from showing that this map is injective. Suppose that for some \( f \in U\langle V \rangle \), we have \( f/1 = 0 \). We will show that \( f = 0 \). By the definition of localization, there must be some \( g \in U\langle V \rangle \) with \( g(\bar{c}) \neq 0 \) and such that \( fg = 0 \). Consider \( X = V_i \), an arbitrary irreducible component of \( V \). Then \( fg \mid_X = 0 \). Since \( \bar{c} \) is in every component of \( V \) and since \( g(\bar{c}) \neq 0 \), \( g \) does not vanish on all of \( X \) and so the set \( \{ \bar{x} \in X : g(\bar{x}) \neq 0 \} \) is a Kolchin-open (hence Kolchin dense) subset of \( X \). Since \( fg = 0 \) on all of \( X \), \( f \) must vanish on a Kolchin dense subset of \( X \) and so \( f \mid_X = 0 \). Since every component of \( V \) contains \( \bar{c} \), \( f \) vanishes on all of \( V \), hence \( f = 0 \). Therefore, \( U\langle V \rangle \) is an integral domain and so \( V \) is irreducible as a \( (U, \partial) \)-variety. \( \square \)
Proposition 3.1.10. Let $(K, \partial) \models \text{DiLF}$. Then $(K, \partial)$ is weakly differentially large.

Proof. Suppose that $(K, \partial) \models \text{DiLF}$. Let $\mathcal{V}$ be a $K$-irreducible differential variety, and suppose $\bar{a} \in \mathcal{V}$ is Kolchin smooth. Then by definition of Kolchin smooth, $(\bar{a}, \ldots, \partial^n(\bar{a}))$ is smooth for $\text{Jet}_n(\mathcal{V})$ for every $n$, and hence $\mathcal{V}(K)$ is dense in $\mathcal{V}$. □

Remark 3.1.11. Let $(K, \partial)$ be a differential field. Suppose that for every $K$-irreducible differential variety $\mathcal{V}$ defined over a tuple $\bar{a}$, there was a formula $\varphi_r(\bar{x})$ such that $\models \varphi_r(\bar{a})$ iff for any basis $\mathcal{B}$ that differentially generate $\mathcal{I}(\mathcal{V}/\bar{a})$, $r$ is the greatest integer such that $\partial^r$ appears in an element of $\mathcal{B}$. Then by Proposition 3.1.5, the class of weakly differentially large fields would be an elementary class, since one would only need to check that $(\bar{a}, \ldots, \partial^i(\bar{a}))$ is smooth for $\text{Jet}_i(\mathcal{V})$ for $i \leq r$ to know that $\bar{a} \in \mathcal{V}$ is Kolchin smooth.

Let $(K, \partial)$ be a differential field. Let

$$K((t)) = \left\{ \sum_{i=-n}^{\infty} a_i t^i : a_i \in K, n \in \mathbb{N} \right\}$$

be the field of formal Laurent series over $K$. We may embed $(K, \partial)$ into $K((t))$ with the so-called “Taylor embedding”:

$$\rho : a \mapsto \sum_{n=0}^{\infty} \frac{\partial^n(a)}{n!} t^n.$$ 

Equipping $K((t))$ with the differential $\partial_t = \frac{d}{dt}$, $\rho : (K, \partial) \hookrightarrow (K((t)), \partial_t)$ becomes an embedding of differential fields. Furthermore, the standard $K$-place $\pi : K[[t]] \to K$ sending $t \mapsto 0$ is a differential homomorphism such that $\pi \circ \rho = \text{id}_K$. The $K$-place $\pi$ corresponds to the standard $t$-adic valuation $v_t : K((t)) \to \mathbb{Z}$, where

$$v_t \left( \sum c_n t^n \right) = \min\{n : c_n \neq 0\}.$$
Note that \( \text{dom}(\pi) = \{ c \in K((t)) : v_t(c) \geq 0 \} = K[[t]] \) is the valuation ring of \( v_t \) with maximal ideal given by \( \{ c \in K((t)) : \pi(c) = 0 \} = \{ c \in K((t)) : v_t(c) > 0 \} \).

**Proposition 3.1.12.** If \( (K, \partial) \models \text{DiLF} \), then \( (K, \partial) \) is existentially closed in the henselization \( (K(t)^h, \partial') \), where \( \partial'(t) = 0 \) and \( \partial' |_{K} = \partial \).

**Proof.** Let \( (F, \partial) \) be a \( |K(t)^h|^+ \)-saturated elementary extension of \( (K, \partial) \). By compactness, it suffices to show that every finitely generated substructure of \( (K(t)^h, \partial) \) embeds over \( K \) into \( (F, \partial) \).

Let \( K(\overline{e}) \) be a finitely generated subfield of \( K(t)^h \). By the primitive element theorem, we may write \( K(\overline{e}) = K(t)[\theta] \), where \( \theta \) is algebraic over \( K(t) \). Let \( f(x, y) \) be a polynomial over \( K \) of minimal degree in \( x \) such that \( f(\theta, t) = 0 \). We may write

\[
f(x, t) = P_n(t)x^n + P_{n-1}(t)x^{n-1} + \ldots + P_0(t),
\]

where each \( P_i(y) \) is a polynomial over \( K \). Since \( f(\theta, t) = 0 \), we have

\[
0 = \partial(P_n(t)\theta^n + P_{n-1}(t)\theta^{n-1} + \ldots + P_0(t))
= P_n^\partial(t)\theta^n +nP_n(t)\theta^{n-1}\partial(\theta) + \ldots + P_0^\partial(t).
\]

Therefore,

\[
\partial(\theta) = -\frac{P_n^\partial(t)\theta^n + \ldots + P_0^\partial(t)}{nP_n(t)\theta^{n-1} + \ldots + P_1(t)},
\]

which is well-defined because \( n \) was chosen to be minimal. It follows that \( \partial(\theta) \in K(t)(\theta) \) and so \( (K(\overline{e}), \partial) \) is a substructure of \( (K(t)^h, \partial) \) and so there is a rational function \( s_1(\overline{x}) \) over \( K \) such that \( \partial(\overline{e}) = s_1(\overline{e}) \).

By general algebraic geometry, there is an irreducible \( K \)-curve \( C \) such that \( \overline{e} \) is Zariski generic for \( C \) over \( K \). Note that the map \( s(\overline{x}) = (\overline{x}, s_1(\overline{x})) \) is defined on some open subset of \( C \) and \( s : C \to \tau(C) \). By largeness, there is a smooth \( K \)-point of
(since \(K\) is existentially closed in \(K(t)^h\) as pure fields, and \(\bar{e}\) is smooth for \(C\)). By differential largeness, for every open \(U \subseteq C\) defined over \(K\), there is \(\bar{a} \in U(K)\) such that \(s(\bar{a}) = (\bar{a}, \partial(\bar{a}))\) and so the differential variety defined by \(V := \{ \bar{x} \in C : s(\bar{x}) = (\bar{x}, \partial(\bar{x}))\}\) has a Kolchin dense set of points. Hence there is a Kolchin generic point \(\bar{d} \in (F, \partial)\), and the map \(\bar{e} \mapsto \bar{d}\) is the required differential field embedding of \((K(\bar{e}), \partial)\) into \((F, \partial)\).

\[
\text{Proposition 3.1.13. Suppose that } (\rho(K), \partial_t) \text{ is existentially closed in } (K((t)), \partial_t), \text{ then } (K, \partial) \models \text{DiLF.}
\]

\textbf{Proof.}\ Since \((K, \partial)\) is isomorphic to \((\rho(K), \partial_t)\) as differential fields, it is enough to show that \((\rho(K), \partial_t) \models \text{DiLF.}\)

Let \(C\) be a smooth, \(K\)-irreducible curve such that \(C(K)\) is non-empty and let \((\bar{x}, s(\bar{x})) : C \to \tau(C)\) be a rational section of the projection onto \(C\) defined over \(K\). Let \(K(C) = K(\bar{e})\) be the function field of \(C\) over \(K\), where \(\bar{e} = (e_1(t), \ldots, e_n(t)) \in K((t))\) is a tuple such that under the canonical \(K\)-place \(\pi : K((t)) \to K\), \(\bar{e} \mapsto P\), where \(P \in C(K)\). Let \(\bar{\partial}\) be the unique derivation on \(K(C)\) extending the derivation on \(K\) such that \(s(\bar{e}) = \bar{\partial}(\bar{e})\). For each \(1 \leq i \leq n\), define the map

\[
e_i(t) \mapsto \sum_{n=0}^{\infty} \frac{\bar{\partial}^n e_i(0)}{n!} t^n,
\]

where \(\bar{\partial}^n e_i(0)\) is the image of \(\bar{\partial}^n(e_i(t))\) under the map \(t \mapsto 0\) (it is straight forward to check that this is always well-defined). This gives a differential field embedding \(\bar{\rho} : (K(C), \bar{\partial}) \to (K((t)), \partial_t)\) extending \(\rho : (K, \partial) \to (K((t)), \partial_t)\). Since \(\bar{e}\) is generic for \(C\) over \(K\), for any Zariski open \(U \subseteq C\) over \(K\), we have \(\bar{e} \in U\) and \(s(\bar{e}) = \partial(\bar{e})\). Since \((\rho(K), \partial_t)\) is existentially closed in \((K((t)), \partial_t)\) and since \(s\) is defined over \(\rho(K)\), for every Zariski open subset \(U \subseteq \rho(C)\) over \(\rho(K)\), there is \(\bar{a} \in U(\rho(K))\) such that \(s(\bar{a}) = \partial(\bar{a})\).
Proposition 3.1.14. Suppose that \((K, \partial) \models \text{DiLF}\). Then \((\rho(K), \partial_t)\) is existentially closed in \((K((t)), \partial_t)\).

Proof. By [70], Theorem 5.2, \((K, \partial)\) is differentially large iff, for any differential field extension \((L, \partial) \supseteq (K, \partial)\), if \(K\) is existentially closed in \(L\) as fields, then \((K, \partial)\) is existentially closed in \((L, \partial)\) as differential fields. Since \((\rho(K), \partial_t)\) is isomorphic to \((K, \partial)\) as a differential field, \((\rho(K), \partial_t)\) is differentially large. Thus, it suffices to show that \(\rho(K)\) is existentially closed in \(K((t))\) as fields. Since \(\rho(K)\) is large as a field, it is enough to show that there is a \(\rho(K)\)-rational \(\rho(K)\)-place \(\mu : K((t)) \to \rho(K) \cup \{\infty\}\).

Observe that the map \(\pi : K((t)) \to K \cup \{\infty\}\) is a \(K\)-rational \(K\)-place. Since \(\rho : K \to \rho(K)\) is an isomorphism, the composition \(\mu = \rho \circ \pi : K((t)) \to \rho(K) \cup \{\infty\}\) works.

Hence we have the following:

Theorem 3.1.15. \((K, \partial)\) is differentially large iff \((\rho(K), \partial_t)\) is existentially closed in \((K((t)), \partial_t)\).

Example 3.1.16. We thank Léo Antoine Philibert Jimenez for pointing out the following non-example: consider the embedding \(\rho : (\mathbb{C}((t)), \partial_t) \hookrightarrow (\mathbb{C}((t))((s)), \partial_s)\). The equation

\[
\partial_s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{tn+1} s^n = \rho \left( \frac{1}{t} \right)
\]

has a solution in \(\mathbb{C}((t))((s))\), but not in \(\rho(\mathbb{C}((t)))\). Hence \((\mathbb{C}((t)), \partial_t)\) is not differentially large.

Question 3.1.17. Is \((\mathbb{C}((t)), \partial_t)\) weakly differentially large? A positive answer would then show that the class of weakly differentially large fields properly contains the...
class of differentially large fields.

**Example 3.1.18.** Let \((K_0, \partial_0) = (\mathbb{C}((t_0)), \partial_{t_0})\) and for each \(i \geq 0\), let \((K_{i+1}, \partial_{i+1}) = (K_i((t_{i+1})), \partial_{t_{i+1}})\), let \(h_{i,i+1} := \rho : K_i \hookrightarrow K_{i+1}\) be the standard Taylor embedding, and let \(h_{i,i}\) be the identity map. For all \(i < j\), let \(h_{ij} : K_i \to K_j\) to be the composition \(h_{i,i+1} \circ \ldots \circ h_{j-1,j}\). Define \((K_\omega, \partial_\omega)\) to be the direct limit \(\lim_{\rightarrow j}(K_i, \partial_i)\). We claim that \((K_\omega, \partial_\omega)\) is differentially large (for general information on direct limits of structures, see [25]).

For each \(i < \omega\), let \(h_i : K_i \to K_\omega\) be the canonical homomorphism (which are embeddings since all the connecting maps are embeddings). By general properties of direct limits, for any tuple \(\bar{a} \in K_\omega\), there is \(i < \omega\) and \(\bar{b} \in K_i\) such that \(h_i(\bar{b}) = \bar{a}\).

Let \(f(x, y, \bar{a}) = 0\) define an absolutely irreducible plane curve, \(C\), over \(K_\omega\) (with parameters \(\bar{a}\)) and suppose that \(C(K_\omega)\) is dense in \(C\). Let \(s : C \to \tau(C)\) be a rational section of the natural projection of the twisted tangent bundle and let \(U \subset C\) be a Zariski-open subset of \(C\). Write \(s(x, y, \bar{b}) = (x, y, s_1(x, y, \bar{b}), s_2(x, y, \bar{b}))\), where \(s_1(x, y, \bar{b})\) and \(s_2(x, y, \bar{b})\) are rational maps defined over a tuple \(\bar{b} \in K_\omega\). Without loss of generality, we may assume that \(U\) is defined over \(\bar{b}\) as well. Let \(i < \omega\) be such that there as \(\bar{a}_i, \bar{b}_i \in K_i\) such that \(C_i = \{(x, y) : f(x, y, \bar{a}_i) = 0\}\) is an absolutely irreducible \(K_i\)-plane curve, \(s(x, y, \bar{b}_i) : C_i \to \tau(C_i)\) is a rational section of the twisted tangent bundle defined over \(\bar{b}_i\), \(h_i(\bar{a}_i) = \bar{a}\) and \(h_i(\bar{b}_i) = \bar{b}\), and \(U_i\) is a Zariski open subset of \(C_i\) defined over \(K_i\). Furthermore, \(C_i(K_i)\) is dense in \(C_i\) and is in bijection with a dense subset of \(C(K_\omega)\). Without loss of generality, we may assume that \((0, 0)\) is a smooth point of \(C_i\). Let \(K_i(e_1(t), e_2(t))\) be the function field of \(C_i\) over \(K_i\) with \(e_j(t)\) a power series over \(K_i\) such that \(e_j(0) = 0, j = 1, 2\). As per the usual trick, \(\partial_i\) extends uniquely to a derivation \(\overline{\partial_i}\) on \(K_i(e_1(t), e_2(t))\) with the property that \(\overline{\partial_i}(e_j(t)) = s_j(e_1(t), e_2(t), \bar{b}_i)\) for \(j = 1, 2\). Now, we may embed \(K_i(e_1(t), e_2(t))\) into
\[ K_{i+1} = K_i(t_{i+1}) \] by the map

\[ e_j(t) \mapsto \sum_{n=0}^{\infty} \frac{\partial^n_j e_j(0)}{n!} t_{i+1}^n =: e'_j, \]

which extends \( \rho : K_i \to K_{i+1} \). Furthermore, the point \( (e_1(t), e_2(t)) \) is generic for \( C_i \) over \( K_i \), so \( (e_1(t), e_2(t)) \in U_i \). By construction, the point

\[ (d_1, d_2) := (h_{i+1}(e'_1), h_{i+1}(e'_2)) \in U(K_\omega) \]

is such that \( (d_1, d_2) \in C \) and \( \partial_\omega(d_1, d_2) = s(d_1, d_2) \). Hence \( (K_\omega, \partial_\omega) \) is differentially large.

**Remark 3.1.19.** By taking the algebraic closure of the last example, we get a differentially closed field.

In [76], the author shows that if \( T \) is a theory of large fields that is model complete, then \( T \) together with “\( \partial \) is a derivation” has a model companion, \( T_D \), and that this model companion is axiomatized uniformly as \( T_D = T \cup \text{DiLF} \) (though they show this in much more generality by allowing many derivations). We reprove this here in the ordinary case for the sake of completeness:

**Proposition 3.1.20.** Let \( T \) be a model complete theory of large fields in the language of rings. Let \( T_\partial \) be the theory \( T \) together with “\( \partial \) is a derivation”. Then \( T_D = T_\partial \cup \text{DiLF} \) is the model companion of \( T_\partial \).

**Proof.** Since \( T \) is model complete, if \( T_\partial \) has a model companion, \( T_D \), then the models of \( T_D \) are precisely the existentially closed models of \( T_\partial \). Suppose that \( (K, \partial) \models T_\partial \cup \text{DiLF} \). We want to see that \( (K, \partial) \) is an existentially closed model of \( T_\partial \). The method here is a standard argument taken from [54]: Let \( (L, \partial) \models T_\partial \) extend \( (K, \partial) \) and suppose that \( \varphi(\bar{x}) \) is a formula with parameters in \( K \) such that \( (L, \partial) \models \varphi(\bar{a}) \) for some \( \bar{a} \in L \). Take \( \psi(\bar{x}_0, \ldots, \bar{x}_n) \) be the formula in the language of rings such
that $\phi(\bar{x}) = \psi(\bar{x}, \partial(\bar{x}), \ldots, \partial^n(\bar{x}))$. Write $\bar{c} = (\bar{a}, \partial(\bar{a}), \ldots, \partial^{n-1}(\bar{a}))$ and let $V$ be the absolutely irreducible $K$-variety for which $\bar{c}$ is generic over $K$. Let $s(\bar{x}_0, \ldots, \bar{x}_{n-1})$ be the unique rational map defined over $K$ such that

$$(\bar{x}_0, \ldots, \bar{x}_{n-1}) \mapsto (\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{x}_1, \ldots, \bar{x}_{n-2}, s(\bar{x}_0, \ldots, \bar{x}_{n-1}))$$

is a section of the projection $\tau(V) \to V$. Since $K \not\preceq L$ as fields, $V(K)$ is dense in $V$. Now, $L \models \psi(\bar{a}, \partial(\bar{a}), \ldots, \partial^{n-1}(\bar{a})), s(\bar{a}, \partial(\bar{a}), \ldots, \partial^{n-1}(\bar{a})))$ and so there is an open subset $U \subseteq V$ (which we may assume is contained in dom($s$)) such that

$$(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in U \Rightarrow K \models \psi(\bar{x}_0, \ldots, \bar{x}_{n-1}, s(\bar{x}_0, \ldots, \bar{x}_{n-1})).$$

As $(K, \partial) \models \text{DiLF}$, there is $\bar{b}_0, \ldots, \bar{b}_{n-1}) \in U(K)$ such that

$$(\bar{b}_0, \ldots, \bar{b}_{n-1}, \bar{b}_1, \ldots, \bar{b}_{n-2}, s(\bar{b}_0, \ldots, \bar{b}_{n-1})) = (\bar{b}_0, \ldots, \bar{b}_{n-1}, \partial(\bar{b}_0), \ldots, \partial(\bar{b}_{n-2}), \partial(\bar{b}_{n-1})).$$

It follows that $K \models \psi(\bar{b}_0, \partial(\bar{b}_0), \ldots, \partial^n(\bar{b}_0))$.

To finish the proof, it suffices to show that every model of $T_\partial$ can be extended to a model of $T_\partial \cup \text{DiLF}$. This is enough because it implies that existentially closed models of $T_\partial$ are models of $T_\partial \cup \text{DiLF}$. So let $(K, \partial) \models T_\partial$. Let $C$ be a smooth, absolutely irreducible $K$-curve, and let $(\bar{x}, s(\bar{x})): C \to \tau(C)$ be a rational section of the projection $\tau(C) \to C$. Suppose that $C(K)$ is non-empty. By largeness, $C(K)$ is dense in $C$ and so there is $L \succ K$ and $\bar{a} \in L$ such that $\bar{a}$ is generic for $C$ over $K$. Since $(\bar{a}, s(\bar{a})) \in \tau(V)(L)$, there is a unique derivation on $K(\bar{a})$ extending $\partial$ sending $\bar{a}$ to $s(\bar{a})$, which may further be extended to some derivation on $L$. By model completeness, repeating this process transfinitely and taking unions gives a model of $T_\partial \cup \text{DiLF}$. 

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**Corollary 3.1.21.** Let $T$ be a model complete theory of large fields and let $T_D$ be the
model companion of $T_\partial$. For every model $(K, \partial) \models T_D$, we have that $C(K) \preceq K$ as fields.

Proof. Since $T$ is model complete, it suffices to show that $C(K)$ is existentially closed in $K$. The argument is a familiar one: let $\varphi(\bar{x})$ be a quantifier-free formula in the language of rings with parameters in $C(K)$. Suppose that $K \models \varphi(\bar{a})$. Without loss of generality, $\bar{a} \not\in C(K)$. Let $V$ be the $C(K)$-Zariski closure of $\{\bar{a}\}$. Since $C(K)$ is relatively algebraically closed in $K$, $V$ is absolutely irreducible. Since $V$ is defined over $C(K)$, we have that $\tau(V) = T(V)$, the usual Zariski tangent bundle of $V$. It follows that the map

$$s(\bar{x}) = (\bar{x}, 0)$$

is a section of the projection $T(V) \to V$. Since $\bar{a}$ is generic for $V$ over $C(K)$, and since $L \models \varphi(\bar{a})$ and $\varphi(\bar{x})$ is a constructible set, there is some Zariski-open $U \subseteq V$ defined over $K$ such that $\bar{x} \in U \Rightarrow \varphi(\bar{x})$. Since $\bar{a}$ is a smooth point of $V$, the axioms give that there is $\bar{b} \in U(K)$ such that $s(\bar{b}) = (\bar{b}, \partial(\bar{b}))$. In particular, $\varphi(\bar{b})$ and $\partial(\bar{b}) = 0$, hence $C(K) \models \varphi(\bar{b})$.

Remark 3.1.22. Some historical remarks: before differentially large fields were introduced in their current generality in [70], they were studied in various special cases elsewhere. In a 1978 paper [72], Singer shows, among other results, that the theory of ordered differential fields (with one derivation) has a model completion, which he calls the theory of closed ordered differential fields (CODF), and gives an axiomatization in the fashion of Blum’s axioms for $DCF_0$ (note that the underlying field of any model of CODF is itself a real closed field). Building on this work, Tressl, in the 2005 paper [76], showed that if $T$ is a complete, model complete theory of large fields, then $T \cup \{“\partial_1, \ldots, \partial_m$ are commuting derivations”$\}$ has a model completion uniformly in $T$, thus simultaneously axiomatizing the theory CODF and the
theory of differentially closed fields in many derivations. We know now that these
fields are precisely the differentially large fields (by definition), though the uniform
axiomatization was not given “geometrically” in the style of Pierce and Pillay \cite{54}
until presented by Tressl and Leon-Sanchez in \cite{70}. Around the same time (2006),
Pillay and Polkowska, \cite{58}, showed that the PAC property for DCF$_0$ is first-order (see
\ref{1.1.3}) and that a differential field $(K, \partial)$ is differentially PAC (or \textit{pseudo-differentially closed} as the authors remark such structures might be called) if for every absolutely irreducible quasi-affine variety $V$ over $K$ and every rational section $s : V \to \tau(V)$, there is $\bar{c} \in V(K)$ such that $s(\bar{c}) = (\bar{c}, \partial(\bar{c}))$ (that is, precisely the geometric axioms for differentially large in the case that the field is PAC). They go on to give elementary invariants of pseudo-differentially closed fields as in Cherlin, van den Dries, and Macintyre’s treatment of PAC fields.

3.2 Large Difference Fields

This final section is a first attempt at adapting some of the differential results in \cite{70} to the setting of fields with an automorphism and to study the relation to ACFA (by which we mean \textit{Algebraically Closed Fields with an Automorphism} and not, sadly, the American Cat Fanciers’ Association$®$).

Let $T$ be the theory structures $(K, \sigma)$ where $K$ is a large field, and $\sigma$ is an automorphism. We define two different axiom schemes for notions of largeness:

\textbf{Definition 3.2.1.} \textit{We will say that $(K, \sigma)$ is a weakly Large Difference Field (w LDF) if:}

- $K$ is large,
- $\sigma : K \to K$ is an automorphism of $K$
- for every absolutely irreducible varieties $V$, $U \subseteq V$ Zariski open, and every rational section $s : V \to V \times V^\sigma$ of the projection onto $V$, if $V(K)$ contains a smooth point, then there is $\bar{a} \in U(K)$ such that $s(\bar{a}) = (\bar{a}, \sigma(\bar{a}))$. 

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Definition 3.2.2. We will say that \((K, \sigma)\) is a Large Difference Field (LDF) if:

- \(K\) is large,
- \(\sigma : K \to K\) is an automorphism of \(K\),
- for every absolutely irreducible variety \(V\), every absolutely irreducible \(W \subseteq V \times V^\sigma\) projecting dominantly onto \(V\) and \(V^\sigma\), and every open \(U \subseteq W\), if \(W(K)\) contains a smooth point, then there is \(\bar{a} \in V(K)\) such that \((\bar{a}, \sigma(\bar{a})) \in U\).

Note that if we were to consider the differential versions of these axioms they would be equivalent. This is because for any \(K\)-variety \(V\) and any \(\bar{a} \in V\), the preimage \(\pi^{-1}(\bar{a}) \subseteq \tau(V)\), where \(\pi : \tau(V) \to V\) is the natural projection onto \(V\), is a linear space, and so we can always find a rational section for any subvariety \(W\) of \(\tau(V)\) projecting onto \(V\). In the difference field case, \(V \times V^\sigma\) we do not have this property, and so the two axiom schema are distinct.

It is clear that if \((K, \sigma)\) is a model of LDF, then \((K, \sigma)\) is a model of \(w\) LDF: if \(V\) is an absolutely irreducible \(K\)-variety, and \(s : V \to V \times V^\sigma\) is a section of the projection, then the image \(s(V) \subseteq V \times V^\sigma\) is a variety projecting dominantly onto \(V\), and so by the axioms for LDF, there is \(\bar{a} \in V(K)\) such that \(s(\bar{a}) = (\bar{a}, \sigma(\bar{a}))\).

Definition 3.2.3. Let \((K, \sigma)\) be a field with an automorphism. We will say that a set is a special \(\sigma\)-variety if it is of the form

\[\{\bar{x} \in V : s(\bar{x}) = (\bar{x}, \sigma(\bar{x}))\},\]

where \(V\) is an absolutely irreducible \(K\)-variety, and \(s : V \to V \times V^\sigma\) is a section of the projection onto \(V\),

The next proposition shows that, for \(w\) LDF, it is enough to consider special \(\sigma\)-varieties arising from absolutely irreducible plane curves:

Proposition 3.2.4. \((K, \sigma) \models w\) LDF iff \(K\) is large, \(\sigma\) is an automorphism, and, for every absolutely irreducible plane curve \(C\) over \(K\), every open subset \(U \subseteq K\), and
every rational section \( s : C \to C \times C^\sigma \), if \( C(K) \) contains a smooth point of \( C \), then there is \((a, b) \in U(K)\) such that \( s(a, b) = (a, b, \sigma(a), \sigma(b)) \).

**Proof.** The forward direction is immediate. Suppose that \( V \) is an absolutely irreducible \( K \)-variety, \( U \subseteq K \) is a Zariski-open subset, and \( s : V \to V \times V^\sigma \) a rational section of the projection onto \( V \). Without loss of generality, we may assume that \( s \) is defined on \( U \) (otherwise just replace \( U \) by the intersection of \( U \) with the domain of \( s \)). Suppose \( V(K) \) contains a smooth point. As \( K \) is large, \( V(K) \) is dense in \( V \) and so \( U(K) \) is dense in \( V \) as well. By general algebraic geometry, for any finite set of points \( P \subseteq V(K) \) there is an absolutely irreducible curve \( C \subseteq V \) such that \( P \subset C(K) \), and if a point in \( P \) is smooth for \( V \), then it is smooth for \( C \) as well.

Thus, we may assume we are in the situation where \( V = C \) is some arbitrary curve, \( s : C \to C \times C^\sigma \) is a section of the projection, and \( U \subseteq C \) is a Zariski-open subset with \( U(K) \) dense in \( C \).

Let \( C' = \{(x, y) : g(x, y) = 0\} \) be a plane curve such that the projection of \( C \) onto the first two coordinates gives a birational equivalence \( \pi : C \to C' \). Observe that \( \pi : C \to C' \) induces a map \( \pi : C \times C^\sigma \to C' \times C'^\sigma \) defined by \( \pi(\bar{x}, \bar{y}) = (\pi(\bar{x}), \pi(\bar{y})) \).

Let \( U' \) be an open subset of \( \pi(U) \) and let \( f(x, y) : C' \to C \) be the inverse of \( \pi \) (in the sense of birational morphisms). Observe that the map \( s' := \pi \circ s \circ f : C' \to C' \times C'^\sigma \) is a rational section of the projection. By assumption there is a point \((a, b) \in U'(K)\) such that \( s'(a, b) = (a, b, \sigma(a), \sigma(b)) \). Consider the map \((f, f^\sigma) : C' \times C'^\sigma \to C \times C^\sigma \), where \( f^\sigma : C'^\sigma \to C^\sigma \) is the map obtained by applying \( \sigma \) to all the coefficients in \( f \).

We have that \((f, f^\sigma)(a, b, \sigma(a), \sigma(b)) = (f(a, b), \sigma(f(a, b))) \in C \times C^\sigma \). Finally, on an open set which we may assume contains \( f(a, b) \in U(K) \), we have \( s = (f, f^\sigma) \circ s' \circ \pi \), and so \( s(f(a, b)) = (f(a, b), \sigma(f(a, b))) \), as required.

**Question 3.2.5.** Can one reduce to plane curves for LDF? Let \((K, \sigma) \models \text{LDF} \) and let \( V \) be an absolutely irreducible \( K \)-variety, and \( W \subseteq V \times V^\sigma \) project dominantly onto \( V \) and \( V^\sigma \). For any curve \( C \subseteq V \), we have that \( C \times C^\sigma \subseteq V \times V^\sigma \). Let
\[ \pi : V \times V^\sigma \to V \] be the projection. Since \( \pi \upharpoonright_W : W \to V \) is dominant, there is an open subset \( U \subseteq \pi(W) \). Without loss of generality, we may choose a curve \( C \subseteq V \) such that \( C \cap U \) is an open subset of \( C \). Now, \( U^\sigma \) is an open subset of \( V^\sigma \cap C^\sigma \).

Let \( X \) be the \( K \)-Zariski closure of \( C \times C^\sigma \cap W \). Then \( X \subseteq C \times C^\sigma \cap W \) projects dominantly onto \( C \) and \( C^\sigma \), and for any \( \bar{a} \in C \) such that \( (\bar{a}, \sigma(\bar{a})) \in X \), we have that \( (\bar{a}, \sigma(\bar{a})) \in W \). Thus we can reduce to the case of arbitrary affine curves. Note that \( X \) is either a curve or all of \( C \times C^\sigma \).

**Remark 3.2.6.** Let \( T \) be the theory of large fields with an automorphism. Then every model \( (K, \sigma) \models T \) can be embedded into a model of LDF: let \( (K, \sigma) \) be a large field equipped with an automorphism. Then \( (K, \sigma) \) may be embedded into a model of ACFA, and ACFA \( \vdash \) LDF.

**Remark 3.2.7.** Let \( (K, \sigma) \) be a large, differential field. Let \( C \) be an absolutely irreducible planar \( K \)-curve such that \( C(K) \) contains a smooth point of \( C \), \( U \subseteq C \) a non-empty Zariski-open subset of \( C \) defined over \( K \), and \( S : C \to C \times C^\sigma \) a \( K \)-rational section of the projection onto \( C \). Write \( s(x, y) = (x, y, s_1(x, y), s_2(x, y)) \).

Let \( (a, b) \in C \) be generic for \( C \) over \( K \) (in some extension of \( K \)). Note that \( (a, b) \in U \) and \( s(a, b) \in C \times C^\sigma \). Let \( K(a, b) \) be the function field of \( C \) over \( K \). Since \( s(a, b) \in C \times C^\sigma(K(a, b)) \), \( \sigma \) extends to an automorphism of \( K(a, b) \) such that \( \sigma(a, b) = (s_1(a, b), s_2(a, b)) \). Now, let \( f : K(a, b) \to K(t)^h \) be an embedding over \( K \) of \( K(a, b) \) into the henselization \( K(t)^h := K(t)^{alg} \cap K((t)) \). The image \( f(K(a, b)) \) is equipped with the automorphism defined by \( \sigma'(a) := f \circ \sigma \circ f^{-1}(a) \). Then any extension \( \tau \) of \( \sigma' \) to \( K(t)^h \) gives an extension \( (K(t)^h, \tau) \geq (K, \sigma) \) with a smooth point \( (a, b) \in U(K(t)^h) \), such that \( \tau(a, b) = (s_1(a, b), s_2(a, b)) \). If one can always find such an extension to the henselization, this construction can be repeated transfinitely and taking unions gives an embedding of \( (K, \sigma) \) into a model of \( w \) LDF.

**Example 3.2.8.** If \( T = ACF_0 \), then \( (K, \sigma) \models T \cup \) LDF iff \( (K, \sigma) \models \) ACFA_0.

Next, we show that a version of Weil restriction for special \( \sigma \)-varieties holds in
the case of finite Galois extensions of fields with an automorphism. Note that the proof actually uses the fact that the extension is Galois. It may be possible to get rid of this assumption and prove this for the case of finite extensions.

**Proposition 3.2.9** (Weil Restriction for Special $\sigma$-varieties). Let $(K, \sigma)$ be a field with an automorphism and let $(L, \sigma')$ be an extension such that $L/K$ is a finite Galois extension and $\sigma'$ is automorphism. Then, for any irreducible $L$-variety $V$ and any rational section $s : V \to V \times V^{\sigma'}$ of the projection onto $V$, there is a (possibly reducible) $K$-variety $W$ and rational section $r : W \to W \times W^{\sigma}$ such that there is a bijection between the sets

\[
\{ \bar{x} \in V(L) : s(\bar{x}) = (\bar{x}, \sigma'(\bar{x})) \}
\]

and

\[
\{ \bar{y} \in W(K) : r(\bar{y}) = (\bar{y}, \sigma(\bar{y})) \}.
\]

**Proof.** First, we recall the classical Weil restriction: write $L = K[e_1, \ldots, e_n]$. Let $V \subseteq \mathbb{A}^l$ be an irreducible $L$-variety and suppose that $V = \mathbb{V}(f_1, \ldots, f_m)$ for some polynomials $f_k \in L[x_1, \ldots, x_l]$. For each variable $x_i$, write $x_i = y_{i,1}e_1 + \ldots + y_{i,n}e_n$. For each of the polynomials $f_i$, we make a substitution

\[
f_k(x_1, \ldots, x_l) = f_k(\ldots, y_{i,1}e_1 + \ldots + y_{i,n}e_n, \ldots)
\]

\[= \sum_{j=1}^{n} F_{k,j}(y_{1,1}, \ldots, y_{1,n}, \ldots, y_{l,1}, \ldots, y_{l,n})e_j,
\]

where each $F_{k,j}$ is a polynomial over $K$. The Weil restriction of $V$ is defined to be

\[
W = \operatorname{Res}_{L/K}(V) := \mathbb{V}(F_{k,j} : 1 \leq k \leq m, 1 \leq j \leq n) \subseteq \mathbb{A}^{nl}.
\]
Note that the classical Weil restriction does not preserve irreducibility. Weil restriction is functorial, so $\text{Res}_{L/K}(V \times V^{\sigma'}) = \text{Res}_{L/K}(V) \times \text{Res}_{L/K}(V^{\sigma'})$ but, unfortunately, $\text{Res}_{L/K}(V^{\sigma'}) \not\cong \text{Res}_{L/K}(V)^{\sigma} =: W^{\sigma}$ in general. Regardless, we will set up a correspondence between the points of $V \times V^{\sigma'}$ and $W \times W^{\sigma}$.

As $\{e_1, \ldots, e_n\}$ is a basis for $L$ over $K$, and $\sigma'$ is an automorphism of $L$, we may define the constants $d_{i,k} \in K$ as those such that $\sigma' - 1(e_i) = \sum_{j=1}^{n} d_{i,j} e_j$.

Suppose now that $(\bar{a}, \bar{b}) \in V \times V^{\sigma'}(L)$ (we don’t assume that $\bar{b} = \sigma'(\bar{a})$). By the usual Weil restriction, $\bar{a}$ corresponds to some point in $W(K)$. We want to find a point in $W^{\sigma}(K)$ corresponding to $\bar{b}$. Let $\pi_i : \mathbb{A}^l \to \mathbb{A}^1$ be the projection onto the $i$-th coordinate. Write $\bar{b} = (b_1, \ldots, b_l)$ and for each $1 \leq i \leq l$, write $b_i = b_{i,1} e_1 + \ldots + b_{i,n} e_n$. Then,

$$\bar{b} \in V^{\sigma'}(L) \iff \sigma'^{-1}(\bar{b}) \in V$$

$$\iff \sum_{j=1}^{n} \sigma^{-1}(b_{i,j}) \left( \sum_{k=1}^{n} d_{j,k} e_k \right) \in \pi_i(V), \forall 1 \leq i \leq l$$

$$\iff \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \sigma^{-1}(b_{i,k}) d_{j,k} \right) e_j \in \pi_i(V), \forall 1 \leq i \leq l$$

$$\iff \sum_{k=1}^{n} \sigma^{-1}(b_{i,k}) d_{j,k} \in \pi_{in+j}(\text{Res}_{L/K}(V)), \forall 1 \leq i \leq l, 1 \leq j \leq n$$

$$\iff \sum_{k=1}^{n} b_{i,k} \sigma(d_{j,k}) \in \pi_{in+j}(\text{Res}_{L/K}(V)^{\sigma}), \forall 1 \leq i \leq l, 1 \leq j \leq n$$

which gives the desired correspondence, independent of $\bar{a}$ or $\bar{b}$.

Now, suppose that we have a rational section $s(\bar{x}) = (\bar{x}, s_1(\bar{x}), \ldots, s_l(\bar{x})) : V \to$
\( V \times V^\sigma \) of the projection onto \( V \). For each \( 1 \leq i \leq l \), write

\[
s_i(\bar{x}) = \frac{q_i(\bar{x})}{r_i(\bar{x})}
\]

for \( q_i(\bar{x}), r_i(\bar{x}) \in L[x_1, \ldots, x_i] \). Making the substitution \( x_k = y_{k,1}e_1 + \ldots + y_{k,n}e_n \) for all \( 1 \leq k \leq l \), we get a rational map

\[
\frac{q_i(\bar{x})}{r_i(\bar{x})} = \sum_{j=1}^n Q_{i,j}(\bar{y}_1, \ldots, y_{1,n}, \ldots, y_{l,1}, \ldots, y_{l,n}) e_j
\]

\[
\sum_{j=1}^n R_{i,j}(\bar{y}_1, \ldots, y_{1,n}, \ldots, y_{l,1}, \ldots, y_{l,n}) e_j
\]

Write \( R_i(\bar{y}) \in L[\bar{y}] \) for the denominator of the right hand side above expression. For each \( 1 \leq i \leq l \), define a polynomial \( P_i(\bar{y}) \) as

\[
P_i(\bar{y}) = \prod_{\tau \in \text{Gal}(L/K)} \frac{\tau(R_i(\bar{y}))}{R_i(\bar{y})} \in L[\bar{y}]
\]

so that \( R_i(\bar{y})P_i(\bar{y}) \) is a polynomial over \( K \). Then

\[
\frac{\sum_{j=1}^n Q_{i,j}(\bar{y}) e_j}{\sum_{j=1}^n R_{i,j}(\bar{y}) e_j} = \frac{\sum_{j=1}^n Q_{i,j}(\bar{y}) e_j}{R_i(\bar{y})} \left( \frac{P_i(\bar{y})}{P_i(\bar{y})} \right)
\]

\[
= \frac{\sum_{j=1}^n Q_{i,j}(\bar{y}) P_i(\bar{y}) e_j}{R_i(\bar{y}) P_i(\bar{y})}
\]

\[
= \frac{1}{R_i(\bar{y}) P_i(\bar{y})} \left( \sum_{j=1}^n Q_{i,j}(\bar{y}) P_i(\bar{y}) e_j \right).
\]

Each \( Q_{i,j}(\bar{y}) P_i(\bar{y}) e_j \) is a polynomial over \( L \), so we may rewrite this as

\[
= \frac{1}{R_i(\bar{y}) P_i(\bar{y})} \left( \sum_{j=1}^n G_{i,j}(\bar{y}) e_j \right)
\]

for some polynomials \( G_{i,j}(\bar{y}) \) defined over \( K \). We now define a rational map for each
\[1 \leq i \leq l, \ 1 \leq j \leq n\] as

\[t_{i,j}(\bar{y}) := \sum_{k=1}^{n} \frac{G_{i,k}(\bar{y})}{R_{i}(\bar{y})P_{i}(\bar{y})} \sigma(d_{j,k}).\]

By construction, the \(t_{i,j}\)'s give a rational section \(t : W \rightarrow W \times W^{\sigma}\). Furthermore, for any \(\bar{a} = (a_1, \ldots, a_l) \in V(L)\) with \(a_i = \sum_{j=1}^{n} a_{i,j} e_j\), we have that

\[(\bar{a}, \sigma'(\bar{a})) = s(\bar{a}) \iff \sigma'(a_i) = \frac{q_i(\bar{a})}{r_i(\bar{a})}\]

\[\iff a_{i,1} e_1 + \cdots + a_{i,n} e_n = \sigma'^{-1}\left(\frac{q_i(\bar{a})}{r_i(\bar{a})}\right)\]

\[\iff a_{i,j} = \sum_{k=1}^{n} \sigma^{-1}\left(\frac{G_{i,k}(a_{1,1}, \ldots, a_{l,n})}{R_{i}(a_{1,1}, \ldots, a_{l,n})P_{i}(a_{1,1}, \ldots, a_{l,n})}\right) d_{j,k}\]

\[\iff \sigma(a_{i,j}) = \sum_{k=1}^{n} \frac{G_{i,k}(a_{1,1}, \ldots, a_{l,n})}{R_{i}(a_{1,1}, \ldots, a_{l,n})P_{i}(a_{1,1}, \ldots, a_{l,n})} \sigma(d_{j,k})\]

\[= t_{i,j}(a_{1,1}, \ldots, a_{l,n}),\]

which completes the proof. \(\square\)

Remark 3.2.10. We remark that the proof of Proposition 3.2.9 can likely be adapted, mutatis mutandis, to give a proof for Weil descent of finite dimensional \(D\)-varieties in the differential case, where it suffices to consider systems of equations arising from a section of the projection \(\tau(V) \rightarrow V\).

**Proposition 3.2.11.** Let \((K, \sigma) \models w \text{LDF}\) and let \((L, \tau)\) be such that \(L/K\) is a finite Galois extension and \(\tau\) extends \(\sigma\). Then \((L, \tau) \models w \text{LDF}\).

**Proof.** Let \(C\) be an irreducible \(L\)-curve such that \(C(L)\) contains a smooth point of \(C\), let \(U \subseteq C\) be a Zariski open subset of \(C\) defined over \(K\), and let \(s : C \rightarrow C \times C^{\tau}\) be a section of the projection defined over \(K\).

It is a general fact that the classical Weil restriction of a quasi-affine variety always exists, and so the is an open subvariety \(U' \subseteq \text{Res}_{L/K}(C)\) corresponding to
Furthermore, Weil restriction preserves smoothness, so $\text{Res}_{L/K}(C)(K)$ is dense in $\text{Res}_{L/K}(C)$. The section $s : C \to C \times C^\sigma$ induces a section $r : \text{Res}_{L/K}(C) \to \text{Res}_{L/K}(C) \times \text{Res}_{L/K}(C)^\sigma$, and since $(K, \sigma)$ is a model of LDF, there is a point $\bar{a} \in U'(K)$ such that $r(\bar{a}) = (\bar{a}, \sigma(\bar{a}))$. By Proposition 3.2.9 $\bar{a}$ corresponds to a point $\bar{b} \in U(L)$ such that $s(\bar{a}) = (\bar{a}, \tau(\bar{a}))$. Hence $(L, \tau)$ is a model of $w$LDF.

**Question 3.2.12.** Is there a version of Weil Descent for general, finite dimensional $\sigma$-varieties? That is, given a difference field $(L, \sigma)$, absolutely irreducible $L$-variety $V$, an absolutely irreducible $L$-variety $W \subseteq V \times V^\sigma$ projecting dominantly onto $V$ and $V^\sigma$, and $L/K$ a finite galois extension such that $\sigma |_K$ is an automorphism, is there a $K$-variety $V'$ and $W' \subseteq V' \times V'^\sigma$ projecting dominantly onto $V'$ such that there is a correspondence between the sets

$$\{\bar{x} \in V(L) : (\bar{x}, \sigma(\bar{x})) \in W\}$$

and

$$\{\bar{y} \in V'(K) : (\bar{y}, \sigma(\bar{y})) \in W'\}?$$

**Remark 3.2.13.** A positive answer to the preceding question should imply that for every model $(K, \sigma)$ of LDF, if $\tau$ is an automorphism of $K^{\text{alg}}$ extending $\sigma$, then $(K^{\text{alg}}, \tau) \models \text{ACFA}$.

Let $T$ be a theory of fields and let $T_\sigma$ be the theory of models of $T$ with an automorphism. If $T_\sigma$ has a model companion, it is usually denoted by $TA$. There are many negative results concerning the existence of $TA$, for example [38, 36, 37]. In particular, if $T$ has the strict order property then $T_\sigma$ has no model companion [38] so, for example, there is no $\text{RCFA}$. However, in the spirit of [76], we make the following conjecture:
Conjecture 3. Let $T$ be a model complete theory of large fields such that $T_\sigma$ has a model companion, $T_A$. Then $T_A = T \cup \text{LDF}$.

We remark that by results mentioned in [36] (see Fact 3.5), if $T$ is model complete and $T_A$ exists, then $T$ eliminates $\exists^\infty$ and does not have the finite cover property.

The difficulty in verifying the aforementioned conjecture seems to be in situations where one needs to extend an automorphism to a field other than the algebraic closure. The same issue arises if one tries to show that every model of $T_A$ embeds into a model of $T \cup \text{LDF}$. In fact, the other direction follows easily by adapting techniques in [54]:

Proposition 3.2.14. Suppose that $T$ is a model complete theory of large fields such that $T_A$ exists. Then $T \cup \text{LDF} \vdash T_A$.

Proof. Note that since $T$ is model complete, it is inductive, and so $T_\sigma$ is inductive as well. It follows that the models of $T_A$ are precisely the existentially closed models of $T_\sigma$.

Let $(K,\sigma) \models T \cup \text{LDF}$. Clearly $(K,\sigma) \models T_\sigma$, so we aim to show that $(K,\sigma)$ is an existentially closed model. Let $(L,\sigma) \models T_\sigma$ be such that $(K,\sigma) \leq (L,\sigma)$ (we will write $\sigma$ to mean both $\sigma^L$ and $\sigma^K$). Since $T$ was assumed to be model complete, $K \preceq L$ as fields, in particular $K$ is relatively algebraically closed in $L$. Let $\varphi(\bar{x})$ be a quantifier-free formula over $K$ in the language of difference rings. Let $\psi(\bar{x}_0,\ldots,\bar{x}_n)$ be a quantifier-free formula in the language of rings such that

$$T_\sigma \vdash \forall\bar{x} (\varphi(\bar{x}) \iff \psi(\bar{x},\sigma(\bar{x}),\ldots,\sigma^n(\bar{x}))).$$

Suppose that $\bar{a} \in L$ is such that $(L,\tau) \models \varphi(\bar{a})$. Let $\bar{c} = (\bar{a},\sigma(\bar{a}),\ldots,\sigma^{n-1}(\bar{a}))$. Let $V$ be the zero-set of all polynomials over $K$ that vanish on $\bar{c}$. Since $K$ is relatively algebraically closed in $L$, $V$ is an absolutely irreducible variety. Furthermore, $\bar{c}$ is a smooth point of $V$ and so by largeness and model completeness, $V(K)$ is dense in $V$.  

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Let $W$ be the absolutely irreducible variety for which $(\bar{c}, \sigma(\bar{c}))$ is generic. As before $W(K)$ is dense in $W$. By construction, $W \subseteq V \times V^\sigma$ and $W$ projects dominantly onto $V$ and $V^\sigma$. Consider the formula

$$
\chi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_0, \ldots, \bar{y}_{n-1}) \iff \psi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_{n-1}) \land \bigwedge_{i=0}^{n-2} (\bar{y}_i = \bar{x}_{i+1}).
$$

Then $(L, \sigma) \models \chi(\bar{c}, \sigma(\bar{c}))$ and so there is some open $U \subseteq W$ defined over $K$ such that $(\bar{x}, \bar{y}) \in U \to \chi(\bar{x}, \bar{y})$ (by the genericity of $(\bar{c}, \sigma(\bar{c}))$ in $W$). By the axioms of LDF, there is $\bar{b} \in V(K)$ such that $(\bar{b}, \sigma(\bar{b})) \in U(K)$, and hence $(K, \sigma) \models \chi(\bar{b}, \sigma(\bar{b}))$, which implies that $(L, \sigma) \models \varphi(\bar{b})$. Hence $(K, \sigma) \models TA$.

On the other hand, suppose that $(K, \sigma) \models TA$. Then $(K, \sigma) \models T_\sigma$ and if we want to show that $(K, \sigma) \models T \cup \text{LDF}$, it suffices to verify the last axiom. Let $V$ be an absolutely irreducible $K$-variety, let $W \subseteq V \times V^\sigma$ project dominantly onto $V$ and $V^\sigma$, and suppose that $U \subseteq W$ is an open subset definable over $K$ such that $U(K)$ is dense in $V$ (by largeness, this is equivalent to $V(K)$ having a smooth point). Let $L \succ K$ be such that $(\bar{a}, \bar{b})$ is generic for $W$ over $K$. Since $(\bar{a}, \bar{b}) \in V \times V^\sigma$, there is an automorphism of $K(\bar{a}, \bar{b})$ extending $\sigma$ taking $\bar{a}$ to $\bar{b}$. To finish the proof, it would suffice to show that one can extend this automorphism to all of $L$.

**Remark 3.2.15.** Let $T$ be a model complete theory of large fields, and let $(M, \sigma) \models T \cup \text{LDF}$. Then $\text{Fix}(\sigma) \subseteq M$, the subfield of $M$ fixed by $\sigma$, is a large field. Let $V$ be an absolutely irreducible variety defined over $\text{Fix}(\sigma)$ and let $U \subseteq V$ be a Zariski open subset (defined, even, over $M$). If $V(M)$ contains a smooth point, then for the open subset $U \times U \cap \Delta$ of the diagonal $\Delta \subseteq V \times V^\sigma = V \times V$, there is $\bar{a} \in V(M)$ such that $(\bar{a}, \sigma(\bar{a})) \in U \times U$. Since $(\bar{a}, \sigma(\bar{a})) \in \Delta$, we have that $\sigma(\bar{a}) = \bar{a} \in U(\text{Fix}(\sigma))$.

**Definition 3.2.16.** Let $T$ be a theory of characteristic 0 fields with an automorphism. We will say that $T$ has almost $\text{QE}$ if, for any model $(K, \sigma)$ and any substructure
A ⊆ K such that A is relatively algebraically closed in K and such that the restriction σ | A is an automorphism of A, T ∪ qftp(A) is complete.

Remark 3.2.17. Note that ACFA has this property: see Theorem 1.3 of [11].

Proposition 3.2.18. Let T be a theory of large fields with almost QE in a language L. Then T ∪ LDF has almost QE in the language L_σ = L ∪ {σ}.

For the proof of 3.2.18, we will use the following fact:

Fact 3.2.19 ([17], Proposition 2.1.9(iii)). Let (F,σ) be a difference field, G a σ-overfield of F and G* an inversive closure of G. If G/F is regular, then G*/F* is regular (where F* is the inversive closure of F in G*).

Since, in characteristic 0, L/K is regular iff K is relatively algebraically closed in L, we have the following corollary:

Corollary 3.2.20. Let (L,σ) be a field with an automorphism and let K ⊂ L be a relatively algebraically closed difference subfield. Then the inversive closure K* of K in L is relatively algebraically closed.

Proof. Since σ is an automorphism of L, L* = L. Apply Fact 3.2.19.

Proof of 3.2.18. Let (A,σ) be relatively algebraically closed in some model of T ∪ LDF such that σ is an automorphism of A. Let (M_i,σ_i) |= T ∪ LDF ∪ qftp_{L_σ}(A), i = 0, 1, be sufficiently saturated. Let B_0 ⊂ M_0 be the inversive closure of the relative algebraic closure of some finitely generated extension of A. Let p = qftp_{L_σ}(B_0/A). We will show that p is finitely satisfiable in M_1. Let ϕ(¯x) ∈ p, and let ψ(¯x_0,...,¯x_n) ∈ L(A) be such that ϕ(¯x) ≡ ψ(¯x,σ(¯x),...,σ^n(¯x)). Let ¯b ∈ B_0 be such that M_0 |= ψ(¯b,σ_0(¯b),...,σ^n_0(¯b)). Let ¯c = (¯b,σ_0(¯b),...,σ^n_0(¯b)) and let V be the absolutely irreducible A-variety for which ¯c is generic and let W ⊆ V × V^σ be the absolutely
irreducible $A$-variety for which $(\bar{c}, \sigma(\bar{c}))$ is generic. Consider the formula

$$\chi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_0, \ldots, \bar{y}_{n-1}) \iff \psi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_{n-1}) \land \bigwedge_{i=0}^{n-2} (\bar{y}_i = \bar{x}_{i+1}),$$

which is true of $(\bar{c}, \sigma(\bar{c}))$. As $(\bar{c}, \sigma(\bar{c}))$ is generic for $W$, there is a Zariski-open $U \subseteq W$, defined over $A$, such that $(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_0, \ldots, \bar{y}_{n-1}) \in U$ implies that $\chi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{y}_0, \ldots, \bar{y}_{n-1})$. By the fact that $T$ has almost QE in $L$, $W(M_1)$ contains a smooth point of $W$. By the axioms for LDF, there is $(\bar{a}_0, \ldots, \bar{a}_{n-1})$ in $M_1$ such that $(\bar{a}_0, \ldots, \bar{a}_{n-1}, \sigma_1(\bar{a}_0), \ldots, \sigma_1(\bar{a}_{n-1})) \in U(M_1)$. It follows that $M_1 \models \chi(\bar{a}_0, \ldots, \bar{a}_{n-1}, \sigma_1(\bar{a}_0), \ldots, \sigma_1(\bar{a}_{n-1}))$ and hence $M_1 \models \psi(\bar{a}_0, \ldots, \sigma_1^{n-1}(\bar{a}_0))$. This completes the proof.

**Example 3.2.21.** Let $K$ be a pseudo-finite field of characteristic 0, let $A := \mathbb{Q}^{alg} \cap K$ and let $T = \text{Th}(K) \cup \text{qftp}_L(A)$, where $L$ is the language of rings. It is well known that $T$ has almost QE in the language $L(A)$. By Proposition 3.2.18, the $L_\sigma(A)$-theory $T \cup \text{LDF} \cup \text{qftp}_{L_\sigma}(A)$, where $\text{qftp}_{L_\sigma}(A)$ implies $\sigma(a) = a$ for all $a \in A$, is complete.

**Remark 3.2.22.** Though we do not attempt to generalize further here, we would like to remark that it may be interesting to adapt the difference/differential setting to the situation of fields of characteristic 0 with free operators as studied by Moosa and Scanlon in [53]. In this paper, the authors introduce a framework for structures that they call $D$-fields, and give geometric axioms for the existentially closed $D$-fields akin to those given by Chatzidakis and Hrushovski for ACFA, [11], and Pierce and Pillay for DCF$_0$, [54]. Indeed, both ACFA and DCF$_0$ occur as special cases of existentially closed $D$-fields, with the appropriate choice of $D$, where $D$ is a free $\mathbb{Q}$-module of finite rank. Appropriately adapting the axiom scheme given in Theorem 4.6 of [53] gives a class of “large $D$-fields” which includes both differentially large fields, and large difference fields.
This section is based on joint work with Quentin Brouette, Anand Pillay, and Françoise Point, [7].

Given an arbitrary differential field \((K, \partial)\) and a homogeneous linear differential equation \(F(y) = 0\) over \(K\) (or equivalently, a logarithmic differential equation on \(\text{GL}_n\)), a corresponding strongly normal (in this case Picard-Vessiot) extension, need not necessarily exist. Kolchin, in [41], show that if \((K, \partial)\) is a differential field such that \(C(K)\) is algebraically closed, then a PV extension of \(F(y) = 0\) over \(K\) exists and is unique up to isomorphism over \(K\). Let \((K, \partial)\) be such a field, and let \((U, \partial)\) be any differentially closed field containing \((K, \partial)\). By Fact 1.1.10, there is a differentially closed field \((\bar{K}, \partial)\) such that \((K, \partial) \subseteq (\bar{K}, \partial) \preceq (U, \partial)\) and such that every element of \((\bar{K}, \partial)\) realized an isolated type over \((K, \partial)\). Since \((\bar{K}, \partial)\) is differentially closed, it contains a fundamental system of solutions of \(L(y) = 0\) (see Section 1.3.4 for terminology). Consider some \(c \in C(\bar{K})\); \(c\) realizes an isolated type \(p(x) \in S^{\text{DCF}_0}(K)\).

Now, in \(\text{DCF}_0\), it is a classical result that the constant field \(C\) has the structure of a pure field, i.e. \(C\) has no induced structure coming from the differential field itself (one says that \(C\) is “stably embedded”). Thus, \(c\) is algebraic in the sense of fields over \(K\), say with minimal polynomial \(P(x) = x^n + \sum_{i=0}^{n-1} d_i x^i = 0\). Suppose that some \(0 \leq j \leq n - 1\) is such that \(\partial(d_j) \neq 0\). Then

\[
\sum_{i=0}^{n-1} [\partial(d_i)c^i + d_i \partial(c^i)] = \partial(d_j)c^j + \sum_{0=i}^{j-1} \partial(d_i)c^i + \sum_{i=j+1}^{n-1} \partial(d_i)c^i = 0,
\]
which contradicts the minimality of $P(x)$. Thus, it follows that $c$ is algebraic over $C(K)$ and so, since $C(K)$ is algebraically closed, $c \in C(K)$. This means that $C(K) = C(\bar{K})$ and so if $\bar{a} \in \bar{K}$ is a fundamental system of solutions for $F(y) = 0$, then $(K(\bar{a}), \partial) \subset (\bar{K}, \partial) \subseteq (U, \partial)$ is a PV extension of $(K, \partial)$ for $F(y) = 0$. In this section, we will call such a PV extension $(K(\bar{a}), \partial) \subseteq (U, \partial)$ an embedded PV extension (embedded in the sense that we were able to find it inside $U$).

In [16], the authors show the existence and uniqueness of PV extensions in the case where $(K, \partial)$ is formally real (or formally $p$-adic) and $C(K)$ is real closed (respectively, $p$-adically closed). To this end, the authors use a theorem of Serre (proven by Serre for this paper specifically) which says that for any linear algebraic group $G$ over a field $k$, there is a bijection $H^1(K, G(K)) \to H^1(k, G(k))$ when $k$ is existentially closed in $K$ and there is a bijection $\text{Gal}(K) \to \text{Gal}(k)$. To apply the theorem, the authors reduce to the case where $(K, \partial)$ is real closed (respectively, $p$-adically closed) and so (in both cases) $C(K) \preceq K$ and $\text{Gal}(K) \cong \text{Gal}(C(K))$ (since both $\mathbb{R}$ and $\mathbb{Q}_p$ are bounded!), satisfying the hypotheses of Serre’s result. Importantly, the authors show that the PV extensions themselves are again formally real or formally $p$-adic.

Extending this work, Pillay and Kamensky in [34] show the following:

**Theorem 4.0.1** (Theorem 1.5, [34]). Let $(K, \partial)$ be a differential field, $G$ an algebraic group defined over $C(K)$, and let

$$
\text{dlog}_G(z) = a
$$

(4.1)

be a logarithmic differential equation on $G$. Suppose that $C(K)$ is large, bounded, and existentially closed in $K$. Then there is a strongly normal extension $(L, \partial)$ of $(K, \partial)$ for (4.1) such that $C(L) = C(K)$ is existentially closed in $L$.

Note that this theorem just says that there is a strongly normal extension, it does not say anything abound such a strongly normal extension being embedded anywhere
in particular. The point of this chapter is to address the following question: if \((K, \partial)\) is a differential field, and \((K_0, \partial)\) is a “sufficiently nice” differential subfield of \((K, \partial)\), when can we find strongly normal extensions of \((K_0, \partial)\) embedded in \((K, \partial)\)? The aforementioned results from \cite{16} and \cite{34} and the example in the first paragraph give us a hint towards this question. Consider the case where \((K, \partial)\) is a real field and \(\mathcal{C}(K)\) is real closed. As discussed above, the authors in \cite{16} show the existence of PV extensions \((L, \partial)\) such that \(L\) is a formally real field. From Remark \ref{3.1.22} \((L, \partial)\) can be embedded into a model of CODF and so if \((U, \partial)\) is a model of CODF, and \((K, \partial)\) is a subfield such that \(\mathcal{C}(K) \preceq \mathcal{C}(K)\), then we may find PV extensions of \(K\) in \(U\). Similarly, if \((K, \partial)\) is a formally \(p\)-adic differential field such that \(\mathcal{C}(K)\) is \(p\)-adically closed, we may extend \((K, \partial)\) to a model of the model companion of \(\text{Th}(\mathbb{Q}_p) \cup \{\partial \text{ is a differential}\}\) (by first extending \(\partial\) to a \(p\)-adic closure of \(K\), which is algebraic, and then extending the derivation). Noting that both \(\mathbb{Q}_p\) and \(\mathbb{R}\) are fields with almost QE, we show here that if we start with a theory of fields \(T\) with almost QE and whose models are large and bounded, then the model companion \(T_D\) of models of \(T\) equipped with a derivation (as introduced in Chapter 3) has the property that one can find embedded PV extensions of of subfields \((K, \partial)\) so long as the constants \(\mathcal{C}(K)\) are existentially closed in the contants of the model. This generalizes the example from the first paragraph, since if \(T = \text{ACF}_0\), then \(T_D = \text{DCF}_0\) by \cite{54}.

First, we introduce a notion of almost QE for differential fields:

**Definition 4.0.2.** Let \(T\) be a theory of differential fields. We will say that \(T\) has almost quantifier elimination (almost QE) if, for any model \((K, \partial) \models T\) and any substructure \(A \subseteq K\) such that \(A\) is relatively algebraically closed in \(K\), \(T \cup \text{qftp}(A)\) is complete.

**Remark 4.0.3.** In Definition \ref{4.0.2} \(\text{qftp}(A)\) is assumed to be in the language of differential rings. Furthermore, the assumption that \(A\) is a substructure of \((K, \partial)\) implies that \(A\) is closed under \(\partial\).
The analogue of Proposition 2.0.2 holds in this case as well:

**Proposition 4.0.4.** Let $T$ be a theory of differential fields. Then $T$ as almost QE (in the sense of Definition 4.0.2) iff $T$ is the model completion of the theory $T_a$.

**Proof.** The proof is identical to the proof of Proposition 2.0.2. \qed

Now, we show that if $T$ is a theory of large fields with almost QE, then we get the differential version of almost QE in $T_D$ for free:

**Proposition 4.0.5.** Let $T$ be a theory of large fields with almost QE in a language $L$ of rings possibly expanded by constants. Then $T_D$ has almost QE in the sense of Definition 4.0.2 in the language $L_0 = L \cup \{\partial\}$.

**Proof.** It suffices to show that for any $A \models (T_D)_{RV}$, $T_D \cup qftp_{L_0}(A)$ is complete. We proceed by back and forth. Let $(M_i, \partial_i) \models T_D \cup qftp_{L_0}(A)$, for $i = 0, 1$ be sufficiently saturated such that $\partial_0 |_A = \partial_1 |_A$. Note that, as $T$ has almost QE in the language $L$, $T \cup qftp_L(A)$ is complete. In particular, $A$ is relatively algebraically closed in both $M_0$ and $M_1$.

Let $(B_0, \partial_0) \subseteq (M_0, \partial_0)$ be a relatively algebraically closed substructure extending $A$ (smaller than the degree of saturation of $M_1$), and let $p = qftp_{L_0}(B_0/A)$. By compactness and saturation of $(M_1, \partial_1)$, it suffices to show that $p$ is finitely satisfiable in $(M_1, \partial_1)$. Let $\varphi(\bar{x}) \in p$. As usual, we let $\psi(\bar{x}, \ldots, \bar{x}_n)$ be a quantifier-free formula in the language $L$ such that $\varphi(\bar{x}) \equiv \psi(\bar{x}, \partial(\bar{x}), \ldots, \partial^n(\bar{x}))$. Let $\bar{b}_0 \in B_0$ be such that $M_0 \models \psi(\bar{b}_0, \ldots, \partial^n(\bar{b}_0))$. Let $\bar{c}_0 = (\bar{b}_0, \ldots, \partial^n(\bar{b}_0))$, let $V$ be the absolutely irreducible $A$-variety for which $\bar{c}$ is generic over $A$, and let $f(\bar{x}_0, \ldots, \bar{x}_{n-1})$ be the unique rational map defined over $A$ such that $f(\bar{b}_0, \partial(\bar{b}_0), \ldots, \partial^n(\bar{b}_0)) = \partial^n(\bar{b}_0)$. It follows that the $A$-rational map

$$s : (\bar{x}_0, \ldots, \bar{x}_{n-1}) \mapsto (\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{x}_1, \ldots, \bar{x}_{n-1}, f(\bar{x}_0, \ldots, \bar{x}_{n-1}))$$
is a section of the projection $\tau(V) \to V$. Now, since $\bar{c}$ is generic for $V$ over $A$, and since $M_0 \models \psi(\bar{c}, f(\bar{c}))$, there is a proper Zariski-open subset $U \subset V$ (which we may assume is defined over $A$ and contained in dom$(s)$) such that $M_0 \models \forall \bar{y}(\bar{y} \in U \to \psi(\bar{y}, f(\bar{y})))$. Observe now that $s$, $V$, $U$ are all defined over $A$ and so, by the axioms of $T_D$, it is enough to show that $V(M_1)$ contains a smooth point. Since $\bar{c}$ is generic for $V$ over $A$, it is necessarily smooth, and so, by the fact that $T$ has almost QE in the language $\mathcal{L}$, $T \cup \text{qftp}_\mathcal{E}(A)$ implies that $V(M_0)$ contains a smooth point. As $M_1 \models T \cup \text{qftp}_\mathcal{E}(A)$, we are done. 

Remark 4.0.6. In [78], van den Dries defines a notion of “differentially bounded”, the differential analogue of algebraically bounded. Question: if $T$ is a theory of large fields with almost QE, is $T_D$ differentially bounded?

Theorem 4.0.7. Let $T$ be a theory of large, bounded fields with almost QE in a language $\mathcal{L}$ (of rings, possibly with extra constants), and let $(U, \partial) \models T_D$. Then, for any subfield $(K, \partial) \subset (U, \partial)$ such that $\mathcal{C}(K) \models T$, any connected algebraic group $G$ defined over $\mathcal{C}(K)$, and any logarithmic differential equation $d\log_G(z) = a$, there is a strongly normal extension $(L, \partial) \supseteq (K, \partial)$ that is unique up to isomorphism over $K$.

To prove Theorem 4.0.7 we require the following facts from [34]:

Fact 4.0.8. Let $(K, \partial)$ be a differential field and let $G$ be a connected algebraic group defined over $\mathcal{C}(K)$. For any logarithmic differential equation over $G$,

$$d\log_G(z) = a$$

we have:

1. If $\mathcal{C}(K)$ is existentially closed in $K$ as fields, then there exists a strongly normal extension for (4.2) (in some differentially closed field containing $K$).

2. If $\mathcal{C}(K)$ is large, bounded, and existentially closed in $K$ as fields, then there is a strongly normal extension $(L, \partial) \supseteq (K, \partial)$ for (4.2) such that $\mathcal{C}(K)$ is
existentially closed in \( L \) as fields. Furthermore, if \( L_1 \) and \( L_2 \) are such strongly normal extensions such that each embed over \( K \) into a field \( (U, \partial) \) with the property that \( C(K) \) is existentially closed in \( U \) as fields, then \( L_1 \) and \( L_2 \) are isomorphic over \( K \) as differential fields.

**Proof of Theorem 4.0.7.** Let \((U, \partial) \models T_D\), and suppose that \((K, \partial) \subseteq (U, \partial)\) is such that \(C(K) \models T\). Let \(G\) be a connected algebraic group defined over \( C(K) \) and let

\[ d\log_G(z) = a \] (4.3)

be a logarithmic differential equation on \( G \).

**Claim 4.0.9.** We may assume that \( K \) is relatively algebraically closed in \( U \).

**Proof of Claim 4.0.9.** Let \( K' \) be the relative algebraic closure of \( K \) inside \( U \). Then \( K' \) is a differential field, and \( C(K') \subseteq C(K)^{alg} \cap U \). By assumption, \( C(K) \subseteq C(U) \), and so \( C(K) \) is algebraically closed in \( U \). Hence \( C(K') = C(K) \). Suppose now that \((L, \partial)\) is a strongly normal extension of \((K', \partial)\). Then \((L, \partial)\) is a strongly normal extension of \((K, \partial)\) as well. \(\square\)

Since \( C(K) \models T \), Fact 4.0.8 gives a strongly normal extension for (4.3), say \((L, \partial)\) such that \(C(K') \preceq L\) as fields. Thus, \(L\) embeds into an elementary extension of \( C(K)\) and so \((L, \partial) \models T_{\forall \partial}\). By Proposition 3.1.20 \((L, \partial)\) can be extended to a model \((L', \partial)\) of \( T_D \). Observe that, as \((K, \partial)\) is relatively algebraically closed in \((U, \partial)\), Proposition 4.0.5 implies that \((L', \partial) \equiv_K (U, \partial)\). Let \(L = K(g)\), where \(g\) is such that \(d\log_G(g) = a\).

**Claim 4.0.10.** Let \( p = qftp_{L, \partial}(g/K) \). Then \( p \) is isolated.

The statement is Lemma 2.2 from [46], but we will include a proof here so that the proof of Theorem 4.0.7 is somewhat more self-contained.

**Proof of Claim 4.0.10.** Note that quantifier elimination in \( \text{DCF}_0 \) implies that \( p = \text{tp}_{\text{DCF}_0}(g/K) \). Let \((M, \partial) \models \text{DCF}_0\) extend \( U \) and \( L' \). Recall that, in \( \text{DCF}_0 \), acl
is just the usual field-theoretic algebraic closure. Now \( L \subseteq L' \) and \( K \) is relatively algebraically closed in \( L' \), so \( K(g) \) is regular over \( K \). It follows \( \text{acl}_M(Kg) \cap K = K \) and so the type \( q = \text{tp}_M(g/K) \) is stationary by Lemma 1.1.8.

Let \( g' \) be some other realization of \( p \) in \( M \), so in particular \( \text{dlog}_G(g') = a \). By Fact 1.3.22, \( g' = gh \) for some \( h \in G(C(M)) \), and so \( g' \in \text{dcl}_M(K,g,C(M)) \). By compactness, there is a formula \( \varphi(x) \in p \) and a partial function \( f(x,y) \) defined over \( A \) such that for any \( g, g' \) realizations of \( p \), there is \( c \in C(M) \) such that \( f(g,c) = g' \).

Since \( \text{DCF}_0 \) is \( \omega \)-stable, Fact 1.1.10 says there is \( M_0 \preceq M \) such that \( M_0 \) is prime over \( K \cup C(M) \) and such that every element of \( M_0 \) realizes an isolated type over \( K \cup C(M) \).

For any \( g' \in M_0 \) realizing \( \varphi(x) \), and any realization \( g \) of \( p \) in \( M \), there is \( c \in C(M) \) such that \( f(g',c) = g \). Since \( \text{tp}_{\text{DCF}_0}(g'/K \cup C(M)) \) is isolated, \( \text{tp}_{\text{DCF}_0}(g/K \cup C(M)) \) is isolated as well. By stationarity, \( \text{tp}_{\text{DCF}_0}(g/K \cup C(M)) \) is the unique, non-forking extension of \( p \), and so, by a compactness argument, \( p \) is isolated as well.

Let \( \psi(x) \) isolate \( p \). For any realization \( g' \) of \( \psi(x) \) (in some differentially closed field), \( K(g) \cong K(g') \) as differential fields, and so \( K(g') \) is a strongly normal extension for (4.3) as well. Since \( (L', \partial) \cong_K (U, \partial) \), we have that \( (U, \partial) \models \exists x \psi(x) \), and any realization generates the required strongly normal extension for (4.3).

For the uniqueness of the embedded strongly normal extension, use Fact 4.0.8 part 2 (this is Theorem 1.4 in [34]).
BIBLIOGRAPHY


2. J. Ax. On the undecidability of power series fields. 


10. Z. Chatzidakis. Simplicity and independence for pseudo-algebraically closed fields. 


