SOME RESULTS IN COMPUTABILITY THEORY

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Abstract

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We consider the question of universality among computable \(\omega\)-branching trees. To this end, we construct a computable \(\omega\)-branching tree \(T_{KP}\) whose paths compute the complete diagrams of the countable \(\omega\)-models of Kripke-Platek set theory (KP). We show that, given a path \(f\) through \(T_{KP}\), representing a model \(M\) of KP, and another computable ill-founded \(\omega\)-branching tree \(T\), if \(f\) fails to compute a path through \(T\), then \(M\) assigns to \(T\) a nonstandard ordinal tree rank. Further, we indicate some circumstances in which, given computable \(\omega\)-branching trees \(T_0\) and \(T_1\), a path through \(T_{KP}\) helps the paths through \(T_0\) compute paths through \(T_1\).

In a different line of work, we consider effective forcing notions. In particular, we define a class of effective forcing notions that are similar to versions of Mathias forcing and Cohen forcing defined in the literature, and prove some results about how these notions relate. As a consequence, we see that the generics for an effective version of Mathias forcing compute generics for an effective version of Hechler forcing, and vice-versa. Later, we focus on a notion of Mathias forcing over a countable Turing ideal, defined by Cholak, Dzhafarov, and Soskova. We show that there are nested Turing ideals for which the Mathias generics for the larger ideal do not all compute Mathias generics for the smaller ideal.
# CONTENTS

Acknowledgments .................................................. iv

Chapter 1: Introduction ........................................... 1
  1.1 Computability-theoretic conventions and notation .......... 2
  1.2 Hierarchies of sets ........................................... 3
    1.2.1 The arithmetical hierarchy ............................... 4
    1.2.2 The hyperarithmetical hierarchy ......................... 4
    1.2.3 The constructible hierarchy ............................ 6
  1.3 Trees .......................................................... 6
    1.3.1 Basis theorems .......................................... 7
    1.3.2 Universal oracles and universal trees ................... 9

Chapter 2: Kripke-Platek set theory and paths through computable \( \omega \)-branching trees ........................................ 13
  2.1 Introduction .................................................. 13
  2.2 Determining which \( \omega \)-branching trees are well-founded .... 15
  2.3 Kripke-Platek set theory ..................................... 19
    2.3.1 Axioms of KP ............................................. 19
    2.3.2 Consequences of KP .................................... 21
    2.3.3 Models of KP ............................................ 23
    2.3.4 Computability theory in an \( \omega \)-model of KP .......... 25
  2.4 Trees in an \( \omega \)-model of KP ................................ 27
    2.4.1 Formalizing trees, paths, and ranks in an \( \omega \)-model of KP 30
  2.5 Using paths through \( T_{KP} \) to compute paths through other trees .... 35
    2.5.1 Determining when an \( \omega \)-model of KP computes a path through a tree \( T \in \omega^{<\omega} \) ............... 36
    2.5.2 Pairs of trees ......................................... 40
  2.6 Using Kleene-Brouwer orderings instead of tree ranks .......... 42
    2.6.1 Linear orderings and well-orderings in an \( \omega \)-model of KP .... 43
    2.6.2 Kleene-Brouwer orderings in an \( \omega \)-model of KP .......... 47

Chapter 3: Effective forcing notions .............................. 51
  3.1 Introduction .................................................. 51
  3.2 Cohen-Mathias-like forcing partial orders .................... 55
  3.3 Mathias generics over countable Turing ideals ............... 71
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CHAPTER 1

INTRODUCTION

The main content of this thesis is divided into two parts, Chapters 2 and 3. In this Chapter (Chapter 1), we summarize the results in Chapters 2 and 3, and then present some necessary background information.

In Chapter 2 we consider the problem of computing infinite paths through computable trees. It is known that the computable binary tree whose paths represent completions of Peano arithmetic (PA) is universal among all computable binary trees; i.e., any completion of PA can compute a path through any ill-founded computable binary tree. The situation is much different for computable ω-branching trees (see, e.g., Theorem 1.3.10). In joint work with Julia Knight and Dan Turetsky, we consider a computable ω-branching tree whose paths code the complete diagrams of countable ω-models of Kripke-Platek set theory (KP). Our goal is to understand the circumstances in which a path through this particular tree can compute a path through another computable ω-branching tree. To accomplish this, we study how a given ω-model of KP understands the concepts of tree rank and Kleene-Brouwer orderings. We show (Theorem 2.5.1) that the complete diagram of a countable ω-model M of KP fails to compute a path through a computable ill-founded ω-branching tree T only if T is assigned a nonstandard ordinal tree rank in M. Next, in Theorem 2.5.3 we show that if M assigns nonstandard ordinal tree rank to two computable trees, then, from the complete diagram of M and any path through the tree of smaller rank, we can compute a path through the tree of greater rank. Finally, in Section 2.6 we verify that, if we had focussed on Kleene-Brouwer orderings, instead of tree
ranks, we would have arrived at analogous results.

In Chapter 3, we study various effective forcing notions, including variants of Cohen, Mathias, and Hechler forcing, each of which corresponds to a collection of generic subsets of the natural numbers. In Section 3.2, we carefully analyze constructions used in [3] and [4], isolating properties that ensure that the generics associated to a given effective forcing notion can compute generics for another. In particular, we show that the generics for the version of Mathias forcing defined in [3] compute generics for a version of Hechler forcing, with computable conditions, and vice-versa. In Section 3.3, we work with a notion of Mathias forcing, with conditions from a countable Turing ideal, defined in [4]. We answer a question from [4], showing that there are nested pairs of Turing ideals, such that the Mathias generics over the larger ideal do not all compute Mathias generics over the smaller ideal.

1.1 Computability-theoretic conventions and notation

We denote by $\omega^\omega$ the set of finite strings of natural numbers, and $2^{\omega} \subseteq \omega^\omega$ the set of finite binary strings. The empty string is denoted by $\lambda$. Given $\sigma \in \omega^\omega$, the length $|\sigma|$ of $\sigma$ is the largest number $n \in \omega$ such that for all $i < n$, $\sigma(i)$ is defined. Given strings $\sigma, \tau \in \omega^\omega$, we write $\sigma \preceq \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for all $i < |\sigma|$. The concatenation of two strings $\sigma, \tau \in \omega^\omega$ is the string $\sigma \ast \tau$ formed by following the sequence $\sigma$ with the sequence of numbers in $\tau$. We fix a sensible way, e.g., using Gödel numbers, to assign a natural number code to every $\sigma \in \omega^\omega$, and every finite set $F \subseteq \omega$; we call these canonical indices.

We denote by $2^\omega$ the set of all functions $f : \omega \rightarrow \{0, 1\}$, and by $\omega^\omega$ the set of all functions $f : \omega \rightarrow \omega$. The elements of $2^\omega$ are the characteristic functions of subsets of $\omega$. For this reason, we often identify $2^\omega$ with the power set of $\omega$. For instance, given $X \subseteq \omega$ and $n \in \omega$, the notation $X(n) = 1$ means that $n \in X$, and $X(n) = 0$ means that $n \notin X$. Given $X \subseteq \omega$, $f : \omega \rightarrow \omega$, and $\sigma \in \omega^\omega$ a string of length $n$, the notation
σ \prec X \text{ means that } \sigma = X|_n, \text{ where } X \text{ is identified with its characteristic function. The notation } \sigma \prec f \text{ means that } \sigma = f|_n.

Cantor space is the set $2^\omega$ with the topology generated by basic clopen sets $[\sigma] = \{X \subseteq \omega : \sigma \prec X\}$, for $\sigma \in 2^{\omega}$. Similarly, Baire space is the set $\omega^\omega$ with the topology generated by basic clopen sets $[\sigma] = \{f : \omega \to \omega : \sigma \prec f\}$, for $\sigma \in \omega^{<\omega}$.

We follow standard conventions for Turing computations (see, e.g., [21]). Given $X \subseteq \omega$, and $e, s, x, y \in \omega$, the notation $\varphi^X_{e,s}(x) \downarrow = y$ means that the $e^{th}$ Turing machine halts, in $s$ steps, on input $x$, with oracle $X$, and outputs $y$. The notation $\varphi^X_e(x) \downarrow$ means that $(\exists s)(\exists y)\varphi^X_{e,s}(x) \downarrow = y$, and $\varphi^X_e(x) \uparrow$ means $\neg(\varphi^X_e(x) \downarrow)$. We assume that if $\varphi^X_{e,s}(x) \downarrow = y$, then the program only makes queries about numbers $n < s$ in the oracle $X$.

Suppose $A, B \subseteq \omega$ are given. We say that $A$ Turing computes $B$, or $B \leq_T A$, if there is some $e \in \omega$ such that $(\forall x)(\varphi^A_e(x) \downarrow = B(x))$. Given $e \in \omega$, and $A \subseteq \omega$, $W^A_e = \{x : \varphi^A_e(x) \downarrow\}$ is the $e^{th}$ r.e. (recursively enumerable) set relative to $A$. We say that $B$ is r.e. relative to $A$ if $B = W^A_e$, for some $e \in \omega$. We say that $A$ is computable, or recursive, if $A \leq_T \emptyset$. If $A$ is r.e. relative to $\emptyset$, we say that $A$ is r.e. Finally, the Turing jump of $A$ is the set $A' = \{x \in \omega : \varphi^A_e(x) \downarrow\}$. For $n > 1$, the $n^{th}$ jump of $A$ is the set $A^{(n)} = (A^{(n-1)})'$. For all $n \in \omega$, $X^{(n)} \prec_T X^{(n+1)}$.

1.2 Hierarchies of sets

We consider sets belonging to several different hierarchies. The arithmetical hierarchy is a hierarchy of subsets of $\omega$ (or $\omega^k$, for $k \in \omega$) generated by the sets $\emptyset^{(n)}$, for $n \in \omega$. The hyperarithmetical hierarchy is a larger collection of subsets of $\omega$ generated by iterating the Turing jump along the computable ordinals. Finally, the constructive hierarchy is a class of sets, containing the hyperarithmetical sets, generated in stages by induction on all ordinals.
1.2.1 The arithmetical hierarchy

Suppose \( X \subseteq \omega^k \), for some \( k \in \omega \). We say that \( X \) is \( \Sigma^0_1 \) if \( X \) is defined by a formula \((\exists y) R(\overline{x}, y)\), where \( R \subseteq \omega^{k+1} \) is recursive; \( X \) is \( \Pi^0_1 \) if its complement is \( \Sigma^0_1 \). Given \( n > 1 \), we say that \( X \) is \( \Sigma^0_n \) if it is defined by a formula \((\exists y) R(\overline{x}, y)\), where \( R \subseteq \omega^{k+1} \) is \( \Pi^0_{n-1} \); \( X \) is \( \Pi^0_n \) if its complement is \( \Sigma^0_n \). Finally, for \( n \geq 1 \), \( X \) is \( \Delta^0_n \) if \( X \) is both \( \Sigma^0_n \) and \( \Pi^0_n \). The arithmetical hierarchy is the collection of all \( \Delta^0_n \) sets, for \( n \geq 1 \). The next result shows that the sets \( \varnothing^{(n)} \) form the backbone of the arithmetical hierarchy.

**Theorem 1.2.1** (Post). Fix \( n \geq 1 \), and \( X \subseteq \omega^k \), for some \( k \in \omega \).

1. \( X \) is \( \Sigma^0_n \) iff \( X \) is r.e. in \( \varnothing^{(n-1)} \);
2. \( X \) is \( \Delta^0_n \) iff \( X \leq_T \varnothing^{(n-1)} \).

1.2.2 The hyperarithmetical hierarchy

We give a very brief introduction to the hyperarithmetical hierarchy. For more detail on the results or definitions discussed here, see \([1]\) or \([17]\). Recall that \( \omega_{CK}^1 \) denotes the first non-computable ordinal; that is, the first ordinal \( \alpha < \omega_1 \) such that there is no computable linear ordering on \( \omega \) with order type \( \alpha \). We have seen that the arithmetical hierarchy is generated by sets \( \varnothing^{(n)} \), for \( n \in \omega \); to get the hyperarithmetical hierarchy, we first define what it means to take the \( \alpha^{th} \) jump of a set, for \( \alpha < \omega_{CK}^1 \). To accomplish this, we define a set \( \mathcal{O} \subseteq \omega \) of natural number notations for the computable ordinals, called Kleene’s \( \mathcal{O} \). We build the set \( \mathcal{O} \) of ordinal notations, a partial ordering \( \prec_{\mathcal{O}} \subseteq \mathcal{O} \times \mathcal{O} \), and a function \( |\cdot| : \mathcal{O} \to \omega_{CK}^1 \) taking notations to the ordinals they represent by induction on \( \alpha < \omega_{CK}^1 \), as follows. First, we assign the notation \( 1 \in \mathcal{O} \) to \( \alpha = 0 \); i.e., \( |1|_{\mathcal{O}} = 0 \). If \( \alpha = \beta + 1 \) is a successor ordinal, then the notations for \( \alpha \) are the numbers \( 2^b \), where \( b \in \mathcal{O} \) and \( |b|_{\mathcal{O}} = \beta \); we define \( c \prec_{\mathcal{O}} 2^b \) if \( c \prec_{\mathcal{O}} b \) or \( c = b \). Finally, if \( \alpha < \omega_{CK}^1 \) is a limit ordinal, the notations for \( \alpha \) are the numbers \( 3 \cdot 5^e \), where \( e \) is an index for a total computable function, and for all \( n \in \omega \), \( \varphi_e(n) \downarrow \varphi_e(n+1) \), and
α = supₙ |φₜ(n)|₁. We let b <₁ 3 ⋅ 5ₑ if b <₁ φₜ(n), for some n ∈ ω.

We define sets H(a), for every a ∈ O, by induction on <₁, as follows. We begin with H(1) = ∅. Given a = 2ₖ in O, we let H(a) = H(b)' if b <₁ φₜ(n), for some n ∈ ω. We let H(a) = {< x, n > : x ∈ H(φₜ(n))}.

**Theorem 1.2.2** (Spector). Given a, b ∈ O, if |a|₁ = |b|₁, then H(a) ≡ₜ H(b).

With Theorem 1.2.2 in mind, we denote by 0ₐ the Turing degree of the sets H(a), for a ∈ O with |a|₁ = α. Given α < ω₁CK and a set X ⊆ ωₖ, for some k ∈ ω, we say that X is Σₐ if X is r.e. in H(a), for some a ∈ O with |a|₁ = α. Such a set X ⊆ ωₖ is Πₐ if its complement is Σₐ. Finally, the ∆ₐ sets are those that are both Σₐ and Πₐ. The hyperarithmetic hierarchy consists of the ∆ₐ sets, for all α < ω₁CK.

We can relativize the results and definitions in the previous paragraphs to any set Y ⊆ ω. The ordinal ω₁ᵢ is the least ordinal not computable from Y. We get a system of notations Oᵢ ⊆ ω for the ordinals less than ω₁ᵢ, and jump sets H(a)ᵢ for every a ∈ Oᵢ. Given two sets X, Y ⊆ ω, we say that X is hyperarithmetical relative to Y, or X ≤ₜ H(a)ᵢ, for some a ∈ Oᵢ.

There is a connection between the hyperarithmetic sets, and sets defined using quantifiers over functions f ∈ ωω. For any k ∈ ω, a set X ⊆ ωₖ is Σ₁ if it has a definition of the form (∃ f)(∀ y)R(x, y, f), where R ⊆ ωₖ₊₁ × ωω is a recursive predicate. The set X is Π₁ if its complement is Σ₁. Finally, X is Δ₁ if it is both Σ₁ and Π₁.

**Theorem 1.2.3** (Kleene). The hyperarithmetic sets are exactly the Δ₁ sets.

It is also true that, given X, Y ⊆ ω, X ≤ₜ Y iff X is Δ₁ relative to Y; i.e., there are Σ₁ predicates P(x, Z), Q(x, Z), each with one free number variable and one free set variable, such that x ∈ X ↔ P(x, Y), and x ∉ X ↔ Q(x, Y).

The last result we mention here is that the set O of ordinal notations is Π₁¹ complete.
Theorem 1.2.4 (Kleene). Given any $\Pi^1_1$ set $X$, there is a total recursive function $f$ such that for all $x$, $x \in X \iff f(x) \in \mathcal{O}$.

1.2.3 The constructible hierarchy

The constructible universe, $L$, is a class of sets first defined by Gödel [9]. The class $L$ is built from sets $L_\alpha$, for every ordinal $\alpha$. The set $L_0$ is the empty set. If $\alpha = \beta + 1$ is a successor ordinal, then $L_\alpha$ is the set whose elements are the definable subsets of $L_\beta$. That is, an element $y \in L_\alpha$ has form $y = \{ x \in L_\beta : (L_\beta, \in) \models \varphi(y, b) \}$, where $\varphi$ is a formula in the language $\{ \in \}$ of set theory, and $b \in L_\beta$ is a finite tuple of parameters from $L_\beta$.

We can relativize the constructible hierarchy to any set $X$. We let $L_0(X)$ be the smallest transitive set with $X$ as an element, and then proceed exactly as in the previous paragraph to get the sets $L_\alpha(X)$, for every ordinal $\alpha$.

1.3 Trees

We present here some well-known information about trees. More specifically, we consider binary, finite-branching, and $\omega$-branching trees, and various theorems about computing paths through such trees.

Definition 1.3.1. A tree is a subset $T \subseteq \omega^\omega$ such that for every $\sigma$, $\tau \in \omega^\omega$, if $\sigma \in T$ and $\tau \subseteq \sigma$, then $\tau \in T$. We often refer to the strings $\sigma$ in a tree $T$ as nodes. If $T \subseteq \omega^\omega$ is a tree, we say $T$ is $\omega$-branching. If, for every $n \in \omega$, there are only finitely many nodes of length $n$ in $T$, we say that $T$ is finite-branching. If $T \subseteq 2^{\omega}$, we say that $T$ is binary. A path through a tree $T$ is a function $f : \omega \to \omega$ such that $f|_n \in T$ for all $n \in \omega$. The set of all paths through a tree $T$ is denoted by $[T]$.

Given a computable tree $T \subseteq \omega^\omega$, a computable index for $T$ is a natural number $e \in \omega$ such that $\varphi_e$ is the characteristic function of $T$, where strings are coded by their Gödel
numbers. If a tree $T$ has no paths, we say that it is well-founded; if it has a path, we say it is ill-founded. The well-founded binary-branching and finite-branching trees are exactly those that are finite; this result is called König’s Lemma.

**Lemma 1.3.2** (König’s Lemma). *Given a finite-branching tree $T \subseteq \omega^\omega$, $T$ is well-founded iff $T$ is infinite.*

Chapter 2 is devoted to understanding how $\omega$-branching trees behave differently from binary and finite-branching trees. As a start, we can easily see that Lemma 1.3.2 cannot be extended to the $\omega$-branching trees. That is, there are infinite $\omega$-branching trees with no paths. Take, for instance, the subtree of $\omega^\omega$ consisting of all possible strings of length 1.

The following notation is used throughout Chapter 2.

**Notation 1.3.3.** Suppose $T \subseteq \omega^\omega$ is a tree. Fix $n \in \omega$ and $\sigma \in T$.

1. Let $T^{\text{ext}} \subseteq T$ be the tree of all nodes that extend to a path through $T$;
2. Let $T^n = \{ \tau \in T : |\tau| = n \}$ be the set of nodes in $T$ of length $n$;
3. Let $T^{\geq n} = \{ \tau \in T : |\tau| \geq n \}$ be the set of nodes in $T$ of length at least $n$;
4. Let $\text{comp}(T, \sigma) = \{ \tau \in T : \tau \leq \sigma \lor \tau \geq \sigma \}$ be the subtree of $T$ consisting of nodes that are comparable with $\sigma$ in the ordering $\leq$;
5. Let $\text{above}(T, \sigma) = \{ \tau \in \omega^\omega : \sigma * \tau \in T \}$ be the tree consisting of nodes that, when concatenated with $\sigma$, result in a node on $T$.

We occasionally combine the notations from 1.3.3. For example, $(\text{comp}(T, \sigma))^n$ means the set of all $\tau \in T$ that extend $\sigma$ and have length $n$, whereas $(\text{above}(T, \sigma))^n$ means the set of all $\tau \in \omega^n$ such that $\sigma * \tau \in T$.

### 1.3.1 Basis theorems

A *basis theorem* for trees is a result showing that all ill-founded trees of a certain type have paths of a certain complexity.
The following result of Jockusch and Soare [13] is known as the Low Basis Theorem.

**Theorem 1.3.4** (Jockusch and Soare). Suppose $T \subseteq 2^{<\omega}$ is an infinite computable binary branching tree. Then $T$ has a low path; that is, a path $f \in [T]$ such that $f' \equiv_T \emptyset'$.

We can relativize Theorem 1.3.4 to computable finite-branching trees. Indeed, any computable ill-founded finite-branching tree $T$ has a path $f \in [T]$ that is low relative to $\emptyset'$. That is, $(f \oplus \emptyset')' \equiv_T \emptyset''$.

The following result implies that, to get a basis theorem for the computable ill-founded $\omega$-branching trees, we have to look further than the hyperarithmetic hierarchy.

**Lemma 1.3.5.** There is a computable $\omega$-branching tree, with paths, but with no hyperarithmetic path.

*Proof.* Harrison [11] showed that there is a computable linear order $H$, with field $\omega$, of order type $\omega^{CK}_1(1 + \eta)$, with no hyperarithmetic infinite descending sequence. Let $T_H \subseteq \omega^{<\omega}$ be the computable tree whose nodes are finite $H$-decreasing sequences. The paths through $T_H$ are exactly the infinite descending sequences in $H$. Thus, $T_H$ has paths, but no hyperarithmetic path.

The following is called the Gandy Basis Theorem.

**Theorem 1.3.6** (Gandy). Suppose $T \subseteq \omega^{<\omega}$ is a computable tree, with paths. Then there is a path $f \in [T]$ such that $f <_h \mathcal{O}$.

The next result, due to Spector, implies that the path $f \in [T]$ guaranteed by the Gandy Basis Theorem satisfies $\omega_1^{f} = \omega_1^{CK}$. That is, $f$ computes only the computable ordinals. For this reason, we often say that such a path $f$ is low for $\omega_1^{CK}$.

**Theorem 1.3.7** (Spector). For any $X \subseteq \omega$, $\omega_1^{X} = \omega_1^{CK} \iff \mathcal{O} <_h X$. 

8
1.3.2 Universal oracles and universal trees

Given a set $X \subseteq \omega$, we say $X$ is universal for the trees belonging to a given collection $C$ of trees, if $X$ computes a path through every ill-founded tree in $C$. Furthermore, if $C$ is a collection of computable trees, we say $X \subseteq \omega$ solves the problem of well-foundedness for trees in $C$ if there is a partial $X$-computable function $f$ such that, given a computable index $e \in \omega$ for a tree $T \in C$, if $T$ is well-founded, then $f(e) = 1$, and if $T$ is ill-founded, then $f(e) = 0$.

Suppose $X \subseteq \omega$ solves the problem of well-foundedness for trees in $C$, and suppose, in addition, that for any $T \in C$ and $\sigma \in T$, $\text{comp}(T, \sigma) \in C$. Then $X$ is universal for $C$. Indeed, fix a tree $T \in C$ with paths. We define an $X$-computable sequence $\sigma_0 \geq \sigma_1 \ldots$ of nodes in $T$ such that $\bigcup_n \sigma_n$ is an infinite path through $T$, as follows. Let $\sigma_0 = \lambda$. Given $\sigma_n$ that extends to a path through $T$, for some $n \in \omega$, use $X$ to find the least number $k$ such that $\sigma_n \ast k \in T$ and $X$ says that $\text{comp}(T, \sigma_n \ast k)$ is ill-founded. Then let $\sigma_{n+1} = \sigma_n \ast k$.

In Proposition 1.3.8, we see that the sets $X \subseteq \omega$ that solve the well-foundedness problem for the computable binary trees, the computable finite-branching trees, and the computable $\omega$-branching trees are exactly those sets $X$ that compute $\emptyset'$, $\emptyset''$, and $\mathcal{O}$, respectively.

**Proposition 1.3.8.** Let $C$ be the collection of all computable binary-branching trees, let $D$ be the collection of all computable finite-branching trees, and let $E$ be the collection of all computable $\omega$-branching trees. Fix $X \subseteq \omega$.

1. The set $X$ solves the problem of well-foundedness for trees in $C$ iff $X \geq_T \emptyset'$.
2. The set $X$ solves the problem of well-foundedness for trees in $D$ iff $X \geq_T \emptyset''$.
3. The set $X$ solves the problem of well-foundedness for trees in $E$ iff $X \geq_T \mathcal{O}$.

**Proof.** (Part 1): If $X \geq_T \emptyset'$, then, given an index $e$ for a computable binary-branching tree $T$, $X$ can determine whether $T$ is finite or infinite, and so can correctly determine
whether $T$ is well-founded or ill-founded. For the other direction, there is a total computable function $g$ such that for all $n \in \omega$, if $n \in \varnothing'$, then $g(n)$ is an index for a finite binary-branching tree, and if $n \notin \varnothing'$, $g(n)$ is an index for an infinite binary-branching tree. For instance, given $n \in \omega$, let $g(n)$ be an index for the tree $T$ consisting of all $\sigma \in 2^{<\omega}$ such that $\varphi_{n,|\sigma|}(n) \uparrow$.

(Part 2): Suppose $X \geq_T \varnothing''$. Then $X$ computes the set $S = \{e : (\exists n)(\forall \sigma \in \omega^n)(\forall s)(\varphi_{e,s}(\sigma) \downarrow = 1)\}$. Given a computable index $e$ for a finite-branching tree $T \subseteq \omega^{<\omega}$, $T$ is well-founded iff $e \in S$. Thus, $X$ can solve the well-foundedness problem for $D$. Next, suppose that $X$ solves the well-foundedness problem for $D$. Recall (see [21]) that the set $FIN = \{e : |W_e| < \infty\}$ is Turing equivalent to $\varnothing''$. We define a total computable function $f$ with the property that, for all $e \in \omega$, $f(e)$ is a computable index for a finite branching tree $T$, such that $T$ is well-founded iff $e \in FIN$. Given $e \in \omega$, let $f(e)$ be an index for the tree $T = \{\sigma \in \omega^{<\omega} : (\forall m < n < |\sigma|)(|W_{e,\sigma(m)}| < |W_{e,\sigma(n)}|)\}$. That is, the nodes $\sigma$ in the tree $T$ track the stages at which numbers are added to $W_e$. If $e \in FIN$, then $W_e$ eventually stops growing, so the tree with index $f(e)$ is finite. If $e \notin FIN$, then the tree with index $f(e)$ consists of a single infinite path.

(Part 3): Kleene and Spector showed that both $O$ and the set $WF$ of indices for computable well-founded $\omega$-branching trees are $\Pi^1_1$-complete (see, e.g., [1]). In particular, $O \equiv_T WF$. Thus, if $X \geq_T O$, then $X$ can sort the indices for the well-founded computable $\omega$-branching trees from those for the ill-founded computable $\omega$-branching trees. For the other direction, note, first, that the set $I$ of computable indices for all computable $\omega$-branching trees is $\Pi^0_3$. If $X$ solves the well-foundedness problem for $\mathcal{E}$, then $X$ also solves the well-foundedness problem for $D$. Thus, by Part 2, $X \geq_T \varnothing'' \geq_T I$. Finally, given $e \in \omega$, $e \in WF$ iff $e \in I$ and $X$ believes that $e$ is an index for a well-founded tree. Thus, $X \geq_T WF \equiv_T O$.

The universal oracles for the computable binary trees are exactly the sets $X \subseteq \omega$
that compute complete extensions of Peano arithmetic (PA). This is implied by results of Scott [18]. To understand these results, we define the notions of Turing ideals and Scott sets, which will be used again in Section 3. A Turing ideal is a collection \( I \) of subsets of \( \omega \), such that, for all \( X, Y \subseteq \omega \),

1. If \( X \in I \) and \( Y \leq_T X \), then \( Y \in I \);
2. If \( X \in I \) and \( Y \in I \), then \( X \oplus Y \in I \), where \( X \oplus Y = \{ 2n : n \in X \} \cup \{ 2n+1 : n \in Y \} \).

Given a countable Turing ideal \( I \), an enumeration of \( I \) is a set \( E \subseteq \omega \times \omega \) such that \( I = \{ X : (\exists e)(\forall n)(n \in X \iff (n, e) \in E) \} \). A Scott set is a Turing ideal \( I \), with the additional property that for any infinite binary tree \( T \) that is coded by some set in \( I \), there is a path \( f \in [T] \) that is also coded by a set in \( I \).

Scott [18] showed that if \( T \) is a complete extension of Peano arithmetic (PA), then the collection of sets representable in \( T \), denoted \( Rep(T) \), is a countable Scott set, where \( X \subseteq \omega \) is in \( Rep(T) \) iff there is some formula \( \varphi(x) \) in the language of arithmetic such that \( (\forall n \in \omega)(n \in X \iff \varphi(n) \in T) \). (Here, by \( n \), we mean the term \( 1 + \cdots + 1 \) defining the natural number \( n \) in the language of arithmetic.)

Furthermore, we have the following, also due to Scott [18]. Given sets \( X, Y \subseteq \omega \), the notation \( X \ll Y \) means that \( Y \) computes a path through every infinite binary \( X \)-computable tree.

**Theorem 1.3.9** (Scott). Fix \( X, Y \subseteq \omega \). The following are equivalent:

1. The set \( Y \gg X \);
2. The set \( Y \) computes an enumeration \( E \) of a Scott set containing \( X \);
3. The set \( Y \) computes a completion \( T \) of Peano arithmetic, with \( X \in Rep(T) \).

We can construct a computable binary tree \( T_{PA} \) whose paths represent the completions of PA. Applying Theorems \[1.3.4\] and \[1.3.9\] to this tree, we get a set \( X \subseteq \omega \) that is universal for the computable binary trees, with \( X' \equiv_T \emptyset' \). By Proposition \[1.3.8\]
this set \( X \) is an example of a universal set that does not solve the well-foundedness problem for computable binary trees. The tree \( T_{PA} \) is often called a \textit{universal tree} among the computable binary trees, since all paths in \( T_{PA} \) are universal for the computable binary trees. We end this section with a result, due to Simpson \cite{20} that implies that there is no computable \( \omega \)-branching tree that behaves analogously to \( T_{PA} \).

**Theorem 1.3.10** (Simpson). Suppose \( T_0 \subseteq \omega^{<\omega} \) is a computable ill-founded tree. There is a computable ill-founded tree \( T_1 \subseteq \omega^{<\omega} \) such that every path through \( T_1 \) computes a path through \( T_0 \), but not every path through \( T_0 \) computes a path through \( T_1 \).

**Proof.** By the Gandy Basis Theorem, there is some \( f \in [T_0] \) such that \( f \prec_h \mathcal{O} \). Fix a computable ill-founded tree \( T \subseteq \omega^{<\omega} \) such that \( f \) does not compute a path through \( T \). (If there were no such tree \( T \), then for any computable tree \( T \subseteq \omega^{<\omega} \), \( T \) is well-founded iff \((\forall i)(\neg \varphi^f_i \in [T])\), and hence the set \( WF \equiv_T \mathcal{O} \) of indices for computable well-founded subtrees of \( \omega^{<\omega} \) would be arithmetic in \( f \), a contradiction.)

Let \( T_1 \subseteq \omega^{<\omega} \) be the \textit{tree product} of \( T_0 \) and \( T \). That is, the strings \( \sigma \in T_1 \) are those strings whose even bits form a string in \( T_0 \), and whose odd bits form a string in \( T \). The tree \( T_1 \) is computable and ill-founded. Furthermore, given any path \( g \in [T_1] \), the even bits of \( g \) are a path through \( T_0 \), and the odd bits of \( g \) are a path through \( T \). From this, it is clear that \( T_1 \) has the desired properties. \( \square \)
2.1 Introduction

In this chapter, we consider computability-theoretic questions about trees. In particular, we explore the ways in which some classical results about binary-branching and finite-branching trees extend to the world of $\omega$-branching trees, with a focus on the question of universality. That is, given a collection of computable $\omega$-branching trees, which oracles can compute a path for each ill-founded tree in the collection?

In Section 1.3.2, we mention several well-known examples of universal oracles. If $X \subseteq \omega$ can tell which computable trees of a given branching type are well-founded, and which are ill-founded, then $X$ is universal for trees of that type. In particular, $\varnothing'$ is universal for computable binary-branching trees, $\varnothing''$ is universal for computable finite-branching trees, and $\mathcal{O}$ is universal for computable $\omega$-branching trees. In the case of computable binary-branching trees, if we relax the constraint that $X$ must be completely correct about which trees are well-founded, we get a wider class of universal oracles. In fact, Scott (Theorem 1.3.9) showed that the universal oracles for the computable binary-branching trees are exactly those oracles $X$ that have the

The work in this chapter is part of a joint project with Julia Knight and Dan Turetsky. We have endeavored to make the chapter self-contained, and have also tried to indicate which of the main results are due to Knight and Turetsky, and which are joint work with Knight and Turetsky.
same Turing degree as a completion of Peano arithmetic (PA). Moreover, there is a single computable binary-branching tree $T_{PA}$ whose paths represent exactly the completions of PA.

In this chapter, we examine how a stronger theory, Kripke-Platek set theory (KP), fares in the role of PA when we consider universality for computable $\omega$-branching trees. Specifically, we construct a computable $\omega$-branching tree $T_{KP}$ whose paths represent the complete diagrams of the countable $\omega$-models of KP. Simpson (Theorem 1.3.10) showed that there is no universal computable $\omega$-branching tree. In particular, $T_{KP}$ has paths that fail to compute paths through other computable ill-founded $\omega$-branching trees. We wish to elucidate how the paths through $T_{KP}$ fail to be universal, and also to find sufficient conditions for a path through $T_{KP}$ to compute a path through another ill-founded, computable $\omega$-branching tree.

This chapter is divided into five further sections. In Section 2.2 we review two objects associated with $\omega$-branching trees; namely, the tree rank function and the Kleene-Brouwer ordering. We see how the question of whether a given $\omega$-branching tree has a path can be reduced to the question of whether it has a tree rank function, or whether its Kleene-Brouwer ordering is a well ordering. In Section 2.3 we introduce Kripke-Platek set theory as the base theory in which we will consider tree rank functions and Kleene-Brouwer orderings. We present the axioms and some known consequences of KP. We also describe several well-known results about models of KP, and see that many computability-theoretic arguments can be carried out in an $\omega$-model of KP. Next, in Section 2.4 we construct a computable tree $T_{KP}$ whose paths represent the complete diagrams of all countable $\omega$-models of KP. We define formulas that refer to trees, tree ranks, and paths in an $\omega$-model of KP, and check that these formulas behave well. In Section 2.5 we use the results from Section 2.4 to determine, given a path $f \in [T_{KP}]$, a collection of computable ill-founded trees through which $f$ can compute a path. We give examples in which such a path $f$ might fail to compute
a path through a given computable ill-founded tree. Furthermore, given any two
computable well-founded trees $T_0$ and $T_1$, we give a condition that guarantees, for a
given path $f \in [T_{KP}]$ and any path $g \in [T_0]$, that $f \oplus g$ computes a path through $[T_1]$.

Finally, in Section 2.6 in order to justify the fact that we overlooked Kleene-Brouwer
orderings in favor of tree rank functions in Sections 2.4 and 2.5 we show that the
Kleene-Brouwer ordering of a tree $T$ is well-ordered in an $\omega$-model of KP iff $T$ has a
tree rank function in that model.

For background on trees, we refer the reader to Section 1.3. In particular, we
make heavy use of the notation presented in 1.3.3

2.2 Determining which $\omega$-branching trees are well-founded

König’s Lemma (Lemma 1.3.2) states that a finitely-branching tree $T$ is well-
founded iff $T$ is finite. Thus, we can reduce the question of whether a given finitely-
branching tree $T$ is well-founded to a question about the natural numbers and $T$
(i.e., is there a number $n \in \omega$ such that $T$ has no nodes of length $n$?). One reason
why completions of Peano arithmetic are well-equipped to compute paths through
computable binary-branching trees is that one can express, in PA, the notion of a
computable tree $T$ having nodes of every finite length, or stopping at some finite
length.

We see in this section that the well-foundedness question for $\omega$-branching trees
is more complicated. We can reduce it to a question about countable ordinals. One
method of doing so is to consider the well-known notion of tree rank (see, for example,
[2, p. 161]).

**Definition 2.2.1.** Let $T \subseteq \omega^{\omega}$ be a tree, and let $\alpha$ be an ordinal. A function
$r : T \rightarrow \alpha + 1$ is a **tree rank function** for $T$ if $r(\lambda) = \alpha$ and
$(\forall \sigma \in T)(r(\sigma) = \bigcup_{\tau \in T_{\omega}^{\alpha}}[r(\tau) + 1])$. In this case, we say that $T$ has **tree rank** $\alpha$. 

15
If a tree has a rank function, then this function is unique (and hence a tree can have at most one ordinal tree rank $\alpha$). Given a tree $T$ with rank function $r : T \to \alpha + 1$, it is helpful to note that $r$ reverses the normal tree ordering on $T$ (i.e., for every $\sigma$, $\tau \in T$ with $\sigma < \tau$, $r(\sigma) > r(\tau)$), and $\alpha$ is the smallest ordinal for which there exists such an order-reversing map on $T$. From these observations, we see that if $T$ has tree rank $\alpha$, witnessed by rank function $r : T \to \alpha + 1$, then $T$ is well-founded, and $\alpha$ is countable. Indeed, if there were a path $f$ through $T$, we could use $f$ and $r$ to get an infinite descending sequence of ordinals below $\alpha$. Furthermore, if $\alpha \geq \omega_1$, we could build a path through $T$ consisting of nodes $\sigma \in T$ with $r(\sigma) \geq \omega_1$.

Thus, all trees with tree rank functions are well-founded. The converse also holds. The following result is well-known.

**Theorem 2.2.2.** Let $T \subseteq \omega^{<\omega}$ be a tree. The following are equivalent:

1. $[T] = \emptyset$;
2. $T$ has rank $\alpha$, for some ordinal $\alpha < \omega_1$.

**Proof.** Suppose, first, that $[T] = \emptyset$. Given $\sigma \in T$, recall that $\text{above}(T, \sigma) = \{\tau \in \omega^{<\omega} : \sigma \ast \tau \in T\}$ denotes the tree of strings $\tau$ that, when they are concatenated with $\sigma$, the result is a string in $T$. It is clear that $T$ has tree rank iff $\text{above}(T, \sigma)$ has tree rank, for all $\sigma \in T$. Furthermore, if $\sigma \in T$ is such that $\text{above}(T, \sigma)$ is unranked, then there is some $n \in \omega$ such that $\sigma \ast n \in T$ and $\text{above}(T, \sigma \ast n)$ is also unranked. Hence, if $T$ were unranked, we could build a path through $T$ from the nodes $\sigma$ with $\text{above}(T, \sigma)$ unranked, a contradiction. We have already argued in the discussion preceding Theorem 2.2.2 that tree ranks, when they exist, must be countable.

For the converse, if $T$ has a tree rank function, we have already shown that $T$ must be well-founded.

Most of this chapter is devoted to understanding how a countable $\omega$-model $\mathcal{M}$ of KP understands tree ranks. We could also have considered the following notion in
Definition 2.2.3. Let $T \subseteq \omega^\omega$ be a tree. The Kleene-Brouwer ordering on $T$ is the linear ordering $L \subseteq T \times T$ defined by $\sigma <_L \tau \iff \sigma > \tau \vee \sigma <_{lex} \tau$, where $\sigma <_{lex} \tau$ means that $\sigma$ and $\tau$ are incomparable, and for the first entry $n \in \omega$ at which they differ, $\sigma(n) < \tau(n)$.

It is well-known that the question of whether a tree $T \subseteq \omega^\omega$ is well-founded is equivalent to the question of whether its Kleene-Brouwer ordering is a well order.

Theorem 2.2.4. Let $T \subseteq \omega^\omega$ be a tree. The following are equivalent:

1. $[T] = \emptyset$;

2. The Kleene-Brouwer ordering on $T$ is well ordered.

Proof. Suppose, first, that $[T] = \emptyset$. Let $<_L \subseteq T \times T$ be the Kleene-Brouwer ordering on $T$. If $<_L$ is not well-ordered, then there is an infinite sequence $(\sigma_n)_{n \in \omega}$ of nodes from $T$, with $\sigma_{n+1} <_L \sigma_n$ for all $n \in \omega$. Since, for any $\sigma$, there are only finitely many nodes $\tau$ with $\tau <_{lex} \sigma$ and $|\tau| \leq |\sigma|$, we know that $\lim_n |\sigma_n| = \infty$. We claim, further, that for every $k \in \omega$, $\lim_n \sigma_n|_k$ exists. The claim is clear for $k = 0$. Suppose the claim holds for $k-1 \geq 0$. Fix $m \in \omega$ such that for all $n \geq m$, $|\sigma_n| \geq k$ and $\sigma_n|_{k-1} = \lim_n \sigma_n|_{k-1}$. For $n > n' \geq m$, we have $\sigma_n(k-1) \leq \sigma_{n'}(k-1) \leq \sigma_m(k-1)$, and hence, since there are only finitely many numbers less than $\sigma_m(k-1)$, $\lim_n \sigma_n|_k$ exists.

Suppose, now, that the Kleene-Brouwer ordering on $T$ is a well ordering of type $\alpha$, for some ordinal $\alpha$. If $[T] \neq \emptyset$, then we could use the Kleene-Brouwer ordering on $T$ to obtain an infinite decreasing sequence below $\alpha$, a contradiction.

To determine whether a finitely-branching tree has a path, we ask questions about the natural numbers. Determining whether an $\omega$-branching tree $T \subseteq \omega^\omega$ has paths involves questions about the countable ordinals (either through tree rank functions or Kleene-Brouwer orderings). It is worthwhile to note that, for a fixed $\omega$-branching
tree \( T \), we actually need only ask about the ordinals less than \( \omega_1^T \), the first ordinal not computable from \( T \).

**Proposition 2.2.5** (Barwise; [2]). Given a tree \( T \in \omega^\omega \), if \( T \) is well-founded, then the tree rank of \( T \) and the order type of the Kleene-Brouwer ordering on \( T \) are both less than \( \omega_1^T \).

**Proof.** Suppose \( T \) has tree rank \( \alpha \), and the Kleene-Brouwer ordering \((L,\prec_L)\) on \( T \) has order type \( \beta \). It is easy to see that \( \alpha \leq \beta \). Indeed, the map taking every ordinal \( \gamma < \alpha \) to the \( \prec_L \)-least node \( \sigma \in T \) assigned ordinal \( \gamma \) by the rank function on \( T \) provides an embedding of \( \alpha \) into the Kleene-Brouwer ordering on \( T \). Finally, the Kleene-Brouwer ordering on \( T \) is computable from \( T \), and hence has order type less than \( \omega_1^T \).

In particular, if \( T \subseteq \omega^\omega \) is computable, we can determine whether \( T \) is well-founded by asking questions just about the computable ordinals. This bound is sharp; given \( \alpha < \omega_1^{CK} \), we can construct a computable well-founded tree \( T \) of rank \( \alpha \). Indeed, if \( \alpha < \omega_1^{CK} \) is computable, and \((L,\prec_L)\) is a computable linear ordering of order type \( \alpha \), the tree \( T_L \subseteq \omega^\omega \) consisting of all \( \prec_L \)-descending sequences has rank \( \alpha \).

The next result will be a useful source of examples later on. Recall (Lemma 1.3.5) that there are computable \( \omega \)-branching trees with paths, but with no hyperarithmetic path.

**Proposition 2.2.6.** Suppose \( T \subseteq \omega^\omega \) is a computable tree with paths, but no hyperarithmetic path. For every \( \alpha < \omega_1^{CK} \), there is a node \( \sigma \in T \) such that \( \text{above}(T, \sigma) \) has tree rank \( \alpha \).

**Proof.** It is clear that, for any \( \alpha < \omega_1^{CK} \), if there is \( \sigma \in T \) such that \( \text{above}(T, \sigma) \) has tree rank \( \alpha \), then, for every \( \beta < \alpha \), there is some \( \tau \in T \) such that \( \text{above}(T, \tau) \) has tree rank \( \beta \). Thus, if the conclusion of Proposition 2.2.6 were false, we could fix a bound \( \alpha_0 < \omega_1^{CK} \) such that for all \( \sigma \in T \), the tree \( \text{above}(T, \sigma) \) has tree rank iff it has rank less than \( \alpha_0 \). Fix a computable linear order \( L \) isomorphic to \( \alpha_0 \).
Then, given $\sigma \in T$, we would have

$\sigma \in T_{\text{ext}}$ iff $(\exists f)(f \in \text{comp}(T, \sigma))$, and

$\sigma \notin T_{\text{ext}}$ iff $(\exists f : \text{above}(T, \sigma) \to L)(\forall \tau, \tau' \in \text{above}(T, \sigma))(\tau < \tau' \rightarrow f(\tau) >_L f(\tau')).$

Thus, both $T_{\text{ext}}$ and its complement are $\Sigma^1_1$ subsets of $T$, implying that $T_{\text{ext}}$ is hyperarithmetic. But then $T$ would have a hyperarithmetic path, a contradiction.

\[\square\]

2.3 Kripke-Platek set theory

In this section, we present the background information about Kripke-Platek set theory (KP) that will be used in our later analysis of computable $\omega$-branching trees. In Subsection 2.3.1, we formally define KP, and fix notation for some formulas defining common set-theoretic operations. In Subsection 2.3.2, we list some well-known consequences of KP. Next, in Subsection 2.3.3, we include some information about models of KP. Finally, in Subsection 2.3.4, we see why we can carry out computability-theoretic arguments in an $\omega$-model of KP.

2.3.1 Axioms of KP

Kripke-Platek set theory is a weak fragment of set theory (ZFC) in the language $L = \{\in\}$. We first define some special classes of $L$-formulas. The collection of $\Delta^0_0$ $L$-formulas is the smallest collection $\mathcal{C}$ of $L$-formulas, containing the atomic formulas, that satisfies the following closure properties:

1. Given $\varphi \in \mathcal{C}$, $\neg \varphi \in \mathcal{C}$;
2. Given $\varphi_0$ and $\varphi_1 \in \mathcal{C}$, the formulas $\varphi_0 \land \varphi_1$ and $\varphi_1 \lor \varphi_1$ are also in $\mathcal{C}$;
3. Given $\varphi \in \mathcal{C}$, the formulas $(\forall x \in y)(\varphi)$ and $(\exists x \in y)(\varphi)$ are also in $\mathcal{C}$.

The $\Sigma$ formulas are those built by closing the $\Delta^0_0$ formulas under unbounded existential quantifiers, finite conjunctions and disjunctions, and bounded quantifiers.
The Π formulas are built from the Δ₀ formulas in the same way as the Σ formulas, except we allow unbounded universal quantifiers, instead of unbounded existential quantifiers. Given a particular L-structure \( \mathcal{M} \), a set \( X \subseteq \mathcal{M} \) is a \( \Sigma (\Pi) \) definable subset of \( \mathcal{M} \) if there is a \( \Sigma (\Pi) \) formula \( \varphi(x) \) such that for all \( a \in \mathcal{M} \), \( a \in X \) iff \( \mathcal{M} \models \varphi(a) \). The set \( X \) is said to be Δ definable if \( X \) is both Σ and Π definable.

The axioms of KP are the (appropriate universal closures) of the following:

1. *Extension*: \( (\forall u)((u \in x \leftrightarrow u \in y) \rightarrow x = y) \);
2. *Foundation Schema*: \( \exists x \varphi(x) \rightarrow (\exists x)(\varphi(x) \land (\forall y \in x)(\neg \varphi(y))) \), for any formula \( \varphi \) in which \( y \) does not occur freely;
3. *Pair*: \( (\exists u)(x \in u \land y \in u \land (\forall z \in u)(z = x \lor z = y)) \);
4. *Union*: \( (\exists u)[(\forall x \in v)(\forall y \in x)(y \in u) \land (\forall y \in u)(\exists x \in v)(y \in x)] \);
5. *Δ₀ Separation Schema*: \( (\exists u)(\forall x)(x \in u \leftrightarrow x \in v \land \varphi(x)) \), for all Δ₀ formulas \( \varphi(x) \) in which \( u \) does not occur freely;
6. *Δ₀ Collection Schema*: \( (\forall x \in u)(\exists y)(\varphi(x, y)) \rightarrow (\exists v)(\forall x \in u)(\exists y \in v)(\varphi(x, y)) \), for all Δ₀ formulas \( \varphi(x, y) \) in which \( v \) does not occur freely.

We can perform many of the basic operations of set theory within a model \( \mathcal{M} \) of KP. Given \( a, b \in \mathcal{M} \), there are elements \( a \cup b \), \( a \cap b \), \( (a, b) \), and \( a \times b \) in \( \mathcal{M} \) representing, respectively, the union, intersection, ordered pair, and cross product of \( a \) and \( b \) in \( \mathcal{M} \). If \( f \in \mathcal{M} \) represents a function, then there are sets \( a, b \in \mathcal{M} \) representing the domain and range of \( f \). If \( R \in \mathcal{M} \) represents a relation, then there is an element of \( \mathcal{M} \) representing the restriction of the given function or relation to \( b \). We can also compose functions and invert bijective functions.

All of the operations and objects from the previous paragraph can be described by Δ₀ formulas. We fix notation here for such formulas. Let \( x = y \cup z \), \( x = y \cap z \), \( x = (y, z) \), and \( x = y \times z \) denote Δ₀ formulas saying, respectively, that \( x \) is the union, intersection, ordered pair, and Cartesian product of \( y \) and \( z \). Let \( x = 1^{\text{st}}y \) and \( x = 2^{\text{nd}}y \) be Δ₀ formulas saying that \( y \) is an ordered pair with first component \( x \), and that \( y \)
is an ordered pair with second component $y$. Let $\text{Reln}(R)$ be a $\Delta_0$ formula asserting that $R$ is a binary relation (that is, $R$ is a set of ordered pairs), and let $\text{Fun}(f)$ be a $\Delta_0$ formula saying that $f$ is a binary relation that is also a function. Let $\text{dom}(f) = a$, $\text{ran}(f) = b$ denote $\Delta_0$ formulas saying that $f$ is a function with domain $a$, and $f$ is a function with range $b$. Similarly, let $\text{field}(R) = a$ be a $\Delta_0$ formula saying that $R$ is a relation with field $a$. Fix $\Delta_0$ formulas $g = f\mid_a$ and $S = R\mid_a$, $h = f \circ g$, and $g = f^{-1}$ saying, respectively, that $g$ is the restriction of a function $f$ to $a$, $S$ is the restriction of a relation $R$ to $a$, $h$ is the composition of functions $f$ and $g$, and $g$ is the inverse of a bijective function $f$. Finally, let $\text{Ord}(a)$ be a $\Delta_0$ formula saying that $a$ is an ordinal; that is, the formula $\text{Ord}(a)$ says that $a$ is transitive, and every element of $a$ is transitive. For more detail on the assertions in this or the previous paragraph, see [2, Ch. 1].

2.3.2 Consequences of KP

We now present some standard results about reasoning in a model $\mathcal{M}$ of KP. For proofs of these results, see [2, Ch. 1].

**Theorem 2.3.1** ($\Sigma$ Collection). Suppose $\mathcal{M}$ is a model of KP, and $\varphi(x,y)$ is a $\Sigma$ formula, possibly with parameters from $\mathcal{M}$. Given $a \in \mathcal{M}$, if $\mathcal{M} \vDash (\forall x \in a)(\exists y)\varphi(x,y)$, then there is some $b \in \mathcal{M}$ such that $\mathcal{M} \vDash (\forall x \in a)(\exists y \in b)\varphi(x,y)$.

The next result is a special case of Theorem 2.3.1.

**Theorem 2.3.2** ($\Sigma$ Replacement). Suppose $\mathcal{M}$ is a model of KP, and $\varphi(x,y)$ is a $\Sigma$ formula, possibly with parameters from $\mathcal{M}$. Given $a \in \mathcal{M}$, if $\mathcal{M} \vDash (\forall x \in a)(\exists! y)\varphi(x,y)$, then there is some $f \in \mathcal{M}$ such that $\mathcal{M} \vDash \text{Fun}(f) \land (\text{dom}(f) = a) \land (\forall (x,y) \in f)\varphi(x,y)$.

Finally, the following theorem can be used to define operations on the ordinals in a model of KP. We shall see some examples of this in Section 2.6.
Theorem 2.3.3 (Σ Recursion). Suppose $\mathcal{M}$ is a model of KP, and $\varphi(\bar{a}, x, f, y)$ is a Σ formula with parameters $\bar{a} \in \mathcal{M}$ such that $\mathcal{M} \models (\forall x)(\forall f)(\exists ! y) \varphi(\bar{a}, x, f, y)$. There is a unique function $F : \mathcal{M} \to \mathcal{M}$, defined by a Σ formula $\psi(\bar{a}, x, y)$ in $\mathcal{M}$ such that for all $x, y \in \mathcal{M}$, $F(x) = y$ iff there is $f \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(\bar{a}, x, f, y)$, and $\mathcal{M} \models f = \{(x', y') : x' \in x \land \psi(\bar{a}, x', y')\}$.

There is a lot of uniformity in the proof of Theorem 2.3.3. The formula $\psi$ defining the function $F$ does not depend on the model $\mathcal{M}$ or on the parameters $\bar{a} \in \mathcal{M}$. That is, given a Σ formula $\varphi(\bar{a}, x, f, y)$, we can effectively determine a Σ formula $\psi(\bar{a}, x, y)$, also without parameters, such that if $\mathcal{M}$ is any model of KP, and $\bar{a} \in \mathcal{M}$ satisfies $\mathcal{M} \models (\forall x)(\forall f)(\exists ! y) \varphi(\bar{a}, x, f, y)$, then $\psi(\bar{a}, x, y)$ defines the unique function $F$ guaranteed by Theorem 2.3.3. Indeed, we can take $\psi(\bar{a}, x, y)$ a Σ formula saying that

$$(\exists f)(Fun(f) \land \text{dom}(f) = x \land (\forall x' \in x) \varphi(\bar{a}, x', f|_{x'}, f(x')))$$

From now on, when we define a function $F : \mathcal{M} \to \mathcal{M}$ by Σ-recursion in a model $\mathcal{M}$ of KP, we may write $\mathcal{M} \models y = F(x)$ or $\mathcal{M} \models z \in F(x)$ as shorthand for the appropriate Σ formulas defining $F$ in $\mathcal{M}$ guaranteed by Theorem 2.3.3. Finally, in practice, we might want the domain of $F$ to be some $\Delta$-definable subset of $\mathcal{M}$, such as the set of ordinals in $\mathcal{M}$.

Corollary 2.3.4 (Σ-recursion on a $\Delta$ definable subset of $\mathcal{M}$). Suppose $\mathcal{M}$ is a model of KP, and $D \subseteq \mathcal{M}$ is a $\Delta$-definable subset of $\mathcal{M} \times \mathcal{M}$ that is closed downwards under $\in^\mathcal{M}$; i.e., if $x \in D$ and $\mathcal{M} \models y \in x$, then $y \in D$. Suppose $\varphi(\bar{a}, x, f, y)$ is a Σ formula with parameters $\bar{a} \in \mathcal{M}$ such that for all $x \in D$, $\mathcal{M} \models (\exists ! y) \varphi(\bar{a}, x, f, y)$. There is a unique function $F : D \to \mathcal{M}$, defined by a Σ formula $\psi(\bar{a}, x, y)$ in $\mathcal{M}$ such that for all $x, y \in \mathcal{M}$, $F(x) = y$ iff $x \in D$ and $\mathcal{M} \models \varphi(\bar{a}, x, f, y)$, where $f \in \mathcal{M}$ is such that $\mathcal{M} \models f = \{(x', y') : x' \in x \land \psi(\bar{a}, x', y')\}$.

22
2.3.3 Models of KP

We present an overview of the properties of models of KP. We see how larger models relate to smaller models, and then we give some explicit examples of models of KP.

Suppose $M$ is a structure in the language $L = \{\in\}$ of set theory, and $N \subseteq M$ is a substructure of $M$. We say $M$ is an end extension of $N$, or $N$ is an initial substructure of $M$, if, for all $x \in N$, $\{y \in N : N \models y \in x\} = \{y \in M : M \models y \in x\}$. In this case, we write $N \subseteq_{\text{end}} M$. Informally speaking, an end extension of a structure $N$ is a superstructure in which no element of $N$ acquires new elements. The following result about end extensions is used throughout the rest of the chapter; for a proof, see [2, Lemma 1.8.4].

Lemma 2.3.5. Suppose $N \subseteq_{\text{end}} M$ are models of KP, $\varphi(\overline{a})$ is a $\Sigma$ formula with parameters from $N$, and $\overline{a} \in N$. If $N \models \varphi(\overline{a})$, then $M \models \varphi(\overline{a})$. Furthermore, if $\varphi$ is $\Delta_0$, then $N \models \varphi(\overline{a})$ iff $M \models \varphi(\overline{a})$.

A structure $M$ in the language $L = \{\in\}$ of set theory is well-founded if, for all $\emptyset \neq X \subseteq M$, there is some $x \in X$ such that for all $y \in X$, $M \models \neg y \in x$. Here, we are quantifying over actual subsets $X \subseteq M$, not sets internal to the model. If a model $M$ of KP is well-founded, then there is a unique set $A$ such that the structure $(A, \in)$ is isomorphic to $M$; furthermore, the isomorphism is unique (see, e.g., [2, Proposition 2.8.1]). Every model $M$ of KP has a largest well-founded initial substructure $N$, called the well-founded part of $M$, which is an end-extension of every other well-founded initial substructure of $M$ [2, Lemma 2.8.2].

The following result is extremely useful. For a proof, see [2, Lemma 2.8.4].

Lemma 2.3.6 (Truncation Lemma). If $M$ is a model of KP, then the well-founded part of $M$ is a model of KP.
Hence, for any model $\mathcal{M}$ of KP, its well-founded part is a model of KP, and is isomorphic to $(A, \in)$, for some set $A$. In general, given a set $A$, we say $A$ is admissible if $A$ is transitive and $(A, \in)$ is a model of KP. An admissible ordinal is an ordinal $\alpha$ such that $L_\alpha$ is an admissible set. The first two admissible ordinals are $\omega$ and $\omega_1^{CK}$; see [2, Corollary 5.5.11]. Sacks [16] showed that the countable admissible ordinals greater than $\omega$ are exactly those of the form $\omega_1^X$ for $X \subseteq \omega$.

The set $L_\omega$ is the collection of all hereditarily finite sets and is the smallest admissible set. The set $L_{\omega_1^{CK}}$ is the smallest admissible set containing $\omega$; that is, if $A$ is admissible and $\omega \in A$, then $L_{\omega_1^{CK}} \subseteq A$ (this follows, for instance, from [2, Theorem 2.5.11]). The subsets of $\omega^n$ for $n \in \omega$ that belong to $L_{\omega_1^{CK}}$ are exactly the hyperarithmetic sets; see [2, Section 4.3].

We can relativize to any $X \subseteq \omega$ the fact that $L_{\omega_1^{CK}}$ is the smallest admissible set containing $\omega$, and all subsets of $\omega$ in $L_{\omega_1^{CK}}$ are hyperarithmetic. That is, given $X \subseteq \omega$, if $\alpha = \omega_1^X$, then $L_\alpha(X)$ is the smallest admissible set containing the characteristic function of $X$ (viewed as an element of $2^\omega$), and the sets $Y \subseteq \omega$ in $L_\alpha(X)$ are exactly those sets hyperarithmetic in $X$.

If $\mathcal{M}$ is a model of KP, we denote by $\text{Ord}(\mathcal{M})$ the set of elements of $\mathcal{M}$ that $\mathcal{M}$ believes are ordinals, i.e., $\text{Ord}(\mathcal{M}) = \{ a \in \mathcal{M} : \mathcal{M} \models \text{Ord}(a) \}$. An element $x \in \text{Ord}(\mathcal{M})$ is a standard ordinal in $\mathcal{M}$ if there is some ordinal $\alpha$ such that

$\{ y \in \mathcal{M} : \mathcal{M} \models y \in x \}$ has order type $\alpha$. The collection of standard ordinals in a model $\mathcal{M}$ of KP is closed downwards; that is, if $x \in \mathcal{M}$ is standard, and $\mathcal{M} \models y \in x$, then $y$ is standard.

We are most interested in $\omega$-models of KP; that is, models $\mathcal{M}$ of KP in which there is a standard ordinal $x \in \text{Ord}(\mathcal{M})$ such that $\{ y \in \mathcal{M} : \mathcal{M} \models y \in x \}$ has order type $\omega$. If $\mathcal{M}$ is an $\omega$-model of KP, we can view $(L_{\omega_1^{CK}}, \in)$ as an initial substructure of $\mathcal{M}$.

Friedman [7] showed that for any $\omega$-model $\mathcal{M}$ of KP containing nonstandard
ordinals, the order type of \( \text{Ord}(\mathcal{M}) \) is \( \alpha(1 + \eta) \), where \( \alpha \geq \omega_1^{CK} \) is the order type of the standard ordinals in \( \mathcal{M} \), and \( \eta \) is the order type of the rationals. There are models \( \mathcal{M} \) of KP for which \( \text{Ord}(\mathcal{M}) \) has order type \( \omega_1^{CK}(1 + \eta) \). Harrison [11] showed that if \( T \subseteq \omega^\omega \) is a computable tree with paths, but no hyperarithmetic path, then the Kleene-Brouwer ordering on \( T \) has order type \( \omega_1^{CK}(1 + \eta) + \beta \), for some \( \beta < \omega_1^{CK} \). This suggests that there could be pairs \( (\mathcal{M}, T) \), where \( T \subseteq \omega^\omega \) is a computable tree, and \( \mathcal{M} \) is a nonstandard \( \omega \)-model of KP, such that \( \mathcal{M} \) believes that the Kleene-Brouwer ordering on \( T \) is well-ordered. We explore this more in Sections 2.5 and 2.6.

2.3.4 Computability theory in an \( \omega \)-model of KP

In this chapter, we wish to use the information from various \( \omega \)-models of KP to answer questions about computable trees. Before we embark on this project, we note that, within any given \( \omega \)-model of KP, we have access to many of the objects typically used in computability theory, and that, given such a model and such an object, we can easily locate the object inside the model.

First, we introduce some notation. Suppose \( \mathcal{M} \) is an \( \omega \)-model of KP, and \( A \) is the unique set isomorphic to the well-founded part of \( \mathcal{M} \), witnessed by an embedding \( f : (A, \in) \to \mathcal{M} \). Given \( x \in A \), we denote by \( x \) the element \( f(x) \in \mathcal{M} \). We say that \( x \) represents \( x \) in \( \mathcal{M} \).

In an arbitrary \( \omega \)-model \( \mathcal{M} \) of KP, we have much of the coding and basic objects from computability theory at our disposal. Indeed, recall, from the previous section, that \( L_{\omega_1^{CK}} \) contains all hyperarithmetic subsets of \( \omega^n \) for \( n \in \omega \), and that every \( \omega \)-model \( \mathcal{M} \) of KP is an end-extension of \( (L_{\omega_1^{CK}}, \in) \). Thus, every hyperarithmetic set \( X \subseteq \omega \) is represented by some \( X \in \mathcal{M} \). For instance, in any \( \omega \)-model \( \mathcal{M} \) of KP, we have access to the pairing function and the enumeration \( E_0 \subseteq \omega \times \omega \times \omega \) of triples \( (e, x, y) \) such that \( \varphi_e(x) \downarrow = y \). With a little more thought, we see that the function \( c : \omega^\omega \to \omega \) taking a string \( \sigma = (n_0, \ldots, n_{k-1}) \) to the code \( 2^{k+1}3^{n_0+1} \cdots p_k^{n_{k-1}+1} \), where
\( p_i \) denotes the \( i \)th prime, is also an element of \( L_{\omega_1^{CK}} \). Hence, every computable (or hyperarithmetic) tree \( T \subseteq \omega^{<\omega} \) is represented in every \( \omega \)-model of KP; so is the enumeration \( E_1 \subseteq \omega \times \omega^{<\omega} \times \omega \times \omega \) of tuples \((e, \sigma, x, y)\) for which \( \varphi_1^e(x) \downarrow = y \).

Given any \( X \in L_{\omega_1^{CK}} \), we would like to be assured that there is a uniform way, given the complete diagram of any countable \( \omega \)-model \( M \) of KP, to locate the representation \( X \in M \) of \( X \) in \( M \). Indeed, this is the case. It suffices to show that every \( X \in L_{\omega_1^{CK}} \) has an absolute definition, as follows.

**Definition 2.3.7.** Given an admissible set \((A, \in)\) and \( a \in A \), we say that \( a \) has an absolute definition in \((A, \in)\) if there is a formula \( \rho(x) \) in the language \( \{\in\} \) of set theory, without parameters, such that \( A \vDash (\forall x)(\rho(x) \iff x = a) \), and for any model \( M \) of KP, if \((A, \in)\) embeds into the well-founded part of \( M \) by some map \( \Gamma : (A, \in) \to M \), then \( M \vDash (\forall x)(\rho(x) \iff x = \Gamma(a)) \).

The result that every \( X \in L_{\omega_1^{CK}} \) has an absolute definition follows from work of Gödel [9]. We include a sketch of the proof.

**Theorem 2.3.8 (Gödel).** Every element \( x \in L_{\omega_1^{CK}} \) has an absolute \( \Sigma \) definition in \((L_{\omega_1^{CK}}, \in)\).

**Proof.** First, we fix a \( \Sigma \) formula \( \varphi(x, y) \) in the language of set theory saying that \( y \) is an ordinal and \( x = L_y \). We may take this formula to be absolute for \( \omega \)-models of KP, in the sense that, for any \( \omega \)-model \( M \) of KP, if \( a \in \text{Ord}(M) \) represents a standard ordinal \( \alpha \) in \( M \), then there is a unique element \( l \in M \) with \( M \vDash \varphi(l, a) \), and \( l \) represents the set \( L_\alpha \).

Next, given any ordinal \( \alpha < \omega_1^{CK} \), there is a \( \Sigma \) formula \( \alpha(x) \) that, in any \( \omega \)-model \( M \) of KP, defines the element representing \( \alpha \). Indeed, if we fix any computable linear ordering \( L \subseteq \omega \times \omega \) with order type \( \alpha \), then \( L \) is a \( \Delta \)-definable subset of \((L_\omega, \in)\) (see, e.g., [2, Theorem 2.2.3]). The formula \( \alpha(x) \) says that \( x \) is an ordinal, and there is a function taking \( x \) isomorphically to \( L \).
We finish the proof by induction on $\alpha < \omega_1^{CK}$. As a base case, note that the elements of $L_\omega$ are exactly the hereditarily finite sets, and all have absolute definitions. Now, fix $\omega < \alpha < \omega_1^{CK}$, and $x \in L_\alpha$, and suppose, inductively, that for every $\gamma < \alpha$, every $y \in (L_\gamma, \epsilon)$ has an absolute $\Sigma$-definition in $(L_\omega^{CK}, \epsilon)$. The interesting case is if $\alpha = \beta + 1$ is a successor ordinal. The set $x \in L_\alpha$ is then a definable subset of $(L_\beta, \epsilon)$. That is, $x = \{ y \in L_\beta : (L_\beta, \epsilon) \models \psi(y, \bar{c}) \}$, for some formula $\psi$ and tuple $\bar{c}$ of constants from $L_\beta$. Fix an absolute definition $c(\bar{u})$ of $\bar{c}$ in $L_\omega^{CK}$. We can take the absolute $\Sigma$-definition of $x$ in $L_\omega^{CK}$ to be
\[
y \in x \iff (\exists u)(\exists v)(\exists w)(\beta(u) \land \varphi(v, u) \land c(w) \land \psi^{(v)}(y, w)),\]
where $\psi^{(v)}$ is the formula obtained from $\psi$ by replacing all quantifiers by quantifiers bounded by $v$. 

2.4 Trees in an $\omega$-model of KP

We now consider the question of how to use the information from an $\omega$-model of KP to compute a path through a computable $\omega$-branching tree. First, we construct a computable $\omega$-branching tree $T_{KP}$ whose paths uniformly compute the complete diagrams of the countable $\omega$-models of KP.

We fix a countable set $C = \{ c_i \}_{i \in \omega}$ of constants. For the purposes of this section, a countable structure $\mathcal{M}$ in the language of set theory is said to be labeled if the universe of $\mathcal{M}$ has been identified with $C$; that is, every element of $\mathcal{M}$ is identified by a unique $c_i \in C$, and every $c_i \in C$ corresponds to some element of $\mathcal{M}$.

For any computable relational language $L'$ extending the language $L = \{ \epsilon \}$ of set theory, we can determine a computable list $(\varphi_i(\bar{u}))_{i \in \omega}$ of all elementary first-order formulas in the language $L'$. Given a countable labeled $L'$-structure $\mathcal{M}$, the complete diagram $D^c(\mathcal{M})$ of $\mathcal{M}$ is the set $D \subseteq \omega$ whose elements are the Gödel codes for the
sequences \((i, j_0, \ldots, j_k)\) such that \(\varphi_i(c_{j_0}, \ldots, c_{j_k})\) holds in \(\mathcal{M}\). It is clear that the choice of labeling for \(\mathcal{M}\) affects the computability-theoretic properties of \(D^c(\mathcal{M})\).

Given any such language \(L'\), and any computable set of sentences \(\Gamma \supseteq KP\) in the language \(L'\), we obtain, via a modified Henkin construction, a computable \(\omega\)-branching tree whose paths represent exactly the complete diagrams of the countable \(\omega\)-models of \(\Gamma\). In particular, we obtain a computable \(\omega\)-branching tree \(T_{KP}\) whose paths represent the complete diagrams of the countable \(\omega\)-models of \(KP\).

**Theorem 2.4.1.** Suppose \(\Gamma \supseteq KP\) is a computable set of elementary first-order sentences in a computable relational language \(L' \supseteq L = \{\varepsilon\}\). There is a computable tree \(T_\Gamma \subseteq \omega^\omega\) and Turing functionals \(\Phi, \Psi\) such that

1. \([T_\Gamma] \neq \emptyset \iff \Gamma \text{ has an } \omega\text{-model;}
2. \forall f \in [T_\Gamma], \Phi(f) \text{ is the complete diagram of a countable } \omega\text{-model } \mathcal{M} \text{ of } \Gamma;
3. \text{For any countable labeled } \omega\text{-model } \mathcal{M} \text{ of } \Gamma, \Psi(D^c(\mathcal{M})) \text{ is a path } f \in [T_\Gamma], \text{ and } \Phi(f) = D^c(\mathcal{M}).

**Proof.** Given any formula \(\varphi(\bar{x})\) in the language \(L \cup C\), we can effectively determine a logically equivalent formula \(\tilde{\varphi}(\bar{x})\) in prenex normal form (that is, \(\tilde{\varphi}\) consists of finitely many blocks of alternating quantifiers followed by a quantifier-free formula). If \(\varphi\) is in prenex normal form, by \(\text{neg}(\varphi)\), we mean a formula in prenex normal form that is equivalent to \(\neg \varphi\) and obtained effectively from an index for \(\varphi\).

Fix a computable enumeration \(\rho_i(\bar{c}_i)_{i \in \omega}\) of the set \(S\) of all elementary first-order \(L \cup C\)-sentences in prenex normal form. Fix a formula \(\omega(x)\) in prenex normal form saying that \(x\) is a limit ordinal and every element of \(x\) is a successor ordinal.

Furthermore, for each \(n \in \omega\), let \(\nu_n(x)\) be a formula in prenex normal form, effectively obtained from \(n\), saying that \(x\) is equal to the finite ordinal \(n\). For instance, \(\nu_0(x)\) and \(\nu_1(x)\) should express, respectively, that \(x = \emptyset\) and \(x = \{\emptyset\}\).

Replace the sentences in \(\Gamma\) by an equivalent computable set \(\Gamma'\) of sentences in prenex normal form. Let \(\Gamma^*\) be the set \(\Gamma' \cup \{c_i \neq c_j : i \neq j\} \cup \{(\exists x)\omega(x)\}\). For any
finite set $F \subseteq S$, we say that $F$ is $\Gamma^*$-acceptable if the set of quantifier-free sentences in $F$ is consistent, if there is no $\rho$ with both $\rho \in F$ and $\text{neg}(\rho) \in F$, and for all $\gamma \in \Gamma^*$, $\text{neg}(\gamma) \notin F$.

Fix a finite $\Gamma^*$-acceptable set of $L \cup C$-sentences $F$. Let $F_\exists$ and $F_\forall$ be the sets of sentences in $F$ of form $(\exists x) \varphi(x, \overline{c})$ and $(\forall x) \varphi(x, \overline{c})$, respectively. Let $i \in \omega$ be the least such that neither $\rho_i(\overline{c})$ nor $\text{neg}(\rho_i(\overline{c}))$ are in $F$. Given any finite subset $F'$ of $S$, we say that $F'$ is an acceptable extension of $F$ if

1. $F \subseteq F'$;
2. $F'$ is $\Gamma^*$-acceptable;
3. For every sentence $(\exists x) \varphi(x, \overline{c})$ in $F_\exists$, there is some $d \in C$ such that $\varphi(d, \overline{c}) \in F'$;
4. For every sentence $(\forall x) \varphi(x, \overline{c})$ in $F_\forall$, if $d \in C$ is the least such that $\varphi(d, \overline{c}) \notin F$, then $\varphi(d, \overline{c}) \in F'$;
5. One of $\rho_i(\overline{c})$ or $\text{neg}(\rho_i(\overline{c}))$ is in $F'$;
6. If $\omega(c) \in F$ for some $c \in C$ and $d \in c$ is in $F$, then $\nu_n(d) \in F'$ for some $n \in \omega$.

The set of finite $\Gamma^*$-acceptable subsets of $S$ is computable, and, given two finite $\Gamma^*$-acceptable subsets $F$ and $F'$ of $S$, we can effectively determine whether $F'$ is an acceptable extension of $F$. We let $T_\Gamma \subseteq \omega^\omega$ be the set of sequences $\sigma = (n_0, \ldots, n_k)$ such that, for $i \leq k$, $n_i$ is the Gödel code of a finite $\Gamma^*$-acceptable set of $L \cup C$-sentences, and, for $i < k$, $F_{i+1}$ is an acceptable extension of $F_i$.

Uniformly effectively in a path $f \in [T_\Gamma]$, we can compute the set $T \subseteq S$ of all $L \cup C$-sentences in prenex normal form coded by $f$. The atomic sentences in $T$ form the atomic diagram of a structure $\mathcal{M}$ in the language $L$ of set theory, with elements labeled by $C$. We can prove, by induction on $L$-formulas, that for any $\varphi(\overline{x})$ in prenex normal form, and any $\overline{c} \in C$, $\varphi(\overline{c}) \in T$ iff $\mathcal{M} \models \varphi(\overline{c})$. From $T$, we can effectively compute the complete diagram $D^\omega(\mathcal{M})$ of $\mathcal{M}$. Our construction guarantees that $\mathcal{M}$ is an $\omega$-model of $\Gamma$. 

29
Conversely, it is clear that from the complete diagram $D^c(M)$ of a countable labeled $\omega$-model $M$ of $\Gamma$, we can uniformly effectively compute a sequence $F_0 \subseteq F_1 \subseteq \ldots$ of $\Gamma^*$-acceptable subsets of $S$ such that $F_{i+1}$ is an acceptable extension of $F_i$ for every $i \in \omega$, and $\bigcup_{i \in \omega} F_i$ is the set of all $L \cup C$ sentences in prenex normal form true in $M$. The sequence $F_0 \subseteq F_1 \subseteq \ldots$ corresponds to some path $f \in T_\Gamma$, and applying the procedure from the previous paragraph to $f$ yields $D^c(M)$.

From now on, we let $T_{KP}$ denote the tree obtained from Theorem 2.4.1 when $\Gamma = KP$. Our goal is to study how well paths through $T_{KP}$ can compute paths through other ill-founded computable trees $T \subseteq \omega^{<\omega}$. First, we note some basic properties of $T_{KP}$. From the construction in Theorem 2.4.1 we see that $T_{KP}$ is $\omega$-branching. If $\sigma \in T_{KP}$ corresponds to some finite set $F$ of $L \cup C$-sentences, then $\sigma$ might have infinitely many successors $\tau \in T_{KP}$ if $F$ contains an unwitnessed existential sentence, or if $F$ contains sentences $\omega(c)$ and $d \in c$ for some $c, d \in C$. We could have cut down on some of the branching in $T_{KP}$ by always witnessing existential sentences with the smallest available constant in $C$, but, in the end, any computable tree $T$ with the properties listed in Theorem 2.4.1 for $\Gamma = KP$ must be infinitely branching. Every path through such a tree $T$ must compute all of the hyperarithmetic sets, since every hyperarithmetic set is represented in every $\omega$-model of KP. If $T$ were only finitely branching, then $T$ would have a path $f \preceq_T \emptyset''$ (indeed, we could take $(f \oplus \emptyset')' \preceq_T \emptyset''$).

2.4.1 Formalizing trees, paths, and ranks in an $\omega$-model of KP

In order to use a path through $T_{KP}$ to compute paths through other computable $\omega$-branching trees, we first formalize the notions of trees, paths and tree rank functions inside a countable $\omega$-model $M$ of KP. Given a computable index for a tree $T \subseteq \omega^{<\omega}$ and the complete diagram $D^c(M)$ of an $\omega$-model $M$ of KP, we want an effective procedure...
to determine whether $\mathcal{M}$ has a path for $T$ and whether $\mathcal{M}$ has a rank function for $T$. We accomplish this by defining formulas $\text{Tree}(x)$, $\text{path}(x,y)$, and $\text{rank}(x,y,z)$ in the language of set theory, without parameters, that express, respectively, in any model $\mathcal{M}$, that $x$ is a subtree of $\omega^{\omega}$, that $x$ is a path through a tree $y$, and that $x$ is a function that assigns rank $z$ to a tree $y$. In Lemmas 2.4.4, 2.4.5 and 2.4.6 we show that each of these formulas is absolute in some sense.

**Definition 2.4.2.** Fix formulas $\rho_0(x)$ and $\rho_1(x)$, without parameters, that are absolute $\Sigma$-definitions of $\omega$ and $\omega^{\omega}$, respectively, in $L_{\omega^1_{CK}}$.

1. We denote by $\text{Tree}(x)$ the $\Sigma$-formula

   $$(\exists y)(\rho_1(y) \land (\forall \sigma \in x)[\sigma \in y \land (\forall \tau \in y)(\tau \subseteq \sigma \rightarrow \tau \in x)]).$$

2. We let $\text{path}(x,y)$ be a $\Sigma$-formula saying

   $$(\exists w, z)[\rho_0(w) \land \rho_1(z) \land \text{Tree}(y) \land \text{Fun}(x)$$
   $$\land (\text{dom}(x) = w) \land (\text{ran}(x) \subseteq w)$$
   $$\land (\forall n \in w)(\exists \sigma \in z)(\sigma = x|_n \land \sigma \in y)].$$

3. We let $\text{rank}(x,y,z)$ be a $\Sigma$-formula saying

   $\text{Fun}(x) \land \text{Tree}(y) \land \text{Ord}(z)$$
   $$\land (\text{dom}(x) = y) \land (\text{ran}(x) = z + 1)$$
   $$\land (\forall \sigma \in y)(x(\sigma) = \{x(\tau) : \tau \in y \land \tau > \sigma\}).$$

4. We denote by $\text{rank}(t) = a$, $\text{rank}(t) < a$, $\text{rank}(t) < \infty$, and $\text{rank}(t) = \infty$ the formulas

   $(\exists r)(\text{rank}(r,t,a))$, $(\exists r)(\exists b \in a)(\text{rank}(r,t,b))$, $(\exists r)(\exists a)(\text{rank}(r,t,a))$, and

   $(\forall r)(\forall a \rightarrow (\exists a)(\text{rank}(r,t,a)))$, respectively.
Before we move on to the results about how the notions of trees, paths, and ranks are absolute between models, we make some observations about how these notions behave in a fixed $\omega$-model $M$ of KP. For instance, it is easy to see that, given $t \in M$ with $M \vDash \text{Tree}(t)$, it cannot be the case that both $M \vDash (\exists f)(\text{path}(f,t))$ and $M \vDash \text{rank}(t) < \infty$. If it were the case that $M \vDash \text{path}(f,t)$ and $M \vDash \text{rank}(r,t,a)$ for some $f, r,$ and $a \in M$, then the range of the restriction of $r$ to the set of finite initial substrings of $f$ would be a set of ordinals in $M$ with no $\in$-least element, violating the Axiom of Foundation.

Furthermore, if, for some $t \in M$, we have $M \vDash \text{rank}(t) < \infty$, then the rank function witnessing it is unique. Indeed, suppose, in $M$, we have functions $r_i : t \to a_i + 1$, for $i = 0, 1$, with $M \vDash \text{rank}(r_i,t,a_i)$. We claim that, in $M$, for all $b \leq a_0$, and all $\sigma \in t$ with $r_0(\sigma) = b$, $r_1(\sigma) = b$. Suppose the claim is true for all $c < b \leq a_0$, and we are given $\sigma \in t$ with $r_0(\sigma) = b$. By Definition 2.4.2, $r_1(\sigma) = \{r_1(\tau) : \tau \in t \land \tau > \sigma\}$. If $\tau \in t$ and $\tau > \sigma$, then $r_0(\tau) < r_0(\sigma) = b$, so, by hypothesis, $r_1(\tau) = r_0(\tau)$. Thus, $r_1(\sigma) = \{r_0(\tau) : \tau \in t \land \tau > \sigma\} = r_0(\sigma)$, as desired. (It follows, by symmetry, that $M \vDash r_0 = r_1$.)

Finally, for later use, we give a characterization of the trees $t \in M$ such that $M \vDash \text{rank}(t) < \infty$. This is a direct consequence of the proof of Theorem 1.9.6 in [2], but we include a proof here for the sake of exposition.

**Lemma 2.4.3.** Let $M$ be an $\omega$-model of KP, and $t \in M$ such that $M \vDash \text{Tree}(t)$. Define a function $F : \text{Ord}(M) \to M$ by $\Sigma$-recursion in $M$ such that for all $a \in \text{Ord}(M)$, $M \vDash F(a) = \{\sigma \in t : (\forall \tau \in t)(\tau > \sigma \to (\exists b \in a)(\tau \in F(b)))\}$. Then $M \vDash \text{rank}(t) < \infty$ iff $M \vDash (\forall \sigma \in t)(\exists a)(\sigma \in F(a))$.

**Proof.** ($\Rightarrow$). Suppose $M \vDash \text{rank}(t) < \infty$, with $M \vDash \text{rank}(r,t,a)$ for some $r, a \in M$. It is clear that we can show, by induction on ordinals $b \leq a$ in $M$, that for all $\sigma \in t$,
if $\mathcal{M} \vDash r(\sigma) \leq b$, then $\mathcal{M} \vDash r(\sigma) \in F(b)$. Thus $\mathcal{M} \vDash (\forall \sigma \in t)(\exists b)(\sigma \in F(b))$.

(\Leftarrow). Suppose $\mathcal{M} \vDash (\forall \sigma \in t)(\exists a)(\sigma \in F(a))$. By $\Sigma$ Collection, there is some fixed $a \in \text{Ord}(\mathcal{M})$ for which $\mathcal{M} \vDash (\forall \sigma \in t)(\exists b \in a)(\sigma \in F(b))$. We can now define, by recursion on ordinals $b \in a$, functions $(r_b)_{b \in a}$, with $\text{dom}(r_b) = F(b)$, such that

$\mathcal{M} \vDash (\forall b \in a)(\forall \sigma \in F(b))(r_b(\sigma) = \{r_c(\tau) : c \in b \land \tau \in t \land \tau \in F(c) \land \tau > \sigma\})$. The function $r = \bigcup_{b \in a} r_b$ belongs to $\mathcal{M}$, and $\mathcal{M} \vDash \text{rank}(r, t, b)$, for some $b \leq a$. \hfill \Box

We are now ready to prove the absoluteness properties of the notions of trees, ranks, and paths defined in Section 2.4.2.

**Lemma 2.4.4.** Suppose $(A, \in)$ is an admissible set, $\mathcal{M}$ is an $\omega$-model of KP, and $e : (A, \in) \to \mathcal{M}$ is a function witnessing that $(A, \in)$ is isomorphic to the well-founded part of $\mathcal{M}$.

1. If $\mathcal{M} \vDash \text{Tree}(t)$ for some $t \in \mathcal{M}$, then $t$ is in the well-founded part of $\mathcal{M}$, and $e^{-1}(t)$ is a tree $T \subseteq \omega^\omega$ in $A$.

2. Conversely, if $T \subseteq \omega^\omega$ is any tree in $A$, then $\mathcal{M} \vDash \text{Tree}(e(T))$.

**Proof.** Let $\rho_0(x), \rho_1(x)$ be as in Definition 2.4.2. Note that the formula $\text{Tree}(x)$ has form $(\exists y)(\rho_1(y) \land \varphi(x, y))$, where $\varphi(x, y)$ is $\Delta_0$.

1. Suppose $\mathcal{M} \vDash \text{Tree}(t)$. Since $\rho_1(x)$ is an absolute definition of $\omega^\omega$ in $L_{\omega_1^{CK}}$, $\rho_1$ defines $e(\omega^\omega)$ in $\mathcal{M}$. Thus, $\mathcal{M}$ believes that $t$ is a subset of $e(\omega^\omega)$, and hence $t$ is in the well-founded part of $\mathcal{M}$. Since $\mathcal{M} \vDash \varphi(t, e(\omega^\omega))$, and $\varphi$ is $\Delta_0$, we have $(A, \in) \vDash \varphi(e^{-1}(t), \omega^\omega)$, and hence $e^{-1}(t)$ is an actual subtree $T \subseteq \omega^\omega$.

2. Suppose, now, that $T \subseteq \omega^\omega$ is a tree in $A$. Thus, $(A, \in) \vDash \varphi(T, \omega^\omega)$, and so $\mathcal{M} \vDash \varphi(e(T), e(\omega^\omega))$, meaning $\mathcal{M} \vDash \text{Tree}(e(T))$, as desired. \hfill \Box

**Lemma 2.4.5.** Suppose $(A, \in)$ is an admissible set, $\mathcal{M}$ is an $\omega$-model of KP, and $e : (A, \in) \to \mathcal{M}$ is a function witnessing that $(A, \in)$ is isomorphic to the well-founded part of $\mathcal{M}$.
1. If $\mathcal{M} \vDash \text{path}(f, t)$ for some $f, t \in \mathcal{M}$, then $f$ and $t$ are in the well-founded part of $\mathcal{M}$, $e^{-1}(t)$ is a tree $T \subseteq \omega^{< \omega}$ in $A$, and $e^{-1}(f)$ is a path through $T$ in $A$.

2. Conversely, given $f, T$ in $A$, if $T$ is a subtree of $\omega^{< \omega}$ and $f \in [T]$, then $\mathcal{M} \vDash \text{path}(e(f), e(T))$.

Proof. Again, let $\rho_0$ and $\rho_1$ be as in Definition 2.4.2. Note that the formula $\text{path}(x, y)$ has form $(\exists w, z)(\rho_0(w) \land \rho_1(z) \land \text{Tree}(y) \land \varphi(x, y, w, z))$, where $\varphi$ is $\Delta_0$.

1. Suppose $\mathcal{M} \vDash \text{path}(f, t)$ for some $f$ and $t$. By Lemma 2.4.4, $t$ represents a subtree $T \subseteq \omega^{< \omega}$ in $A$. Furthermore, we have $\mathcal{M} \vDash \varphi(f, t, e(\omega), e(\omega^{< \omega}))$, so, in $\mathcal{M}$, $f$ is a function with domain $e(\omega)$ and range a subset of $e(\omega)$, where $e(\omega)$ is in the well-founded part. Hence, $f$ is also well-founded. Since $\varphi$ is $\Delta_0$, we have $(A, \varepsilon) \models \varphi(e^{-1}(f), T, \omega, \omega^{< \omega})$, and hence $e^{-1}(f)$ is a path through $T$.

2. Suppose, now, that $T \subseteq \omega^{< \omega}$ is a tree in $A$ and $f \in A$ is a path through $T$. Then $(A, \varepsilon) \models \varphi(f, T, \omega, \omega^{< \omega})$, so $\mathcal{M} \models \varphi(e(f), e(T), e(\omega), e(\omega^{< \omega}))$. Since the definitions $\rho_0$ and $\rho_1$ of $\omega$ and $\omega^{< \omega}$ are absolute, we also have $\mathcal{M} \vDash \text{path}(e(f), e(T))$. $\square$

In the following Lemma, the second item is due to Barwise [2, p. 161].

**Lemma 2.4.6.** Suppose $(A, \varepsilon)$ is an admissible set, $\mathcal{M}$ is an $\omega$-model of KP, and $e : (A, \varepsilon) \to \mathcal{M}$ is a function witnessing that $(A, \varepsilon)$ is isomorphic to the well-founded part of $\mathcal{M}$.

1. Suppose $\mathcal{M} \vDash \text{rank}(r, t, a)$ for some $r, t, a$. The element $t \in \mathcal{M}$ represents a tree $T \subseteq \omega^{< \omega}$ in $A$. If $a$ is a standard ordinal in $\mathcal{M}$, then $r$ is also in the well-founded part of $\mathcal{M}$, and $e^{-1}(r) : T \to e^{-1}(a + 1)$ is a rank function for $T$.

2. Suppose $T$ is a tree in $A$, and $T$ has rank some ordinal $\alpha$. The ordinal $\alpha$ and the rank function $r : T \to \alpha + 1$ of $T$ both belong to $A$, and $\mathcal{M} \vDash \text{rank}(e(r), e(T), e(\alpha))$.

Proof. Consider the formula $\text{rank}(x, y, z)$ defined in 2.4.2. The formula $\text{rank}(x, y, z)$ is equivalent to a formula of the form $\text{Tree}(y) \land \varphi(x, y, z)$, where $\varphi$ is $\Delta_0$.

1. Suppose $\mathcal{M} \vDash \text{Tree}(t) \land \varphi(r, t, a)$, for some $r, t, a \in \mathcal{M}$. By Lemma 2.4.4, we have that $t$ is in the standard part of $\mathcal{M}$, and represents a subtree $T$ of $\omega^{< \omega}$ in
A. Suppose that \( a \in \mathcal{M} \) is well-founded. Then, according to \( \mathcal{M} \), \( r \) is a function from \( t \) to \( a + 1 \), so \( r \) must also be in the well-founded part of \( \mathcal{M} \). We then have \((A,\epsilon) \models \text{rank}(e^{-1}(r),T,e^{-1}(a))\), so \( e^{-1}(r): T \to e^{-1}(a + 1) \) is a rank function for \( T \).

2. Suppose, now, that \( T \in A \) is a subtree of \( \omega^\omega \), and \( T \) has rank \( \alpha \), for some ordinal \( \alpha \). Recall, from Proposition 2.2.5, that we must have \( \alpha < \omega^T_1 \). The set \( L_{\omega^T_1}(T) \) is the smallest admissible set containing \( T \), and hence a subset of \( A \). Thus, \( \alpha \in A \). We must show that the rank function for \( T \) is in \( A \). It is clear that if there is some \( r \in A \) with \((A,\epsilon) \models \text{rank}(r,T,\alpha)\), then this \( r \) is the actual rank function for \( T \). Define the function \( F \) with domain \( \text{Ord}(A) \) as in Lemma 2.4.3. It suffices to show that \((A,\epsilon) \models (\forall \sigma \in T)(\exists \beta)(\sigma \in F(\beta))\). For any \( \sigma \in T \), if \( \sigma \) has rank \( \beta \leq \alpha \) in the real world, then \( \sigma \in F(\beta) \). Thus the rank function \( r \) for \( T \) lives in \( A \). By Lemma 2.4.4, \( \mathcal{M} \models \text{Tree}(e(T)) \). Since \( \varphi \) is \( \Delta_0 \), and \((A,\epsilon) \models \varphi(r,T,\alpha)\), we have \( \mathcal{M} \models \text{rank}(e(r),e(T),e(\alpha)) \).

\[ \square \]

2.5 Using paths through \( T_{KP} \) to compute paths through other trees

We defined \( T_{KP} \) in the beginning of Section 2.4 in an attempt to build an analogy between the world of computable binary branching trees and that of computable \( \omega \)-branching trees. We already knew, from Theorem 1.3.10, that our analogy was flawed; the computable binary branching tree \( T_{PA} \) is universal among computable binary branching trees, while no computable \( \omega \)-branching tree is universal among computable \( \omega \)-branching trees. In Section 2.5.1, we apply the analysis in Section 2.4.1 to show explicitly how the tree \( T_{KP} \) fails to be universal. In Section 2.5.2, we try to rescue our analogy somewhat. Specifically, we describe a situation in which the complete diagram of an \( \omega \)-model of KP helps to reduce the problem of finding paths through a given computable \( \omega \)-branching tree to the problem of finding paths through another computable \( \omega \)-branching tree.
2.5.1 Determining when an \( \omega \)-model of KP computes a path through a tree \( T \subseteq \omega^\omega \)

The following result implies that the only way a path \( f \in [T_{\text{KP}}] \) (corresponding to the complete diagram \( D^c(M) \) of a countable labeled \( \omega \)-model \( M \) of KP) can fail to compute a path through some other ill-founded computable tree \( T \subseteq \omega^\omega \) is when \( M \) assigns \( T \) a nonstandard ordinal tree rank.

**Theorem 2.5.1** (With Knight and Turetsky). *There is a uniform effective procedure that, given a computable index of a tree \( T \subseteq \omega^\omega \) and the complete diagram \( D^c(M) \) of a countable labeled \( \omega \)-model \( M \) of KP, decides whether

1. \( M \models (\exists f)\operatorname{path}(f, T) \),
2. \( M \models \operatorname{rank}(T) = \infty \), or
3. \( M \models \operatorname{rank}(T) < \infty \).

(Note that if the first case holds, then the second does, as well.) If either of the first two cases is true, we can uniformly effectively compute a path through \( T \) from a computable index for \( T \) and \( D^c(M) \).

*Proof.* Let \( (\nu_n(x))_{n \in \omega} \), \( (\tau_e(x))_{e \in \omega} \), and \( (\gamma_\sigma(x))_{\sigma \in \omega^\omega} \) be computable lists of formulas without parameters in the language of set theory such that for every \( \omega \)-model \( M \) of KP and every \( x \in M \), \( M \models \nu_n(x) \) iff \( x \) is the representation \( n \) of \( n \) in \( M \), \( M \models \tau_e(x) \) iff \( x \) represents a subtree \( T \subseteq \omega^\omega \) with computable index \( e \), and \( M \models \gamma_\sigma(x) \) iff \( x \) is the representation \( \sigma \) of \( \sigma \) in \( M \). The formulas \( (\tau_e(x))_{e \in \omega} \) can be built effectively from the formulas \( (\nu_n(x))_{n \in \omega} \) using the fact that the set \( \omega^\omega \), the set of computable indices for trees, the set \( E_0 = \{(i, x, y) : \varphi_i(x) \downarrow = y\} \), and the coding function \( c : \omega^\omega \to \omega \) defined in Section 2.3.4 all have absolute \( \Sigma \)-definitions in \( (L_{\omega_1^{CK}, \epsilon}, \epsilon) \), by Theorem 2.3.8. Similarly, the formulas \( (\gamma_\sigma(x))_{\sigma \in \omega^\omega} \) are obtained effectively from the formulas \( (\nu_n(x))_{n \in \omega} \) and an absolute definition for the coding function \( c : \omega^\omega \to \omega \).

Fix a \( \Sigma \) formula \( \psi(u, x, y) \) without parameters such that for every \( \omega \)-model \( M \) of KP, and every \( t \in M \) such that \( M \models \text{Tree}(t) \), the formula \( \psi(t, x, y) \) defines a function.
$F : \text{Ord}(\mathcal{M}) \to \mathcal{M}$ such that for all $a \in \text{Ord}(\mathcal{M})$, 
$\mathcal{M} \models F(a) = \{ \sigma \in t : (\forall \tau \in t)(\tau > \sigma \to (\exists b \in a)(\tau \in F(b))) \}$. This is the function $F$ defined in the statement of Lemma 2.4.3. The existence of the formula $\psi(u, x, y)$ follows from the discussion of uniformity immediately after the statement of Theorem 2.3.3 ($\Sigma$ recursion).

Given any computable index for a tree $T \subseteq \omega^\omega$ and the complete diagram $D^c(\mathcal{M})$ of any countable labeled $\omega$-model $\mathcal{M}$ of KP, we can effectively find the number $i \in \omega$ such that the representation $T$ of $T$ in $\mathcal{M}$ is coded by the constant $c_i$. To determine which of the three cases in the statement of Theorem 2.5.1 is true, we use the formulas defined in Section 2.4.1. Note that if the first case holds, the second does, too.

If the first case holds, and $c_i$ is the constant coding $T$ in $\mathcal{M}$ as above, we search for the least $j \in \omega$ such that the code for the formula $\text{path}(c_j, c_i)$ belongs to $D^c(\mathcal{M})$. By Lemma 2.4.5, $c_j$ codes an actual path $f$ through $T$. It is clear that we can uniformly effectively compute $f(n)$ for all $n \in \omega$, using $D^c(\mathcal{M})$ and the computable list $\nu_n(x)$ of formulas defining the elements of $\omega$ from above.

Next suppose that the second case holds. Let $S = \{ \sigma \in T : \mathcal{M} \models (\forall a)(\sigma \notin F(a)) \}$, where $F$ is the function defined in $\mathcal{M}$ by the $\Sigma$ formula $\psi(T, x, y)$. It is clear that we can compute $S$ from a computable index for $T$ and $D^c(\mathcal{M})$, using the formulas $(\gamma_\sigma(x))_{\sigma \in \omega^\omega}$. By Lemma 2.4.3, $S \neq \emptyset$. In fact, the definition of $F$ implies that $S$ is a subtree of $T$. We claim that $S$ has no dead ends. Suppose there is a dead end, and fix some $\sigma \in S$ that has no successor $\tau$ in $S$. Thus, $\mathcal{M} \models (\forall \tau \in (\text{comp}(T, \sigma))^{|\sigma|+1})(\exists b)(\tau \in F(b))$. By $\Sigma$ Collection (2.3.1), there is some $c \in \mathcal{M}$ such that $\mathcal{M} \models (\forall \tau \in (\text{comp}(T, \sigma))^{|\sigma|+1})(\exists b \in c)(\tau \in F(b))$. But then $\mathcal{M} \models \sigma \in F(a)$, where $a \in \mathcal{M}$ is the supremum of the ordinals in $c$, a contradiction.

From $S$, we can uniformly effectively compute a path through $S$, since $S$ is nonempty and has no dead ends.

Finally, if the third case holds, then it may not be the case that $D^c(\mathcal{M})$ computes
a path through $T$, even if $T$ is ill-founded. We discuss this further in the remainder of Section 2.5.1.

Let $\mathcal{P}$ be the set of pairs $(\mathcal{M}, T)$, where $\mathcal{M}$ is a countable labeled $\omega$-model of KP, and $T \subseteq \omega^{<\omega}$ is a computable tree with paths. We can show that each of the three cases mentioned in 2.5.1 applies to some pair $(\mathcal{M}, T)$ in $\mathcal{P}$. The first case in Theorem 2.5.1 applies to all pairs $(\mathcal{M}, T) \in \mathcal{P}$ such that $T$ has a hyperarithmetic path, for if $h$ is a hyperarithmetic path through $T$, then $h$ is represented in every countable labeled $\omega$-model $\mathcal{M}$ of KP. There are pairs $(\mathcal{M}, T) \in \mathcal{P}$ for which the second case of Theorem 2.5.1 applies, but not the first; indeed, this is true whenever $\mathcal{M}$ is a labeled copy of $(L^{\omega_1^{CK}}, \epsilon)$, and $T \subseteq \omega^{<\omega}$ is a computable tree with paths but no hyperarithmetic path. Since the only subsets of $\omega$ in $L^{\omega_1^{CK}}$ are the hyperarithmetic sets, the first case does not hold. Furthermore, Lemma 2.4.6 implies that if $\mathcal{M}$ is a copy of $(L^{\omega_1^{CK}}, \epsilon)$, then $\mathcal{M} \models rank(T) = \infty$. Finally, there are pairs $(\mathcal{M}, T) \in \mathcal{P}$ where $\mathcal{M} \models rank(T) < \infty$, but the rank of $T$ in $\mathcal{M}$ is a nonstandard ordinal. By Theorem 1.3.6 there is a path $f \in [T_{KP}]$ such that $f \equiv_T O$. As in the proof of Theorem 1.3.10 we see that there is some computable tree $T \subseteq \omega^{<\omega}$, with paths, such that $f$ does not compute a path through $T$. Let $\mathcal{M}$ be the countable labeled model whose complete diagram $D^c(\mathcal{M})$ corresponds to $f$ as in Theorem 2.4.1. This model $\mathcal{M}$ has nonstandard ordinals; indeed, since $\omega^f_1 = \omega^{CK}_1$ (see Theorem 1.3.7), we see that $Ord(\mathcal{M})$ must have order type $\omega^{CK}_1(1 + \eta)$. Since $f \equiv_T D^c(\mathcal{M})$ does not compute a path through $T$, by Theorem 2.5.1 we must have $\mathcal{M} \models rank(T) < \infty$. The rank of $T$ in $\mathcal{M}$ is not a standard ordinal, for then Lemma 2.4.6 would imply that $T$ has tree rank in the real world.

Theorem 2.5.1 implies that for any countable $\omega$-model $\mathcal{M}$ of KP, and any computable ill-founded $\omega$-branching tree $T \subseteq \omega^{<\omega}$, if $D^c(\mathcal{M})$ fails to compute a path through $T$, then $\mathcal{M}$ assigns a nonstandard ordinal tree rank to $T$. The next result provides a source of examples of computable ill-founded trees that are assigned a
nonstandard rank in some countable $\omega$-model of KP.

**Proposition 2.5.2** (Knight and Turetsky). Suppose $T_0, \ldots, T_{k-1}$ are computable ill-founded subtrees of $\omega^\omega$ without hyperarithmetic paths. There are nodes $\sigma_i \in (T_i)^{ext}$, for $i = 0, \ldots, k-1$, and a countable labeled $\omega$-model $\mathcal{M}$ of KP, such that $\mathcal{M} \models \text{rank}(above(T_0, \sigma_0)) < \cdots < \text{rank}(above(T_{k-1}, \sigma_{k-1})) < \infty$.

**Proof.** By Proposition 2.2.6 and Lemma 2.4.6 there are nodes $\sigma_i \in T_i$, $i = 0, \ldots, k-1$ such that $(L_{\omega CK}^\omega, \in) \models \text{rank}(above(T_0, \sigma_0)) < \cdots < \text{rank}(above(T_{k-1}, \sigma_{k-1})) < \infty$; however, these nodes do not extend to paths through their respective trees.

Let $L' = \{\in\} \cup \{S_0, \ldots, S_{k-1}\}$ be the language of set theory with additional unary relation symbols $S_0, \ldots, S_{k-1}$. Let $\Gamma \supseteq \text{KP}$ be a computable collection of $L'$-sentences saying, for $i = 0, \ldots, k-1$, that $S_i$ is a node on $T_i$, and $\text{rank}(above(T_0, S_0)) < \cdots < \text{rank}(above(T_{k-1}, S_{k-1})) < \infty$. (To get the sentences in $\Gamma$, we use the absolute $\Sigma$-definitions of the trees $T_i$, $i = 0, \ldots, k-1$, in $(L_{\omega CK}^\omega, \in)$ guaranteed by Theorem 2.3.8 and the formulas defined in Section 2.4.1.) Apply Theorem 2.4.1 to obtain a computable tree $T_\Gamma$ and a Turing functional $\Phi$ such that for all $f \in [T_\Gamma]$, $\Phi(f) = D^c(M)$ for some countable labeled $L'$-structure $M$ that is an $\omega$-model of $\Gamma$. From our earlier observation, there is a path $f \in [T_\Gamma]$ corresponding to an expansion of $(L_{\omega CK}^\omega, \in)$, hence $[T_\Gamma] \neq \emptyset$.

Suppose, for contradiction, that for every choice of nodes $\sigma_i \in (T_i)^{ext}$, $i = 0, \ldots, k-1$, and every countable labeled $\omega$-model $\mathcal{M}$ of KP, $\mathcal{M} \not\models \text{rank}(above(T_0, \sigma_0)) < \cdots < \text{rank}(above(T_{k-1}, \sigma_{k-1})) < \infty$. Then, for any $\sigma \in T_0$, $\sigma \in (T_0)^{ext}$ iff $(\exists f)(f \in [\text{comp}(T_0, \sigma)])$, and $\sigma \notin (T_0)^{ext}$ iff $(\exists f)(f \in [T_\Gamma] \land \Phi(f) = D^c(M) \land M \models S_0(\sigma))$. This implies that $(T_0)^{ext}$ is hyperarithmetic, and hence $T_0$ has a hyperarithmetic path, a contradiction. \qed
2.5.2 Pairs of trees

We can show, using Theorem 1.3.9, that a set \( X \subseteq \omega \) has PA degree iff \( X \) computes a function \( f : \omega \times \omega \to \{0, 1\} \) such that whenever \( e_0, e_1 \in \omega \) are computable indices for binary trees \( T_0, T_1 \subseteq 2^{\omega} \), if at least one of \( T_0 \) or \( T_1 \) has a path, then \( T_{f(e_0,e_1)} \) has a path. We can use the function \( f \) to uniformly compute a path through any infinite computable binary tree \( T \subseteq 2^{\omega} \). Indeed, given \( \sigma \in T \) that extends to a path through \( T \), if both \( \sigma \upharpoonright 0 \) and \( \sigma \upharpoonright 1 \) are in \( T \), we can find, effectively in \( \sigma \) and a computable index for \( T \), indices \( e_i \) for the subtrees \( T_i \) of \( T \) consisting of the nodes of \( T \) comparable with \( \sigma \upharpoonright i \), for \( i = 0, 1 \). Since \( T \) is binary, and \( \sigma \in T \) extends to a path through \( T \), at least one of \( T_0 \) or \( T_1 \) has a path. If \( f(e_0,e_1) = i \), then \( T_i \) has a path, and so \( \sigma \upharpoonright i \in T \) extends to a path through \( T \).

Suppose, now, that \( X \) computes the complete diagram \( D^c(\mathcal{M}) \) of a countable labeled \( \omega \)-model \( \mathcal{M} \) of KP. We claim that \( D^c(\mathcal{M}) \) computes a total function \( f : \omega \times \omega \to \{0, 1\} \) such that, given indices \( e_0, e_1 \in \omega \) for computable \( \omega \)-branching trees \( T_0, T_1 \subseteq \omega^{<\omega} \), if at least one of \( T_0 \) or \( T_1 \) has a path, then \( T_{f(e_0,e_1)} \) has a path. Since \( \mathcal{M} \) is an \( \omega \)-model, \( D^c(\mathcal{M}) \) can compute the set of pairs \( (e_0,e_1) \) of computable indices for \( \omega \)-branching trees. Given a pair \( (e_0,e_1) \) of computable indices for \( \omega \)-branching trees \( T_0, T_1 \), Theorem 2.5.1 implies that \( D^c(\mathcal{M}) \) can tell, for \( i = 0, 1 \), whether \( T_i \) is ranked or unranked in \( \mathcal{M} \). Moreover, if both \( T_0 \) and \( T_1 \) are ranked in \( \mathcal{M} \), we can effectively search \( D^c(\mathcal{M}) \) to tell whether \( \mathcal{M} \models rank(T_0) = rank(T_1) \) or \( \mathcal{M} \models rank(T_0) < rank(T_1) \), for some \( i = 0, 1 \). If \( T_i \) is unranked in \( \mathcal{M} \) for some \( i = 0, 1 \), then let \( f(e_0,e_1) \) be the least such \( i \). If \( T_0 \) and \( T_1 \) are assigned equal rank in \( \mathcal{M} \), then let \( f(e_0,e_1) = 0 \). Finally, if \( \mathcal{M} \models rank(T_1) < rank(T_{1-i}) \) for some \( i = 0, 1 \), we let \( f(e_0,e_1) = 1 - i \). If at least one of \( T_0, T_1 \) has a path, then \( T_{f(e_0,e_1)} \) must have a path. If one of the trees is unranked in \( \mathcal{M} \), then that tree has a path (see Theorem 2.5.1), and \( f \) picks \( i \) such that \( T_i \) has a path. Suppose both trees are ranked in \( \mathcal{M} \). At least one of the trees must have nonstandard rank, for (see Lemma 2.4.6) any tree
with standard rank in $\mathcal{M}$ must be well-founded. Thus, the tree assigned greatest rank in $\mathcal{M}$ has a path, and $f$ picks a tree with a path.

Unfortunately, the $D^c(\mathcal{M})$-computable function $f$ described in the previous paragraph need not compute a path through any computable ill-founded $\omega$-branching tree $T \subseteq \omega^\omega$. Given a computable ill-founded tree $T$ that is ranked in $\mathcal{M}$ and a node $\sigma \in T$ that extends to a path, $\sigma$ might have infinitely many successors in $T$, and $f$ (indeed, $D^c(\mathcal{M})$) cannot accurately select one successor among these infinitely many that extends to a path. In the next theorem, we make the best of this situation, showing that for some pairs $T_0$, $T_1 \subseteq \omega^\omega$ of computable trees ranked in $\mathcal{M}$, we can uniformly compute a path through $T_1$ from $D^c(\mathcal{M})$ and any path through $T_0$.

**Theorem 2.5.3.** Suppose $\mathcal{M}$ is a countable labeled $\omega$-model of KP, and $T_0$, $T_1 \subseteq \omega^\omega$ are computable ill-founded trees such that $\mathcal{M} \models \text{rank}(T_0) < \text{rank}(T_1) < \infty$. Given any path $f \in [T_0]$, we can compute a path $g \in [T_1]$ from $D^c(\mathcal{M}) \oplus f$. Moreover, the reduction is uniform in $D^c(\mathcal{M})$, computable indices for $T_0$ and $T_1$, and the path $f \in [T_0]$.

**Proof.** Much as in the proof of Theorem 2.5.1, we effectively search $D^c(\mathcal{M})$ to find the constants coding $T_0$, $T_1$, and the rank functions $r_0$, $r_1$ for $T_0$ and $T_1$ in $\mathcal{M}$. Lemma 2.4.6 implies that, for $i = 0, 1$, given $\sigma \in T_i$, the ordinal $r_i(\sigma) \in \text{Ord}(\mathcal{M})$ is standard iff $\sigma$ does not extend to a path through $T_i$.

We compute a path $g \in [T_1]$ inductively, as follows. Given $\sigma = g|_s \in T_1$ such that $\sigma$ extends to a path through $T_1$, and $\mathcal{M} \models r_0(f|_s) < r_1(\sigma)$, let $g|_{s+1}$ be the least $\tau \in T_1$ such that $|\tau| = s + 1$, $\tau > \sigma$ and $\mathcal{M} \models r_0(f|_{s+1}) < r_1(\tau)$. To see that such a $\tau$ exists, fix $\alpha \in \text{Ord}(\mathcal{M})$ such that $\mathcal{M} \models \alpha = r_0(f|_{s+1})$. If, for every $\tau \in T_1$ such that $|\tau| = s + 1$ and $\tau > \sigma$, we had $\mathcal{M} \models r_1(\tau) \leq \alpha$, then we would have $\mathcal{M} \models r_1(\sigma) \leq \alpha + 1 = r_0(f|_{s+1}) + 1 \leq r_0(f|_s)$, contradicting the inductive hypothesis. Furthermore, $\alpha \in \text{Ord}(\mathcal{M})$ is nonstandard, since $f|_{s+1}$ extends to a path through $T_0$. Hence any $\tau \in T_1$ with $\mathcal{M} \models \alpha < r_1(\tau)$ extends to a path through $T_1$. □
It is interesting to consider how much of the computational power of $D^c(\mathcal{M})$ is really necessary in Theorem 2.5.3. At worst, it could be the case that whenever two computable trees $T_0$ and $T_1$ with paths are ranked in some $\omega$-model $\mathcal{M}$ of KP, just as in the hypothesis of Theorem 2.5.3, it is actually the case that every path through $T_0$ already computes a path through $T_1$, without the help of $D^c(\mathcal{M})$. We can show that this is false, using results of Goncharov, Knight, Harizanov, and Shore [10]. In particular, the authors of [10] showed that the Turing degrees of the $\Pi_1^1$ paths through $\mathcal{O}$ are exactly the Turing degrees of the well-founded parts of computable Harrison orderings. Furthermore, they showed that there is a $\Pi_1^1$ path through $\mathcal{O}$ that does not compute $\varnothing'$. For our example, fix a computable Harrison ordering $H$, with no hyperarithmetic descending sequence, whose well-founded part, $W$, does not compute $\varnothing'$. Let $T_H \subseteq \omega^{<\omega}$ be the computable $\omega$-branching tree whose nodes represent all of the finite descending sequences in $H$. The tree $T_H$ is ill-founded and has no hyperarithmetic path. From $W$, we can compute an infinite descending sequence in $H$ below any $a \in H - W$. Thus, below any $a \in H - W$, there is an infinite descending sequence that does not compute $\varnothing'$. In terms of $T_H$, this means that for any $\sigma \in T_H$ that extends to a path, there is $f \in \text{comp}(T_H, \sigma)$ with $f \not\leq_T \varnothing'$. Fix any other ill-founded computable tree $T \subseteq \omega^{<\omega}$ with no hyperarithmetic path. By Proposition 2.5.2, there is a countable labeled $\omega$-model $\mathcal{M}$ of KP and there are extendible nodes $\sigma \in T_H$, $\tau \in T$ for which $\mathcal{M} \models \text{rank}(\text{above}((T_H, \sigma)) < \text{rank}(\text{above}(T, \tau))) < \infty$. Pick any $f \in \text{above}(T_H, \sigma)$ with $f \not\leq_T \varnothing'$. On its own, $f$ cannot compute a path through $\text{above}(T, \tau)$. However, Theorem 2.5.3 implies that $D^c(\mathcal{M}) \oplus f$ does compute a path through $\text{above}(T, \tau)$.

2.6 Using Kleene-Brouwer orderings instead of tree ranks

We have seen that, for a given computable tree $T \subseteq \omega^{<\omega}$ with paths, and an $\omega$-model $\mathcal{M}$ of KP, the only situation in which the complete diagram of $\mathcal{M}$ may fail
to compute a path through $T$ is that in which $\mathcal{M}$ assigns $T$ a nonstandard ordinal rank. It is reasonable to wonder whether we might have gotten a clearer picture of the situation, or been able to compute paths through more trees, by considering Kleene-Brouwer orderings instead of tree ranks. In Section 2.6.2 we show that if $T \subseteq \omega^\omega$ is a computable tree whose Kleene-Brouwer ordering is not well-ordered in a given countable labeled $\omega$-model $\mathcal{M}$ of KP, then we can use the information about the Kleene-Brouwer ordering on $T$, recorded in the complete diagram of $\mathcal{M}$, to compute a path through $T$. We also show that, given a computable tree $T \subseteq \omega^\omega$ and an $\omega$-model $\mathcal{M}$ of KP, $T$ has a tree rank function in $\mathcal{M}$ iff the Kleene-Brouwer ordering on $T$ is isomorphic to an ordinal in $\mathcal{M}$. These results require some basic facts about linear orderings in an $\omega$-model of KP, which we have collected in Section 2.6.1.

2.6.1 Linear orderings and well-orderings in an $\omega$-model of KP

In this section, we use the notation $y = l_x$ as shorthand for a $\Delta_0$ formula saying that expresses, in an $\omega$-model $\mathcal{M}$ of KP, that $l$ is a linear order, $x$ is in the field of $l$, and $y$ is the restriction of $l$ to the set of elements of the domain of $l$ strictly $<_l$-less than $x$. Furthermore, we fix a $\Sigma$-formula $Iso(l,a)$ saying that $l$ is a linear ordering, $a$ is an ordinal, and $l$ is isomorphic to $a$. We say that $l \in \mathcal{M}$ is well-ordered in $\mathcal{M}$ if $\mathcal{M} \models (\exists a)Iso(l,a)$.

In many ways, an $\omega$-model of KP handles linear orderings and well-orderings in a sensible manner. Lemmas 2.6.1, 2.6.2 and 2.6.3 are evidence of this.

**Lemma 2.6.1.** Suppose $\mathcal{M}$ is an $\omega$-model of KP, and $l$ is a linear ordering in $\mathcal{M}$. If $\mathcal{M} \models (\exists a)Iso(l,a)$, then the ordinal $a \in \text{Ord}(\mathcal{M})$ and the isomorphism in $\mathcal{M}$ witnessing this are unique.

**Proof.** Fix $a_i \in \text{Ord}(\mathcal{M})$ and isomorphisms $f_i : l \to a_i$ in $\mathcal{M}$ witnessing that $l$ is well-ordered in $\mathcal{M}$, for $i = 0, 1$. We show that $a_0 = a_1$ and $f_0 = f_1$. Let $g = f_1 \circ f_0^{-1}$. 43
In \( M \), the function \( g \) is an order-preserving bijection with domain \( a_0 \) and range \( a_1 \). Suppose there is some \( b < a_0 \) such that \( g(b) \neq b \). Let \( b_0 < a_0 \) be the least such (we are using Foundation here). Then, for some \( c < a_0 \), we have \( c > b_0 \) and \( g(c) = b_0 < g(b_0) \), contradicting the fact that \( g \) is order-preserving. Hence, for all \( b \in a_0 \), \( g(b) = b \). This implies that \( a_0 = a_1 \) and \( g = id_{a_0} \), hence \( f_0 = f_1 \).

Lemma 2.6.2. Suppose \( M \) is an \( \omega \)-model of KP, and \( l \in M \) is a linear ordering in \( M \). If there is some \( x \) in the domain of \( l \) for which \( M \not\models (\exists a)(\exists y)(y = l_x \land Iso(y, a)) \), then there is no \( l \)-least such \( x \).

Proof. Suppose, for contradiction, that \( x_0 \in M \) is the \( l \)-least element of the domain of \( l \) for which \( M \not\models (\exists a)(\exists y)(y = l_x \land Iso(y, a)) \). Then
\[
M \models (\forall x \in l_{x_0})(\exists a)(\exists y)(y = l_x \land Iso(y, a)),
\]
where the uniqueness follows from Lemma 2.6.1. By \( \Sigma \)-Replacement, there is a function \( f \in M \), with domain \( l_{x_0} \), taking each \( x <_l x_0 \) to the ordinal isomorphic to \( l_x \). The range \( b \) of \( f \) is an element of \( \text{Ord}(M) \), and \( f \) witnesses that \( M \models Iso(l_{x_0}, b) \). This contradicts our original assumption about \( x_0 \).

Lemma 2.6.3. Suppose \( l \in M \) is a linear order with domain \( a \in M \), and \( b \in M \) is a subset of \( a \). If \( l \) is well-ordered in \( M \), then the restriction of \( l \) to \( b \) is also well-ordered in \( M \).

Proof. It is sufficient to consider restrictions of an ordinal to some subset. Suppose \( \alpha \in \text{Ord}(M) \) and \( b \in M \) is a subset of \( \alpha \). We show that \( (b, \in|_b) \) is well-ordered in \( M \). Using \( \Sigma \)-Recursion, we get a function \( f \in M \) with domain \( b \) such that
\[
M \models (\forall x \in b)(f(x) = \bigcup_{y < x} f(y) \cup \{ f(y) \}).
\]
Let \( \gamma = \bigcup_{x \in b} f(x) \). The function \( f \) maps \( (b, \in|_b) \) isomorphically to \( \gamma \).

We now consider the operations of ordinal addition and multiplication in an \( \omega \)-model of KP.
Definition 2.6.4. Suppose $\mathcal{M}$ is an $\omega$-model of KP, and $\alpha, \beta \in \text{Ord}(\mathcal{M})$.

1. Let $\alpha + \beta$ be the linear ordering in $\mathcal{M}$ with domain $(\alpha \times 0) \cup (\beta \times 1)$ such that $(\gamma, i) < (\delta, j)$ if $i < j$ or if $i = j$ and $\gamma < \delta$.

2. Let $\alpha \cdot \beta$ be the linear ordering in $\mathcal{M}$ with domain $\alpha \times \beta$ such that $(\gamma_0, \delta_0) < (\gamma_1, \delta_1)$ if $\delta_0 < \delta_1$ or $\delta_0 = \delta_1$ and $\gamma_0 < \gamma_1$.

Lemma 2.6.5. Suppose $\mathcal{M}$ is an $\omega$-model of KP. Given $\alpha, \beta \in \text{Ord}(\mathcal{M})$, the ordering $\alpha + \beta$ is well-ordered in $\mathcal{M}$.

Proof. We define a $\Sigma$ function symbol $F_\alpha : \text{Ord}(\mathcal{M}) \to \text{Ord}(\mathcal{M})$ by $\Sigma$-Recursion on ordinals, as follows:

1. $F_\alpha(0) = \alpha$;
2. $F_\alpha(\beta + 1) = F_\alpha(\beta) \cup \{F_\alpha(\gamma)\}$;
3. $F_\alpha(\lambda) = \bigcup_{\beta < \lambda} F_\alpha(\beta)$ for $\lambda$ a limit ordinal.

It is easy to see that for all $\gamma < \gamma' \in \text{Ord}(\mathcal{M})$, we have $\alpha \leq F_\alpha(\gamma) < F_\alpha(\gamma')$. Since $F_\alpha$ is defined by a $\Sigma$ formula, for every $\beta \in \text{Ord}(\mathcal{M})$, we have $F_\alpha|_\beta \in \mathcal{M}$, by $\Sigma$ Replacement.

Define a function $G \in \mathcal{M}$ with domain $\alpha + \beta$, as follows:

1. $G(\gamma, 0) = \gamma$;
2. $G(\gamma, 1) = F_\alpha|_\beta(\gamma)$.

From the remarks following the definition of $F_\alpha$, it is easy to see that $G$ is order-preserving. Let $\delta = \bigcup_{\gamma < \beta} (G(\gamma))$. $G$ maps the ordering $\alpha + \beta$ isomorphically onto a subset of $\delta$. Hence, by Lemma 2.6.3, $\alpha + \beta$ is well-ordered in $\mathcal{M}$.

Lemma 2.6.6. Suppose $\mathcal{M}$ is an $\omega$-model of KP. Given $\alpha, \beta \in \text{Ord}(\mathcal{M})$, the ordering $\alpha \cdot \beta$ is well-ordered in $\mathcal{M}$.

Proof. We first define a $\Sigma$ function symbol $F_\alpha : \text{Ord}(\mathcal{M}) \to \text{Ord}(\mathcal{M})$ by $\Sigma$-Recursion on ordinals such that $F_\alpha(\beta) = \bigcup_{\gamma < \beta} F_\alpha(\gamma) \cup \{F_\alpha(\gamma)\}$ for all $\beta \in \text{Ord}(\mathcal{M})$. For any $\beta \in \text{Ord}(\mathcal{M})$, the restriction $F_\alpha|_\beta$ belongs to $\mathcal{M}$. Furthermore, for every $\beta \in \text{Ord}(\mathcal{M})$,
there is a sequence in $\mathcal{M}$ of the form $(F_\alpha(\gamma) + \delta)_{(\delta, \gamma) \in \alpha \cdot \beta}$. Each of the linear orderings $F_\alpha(\gamma) + \delta$ in the sequence is well-ordered in $\mathcal{M}$, by Lemma 2.6.5. Hence, applying Lemma 2.6.8 there is a map $G \in \mathcal{M}$ with domain $\alpha \cdot \beta$ taking every $(\delta, \gamma) \in \alpha \cdot \beta$ to the ordinal in $\mathcal{M}$ isomorphic to $F_\alpha(\gamma) + \delta$. It is easy to see that $G$ is order-preserving.

If $\eta = \bigcup_{(\delta, \gamma) \in \alpha \cdot \beta} G(\delta, \gamma)$, then $G$ maps $\alpha \cdot \beta$ isomorphically to a subset of $\eta$, hence, by Lemma 2.6.3 $\alpha \cdot \beta$ is well-ordered in $\mathcal{M}$. □

Finally, we include some information about sequences and sums of linear orderings.

**Definition 2.6.7.** Suppose $\mathcal{M}$ is an $\omega$-model of KP.

1. Suppose $f$ is a function in $\mathcal{M}$ with domain $b \in \mathcal{M}$. We say $f$ is a sequence of linear orderings in $\mathcal{M}$ if, for every $a \in b$, $f(a) = l_a$, for some linear ordering $l_a \in \mathcal{M}$. In this case, we write $f = (l_a)_{a \in b}$.

2. Suppose $(l_a)_{a \in b}$ is a sequence of linear orderings in $\mathcal{M}$, and $b$ is linearly ordered by $\prec_b \in \mathcal{M}$. The sum of $(l_a)_{a \in b}$, denoted $\Sigma_{a \in b} l_a$, is the linear ordering $l \in \mathcal{M}$ with domain $\{(a, x) : a \in b \land x \in l_a\}$ defined by $(a_0, x_0) < (a_1, x_1)$ if $a_0 <_b a_1$ or $a_0 = a_1 = a$ and $x_0 <_{l_a} x_1$.

**Lemma 2.6.8.** Suppose $\mathcal{M}$ is an $\omega$-model of KP, and $f = (l_a)_{a \in b}$ is a sequence of linear orderings in $\mathcal{M}$. If $l_a$ is well-ordered in $\mathcal{M}$ for every $a \in b$, then there is a function $h \in \mathcal{M}$ with domain $b$ such that for every $a \in b$, $h(a) : \text{dom}(l_a) \to \alpha_a$ is an isomorphism from $l_a$ to an ordinal $\alpha_a \in \text{Ord}(\mathcal{M})$.

**Proof.** In $\mathcal{M}$, for every $a \in b$, there is a unique function mapping $l_a$ isomorphically to some ordinal. We obtain the function $h \in \mathcal{M}$ by applying $\Sigma$-Replacement to this fact. □

**Lemma 2.6.9.** Suppose $\mathcal{M}$ is an $\omega$-model of KP, $(l_a)_{a \in b}$ is a sequence of linear orderings in $\mathcal{M}$, and $b \in \mathcal{M}$ is linearly ordered by $\prec_b \in b \times b$. If the orderings $(l_a)_{a \in b}$ and $\prec_b$ are all well-ordered in $\mathcal{M}$, then so is their sum $\Sigma_{a \in b} l_a$.

**Proof.** Fix $\alpha, \beta \in \text{Ord}(\mathcal{M})$ such that $(b, \prec_b)$ has order type $\beta$ in $\mathcal{M}$ and for every $a \in b$, $l_a$ has order type at most $\alpha$ in $\mathcal{M}$. We can see that $\alpha$ exists because of Lemma 2.6.8.
We claim that $\Sigma_{aebl_a} \subset b \subset \alpha \cdot \beta$, and, hence, is well-ordered by Lemmas 2.6.3 and 2.6.6. Applying Lemma 2.6.8, we get a new sequence of functions $(f_a : l_a \to \alpha)_{aeb}$ in $\mathcal{M}$ such that for all $a \in b$, $f_a$ maps $l_a$ isomorphically to an initial segment of $\alpha$. Fix $g : b \to \beta$ witnessing that $<_b$ has order type $\beta$ in $\mathcal{M}$. It is clear that the function $G : \Sigma_{aebl_a} \to \alpha \cdot \beta$ defined by $(a,x) \mapsto (f_a(x),g(a))$ is an embedding of $\Sigma_{aebl_a}$ into $\alpha \cdot \beta$ in $\mathcal{M}$. \hfill \Box

2.6.2 Kleene-Brouwer orderings in an $\omega$-model of KP

In Section 2.5.1, we showed that there is a uniform effective procedure, that, given an index for a computable tree $T \subseteq \omega^\omega$, and the complete diagram of a countable labeled $\omega$-model $\mathcal{M}$ of KP in which $T$ has no tree rank function, computes a path through $T$. There is a similar procedure that works when the Kleene-Brouwer ordering on $T$ is not well-ordered in $\mathcal{M}$.

Indeed, given an index for such a tree $T$ and the complete diagram $D^c(\mathcal{M})$ for such a model $\mathcal{M}$, we can effectively search $D^c(\mathcal{M})$ to locate the constant in $\mathcal{M}$ that represents the Kleene-Brouwer ordering $H$ on $T$. Given any $\sigma \in T$, we can also effectively determine, from $D^c(\mathcal{M})$, whether $\mathcal{M}$ believes that $H_\sigma$ is well-ordered. (Here, we follow the notation introduced in Section 2.6.1. That is, for any $\sigma \in T$, we denote by $H_\sigma$ the restriction of $H$ to the set of strings strictly $<_H$-less than $\sigma$.) To get a path through $T$, we first use $D^c(\mathcal{M})$ to compute a sequence $(\sigma_n)_{n \in \omega}$ of strings, where, for each $n \in \omega$, $\sigma_n \in T$ is the lexicographically-least string in $\sigma \in T$ of length $n$ such that $\mathcal{M}$ believes $H_\sigma$ is not well-ordered. Note that $\sigma_0 = \lambda$ is well-defined, since, by assumption, $H$ is not well-ordered in $\mathcal{M}$. Suppose, inductively, that for every $i \leq n$, the strings $\sigma_i$ are defined, and $\lambda = \sigma_0 < \cdots < \sigma_n$. We claim that $\sigma_{n+1}$ is defined, and extends $\sigma_n$. First, note that we can write $H_{\sigma_n}$ as the sum of finitely many orderings $H_\tau$, for some $\tau <_{\text{lex}} \sigma_n$, and all of the orderings $H_{\sigma_n \cdot k}$, for each $k \in \omega$ with $\sigma_n \cdot k \in T$. Each of the orderings $H_\tau$ is a sub-ordering of some $H_{\tau'}$, where $|\tau'| \leq n$.
and \( \tau' < \sigma_i \), for some \( i \leq n \). The ordering \( H_{\nu} \) is well-ordered in \( \mathcal{M} \) by assumption. Thus, by Lemma \[2.6.3\] so is \( H_\tau \). If, in addition, each of orderings \( H_{\sigma_{n+k}} \) in the sum were well-ordered in \( \mathcal{M} \), then, by Lemma \[2.6.9\] \( H_{\sigma_n} \) would also be well-ordered, a contradiction. It follows that \( \sigma_n \) has an extension \( \sigma \) of length \( n+1 \) in \( T \) such that \( H_\sigma \) is not well-ordered in \( \mathcal{M} \). Hence, \( \sigma_{n+1} \) is well-defined, and is equal to the leftmost such \( \sigma \). Thus, the entire sequence \((\sigma_n)_{n<\omega}\) is well-defined, and the union of the sequence is an infinite path through \( T \).

We now have two methods for using the complete diagram of an \( \omega \)-model \( \mathcal{M} \) of KP to compute a path through a computable tree \( T \subseteq \omega^{<\omega} \). One method, described in Section \[2.5.1\] applies when \( \mathcal{M} \) does not rank \( T \); the other, detailed in the previous paragraph, applies when the Kleene-Brouwer ordering on \( T \) is not well-ordered in \( \mathcal{M} \). In the remainder of this section, we show that these two methods are equally applicable. That is, for a given computable tree \( T \subseteq \omega^{<\omega} \), the collection of \( \omega \)-models of KP in which \( T \) is ranked is equal to the collection of models in which the Kleene-Brouwer ordering on \( T \) is well-ordered.

**Theorem 2.6.10** (With Knight and Turetsky). *Suppose \( \mathcal{M} \) is an \( \omega \)-model of KP, and \( T \subseteq \omega^{<\omega} \) is a tree in \( \mathcal{M} \). If the Kleene-Brouwer ordering \( \triangleleft_{L} \subseteq T \times T \) on \( T \) is well-ordered in \( \mathcal{M} \), then \( T \) is ranked in \( \mathcal{M} \).*

*Proof*. Suppose \( f : T \to \alpha \) is the well-ordering of \( \triangleleft_{L} \) in \( \mathcal{M} \). Define a \( \Sigma \) function symbol \( F : \text{Ord}(\mathcal{M}) \to \mathcal{M} \) by \( \Sigma \)-recursion, as follows:

\[
F(\gamma) = \{ \sigma \in T : (\forall \tau \in T)(\tau > \sigma \rightarrow (\exists \beta < \gamma)(\tau \in F(\beta))) \}.
\]

Notice that \( \gamma_0 < \gamma_1 \) implies \( F(\gamma_0) \subseteq F(\gamma_1) \). By Lemma \[2.4.3\] it is enough to show that \( \mathcal{M} \models (\forall \tau \in T)(\exists \gamma)(\tau < L F(\gamma)) \). Indeed, we claim that \( \mathcal{M} \models (\forall \tau \in T)(\tau < L F(f(\tau))) \). Suppose not. Then, since \( \triangleleft_{L} \) is well-ordered by \( f \) in \( \mathcal{M} \), we can apply Foundation to obtain some \( \tau_0 \in T \) such that \( \tau_0 \notin F(f(\tau_0)) \), but \( \mathcal{M} \models (\forall \tau < L \tau_0)(\tau < L F(\tau)) \). But
then $M \models (\forall \tau \in T)(\tau > \tau_0 \rightarrow (\exists \gamma < f(\tau_0))(\tau \in F(\gamma)))$, and hence $\tau_0 \in F(f(\tau_0))$, a contradiction.

We now show that if a computable tree $T \subseteq \omega^{<\omega}$ is ranked in an $\omega$-model $M$ of KP, then the Kleene-Brouwer ordering on $T$ is well-ordered in $M$. In fact, we exhibit a collection of linear orderings associated to $T$, including the Kleene-Brouwer ordering on $T$, that must all be well-ordered in $M$.

By the tree ordering on $T$, we mean the partial ordering $\sigma <_L \tau \iff (\forall \tau \in T)\exists \gamma < f(\tau_0))(\tau \in F(\gamma))$, A linearization of the tree ordering on $T$ is a total ordering $<_L \subseteq T \times T$ extending the tree ordering on $T$. If $<_L$ is any linear ordering on $T$ and $U, V \subseteq T$, we write $U <_L V$ to mean that for every $\sigma \in U$ and $\tau \in V$, $\sigma <_L \tau$.

**Definition 2.6.11.** Given a tree $T \subseteq \omega^{<\omega}$ and a linear ordering $L$ with domain $T$, we say $L$ is determined by least differences if, for every $\sigma \in T$, and $m \neq n \in \omega$ with $\sigma \ast m, \sigma \ast n \in T$, we have either $(\text{comp}(T, \sigma \ast m))^{\leq |\sigma| + 1} <_L (\text{comp}(T, \sigma \ast n))^{\leq |\sigma| + 1}$ or $(\text{comp}(T, \sigma \ast n))^{\leq |\sigma| + 1} <_L (\text{comp}(T, \sigma \ast m))^{\leq |\sigma| + 1}$.

That is, if $\sigma$ and $\tau$ are incomparable nodes on a tree $T$ and $<_L$ is a linear ordering on $T$ determined by least differences, to decide the $<_L$-ordering of $\sigma$ and $\tau$, we only need to know the $<_L$-ordering of the restrictions of $\sigma$ and $\tau$ to the first level where they differ. The Kleene-Brouwer ordering on a tree $T$ is an example of an ordering determined by least differences.

**Theorem 2.6.12 (With Knight and Turetsky).** Suppose $M$ is an $\omega$-model of KP, and $T \subseteq \omega^{<\omega} \in M$ is a tree ranked in $M$. Suppose, further, that $<_L \in M$ is a linearization of the tree ordering on $T$ that is determined by least differences. If $L|_{T^n}$ is well-ordered in $M$ for every $n \in \omega$, then $L$ is well-ordered in $M$.

**Proof.** Let $r \in M$ be the tree rank function for $T$ in $M$. Given $\sigma \in T$, denote by $S(\sigma)$ the set $\{\tau \in T : \tau \geq \sigma\}$. Let $f \in M$ be the function with domain $T$ such that
\[ f(\sigma) = L\big|_{S(\sigma)}(\sigma) \] for all \( \sigma \in T \). The existence of \( f \) in \( M \) follows from \( \Sigma \)-Replacement and the fact that the orderings \( L\big|_{S(\sigma)}(\sigma) \) are uniformly \( \Delta_0 \)-definable from \( L \) and \( \sigma \). We claim that for all \( \sigma \in T \), \( L\big|_{S(\sigma)}(\sigma) \) is well-ordered in \( M \). We prove this by induction on the ordinal rank \( r(\sigma) \). Suppose the claim is true for every \( \sigma \in T \) with \( r(\sigma) < \alpha \), for some ordinal \( \alpha \in \text{Ord}(M) \). Fix \( \sigma \in T \) of rank \( \alpha \). It is enough to show that \( L\big|_{S(\sigma)}(\sigma) \) is isomorphic to some \( \gamma \in \text{Ord}(M) \), for then \( L\big|_{S(\sigma)}(\sigma) \) has order type \( \gamma + 1 \).

Consider the sum \( (S, <_S) \) of the orderings \( (L\big|_{S(\tau)}(\tau))_{\tau \in \text{comp}(T, \sigma)}\big|_{\sigma}^{\sigma+1} \), where the index set \( \text{comp}(T, \sigma)\big|_{\sigma}^{\sigma+1} \) of immediate successors of \( \sigma \) is ordered by \( L \). We can use the function \( f \) to show that \( S \) exists in \( M \). By assumption, \( T\big|_{\sigma}^{\sigma+1} \) is well-ordered under \( L \). Hence, \( \text{comp}(T, \sigma)\big|_{\sigma}^{\sigma+1} \) is also well-ordered under \( L \), by Lemma \( 2.6.3 \). Furthermore, given \( \tau \in \text{comp}(T, \sigma)\big|_{\sigma}^{\sigma+1} \), since \( r(\tau) < \alpha \), our inductive hypothesis implies that \( L\big|_{S(\tau)}(\sigma) \) is well-ordered. Thus, by Lemma \( 2.6.9 \), \( S \) is well-ordered in \( M \).

Elements of \( S \) are pairs \( (\tau, \gamma) \), with \( \tau \in \text{comp}(T, \sigma)\big|_{\sigma}^{\sigma+1} \) and \( \gamma \in S(\tau) \). Because \( L \) is determined by least differences, it is easy to see that the function \( g : S \to L\big|_{S(\sigma)}(\sigma) \setminus \{\sigma\} \) defined by \( (\tau, \gamma) \mapsto \gamma \) is an isomorphism. Hence, \( L\big|_{S(\sigma)}(\sigma) \setminus \{\sigma\} \) is well-ordered in \( M \).

The Kleene-Brouwer ordering on a tree \( T \in \omega^\omega \) is a linearization of the tree ordering determined by least differences. Furthermore, for any \( n \in \omega \), the order type of the restriction of the Kleene-Brouwer ordering on \( T \) to \( T^n \) is either \( \omega \) or some \( k \in \omega \). Thus, applying Theorem \( 2.6.12 \), we have the following.

**Corollary 2.6.13.** Suppose \( T \subseteq \omega^\omega \) is a computable tree ranked in an \( \omega \)-model \( M \) of KP. Then the Kleene-Brouwer ordering on \( T \) is well-ordered in \( M \).
CHAPTER 3

EFFECTIVE FORCING NOTIONS

3.1 Introduction

In this chapter, we study the computability-theoretic properties of the generic objects associated with various types of effective forcing. While forcing was originally developed by set theorists as a means to prove independence results, some of the ideas from set-theoretic forcing have been adapted in order to solve problems in computability theory. We refer to the types of forcing used in computability theory as **effective forcing notions**.

In general, an effective forcing notion is a countable collection \( P \) of **forcing conditions** partially ordered by an **extension relation**, \( \leq_P \). Furthermore, each condition \( p \in P \) corresponds to some set \([p] \subseteq 2^\omega\), which we call the **interpretation of** \( p \).

Typically, if \( p \leq_P q \) (\( p \) extends \( q \)), then \([p] \subseteq [q]\). In this chapter, we use \( 2^\omega \) interchangeably with \( P(\omega) \); subsets of \( \omega \) are identified with their characteristic functions. We imagine using \((P, \leq_P)\) to construct a set \( G \in 2^\omega \), in stages. That is, if, at some stage of the construction, we select a condition \( q \in P \), then we commit to the constraint that \( G \in [q] \). At a later stage, if we select an extension \( p \leq_P q \), then, since \([p] \subseteq [q]\), we have further narrowed the possibilities for which element of \( 2^\omega \) the set \( G \) could be.

Since \( P \) is countable, we may have a sensible way of assigning natural number codes to the elements of \( P \). If this is the case, we can consider sets of conditions from \( P \) of varying levels of complexity, and we can insist that for all sets \( C \subseteq P \) of some
fixed complexity, there is some \( p \in \mathbb{P} \) such that \( G \in [p] \) and either \( p \in \mathcal{C} \), or else there is no \( q \in \mathbb{P} \) such that \( q \leq_P p \) and \( q \in \mathcal{C} \). A set \( G \) that is constructed in this way is called \textit{generic} for \((\mathbb{P}, \leq_P)\).

Kleene and Post used an effective version of \textit{Cohen forcing} to show that there are sets \( A_0, A_1 \preceq_T \emptyset' \) that are Turing incomparable. The conditions in Cohen forcing are strings \( \sigma \in 2^{<\omega} \), with interpretations \([\sigma] = \{ X \in 2^\omega : \sigma < X \}\). Strings are coded by their Gödel numbers. In the standard Kleene-Post argument, a generic set \( G \preceq_T \emptyset' \) is built in stages from Cohen forcing conditions, so that for every \( \Sigma^0_1 \) set \( \mathcal{C} \) of strings, there is a string \( \sigma \in 2^{<\omega} \), such that \( G \in [\sigma] \), and either \( \sigma \in \mathcal{C} \), or \( \sigma \) has no extension in \( \mathcal{C} \). The sets \( A_0 = \{ n \in \omega : 2n \in G \} \) and \( A_1 = \{ n \in \omega : 2n - 1 \in G \} \) are Turing incomparable.

More recently, effective versions of \textit{Mathias forcing} have been used to solve problems in computability theory and reverse mathematics. Conditions for Mathias forcing are pairs \((F, S)\), where \( F \subseteq \omega \) is finite, \( S \subseteq \omega \) is infinite, and \( \max(F) < \min(S) \). The set \( F \) represents a finite initial segment of the set \( G \) being built, and the set \( S \), often called the reservoir, is a constraint on what further natural numbers we may add to \( G \). The interpretation of a condition \((F, S)\) is the set \( \{ X \in 2^\omega : F \subseteq X \subseteq F \cup S \} \). Given conditions \((F, S)\) and \((F', S')\), then \((F, S) \preceq (F', S')\) iff \([\mathcal{C}(F, S)] \subseteq [\mathcal{C}(F', S')]\).

In effective Mathias forcing, we consider conditions \((F, S)\), where the reservoir set \( S \) is an element of a fixed countable collection of subsets of \( \omega \) (for example, we may insist that \( S \) is an element of a given countable Turing ideal).

Effective Mathias forcing is often used to construct a set \( G \) that is \textit{cohesive} for a given countable collection \((X_i)_{i \in \omega}\) of sets, i.e., a set \( G \subseteq \omega \) such that for every \( i \in \omega \), either \( G \equiv^* X_i \), or \( G \equiv^* \overline{X}_i \). For instance, Cholak, Jockusch, and Slaman \[5\] used a variant of Mathias forcing, with computable reservoir sets, to construct a low\(_2\) set that is cohesive for the collection of all computable sets, simplifying a construction of Jockusch and Stephan \[14\].
Variants of effective Mathias forcing are often used when constructing infinite homogeneous sets for an instance of Ramsey’s Theorem. For instance, in Seetapun’s Theorem [19], Mathias forcing with conditions from a countable Scott set is used to prove that, for any computable coloring \( f : [\omega]^2 \to 2 \), and any countable collection \((X_i)_{i \in \omega}\) of sets, there is an infinite homogeneous set \( G \) for \( f \) that avoids the cone above each \( X_i \).

We shall also consider an effective version of Hechler forcing, in Section 3.2. Hechler forcing conditions are pairs \((\sigma, f)\), where \( \sigma \in \omega^{< \omega} \), and \( f : \omega \to \omega \). (Note that any function \( f : \omega \to \omega \) can be identified with the subset of \( \omega \) whose elements code the pairs \((x, y)\), with \( f(x) = y \).) The interpretation of a condition \((\sigma, f)\) is the set \( \{ X \in \omega^\omega : \sigma < X \land (\forall n \geq |\sigma|)(X(n) \geq f(n)) \} \). Hechler forcing can be used to produce a function \( G \in \omega^\omega \) that dominates a given countable collection of functions. This type of forcing has been used less in computability theory than Cohen or Mathias forcing, although Gerdes [8] used a type of Hechler forcing to show that any set with a modulus function has a uniform modulus function.

While effective forcing has been applied to solve existing problems in computability theory, there is also an interesting body of work in which the degree-theoretic properties of the generics associated to various types of effective forcing are studied, for their own sake. Jockusch [12] extensively studied the properties of the Cohen \( n \)-generics. In the same tradition, Cholak, Dzhafarov, Hirst, and Slaman [3] studied the \( n \)-generics for a version of Mathias forcing with computable conditions. Later, Cholak, Dzhafarov, and Soskova [4] expanded the study of Mathias forcing to a version in which the conditions belong to a fixed countable Turing ideal.

Our work in this chapter follows the tradition of that mentioned in the previous paragraph. In particular, we wish to compare the generics associated to different types of forcing. More specifically, when is it the case that the generics for one type of effective forcing can compute generics for another? In Section 3.2 we define a
version of Hechler forcing with computable conditions, and show, using constructions from [3] and [4], that the generics for this Hechler forcing can compute generics for the Mathias forcing of [3], and vice-versa. Along the way, we define a general collection of notions of forcing, which we call Cohen-Mathias-like forcing partial orders, and prove some results comparing the generics for different Cohen-Mathias-like forcing partial orders. In Section 3.3, we answer a question from [4]. Namely, we show that there are nested countable Turing ideals $I \subseteq J$ for which the problem of finding a Mathias generic for $I$ (in the sense of [4]) cannot be reduced to the problem of finding a Mathias generic for $J$. This result is surprising, as it is shown, in [4], that if $I$ is the ideal of all computable sets, such a reduction exists.

We end this section with some terminology used in Sections 3.2 and 3.3. Suppose $\mathcal{P} = (\mathbb{P}, \leq_P)$ is a forcing partial order, and for every $p \in \mathbb{P}$, the interpretation $[p]$ of $p$ is a subset of $2^\omega$. Fix a set $G \subseteq \omega$. Given $p \in \mathbb{P}$, we say that $G$ satisfies $p$ if $G \in [p]$. Given a set $\mathcal{C} \subseteq \mathbb{P}$ of forcing conditions, we say that $G$ meets $\mathcal{C}$ if $G$ satisfies some condition $p \in \mathcal{C}$. We say that $G$ avoids $\mathcal{C}$ if $G$ satisfies a condition $p \in \mathbb{P}$ that has no extension in $\mathcal{C}$, i.e., for every $q \leq_P p$, $q \notin \mathcal{C}$. A set $\mathcal{C} \subseteq \mathbb{P}$ is dense if, for any $p \in \mathbb{P}$, there is some $q \leq_P p$ with $q \in \mathcal{C}$. Note that it is impossible for a set $G \subseteq \omega$ to avoid a dense set $\mathcal{C} \subseteq \mathbb{P}$.

Finally, we introduce some notation about finite sets. Given a finite set $F \subseteq \omega$, and $X \in 2^\omega$, we write $F < X$ if $\sigma_F < X$, where $\sigma_F \in 2^{<\omega}$ is the binary string of length $k = max(F) + 1$, with $\sigma(i) = 1 \leftrightarrow i \in F$, for $i < k$. Similarly, if $F \subseteq \omega$ and $F' \subseteq \omega$ are finite, coded by $\sigma_F, \sigma_{F'} \in 2^{<\omega}$, respectively, then by $F \leq F'$ we mean that $\sigma \leq \sigma'$. Lastly, given sets $F, F' \subseteq \omega$, not necessarily finite, the notation $F < F'$ means that $max(F) < min(F')$.  

54
3.2 Cohen-Mathias-like forcing partial orders

The motivating question for the work in this section involves a notion of Hechler forcing with computable conditions, analogous to the notion of Mathias forcing with computable conditions defined in [3]. In particular, we ask whether the generics associated to this type of Hechler forcing can compute generics for the Mathias forcing in [3], and vice-versa. We show that this is the case. The two directions (computing a Mathias generic from a Hechler generic, and computing a Hechler generic from a Mathias generic) are proved using constructions that very closely mirror an argument from [4]. Rather than proving one direction, only then to prove the other in a similar way, we obtain both results from a single metatheorem. For this purpose, we define and study a general collection of forcing partial orders that we call Cohen-Mathias-like forcing partial orders. We show that the Cohen generics from [12], the Mathias generics from [4], and a specific type of Hechler generics all arise from Cohen-Mathias-like forcing partial orders $P_{\text{Cohen}}$, $P_{\text{Mathias}}$, and $P_{\text{Hechler}}$, respectively. In Theorem 3.2.8 we verify, using a construction from [3], that the partial order $P_{\text{Cohen}}$ is, in some sense, the weakest of the Cohen-Mathias-like forcing partial orders. Next, in Theorem 3.2.14 we show, using a construction from [4], that the generics for a particular sub-class of the Cohen-Mathias-like forcing partial orders (containing $P_{\text{Mathias}}$ and $P_{\text{Hechler}}$) are the strongest of the Cohen-Mathias-like forcing partial orders. Finally, from Theorem 3.2.14 we obtain the desired result relating Hechler generics and Mathias generics (Corollary 3.2.19).

We limit ourselves to countable forcing partial orders, where the interpretation of a forcing condition is a nonempty subset of $2^\omega$.

**Definition 3.2.1** (Countable forcing partial orders). A countable forcing partial order consists of a countable partial order $(\mathbb{P}, \leq)$ and an interpretation function $[-] : \mathbb{P} \to \mathcal{P}(2^\omega)$ such that
1. For all $p \in \mathbb{P}$, $[p] \neq \emptyset$;

2. For all $p, q \in \mathbb{P}$, $p \leq q \iff [p] \subseteq [q]$.

Next, we define the Cohen-Mathias-like forcing partial orders, and their associated generics.

**Definition 3.2.2** (Cohen-Mathias-like forcing partial orders). A forcing partial order $\mathcal{P} = (\mathbb{P}, \leq)$ with interpretation function $[-] : \mathbb{P} \rightarrow \mathcal{P}(2^\omega)$ is *Cohen-Mathias-like* if the following hold:

1. All elements of $\mathbb{P}$ are pairs $(D, e)$, where $D \subseteq \omega$ is a finite set, and $e \in \omega$;

2. $(\mathbb{P}, \leq)$ is $\Pi^0_2$, when the first component $D$ of each pair $(D, e) \in \mathbb{P}$ is identified with its canonical index as a finite set;

3. If $(D, e) \leq (F, i)$, then $D \geq F$;

4. Given $p = (D, e) \in \mathbb{P}$ and $X \in 2^\omega$, $X \in [p] \iff D < X$ and for all $n \geq \max(D)$, $(X \cap \{0, \ldots, n\}, e) \leq (D, e)$;

5. If $(D, e) \leq (F, e)$ and $D \geq E \geq F$, then $(D, e) \leq (E, e) \leq (F, e)$;

6. For every finite $D \subseteq \omega$, there is an $e \in \omega$ such that $(D, e) \in \mathbb{P}$;

7. For every $(D, e) \in \mathbb{P}$, there are arbitrarily large $n \in \omega$ such that $(D \cup \{n\}, e) \leq (D, e)$;

8. There is an effective procedure $\Phi$ such that whenever $(D, e) \in \mathbb{P}$ and $F \subseteq \omega$ is finite, $\Phi((D, e), F) \downarrow = 1 \iff (F, e) \leq (D, e)$.

**Definition 3.2.3** (Generics for a Cohen-Mathias-like forcing partial order). Fix a Cohen-Mathias-like forcing partial order $\mathcal{P} = (\mathbb{P}, \leq)$, $G \in 2^\omega$, and a set $\mathcal{C} \subseteq \mathbb{P}$.

1. $G$ satisfies a condition $p \in \mathbb{P}$ if $G \in [p]$;

2. $G$ meets $\mathcal{C}$ if $G$ satisfies a condition in $\mathcal{C}$.

3. $G$ avoids $\mathcal{C}$ if $G$ satisfies a condition $p$ such that for every $q \leq p$, $q \notin \mathcal{C}$.

4. $G$ is $n$-generic (with respect to $\mathcal{P}$) if $G$ meets or avoids every $\Sigma^0_n$ set $\mathcal{C} \subseteq \mathbb{P}$.
Before we consider specific examples of Cohen-Mathias-like forcing partial orders, we make a few general observations. Suppose $\mathcal{P} = (\mathbb{P}, \leq)$ is any Cohen-Mathias-like forcing partial order. First, Parts (3) and (4) of Definition 3.2 suggest that, given conditions $p_0 \leq p_1 \in \mathbb{P}$, with $p_i = (D_i, e_i)$, the first components of the conditions $p_0$ and $p_1$ behave like Cohen forcing conditions; that is, $D_0 \supseteq D_1$, and for any $X \in [p_i]$, for $i = 0, 1$, $D_i \cap X$. For such conditions $p_0$, $p_1$, if the second components of $p_0$ and $p_1$ are the same (i.e., if $e_0 = e_1$), we think of $p_0$ as a finite extension of $p_1$.

Several of the items in Definition 3.2 involve finite extensions. In particular, while the entire partial order $\leq$ is $\Pi^0_2$, Part (8) of Definition 3.2 implies that, given $(D, e) \in \mathbb{P}$, the set of finite extensions $(F, e) \leq (D, e)$ of $(D, e)$ is only $\Sigma^0_1$. We can also show that, given $(D_0, e_0) \leq (D_1, e_1) \in \mathbb{P}$, not necessarily a finite extension, it must also be true that $(D_0, e_1) \leq (D_1, e_1)$. To see why, note that, by Part (4) of Definition 3.2, $(D_0, e_0) \leq (D_1, e_1)$ implies $\emptyset \notin [(D_0, e_0)] \subseteq [(D_1, e_1)]$. We also know that $D_0 \supseteq D_1$. Fix $X \in [(D_0, e_0)]$. Since $X$ is also an element of $[(D_1, e_1)]$, it follows, from Part (4) of Definition 3.2, that $(D_0, e_1) = (X \cap \{0, \ldots, \max(D_0)e_1\}, e_1) \leq (D_1, e_1)$.

Finally, we introduce a uniform method of approximating jumps, and check that this method works well with Definition 3.2. Let

$$J(Z, s, x) = \begin{cases} 0 & \text{if } \varphi^Z_{s, s}(x) \uparrow, \\ 1 & \text{if } \varphi^Z_{s, s}(x) \downarrow. \end{cases}$$

That is, $J$ is a total computable functional that correctly approximates $Z'$ in the limit, for any set $Z \subseteq \omega$. We inductively define a method for approximating successive jumps of a set $Z \subseteq \omega$, as follows. Let $Z'[s](x) = J(Z, s, x)$. For $n \geq 2$, and $s_0 < \cdots < s_{n-1}$, let $Z^{(n)}[s_0, \ldots, s_{n-1}](x) = J(Z^{(n-1)}[s_1, \ldots, s_{n-1}], s_0, x)$.

Given any $n \geq 1$ and $s_0 < \cdots < s_{n-1}$, the approximation $Z^{(n)}[s_0, \ldots, s_{n-1}](x)$ is a total uniformly $Z$-computable function of $x$. Since we will consider various $\Sigma^0_n$ sets of forcing conditions, we also define a uniform way to approximate a given $\Sigma^0_n$ set.
$W^{\varnothing(n-1)}_\omega$, for $n \geq 2$. Let $W^{\varnothing(n-1)}_\omega[s_0, \ldots, s_{n-2}] := W^{\varnothing(n-1)}_\omega[s_0, \ldots, s_{n-2}] \upharpoonright s_0$.

By our conventions on use, if $W^{\varnothing(n-1)}_\omega[s_0] \upharpoonright_x = W^{\varnothing(n-1)}_\omega \upharpoonright_x$, and

$\varnothing^{(n-1)}[s_0, \ldots, s_{n-2}] \upharpoonright_s = \varnothing^{(n-1)} \upharpoonright_s$, then $W^{\varnothing(n-1)}_\omega[s_0, \ldots, s_{n-2}] \upharpoonright_x = W^{\varnothing(n-1)}_\omega \upharpoonright_x$.

We shall use these approximations in conjunction with the conditions from a given Cohen-Mathias-like forcing partial order, in the following sense.

**Lemma 3.2.4.** Suppose $\mathcal{P} = (\mathbb{P}, \leq_\mathcal{P})$ is a Cohen-Mathias-like forcing partial order. Fix $Z \subseteq \omega$, $n \geq 1$ and $x \in \omega$. For any condition $(D, e) \in \mathbb{P}$, there are natural numbers $D < s_0 < \cdots < s_{n-1}$ such that $(D \cup \{s_0, \ldots, s_{n-1}\}, e) \leq_\mathcal{P} (D, e)$, and

$Z^{(n)}[s_0, \ldots, s_{n-1}] \upharpoonright_x = Z^{(n)} \upharpoonright_x$.

**Proof.** For $n = 1$, the approximation $Z^{(1)}[s](x)$ is correct for all sufficiently large $s \in \omega$. Using Part (7) of Definition 3.2.2, we get an extension $(D \cup \{s\}, e) \leq_\mathcal{P} (D, e)$, where $s$ is large enough so that $Z^{(1)}[s] \upharpoonright_x = Z' \upharpoonright_x$. Fix $n \geq 2$, and assume that Lemma 3.2.4 is true for $n - 1$. Given $x \in \omega$ and a condition $(D, e) \in \mathbb{P}$, pick $D < s_0$ so that $(D \cup \{s_0\}, e) \leq_\mathcal{P} (D, e)$ and $(\forall y < x)J(Z^{(n-1)}, s_0, y) = Z^{(n)}(y)$. (We are using Part (7) of Definition 3.2.2 again.) Next, by our assumption, we can find $s_1 < \ldots s_{n-1}$, larger than $s_0$, so that $(D \cup \{s_0, s_1, \ldots, s_{n-1}\}, e) \leq_\mathcal{P} (D \cup \{s_0\}, e)$ and $Z^{(n-1)}[s_1, \ldots, s_{n-1}] \upharpoonright_{s_0} = Z^{(n-1)} \upharpoonright_{s_0}$. The condition $(D \cup \{s_0, s_1, \ldots, s_{n-1}\}, e) \leq_\mathcal{P} (D, e)$ has the desired properties.

In Examples 3.2.5 and 3.2.6 we verify that the Cohen generics and Mathias generics defined, respectively, in [12], [3], arise from Cohen-Mathias-like forcing partial orders, which we call $\mathcal{P}_{\text{Cohen}}$ and $\mathcal{P}_{\text{Mathias}}$.

**Example 3.2.5** (Cohen forcing). In the literature (see, e.g., Jockusch [12]), a Cohen forcing condition is a string $\sigma \in 2^\omega$. Given such a forcing condition $p = \sigma$, its interpretation is the set $[p] = \{X \in 2^\omega : \sigma \prec X\}$. A set $G \in 2^\omega$ is *Cohen $n$-generic*, for $n \in \omega$, if $G$ meets or avoids every $\Sigma^0_n$ set of Cohen forcing conditions.
We define a Cohen-Mathias-like forcing partial order \( P_{\text{Cohen}} \) whose \( n \)-generics are exactly the traditional Cohen \( n \)-generics, for every \( n \in \omega \), as follows. The set \( P_{\text{Cohen}} \) of forcing conditions is the set of all \((D, e)\), where \( D \subseteq \omega \) is finite, and \( e \in \omega \). Given such a condition \((D, e)\), its interpretation \([((D, e))]\) is the set \( \{ X \in 2^\omega : D \subseteq X \} \). The partial ordering \( \leq_{\text{Cohen}} \) induced by the interpretation function is characterized by \((D, e) \leq (F, i) \iff D \supseteq F\). It is not hard to check that \( P_{\text{Cohen}} \) is a Cohen-Mathias-like forcing partial order, and that for every \( n \in \omega \), the \( n \)-generics for this forcing partial order are exactly the usual Cohen \( n \)-generics.

**Example 3.2.6** (Mathias forcing with computable conditions).

The authors of [3] considered a version of Mathias forcing with conditions \((D, S)\), where \( D \subseteq \omega \) is finite, \( S \subseteq \omega \) is infinite and computable, and \( \text{max}(D) < \text{min}(S) \). A code for such a condition is a pair \((e_0, e_1) \in \omega \times \omega\), where \( e_0 \) is the canonical code for \( D \) as a finite set, and \( e_1 \) is any index of a Turing machine that computes the characteristic function of \( S \). The interpretation of a condition \( p = (D, S) \) is the set \([p] = \{ X \in 2^\omega : D \subseteq X \subseteq D \cup S \}\), and, given any two conditions \( p_0 \) and \( p_1 \), \( p_0 \) extends \( p_1 \) iff \([p_0] \subseteq [p_1]\). A collection \( C \) of computable Mathias conditions is \( \Sigma^0_n \), for some \( n \in \omega \), if there is a \( \Sigma^0_n \) set \( D \) of codes for computable Mathias conditions such that \( C \) is exactly the collection of computable Mathias conditions represented by the codes in \( D \). A set \( G \subseteq \omega \) is Mathias \( n \)-generic, for \( n \in \omega \), if \( G \) meets or avoids every \( \Sigma^0_n \) set of computable Mathias conditions.

We define a Cohen-Mathias-like forcing partial order \( P_{\text{Mathias}} \) that yields the same collection of \( n \)-generics, for every \( n \in \omega \), as the notion described in the preceding paragraph. Let \( P_{\text{Mathias}} \) be the collection of all pairs \((D, e)\), such that \( D \subseteq \omega \) is finite, and \( e \in \omega \) is the index of a Turing machine that computes the characteristic function of an infinite set. We do not require that \( e \) be a computable index for a set whose elements are all larger than \( \text{max}(D) \). Given a condition \((D, e) \in P_{\text{Mathias}}\), its interpretation \([((D, e))]\) is the set
\{X \in 2^{\omega} : D \triangleleft X \land (\forall n \geq \max(D))(n \in X \rightarrow \varphi_e(n) \downarrow = 1)\}. The induced partial ordering on the forcing conditions can be described as follows. Suppose we are given two conditions \((D, e) \in \mathbb{P}_{\text{Mathias}}, \) where \(e\) is an index for a set \(S \subseteq \omega,\) and \(i\) is an index for a set \(T \subseteq \omega.\) Then \((D, e) \leq_{\text{Mathias}} (F, i)\) iff

1. \(D \supseteq F;\)
2. For every \(n \in D - F, n \in T;\) and
3. The set of elements of \(S\) greater than \(\max(D)\) is a subset of \(T.\)

It is not difficult to see that \(\mathbb{P}_{\text{Mathias}}\) is a Cohen-Mathias-like forcing partial order. We can also see that \(\mathbb{P}_{\text{Mathias}}\) generates the same collection of \(n\)-generics as in \([3]\).

In the next example, we define a type of Hechler forcing \(\mathbb{P}_{\text{Hechler}},\) also a Cohen-Mathias-like forcing partial order, that is analogous to the Mathias forcing defined in \([3]\). One peculiarity of the particular forcing \(\mathbb{P}_{\text{Hechler}}\) is as follows. While the generics associated to the standard Hechler forcing, as described in Section \([3.1]\), are generally functions \(G : \omega \rightarrow \omega,\) not necessarily strictly increasing, a generic \(G : \omega \rightarrow \omega\) for \(\mathbb{P}_{\text{Hechler}}\) turns out to be a strictly increasing function, which we identify with the infinite set \(\{G(0), G(1), \ldots\} \subseteq \omega.\)

**Example 3.2.7** (Hechler forcing with computable conditions). We define the Cohen-Mathias-like forcing partial order \(\mathbb{P}_{\text{Hechler}}\) as follows. The set \(\mathbb{P}_{\text{Hechler}}\) of conditions is the set of pairs \((D, e),\) where \(D \subseteq \omega\) is finite, and \(e \in \omega\) is the index of a total computable function \(f : \omega \rightarrow \omega.\) Given a condition \((D, e) \in \mathbb{P}_{\text{Hechler}},\) its interpretation \([\!(D, e)\!]\) is the set \(\{X \in 2^{\omega} : D \triangleleft X \land (\forall n \geq |D|)(p_X(n) \geq \varphi_e(n))\}.\) (Here, \(p_X\) means the principal function of \(X,\) or the function that lists all elements of \(X\) in increasing order.) It is clear that \(\mathbb{P}_{\text{Hechler}}\) satisfies the requirements of Definition \([3.2.2].\)

The authors of \([3]\) showed that, for \(n \geq 3,\) every Mathias \(n\)-generic computes a Cohen \(n\)-generic. We now check that this result lifts to the setting of Cohen-Mathias-
like forcing partial orders. This verification does not require any new techniques; our proof proceeds along the same lines as in [3], mutatis mutandis.

**Theorem 3.2.8.** Fix \( n \geq 3 \). If \( \mathcal{P} \) is a Cohen-Mathias-like forcing partial order and \( G \) is \( n \)-generic with respect to \( \mathcal{P} \), then \( G \) computes a Cohen \( n \)-generic.

**Proof.** We define an effective procedure \( \Gamma \) which takes as input any finite set \( F \subseteq \omega \) and outputs a string \( \Gamma(F) \in 2^{<\omega} \). We ensure that if \( F/\mathcal{P} \) witnesses \( \mathcal{P} \)-genericity, then \( \Gamma(F) \) witnesses \( \mathcal{P} \)-genericity. Given any \( \mathcal{P} \)-\( n \)-generic \( G \), we show that \( H := \bigcup_{m \in \omega} \Gamma(G \upharpoonright m) \) is Cohen \( n \)-generic.

**Definition of \( \Gamma \):** Let \( \Gamma(\varnothing) = \lambda \). Suppose we have defined \( \Gamma(F) \) for every finite set \( F \) of size at most \( k \). Given \( F = \{s_0 < s_1 < \cdots < s_k\} \) of size \( k + 1 \), if \( k + 1 \) is not a multiple of \( n \), then let \( \Gamma(F) = \Gamma(\{s_0, \ldots, s_{k-1}\}) \). Suppose now that \( k + 1 \) is a multiple of \( n \). Write \( F = \tilde{F} \cup T \), where \( T = \{t_0 < t_1 < \cdots < t_{n-1}\} \) consists of the last \( n \) elements of \( F \). Suppose \( \Gamma(\tilde{F}) = \sigma_0 \). Search for \( e \leq k \) such that

1. no initial segment of \( \sigma_0 \) belongs to \( W^{(n-1)}_e[t_0, \ldots, t_{n-1}] \) and 
2. there is a \( \tau \geq \sigma_0 \) such that \( \tau \in W^{(n-1)}_e[t_0, \ldots, t_{n-1}] \).

If no such \( e \) is found, then let \( \Gamma(F) = \Gamma(\tilde{F}) = \sigma_0 \). Otherwise, let \( \Gamma(F) = \tau \) where \( \tau \) is the least string satisfying the above conditions for the least \( e \) found. This completes the definition of \( \Gamma \).

**Verification:** Suppose \( G \) is \( n \)-generic with respect to \( \mathcal{P} \). We show that \( H := \bigcup_{m \in \omega} \Gamma(G \upharpoonright m) \) is Cohen \( n \)-generic. Fix \( e \in \omega \). We must show that \( H \) meets or avoids the set \( W^{(n-1)}_e \), where we think of the elements of \( W^{(n-1)}_e \) as codes for binary strings. Assume inductively that for all \( i < e \), \( H \) either meets or avoids \( W^{(n-1)}_{i} \), and \( m \) is the least number such that for all \( i < e \), if \( H \) meets \( W^{(n-1)}_{i} \), then some initial
substring of $\Gamma(G \upharpoonright m)$ belongs to $W^{(n-1)}_i$. Let

$$\mathcal{C}_e = \{(D,j) \in \mathbb{P} : D \supseteq G_m \land (\exists \sigma)(\sigma \in W^{(n-1)}_e \land \sigma \not\in \Gamma(D))\}.$$ 

Since $\mathbb{P}$ is $\Pi^0_2$ and $n \geq 3$, $\mathcal{C}_e$ is $\Sigma^0_n$ definable. Thus $G$ meets or avoids $\mathcal{C}_e$. Suppose $G$ meets $\mathcal{C}_e$ via $(D,j)$. Then some initial substring of $\Gamma(D)$ (and hence an initial substring of $H$) belongs to $W^{(n-1)}_e$, and we are done.

Suppose, now, that $G$ avoids $\mathcal{C}_e$ via $(D,j)$. We may assume that the number of elements in $D$ is a multiple of $n$. We claim that $H$ avoids $W^{(n-1)}_e$ via $\Gamma(D)$. If not, let $\sigma \in W^{(n-1)}_e$ be the least such that $\sigma \not\in \Gamma(D)$. Choose $s_0 \in \omega$ with $(D \cup \{s_0\},j) \leq (\tau,j)$, large enough so that $\sigma \in W^{(n-1)}_i \upharpoonright s_0$ and for each $i < e$, if $\Gamma(G)$ meets $W^{(n-1)}_i$ then there is $\sigma_i \in W^{(n-1)}_i \upharpoonright s_0$ such that $\sigma_i \not\in \Gamma(\tau)$. Next choose $s_1 < \cdots < s_{n-1}$ with $(D \cup \{s_0, \ldots, s_{n-1}\},j) \leq (D \cup \{s_0\},j) \leq (D,j)$ such that for all $i \leq e$, $W^{(n-1)}_i \upharpoonright s_0 = W^{(n-1)}_i[s_0, \ldots, s_{n-1}]$. Then $(D \cup \{s_0, \ldots, s_{n-1}\},j) \in \mathcal{C}_e$, contradicting the fact that $(D,j)$ has no extension in $\mathcal{C}_e$.

Theorem 3.2.8 suggests a way of structuring the collection of all Cohen-Mathias-like forcing partial orders, with $\mathcal{P}_{Cohen}$ at the bottom of the collection. We make this notion more precise.

**Definition 3.2.9.** Given two Cohen-Mathias-like forcing partial orders $\mathcal{P} = (\mathbb{P}, \leq_{\mathcal{P}})$ and $\mathcal{Q} = (\mathbb{Q}, \leq_{\mathcal{Q}})$, we say that $\mathcal{P}$ is stronger than $\mathcal{Q}$ (written $\mathcal{Q} \not\leq \mathcal{P}$) if for every $m \in \omega$ there is an $n \in \omega$ such that every $n$-generic with respect to $\mathcal{P}$ computes an $m$-generic with respect to $\mathcal{Q}$. If $\mathcal{Q} \not\leq \mathcal{P}$, and $\neg(\mathcal{P} \not\leq \mathcal{Q})$, we say that $\mathcal{P}$ is strictly stronger than $\mathcal{Q}$.

The previous theorem can be restated in this framework, as follows:

**Corollary 3.2.10.** For every Cohen-Mathias-like forcing partial order $\mathcal{P}$, $\mathcal{P}_{Cohen} \not\leq \mathcal{P}$.
Jockusch [12] showed that every Cohen $n$-generic $G$ satisfies $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. Furthermore, for every $n$, $\emptyset^{(n)}$ computes a Cohen $n$-generic. Thus, for every $n$, there is a Cohen $n$-generic $G$ that is low$_n$, i.e., $G^{(n)} \equiv_T \emptyset^{(n)}$. On the other hand, as noted in [3], for $m \geq 3$, every Mathias $m$-generic $H$ is high, i.e., $H' \geq_T \emptyset''$. Hence, there is no $n$ such that every Cohen $n$-generic computes a Mathias $3$-generic. In the terminology introduced in Definition 3.2.9 this means that $\mathcal{P}_{\text{Mathias}}$ is strictly stronger than $\mathcal{P}_{\text{Cohen}}$.

There are several questions we might ask about the structure of the Cohen-Mathias-like forcing partial orders under the relation $\preceq$. The rest of this section is devoted to showing that certain Cohen-Mathias-like forcing partial orders, including $\mathcal{P}_{\text{Mathias}}$ and $\mathcal{P}_{\text{Hechler}}$, are maximal with respect to $\preceq$. Indeed, the following property guarantees that a given Cohen-Mathias-like forcing partial order is stronger than all other Cohen-Mathias-like forcing partial orders.

**Definition 3.2.11.** Fix a Cohen-Mathias-like forcing partial order $\mathcal{P} = (\mathbb{P}, \leq)$. We say that $\mathcal{P}$ produces nice $\Pi^0_2$ approximations if there is $n \in \omega$ such that for every $\Pi^0_2$ set $X \subseteq \omega$, there is a computable $\Phi$ such that

1. For every pair $(x, D)$, with $x \in \omega$ and $D \subseteq \omega$ finite, $\Phi(x, D) \downarrow \in \{0, 1\}$;
2. If $G$ is $\mathcal{P}$-$n$-generic, then for all $x \in \omega$, $\lim_s \Phi(x, G \cap \{0, \ldots, s\}) = X(x)$;
3. For all $x \in X$, for every condition $p = (D, e) \in \mathbb{P}$, the set
   $$\mathcal{C}_X(x, p) = \{(D, i) \in \mathbb{P} : (D, i) \preceq p \land (\forall (E, j) \leq (D, i))(|E| > |D| \rightarrow \Phi(x, E) \downarrow = 1)\}$$

is nonempty.

**Lemma 3.2.12.** The partial order $\mathcal{P}_{\text{Mathias}}$ produces nice $\Pi^0_2$ approximations, witnessed by $n = 3$.

**Proof.** Suppose $X \subseteq \omega$ is $\Pi^0_2$ with definition $x \in X \leftrightarrow (\forall y)(\exists z)R(x, y, z)$, for some recursive relation $R$. 63
Given $x \in \omega$ and $D \subseteq \omega$ finite, let

$$
\Phi(x, D) = \begin{cases} 
1 & \text{if } (\forall y < |D|)(\exists z < \text{max}(D))[R(x, y, z)] \\
0 & \text{otherwise}
\end{cases}.
$$

It is clear that $\Phi$ satisfies the first item in Definition 3.2.11. For the second part, fix a $P_{\text{Mathias}}$ 3-generic $G$, and any $x \in \omega$. If $x \notin X$, there is some $y \in \omega$ such that $(\forall z)[\neg R(x, y, z)]$. For any $s$ such that $|G \cap \{0, \ldots, s\}| > y$, we have

$$
\Phi(x, G \cap \{0, \ldots, s\}) = 0.
$$

Since $G$ is infinite, it follows that $\lim_s \Phi(x, G \cap \{0, \ldots, s\}) = 0$.

Suppose now that $x \in X$. Let

$$
C = \{(D, i) \in P_{\text{Mathias}} : (\forall (E, j) \leq (D, i))[|E| > |D| \rightarrow \Phi(x, E) = 1]\}.
$$

We claim that $G$ meets $C$, and hence $\lim_s \Phi(x, G \cap \{0, \ldots, s\}) = 1$. At first glance, it appears to be $\Pi^0_3$ to decide whether a given pair $(D, i)$ belongs to $C$, since the above definition of $C$ quantifies over all extensions $(E, j) \leq (D, i)$. However, using (2) from Definition 3.2.1 and (3) from Definition 3.2.2, we see that, given $(E, j) \leq (D, i) \in P_{\text{Mathias}}$, it must also be true that $(E, i) \leq (D, i)$. Hence,

$$
(D, i) \in C \leftrightarrow (D, i) \in P_{\text{Mathias}} \land (\forall E \geq D)[(E, i) \leq (D, i) \rightarrow \Phi(x, E) = 1].
$$

From the above, and applying (7) from Definition 3.2.2, we see that $C$ is actually $\Pi^0_2$. Since $G$ is Mathias 3-generic, $G$ meets or avoids $C$. Furthermore, $C$ is dense, and so $G$ meets $C$. Suppose we are given any condition $(D, i) \in P_{\text{Mathias}}$, where $i$ is a computable index for an infinite set $S \subseteq \omega$. We may assume that $\text{max}(D) < \text{min}(S)$. We locate an extension $(D, e) \leq (D, i)$, where $e$ is an index for $\tilde{S} = \{s_1 < s_2 < \ldots\} \subseteq S$, and $(D, e) \in C$, as follows. Given $y \in \omega$, let $w(y) := \mu(z)(R(x, y, z))$. Since $x \in X$, $w$ is a total recursive function.
1. \( s_1 = \mu(s \in S)[(\forall y < |D|)(w(y) < s)] \) and
2. For \( n \geq 1, s_{n+1} = \mu(s \in S)[s > s_n \land (\forall y < |D| + n)(w(y) < s)] \).

Finally, for the third part of Definition \[3.2.11\] note that, given \( x \in X \), and \( p \in P_{\text{Mathias}} \), the set \( C_X(x,p) \) is simply the set of conditions above \( p \) in the set \( C \) defined in the previous paragraph. Since \( C \) is dense, \( C_X(x,p) \neq \emptyset \).

\[\square\]

**Lemma 3.2.13.** The partial order \( P_{\text{Hechler}} \) produces nice \( \Pi^0_2 \) approximations, witnessed by \( n = 3 \).

**Proof.** Define \( \Phi \) exactly as in Lemma \[3.2.12\]. The verification that \( \Phi \) satisfies the requirements of Definition \[3.2.11\] proceeds very much as in Lemma \[3.2.12\] replacing, when necessary, the word “Mathias” with the word “Hechler.” Suppose we have arrived at the point where we need to show that the set

\[
C = \{(D, i) \in P_{\text{Hechler}} : (\forall (E, j) \leq (D, i))(|E| > |D| \rightarrow \Phi(x, E) = 1)\}
\]

is dense. Fix any \( (D, i) \in P_{\text{Hechler}} \), where \( i \) is an index for a total computable function \( f \). Let \( w(y) \) be as in Lemma \[3.2.12\]. Let \( e \in \omega \) be an index for the computable function \( \bar{f} \), with \( \bar{f}(n) = \max\{f(n), w(0), \ldots, w(n)\} \). Then \( (D, e) \leq (D, i) \) is an extension of \( (D, i) \) in \( C \).

\[\square\]

We now show that the Cohen-Mathias-like forcing partial orders that produce nice \( \Pi^0_2 \) approximations are on top with respect to \( \preceq \). The authors of [3] proved that if \( G \) is Mathias \( n \)-generic for conditions in a countable Turing ideal, then \( G \) computes a Mathias \( n \)-generic for computable conditions. Theorem \[3.2.14\] is proved using a construction very similar to the one in [3].

**Theorem 3.2.14.** Suppose \( P = (P, \leq_P) \) and \( Q = (Q, \leq_Q) \) are Cohen-Mathias-like forcing partial orders and \( P \) produces nice \( \Pi^0_2 \) approximations. Then \( Q \preceq P \).
Proof. Given \( m \in \omega \), let \( n \geq \max(m, 3) \) be large enough to witness that \( \mathcal{P} \) produces nice \( \Pi^0_2 \) approximations. Suppose \( G \) is \( n \)-generic with respect to \( \mathcal{P} \). We show that \( G \) computes an \( n \)-generic with respect to \( \mathcal{Q} \). As in the proof of Theorem 3.2.8, we define a total computable functional \( \Gamma \) that takes as input a finite set \( F \) and outputs a finite set \( \Gamma(F) \), with the property that \( \Gamma(F) \leq \Gamma(\bar{F}) \) whenever \( F \leq \bar{F} \). We then show that \( H := \bigcup_i \Gamma(G \cap \{0, \ldots, s\}) \) is \( n \)-generic with respect to \( \mathcal{Q} \). That is, given any \( \Sigma^0_n \) set \( \mathcal{C} \) of \( \mathcal{Q} \)-conditions, we must show that \( H \) meets or avoids \( \mathcal{C} \). We note that, given an index for any \( \Sigma^0_n \) set \( \mathcal{C} \), possibly containing numbers that do not code \( \mathcal{Q} \)-conditions, we can effectively pass to the index of the \( \Sigma^0_n \) subset \( \mathcal{D} \subseteq \mathcal{C} \) of all elements of \( \mathcal{C} \) that do code \( \mathcal{Q} \)-conditions. Given this, we are justified in assuming, in our construction, that the \( e \)-th \( \Sigma^0_n \) set \( W^e_{\varphi(n-1)} \) contains only codes for \( \mathcal{Q} \)-conditions.

Let \( \Phi_0 \) be a total computable functional witnessing Definition 3.2.11 for \( \mathcal{P} \) when \( X = \mathcal{Q} \), and let \( \Phi_1 \) be such a witness for \( \mathcal{P} \) when \( X \) is the set of pairs \( (\alpha_0, \alpha_1) \in \mathcal{Q} \times \mathcal{Q} \) with \( \alpha_0 \geq \mathcal{Q} \alpha_1 \). Let \( \Phi_2 \) witness part (7) of Definition 3.2.2 for \( \mathcal{Q} \). We (non-uniformly) fix a ”starting condition” \( \beta \in \mathcal{Q} \). We may assume that \( \Phi_0(\beta, F) \downarrow 1 \) for all finite sets \( F \subset \omega \). Given a sequence \( \bar{\alpha} = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1}) \), with \( D_i \subseteq \omega \) finite, and \( e_i \in \omega \), for \( i = 0, \ldots, k-1 \), we say \( \bar{\alpha} \) is good if \( \bar{\alpha} \) is a \( \leq \mathcal{Q} \)-decreasing sequence of elements of \( \mathcal{Q} \).

Given a finite set \( F \subset \omega \), we say \( F \) thinks \( \bar{\alpha} \) is good if

1. For every \( i \leq k-1 \), \( \Phi_0((D_i, e_i), F) \downarrow 1 \) and
2. For every \( i < j \leq k-1 \), \( \Phi_1((D_i, e_i), (D_j, e_j), F) \downarrow 1 \).

That is, according to the approximations \( \Phi_0 \) and \( \Phi_1 \) with input \( F \), \( \bar{\alpha} \) looks like a decreasing sequence of actual forcing conditions from \( \mathcal{Q} \).

Construction 3.2.15. We define \( \Gamma \) and an intermediary functional \( \Psi \) by induction on the size of \( F \), as follows. We let \( \Psi(\emptyset) = \beta \), where \( \beta = (D_0, e_0) \) is the starting condition in \( \mathcal{Q} \). Let \( \Gamma(\emptyset) = D_0 \). Now suppose that we have defined \( \Psi(F) \) and \( \Gamma(F) \) for all \( F \subset \omega \) of size at most \( s \geq 0 \). Given a set \( F \) of size \( s+1 \), if \( s+1 \) is not a multiple
of \( n \) then let \( \Psi(F) = \Psi(\tilde{F}) \) and \( \Gamma(F) = \Gamma(\tilde{F}) \), where \( \tilde{F} = F \setminus \{\max(F)\} \). If \( s + 1 \) is a multiple of \( n \), rewrite \( F = E \cup \{t_0 < \cdots < t_{n-1}\} \), where the \( t_i \) are the last \( n \) elements of \( F \). Suppose \( \Psi(E) = \overline{\alpha} = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1}) \), and \( \Gamma(E) = D_{k-1} \).

First, ask whether \( F \) thinks \( \overline{\alpha} \) is good. If not, let \( 0 < i \leq k-1 \) be the least such that either \( \Phi_0(((D_i, e_i), F) \downarrow \neq 1 \) or there is \( 0 \leq i' < i \) such that \( \Phi_1(((D_{i'}, e_{i'}), (D_i, e_i), F) \downarrow \neq 1 \). Then let \( \Psi(F) = (D_0, e_0), \ldots, (D_{i-1}, e_{i-1}), (D_{k-1}, e_{i-1}) \) and \( \Gamma(F) = D_{k-1} \).

If \( F \) thinks \( \overline{\alpha} \) is good, we try to extend the sequence \( \overline{\alpha} \) to meet a new \( \Sigma_0^0 \) set of \( \mathbb{Q} \)-conditions. Search for \( i \leq s \) such that

1. No element of the sequence \( \overline{\alpha} \) is in \( W^{\varnothing(n-1)}_i [t_0, \ldots, t_{n-1}] \), and
2. There is \((D, e) \in W^{\varnothing(n-1)}_i [t_0, \ldots, t_{n-1}] \) such that
   
   (a) \( \Gamma(E) = D_{k-1} \leq D \),
   (b) \( F \) thinks the sequence \((\overline{\alpha}, (D, e)) \) is good, and
   (c) For every \( i \leq k-1 \), \( \Phi_2(((D_i, e_i), D) \downarrow = 1 \) in at most \( \max(F) \) steps.

If no such \( i \) is found, then let \( \Psi(F) = \Psi(E) = \overline{\alpha} \) and \( \Gamma(F) = \Gamma(E) = D_{k-1} \). Otherwise, let \( \Psi(F) = (\overline{\alpha}, (D, e)) \), where \((D, e) \) is the least witness for the least \( i \leq s \) satisfying (1) and (2) above, and let \( \Gamma(F) = D \).

This completes the construction.

Before we verify that the construction works, we prove some facts about \( \Gamma \) and \( \Psi \).

**Claim 3.2.16.** Suppose \( F \subset \omega \) is finite. If \( \Psi(F) = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1}) \), then for all \( 0 \leq i < j \leq k-1 \), if \((D_i, e_i) \in \mathbb{Q} \), then \((D_i, e_i) \geq_{\mathbb{Q}} (D_j, e_j) \).

**Proof.** If \( F = \varnothing \), then \( \Psi(F) = \beta \) and the claim holds trivially. Suppose now the claim is true for all \( F \subset \omega \) of size at most \( s \geq 0 \). Write \( F = E \cup \{t_0 < \cdots < t_{n-1}\} \) as in the construction. We may assume \(|F| \) is a multiple of \( n \), as otherwise \( \Psi(F) = \Psi(E) \) and we are done. Suppose \( \Psi(E) = \overline{\alpha} = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1}) \). If \( F \) does not think \( \overline{\alpha} \) is good, then
\( \Psi(F) = (D_0,e_0),\ldots,(D_{i-1},e_{i-1}),(D_{k-1},e_{k-1}) \) for some \( 0 < i \leq k - 1 \) and the claim holds by inductive hypothesis. If \( F \) thinks \( \overline{\alpha} \) is good, then either \( \Psi(F) = \Psi(E) \), \( \Psi(F) = (\Psi(E),(D'_0,e'_0)) \), or \( \Psi(F) = (\Psi(E),(D'_0,e'_0),(D'_1,e'_1)) \) for some new pairs \( (D'_i,e'_i), i = 0,1 \). In the latter two cases, we have ensured the claim holds using the functional \( \Phi_2 \).

Given \( p = (F,i) \in \mathbb{P} \) and a sequence \( \overline{\alpha} = (D_0,e_0),\ldots,(D_{k-1},e_{k-1}) \), say that \( (\overline{\alpha},p) \) is a good pair if \( \overline{\alpha} \) is a good sequence, \( \overline{\alpha} \) is an initial subsequence of \( \Psi(F) \), and for all \( p' = (F',i') \leq_p p \) with \( |F'| > |F| \), \( F' \) thinks that \( \overline{\alpha} \) is good. In particular, this implies that for every \( p' = (F',i') \leq_p p \), \( \overline{\alpha} \) is an initial subsequence of \( \Psi(F') \). Given any \( p_0 = (F_0,i_0) \in \mathbb{P} \), if \( \Psi(F_0) = \overline{\alpha} \) is a good sequence, then, applying part (3) of Definition \[3.2.11\] finitely many times, we can obtain an extension \( p_1 = (F_0,i_1) \leq p_0 \) such that for every \( p = (F,i) \leq p_1 \) with \( |F| > |F_0| \), \( F \) thinks that \( \overline{\alpha} \) is good. It follows that \( (\overline{\alpha},p_1) \) is a good pair. We use this fact in the next claim.

**Claim 3.2.17.** The set \( C = \{ p = (F,i) \in \mathbb{P} : (\Psi(F),p) \text{ is a good pair} \} \) is a dense \( \Sigma_3^0 \) set of \( \mathbb{P} \)-conditions.

**Proof.** For density, according to the preceding paragraph, it suffices to show that the set \( \tilde{C} = \{ p = (F,i) \in \mathbb{P} : \Psi(F) \text{ is good} \} \) is dense, for, given \( p = (F,i) \in \tilde{C} \), we can always extend to \( p' = (F,i') \leq p \) such that \( (\Psi(F),p') \) is a good pair. Fix \( p_0 = (F_0,i_0) \in \mathbb{P} \). Let \( G \) be a \( \mathcal{P} \)-\( n \)-generic set that satisfies \( p_0 \). Suppose \( \Psi(F_0) = (D_0,e_0),\ldots,(D_{k-1},e_{k-1}) \). If this sequence is good, then we are done.

Otherwise, let \( 0 \leq j \leq k - 1 \) be the greatest such that \( (D_0,e_0),\ldots,(D_j,e_j) \) is an initial segment of \( \Psi(G \cap \{0,\ldots,t\}) \) for all \( t \geq \text{max}(F_0) \). (Because we ensured that the initial condition \( \beta \) is an initial segment of \( \Psi(F) \) for every finite \( F \subset \omega \), the number \( j \) is well-defined.) The sequence \( (D_0,e_0),\ldots,(D_j,e_j) \) is good, because \( \mathcal{P} \) produces nice \( \Pi_3^0 \) approximations, and \( G \) is sufficiently generic to witness this. Let \( t_1 > \text{max}(F_0) \) be the least such that \( (D_0,e_0),\ldots,(D_j,e_j) \) is an initial segment of
\(\Psi(G \cap \{0, \ldots, t_1\})\) but \((D_0, e_0), \ldots, (D_{j+1}, e_{j+1})\) is not. Then \(\Psi(G \cap \{0, \ldots, t_1\})\) must have form \((D_0, e_0), \ldots, (D_j, e_j), (D, e_j)\), for some \(D \subseteq \omega\) with \((D, e_j) \leq (D_j, e_j)\). Hence \(p_t = (G \cap \{0, \ldots, t_1\}, i_0) \leq (F_0, i_0) = p_0\) is an extension of \(p_0\) in \(\mathcal{C}\).

To see that \(\mathcal{C}\) is \(\Sigma^0_3\), it is enough to show that the set of all good pairs is \(\Sigma^0_3\). Given a pair \((\overline{\alpha}, p)\), where \(p = (F, i)\), \((\overline{\alpha}, p)\) is a good pair iff \(\overline{\alpha}\) is good, \(p \in \mathbb{P}\), \(\overline{\alpha} \leq \Psi(F)\), and for every \((F', i') \leq_p (F, i)\) with \(|F'| > |F|\), \(F'\) thinks \(\overline{\alpha}\) is good. This definition quantifies over all extensions \((F', i') \leq_p (F, i)\), which might make \(\mathcal{C}\) a \(\Sigma^0_4\) set, at best. However, according to our remarks following Definition \(3.2.2\), if \((F', i') \leq_p (F, i)\), then \((F', i) \leq_p (F, i)\). Furthermore, the set of finite \(F' \subseteq \omega\) such that \((F', i) \leq_p (F, i)\) is only \(\Sigma^0_1\). This implies that \(\mathcal{C}\) is actually \(\Sigma^0_3\).

\(\square\)

Suppose, now, that \(G\) is \(n\)-generic with respect to \(\mathcal{P}\), and \(H = \bigcup \Gamma(G \cap \{0, \ldots, s\})\).

**Claim 3.2.18.** Suppose \(\overline{\alpha} = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1})\) is an initial segment of the sequences \(\Psi(G \cap \{0, \ldots, s\})\) for all sufficiently large \(s \in \omega\). Then

1. \(\overline{\alpha}\) is good and
2. For all \(i \leq k - 1\), \(H\) satisfies \((D_i, e_i)\).

**Proof.** Part (1) follows from Claim \(3.2.17\). Indeed, for infinitely many \(s \in \omega\), \(\Psi(G \cap \{0, \ldots, s\})\) is a good sequence. For Part (2), fix \(i \leq k - 1\). According to Part (4) of Definition \(3.2.2\), to show that \(H\) satisfies \((D_i, e_i)\), we must show that for all \(t \geq |D_i|\), \((H \cap \{0, \ldots, t\}, e_i) \leq_Q (D_i, e_i)\). Choose \(s \in \omega\) large enough so that

\[
\Psi(G \cap \{0, \ldots, s\}) = (D_0, e_0), \ldots, (D_k, e_k), (D_{k+1}, e_{k+1}), \ldots, (D_{k+j}, e_{k+j}),\]

and \(H \cap \{0, \ldots, t\} \leq D_{k+j} = \Gamma(G \cap \{0, \ldots, s\})\). By Claim \(3.2.16\), we have \((D_i, e_i) \geq_Q (D_{k+j}, e_{k+j})\). Since \(D_i \leq H \cap \{0, \ldots, t\} \leq D_{k+j}\), it follows from item (5) of Definition \(3.2.2\) that \((D_i, e_i) \geq_Q (H \cap \{0, \ldots, t\}, e_i)\). \(\square\)

**Verification of Construction 3.2.15**
Fix $e \in \omega$. Assume, inductively, that for all $i < e$, $H = \bigcup_i \Gamma(G \cap \{0, \ldots, s\})$ meets or avoids $W^{\varphi^{(n-1)}}_i$, that there is a good sequence $\overline{\alpha} = (D_0, e_0), \ldots, (D_{k-1}, e_{k-1})$, and $t \in \omega$ such that

1. $\overline{\alpha}$ is an initial subsequence of $\Psi(G \upharpoonright s)$ for all $s \geq t$ and
2. For all $i < e$, at least one of the following is true:
   
   (a) $H$ meets $W^{\varphi^{(n-1)}}_i$ and some element of $\overline{\alpha}$ is in $W^{\varphi^{(n-1)}}_i$;
   
   (b) $H$ avoids $W^{\varphi^{(n-1)}}_i$ and some element of $\overline{\alpha}$ has no extension in $W^{\varphi^{(n-1)}}_i$.

Let $C \subseteq \mathbb{P}$ be the set

$$
\{p = (F, i) \in \mathbb{P} : ((\Psi(F), p) \text{ is a good pair}) \land (\text{some element of } \Psi(F) \text{ is in } W^{\varphi^{(n-1)}}_i)\}.
$$

The set of good pairs is $\Sigma^0_3$, and $n \geq 3$, so $C$ is a $\Sigma^0_n$ subset of $\mathbb{P}$. If $G$ meets $C$, then we are done. Suppose, now, that $G$ avoids $C$ via a condition $p_0 = (F_0, i_0)$. We may assume that $\operatorname{max}(F_0) \leq t$. Consider the set

$$
\tilde{C} = \{p = (F, i) \in \mathbb{P} : p \leq p_0 \land (\Psi(F), p) \text{ is a good pair}\}.
$$

By Claim 3.2.17, $\tilde{C}$ is dense above $p_0$. Since $G$ satisfies $p_0$, $G$ meets $\tilde{C}$ via some condition $p_1 = (F_1, i_1) \leq p_0$. The sequence $\Psi(F_1) = \overline{\alpha}, (D'_0, e'_0), \ldots, (D'_j, e'_j)$ is good and an initial subsequence of $\Psi(G \cap \{0, \ldots, s\})$ for all $s > \operatorname{max}(F_1)$. Suppose, for contradiction, that $(D'_j, e'_j)$ has an extension in $W^{\varphi^{(n-1)}}_{e'}$. Choose $D_1 < t_0 < \cdots < t_{n-1}$ with $p_2 = (D_1 \cup \{t_0, \ldots, t_{n-1}\}, i_1) \leq p_1$ large enough to ensure that $\Psi(D_1 \cup \{t_0, \ldots, t_{n-1}\})$ is a good sequence containing an element of $W^{\varphi^{(n-1)}}_{e'}$. We may then extend to $p_3 = (D_1 \cup \{t_0, \ldots, t_{n-1}\}, i_3) \leq p_2$ so that $(\Psi(D_1 \cup \{t_0, \ldots, t_{n-1}\}), p_3)$ is a good pair. But then $p_3$ is an extension of $p_0$ belonging to $C$, a contradiction. \hfill \Box

**Corollary 3.2.19.** Fix $n \geq 3$. Every Hechler $n$-generic computes a Mathias $n$-generic, and every Mathias $n$-generic computes a Hechler $n$-generic.

**Proof.** In Lemmas 3.2.12 and 3.2.13 we showed that $P_{\text{Mathias}}$ and $P_{\text{Hechler}}$ produce nice $\Pi^0_2$ approximations, witnessed by $n = 3$. Corollary 3.2.19 then follows from the proof of Theorem 3.2.14. \hfill \Box
While we developed the framework of Cohen-Mathias-like forcing partial orders to streamline the process of proving Corollary 3.2.19, there are several remaining questions purely about the degree structure of the collection of all Cohen-Mathias-like forcing partial orders induced by the relation $\preceq$.

We have seen that $P_{\text{Hechler}}$ and $P_{\text{Mathias}}$ are at the top of the Cohen-Mathias-like forcing partial orders, $P_{\text{Cohen}}$ is at the bottom, and that $P_{\text{Hechler}}$ and $P_{\text{Mathias}}$ are strictly stronger than $P_{\text{Cohen}}$. It is natural to ask whether there is a Cohen-Mathias-like forcing partial order that lies strictly between the top and the bottom elements. In a similar vein, we wonder whether there are two Cohen-Mathias-like forcing partial orders that are incomparable with respect to $\preceq$.

In a different direction, note that if $P$ is a Cohen-Mathias-like forcing partial order that produces nice $\Pi^0_2$ approximations, then, for every $G \subseteq \omega$, if $G$ is sufficiently generic with respect to $P$, $G$ is high. We do not know whether, given a Cohen-Mathias-like forcing partial order $P$ for which all sufficiently generic sets $G$ are high, $P$ must be on top of the degree structure induced by $\preceq$.

### 3.3 Mathias generics over countable Turing ideals

Cholak, Dzhafarov, and Soskova [4] introduced a notion of Mathias forcing in which the forcing conditions belong to a countable Turing ideal $\mathcal{I}$. In particular, they asked whether, for any nested pair $\mathcal{I} \subseteq \mathcal{J}$ of countable Turing ideals, the Mathias generics from $\mathcal{J}$ can all compute Mathias generics from $\mathcal{I}$. Toward a positive answer, they showed that this is so when $\mathcal{I} = \text{COMP}$, the ideal consisting of the computable sets. In this section, we give a negative answer to their general question.

We briefly recall the type of Mathias forcing defined in [4]. Given a countable Turing ideal $\mathcal{I}$, a Mathias $\mathcal{I}$-condition is a Mathias condition $(F, S)$, with reservoir set $S \in \mathcal{I}$. The interpretation of an $\mathcal{I}$-condition and the partial ordering on $\mathcal{I}$-conditions is exactly as described in Section 3.1. An exact pair for a countable Turing ideal $\mathcal{I}$ is
a pair \((A_0, A_1)\) of sets such that \(\mathcal{I} = \{X \subseteq \omega : X \leq_T A_0 \land X \leq_T A_1\}\). Every countable Turing ideal has an exact pair. In [4], exact pairs are used to assign codes to the Mathias \(\mathcal{I}\)-conditions, for a given countable ideal \(\mathcal{I}\).

**Definition 3.3.1.** Let \(\mathcal{I}\) be a countable Turing ideal, and fix an exact pair \((A_0, A_1)\) for \(\mathcal{I}\).

1. If \((F, S)\) is a Mathias \(\mathcal{I}\)-condition, a *code* for \((F, S)\) is a triple \((F, e_0, e_1)\), where \(\varphi_{e_i}^{A_i} = \chi_S\), for \(i = 0, 1\).

2. A set \(C\) of Mathias \(\mathcal{I}\)-conditions is \(\Sigma^0_n(\mathcal{I})\) if there is a \(\Sigma^0_n(A_0 \oplus A_1)\) set of codes for \(\mathcal{I}\)-conditions representing exactly the conditions in \(C\).

3. A set \(G \subseteq \omega\) is \(n\)-\(\mathcal{I}\)-generic, for \(n \in \omega\), if \(G\) meets or avoids every \(\Sigma^0_n(\mathcal{I})\) set of \(\mathcal{I}\)-conditions.

Given an ideal \(\mathcal{I}\) with exact pair \((A_0, A_1)\), the set of codes for all Mathias \(\mathcal{I}\)-conditions is \(\Sigma^0_3(A_0 \oplus A_1)\). For this reason, it is customary only to consider Mathias \(n\)-\(\mathcal{I}\)-generics for \(n \geq 3\). We also note that a countable Turing ideal \(\mathcal{I}\) has many different exact pairs. Thus, the definition of Mathias \(n\)-\(\mathcal{I}\)-genericity depends on the exact pair chosen to code the ideal \(\mathcal{I}\). Fortunately, our results do not depend on these choices, and so we move all considerations about exact pairs to the sidelines from now on.

Cholak, Dzhafarov, and Soskova [4] showed that, for any countable ideal \(\mathcal{J}\) and \(n \geq 3\), every Mathias \(n\)-\(\mathcal{J}\)-generic computes a Mathias \(n\)-\(COMP\)-generic, where \(COMP\) is the Turing ideal containing only the computable sets. Furthermore, they asked the following.

**Question 3.3.2.** Suppose \(\mathcal{I}\) and \(\mathcal{J}\) are countable Turing ideals, and \(\mathcal{I} \subseteq \mathcal{J}\). Is there some Mathias \(n \in \omega\) such that every Mathias \(n\)-\(\mathcal{J}\)-generic computes a Mathias \(3\)-\(\mathcal{I}\)-generic?

We give a negative answer to Question 3.3.2. In particular, we prove the following.
Theorem 3.3.3. Suppose $\mathcal{J}$ is a countable Turing ideal containing a $\Delta^0_n$ set $X \subseteq \omega$ of PA degree, for some $n \in \omega$. For any set $G \subseteq \omega$, there is a sub-ideal $\mathcal{I} \subseteq \mathcal{J}$ such that $G$ computes no Mathias max$(n, 3)$-$\mathcal{I}$-generic.

Before proving Theorem 3.3.3, let us see that it provides an answer to Question 3.3.2. Fix a countable Turing ideal $\mathcal{J}$ that contains a $\Delta^0_2$ set $X$ of PA degree (indeed, we could take $X$ to be a low PA degree). Let $G \subseteq \omega$ be Mathias $m$-$\mathcal{J}$-generic, for every $m \in \omega$. Theorem 3.3.3 implies that there is a sub-ideal $\mathcal{I} \subseteq \mathcal{J}$ such that $G$ computes no Mathias $3$-$\mathcal{I}$-generic. The pair $\mathcal{I} \subseteq \mathcal{J}$ of ideals gives a negative answer to Question 3.3.2.

In order to prove Theorem 3.3.3, we need some intermediary results. Given $\mathcal{J}, X,$ and $G$ as in Theorem 3.3.3, we shall see that there are uncountably many sub-ideals of $\mathcal{J}$, with pairwise disjoint collections of Mathias max$(n, 3)$-generics. The given set $G$ can only compute countably many sets, and so cannot possibly compute a Mathias max$(n, 3)$-generic, for every sub-ideal $\mathcal{I} \subseteq \mathcal{J}$.

The first intermediary result is due to Turetsky [22]. Recall, from Section 1.3 that, given $X, Y \subseteq \omega$, we say $X$ has PA degree relative to $Y$, or $Y \ll X$, if $X$ computes a path through every infinite binary $Y$-computable tree. Scott [18] showed that, for any $X, Y \subseteq \omega, Y \ll X$ iff $X$ computes an enumeration of a Scott set containing $Y$.

Theorem 3.3.4 (Turetsky). Given a Scott set $\mathcal{I}$, there is a collection $(\mathcal{I}_t)_{t \in \mathbb{R}}$ such that

1. For all $t \in \mathbb{R}$, $\mathcal{I}_t$ is a Scott set contained in $\mathcal{I}$;
2. For all $s < t \in \mathbb{R}$, $\mathcal{I}_s$ is a proper subset of $\mathcal{I}_t$;
3. For all $s < t \in \mathbb{R}$, there is $X \in \mathcal{I}_t$ such that $X$ has PA degree relative to $Y$, for every $Y \in \mathcal{I}_s$.

Proof. First, we construct a sequence $(X_q)_{q \in \mathbb{Q}}$ of elements of $\mathcal{I}$ such that for every $p < q \in \mathbb{Q}, X_p \ll X_q$. Fix an enumeration $(q_n)_{n \in \omega}$ of $\mathbb{Q}$. Let $X_{q_0}$ be any set of PA
degree in $\mathcal{I}$. Suppose, inductively, we have chosen sets $X_{q_0}, \ldots, X_{q_n}$ in $\mathcal{I}$, for some $n \geq 0$, such that for all $i < j \leq n$, if $q_i < q_j$, then $X_{q_i} \ll X_{q_j}$. Consider $q_{n+1}$. If $q_{n+1} < q_i$, for every $i \leq n$, let $q = \min \{q_i : i < n\}$. The set $X_q$ has PA degree, and so it computes an enumeration of a Scott set, all of whose elements belong to $\mathcal{I}$. Let $X_{q_{n+1}}$ be any element of this Scott set that has PA degree. Now, suppose $q_{n+1} > q_i$, for every $i \leq n$. Let $Y = \bigoplus_{i \in \omega} X_i$. Since $\mathcal{I}$ is a Scott set, there is some $Z \in \mathcal{I}$ such that $Y \ll Z$. Let $X_{q_{n+1}}$ be any such set $Z$. Finally, suppose that $q_{n+1}$ is directly between $q_i < q_j$, for some $i, j \leq n$. Since $X_{q_i} \ll X_{q_j}$, $X_{q_j}$ computes an enumeration of a Scott set $\mathcal{S}$ containing $X_{q_i}$. Let $X_{q_{n+1}}$ be any set $Y \in \mathcal{S}$ such that $X_{q_i} \ll Y$. Since $X_{q_j}$ computes an enumeration of $\mathcal{S}$, we also have that $Y \ll X_{q_j}$.

We define the collection $(I_t)_{t \in \mathbb{R}}$ from the sequence $(X_q)_{q \in \mathbb{Q}}$, as follows. Given $t \in \mathbb{R}$, let $I_t = \{Y \subseteq \omega : (\exists q \in \mathbb{Q})(q < t \land Y \leq_T X_q)\}$. It is easy to check that $(I_t)_{t \in \mathbb{R}}$ satisfies Parts (1) and (2) of Theorem 3.3.4. For Part (3), given $s < t \in \mathbb{R}$, find $p \in \mathbb{Q}$ with $s < p < t$. We claim that $X_p \in I_t$ has PA degree relative to every set in $I_s$. Indeed, given $Y \in I_s$, there is $q \in \mathbb{Q}$ with $q < s$ and $Y \leq_T X_q$. Then $Y \leq_T X_q \ll X_p$, so $Y \ll X_p$.

Next, we present a condition on nested pairs of countable ideals $\mathcal{I} \subseteq \mathcal{I}'$ that guarantees that the Mathias generics for $\mathcal{I}$ are distinct from those for $\mathcal{I}'$.

**Lemma 3.3.5.** Suppose $\mathcal{I} \subseteq \mathcal{I}'$ are countable Turing ideals, and $X \in \mathcal{I}'$ is $\Delta^0_n$, for some $n \in \omega$. If neither $X$ nor $X$ has any infinite subset in $\mathcal{I}$, then the collection of Mathias $\max(n,3)\mathcal{I}$-generics is disjoint from the collection of Mathias $\max(n,3)\mathcal{I}'$-generics.

**Proof.** First, note that for any Mathias $\max(n,3)\mathcal{I}'$-generic $G$, either $G \Vdash X$, or $G \Vdash \bar{X}$. To see this, let $\mathcal{C}$ be the set of all Mathias $\mathcal{I}'$-conditions $(F,S)$ such that $S \subseteq X$ or $S \subseteq \bar{X}$. Since $X \in \mathcal{I}'$, and the set of all $\mathcal{I}'$-conditions is $\Sigma^0_3(\mathcal{I}')$, the set $\mathcal{C}$ is also $\Sigma^0_3(\mathcal{I}')$. Furthermore, $\mathcal{C}$ is dense among the $\mathcal{I}'$-conditions, for, given any

74
$\mathcal{I}'$-condition $(F,S)$, both of $S \cap X, S \cap X$ are in $\mathcal{I}'$, and at least one is infinite. Thus, if $G$ is Mathias $\text{max}(n,3)\mathcal{I}'$-generic, then $G$ meets $C$, and so either $G \not\subseteq X$ or $G \not\subseteq X$.

Next, we claim that for any Mathias $\text{max}(n,3)\mathcal{I}$-generic $G$, both $G \cap X$ and $G \cap X$ are infinite. Given $k \in \omega$, let $D_k$ be the set of all Mathias $\mathcal{I}$-conditions $(D,S)$ such that $D$ contains at least $k$ elements from $X$, and at least $k$ elements from $X$. Although $X \not\in \mathcal{I}$, $X$ is $\Delta^0_n$, so, for every $k \in \omega$, $D_k$ is $\Sigma^0_{\text{max}(n,3)}(\mathcal{I})$. For every $k \in \omega$, the set $D_k$ is dense in the Mathias $\mathcal{I}$-conditions. Indeed, $D_0$ is the set of all Mathias $\mathcal{I}$-conditions. Assume, inductively, that $D_k$ is dense, for some $k \geq 0$. Fix any Mathias $\mathcal{I}$-condition $p_0 = (D_0,S_0)$. Let $(D_1,S_1)$ be an extension of $p_0$ in $D_k$. By assumption, $S_1 \notin X$ and $S_1 \notin X$. Hence, we can extend $p_1$ to a Mathias $\mathcal{I}$-condition $p_2 = (D_2,S_2)$ belonging to $D_k + 1$. If $G$ is Mathias $\text{max}(n,3)\mathcal{I}$-generic, then $G$ meets $D_k$, for every $k \in \omega$. This implies that both $G \cap X$ and $G \cap X$ are infinite. ⌦

To generate examples of ideals satisfying the hypotheses of Lemma 3.3.5, we use some facts about Martin-Löf randomness (see, for instance, [6], or [15]). We present the relevant definitions here. Let $\lambda$ be the Lebesgue measure on $2^\omega$, where the measure of the basic clopen set $\{ X \in 2^\omega : \sigma \prec X \}$ is $2^{-|\sigma|}$, for every $\sigma \in 2^\omega$. A sequence $(U_n)_{n \in \omega}$ of open sets in $2^\omega$ is uniformly c.e. if there is a total computable function $f$ such that for every $n \in \omega$, $U_n = \{ X \in 2^\omega : (\exists \sigma \in W_{f(n)})(\sigma \prec X) \}$, where the elements of the c.e. sets $W_e$ are viewed as binary strings. A Martin-Löf test is a uniformly c.e. sequence $(U_n)_{n \in \omega}$ such that for all $n$, $\lambda(U_n) \leq 2^{-n}$. Finally, a set $X \subseteq \omega$ is Martin-Löf random, written $X \in \text{MLR}$, if, for every Martin-Löf test $(U_n)_{n \in \omega}$, $X \not\in \cap_n U_n$. One well-known fact about Martin-Löf randomness is that there is a universal Martin-Löf test, that is, a Martin-Löf test $(U_n)_{n \in \omega}$ such that $\cap_{n \in \omega} U_n = 2^\omega - \text{MLR}$.

In the following, we relativize the notion of Martin-Löf randomness to an oracle $Y \subseteq \omega$.

**Lemma 3.3.6.** Suppose $Y$ is infinite, and $X \subseteq \omega$ is Martin-Löf random relative to $Y$. Then $Y \not\subseteq X$ and $Y \not\subseteq X$. 

75
Proof. Suppose \( Y = \{ y_0 < y_1 < \ldots \} \). We define a uniformly \( Y \)-c.e. sequence \((U_n)_{n \in \omega}\), as follows. Let \( U_0 = \{ X \in 2^\omega : X(y_0) = 1 \} \). Given \( U_n \), for some \( n \in \omega \), let \( U_{n+1} = \{ X \in U_n : X(y_{n+1}) = 1 \} \). It is easy to see that \((U_n)_{n \in \omega}\) is a Martin-Löf test relative to \( Y \), and that \( \bigcap_n U_n = \{ X \in 2^\omega : Y \subseteq X \} \). If \( X \) is Martin-Löf random relative to \( Y \), then \( X \in \overline{U_n} \), for some \( n \in \omega \), implying that \( Y \not\subseteq X \). To see that \( Y \not\subseteq X \), note that, if \( Y \) is Martin-Löf random relative to \( X \), then \( Y \) is also Martin-Löf random relative to \( X \). We now have what we need to prove Theorem 3.3.3.

We now have what we need to prove Theorem 3.3.3. Suppose \( J \) is a countable Turing ideal, and \( X \in J \) is a \( \Delta^0_n \) set of PA degree. As we have already noted, it is enough to show that there are uncountably many Turing ideals \( \mathcal{I} \subseteq J \) whose collections of Mathias \( \text{max}(n,3) \)-generics are pairwise disjoint. Since \( X \in J \) has PA degree, \( X \) computes an enumeration of a Scott set \( \mathcal{I} \subseteq J \), where every set in \( \mathcal{I} \) is \( \Delta^0_n \). Apply Theorem 3.3.4 to \( \mathcal{I} \). We claim that for any two distinct elements \( \mathcal{I}_s \subseteq \mathcal{I}_t \) of the resulting collection \( (\mathcal{I}_t)_{t \in \mathbb{Q}} \) of sub-Scott sets of \( \mathcal{I} \), the collection of Mathias \( \text{max}(n,3) \)-\( \mathcal{I}_s \)-generics is disjoint from the collection of Mathias \( \text{max}(n,3) \)-\( \mathcal{I}_t \)-generics. Fix \( p < q \in \mathbb{Q} \), with \( s < p < q < t \), and let \( X_p \ll X_q \) be as in the proof of Theorem 3.3.4. Let \((U_n)_{n \in \omega}\) be a universal Martin-Löf test relative to \( X_p \). Then \( C = \overline{U_1} \subseteq 2^\omega \) is a nonempty closed set containing only sets that are Martin-Löf random relative to \( X_p \). The set \( C \) is the set of paths through some infinite binary \( X_p \)-computable tree, and hence \( X_q \) computes an element \( X \in C \). This set \( X \in \mathcal{I}_s \) is random relative to every set \( Y \in \mathcal{I}_s \). By Lemma 3.3.6, neither \( X \) nor \( X \) has an infinite subset in \( \mathcal{I}_s \). Finally, by Lemma 3.3.5, the set of Mathias \( \text{max}(n,3) \)-\( \mathcal{I}_s \)-generics is disjoint from the set of Mathias \( \text{max}(n,3) \)-\( \mathcal{I}_t \)-generics. This completes the proof of Theorem 3.3.3.


