ON THE COORDINATE RING OF A PROJECTION OF A DEGREE TWO VERONESE VARIETY

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by
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Abstract

by

Whitney Liske

The goal of this thesis is to study Rees algebra $\mathcal{R}(I)$ and the special fiber ring $\mathcal{F}(I)$ for a family of ideals. Given a map between projective spaces parameterizing a variety, the Rees algebra is the coordinate ring of the graph and the special fiber ring is the coordinate ring of the image. We will compute the defining ideal of these algebras. Let $R = k[x_1, \ldots, x_d]$ for $d \geq 4$ be a polynomial ring with homogeneous maximal ideal $m$. We study the $R$-ideals $I$ which are $m$-primary, Gorenstein, generated in degree 2, and have a Gorenstein linear resolution. The defining ideal of the Rees algebra will be of fiber type. That is, the defining ideal of the Rees algebra is generated by the defining ideals of the special fiber ring and of the symmetric algebra. The defining ideal of the symmetric algebra is well understood, so we concentrate on computing the defining ideal of the special fiber ring. In Chapter 4, the defining ideal of the special fiber ring $\mathcal{F}(I)$ will be given as a sub-ideal of the $2 \times 2$ minors of a symmetric matrix of variables modeled after the defining ideal of $\mathcal{F}(m^2)$. In Chapter 5, the defining ideal of the special fiber ring of $I$ will be described as a saturation of the maximal minors of the Jacobian dual. We include both descriptions of the defining ideal in this manuscript because while the methods in Chapter 4 give explicit polynomial generators of the defining ideal, the methods in Chapter 5 are more likely to generalize to the $m$-primary Gorenstein ideals having a Gorenstein linear resolution.
<table>
<thead>
<tr>
<th>CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figures .................................................................................................. iii</td>
</tr>
<tr>
<td>Tables ...................................................................................................... iv</td>
</tr>
<tr>
<td>Symbols ..................................................................................................... v</td>
</tr>
<tr>
<td>Acknowledgments ...................................................................................... vii</td>
</tr>
<tr>
<td>Chapter 1: Introduction .......................................................................... 1</td>
</tr>
<tr>
<td>1.1 Main Results ....................................................................................... 4</td>
</tr>
<tr>
<td>Chapter 2: Preliminaries .......................................................................... 7</td>
</tr>
<tr>
<td>2.1 The Rees Algebra and Ideals with a Gorenstein Linear Resolution ........ 7</td>
</tr>
<tr>
<td>2.2 Gröbner Bases and the Hilbert function ........................................... 11</td>
</tr>
<tr>
<td>2.3 Some Linear Algebra .......................................................................... 16</td>
</tr>
<tr>
<td>Chapter 3: The Family ............................................................................ 18</td>
</tr>
<tr>
<td>3.1 Excluded Dimensions ......................................................................... 20</td>
</tr>
<tr>
<td>Chapter 4: Symmetric Matrix of Variables ............................................. 23</td>
</tr>
<tr>
<td>4.1 A Candidate for the Defining Ideal ................................................... 23</td>
</tr>
<tr>
<td>4.2 Degree of Generators of $I(X)$ .......................................................... 32</td>
</tr>
<tr>
<td>4.3 Hilbert Functions in Degree 2 ............................................................. 35</td>
</tr>
<tr>
<td>4.4 Hilbert Functions in Degree 3 ............................................................. 43</td>
</tr>
<tr>
<td>4.4.1 The Hilbert Function of $I(X)$ ......................................................... 43</td>
</tr>
<tr>
<td>4.4.2 The Hilbert Function of $\Lambda$, Multiples ....................................... 44</td>
</tr>
<tr>
<td>4.4.3 The Hilbert Function of $\Lambda$, Gröbner Basis ................................ 62</td>
</tr>
<tr>
<td>4.5 Main Theorem 1 .................................................................................. 90</td>
</tr>
<tr>
<td>Chapter 5: Jacobian Dual .......................................................................... 91</td>
</tr>
<tr>
<td>Chapter 6: Open Questions ....................................................................... 98</td>
</tr>
<tr>
<td>Bibliography ............................................................................................. 101</td>
</tr>
</tbody>
</table>
FIGURES

4.1  $S_{1,j}$ Visualization .......................................................... 49
4.2  $S_{i,j}$ Visualization ........................................................... 52
4.3  $L_{1,j}$ Visualization ............................................................ 55
4.4  $L_{i,j}$ Visualization ............................................................ 58
TABLES

4.1 Functions on $A_1$ .............................................................. 28
4.2 Generators of $\Lambda_0$ in $M_{i,j,k,l}$ ............................................. 38
4.3 Generators of $\Lambda_0$ in $M_{i,j,k,l}$ ............................................. 40
4.4 Counting Generators ............................................................ 41
4.5 Structure of Generators .......................................................... 42
4.6 Subsets of $K$ when $d = 4$ .................................................... 46
SYMBOLS

\( k \) a field
\( R \) a polynomial ring in \( d \) variables over a field \( k \)
\( m \) maximal homogenous ideal of \( R \)
\( I \) \( m \)-primary Gorenstein ideal with a Gorenstein linear resolution
\( \mu(I) \) number of minimal generators of \( I \)
\( \delta \) degree of generators of \( I \)
\( \Psi \) projective map parameterizing a variety
\( W \) a polynomial ring in \( n = \left(\frac{d+1}{2}\right) - 1 \) variables over the field \( k \)
\( n \) maximal homogenous ideal of \( W \)
\( S \) a polynomial ring in \( \mu(I) \) variables over the ring \( R \)
\( U \) a polynomial ring in \( \mu(m^2) \) variables over the field \( k \)
\( \phi \) presentation matrix of \( I \)
\( B = B(\phi) \) Jacobian dual of \( \phi \)
\( S(I) \) symmetric algebra of \( I \)
\( \mathcal{L} \) defining ideal of the symmetric algebra of \( I \)
\( \mathcal{R}(I) \) Rees algebra of \( I \)
\( \mathcal{J} \) defining ideal of the Rees algebra of \( I \)
\( \mathcal{F}(I) \) special fiber ring of \( I \)
\( I(X) \) defining ideal of the special fiber ring of \( I \)
\( I(Y) \) defining ideal of the special fiber ring of \( m^2 \)
\( I_i(M) \) ideal of \( i \times i \) minors of the matrix \( M \)
\( \mathcal{M} \) \( d \times d \) symmetric matrix of variables \( w_{i,j} \) with 0 in row \( d \), column \( d \)
$\mathcal{M}_{i,j,k,l}$ 4×4 principal submatrix of $\mathcal{M}$

$\mathcal{N}$ $d \times d$ symmetric matrix of variables $u_{i,j}$

$m^M$ 2×2 submatrix of $\mathcal{M}$

$m^M$ 2×2 minor of $\mathcal{M}$

$A_i$ set of 2×2 minors of $\mathcal{M}$ with $i$ entries in the diagonal of $\mathcal{M}$

$\omega$ monomial ordering on $W$

$S(f,g)$ S-polynomial of $f$ and $g$

$\text{lm}(f)$ leading monomial of $f$

$HF_A(i)$ Hilbert function of the ring/ideal $A$ in degree $i$

$\text{in}J$ initial ideal of the ideal $J$

$\Lambda$ candidate for defining ideal of $\mathcal{F}(I)$

$L$ degree 2 graded component of $\text{in}\Lambda$

$H$ degree 3 graded component of $\text{in}\Lambda$

$K$ degree 3 multiples of $L$

$G$ $H \setminus K$

$S_{i,j}$ set $\{w_{k,l} \mid w_{k,l}w_{i,j} \in L \text{ with } w_{k,l} \leq_\omega w_{i,j}\}$

$s_{i,j}$ integer $|S_{i,j}|$

$K^{(k,l)}_{i,j}$ subset of $K$

$k^{(k,l)}_{i,j}$ integer $|K^{(k,l)}_{i,j}|$

$\tau$ an ordering of the sets $K^{(k,l)}_{i,j}$

$L_{i,j}$ maximal $K^{(k,l)}_{i,j}$ with respect to $\tau$ for fixed $i,j$

$l_{i,j}$ integer $|L_{i,j}|$

$T$ integer $\sum k^{(k,l)}_{i,j}$

Fitt$_jM$ $j^{th}$ Fitting ideal of a module $M$
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CHAPTER 1

INTRODUCTION

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$ with homogeneous maximal ideal $m$. Let $I$ be a homogeneous $R$-ideal generated in degree $\delta$. In this manuscript we study the Rees algebra

$$\mathcal{R}(I) = R[It] = \bigoplus_{j \geq 0} I^j$$

and the special fiber ring

$$\mathcal{F}(I) = \mathcal{R}(I) \otimes k = \bigoplus_{j \geq 0} I^j / mI^j.$$

The Rees algebra is an important object in commutative algebra, algebraic geometry, elimination theory, intersection theory, and geometric modeling, to name a few. It holds asymptotic information about the powers of an ideal. The Rees algebra provides a natural way to look at the multiplicity of a ring. In algebraic geometry, it provides the algebraic description of the blowup of the scheme $\text{Spec}(R)$ along the closed subscheme $V(I)$. In addition, the special fiber ring gives the coordinate ring of algebraic varieties, such as Segre, Veronese, Grassmanian, Pfaffian, and determinantal varieties. These classical algebraic varieties are of fundamental importance as they are ubiquitous and play an important role in almost all fields of mathematics as well as in theoretical computer science, complexity theory, signal processing, phylogenetics, and algebraic statistics.

One of the main problems in this area is to describe $\mathcal{R}(I)$ and $\mathcal{F}(I)$ in terms of generators and relations (\cite{7}, \cite{32}, \cite{24}, \cite{19}, \cite{29}, \cite{31}, \cite{30}, \cite{21}). That is, for an ideal $I = (g_1, \ldots, g_n)$ minimally generated by forms $g_i$ of the same degree, we want to study the kernels $\mathcal{J}$ and $I(X)$ of the maps $S = R[w_1, \ldots, w_n] \to \mathcal{R}(I), \ w_i \to g_i t$ and $W = k[w_1, \ldots, w_n] \to$
\( F(I), \ w_i \rightarrow g_i \). Then \( R(I) = S/J \) and \( F(I) = W/I(X) \). We call \( J \) and \( I(X) \) the defining ideals of \( R(I) \) and \( F(I) \). The generators of \( J \) and \( I(X) \) are referred to as defining equations. Since the generators of \( I \) are of the same degree, they define a rational map between projective spaces. The Rees algebra and special fiber ring are the coordinate rings of the graph and image of this map, respectively. Additionally, the defining equations are the implicit equations of the variety defined by the graph (or image) of the map. Finding the implicit equations of a variety is also of interest to applied mathematicians, where it is known as the implicitization problem ([4], [5], [6], [15], [9]). In particular, for \( d = 2 \) or \( d = 3 \) the implicitization problem has significant applications to computer aided modeling.

Finding the defining ideal of the Rees algebra is a challenging problem and is still open for many classes of ideals. For instance, we are still not able to understand the defining ideal of the Rees algebra for three generated ideals in a polynomial ring in two variables i.e. curves in \( \mathbb{P}^2 \) ([3], [1], [25]), height two perfect ideals ([16], [28], [27]), and height three Gorenstein ideals ([22]) without imposing additional conditions.

Another algebra closely related to the Rees algebra is the symmetric algebra \( S(I) \). Once again, we present the symmetric algebra using the map \( S = R[w_1, \ldots, w_n] \rightarrow S(I), \ w_i \rightarrow g_i \) and study its kernel \( L \). We call \( L \) the defining ideal of the symmetric algebra and can write \( S(I) = S/L \). Unlike the Rees algebra, the defining equations of the symmetric algebra are well understood and can be obtained from a minimal presentation matrix of \( I \). In particular if \( w = [w_1, \ldots, w_n] \) and \( \phi \) is a minimal presentation matrix of \( I \), then \( L = I_1(w \cdot \phi) \). Notice in particular that these generators are homogeneous polynomials of degree one in the \( w \)'s. It turns out that we always have the inclusion \( L \subseteq J \). When we have equality, that is when \( J \) is generated only by homogeneous degree one polynomials in the \( w \)'s, the Rees algebra is said to be of linear type. Several nice classes of ideals are of linear type including complete intersections.

For ideals which are not linear type it is often mysterious how to obtain higher degree generators of \( J \). One way to obtain these generators is to focus on ideals which are linearly
presented. This allows one to find a unique matrix $B$ with linear entries in $W$ such that $\nu \cdot \varphi = x \cdot B$. This matrix is known as the Jacobian dual matrix of $\varphi$. One can see from Cramer’s rule that the $d \times d$ minors of $B$ will be contained in the defining ideal $J$. In some cases, we are able to obtain all generators of $J$ from the defining equations of the symmetric algebra and the maximal minors of the Jacobian dual. If that is the case, $J = (\mathcal{L}, I_d(B))$ and we say that the Rees algebra is of expected form. One class of ideals which are of expected form are linearly presented height two perfect ideals satisfying the condition $G_d$. The condition $G_d$ is a mild assumption on the local number of generators of $I$ which appears frequently throughout the study of Rees algebras and in the study of linkage. We say $I$ satisfies $G_d$ provided for every $p \in V(I)$ with $\dim R_p < d$, we have $\mu(I_p) \leq \dim R_p$. This property can be easily checked by computing the height of Fitting ideals of $I$. The property $G_d$ is always satisfied if $I$ is $m$ primary or if $I$ is a generic perfect ideal of height two or a generic Gorenstein ideal of height three.

Another way to obtain higher degree generators of $J$ is through the special fiber ring. We do this by using the bigraded structure of the Rees algebra. Indeed we can assign the variables $x_i$ the degree $(1,0)$ and the variables $w_i$ the degree $(0,1)$. In this setting, $\mathcal{F}(I)$ is simply the degree $(0, \ast)$ component of $\mathcal{R}(I)$, and in particular its defining ideal $I(X)$ is contained in $J$. When $J = \mathcal{L} + I(X)S$, that is when the defining ideal of the Rees algebra is generated by the defining ideals of the symmetric algebra and special fiber ring, the Rees algebra is said to be of fiber type. Since the special fiber ring gives the implicit equations of a variety embedded in projective space, it is of interest in its own right. For the ideals considered in this thesis, the Rees algebra will be of fiber type and the special fiber ring will provide all higher degree generators of $J$.

One result of particular importance to this thesis is by Kustin, Polini, and Ulrich where they compute the defining ideal of the Rees algebra for linearly presented height three Gorenstein ideals satisfying $G_d$. They show in particular that the Rees algebra is fiber type. Further, if the dimension of $R$ is odd, then $\mathcal{R}(I)$ is of expected form. We will attempt
to generalize this result in the following sense. Notice that since $I$ is linearly presented and the resolution for $I$ is short, the minimal free resolution for $I$ has linear maps everywhere except for the first and last ones. Gorenstein ideals for which the only non-linear maps in the minimal free resolution are the first and the last are said to have a *Gorenstein linear* resolution. By a paper of Eisenbud, Huneke, and Ulrich ([12]) all $m$-primary ideals with a Gorenstein linear resolution are fiber type. Hence, determining the defining ideal of the Rees algebra is reduced to the problem of finding the defining ideal of the special fiber ring. We wish to compute the defining ideal of the Rees algebra for $m$-primary Gorenstein ideals with a Gorenstein linear resolution.

To understand the defining ideal of the Rees algebra for such ideals, a natural place to start is to study those generated in degree $\delta = 2$. By results of Miller and Villarreal [26] and Hong, Simis, and Vasconcelos [18] we obtain a description of the generators of $I$. In particular, these ideals have exactly one less generator than $m^2$ and we know an explicit generating set up to change of variables. We utilize this structure to compute the defining ideals of the Rees algebra and special fiber ring for this family of ideals. In particular, the fact that these ideals have exactly one less generator than $m^2$ means that they are the ideal associated to the projection of a degree two Veronese variety. Further, the fact that they are Gorenstein means that we project from a general point. The main results of this thesis are summarized below.

1.1 Main Results

In Chapter 4 we use the close relationship between $I$ and $m^2$ to construct the defining ideal of $\mathcal{F}(I)$ in a similar way to the defining ideal of $\mathcal{F}(m^2)$. The defining ideal of $\mathcal{F}(m^2)$ is well known to be the ideal of $2 \times 2$ minors of a symmetric matrix of variables. In Section 4.1 we construct an ideal $\Lambda$ whose generators are linear combinations of $2 \times 2$ minors of a symmetric matrix of variables with a zero. The first main theorem, found in Section 4.5 is that $\Lambda$ is indeed the defining ideal of the special fiber ring of $I$. 

4
Theorem 4.6. Adopt Data 4.5. The ideal $\Lambda$ is the defining ideal of $\mathcal{F}(I)$, that is

$$\Lambda = I(X).$$

Further

$$\mathcal{J} = \mathcal{L} + \Lambda S$$

is the defining ideal of $\mathcal{R}(I)$.

To prove this theorem, we first show in Section 4.1 that $\Lambda \subseteq I(X)$. For the opposite inclusion, we show in Section 4.2 that $I(X)$ is generated in degrees two and three. This will follow after applying Tor to the short exact sequence $0 \to \mathcal{F}(I) \to \mathcal{F}(m^2) \to k(-1) \to 0$ and computing generators of the defining ideal of $\mathcal{F}(m^2)$ as a $W$-module. We then compute the Hilbert function of $I(X)$ and $\Lambda$ in degrees two and three. The degree two calculation is in Section 4.3 and the degree 3 calculation is in Section 4.4. The Hilbert function of $I(X)$ is easy to compute in any degree since $I^k = m^{2k}$ for $k \geq 2$. Calculating the Hilbert function of $\Lambda$ is laborious, but mostly straightforward.

In Chapter 5, we relate the defining ideal of the special fiber ring to the Jacobian dual. In this section we first study the rational map defined by the generators of $I$, namely

$$\Psi: \mathbb{P}^{d-1}_k \dashrightarrow \mathbb{P}^{n-1}_k.$$  

Since $I$ is $m$-primary the map $\Psi$ is regular and defined everywhere [14]. The first theorem of this chapter states that $\Psi$ is biregular, that is the fiber consists of a single point. We also compute the multiplicity of the special fiber ring, which is the same as that of the Veronese variety.

The rest of the chapter focuses on giving an alternate description of the defining ideal of the special fiber ring of $I$. We show that the defining ideal of the Rees algebra is of the expected form up to radical. That is, $\mathcal{J} = \sqrt{\mathcal{L} + I_d(B)}$. This holds for any $m$-primary
Gorenstein ideal with a Gorenstein linear resolution. If $\delta = 2$ we are able to further deduce that $I(X)$ is a saturation of the ideal of maximal minors of the Jacobian dual with respect to the homogeneous maximal ideal of $W$. That is,

**Theorem 5.3** Adopt Data 5.3 The defining ideal of the special fiber ring is

$$I(X) = I_d(B) : n^\infty.$$  

In particular, the defining ideal of the Rees algebra is

$$\mathcal{J} = \mathcal{L} + (I_d(B) : n^\infty)S.$$  

We include both descriptions of the defining ideal $I(X)$ because they have different advantages. The advantage of the description $I(X) = \Lambda$ in Chapter 4 is that it gives an explicit description of the polynomials which generate $I(X)$. However, the proof relies heavily on the structure of the generators of $I$. The description $I(X) = I_d(B) : n^\infty$, on the other hand, has the potential to be generalized to the class of $m$-primary Gorenstein ideals with a Gorenstein linear resolution. Indeed, several of the steps of the proof do not require $I$ to be generated in degree 2.

In the final chapter we describe open questions related to this thesis, some of which are ongoing projects.
CHAPTER 2

PRELIMINARIES

In this chapter, we will provide the definitions and results from commutative algebra which are necessary to prove and understand the results of this thesis. For notation and theory which may not be explicitly stated good general reference are the books of Eisenbud [10] and Ene and Herzog [11]. We will assume that $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$. Let $I$ be a homogeneous $R$-ideal which is minimally generated by $n$ forms $g_1, \ldots, g_n$ of the same degree $\delta$. That is $I = (g_1, \ldots, g_n)$.

2.1 The Rees Algebra and Ideals with a Gorenstein Linear Resolution

**Definition 2.1.** Let $t$ be an indeterminate. The **Rees algebra** of the ideal $I$, denoted $\mathcal{R}(I)$, is the graded subalgebra of $R[t]$ generated by $It$. That is,

$$\mathcal{R}(I) = R[It] = \bigoplus_{j \geq 0} I^j t^j.$$ 

The Rees algebra provides an algebraic realization for the classical notion in algebraic geometry of blowing up $\text{Spec}(R)$ along the subscheme $V(I)$. Additionally, and most fundamental to this thesis, given a map $\psi : \mathbb{P}^{d-1} \to \mathbb{P}^{n-1}$ parameterizing a variety $X$ the Rees algebra gives the coordinate ring of the graph of $\psi$. In this setting, we have $\psi = [g_1 : \ldots : g_n]$ where the $g_i$ are forms of degree $\delta$ in $d$ variables and $I = (g_1, \ldots, g_n)$.

We will study the Rees algebra $\mathcal{R}(I)$ as the image of a larger polynomial ring $S =$
\( R[w_1, \ldots, w_n] \) via the short exact sequence

\[
0 \rightarrow \mathcal{J} \rightarrow S \xrightarrow{\xi} \mathcal{R}(I) \rightarrow 0
\]

where \( \xi(w_i) = g_i t \)

**Definition 2.2.** The ideal \( \mathcal{J} \) is called the **defining ideal** of the Rees algebra.

A closely related algebra is the symmetric algebra \( \mathcal{S}(I) \) of the ideal \( I \). We can also study the symmetric algebra as the image of the polynomial ring \( S \) via the short exact sequence

\[
0 \rightarrow \mathcal{L} \rightarrow S \xrightarrow{\zeta} \mathcal{S}(I) \rightarrow 0
\]

where \( \zeta(w_i) = g_i \).

**Definition 2.3.** The ideal \( \mathcal{L} \) is called the **defining ideal** of the symmetric algebra.

Unlike the Rees algebra, the defining ideal of the symmetric algebra is well understood. Consider a presentation of the ideal \( I \) with respect to the generators \( g_1, \ldots, g_n \) as below

\[
R^m \xrightarrow{\varphi} R^n \rightarrow I \rightarrow 0.
\]

The defining ideal of the symmetric algebra is \( \mathcal{L} = I_1(\underline{w} \cdot \varphi) \). That is, \( \mathcal{L} \) is generated by the entries of the vector \( \underline{w} \cdot \varphi \) where \( \underline{w} = [w_1, \ldots, w_n] \). Indeed

\[
\mathcal{S}(I) \cong S/\mathcal{L} \cong R[w_1, \ldots, w_n]/I_1(\underline{w} \cdot \varphi).
\]

Notice that the map \( \xi \) factors through the symmetric algebra. Hence, we have \( \mathcal{L} \subseteq \mathcal{J} \)
and we obtain the short exact sequence

\[ 0 \longrightarrow A \longrightarrow S(I) \longrightarrow \mathcal{R}(I) \longrightarrow 0 \]

where \( A \cong \mathcal{J}/\mathcal{L} \). Indeed, the ideal \( A \) measures how different the Rees algebra \( \mathcal{R}(I) \) and the symmetric algebra \( S(I) \) are. It turns out that \( A \) is the torsion submodule of the symmetric algebra with respect to \( m \). That is, \( \mathcal{J} = 0 : S(I) \ m^\infty \). Let \( A_t \) be the component of \( A \) in \( S_t(I) \).

In particular \( A = \bigoplus_{t \geq 2} A_t \).

**Definition 2.4.** The Rees algebra is said to be of linear type provided \( \mathcal{L} = \mathcal{J} \), that is when the symmetric algebra and Rees algebra are isomorphic.

One important class of ideals for which the Rees algebra is of linear type are complete intersection ideals. That is, ideals for which the minimal number of generators of \( I \) is equal to the height of \( I \).

Another algebra that will play an important role is the special fiber ring.

**Definition 2.5.** The special fiber ring of \( I \) is the graded ring

\[ \mathcal{F}(I) = \mathcal{R}(I) \otimes_R k \cong \bigoplus_{j \geq 0} I^j/m^j. \]

A polynomial presentation for the special fiber ring can be obtained by tensoring the presentation of the Rees algebra with \( k \) over \( R \). That is,

\[ S \otimes_R k \longrightarrow \mathcal{R}(I) \otimes_R k \longrightarrow 0 \]

\[ 0 \longrightarrow I(X) \longrightarrow W = k[w_1, \ldots, w_n] = S \otimes_R k \longrightarrow \mathcal{F}(I) = \mathcal{R}(I) \otimes_R k \longrightarrow 0 \]

**Definition 2.6.** The ideal \( I(X) \) is called the defining ideal of the special fiber ring.
In the case where $I$ is generated by homogeneous forms of the same degree, we have $\mathcal{F}(I) \cong k[g_1t, \ldots, g_n] \subseteq \mathcal{R}(I)$ and $I(X) \subseteq \mathcal{J}$. To see this, consider the bigrading on $\mathcal{R}(I)$ such that $\deg(x_i) = (1,0)$ and $\deg(w_i) = (0,1)$. Then notice $\mathcal{F}(I)$ is the degree $(0, \ast)$ component of $\mathcal{R}(I)$ and $I(X)$ is the degree $(0, \ast)$ component of $\mathcal{J}$. In special cases, the defining ideals of the special fiber ring and the symmetric algebra completely generate the defining ideal of the Rees algebra. If this is the case, the ideal is said to be of fiber type.

**Definition 2.7.** An ideal is of fiber type provided $\mathcal{J} = \mathcal{L} + I(X)S$.

For a linearly presented matrix $I$ with minimal presentation matrix $\varphi$, there is a unique $d \times m$ matrix $B(\varphi)$ with linear entries in $k[w_1, \ldots, w_n]$ satisfying

$$w \cdot \varphi = x \cdot B(\varphi)$$

where $w = [w_1 \cdots w_n]$ and $x = [x_1 \cdots x_n]$.

**Definition 2.8.** The matrix $B(\varphi)$ is called the Jacobian dual matrix of $\varphi$.

We will simply write $B$ when the presentation matrix is clear. The minors of the Jacobian dual are a typical source for higher degree generators of $\mathcal{J}$. Indeed, the maximal minors of $B$ satisfy the inclusion $I_d(B) \subseteq \mathcal{J}$.

**Definition 2.9.** The ideal $\mathcal{J}$ is of expected form provided $\mathcal{J} = (\mathcal{L}, I_d(B))$.

We will use the following theorem from [13] as our definition for an ideal having a Gorenstein linear resolution. For these ideals, the maps of the minimal free resolution will be linear except for the first and last ones. Ideals which are $m$-primary, Gorenstein and have a Gorenstein linear resolution will be of particular importance to this thesis. Indeed, we wish to find the defining ideal of the Rees algebra for such ideals.

**Theorem** (El Khoury, Kustin, (Proposition 2.7) [13]). Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$. Let $I$ be an $m$-primary homogeneous Gorenstein ideal and let $\delta$ be an integer.
(a) the minimal homogeneous resolution of $R/I$ by free $R$-modules is \textbf{Gorenstein linear}, and $I$ is generated by forms of degree $\delta$.

(b) the minimal homogeneous resolution of $R/I$ by free $R$-modules has the form

$$0 \to R(-2\delta - d + 2) \to R(-\delta - d + 2)^{\beta_2} \to \cdots \to R(-\delta - 1)^{\beta_1} \to R,$$

with

$$\beta_i = \frac{2\delta + d - 2}{\delta + i - 1} \binom{\delta + d - 2}{i - 1} \binom{\delta + d - i - 2}{\delta - 1},$$

for $1 \leq i \leq d - 1$.

(c) all of the minimal generators of $I$ have degree $\delta$ and the socle of $R/I$ has degree $2\delta - 2$.

In other texts, these ideals are sometimes referred to as extremal Gorenstein ideals. One can see the equivalence of (b) and (c) of the Theorem by studying the resolution of the socle $I:_R m$ via the mapping cone. Kustin and El Khoury list several other statements which are equivalent to those listed above, including some based on the Macaulay inverse system associated to $I$. In a series of papers, they compute an explicit minimal free resolution of these ideals. We hope that understanding the resolution of $I$ and the structures of its generators will aid in computing the defining ideal of the Rees algebra for these ideals. For now, at least, the resolutions of these ideals are linear for at least $[n/2]$ which allows us to take advantage of a highly nontrivial result, Theorem 9.1(c) in [12], by Eisenbud, Huneke, and Ulrich to show that all such ideals will be fiber type. This result gives rise to a large class of ideals which are fiber type.

**Corollary 2.1.** If $I$ is an $m$-primary Gorenstein ideal with a Gorenstein linear resolution and generated in degree $\delta$, $I$ is of fiber type and $A$ is annihilated by $m$.

2.2 Gröbner Bases and the Hilbert function

In Chapter 4 many of the proofs rely on the theory of Hilbert functions and Gröbner bases. We use these ideas, in particular, to help describe the graded components of the defining ideal of the special fiber ring. The Hilbert function (as well as the closely related Hilbert polynomial and Hilbert series) provide a way to study a graded algebras by
breaking them up into their graded components. They also allow one to calculate the multiplicity of a ring, an important invariant throughout commutative algebra. Let \( A = \oplus_{i \geq 0} A_i \) be a graded \( k \)-algebra. The Hilbert function of \( A \) in degree \( i \) is \( HF_A(i) = \dim_k A_i \). For a polynomial ring, the Hilbert function simply counts the number of monomials in any given degree. Indeed, if \( R = k[x_1, \ldots, x_d] \) is a polynomial ring in \( d \) variables over \( k \), the Hilbert function is

\[
HF_R(i) = \binom{d + i - 1}{d - 1} = \binom{d + i - 1}{i}.
\]

For monomial ideals, we can again count the dimension of its graded components easily. Further, if \( I \) is a graded monomial ideal, we very clearly have the relationship

\[
HF_{R/I}(i) = HF_R(i) - HF_I(i).
\]

We would also like to be able to compute \( HF_{R/I}(i) \) when \( I \) is not monomial. To do this, we need the notions of Gröbner bases and initial ideals. A Gröbner basis is a special kind of generating set for \( I \) and an initial ideal is a monomial ideal associated to \( I \) which will have the same Hilbert function.

Gröbner bases are fundamental in elimination theory and provide a way of studying the intersection of ideals. By using a Gröbner basis as opposed to another generating set, we are often able to obtain more information about the ideal and its associated variety. Gröbner bases always exist and can be effectively computed for any specific finitely generated ideal. However, it is often laborious to compute them for large classes of ideals and determine when the algorithm terminates. An important fact about Gröbner bases for an ideal \( I \) is that they are not unique. In particular they depend greatly on a monomial ordering on the variables of \( R \). There are, however, a finite number of minimal Gröbner bases for any ideal \( I \). One monomial ordering, that will be of fundamental importance to this thesis, is the reverse lexicographic order.

**Definition 2.10.** Let \( R = k[x_1, \ldots, x_d] \) be a polynomial ring over a field \( k \). Let \( <_{\text{rlex}} \) be the
ordering on the monomials of $R$ such that $x_1^{a_1}x_2^{a_2}\ldots x_d^{a_d} \prec_{\text{rlex}} x_1^{b_1}x_2^{b_2}\ldots x_d^{b_d}$ provided either

\begin{itemize}
  \item $\sum_{i=1}^{d} a_i < \sum_{i=1}^{d} b_i$
  \item $\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i$ and the right-most nonzero component of $(a_1 - b_1, \ldots, a_d - b_d)$ is positive.
\end{itemize}

**Example 2.1.** Let $R = \mathbb{C}[x_1, x_2]$. Write $\prec$ for $\prec_{\text{rlex}}$. Notice that if $m, n$ are monomials such that the degree of $m$ is less than the degree of $n$, we have $m \prec n$. Further, for degrees less than or equal to three we have the following ordering

\[
x_1 > x_2 > x_3
\]

\[
x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2
\]

\[
x_1^3 > x_1^2x_2 > x_1x_2^2 > x_2^3 > x_1^2x_3 > x_1x_2x_3 > x_2^2x_3 > x_1^2x_3 > x_2x_3^2 > x_3^3.
\]

After fixing a monomial ordering, one can associate a monomial ideal to an ideal $I$ which is known as the initial ideal. The initial ideal is of particular importance since the monomials in $S$ which are not in the initial ideal form a $k$ basis of $R/I$. One major advantage of passing to the initial ideal is that the dimension and other properties are often easier to compute in the monomial case. In particular, the Krull dimension and Hilbert functions of $I$ and its initial ideal will be equal. Additionally, initial ideals give information about nonzero divisors. It turns out that nonzero divisors on $R/\text{in}(I)$ will also be nonzero divisors on $R/I$.

**Definition 2.11.** Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$. Let $\prec$ be a monomial order on $R$, and $f \in R$. Let $\text{lm}(f)$ largest monomial of $f$ with respect to the monomial order $\prec$ and call it the leading monomial. Let $I$ a nonzero $R$-ideal. The initial ideal of $I$ is the monomial ideal

\[
\text{in}(I) = < \{\text{lm}(f) \mid f \in I, \ f \neq 0\} >
\]
We say a sequence $g_1,\ldots,g_n$ of elements in $I$ is a **Gröbner basis** of $I$ with respect to $<$ provided

$$\text{in}(I) = \langle \text{lm}(g_1), \text{lm}(g_2), \ldots, \text{lm}(g_n) \rangle.$$ 

Further, the elements of a Gröbner basis form a generating set for $I$.

It turns out that $\dim_k R/I = \dim_k R/\text{in}(I)$ and in particular $H_F(i) = H_{\text{in}(I)}(i)$. Indeed the initial ideal forms a $k$-basis for $R/I$, we see that

$$H_{F(R/I)}(i) = H_{F(R/\text{in}(I))}(i) = H_{F(R)}(i) - H_{F(\text{in}(I))}(i) = H_{F(R)}(i) - H_F(i).$$

It is important to notice that if you take a typical generating set for $I$ it is unlikely to be a Gröbner basis since initial ideal is generated by the leading monomials of all $f \in I$ not just the leading monomials of the generators. So in particular, one needs to know how to take a generating set of $I$ and obtain a Gröbner basis. We do this by computing what are known as S-polynomials (or S-pairs).

**Definition 2.12.** Let $R = k[x_1,\ldots,x_d]$ be a polynomial ring over a field $k$. Let $<$ be a monomial order on $R$ and $f, g \in R$. Let $a$ be the least common multiple of the monomials $\text{lm}(f)$ and $\text{lm}(g)$. Let $c_f$ be the coefficient of $\text{lm}(f)$ in $f$ and $c_g$ be the coefficient of $\text{lm}(g)$ in $g$. The polynomial

$$S(f,g) = \frac{a}{c_f\text{lm}(f)}f - \frac{a}{c_g\text{lm}(g)}g$$

is called the **S-Polynomial** of $f$ and $g$ with respect to $<$. Notice that if $f,g$ are elements of some $R$-ideal $I$, then so is $S(f,g)$.

Now, we hope to find a Gröbner basis which is finite. To this end, we do not want to include any polynomials which are not necessary. Once we compute the S-polynomial we will then reduce with respect to the other polynomials and include it only if it is nonzero.

**Definition 2.13.** Let $R = k[x_1,\ldots,x_d]$ be a polynomial ring over a field $k$. Let $<$ be a monomial order on $R$. Let $f$ and $g_1,\ldots,g_n$ be polynomials in $R$ with $g_i \neq 0$. By the division
algorithm there exist polynomials \( q_1, \ldots, q_n, r \) in \( R \) satisfying

\[
f = q_1 g_1 + \cdots + q_n g_n + r
\]

such that no monomial term of \( r \) is in the ideal generated by \( \text{lcm}(g_1), \ldots, \text{lcm}(g_n) \) and \( \text{lcm}(f) \geq \text{lcm}(q_i g_i) \) for all \( i \). We call \( r \) the \textbf{remainder} of \( f \) with respect to \( g_1, \ldots, g_n \).

The \textbf{Buchberger Algorithm} allows one to compute a Gröbner basis of \( I \) starting from a finite system of generators \( \mathcal{G} \) of \( I \).

- **STEP 1**: For each pair of distinct elements of \( \mathcal{G} \) compute the S-polynomial, and its remainder with respect to \( \mathcal{G} \).
- **STEP 2**: If all S-polynomials reduce to 0, then the algorithm ends and \( \mathcal{G} \) is a Gröbner basis of \( I \). Otherwise we add one of the nonzero remainders to the system of generators, call the new system of generators \( \mathcal{G} \) again and return to step 1.

The S-polynomial is of particular importance to this thesis. As mentioned earlier, we can use Gröbner bases to find the initial ideal. The initial ideal is helpful in understanding the Hilbert functions of an ideal since they help to compute a \( k \) basis of \( R/I \). The Hilbert function is an important invariant which measures the growth of the dimension of the homogeneous components of the algebra. So in particular, when the S-polynomial reduces to zero or has a leading term in common with another element of a partial Gröbner basis, this tells us that we have not obtained a new element in the \( k \) basis. If on the other hand the S-polynomial does not reduce to zero, its leading monomial will be a part of the Gröbner basis. Since the candidate \( \Lambda \), for which we will compute a partial Gröbner basis is included in an ideal which has a lot of known structure, we will not be obliged to show that we have computed all elements of the Gröbner basis. That is, we do not have to show that all S-polynomials reduce to zero which is often one of the hardest steps in showing that a generating set is indeed a Gröbner basis.
2.3 Some Linear Algebra

In Chapter 4, we construct a candidate ideal $\Lambda$ which we aim to show is the defining ideal of the special fiber ring for the family studied in this thesis. To do this, we rely on some linear algebra facts about determinants of $2 \times 2$ matrices. The necessary facts are listed in this section. We begin by formalizing the notion of an adjugate matrix, this is simply the transpose of the cofactor matrix. The construction of this matrix is classical and is used in computing the inverse of a matrix.

**Definition 2.14.** Let $A$ be an $n \times n$ square matrix.

- The $(i, j)$ minor of $A$ denoted $M_{i,j}$ is the determinant of the $n-1 \times n-1$ matrix obtained by deleting row $i$ and column $j$ from $A$.

- The adjugate of $A$ is the $n \times n$ matrix whose $(i, j)$ entry is $(-1)^{i+j}M_{j,i}$.

For a $2 \times 2$ matrix we have

$$
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
$$

For the purposes of this thesis, we will use the adjugate matrix to understand how the determinant of a $2 \times 2$ square matrix changes with addition.

**Theorem** (Matrix Determinant Lemma). Let $A$ be a square matrix and $u$ and $v$ column vectors. Then

$$
\det(A + uv^T) = \det A + v^T \text{adj}(A)u.
$$

We will be using the matrix determinant lemma in the $2 \times 2$ case where the matrices $A$ and $A + uv^T$ differ by in only one of the entries. For example,

$$
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d-e \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ e \end{bmatrix}.
$$
In particular $B = A + uv^T$ so we can describe the determinant using the lemma. Indeed, 
$\det B = \det A - ae$ and 

$$v^T \text{adj}(A) u = \begin{bmatrix} 0 & e \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & e \end{bmatrix} \begin{bmatrix} 0 \\ -a \end{bmatrix} = -ae.$$ 

Another important idea of linear algebra is that of the characteristic polynomial of a square matrix. Indeed, if $A$ is an $n \times n$ square matrix, the characteristic polynomial of $A$ is the degree $n$ polynomial 

$$p_A(t) = \det(t \cdot \text{Id}_n - A)$$

where $\text{Id}_n$ is the $n \times n$ identity matrix. The characteristic polynomial most obviously detects eigenvalues of a matrix but is common throughout the study of linear algebra. Indeed the eigenvalues of a matrix will be the roots of the characteristic polynomial. For the $2 \times 2$ case, the characteristic polynomial also allows one to read off both the trace and determinant of the matrix. Indeed, the characteristic polynomial of a $2 \times 2$ matrix $A$ is simply 

$$p_A(t) = \det(t \cdot \text{Id}_2 - A) = t^2 - \text{tr}(A)t + \det(A).$$

We will be using the characteristic polynomial in the $2 \times 2$ to study the determinants of two matrices $A$ and $B$ differ by $t \cdot \text{Id}_2$. For example 

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} a-e & b \\ c & d-e \end{bmatrix}.$$ 

Indeed, $\det B = \det A - (a + d)e + e^2 = p_A(-e)$. 

17
CHAPTER 3

THE FAMILY

The primary goal of this chapter is to formally define the family of ideals to be studied in this thesis. We will also fix the notation to be used throughout the manuscript. As mentioned in the introduction, we are interested in computing the defining ideal of the Rees algebra for \( m \)-primary Gorenstein ideals which have a Gorenstein linear resolution.

In pursuit of this goal, this thesis will focus on such ideals which are generated in the smallest degree, \( \delta = 2 \).

**Data 3.1.** Let \( R = k[x_1, \ldots, x_d] \) be a polynomial ring over a field \( k \).

Let \( m = (x_1, \ldots, x_d) \) be the maximal homogenous ideal of \( R \).

Let \( I \) be an \( m \)-primary Gorenstein ideal generated in degree \( \delta = 2 \) having a Gorenstein linear resolution.

The following theorems from the literature will give information about the structure of these ideals and their Rees algebras.

**Theorem** (Miller, Villarreal, (Theorem 2.1) [26]). Let \( \mu(J) \), the number of minimal generators of an ideal \( J \). If \( I = \oplus I_i \) is a graded Gorenstein ideal of codimension \( g \geq 3 \) and initial degree \( \delta \geq 2 \), then

\[
\mu(I_\delta) \leq \binom{\delta + g - 1}{g - 1} - \binom{\delta + g - 3}{g - 1}.
\]

Equality holds if \( I \) has a Gorenstein linear resolution.

**Theorem** (Hong, Simis, Vasconcelos, (Theorem 3.1) [18]). Let \( R = k[x_1, \ldots, x_d] \) be a polynomial ring over a field \( k \) and \( m = (x_1, \ldots, x_d) \) be the maximal homogenous ideal of \( R \). An
ideal $I$ of $R$ said to be **submaximally generated** generated by quadrics if it is codimension $d$ and minimally generated by $\binom{d+1}{2} - 1$ quadrics. If $I$ is Gorenstein and submaximally generated by quadrics, then up to a change of variables,

$$I = \langle x_i x_j, x_k^2 - x_d^2 \mid 1 \leq i < j \leq d, k = 1, \ldots, d-1 \rangle.$$

Further, $I^2 = m^4$.

We can now combine these results to get a structure theorem for the ideals defined in Data 3.1.

**Corollary 3.1.** Adopt Data 3.1. After a change of variables,

$$I = \langle x_i x_j, x_k^2 - x_d^2 \mid 1 \leq i < j \leq d, k = 1, \ldots, d-2 \rangle.$$

**Proof.** We apply the theorem of [26] with $g = d$ and $\delta = 2$. That is,

$$\mu(I) = \binom{\delta + d - 1}{d-1} - \binom{\delta + d - 3}{d-1}$$

$$= \binom{2 + d - 1}{d-1} - \binom{2 + d - 3}{d-1}$$

$$= \binom{d + 1}{2} - 1$$

Thus, $I$ is submaximally generated by quadrics. Now we apply the theorem of [18] and up to a change of variables

$$I = \langle x_i x_j, x_k^2 - x_d^2 \mid 1 \leq i < j \leq d, k = 1, \ldots, d-1 \rangle.$$

**Remark 3.1.** Adopt Data 3.1. The Rees algebra $\mathcal{R}(I)$ is fiber type. This follows from Theorem 9.1(c) of [12], stated in Section 2.1 of this thesis.
We wish to compute the defining ideal of the Rees algebra for all ideals $I$ as in Data 3.1.

Since the Rees algebra is fiber type, we may focus our efforts on computing the defining ideal of the special fiber ring. We will now set the notation needed to compute the defining ideals of the blow up algebras.

**Data 3.2.** Adopt Data 3.1 and let $d \geq 4$.

- Let $W = k[w_{i,j} | 1 \leq i, j \leq d]/ < w_{i,j} - w_{j,i}, w_{d,d}>$ and $S = R[w_{i,j} | 1 \leq i, j \leq d]/ < w_{i,j} - w_{j,i}, w_{d,d}>$ be polynomial rings. As an abuse of notation, we may write $W = k[w_{i,j}]$ and $S = R[w_{i,j}]$.

- Let $\omega$ be the reverse lexicographic ordering on $W$ where the variables satisfy $w_{i,j} <_\omega w_{k,l}$ if either of the following hold:
  - $\max\{i, j\} < \max\{k, l\}$
  - $\max\{i, l\} = \max\{k, l\}$ and $\min\{i, j\} < \min\{k, l\}$

- Let $g_{i,j} = \begin{cases} x_i x_j & i \neq j \\ x_i^2 - x_j^2 & i = j \end{cases}$ be the minimal generators of $I$ from Corollary 3.1.

- Let $x = [x_1, \ldots, x_d]$ be a $1 \times d$ row vector.

- Let $w = [w_{1,1}, \ldots, w_{d-1,d}]$ be a $1 \times n$ row vector where the variables are listed in increasing order coming from $\omega$ such that $n = \binom{d+1}{2} - 1$. Notice $w_{i,j} = w_{j,i}$ appears only once in $w$.

- Let $g = [g_{1,1}, \ldots, g_{d-1,d}]$ be a $1 \times n$ row vector ordered so that $g_{i,j}$ appears before $g_{k,l}$ if $w_{i,j} <_\omega w_{k,l}$.

- Let $\varphi$ be a presentation matrix of $g$.

- Let $J$ be the defining ideal of the Rees algebra $R(I)$.

- Let $I(X)$ be the defining ideal of the special fiber ring $\mathcal{F}(I)$.

- Let $L = I_1(w \cdot \varphi)$ be the defining ideal of the symmetric algebra $S(I)$.

### 3.1 Excluded Dimensions

In Data 3.2, we restricted the dimension of the polynomial ring $R$ to have dimension $d \geq 4$. This is for two main reasons. First, throughout commutative algebra, ideals which are height two or Gorenstein of height three often have special behavior. This is no exception.
In particular the cases \( d = 2 \) and \( d = 3 \) have vastly different behavior from each other as well as from the \( d \geq 4 \) cases. It is maybe more surprising that the \( d \geq 4 \) cases have such similar behavior. The second reason that the \( d = 2 \) and \( d = 3 \) cases have been excluded from this research is that the defining ideal of the Rees algebra in these cases are known. We will give a brief description of those defining ideals below.

In the \( d = 2 \) case, we have a height two Gorenstein ideal in a polynomial ring. Since height two Gorenstein ideals are complete intersections, the Rees algebra is linear type. That is, the Rees algebra and symmetric algebras are equal

\[
\mathcal{R}(I) = S(I) \quad \text{and} \quad \mathcal{J} = \mathcal{L}.
\]

In the \( d = 3 \) case, we have \( R = k[x_1, x_2, x_3] \) and \( I \cong (x_1x_2, x_1x_3, x_2x_3, x_1^2 - x_3^2, x_2^2 - x_3^2) \). The ideal \( I \) is a linearly presented, height three, \( m \)-primary, Gorenstein ideal. Just as the defining ideal of the Rees algebra is known to be linear type for all complete intersections, the defining ideal for grade three Gorenstein ideals is also known in a more general setting than the scope of this thesis. See [22] for more details.

**Theorem** (Kustin, Polini, Ulrich, (Theorem 9.1) [22]). Let \( I \) be a linearly presented height three Gorenstein ideal that satisfies \( G_d \). Then the defining ideal of the Rees ring of \( I \) is

\[
\mathcal{J} = (\mathcal{L}, I_d(B), C(\phi))
\]

and the defining ideal of the special fiber ring is

\[
I(X) = (I_d(B), C(\phi))
\]

where \( \phi \) is a minimal homogeneous alternating presentation matrix for \( I \). In particular, \( I \) is of fiber type and, if \( d \) is odd, then \( C(\phi) = 0 \) and \( \mathcal{R}(I) \) has expected form.

The condition \( G_d \) appears frequently throughout the study of Rees algebras. In partic-
ular for those which are not linear type. We say $I$ satisfies $G_d$ provided for every $p \in V(I)$ with $\dim R_p < d$, we have $\mu(I_p) \leq \dim R_p$. The property $G_d$ is always satisfied if $R/I$ is 0-dimensional or if $I$ is a generic perfect ideal of height two or a generic Gorenstein ideal of height three. In particular, $m$-primary ideals satisfy $G_d$. The ideal $C(\varphi)$ mentioned in the theorem is called the content ideal and is described in Definition 8.1 [22]. In this setting, the content ideal is simply zero and the Rees algebra has expected form. That is,

$$\mathcal{J} = (\mathcal{L}, I_d(B)).$$
In this chapter we continue to explore the Rees algebra and special fiber ring for m-primary, Gorenstein ideals, generated in degree $\delta = 2$, which have a Gorenstein linear resolution. We will adopt the notation from Data 3.2. The goal of this chapter, is to compute explicit polynomial generators for $I(X)$, the defining ideal of the special fiber ring of $I$. We achieve this goal by studying the special fiber ring of $m^2$, whose defining ideal is well known. We then exploit the relationship $m^2 = (I, x_0^2)$ to compute $I(X)$.

In Section 4.1, we construct a candidate ideal $\Lambda$ consisting of particular linear combinations of minors of a symmetric matrix. We then confirm the inclusion $\Lambda \subseteq I(X)$. In Section 4.2, we use the Tor functor on the short exact sequence $0 \to \mathcal{F}(I) \to \mathcal{F}(m^2) \to k(-1) \to 0$ to compute the degrees of the generators of $I(X)$. In particular, we show that $I(X)$ is generated in degrees two and three. To confirm the desired equality, we compute the Hilbert functions of $I(X)$ and $\Lambda$ in those degrees. We show $HF_{\Lambda}(2) = HF_{I(X)}(2)$ in Section 4.3 and show $HF_{\Lambda}(3) = HF_{I(X)}(3)$ in Section 4.4. We highlight the main theorem in Section 4.5 and complete the proof.

4.1 A Candidate for the Defining Ideal

In this section, we will construct a candidate for the defining ideal of $\mathcal{F}(I)$ using the defining ideal of $\mathcal{F}(m^2)$ as a model. We will begin by setting notation and describing the defining ideal of $\mathcal{F}(m^2)$.

Data 4.1. Adopt Data 3.2
• Let \( U = k[u_{i,j}] / < u_{i,j} - u_{j,i} > \) be a polynomial ring. As an abuse of notation, we may write \( U = k[u_{i,j}] \).

• Let \( \varepsilon : W \to U \) be the map such that \( \varepsilon(w_{i,j}) = \begin{cases} u_{i,j} & i \neq j \\ u_{i,i} - u_{d,d} & i = j \end{cases} \)

• Let \( \phi_U : U \to F(m^2) \) be the map \( \phi_U(u_{i,j}) = x_i x_j \).

• Let \( \phi_W : W \to F(I) \) be the map \( \phi_W(w_{i,j}) = \begin{cases} x_i x_j & i \neq j \\ x_i^2 - x_d^2 & i = j \end{cases} \).

• Let \( I(Y) \) be the defining ideal of \( F(m^2) \).

Notice that \( I(X) = \ker(\phi_W) \) and \( I(Y) = \ker(\phi_U) \). Further, \( \phi_W \) factors through \( U \). That is, \( \phi_W = \phi_U \circ \varepsilon \). We will adopt the following convention for variables of the rings \( W \) and \( U \).

**Remark 4.1.** Notice that \( w_{i,j} = w_{j,i} \) and \( u_{i,j} = u_{j,i} \) for all \( i, j \). In this thesis, if the relative size of \( i, j \) is known, we will write the variables as \( w_{i,j} \) and \( u_{i,j} \) with \( i \leq j \).

The defining ideal of \( F(m^2) \) are well known (for example [24], [8]) to be \( I(Y) = I_2(N) \), where \( N = (u_{i,j}) \) is a \( d \times d \) symmetric matrix. The matrix \( N \) is written below when \( d = 4 \) to aid in the visualization of the construction of \( \Lambda \).

**Example 4.1.** Case: \( d = 4 \)

\[
N = \begin{bmatrix}
  u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\
  u_{1,2} & u_{2,2} & u_{2,3} & u_{2,4} \\
  u_{1,3} & u_{2,3} & u_{3,3} & u_{3,4} \\
  u_{1,4} & u_{2,4} & u_{3,4} & u_{4,4}
\end{bmatrix}
\]

\[
\phi_U(N) = \begin{bmatrix}
  x_1^2 & x_1 x_2 & x_1 x_3 & x_1 x_4 \\
  x_1 x_2 & x_2^2 & x_2 x_3 & x_2 x_4 \\
  x_1 x_3 & x_2 x_3 & x_3^2 & x_3 x_4 \\
  x_1 x_4 & x_2 x_4 & x_3 x_4 & x_4^2
\end{bmatrix}
\]

The containment \( I_2(N) \subseteq I(Y) \) is clear from the example. Equality comes, in particular since \( I_2(N) \) is prime and has the correct height. The \( 2 \times 2 \) minors of \( N \) even form a Gröbner basis for \( I(Y) \) under any diagonal ordering. Following the example of \( F(m^2) \), we will now define a symmetric matrix \( M \) such that the entries of \( \phi_W(M) \) are the generators of \( I \). Let \( I_{d_d} \) for the \( d \times d \) identity matrix and let \( M = \varepsilon^{-1}(N - u_{d,d} I_{d_d}) \).
Example 4.2. Case: \( d = 4 \)

\[
\mathcal{M} = \begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,3} & w_{1,4} \\
    w_{1,2} & w_{2,2} & w_{2,3} & w_{2,4} \\
    w_{1,3} & w_{2,3} & w_{3,3} & w_{3,4} \\
    w_{1,4} & w_{2,4} & w_{3,4} & 0
\end{bmatrix}
\]

\[
\varphi_W (\mathcal{M}) = \begin{bmatrix}
    x_1^2 - x_4^2 & x_1 x_2 & x_1 x_3 & x_1 x_4 \\
    x_1 x_2 & x_2^2 - x_4^2 & x_2 x_3 & x_2 x_4 \\
    x_1 x_3 & x_2 x_3 & x_3^2 - x_4^2 & x_3 x_4 \\
    x_1 x_4 & x_2 x_4 & x_3 x_4 & 0
\end{bmatrix}
\]

Since \( I(Y) = I_2 (N) \), one might wonder if \( I(X) = I_2 (\mathcal{M}) \). Unfortunately, this equality does not hold. Indeed when \( d = 4 \), as in the example, it is clear that \( w_{3,4} \notin I_2 (\mathcal{M}) \) while \( w_{3,4} \notin I_2 (\mathcal{M}) \). Since \( I(X) \) is a prime ideal, \( I_2 (\mathcal{M}) \) cannot be \( I(X) \). Although \( I(X) \) is not simply the ideal of \( 2 \times 2 \) minors of \( \mathcal{M} \), this matrix is incredibly helpful in finding generators of \( I(X) \). Since \( \varphi_W = \varphi_U \circ \varepsilon \) we have

\[
I(X) = \ker (\varphi_W) = \ker (\varphi_U \circ \varepsilon) = \varepsilon^{-1} (\ker (\varphi_U) \cap \im (\varepsilon)).
\]

This relationship gives rise to the following criterion which allows us to check if an element is in the defining ideal.

Criteria 4.1. If \( f \in W \) with \( \varepsilon(f) \in I_2 (N) \) then \( f \in I(X) = \ker (\varphi_W) \).

This criteria will be key in confirming the inclusion \( \Lambda \subseteq I(X) \), once we define the candidate ideal \( \Lambda \). The elements \( f \in W \) that will be studied are linear combinations of \( 2 \times 2 \) minors of \( \mathcal{M} \). We will now set notation for minors and submatrices.

Definition 4.1. Let \( B \in \{ \mathcal{M}, N \} \) be a \( d \times d \) symmetric matrix.

- Let \( m_{(i,j),(k,l)}^B \) be the \( 2 \times 2 \) submatrix of \( B \) consisting of the rows \( i, j \) and the columns \( k, l \).
- Let \( m_{(i,j),(k,l)}^B \) be the unsigned minor of the \( 2 \times 2 \) submatrix \( m_{(i,j),(k,l)}^B \).
- Let \( s \in \{ 0, 1, 2 \} \) and \( A_s = \{ m_{(i,j),(k,l)}^\mathcal{M} \} \) such that \( |\{ i, j, k, l \}| = 4 - s \). That is, the subset of the set of \( 2 \times 2 \) minors of \( \mathcal{M} \) containing of exactly \( s \) variables in the diagonal of \( \mathcal{M} \).
• We say a submatrix \( B' \) of \( B \) is **principal** if it is a square matrix such that the set of diagonal entries of \( B' \) is a subset of the diagonal entries of \( B \).

• We write \( B_{i,j,k,l} \) for the \( 4 \times 4 \) principal submatrix of \( B \) obtained by selecting rows and columns \( \{i, j, k, l\} \) from \( B \).

• We say that the submatrices \( m^B \) and \( n^B \) are **complementary** if there is a \( 4 \times 4 \) principal submatrix \( B_{i,j,k,l} \) of \( B \) such that \( m^B \) and \( n^B \) are \( 2 \times 2 \) submatrices of \( B_{i,j,k,l} \) which share no rows and share no columns.

• We say that minors are **complementary** if they are obtained from complementary submatrices.

**Remark 4.2.** Subscripts and superscripts of minors and submatrices may be omitted if not needed for a proof or clear from the context.

Notice that \( A_0 \sqcup A_1 \sqcup A_2 \) partition the set of \( 2 \times 2 \) minors of \( M \). We use the sets \( A_s \) to distinguish between the \( 2 \times 2 \) minors since the diagonal of \( M \) has different behavior under \( \varphi_W \) than the rest of the matrix.

**Example 4.3.** The red and orange submatrices in (A) are complementary, but the red and orange submatrices in (B) are not complementary because they share a column. The minor obtained from the orange submatrix in (B) is an element in \( A_0 \). The other minors obtained from colored submatrices are elements of \( A_1 \).

\[
(A): \begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,4} \\
w_{2,1} & w_{2,2} & w_{2,3} & w_{2,4} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,4} \\
w_{1,4} & w_{2,4} & w_{3,4} & w_{4,4}
\end{bmatrix}
\]

\[
(B): \begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,4} \\
w_{2,1} & w_{2,2} & w_{2,3} & w_{2,4} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,4} \\
w_{1,4} & w_{2,4} & w_{3,4} & w_{4,4}
\end{bmatrix}
\]

When \( d = 4 \) all \( 2 \times 2 \) minors will have unique complements as there is only one \( 4 \times 4 \) submatrix of \( M \), namely \( M \) itself. If \( m^M_{\{i,j\},\{k,l\}} \in A_0 \), it will have a unique complement regardless of the value of \( d \). This is because \( |\{i,j,k,l\}| = 4 \) so there is only one \( 4 \times 4 \) principal submatrix of \( M \) containing \( m^M_{\{i,j\},\{k,l\}} \) as a submatrix. If \( d > 4 \) and \( m^M_{\{i,j\},\{k,l\}} \) is in \( A_1 \) or \( A_2 \), then it will not have a unique complement. This is because \( |\{i,j,k,l\}| < 4 \) so
there are multiple $4 \times 4$ principal submatrices of $\mathcal{M}$ containing $m^\mathcal{M}_{\{i,j\},\{k,l\}}$ as a submatrix. There will be a different complement for $m^\mathcal{M}_{\{i,j\},\{k,l\}}$ associated to each different principal submatrix.

**Lemma 4.1.** If $m^\mathcal{M}, n^\mathcal{M}$ are complementary submatrices, then $m^\mathcal{M} \in A_s$ if and only if $n^\mathcal{M} \in A_s$.

**Proof.** Let $m = m^\mathcal{M}$ and $n = n^\mathcal{M}$ be complementary. Let $\mathcal{M}_{i,j,k,l}$ be a $4 \times 4$ principal submatrix containing $m, n$ as submatrices. Define the sets $R_m, R_n, C_m, C_n$ to be the sets of row indices for $m, n$ and column indices for $m, n$ respectively. Let $a = |R_m \cap C_m|$ and let $b = |R_n \cap C_n|$. So $m \in A_a$ and $n \in A_b$. Since $m$ and $n$ are complementary, notice that $R_m \cup C_m = \{i, j, k, l\} \setminus \{R_n \cap C_n\}$. So in particular $|R_m \cup C_m| = 4 - b$ and thus $|R_m \cap C_m| = b$. Therefore $a = b$ and $m \in A_s$ if and only if $n \in A_s$ as claimed. 

We now begin to construct $\Lambda$, the candidate for the defining ideal. Generators of $\Lambda$ will arise in three ways, those coming from $A_0, A_1$ and $A_2$. The ideal $\Lambda_0$ will be generated by all minors in the set $A_0$. Since minors in $A_0$ do not involve the diagonal, there will be no obstructions to the minors being contained in $I(X)$. Let $\Lambda_0$ be the $W$-ideal

$$\Lambda_0 = < \{ m \mid m \in A_0 \} > .$$

The ideal $\Lambda_1$ will be generated by the forms $m \pm n$, where $m, n$ are complementary minors in the set $A_1$. The sign will be determined by the locations of the diagonal entries in the $2 \times 2$ submatrix. In particular, it will not matter where the diagonal entry was in $\mathcal{M}$.

**Definition 4.2.** Let $r_{\text{diag}} : A_1 \to \{1, 2\}$, $c_{\text{diag}} : A_1 \to \{1, 2\}$, $D : A_1 \to \{2, 3, 4\}$ be functions such that if the diagonal entry is in the $a^{th}$ row and $b^{th}$ column of $m$, then

$$r_{\text{diag}}(m) = a$$

$$c_{\text{diag}}(m) = b$$
$D(m) = a + b.$

The four possibilities are listed in Table 4.1.

**TABLE 4.1**

FUNCTIONS ON $A_1$

<table>
<thead>
<tr>
<th>$m_1 = \begin{bmatrix} w_{j,j} &amp; w_{i,j} \ w_{j,k} &amp; w_{i,k} \end{bmatrix}$</th>
<th>$m_2 = \begin{bmatrix} w_{i,k} &amp; w_{i,j} \ w_{j,k} &amp; w_{j,j} \end{bmatrix}$</th>
<th>$m_3 = \begin{bmatrix} w_{i,j} &amp; w_{j,j} \ w_{i,k} &amp; w_{j,k} \end{bmatrix}$</th>
<th>$m_4 = \begin{bmatrix} w_{i,j} &amp; w_{i,k} \ w_{j,j} &amp; w_{j,k} \end{bmatrix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{\text{diag}}(m_1) = 1$</td>
<td>$r_{\text{diag}}(m_2) = 2$</td>
<td>$r_{\text{diag}}(m_3) = 1$</td>
<td>$r_{\text{diag}}(m_4) = 2$</td>
</tr>
<tr>
<td>$c_{\text{diag}}(m_1) = 1$</td>
<td>$c_{\text{diag}}(m_2) = 2$</td>
<td>$c_{\text{diag}}(m_3) = 2$</td>
<td>$c_{\text{diag}}(m_4) = 1$</td>
</tr>
<tr>
<td>$D(m_1) = 1 + 1 = 2$</td>
<td>$D(m_2) = 2 + 2 = 4$</td>
<td>$D(m_3) = 1 + 2 = 3$</td>
<td>$D(m_4) = 2 + 1 = 3$</td>
</tr>
</tbody>
</table>

Each element $m \in A_1$ has exactly one variable in the diagonal of $\mathcal{M}$. The index of the diagonal will be referred to as the **repeated index**. For each entry in the table above, the repeated index is $j$.

Let $\Lambda_1$ be the $W$-ideal

$$\Lambda_1 = \langle \{ m + (-1)^{D(m)+D(n)+1}n \mid m, n \in A_1 \text{ are complementary} \} \rangle.$$  

The ideal $\Lambda_2$ will be generated by certain linear combinations of minors coming from $A_2$. Indeed, the obstruction to the sum of complementary minors $m + n$ lying inside $I(X)$ will be related to the trace of $\mathcal{M}_{i,j,k,l}$, the $4 \times 4$ principal submatrix containing $m$ and $n$. To sidestep this obstruction we will introduce the notion of a pair of complementary minors.
**Definition 4.3.** We say \((m_1, n_1), (m_2, n_2)\) are a pair of complementary minors or (p.c.m.) provided:

- \(m_1, n_1, m_2, n_2 \in A_2\)
- \(m_1, n_1\) are complementary
- \(m_2, n_2\) are complementary
- There is a \(4 \times 4\) principal submatrix \(M_{i,j,k,l}\) containing \(m_1, n_1, m_2, n_2\) as submatrices.

Let \(\Lambda_2\) be the \(W\)-ideal

\[\Lambda_2 = \langle \{(m_1 + n_1) - (m_2 + n_2) \mid (m_1, m_1), (m_2, m_2)\} \rangle\]

The candidate \(\Lambda\) for the defining ideal of special fiber ring will be the sum of these \(\Lambda_i\).

\[\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2\]

In the following lemmas we will show that the \(\Lambda_i\) are all subideals of \(I(X)\). Hence their sum \(\Lambda\) is likewise contained in \(I(X)\).

**Lemma 4.2.** Adopt Data 3.2 The ideal

\[\Lambda_0 = \langle \{m \mid m \in A_0\} \rangle\]

is a subideal of \(I(X)\), the defining ideal of \(F(I)\).

**Proof.** Let \(m^M_{\{i,j\},\{k,l\}} \in A_0\). Since, \(\varepsilon(m^M_{\{i,j\},\{k,l\}}) = m^N_{\{i,j\},\{k,l\}} \in I_2(N)\) by Criteria 4.1 we have \(m^M_{\{i,j\},\{k,l\}} \in I(X)\) and \(\Lambda_0 \subset I(X)\).

**Lemma 4.3.** Adopt Data 3.2 The ideal

\[\Lambda_1 = \langle \{m + (-1)^{D(m)+D(n)+1}n \mid m, n \in A_1\ \text{are complementary}\} \rangle\]

is a subideal of \(I(X)\), the defining ideal of \(F(I)\).
Proof. Let \( m^M_{i,j},(k,j) \in A_1 \). Let \( M_{i,j,k,l} \) be a principal submatrix containing \( m^M_{i,j},(k,j) \). Let \( n^M_{i,l},(k,l) \) be the complement of \( m^M_{i,j},(k,j) \) in the principal submatrix \( M_{i,j,k,l} \). Notice that \( j \) is the repeated index of \( m^M_{i,j},(j,k) \) and \( l \) is the repeated index of \( n^M_{i,l},(k,l) \). Let

\[
\begin{align*}
r_1 &= \begin{bmatrix} u_{d,d} \\ 0 \end{bmatrix}, \\
r_2 &= \begin{bmatrix} 0 \\ u_{d,d} \end{bmatrix}, \\
c_1 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
c_2 &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\end{align*}
\]

be vectors.

Let \( a = r_{\text{diag}}(m^M_{i,j},(k,j)) \) and \( b = r_{\text{diag}}(m^M_{i,j},(k,j)) \). Notice that the matrices \( m^N_{i,j},(k,j) + r_a c_b^T \) and \( e(m^M_{i,j},(k,j)) \) are equal. By the Matrix Determinant Lemma in Section 2.3, we have

\[
\begin{align*}
e(m^M_{i,j},(k,j)) &= e(\det(m^M_{i,j},(k,j))) \\
&= \det(e(m^M_{i,j},(k,j))) \\
&= \det(m^N_{i,j},(k,j) + r_a c_b^T)
\end{align*}
\]

By the Matrix Determinant Lemma in Section 2.3, we have

\[
\begin{align*}
e(m^M_{i,j},(k,j)) &= m^N_{i,j},(k,j) + c_b^T \text{adj}(m^N_{i,j},(k,j)) r_a \\
&= m^N_{i,j},(k,j) + (-1)^{a+b+1} u_{i,k} u_{d,d} \\
&= m^N_{i,j},(k,j) + (-1)^{D(m^M_{i,j},(k,j)) + 1} u_{i,k} u_{d,d}
\end{align*}
\]

and similarly

\[
\begin{align*}
e(n^M_{i,l},(k,l)) &= n^N_{i,l},(k,l) + (-1)^{D(n^M_{i,l},(k,l)) + 1} u_{i,k} u_{d,d}
\end{align*}
\]

We drop the subscripts and let \( m^M = m^M_{i,j},(k,j) \), \( n^M = n^M_{i,l},(k,l) \), \( m^N = m^N_{i,j},(k,j) \), and
\[ n^N = n^N_{\{i,d\},\{k,l\}}. \] Let \( g = m^M + (-1)^{D(m^M)+D(n^M)+1}n^M. \) Now we have

\[
\varepsilon(g) = \varepsilon(m^M + (-1)^{D(m^M)+D(n^M)+1}n^M)
\]
\[
= \varepsilon(m^M) + (-1)^{D(m^M)+D(n^M)+1}\varepsilon(n^M)
\]
\[
= [m^N + (-1)^{D(m^M)+1}u_{i,k}u_{d,d}] + (-1)^{D(m^M)+D(n^M)+1}[n^N + (-1)^D(n^M)+1]u_{i,k}u_{d,d}
\]
\[
= m^N + (-1)^{D(m^M)+D(n^M)+1}n^N + [(-1)^D(m^M)+1 + (-1)^D(n^M)]u_{i,k}u_{d,d}
\]
\[
= m^N + (-1)^{D(m^M)+D(n^M)+1}n^N.
\]

Since \( m^N + (-1)^{D(m^M)+D(n^M)+1}n^N \) is in \( I_2(N) \), by Criteria 4.1 we have \( g \in I(X) \) and \( \Lambda_1 \subset I(X) \).

**Lemma 4.4.** Adopt Data 3.2 The ideal

\[
\Lambda_2 = \langle (m_1 + n_1) - (m_2 + n_2) \mid (m_1, m_1), (m_2, m_2) \text{ are p.c.m.} \rangle
\]

is a subideal of \( I(X) \), the defining ideal of \( F(I) \).

**Proof.** Let \( Id_2 \) be the 2x2 identity matrix and let \( m^M_{\{i,j\},\{i,j\}} \in A_2 \). The submatrices \( \varepsilon(m^M_{\{i,j\},\{i,j\}}) \) and \( m^N_{\{i,j\},\{i,j\}} - u_{d,d}Id_2 \) of \( N \) are equal. Recall from Section 2.3 that the characteristic polynomial of a 2x2 matrix \( m \) is \( p_m(t) = \det(t \cdot Id_2 - m) = t^2 - \text{Tr}(m) \cdot t + \det(m) \). Therefore

\[
\varepsilon(m^M_{\{i,j\},\{i,j\}}) = \varepsilon(\det(m^M_{\{i,j\},\{i,j\}}))
\]
\[
= \det(\varepsilon(m^M_{\{i,j\},\{i,j\}}))
\]
\[
= \det(-u_{d,d}Id + m^N_{\{i,j\},\{i,j\}})
\]
\[
= p_{m^N_{\{i,j\},\{i,j\}}}(-u_{d,d})
\]
\[
= u_{d,d}^2 - \text{Tr}(m^N_{\{i,j\},\{i,j\}})(-u_{d,d}) + m^N_{\{i,j\},\{i,j\}}
\]
\[
= u_{d,d}^2 - \text{Tr}(m^N_{\{i,j\},\{i,j\}})(u_{d,d}) + m^N_{\{i,j\},\{i,j\}}
\]
Let \((m_1^M, n_1^M), (m_2^M, n_2^M)\) be a pair of complementary minors in \(M_{i, j, k, l}\). Now,

\[
\varepsilon(m_1^M + n_1^M) = \varepsilon(m_1^M) + \varepsilon(n_1^N)
\]

\[
= u_{d,d}^2 - \text{Tr}(m_1^N)(u_{d,d}) + m_1^N + u_{d,d}^2 - \text{Tr}(n_1^N)(u_{d,d}) + n_1^N
\]

\[
= 2u_{d,d}^2 - \text{Tr}(N_{i,j,k,l})(u_{d,d}) + m_1^N + n_1^N
\]

and similarly

\[
\varepsilon(m_2^M + n_2^M) = 2u_{d,d}^2 - \text{Tr}(N_{i,j,k,l})(u_{d,d}) + m_2^N + n_2^N.
\]

Therefore

\[
\varepsilon((m_1^M + n_1^M) - (m_2^M + n_2^M)) = \varepsilon(m_1^M + n_1^M) - \varepsilon(m_2^M + n_2^M)
\]

\[
= (2u_{d,d}^2 - \text{Tr}(N_{i,j,k,l})(u_{d,d}) + m_1^N + n_1^N)
\]

\[
- (2u_{d,d}^2 - \text{Tr}(N_{i,j,k,l})(u_{d,d}) + m_2^N + n_2^N)
\]

\[
= (m_1^N + n_1^N) - (m_2^N + n_2^N).
\]

Since \((m_1^N + n_1^N) - (m_2^N + n_2^N)\) is in \(I_2(N)\), by Criteria [4.1] we have \((m_1 + n_1) - (m_2 + n_2) \in I(X)\) and \(\Lambda_2 \subset I(X)\).

\[\square\]

**Theorem 4.1.** Adopt Data 3.2. The ideal \(\Lambda\) is a subideal of \(I(X)\), the defining ideal of \(\mathcal{F}(I)\).

**Proof.** This statement follows directly from Lemmas 4.2, 4.3, 4.4. \[\square\]

### 4.2 Degree of Generators of \(I(X)\)

The goal of this section is to bound the degree of the generators of \(I(X)\). This will allow us to conclude the equality of the ideals \(\Lambda\) and \(I(X)\) after we compare the Hilbert
functions of these ideals in low degrees.

**Theorem 4.2.** Adopt Data 4.1. The ideal \( I(X) \) is generated in degrees 2 and 3.

**Proof.** We will show that \([\text{Tor}^W_1(F(I), k)]_j = 0\) for all \( j \geq 4 \) which will allow us to conclude the statement of the theorem. We begin by setting the degrees of the generators of \( I \) and \( m^2 \) to be in degree 1. Since \( I^2 = m^4 \), we have the short exact sequence

\[
0 \to F(I) \to F(m^2) \to k(-1) \to 0.
\]

Applying the long exact sequence of Tor to this sequence, we have

\[
\cdots \to \text{Tor}^W_2(k(-1), k) \to \text{Tor}^W_1(F(I), k) \to \text{Tor}^W_1(F(m^2), k) \to \cdots.
\]

As this is a graded sequence, it is enough to show that for \( j \geq 4 \) we have \([\text{Tor}^W_2(k(-1), k)]_j = 0\) and \([\text{Tor}^W_1(F(m^2), k)]_j = 0\).

Notice that \( \text{Tor}^W_i(k(-1), k) = \text{Tor}^W_i(k, k)(-1) \). Since \( \text{Tor}^W_i(k, k) \) can be computed with the Koszul complex, we see \([\text{Tor}^W_i(k, k)]_j = 0\) for all \( j \neq i \). Therefore \([\text{Tor}^W_i(k, k)(-1)]_j = 0\) for all \( j \neq i + 1 \). In particular, \([\text{Tor}^W_2(k, k)(-1)]_j = [\text{Tor}^W_2(k(-1), k)]_j = 0\) for all \( j \geq 4 \).

We will now compute \([\text{Tor}^W_1(F(m^2), k)]_j\). That is, the degrees of the generators of the defining ideal of \( F(m^2) \) over the ring \( W \). Consider the commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\epsilon} & U \\
\downarrow{\phi_W} & & \downarrow{\phi_U} \\
F(I) & \xleftarrow{\ast} & F(m^2).
\end{array}
\]

We may view \( U \) as a polynomial ring over \( W \), that is, \( U \cong W[u_{d,d}] \). Notice that \( \phi_U(u_{d,d}^2) = x_d^4 \in m^4 = I^2 \) so there are \( a_i \in k \) and \( f_i \in I^2 \) such that \( \phi_U(u_{d,d}^2) = \sum a_if_i \). Since \( \phi_w \) is surjective, we can find degree two elements \( g_i \in W \) so that \( \phi_W(g_i) = f_i \). Let \( G = u_{d,d}^2 - \sum a_ig_i \). Notice that \( u_{d,d} \) is integral over \( W \) with degree two equation of integrality \( G \). So in particular
\[ \mathcal{F}(m^2) \] is integral over \( \mathcal{F}(I) \) and \( \mathcal{F}(m^2) = U/I(Y) \) is a \( U/(G) \) module. Let \( \overset{\longrightarrow}{w} \) represent going modulo \( G \). Indeed, by the first isomorphism theorem we have

\[
\begin{array}{cccc}
0 & \rightarrow & I(Y) & \rightarrow \mathcal{W}[u_{d,d}] & \rightarrow \mathcal{F}(m^2) \\
0 & \rightarrow & \overset{\longrightarrow}{I(Y)} & \rightarrow \mathcal{W}[u_{d,d}] & \\
\end{array}
\]

and in particular \( \mathcal{F}(m^2) = \frac{W[u_{d,d}]}{I(Y)} \). Since \( W[u_{d,d}] = W \oplus W_{d,d} \) is free over \( W \), we have \( \mathcal{F}(m^2) = \frac{W \oplus W_{d,d}}{I(Y)} \). Recall that the \( 2 \times 2 \) minors of \( \mathcal{N} \) form a generating set over \( U \). We will study the presentation matrix \( \phi \) of this generating set in the \( W \) resolution of \( \mathcal{F}(m^2) \)

\[
W \oplus W_{d,d} \overset{\phi}{\rightarrow} \mathcal{F}(m^2).
\]

Recall that \( \varepsilon : W \hookrightarrow U \) is the map such that \( \varepsilon(w_{i,j}) = \begin{cases} u_{i,j} & i \neq j \\ u_{i,i} - u_{d,d} & i = j \end{cases} \). If a \( 2 \times 2 \) minor \( u_{i,j}u_{k,l} - u_{i,k}u_{j,l} \) has distinct \( i, j, k, l, \) over \( W \oplus W[u_{d,d}] \) the generator will be \( w_{i,j}w_{k,l} - w_{i,k}w_{j,l} \). In particular, it will contribute the column \( \begin{bmatrix} w_{i,j}w_{k,l} - w_{i,k}w_{j,l} \\ 0 \end{bmatrix} \) to \( \phi \). For the next two cases, notice that if \( a = d \) the proof is the same except for setting \( w_{a,a} = 0 \). If a \( 2 \times 2 \) minor \( u_{i,j}u_{a,a} - u_{i,a}u_{j,a} \) has distinct \( i, j, a, \) over \( W \oplus W[u_{d,d}] \) the generator will be \( w_{i,j}(w_{a,a} + u_{d,d}) - w_{i,a}w_{j,a} = (w_{i,j}w_{a,a} - w_{i,a}w_{j,a}) + w_{i,j}u_{d,d} \). In particular, it will contribute the column \( \begin{bmatrix} w_{i,j}w_{a,a} - w_{i,a}w_{j,a} \\ w_{i,j} \end{bmatrix} \) to \( \phi \). If a \( 2 \times 2 \) minor \( u_{a,a}u_{b,b} - u_{a,b}^2 \) has distinct \( a, b, \) over \( W \oplus W[u_{d,d}] \) the generator will be

\[
(w_{a,a} + u_{d,d})(w_{b,b} + u_{d,d}) - w_{a,b}^2 = w_{a,a}w_{b,b} + w_{a,a}u_{d,d} + w_{b,b}u_{d,d} + u_{d,d}^2 - w_{a,b}^2
\]

\[
= w_{a,a}w_{b,b} + w_{a,a}u_{d,d} + w_{b,b}u_{d,d} + \sum a_i g_i - w_{a,b}^2
\]

\[
= (w_{a,a}w_{b,b} + \sum a_i g_i - w_{a,b}^2) + (w_{a,a} + w_{b,b})u_{d,d}.
\]
In particular, it will contribute the column
\[
\begin{bmatrix}
w_{a,a}w_{b,b} + \sum a_i g_i - w_{a,b}^2 \\
w_{a,a} + w_{b,b}
\end{bmatrix}
\]
to $\phi$. Since each column of $\phi$ has degree 2, we have $[\Tor_1^W(\mathcal{F}(m^2, k))]_j = 0$ for all $j \neq 2$. In particular, $[\Tor_1^W(\mathcal{F}(m^2, k))]_j = 0$ for all $j \geq 4$.

Thus $[\Tor_1^W(\mathcal{F}(I), k)]_j = 0$ for all $j \geq 4$ and $I(X)$ is generated in degrees 2 and 3, as claimed.

4.3 Hilbert Functions in Degree 2

In Section 4.1, we proposed a candidate $\Lambda$ for the defining ideal of $\mathcal{F}(I)$. We also concluded that $I(X)$ was generated in degrees 2 and 3 in Section 4.2. The goal of this section is to compute the Hilbert function for $I(X)$ and the candidate $\Lambda$ in degree 2 and show that they are equal. We begin by setting some notation for the section.

**Data 4.2.** Adopt Data 4.1.

- Let $HF_A(d)$ be Hilbert function of a ring or ideal $A$ in degree $d$.
- Let $\text{Im}(f)$ be the leading monomial $f$ from the ordering $\omega$ defined in Data 3.2.
- Let $\text{in}J$ be the initial ideal of an ideal $J$.
- Let $L_0 = \{\text{Im}(m) \mid m \in A_0\}$.
- Let $L_1 = \{\text{Im}(m + (-1)^D(m) + D(n) + 1)n) \mid m, n \in A_1$ are complementary $\}$.  
- Let $L_2 = \{\text{Im}((m_1 + n_1) - (m_2 + n_2)) \mid (m_1, m_1), (m_2, m_2)$ p.c.m $\}$.  

We will begin by computing the Hilbert function for $I(X)$ in degree 2.

**Proposition 4.1.** Adopt Data 4.2 The Hilbert function of $I(X)$ in degree 2 is

\[
HF_{I(X)}(2) = \frac{2(d + 2)(d + 1)(d)(d - 3)}{4!}.
\]
Proof. Since $\mathcal{F}(I) = W/I(X)$, we have $HF_{I(X)}(2) = HF_W(2) - HF_{\mathcal{F}(I)}(2)$. Further, $W$ is a polynomial ring in $n = \binom{d+1}{2} - 1$ variables, we have
\[
HF_W(2) = \left( \binom{2}{2} + \binom{d+1}{2} - 2 \right) = \binom{d+1}{2} = \binom{d+2}{4}.
\]
Since $I^2 = m^4$ we have
\[
HF_{\mathcal{F}(I)}(2) = \dim_k \left( \frac{I^2}{mI^2} \right) = \mu(I^2) = \mu(m^4) = HF_R(4) = \binom{d+3}{4}.
\]
This allows us to conclude the claimed equality
\[
HF(I(X), 2) = HF(W, 2) - HF(\mathcal{F}(I), 2)
= 3 \binom{d+2}{4} - \binom{d+3}{4}
= \frac{3(d+2)(d+1)(d)(d-1)}{4!} - \frac{(d+3)(d+2)(d+1)(d)}{4!}
= \frac{2(d+2)(d+1)(d)(d-3)}{4!}.
\]

We will now compute the Hilbert function of $\Lambda$ in the following way:

- We show $|L_0|$ is at least $2 \binom{d}{4}$.
- We show $|L_1|$ is at least $\frac{d(d-1)(d-3)}{2}$.
- We show $|L_2|$ is at least $\frac{d(d-3)}{2}$.
- We observe if $w_{i,j}w_{k,l}$ is any monomial term of a generator of $\Lambda$ then $|\{i,j,k,l\}| = 4-i$. In particular, the sets $L_i$ are disjoint.
- We conclude that
\[
HF_{\Lambda}(2) \geq HF_{\Lambda_0}(2) + HF_{\Lambda_1}(2) + HF_{\Lambda_1}(2) \geq |L_0| + |L_1| + |L_2| \geq HF_{\Lambda}(2)
\]
Hence $HF_{\Lambda}(2) = HF_{I(X)}(2)$. 

36
Remark 4.3. We will adhere to the following conventions.

• For the rest of the manuscript, if a leading monomial is known, it will be listed in blue.
• Since \( w_{i,j} = w_{j,i} \), if the relative size of \( i, j \) is known, we will write the variable as \( w_{i,j} \) with \( i \leq j \).

Example 4.4. We are going to be calculating leading terms, so it is helpful to have an idea of how the ordering \( \omega \) on the monomials of \( W \) relates to the matrix. Recall \( \omega \) is the reverse lexicographic ordering on \( W \) where the variables satisfy \( w_{i,j} \leq w_{k,l} \) if either of the following hold:

\[
\begin{align*}
\text{o} & \quad \max\{i,j\} < \max\{k,l\} \\
\text{o} & \quad \max\{i,l\} = \max\{k,l\} \text{ and } \min\{i,j\} < \min\{k,l\}
\end{align*}
\]

For \( d = 4 \), the positions of \( M \) have been labeled from largest (1) to smallest (9)

\[
\begin{bmatrix}
9 & 8 & 6 & 3 \\
8 & 7 & 5 & 2 \\
6 & 5 & 4 & 1 \\
3 & 2 & 1 & \end{bmatrix}
\]

To compare degree two monomials \( w_{i,j}w_{k,l} \) and \( w_{i',j'}w_{k',l'} \):

1. Identify the smallest variable. Without loss of generality, assume it is \( w_{i,j} \).
2. If \( w_{i,j} \neq w_{i',j'} \) and \( w_{i,j} \neq w_{k',l'} \), then \( w_{i,j}w_{k,l} \leq \omega w_{i',j'}w_{k',l'} \).
3. Otherwise, assume without loss of generality that \( w_{i,j} = w_{i',j'} \). Then \( w_{i,j}w_{k,l} \leq \omega w_{i',j'}w_{k',l'} \) if and only if \( w_{k,l} \leq \omega w_{k',l'} \).

In particular, for any \( 2 \times 2 \) minor of \( m \) monomial coming from the diagonal will always be smaller than monomial coming from the antidiagonal since the smallest of the variables involved will come from the top left corner of the submatrix.
**Lemma 4.5.** Adopt Data[4.2] The Hilbert function of $\Lambda_0$ in degree 2 satisfies the inequality

\[
HF_{\Lambda_0}(2) \geq 2\binom{d}{4}.
\]

**Proof.** Recall that

\[
\Lambda_0 = \{ m \mid m \in \Lambda_0 \}.
\]

We will show that for each $4 \times 4$ principal submatrix $M_{i,j,k,l}$ of $\mathcal{M}$ there are at least two distinct elements of the set $L_0$. Without loss of generality assume $1 \leq i < j < k < l \leq d$. The three candidates for generators of $\Lambda_0$ coming from $M_{i,j,k,l}$ are listed in Table 4.3.

**TABLE 4.2**

**GENERATORS OF $\Lambda_0$ IN $\mathcal{M}_{i,j,k,l}$**

<table>
<thead>
<tr>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
</tr>
<tr>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
</tr>
<tr>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
</tr>
<tr>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
</tr>
<tr>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
</tr>
<tr>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,i}$ $w_{j,k}$ $w_{j,l}$</td>
</tr>
<tr>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,i}$ $w_{k,j}$</td>
</tr>
<tr>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,i}$</td>
</tr>
</tbody>
</table>

$f_A = w_{i,j}w_{k,l} - w_{i,k}w_{j,l}$  
$f_B = w_{i,j}w_{k,l} - w_{i,l}w_{j,k}$  
$f_C = w_{i,k}w_{j,l} - w_{i,l}w_{j,k}$
Notice, however that $f_C = f_B - f_A$. Further, since $\text{Im}(f_A) \neq \text{Im}(f_B)$, we have two distinct elements of $L_0$ for each $M_{i,j,k,l}$. Since each term of the generators require all four indices $i,j,k,l$, there is no overlap arising from any two distinct choices of principal submatrices $M_{i,j,k,l}$. As there are $\binom{d}{4}$ options for principal submatrices, we have $2\binom{d}{4} \leq |L_0|$. Therefore the desired inequality $HF_{\Lambda_0}(2) = |L_0| \geq 2\binom{d}{4}$ holds. \hfill $\square$

**Lemma 4.6.** Adopt Data 4.2. The Hilbert function of $\Lambda_1$ in degree 2 satisfies the inequality $HF_{\Lambda_1}(2) \geq (d - 3)\binom{d}{2}$.

*Proof.* Recall that generators of $\Lambda_1$ are of the form $m \pm n$, where the submatrices $m,n \in A_1$ are complementary. To prove the claim, we will show that for each choice of non-repeated indices there are at least $d - 3$ elements of $L_1$.

First we will make some observations about elements of $L_1$. Let $m,n \in A_1$ be complementary submatrices such that $m,n$ are submatrices of $M_{i,j,k,l}$ such that $k$ is the repeated index of $m$ and $l$ is the repeated index of $n$. Then $m = m_{\{i,k\},\{j,k\}}$ and $n = n_{\{i,l\},\{j,l\}}$. The generator of $\Lambda_1$ involving $m,n$ is $(w_{i,j}w_{i,k} - w_{j,k}w_{i,k}) \pm (w_{i,j}w_{i,l} - w_{j,l}w_{i,l})$. To determine the leading monomial, we need to know the relative size of $i,j,k,l$. Without loss of generality we may assume that $i < j$ and $k < l$. With these assumptions, $w_{i,j}w_{i,k} < w_{i,j}w_{i,l}$ and $w_{j,k}w_{i,k} < w_{j,l}w_{i,l}$. In particular, the leading monomial will never involve the smallest repeated index $k$. We now have three cases:

(A) If $i < j < l$ the smallest variable is $w_{i,j}$ and the leading monomial is $w_{j,l}w_{i,l}$.

(B) If $i < l < j$ the smallest variable is $w_{i,l}$ and the leading monomial is $w_{i,j}w_{i,l}$.

(C) If $l < i < j$ the smallest variable is $w_{i,l}$ and the leading monomial is $w_{j,l}w_{i,l}$.

Let $a = \min\{\{1,\ldots,d\} \setminus \{i,j\}\}$ and let $b \in \{1,\ldots,d\} \setminus \{i,j,a\}$. If $i < b < j$, then $w_{i,j}w_{b,b} \in L_1$ is obtained by the complementary minors in $M_{i,j,a,b}$ such that $m,n$ have repeated indices $a,b$. Otherwise $w_{j,b}w_{i,b} \in L_1$. Notice $|\{1,\ldots,d\} \setminus \{i,j,a\}| = d - 3$ and there are $\binom{d}{2}$
options for non-repeated indices $i, j$. Thus $|L_1| \geq (d-3){d\choose 2}$. Hence $H\Lambda_1(2) \geq (d-3){d\choose 2}$ as claimed.

**Lemma 4.7.** Adopt Data 4.2 The Hilbert function of $\Lambda_2$ in degree 2 satisfies the inequality

$$H\Lambda_2(2) \geq \frac{d(d-3)}{2}.$$ 

**Proof.** Recall that the generators of $\Lambda_2$ are of the form $(m_1+n_1) - (m_2+n_2)$ where $(m_1,n_1), (m_2,n_2) \in A_2$ are a pair of complementary minors. Let $(m_1,n_1), (m_2,n_2)$ be a pair of complementary minors. Let $M_{i,j,k,l}$ be the $4 \times 4$ principal submatrix containing $m_1,n_1,m_2,n_2$ as submatrices. If $l = d$ the proof is unchanged except for replacing the variable $w_{l,l}$ with a zero. We may assume without loss of generality that $i < j < k < l$. Then there are three ways to choose complementary minors in $A_2$, listed in the table below:

<table>
<thead>
<tr>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
<td>$w_{i,i}$ $w_{i,j}$ $w_{i,k}$ $w_{i,l}$</td>
</tr>
<tr>
<td>$w_{i,j}$ $w_{j,j}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,j}$ $w_{j,k}$ $w_{j,l}$</td>
<td>$w_{i,j}$ $w_{j,j}$ $w_{j,k}$ $w_{j,l}$</td>
</tr>
<tr>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,k}$ $w_{k,l}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,k}$ $w_{k,l}$</td>
<td>$w_{i,k}$ $w_{j,k}$ $w_{k,k}$ $w_{k,l}$</td>
</tr>
<tr>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,l}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,l}$</td>
<td>$w_{i,l}$ $w_{j,l}$ $w_{k,l}$ $w_{l,l}$</td>
</tr>
</tbody>
</table>

There are $\left(\begin{array}{c}3 \\ 2\end{array}\right) = 3$ choices of generators of $\Lambda_2$, listed below.
\[(AB) \ w_{i,i}w_{j,j} - w_{i,j}^2 + w_{k,k}w_{l,l} - w_{k,l}^2 - w_{i,i}w_{k,k} + w_{i,k}^2 - w_{j,j}w_{l,l} + w_{j,l}^2\]

\[(AC) \ w_{i,i}w_{j,j} - w_{i,j}^2 + w_{k,k}w_{l,l} - w_{k,l}^2 - w_{i,i}w_{l,l} + w_{i,l}^2 - w_{j,j}w_{k,k} + w_{j,k}^2\]

\[(BC) \ w_{i,i}w_{k,k} - w_{i,k}^2 + w_{j,j}w_{l,l} - w_{j,l}^2 - w_{i,i}w_{l,l} + w_{i,l}^2 - w_{j,j}w_{k,k} + w_{j,k}^2\]

Notice that the leading monomials are of the form $w_{s,t}^2$ with $2 \leq s < t$ and $t \geq 4$. For each fixed $t$ there are $t - 2$ options for $s$, summarized in the table below:

<table>
<thead>
<tr>
<th>Table 4.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>COUNTING GENERATORS</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$</th>
<th>$s$</th>
<th># of Generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2,3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2,3,4</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2,3,4,5</td>
<td>4</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$d$</td>
<td>2,3,4, $\ldots$, $d-1$</td>
<td>$d-2$</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>$\frac{d(d-3)}{2}$</td>
</tr>
</tbody>
</table>

We obtain a new element of $L_2$ for each choice of $s,t$. Since there are

$$2 + 3 + 4 + \cdots + (d - 2) = \frac{d(d-3)}{2}$$

choices of $s,t$, we have $|L_2| \geq \frac{d(d-3)}{2}$. Further the desired inequality $HF_{\Lambda_2}(2) \geq \frac{d(d-3)}{2}$ holds.
Remark 4.4. Every monomial of a generator \( f \in \Lambda_i \) requires exactly \( 4-i \) indices, and in particular every element of \( L_i \) requires exactly \( 4-i \) indices. Up to reordering \( i, j, k, l \) and potentially setting \( w_{d,d} = 0 \), the generators of \( \Lambda_i \) are listed in the table below:

TABLE 4.5

STRUCTURE OF GENERATORS

<table>
<thead>
<tr>
<th>( \Lambda_0 )</th>
<th>( w_{i,j}w_{k,l} - w_{i,k}w_{j,l} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda_1 )</td>
<td>((w_{i,j}w_{k,k} - w_{j,k}w_{i,k}) \pm (w_{i,j}w_{l,l} - w_{j,l}w_{i,l}))</td>
</tr>
<tr>
<td>( \Lambda_2 )</td>
<td>( w_{i,i}w_{j,j} - w_{i,j}^2 + w_{k,k}w_{l,l} - w_{k,l}^2 - w_{i,i}w_{k,k} + w_{i,k}^2 - w_{j,j}w_{l,l} + w_{j,l}^2 )</td>
</tr>
</tbody>
</table>

Theorem 4.3. Adopt Data 4.2. The Hilbert functions of \( \Lambda \) and \( I(X) \) agree in degree two, that is

\[
HF_{\Lambda}(2) = HF_{I(X)}(2).
\]

Proof. Since \( \Lambda \subseteq I(X) \) we have \( HF_{\Lambda}(2) \leq HF_{I(X)}(2) \). For the converse, notice that by Remark 4.4 the generators of \( L_i \) are disjoint. Therefore

\[
HF_{\Lambda}(2) = HF_{\Lambda_0}(2) + HF_{\Lambda_1}(2) + HF_{\Lambda_2}(2)
\]

\[
= |L_0| + |L_1| + |L_2|
\]

\[
\geq 2\binom{d}{4} + \frac{d(d-1)(d-3)}{2} + \frac{d(d-3)}{2}
\]

\[
= 2(d+2)(d+1)(d)(d-3)
\]

\[
= 4!
\]

\[
= HF_{I(X)}(2).
\]
Thus $HF_{I(X)}(2) \leq HF_{\Lambda}(2) \leq HF_{I(X)}(2)$ and the Hilbert functions of $I(X)$ and $\Lambda$ agree in degree 2, as claimed.

4.4 Hilbert Functions in Degree 3

The purpose of this section is to compute the Hilbert function of the defining ideal $I(X)$ and the candidate ideal $\Lambda$ in degree three. In particular, we will show that their Hilbert functions are equal in degree 3. Computing the Hilbert function of $I(X)$ in degree 3 is very similar to the degree 2 computation. Computing the Hilbert function of $\Lambda$ in degree 3 is considerably more complicated than the degree 2 computation. This will be done in two steps. First, we count the degree three multiples of the degree two elements of the sets $L_0, L_1, L_2$ defined in Data 4.2. This must be done carefully to avoid double counting. Second, we calculate the new elements which arise as a part of a minimal Gröbner basis of $\Lambda$.

4.4.1 The Hilbert Function of $I(X)$

We will begin by calculating the Hilbert function of $I(X)$ in degree 3.

**Proposition 4.2.** Adopt Data 4.2. The Hilbert function of $I(X)$ in degree 3 is

$$HF_{I(X)}(3) = \frac{14d^6 + 30d^5 - 40d^4 - 210d^3 - 334d^2 - 180d}{6!}.$$ 

**Proof.** Since $\mathcal{F}(I) = W/I(X)$, we have $HF_{I(X)}(3) = HF_W(3) - HF_{\mathcal{F}(I)}(3)$. Indeed, $W$ is a polynomial ring in $\binom{d+1}{2} - 1$ variables. Therefore

$$HF(W,3) = \binom{3 + \binom{d+1}{2}}{3} - 2 = \binom{\binom{d+1}{2}}{3} + 1 = \binom{d+2}{4} \binom{\binom{d+1}{2}}{2} + 1.$$ 

43
Since $I^3 = m^6$ we have

$$HF_{\mathcal{F}(I)}(3) = \dim_k \left( \frac{I^3}{m I^3} \right) = \mu(I^3) = \mu(m^6) = HF_k(6) = \binom{d+5}{6}. $$

This allows us to conclude the claimed equality

$$HF_{I(X)}(3) = HF_W(3) - HF_{\mathcal{F}(I)}(3)$$

$$= \binom{d+2}{4} \left( \binom{d+1}{2} + 1 \right) - \binom{d+5}{6}$$

$$= \frac{14d^6 + 30d^5 - 40d^4 - 210d^3 - 334d^2 - 180d}{6!}.$$

4.4.2 The Hilbert Function of $\Lambda$, Multiples

We will begin the computation for the Hilbert function of $\Lambda$ in degree 3 by finding the number of unique degree three multiples of the elements of $L_0, L_1, L_2$ defined in Data 4.2. We must pay special care to avoid double counting. To this end, we will begin with a considerable number of definitions and examples illustrating them.

**Data 4.3. Adopt Data 4.2**

- Let $L = L_0 \cup L_1 \cup L_2$.
- Let $S_{i,j} = \{ w_{k,l} \mid w_{k,l}w_{i,j} \in L \text{ and } w_{k,l} \preceq w_{i,j} \}$.
- Let $s_{i,j} = |S_{i,j}|$.

We will use the sets $S_{i,j}$ as a tool to avoid double counting. Although the sets $S_{i,j}$ cannot partition $L$, as they are not even subsets of $L$, they play a similar role. Indeed, the closely related sets $S'_{i,j} = \{ w_{i,j}w_{k,l} \mid w_{k,l} \in S_{i,j} \}$ would be a true partition of $L$. 44
Example 4.5. Let $d = 4$. Then,

\[ L_0 = \{w_{1,3}w_{2,4}, w_{2,3}w_{1,4}\} \]
\[ L_1 = \{w_{2,4}w_{3,4}, w_{1,4}w_{3,4}, w_{1,4}w_{2,4}, w_{3,3}w_{2,4}, w_{2,3}w_{2,4}, w_{3,3}w_{1,4}\} \]
\[ L_2 = \{w_{2,4}^2, w_{3,4}^2\} \]
\[ S_{3,4} = \{w_{3,4}, w_{2,4}, w_{1,4}\} \]
\[ S_{2,4} = \{w_{2,4}, w_{1,4}, w_{3,3}, w_{2,3}, w_{1,3}\} \]
\[ S_{1,4} = \{w_{3,3}, w_{1,4}\} \]

In particular, $s_{3,4} = 3$, $s_{2,4} = 5$ and $s_{1,4} = 2$ and all other $s_{i,j} = 0$.

Criteria 4.2. Since elements of $L$ arise from antidiagonals of the matrix $M$, if $w_{k,l} \in S_{i,j}$ then $w_{i,j}w_{k,l}$ is an antidiagonal of $M$.

Notice that this criteria is not sufficient for inclusion in $L$. Since generators of $\Lambda_1, \Lambda_2$ are obtained by linear combinations of minors, there are some antidiagonals of $M$ which will never appear in $L$. We will now define some disjoint subsets of $K$ along with a well ordering on them.

Data 4.4. Adopt Data 4.3

- Let $K = \{w_{i,j} f \mid f \in L\}$.
- Let $K_{(i,j)}^{(k,l)} = \{w_{s,t}w_{k,l}w_{i,j} \mid w_{k,l} \in S_{i,j}\}$.
- Let $\tau$ be an ordering of sets $K_{(i,j)}^{(k,l)}$ so that $K_{(i,j)}^{(k_1,l_1)} >_{\tau} K_{(i,j)}^{(k_2,l_2)}$ provided one of the following hold
  \begin{itemize}
  \item $w_{i_1,j_1} >_{\omega} w_{i_2,j_2}$
  \item $w_{i_1,j_1} =_{\omega} w_{i_2,j_2}$ and $w_{k_1,l_1} >_{\omega} w_{k_2,l_2}$
  \end{itemize}
- Let $K_{(i,j)}^{(k,l)} = K_{(i,j)}^{(k,l)} \setminus \{w_{s,t}w_{k,l}w_{i,j} \mid w_{s,t}w_{k,l}w_{i,j} \in K_{(i,j)}^{(k',l')} \text{ and } K_{(i,j)}^{(k',l')} >_{\tau} K_{(i,j)}^{(k,l)}\}$.
- Let $k_{(i,j)}^{(k,l)} = |K_{(i,j)}^{(k,l)}|$. 

45
• Let $\tau$ be an ordering of sets $K_{(i,j)}^{(k,l)}$ so that $K_{(i_1,j_1)}^{(k_1,l_1)} \geq_{\tau} K_{(i_2,j_2)}^{(k_2,l_2)}$ precisely when $K_{(i_1,j_1)}^{(k_1,l_1)} >_{\tau} K_{(i_2,j_2)}^{(k_2,l_2)}$.

• Let $L_i = K_{(i,j)}^{(k,l)}$ such that $K_{(i,j)}^{(k,l)} \geq_{\tau} K_{(i,j)}^{(k',l')}$ for all $k',l'$ with $w_{k'l'} \in S_{i,j}$.

• Call the sets $L_{i,j}$ the distinguished $K$-subsets.

• Let $l_{i,j} = |L_{i,j}|$.

Notice that the sets $K_{(i,j)}^{(k,l)}$ are disjoint and contained in $K$. In particular $|K| = \sum k_{(i,j)}^{(k,l)}$, we will compute this number.

Example 4.6. Let $d = 4$, then

| TABLE 4.6 |
| SUBSETS OF $K$ WHEN $d = 4$ |

<table>
<thead>
<tr>
<th>Generator $f \in L$</th>
<th>$k, l$</th>
<th>$i, j$</th>
<th>$w_{s,t}$ such that $w_{s,t} f \in K_{(i,j)}^{(k,l)}$</th>
<th>$k_{(i,j)}^{(k,l)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{3,4}^2$</td>
<td>3, 4</td>
<td>3, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}, w_{4,3}$</td>
<td>9</td>
</tr>
<tr>
<td>$w_{2,4} w_{3,4}$</td>
<td>2, 4</td>
<td>3, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}, w_{4,3}$</td>
<td>8</td>
</tr>
<tr>
<td>$w_{1,4} w_{3,4}$</td>
<td>1, 4</td>
<td>3, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}, w_{4,3}$</td>
<td>7</td>
</tr>
<tr>
<td>$w_{2,4}^2$</td>
<td>2, 4</td>
<td>2, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}, w_{4,3}$</td>
<td>8</td>
</tr>
<tr>
<td>$w_{1,4} w_{2,4}$</td>
<td>1, 4</td>
<td>2, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}, w_{4,3}$</td>
<td>7</td>
</tr>
<tr>
<td>$w_{3,3} w_{2,4}$</td>
<td>3, 3</td>
<td>2, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}$</td>
<td>6</td>
</tr>
<tr>
<td>$w_{2,3} w_{2,4}$</td>
<td>2, 3</td>
<td>2, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}$</td>
<td>5</td>
</tr>
<tr>
<td>$w_{1,3} w_{2,4}$</td>
<td>1, 3</td>
<td>2, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}$</td>
<td>4</td>
</tr>
<tr>
<td>$w_{3,3} w_{1,4}$</td>
<td>3, 3</td>
<td>1, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}$</td>
<td>7</td>
</tr>
<tr>
<td>$w_{2,3} w_{1,4}$</td>
<td>2, 3</td>
<td>1, 4</td>
<td>$w_{1,1}, w_{1,2}, w_{2,2}, w_{1,3}, w_{2,3}, w_{3,3}$</td>
<td>6</td>
</tr>
</tbody>
</table>
The distinguished \( K \)-subsets are \( L_{3,4} = K_{\{3,4\}} \), \( L_{2,4} = K_{\{2,4\}} \), and \( L_{1,4} = K_{\{3,3\}} \). Additionally \( l_{3,4} = 9 \), \( l_{2,4} = 8 \) and \( l_{1,4} = 7 \) and all other \( l_{i,j} = 0 \). In particular, \( 67 = |K| = \sum_{i,j,k,l} k_{\{i,j\}} \).

Now that we have some notation, we may begin to compute \( |K| \). Most of the work is to describe \( l_{i,j} \) and \( s_{i,j} \) properly. We will then show that if we fix \( i, j \) and let \( T_{i,j} = \sum_{k,l} k_{\{i,j\}} \) can be expressed as consecutive sum of \( s_{i,j} \) integers, the largest of which is \( l_{i,j} \). That is, \( T_{i,j} = l_{i,j} + (l_{i,j} - 1) + \cdots + (l_{i,j} - s_{i,j} + 1) = \frac{1}{2}(s_{i,j})(l_{i,j} + (l_{i,j} - s_{i,j} + 1)) = \frac{1}{2}(s_{i,j})(2l_{i,j} - s_{i,j} + 1) \).

See Example 4.6 above for intuition.

We will now describe the sets \( S_{i,j} \) and compute \( s_{i,j} \).

**Lemma 4.8.** Adopt Data 4.3 The sets \( S_{1,2}, S_{1,3}, S_{2,3} \) are empty. In particular, \( s_{1,2} = s_{1,3} = s_{2,3} = 0 \).

**Proof.** Notice by Criteria 4.2 if \( w_{k,l} \in S_{i,j} \) then \( w_{i,j}w_{k,l} \) is an antidiagonal of the matrix. Further, if \( w_{k,l} \in S_{i,j} \), then \( w_{k,l} \preceq w_{i,j} \). We may assume without loss of generality that \( k \leq l \).

We will show that any degree two monomials of \( W \) which satisfy both of these properties cannot be in \( L \). To show this, we use Remark 4.4 to show that any time \( w_{i,j}w_{k,l} \) is a candidate it always appears with a larger monomial.

**Case 1:** We will show \( S_{1,2} \) is empty. Antidiagonals of the matrix involving \( w_{1,2} \) are of the form \( w_{1,2}w_{1,j} \) for \( j \geq 2 \). Since only \( w_{1,2} \preceq w_{1,2} \), we have \( S_{1,2} \subseteq \{ w_{1,2} \} \).

According to Remark 4.4, if \( w_{1,2}^2 \in L \) it must be in \( L_2 \). That is, it is the leading monomial of a generator of \( \Lambda_2 \). Again by Remark 4.4, any time \( w_{1,2}^2 \) is a term of a generator of \( \Lambda_2 \), it appears together with the larger monomial \( w_{s,t}^2 \) with \( 2 < s < t \leq d \). Thus \( w_{1,2}^2 \notin L \) and \( S_{1,2} = \emptyset \).

**Case 2:** We will show \( S_{1,3} \) is empty. Antidiagonals of the matrix involving \( w_{1,3} \) are of the form \( w_{1,3}w_{j,k} \) with \( j = 1, 2 \) and \( k \geq 2 \). Since only \( w_{1,2}, w_{2,2}, w_{1,3} \preceq w_{1,3} \), we have \( S_{1,3} \subseteq \{ w_{1,2}, w_{2,2}, w_{1,3} \} \).

According to Remark 4.4, if \( w_{1,2}w_{1,3} \in L \) it must be in \( L_1 \). However, any time \( w_{1,2}w_{1,3} \) is a term of a generator of \( \Lambda_1 \), it appears together with the larger monomial \( w_{2,k}w_{3,k} \) with \( k > 3 \). Thus \( w_{1,2}w_{1,3} \notin L \).
Also according to Remark 4.4, if $w_{2,2}w_{1,3} \in L$ it must be in $L_1$. However, any time $w_{2,2}w_{1,3}$ is a term of a generator of $\Lambda_1$, it appears together with the larger monomial $w_{1,k}w_{3,k}$ with $k > 3$. Thus $w_{2,2}w_{1,3} \notin L$.

Also according to Remark 4.4, if $w^2_{1,3} \in L$ it must be in $L_2$. However, any time $w^2_{1,3}$ is a term of a generator of $\Lambda_2$, it appears together with the larger monomial $w^2_{s,t}$ with $1 < s \leq t$ and $3 < t \leq d$. Thus $w^2_{1,3} \notin L$ and $S_{1,3} = \emptyset$.

**Case 3:** We will show $S_{2,3}$ is empty. Antidiagonals of the matrix involving $w_{2,3}$ are of the form $w_{2,3}w_{j,k}$ with $j = 1, 2$ and $k \geq 3$. Since only $w_{1,3}, w_{2,3} \leq w_{2,3}$ we have $S_{2,3} \subseteq \{w_{1,3}, w_{2,3}\}$.

According to Remark 4.4, if $w_{1,3}w_{2,3} \in L$ it must be in $L_1$. However, any time $w_{1,3}w_{2,3}$ is a term of a generator of $\Lambda_1$ it appears together with the larger monomial $w_{1,k}w_{2,k}$ with $k > 3$. Thus $w_{1,3}w_{2,3} \notin L$.

Also according to Remark 4.4, if $w^2_{2,3} \in L$ it must be in $L_2$. However, any time $w^2_{2,3}$ is a term of a generator of $\Lambda_2$, it appears together with the larger monomial $w^2_{s,t}$ with $s \leq t$ and $3 < t \leq d$. Thus $w^2_{2,3} \notin L$ and $S_{2,3} = \emptyset$.

**Lemma 4.9.** Adopt Data 4.3. The sets $S_{i,i}$ are empty. In particular, $s_{i,i} = 0$.

**Proof.** Notice by Criteria 4.2, if $w_{k,l} \in S_{i,j}$ then $w_{i,j}w_{k,l}$ is an antidiagonal of the matrix. Additionally, if $w_{k,l} \in S_{i,j}$ then $w_{k,l} \leq w_{i,j}$. We will show that there are no candidates satisfying both properties.

**Case 1:** Since $w_{1,1}$ is never a part of an antidiagonal, $S_{1,1} = \emptyset$.

**Case 2:** Let $i > 1$. Antidiagonals of the matrix involving $w_{i,j}$ are of the form $w_{i,i}w_{k,l}$ for $k < i < l$. Since $l > i$, we have $w_{k,l} \geq w_{i,i}$ and $S_{i,i} = \emptyset$.

For the next two Propositions, we will describe elements $w_{k,l} \in S_{i,j}$. We will assume, without loss of generality that $k \leq l$.

**Proposition 4.3.** Adopt Data 4.3. For $j \geq 4$, we have $w_{k,l} \in S_{1,j}$ if and only if $2 \leq k \leq l < j$.
and $2 < l$. In particular,

$$s_{1,j} = \frac{j(j-3)}{2}.$$

**Proof.** We will use the following figure to represent the locations of the variables $w_{k,l} = w_{l,k}$ in the matrix $M$. Indeed, we claim that $w_{k,l} \in S_{1,j}$ if and only if they are located in a position which is shaded blue. The location of $w_{1,j}$ is indicated by a circle.

![Figure 4.1](image.png)

Figure 4.1. $S_{1,j}$ Visualization

$(\implies)$ We will show that if $w_{k,l} \in S_{1,j}$ then $w_{k,l}$ must be in a position shaded with blue. By Criteria 4.2, if $w_{k,l} \in S_{1,j}$ then $w_{k,l}w_{1,j}$ is an antidiagonal. Thus, the variable marked with an “A” cannot be in $S_{1,j}$. Since variables located in positions marked with a “B” are larger than $w_{1,j}$ by the $\omega$ ordering, they also cannot be in $S_{1,j}$. Variables located in positions marked with a “C_i” satisfy Criteria 4.2 and are less than or equal to $w_{1,j}$ in the $\omega$ ordering. We will show that if $w_{k,l}$ is labeled with a “C_i” then $w_{1,j}w_{k,l} \not\in L$ and hence $w_{k,l} \not\in S_{1,j}$. To do this, we use remark 4.4 to show that any candidate $w_{1,j}w_{k,l}$ must always appear in a generator of $\Lambda$ with a larger monomial.
Variables located in positions marked with a “C1” are \( w_{1,k} \) for \( 1 < k < j \). According to Remark 4.4, if \( w_{1,k} w_{1,j} \in L \) it must be in \( L_1 \). That is, it is the leading monomial of a generator of \( \Lambda_1 \). However, any time \( w_{1,k} w_{1,j} \) is a term of a generator of \( \Lambda_1 \), it appears together with the larger monomial \( w_{l,k} w_{l,j} \) with \( l > 1 \). Thus \( w_{1,k} w_{1,j} \notin L \) and \( w_{1,k} \notin S_{1,j} \).

The only variable located in a position marked with a “C2” is \( w_{2,2} \). According to Remark 4.4, if \( w_{2,2} w_{1,j} \in L \) it must be in \( L_1 \). However, any time \( w_{2,2} w_{1,j} \) is a term of a generator of \( \Lambda_1 \), it appears together with the larger monomial \( w_{l,1} w_{l,j} \) with \( l > 2 \). Thus \( w_{2,2} w_{1,j} \notin L \) and \( w_{2,2} \notin S_{1,j} \).

The only variable located in a position marked with a “C3” is \( w_{1,j} \). According to Remark 4.4, if \( w_{1,1} w_{1,j} \in L \) it must be in \( L_2 \). However, any time \( w_{1,1} w_{1,j} \) is a term of a generator of \( \Lambda_2 \), it appears together with the larger monomial \( w_{s,1} w_{s,j} \) with \( 1 < s \). Thus \( w_{1,1} w_{1,j} \notin L \) and \( w_{1,j} \notin S_{1,j} \).

Therefore \( S_{1,j} \) is contained in the set of variables located in the blue positions. That is, if \( w_{k,l} \in S_{1,j} \) then \( 2 \leq k \leq l < j \) and \( 2 < l \).

(\( \iff \)) Now we must show that every variable located in a blue position is in \( S_{1,j} \). Let \( w_{k,l} \) be in a blue position with \( k \leq l \). Notice that \( w_{k,l} \preceq w_{1,j} \). To show \( w_{k,l} \in S_{1,j} \) we will find a generator of \( \Lambda \) with \( w_{k,l} w_{1,j} \) as its leading monomial. To do this we will break into cases: \( k \neq l \) and \( k = l \).

**Case 1:** Let \( 2 \leq k < l < j \). Let \( f \) be the element in \( \Lambda_0 \) obtained by selecting rows \( k \) and \( j \) and columns 1 and \( l \) from \( M \). Then we have \( f = \det \begin{bmatrix} w_{1,k} & w_{k,l} \\ w_{1,j} & w_{j,l} \end{bmatrix} = w_{1,k} w_{j,l} - w_{k,l} w_{1,j} \). Therefore \( w_{k,l} w_{1,j} \in L \) and \( w_{k,l} \in S_{1,j} \).

**Case 2:** Let \( 2 < k = l < j \). Let \( g \) be the element in \( \Lambda_1 \) obtained by selecting the 4 × 4 principal submatrix \( M_{1,2,k,j} \) and then choosing the complementary submatrices in red and orange as follows.
Then we have \( g = w_{1,k}w_{k,j} - w_{k,k}w_{1,j} - w_{1,2}w_{2,j} + w_{2,2}w_{1,j} \). Therefore \( w_{k,k}w_{1,j} \in L \) and \( w_{k,k} \in S_{1,j} \).

Thus \( S_{1,j} \) is exactly the set of variables located in blue positions, that is \( w_{k,l} \in S_{1,j} \) if and only if \( 2 \leq k, l < j \) and \( 2 < l \).

Finally, we count the size of the set \( S_{1,j} \). Notice that the variables in the blue positions that are labeled with a “D” are also located below the diagonal since \( w_{k,l} = w_{l,k} \). Therefore we ignore those positions to avoid double counting. The only blue positions not labeled with a “D” are in rows \( r \) where \( 2 < r < j \). For those rows there are exactly \( r - 1 \) variables in said positions. Thus we have \( s_{1,j} = 2 + 3 + \cdots + j - 2 = \frac{j(j-3)}{2} \), as claimed.

\[ \square \]

**Proposition 4.4.** Adopt Data 4.3. For the values \( 1 < i < j \leq d \) and \( j \geq 4 \), we have \( w_{k,l} \in S_{i,j} \) with \( k \leq l \) if and only if either \( i < l < j \) or \( l = j \) and \( k \leq i \). In particular

\[ s_{i,j} = \frac{(j-i)(i+j-1)}{2}. \]

**Proof.** We will use the following figure to represent the locations of the variables \( w_{k,l} = w_{l,k} \) in the matrix \( M \). Indeed, we claim that \( w_{k,l} \in S_{i,j} \) if and only if they are located in a position which is shaded blue. The location of \( w_{i,j} \) is indicated by a circle.
(⇒) We will show that if \( w_{k,l} \in S_{i,j} \) then \( w_{k,l} \) is in a shaded position. By Criteria 4.2, if \( w_{k,l} \in S_{i,j} \) then \( w_{k,l} w_{i,j} \) is an antidiagonal. Thus variables located in positions marked with an “A” cannot be in \( S_{i,j} \). Variables in a position labeled with a “B” are larger than \( w_{i,j} \) by the \( \omega \) ordering and also cannot be in \( S_{i,j} \). Therefore \( S_{i,j} \) is contained in the set of variables located in the shaded positions. That is, if \( w_{k,l} \in S_{i,j} \) with \( k \leq l \) then \( i < l < j \) or \( [l = j \text{ and } k \leq i] \).

(⇐) Now we will show that every variable located in a shaded position is in \( S_{i,j} \). Let \( w_{k,l} \) be in a shaded position with \( k \leq l \). Indeed \( w_{k,l} \leq \omega w_{i,j} \). To show \( w_{k,l} \in S_{i,j} \) we will find a generator of \( \Lambda \) with \( w_{k,l} w_{i,j} \) as its leading monomial. To do this we will break into cases. The first three cases account for when \( i < l < j \) and the last three cases account for when \( [l = j \text{ and } k \leq i] \).

**Case 1:** Let \( i < l < j \) and \( k = l \). Let \( g_1 \) be the element in \( \Lambda_1 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,i,l,j} \) and choosing the complementary minors in red and orange as follows.
Then we have $g_1 = w_{1,1}w_{i,j} - w_{1,i}w_{1,j} - w_{l,j}w_{i,j} + w_{i,j}w_{l,j}$. Therefore $w_{i,j} = w_{k,l} \in S_{i,j}$.

**Case 2:** Let $i < l < j$ and $k = i$. Let $g_2$ be the element in $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $\mathcal{M}_{1,i,l,j}$ and then choosing the complementary minors in red and orange as follows.

\[
\begin{bmatrix}
  w_{1,1} & w_{1,i} & w_{1,l} & w_{1,j} \\
  w_{1,i} & w_{i,i} & w_{i,l} & w_{i,j} \\
  w_{1,l} & w_{i,l} & w_{l,l} & w_{l,j} \\
  w_{1,j} & w_{i,j} & w_{l,j} & w_{j,j}
\end{bmatrix}
\]

Then we have $g_2 = w_{1,1}w_{i,j} - w_{1,i}w_{1,j} - w_{i,j}w_{i,j} + w_{i,j}w_{l,j}$. Therefore $w_{i,j} = w_{k,l} \in S_{i,j}$.

**Case 3:** Let $i < l < j$ with $k \neq i$ and $k \neq l$. Let $f$ be the element in $\Lambda_0$ obtained by selecting rows $k$ and $j$ and columns $i$ and $l$. Then we have $f = \det \begin{bmatrix} w_{i,k} & w_{k,l} \\ w_{i,j} & w_{j,l} \end{bmatrix} = w_{i,k}w_{j,l} - w_{k,j}w_{i,j}$. Therefore $w_{k,l} \in S_{i,j}$.

**Case 4:** Let $j = l$ and $k < i < j$. Let $a = \min \{\{1, 2, \ldots, j - 1\} \setminus \{i, k\}\}$. Let $g_3$ be the element in $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $\mathcal{M}_{a,i,k,j}$ and then choosing the complementary minors in red and orange as follows.

\[
\begin{bmatrix}
  w_{a,a} & w_{a,k} & w_{m,i} & w_{a,j} \\
  w_{a,k} & w_{k,k} & w_{k,i} & w_{k,j} \\
  w_{a,i} & w_{k,i} & w_{i,i} & w_{i,j} \\
  w_{a,j} & w_{k,j} & w_{i,j} & w_{j,j}
\end{bmatrix}
\]

Then we have $g_3 = w_{a,a}w_{k,i} - w_{a,k}w_{a,k} \pm (w_{i,j}w_{k,j} - w_{k,j}w_{i,j})$. Therefore $w_{k,j} = w_{k,l} \in S_{i,j}$.

**Case 5:** Let $l = j$ and $k = i$ with $i > 2$. Let $h_1$ be the element in $\Lambda_2$ obtained by selecting the $4 \times 4$ principal submatrix $\mathcal{M}_{1,2,i,j}$ and then choosing the pair of complementary minors
in red, orange, green and purple as follows.

\[
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,i} & w_{1,j} \\
w_{1,2} & w_{2,2} & w_{2,i} & w_{2,j} \\
w_{1,i} & w_{2,i} & w_{i,i} & w_{i,j} \\
w_{1,j} & w_{2,j} & w_{i,j} & w_{j,j}
\end{bmatrix}
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,i} & w_{1,j} \\
w_{1,2} & w_{2,2} & w_{2,i} & w_{2,j} \\
w_{1,i} & w_{2,i} & w_{i,i} & w_{i,j} \\
w_{1,j} & w_{2,j} & w_{i,j} & w_{j,j}
\end{bmatrix}
\]

Then we have \( h_1 = w_{i,j}^2 + w_{1,2}^2 - w_{1,j}^2 = w_{1,2} w_{i,j} + w_{1,1} w_{j,j} + w_{1,1} w_{j,j} - w_{1,2} w_{j,j} \). Therefore \( w_{i,j} = w_{k,l} \in S_{i,j} \).

**Case 6:** Let \( l = j \) and \( k = i = 2 \). Let \( h_2 \) be the element in \( \Lambda_2 \) obtained by selecting the 4 \( \times \) 4 principal submatrix \( M_{1,2,3,j} \) and then choosing the pair of complementary minors in red, orange, green and purple as follows.

\[
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,j} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,j} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,j} \\
w_{1,j} & w_{2,j} & w_{3,j} & w_{j,j}
\end{bmatrix}
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,j} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,j} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,j} \\
w_{1,j} & w_{2,j} & w_{3,j} & w_{j,j}
\end{bmatrix}
\]

Then, \( h_2 = w_{2,j}^2 + w_{1,3}^2 - w_{1,j}^2 = w_{1,2} w_{3,3} + w_{2,3} w_{3,3} - w_{2,3} w_{j,j} + w_{1,1} w_{j,j} \). Therefore \( w_{2,j} = w_{k,l} \in S_{i,j} \).

Thus \( S_{i,j} \) is exactly the set of variables located in shaded positions, that is \( w_{k,l} \in S_{i,j} \) with \( k \leq l \) if and only if either \( i < l < j \) or \( l = j \) and \( k \leq i \).

Finally, we count the size of the set \( S_{i,j} \). Notice that the variables in the shaded positions that are labeled with a “D” are also located below the diagonal since \( w_{k,l} = w_{l,k} \). We ignore them to avoid double counting. Notice that there are only shaded positions not labeled with a “D” in rows \( r \) where \( i < r \leq j \). In row \( r = j \) there are \( i \) such variables. For rows \( i < r < j \) there are exactly \( r \) such variables. Thus there are \( (i+1) + \cdots + (j-1) + i \) elements of \( S_{i,j} \). Therefore \( s_{i,j} = i + (i+1) + \cdots + (j-1) = \frac{(j-i)(i+j-1)}{2} \), as claimed.
Now that we have counted $s_{i,j}$ we must also count $l_{i,j}$. Refer to Data 4.4 for definitions. To count $l_{i,j}$ we will describe elements of $K_{(i,j)}$ and in particular $L_{i,j}$. We also notice that for any fixed $i, j$ the integers $k_{(i,j)}$ are consecutive. If $S_{i,j} = \varnothing$, then in particular $l_{i,j} = 0$, so we focus on computing $l_{i,j}$ in the cases where they are nonzero.

**Lemma 4.10.** Adopt Data 4.4. For $j \geq 4$ the size of the set $L_{1,j}$ is

$$l_{1,j} = \frac{d^2 - 2dj + 3d + 2j^2 - 4j + 2}{2}.$$  

**Proof.** We will use the following figure to represent the locations of the variables $w_{k,l}$ in the matrix $M$. Notice that the location of $w_{1,j}$ is indicated by a circle and the blue shaded elements are exactly those in $S_{1,j}$.

![Figure 4.3. $L_{1,j}$ Visualization](image)

First we will show that the variables located in a position labeled with a “$B_1$” are too large to divide any element of $L_{1,j}$. Let $w_{m,n}$ be a variable in a position labeled with a “$B_1$”.

55
Refer to Figure 4.2 to see \( w_{1,j} \in S_{m,n} \). So any degree three multiple of \( w_{1,j}w_{m,n} \) will not be in \( L_{1,j} \) since \( L_{1,j} < K_{(1,j)} \). Therefore \( w_{m,n} \) will not divide any element of \( L_{1,j} \).

Let \( w_{1,n} \) be a variable in a position labeled with a “\( B_2 \)”. To see \( S_{1,j} \in S_{1,n} \) notice that the blue region in Figure 4.3 will simply be stretched. Suppose \( w_{k,l}w_{1,n} \in K \). Then it is in some set \( K_{\{*,*,\}j} \). However, it will not be in \( K_{\{k,l\}1,j} \) since \( K_{\{k,l\}1,j} < K_{\{1,n\}} \). Therefore \( w_{1,n} \) will not divide any element of \( L_{1,j} \).

We will now show that variables in the shaded positions divide an element of \( L_{1,j} \). We show that for any \( w_{k,l} \in S_{1,j} \) and \( w_{s,t} \) in a yellow or green position \( w_{k,l}w_{s,t}w_{1,j} \in K_{\{k,l\}} \).

In particular, for the maximal \( w_{k,l} \in S_{1,j} \) with respect to \( \omega \) and \( w_{s,t} \) in a yellow or green position \( w_{k,l}w_{s,t}w_{1,j} \in K_{\{k,l\}} = L_{1,j} \). Further, for the maximal \( w_{k,l} \in S_{1,j} \) with respect to \( \omega \) and \( w_{s,t} \) in a blue position \( w_{k,l}w_{s,t}w_{1,j} \in K_{\{k,l\}} = L_{1,j} \).

Let \( w_{k,l} \in S_{1,j} \) and \( w_{s,t} \) be a variable in a yellow position. Notice \( w_{1,j}w_{k,l}w_{s,t} \in K \). To show that it is in \( K_{\{k,l\}} \) we must show that it is not in a larger \( K_{\{*,*,\}} \). Since \( w_{s,t} \) is never in an antidiagonal with \( w_{k,l} \) or \( w_{1,j} \), we have \( w_{1,j},w_{k,l} \notin S_{s,t} \). Thus \( w_{1,j}w_{k,l}w_{s,t} \notin K_{\{*,*,\}} \) and \( w_{k,l}w_{s,t}w_{1,j} \in K_{\{k,l\}} \).

Indeed, \( w_{s,t} \) divides an element of \( K_{\{k,l\}} \) for every \( w_{k,l} \in S_{1,j} \), including the maximal one. Thus \( w_{s,t} \) divides an element of \( L_{1,j} \).

Let \( w_{k,l} \in S_{1,j} \) and \( w_{s,t} \) be a variable in a green position. Notice \( w_{1,j}w_{k,l}w_{s,t} \in K \). Refer to figure 4.1 to see \( w_{s,t} \notin S_{1,j} \). Further, \( w_{1,j} \) is the largest of the three variables with respect to \( \omega \). Thus \( w_{1,j}w_{k,l}w_{s,t} \in K_{\{k,l\}} \). Indeed, \( w_{s,t} \) divides an element of \( K_{\{k,l\}} \) for every \( w_{k,l} \in S_{1,j} \), including the maximal one. Thus \( w_{s,t} \) divides an element of \( L_{1,j} \).

Let \( w_{k,l} \in S_{1,j} \) and \( w_{s,t} \) be a variable in a blue position. Notice \( w_{1,j}w_{k,l}w_{s,t} \in K \). Refer to figure 4.1 to see that variables in blue positions are exactly the variables in \( S_{1,j} \). Since both \( w_{k,l},w_{s,t} \in S_{1,j} \), the monomial \( w_{1,j}w_{k,l}w_{s,t} \in K_{\{k,l\}} \) if and only if \( w_{s,t} \leq \omega w_{k,l} \). If \( w_{k,l} \) is maximal, then in particular \( w_{s,t} \leq \omega w_{k,l} \) and \( w_{1,j}w_{k,l}w_{s,t} \in L_{1,j} \). Hence, \( w_{s,t} \) will always divide an element of \( L_{1,j} \).

We have shown that if \( w_{k,l} \in S_{1,j} \) is maximal and \( w_{s,t} \) is in a shaded position, then \( w_{1,j}w_{k,l}w_{s,t} \in L_{1,j} \). Further, if \( w_{s,t} \) is not in a shaded position then it is in a “\( B \)” position.
and does not divide any element of $L_{1,j}$. To compute $l_{1,j}$, we ignore everything above the diagonal to avoid double counting. Recall that $W$ has $\binom{d+1}{2} - 1$ variables. We eliminate all variables labeled with a “B”, in particular we eliminate a rectangle of size $(j-1)(d-j+1)$ except for one variable. Therefore, we have the claimed value

$$l_{1,j} = \left(\frac{d+1}{2}\right) - 1 - [(j-1)(d-j+1) - 1] = \frac{d^2 - 2dj + 3d + 2j^2 - 4j + 2}{2}.$$ 

\[ \square \]

**Corollary 4.1.** Adopt Data 4.4. If $K_{\{1,j\},\{k_1,l_1\}} >_\tau K_{\{1,j\},\{k_2,l_2\}}$ are consecutive, then

$$k_{\{k_2,l_2\},\{1,j\}} - 1 = k_{\{k_1,l_1\},\{1,j\}}.$$

**Proof.** First notice that if $w_{s,t}$ is in a yellow or green position of Figure 4.3, then $w_{s,t}w_{1,j}w_{k_a,l_a} \notin K_{\{1,j\},\{k_1,l_1\}}$ for $a = 1, 2$ from the proof of Theorem 4.10.

Let $w_{s,t}$ be in a blue position of Figure 4.3. Notice that if $w_{s,t} \leq \omega w_{k_1,l_1}$, then $w_{s,t}w_{1,j}w_{k_a,l_a} \in K_{\{1,j\},\{k_a,l_a\}}$ for $a = 1, 2$. Further if $w_{s,t} \geq \omega w_{k_2,l_2}$, then $w_{s,t}w_{1,j}w_{k_a,l_a} \notin K_{\{1,j\},\{k_a,l_a\}}$ for $a = 1, 2$. So the only case to consider is $w_{s,t} = w_{k_2,l_2}$. Indeed, $w_{1,j}w_{k_2,l_2} \in K_{\{1,j\},\{k_2,l_2\}}$ but $w_{1,j}w_{k_1,l_1}w_{k_2,l_2} \notin K_{\{1,j\},\{k_1,l_1\}}$ since it is in the larger set $K_{\{1,j\},\{k_1,l_1\}}$. Therefore

$$k_{\{k_2,l_2\},\{1,j\}} - 1 = k_{\{k_1,l_1\},\{1,j\}}.$$

\[ \square \]

**Lemma 4.11.** Adopt Data 4.4. For $j \geq 4$ and $1 < i < j$, the size of the set $L_{i,j}$ is

$$l_{i,j} = \frac{d^2 - 2dj + 3d + 2j^2 - 4j + 2i}{2}.$$

**Proof.** We will use the following figure to represent the locations of the variables $w_{k,l} = w_{l,k}$ in the matrix $M$. Notice that the location of $w_{i,j}$ is indicated by a circle and the blue shaded
elements are exactly those in $S_{i,j}$.

First we will show that the variables in the positions labeled with a “B” are too large to divide any element of $L_{i,j}$. Let $w_{s,t}$ be a variable in a position labeled with a “B1”. Refer to Figure 4.2 to see $S_{i,j} \subset S_{s,t}$. Suppose $w_{k,l}w_{i,j}w_{s,t} \in K$. However, it will not be in $K_{\{s,t\}}^{\{*,*\}}$ since $K_{\{s,t\}}^{\{*,*\}} <_\tau K_{\{i,j\}}^{\{*,*\}}$. Therefore $w_{s,t}$ will not divide any element of $L_{i,j}$. Now let $w_{1,t}$ be a variable in a position labeled with a “B2”. Refer to Figure 4.1 to see $w_{1,j} \in S_{1,t}$. So any monomial $w_{k,l}w_{i,j}w_{1,t}$ will be not be in $L_{i,j}$ since $K_{\{i,j\}}^{\{*,*\}} <_\tau K_{\{1,t\}}^{\{*,*\}}$. Therefore $w_{1,t}$ will not divide any element of $L_{i,j}$.

We will now show that variables in the shaded positions divide an element of $L_{i,j}$. We show that for any $w_{k,l} \in S_{i,j}$ and $w_{s,t}$ in a yellow or green position $w_{k,l}w_{s,t}w_{i,j} \in K_{\{i,j\}}^{\{k,l\}}$. In particular, for the maximal $w_{k,l} \in S_{i,j}$ with respect to $\omega$ and $w_{s,t}$ in a yellow or green position $w_{k,l}w_{s,t}w_{i,j} \in K_{\{i,j\}}^{\{k,l\}} = L_{i,j}$. Further, for the maximal $w_{k,l} \in S_{i,j}$ with respect to $\omega$ and $w_{s,t}$ in a blue position $w_{k,l}w_{s,t}w_{i,j} \in K_{\{i,j\}}^{\{k,l\}} = L_{i,j}$.
Let \( w_{k,l} \in S_{i,j} \) and \( w_{s,t} \) be a variable in a yellow position. Notice \( w_{i,j}w_{k,l}w_{s,t} \in K \). To show that it is in \( K_{i,j} \) we must show that it is not in a larger \( K_{i,j} \). Notice that \( w_{s,t} \) is never in an antidiagonal with \( w_{k,l} \) or \( w_{i,j} \). Thus \( w_{i,j}w_{k,l} \notin S_{s,t} \) and so \( w_{i,j}w_{k,l}w_{s,t} \notin K_{i,j} \). Indeed, \( w_{i,j}w_{k,l}w_{s,t} \in K_{i,j} \). Therefore \( w_{s,t} \) divides an element of \( K_{i,j} \) for every \( w_{k,l} \in S_{i,j} \), including the maximal one.

Let \( w_{k,l} \in S_{i,j} \) and \( w_{s,t} \) be a variable in a green position. Notice \( w_{i,j}w_{k,l}w_{s,t} \in K \). Additionally, \( w_{s,t} < \omega w_{k,l} \leq \omega w_{i,j} \) and therefore \( w_{i,j}w_{k,l}w_{s,t} \in K_{i,j} \). Therefore \( w_{s,t} \) divides an element of \( K_{i,j} \) for every \( w_{k,l} \in S_{i,j} \).

Let \( w_{k,l} \in S_{i,j} \) and \( w_{s,t} \) be a variable in a blue position. Notice \( w_{i,j}w_{k,l}w_{s,t} \in K \). Refer to figure 4.2 to see that these are the variables in \( S_{i,j} \). Indeed, \( w_{i,j}w_{k,l}w_{s,t} \in K_{i,j} \) if and only if \( w_{s,t} \leq \omega w_{k,l} \). If \( w_{k,l} \) is maximal, then in particular \( w_{s,t} \leq \omega w_{k,l} \) and \( w_{i,j}w_{k,l}w_{s,t} \in L_{i,j} \). Hence, \( w_{s,t} \) will always divide an element of \( L_{i,j} \).

We have shown that if \( w_{k,l} \in S_{i,j} \) is maximal, and \( w_{s,t} \) is in a shaded position, then \( w_{i,j}w_{k,l}w_{s,t} \in L_{i,j} \). Further, if \( w_{s,t} \) is not in a shaded position then it is in a “B” position and does not divide any element of \( L_{i,j} \). To compute \( l_{i,j} \), we ignore everything above the diagonal to avoid double counting. Recall that \( W \) has \( (d+1)/2 \) variables. We eliminate all variables labeled with a “B”, in particular we eliminate a rectangle of size \((j-1)(d-j+1)\) except for \( i \) variables. Therefore, we have the claimed value

\[
l_{i,j} = \left( \frac{d+1}{2} \right) - 1 - \left[ (j-1)(d-j+1)-i \right] = \frac{d^2 - 2dj + 3d + 2j^2 - 4j + 2i}{2}.
\]

\[
\square
\]

**Corollary 4.2.** Adopt Data 4.4 If \( K_{i,j}^{(k_2,l_2)} \succ K_{i,j}^{(k_1,l_1)} \) are consecutive, then

\[
k_{i,j}^{(k_2,l_2)} - 1 = k_{i,j}^{(k_1,l_1)}.
\]

**Proof.** First notice that if \( w_{s,t} \) is in a yellow or green position of Figure 4.3, then \( w_{s,t}w_{i,j}w_{k_0,l_0} \in

59
Adopt Data 4.4. The size of the set $K$ is Theorem 4.4. that integer.

Fix $k$ consecutive sum is $k$ such that the largest number in the consecutive sum is $k$. Indeed, $w_{i,j}w_{k_1,l_1}^2 \in K^{(k_1,l_1)}$ since it is in the larger set $K^{(k_2,l_2)}$. Therefore

$$k^{(k_2,l_2)}_{(i,j)} - 1 = k^{(k_1,l_1)}_{(i,j)}. \tag*{\square}$$

We are finally able to calculate the size of $K$. We have shown that for a fixed $i, j$, the values $k^{(k,l)}_{(i,j)}$ are consecutive integers and calculated the value of the largest such $k^{(k_1,l_1)}_{(i,j)} = l_{i,j}$. We also know that there are exactly $s_{i,j}$ many sets of the form $K^{(k,l)}_{(i,j)}$ and have computed that integer.

**Theorem 4.4.** Adopt Data 4.4 The size of the set $K$ is

$$|K| = \frac{14d^6 + 30d^5 - 40d^4 - 330d^3 - 694d^2 + 1740d - 4320}{6!}.$$

**Proof.** Fix $i, j$ and let $T_{i,j} = \sum k^{(k,l)}_{(i,j)}$. By Corollaries 4.1 and 4.2 we have that $k^{(k_2,l_2)}_{(i,j)} = k^{(k_1,l_1)}_{(i,j)} + 1$ for $w_{k_1,l_1} < w_{k_2,l_2}$ consecutive in $S_{i,j}$. Further, there are exactly $s_{i,j}$ many sets of the form $K^{(k,l)}_{(i,j)}$. The largest set of the form $K^{(k,l)}_{(i,j)}$ is $L_{i,j}$ and has cardinality $l_{i,j}$. So to compute the value $T_{i,j} = \sum k^{(k,l)}_{(i,j)}$, we must compute the consecutive sum of integers such that the largest number in the consecutive sum is $l_{i,j}$ and the smallest number in the consecutive sum is $l_{i,j} - s_{i,j} + 1$. That is,

$$T_{i,j} = \sum k^{(k,l)}_{(i,j)} = \frac{(s_{i,j})(l_{i,j} + (l_{i,j} - s_{i,j} + 1))}{2} = \frac{(s_{i,j})(2l_{i,j} - s_{i,j} + 1)}{2}.$$ 

The computations below are simply an algebraic manipulation of finite sums and some details will be omitted. To compute these, we treat the cases $T_{1,j}$ and $T_{i,j}$ for $i \neq 1$ separately.
Substituting in the values for $s_{1,j}, s_{i,j}, l_{1,j}, l_{i,j}$ in Lemmas 4.3, 4.4, 4.10 and 4.11 we see the following.

\[
T_{1,j} = \frac{\frac{j(j-3)}{2} \left[ 2\left( \frac{d^2 - 2d j + 3d + 2j^2 - 4j + 2}{2} \right) - \frac{j(j-3)}{2} + 1 \right]}{2} = \frac{j(j-3)\left[ 2(d^2 - 2dj + 3d + 2j^2 - 4j + 2) - j(j-3) + 2 \right]}{8}
\]

\[
T_{i,j} = \frac{\frac{(j-i)(i+j-1)}{2} \left[ 2\left( \frac{d^2 - 2d j + 3d + 2j^2 - 4j + 2}{2} \right) - \frac{(j-i)(i+j-1)}{2} + 1 \right]}{2} = \frac{(j-i)(i+j-1)\left[ 2(d^2 - 2dj + 3d + 2j^2 - 4j + 2i) - (j-i)(i+j-1) + 1 \right]}{8}
\]

\[
= \frac{(-2)^i}{8} \left[ \frac{(-2d^2 + 4d j - 6d + 4j)^2}{8} + \frac{(2d^2 - 4d j + 6d + 4j^2 - 8j + 2)i}{8} + \frac{(2d^2 j^2 - 2d^2 j - 4d j^3 + 10d j^2 - 6d j + 2j^4 - 8j^3 + 8j^2 - 2j)}{8} \right]
\]

Let $T_j = \sum_{i<j} T_{i,j}$. Then we have

\[
T_j = T_{1,j} + \sum_{i=2}^{j-1} T_{i,j}
\]

\[
= \left( \frac{32}{120} \right) j^5 + \left( \frac{-40d - 105}{120} \right) j^4 + \left( \frac{20d^2 + 240d + 70}{120} \right) j^3 + \left( \frac{-30d^2 + 10d + 45}{120} \right) j^2 + \left( \frac{-50d^2 - 150d - 162}{120} \right) j
\]

Finally, we see

\[
|K| = \sum T_j = \frac{14d^6 + 30d^5 - 40d^4 - 330d^3 - 694d^2 + 1740d - 4320}{6!}
\]
as claimed. \qed
4.4.3 The Hilbert Function of $\Lambda$, Gröbner Basis

The purpose of this subsection is to continue the computation of $HF_\Lambda(3)$. In the previous subsection, we computed the size of $K$. Recall that $K$ is the set of degree three monomials which are multiples of the leading monomials of generators of $\Lambda$. The goal of this subsection is to find degree three monomials in the initial ideal of $\Lambda$ which were not counted in $K$. We obtain these monomials from S-polynomials to ensure the leading monomial is in the initial ideal. After, we show that they do not appear in $K$. We define a set $G$ below to collect these monomials. The sets $G$ and $K$ will be disjoint so in particular $HF_\Lambda(3) \geq |G| + |K|$. Moreover, we will show that $HF_\Lambda(3) = |G| + |K| = HFI(X)(3)$. After proving this equality we will be able to conclude that the candidate $\Lambda$ is indeed the defining ideal of the special fiber ring of $I$.

**Data 4.5.** Adopt Data 4.4. Let $H$ be the degree three graded component of the initial ideal of $\Lambda$. Let $G = H \setminus K$ and let

\[
G_1 = \{w_{1,3}w_{2,3}^2, w_{1,3}^2w_{2,3}, w_{2,2}w_{1,3}w_{2,3}, w_{2,3}^3, w_{1,2}w_{1,d}^2, w_{2,2}w_{1,d}^2\}
\]

\[
G_2 = \{w_{1,i}w_{1,j}w_{1,k} \mid 3 \leq i \leq j \leq k \leq d\}
\]

\[
G_3 = \{w_{1,3}w_{2,3}w_{i,j} \mid 3 \leq i \leq j \leq d\}
\]

\[
G_4 = \{w_{2,3}^2w_{i,j} \mid 3 \leq i \leq j \leq d\}.
\]

be sets of degree three monomials of $W$.

Notice that the sets $G_i$ are disjoint. We will show that $G_i \subset G$ and hence $|G| \geq \sum |G_i|$. For readability, all leading monomials will once again be listed in blue. The general proof strategy for the containments $G_i \subset G$ is as follows. Let $w_{i,j}w_{k,l}w_{s,t} \in G_i$. Then construct an element $F \in \Lambda$, obtained by S-polynomials, such that the $\text{Im}(F) = w_{i,j}w_{k,l}w_{s,t}$. Finally, show $w_{i,j}w_{k,l}w_{s,t} \notin K$ and conclude $w_{i,j}w_{k,l}w_{s,t} \in G$.

We will now show that $G_1 \subset G$. We will break the proof into two lemmas. The first
lemma will show the containment of the first four elements in the definition above and the second lemma will show the containment of the last two elements.

**Lemma 4.12.** Adopt Data 4.5. The elements \(w_{1,3}w_{2,3}^2, w_{1,3}^2w_{2,3}, w_{2,2}w_{1,3}w_{2,3}, w_{2,3}^3\) can be obtained as the leading monomial of an element of a minimal Gröbner Basis of \(\Lambda\). That is,

\[\{w_{1,3}w_{2,3}^2, w_{1,3}^2w_{2,3}, w_{2,2}w_{1,3}w_{2,3}, w_{2,3}^3\} \subset G.\]

**Proof.** To begin, we list six generators of \(\Lambda\) that will be involved in the S-polynomials used to show the desired monomials are elements of \(G\).

Let \(f_1\) be the element of \(\Lambda_0\) obtained by selecting rows 1 and 3 and columns 2 and \(d\). Let \(f_2\) be the element of \(\Lambda_0\) obtained by selecting rows 1 and \(d\) and columns 2 and 3. More precisely,

\[
f_1 = \det\begin{bmatrix} w_{1,2} & w_{1,d} \\ w_{2,3} & w_{3,d} \end{bmatrix} = w_{1,2}w_{3,d} - w_{2,3}w_{1,d}
\]

\[
f_2 = \det\begin{bmatrix} w_{1,2} & w_{1,3} \\ w_{2,d} & w_{3,d} \end{bmatrix} = w_{1,2}w_{3,d} - w_{1,3}w_{2,d}.
\]

Let \(g_1, g_2, g_3\) be elements of \(\Lambda_1\) obtained by selecting the \(4 \times 4\) principal submatrix \(M_{1,2,3,d}\) and then choosing the complementary minors in red and orange as follows.
More precisely,

\[ g_1 = w_{1,2}w_{3,3} - w_{1,3}w_{2,3} + w_{1,d}w_{2,d} \]
\[ g_2 = w_{1,2}w_{2,3} - w_{2,2}w_{1,3} - w_{1,d}w_{3,d} \]
\[ g_3 = w_{1,1}w_{3,d} - w_{1,3}w_{1,d} + w_{2,2}w_{2,d} - w_{2,2}w_{3,d} \].

Let \( h \) be the element of \( \Lambda_2 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,3,d} \) and then choosing the pair of complementary minors in red, orange, green, and purple as follows.

\[
\begin{bmatrix}
  w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
  w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
  w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
  w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\begin{bmatrix}
  w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
  w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
  w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
  w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\]

More precisely,

\[ h = w_{2,d}^2 + w_{1,3}^2 - w_{1,d}^2 - w_{2,3}^2 - w_{1,1}w_{3,3} + w_{2,2}w_{3,3} \].

**Claim:** The monomial \( w_{1,3}w_{2,3}^2 \) is in \( G \).

The monomial \( w_{1,3}w_{2,3}^2 \) arises as the leading monomial of the S-polynomial between \( f_1 \) and \( g_1 \). Indeed,

\[ S(f_1,g_1) = w_{2,d}f_1 + w_{2,3}g_1 = w_{1,2}w_{2,d}w_{3,d} + w_{1,2}w_{2,3}w_{3,3} - w_{1,3}w_{2,3}^2. \]

To show \( w_{1,3}w_{2,3}^2 \in G \), we must show \( w_{1,3}w_{2,3}^2 \notin K \). That is, no degree two divisor is in \( L \), the set of leading monomials of generators of \( \Lambda \).

According to Remark 4.4, if \( w_{1,3}w_{2,3} \in L \) it must be in \( L_1 \). Again from Remark 4.4 any time \( w_{1,3}w_{2,3} \) is a term of a generator of \( \Lambda_1 \), it appears together with the larger monomial \( w_{1,k}w_{2,k} \) where \( k > 3 \). Thus \( w_{1,3}w_{2,3} \notin L \).
Again by Remark 4.4, if \( w_{2,3}^2 \in L \) it must be in \( L_2 \). However, any time \( w_{2,3}^2 \) is a term of a generator of \( \Lambda_2 \), it appears together with the larger monomial \( w_{i,j}^2 \) where \( j \geq 4 \). Thus \( w_{2,3}^2 \notin L \). Hence, \( w_{1,3}w_{2,3}^2 \notin K \). Therefore \( w_{1,3}w_{2,3}^2 \in G \), as claimed.

**Claim:** The monomial \( w_{1,3}^2w_{2,3} \) is in \( G \).

The monomial \( w_{1,3}^2w_{2,3} \) arises as the leading monomial of the S-polynomial between \( f_2 \) and \( g_1 \). Indeed,

\[
S(f_2, g_1) = w_{1,d}f_2 + w_{1,3}g_1 = w_{1,2}w_{1,d}w_{3,d} + w_{1,2}w_{1,3}w_{3,3} - w_{1,3}^2w_{2,3}.
\]

To show \( w_{1,3}^2w_{2,3} \in G \), we must show \( w_{1,3}^2w_{2,3} \notin K \). That is, no degree two divisor is in \( L \).

We saw above that \( w_{1,3}w_{2,3} \notin L \).

According to Remark 4.4, if \( w_{1,3}^2 \in L \) it must be in \( L_2 \). However, any time \( w_{1,3}^2 \) is a term of a generator of \( \Lambda_2 \), it appears together with the larger monomial \( w_{i,j}^2 \) where \( j \geq 4 \). Thus \( w_{1,3}^2 \notin L \). Hence \( w_{1,3}^2w_{2,3} \notin K \). Therefore \( w_{1,3}w_{2,3} \in G \), as claimed.

**Claim:** The monomial \( w_{2,2}w_{1,3}w_{2,3} \) is in \( G \).

The monomial \( w_{2,2}w_{1,3}w_{2,3} \) arises as the leading monomial of the S-polynomial between \( f_1 \) and \( g_2 \). Indeed,

\[
S(f_1, g_2) = w_{3,d}f_1 - w_{2,3}g_2 = w_{1,2}w_{3,d}^2 - w_{1,2}w_{2,3}^2 + w_{2,2}w_{1,3}w_{2,3}.
\]

To show \( w_{2,2}w_{1,3}w_{2,3} \in G \), we must show \( w_{2,2}w_{1,3}w_{2,3} \notin K \). We saw above that \( w_{1,3}w_{2,3} \notin L \).

By Criteria 4.2, since \( w_{2,2}w_{2,3} \) is not an antidiagonal, it is not in \( L \).

According to Remark 4.4, if \( w_{2,2}w_{1,3} \in L \) it must be in \( L_1 \). However, any time \( w_{2,2}w_{1,3} \) is a term of a generator of \( \Lambda_1 \), it appears together with the larger monomial \( w_{1,k}w_{3,k} \) where \( k \geq 4 \). Thus \( w_{2,2}w_{1,3} \notin L \). Hence \( w_{2,2}w_{1,3}w_{2,3} \notin K \). Therefore \( w_{2,2}w_{1,3}w_{2,3} \in G \), as claimed.

**Claim:** The monomial \( w_{2,3}^3 \) is in \( G \).

The monomial \( w_{2,3}^3 \) arises from the S-polynomial between \( g_3 \) and \( h \) after a reduction
with $f_1$. Let

$$F = S(g_3, h)$$

$$= w_{2,d}g_3 - w_{2,3}h$$

$$= w_{1,1}w_{2,d}w_{3,d} - w_{1,3}w_{1,d}w_{2,d} - w_{2,2}w_{2,d}w_{3,d} + w_{1,1}w_{2,3}w_{3,3} - w_{1,3}w_{2,3} + w_{2,3}w_{1,d}^2 + w_{2,3}^3.$$

Notice that the leading monomial of $F$ is $w_{2,3}w_{1,d}^2$ and the leading monomial of $f_1$ is $w_{2,3}w_{1,d}$. Now reduce $F$ using $f_1$. Then

$$S(F, f_1) = F + w_{1,d}f_1 = w_{1,1}w_{2,d}w_{3,d} - w_{1,3}w_{1,d}w_{2,d} - w_{2,2}w_{2,d}w_{3,d} + w_{1,1}w_{2,3}w_{3,3} - w_{2,2}w_{2,3}w_{3,3} - w_{1,3}w_{2,3} + w_{2,3}^3 + w_{1,2}w_{1,d}w_{3,d}.$$

To show $w_{2,3}^3 \in G$, we must show $w_{2,3}^3 \not\in K$. We saw above that $w_{2,3}^2 \not\in L$. Hence $w_{2,3}^3 \not\in K$. Therefore $w_{2,3}^3 \in G$, as claimed.

**Lemma 4.13.** Adopt Data 4.5. The elements $w_{1,2}w_{1,d}^2, w_{2,2}w_{1,d}^2$ can be obtained as the leading monomial of an element of a minimal Gröbner basis of $\Lambda$. That is,

$$\{w_{1,2}w_{1,d}^2, w_{2,2}w_{1,d}^2\} \subset G.$$

**Proof.** To begin, we list five generators of $\Lambda$ that will be involved in the S-polynomials used to show the desired monomials are elements of $G$.

Let $f$ be the element of $\Lambda_0$ obtained by selecting rows 1 and $d$ and columns 2 and 3.
More precisely,

$$f = \det \begin{bmatrix} w_{1,2} & w_{1,3} \\ w_{2,d} & w_{3,d} \end{bmatrix} = w_{1,2}w_{3,d} - w_{1,3}w_{2,d}. $$

Let $g_1, g_2, g_3$ be elements of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,d}$ and then choosing the complementary minors in red and orange as follows.

\[ g_1 : \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix}, \quad g_2 : \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix}, \quad g_3 : \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix} \]

More precisely,

\[ g_1 = w_{1,1}w_{2,3} - w_{1,2}w_{1,3} + w_{2,2}w_{3,d} \]
\[ g_2 = w_{1,1}w_{3,d} - w_{1,3}w_{1,d} + w_{2,3}w_{2,d} - w_{2,2}w_{3,d} \]
\[ g_3 = w_{1,2}w_{2,3} - w_{2,2}w_{1,3} - w_{1,d}w_{3,d}. \]

Let $h$ be the element of $\Lambda_2$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,d}$ and then choosing the pair of complementary minors in red, orange, green and purple as follows.

\[ \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix}, \quad \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix}, \quad \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\ w_{1,d} & w_{2,d} & w_{3,d} & 0 \end{bmatrix} \]

More precisely,

$$h = w_{3,d}^2 + w_{1,2}^2 - w_{1,d}^2 - w_{2,3}^2 - w_{1,1}w_{2,2} + w_{2,2}w_{3,3}. $$
Claim: The monomial $w_{1,2}w_{1,d}^2$ is in $G$.

The monomial $w_{1,2}w_{1,d}^2$ arises from the S-polynomial between $f$ and $g_1$ after a reduction with $h$. Let

$$F_1 = S(f, g_1) = w_{3,d}f + w_{1,3}g_1 = w_{1,2}w_{3,d}^2 + w_{1,1}w_{1,3}w_{2,3} - w_{1,2}w_{1,3}^2.$$

Notice that the leading monomial of $F_1$ is $w_{1,2}w_{3,d}^2$ and the leading monomial of $h$ is $w_{3,d}^2$.

Now reduce $F_1$ using $h$. Then

$$S(F_1, h) = w_{1,2}h - F_1$$

$$= w_{1,2} - w_{1,2}w_{1,d}^2 - w_{1,2}w_{2,3}^2 - w_{1,1}w_{1,2}w_{2,2}$$
$$+ w_{1,2}w_{2,2}w_{3,3} - w_{1,1}w_{1,3}w_{2,3} + w_{1,2}w_{1,d}^2.$$

To show $w_{1,2}w_{1,d}^2 \in G$, we must show $w_{1,2}w_{1,d}^2 \notin K$.

According to Remark 4.4 if $w_{1,2}w_{1,d} \in L$ it must be in $L_1$. However, any time $w_{1,2}w_{1,d}$ is a term of a generator of $A_1$ it appears together with the larger monomial $w_{2,k}w_{k,d}$ where $k > 2$. Thus $w_{1,2}w_{1,d} \notin L$.

Again by Remark 4.4 if $w_{1,d}^2 \in L$ it must be in $L_2$. However, any time $w_{1,d}^2$ is a term of a generator of $A_2$ it appears together with the larger monomial $w_{k,d}^2$ where $k > 1$. Thus $w_{1,d}^2 \notin L$. Hence $w_{1,2}w_{1,d}^2 \notin K$. Therefore $w_{1,2}w_{1,d}^2 \in G$, as claimed.

Claim: The monomial $w_{2,2}w_{1,d}^2$ is in $G$.

The monomial $w_{2,2}w_{1,d}^2$ arises from the S-polynomial between $g_1$ and $g_2$ after reductions with $g_3$ and $h$. Let

$$F_2 = S(g_1, g_2)$$
$$= w_{2,2}g_1 - w_{3,d}g_2$$
$$= w_{1,1}w_{2,2}w_{2,3} - w_{1,2}w_{2,2}w_{1,3} - w_{1,1}w_{3,d}^2 + w_{1,3}w_{1,d}w_{3,d} + w_{2,2}w_{3,d}^2.$$
Notice that the leading monomial of $F_2$ is $w_{1,3}w_{1,d}w_{3,d}$ and the leading monomial of $g_3$ is $w_{1,d}w_{3,d}$. Now reduce $F_2$ using $g_1$. Let

$$F_3 = S(F_2, g_3)$$

$$= F_2 + w_{1,3}g_3$$

$$= w_{1,1}w_{2,2}w_{2,3} - w_{1,2}w_{2,2}w_{1,3} - w_{1,1}w_{3,d}^2 + w_{2,2}w_{3,d}^2 + w_{1,2}w_{1,3}w_{2,3} - w_{2,2}w_{1,3}^2.$$

Notice that the leading monomial of $F_3$ is $w_{2,2}w_{3,d}^2$ and the leading monomial of $h$ is $w_{3,d}^2$. Now reduce $F_3$ using $h$.

Then

$$S(F_3, h) = F_3 - w_{2,2}h$$

$$= w_{1,1}w_{2,2}w_{2,3} - w_{1,2}w_{2,2}w_{1,3} - w_{1,1}w_{3,d}^2 + w_{1,2}w_{1,3}w_{2,3} - w_{2,2}w_{1,3}^2$$

$$- w_{1,2}^2w_{2,2} + w_{2,2}w_{1,d}^2 + w_{2,2}w_{2,3}^2 + w_{1,1}w_{2,2}^2 - w_{2,2}^2w_{3,3}.$$

To show $w_{2,2}w_{1,d}^2 \in G$, we must show $w_{2,2}w_{1,d}^2 \notin K$. We saw above that $w_{2,2}^2w_{1,d} \notin L$.

According to Remark 4.4 if $w_{2,2}w_{1,d} \in L$ it must be in $L_1$. However, any time $w_{2,2}w_{1,d}$ is a term of a generator of $\Lambda_1$ it appears together with the larger monomial $w_{1,k}w_{k,d}$ where $k > 2$. Thus $w_{2,2}w_{1,d} \notin L$. Hence $w_{2,2}w_{1,d} \notin K$. Therefore $w_{2,2}w_{1,d}^2 \in G$, as claimed.

We are now ready to conclude $G_1 \subset G$ and count its size.

**Proposition 4.5.** Adopt Data 4.4 4.5 There is an inclusion of sets

$$G_1 = \{ w_{1,3}w_{2,3}^2, w_{1,3}^2w_{2,3}, w_{2,2}w_{1,3}w_{2,3}, w_{2,2}^3, w_{1,2}w_{1,d}^2, w_{2,2}w_{1,d}^2 \} \subset G.$$

Further $|G_1| = 6$.

**Proof.** This follows directly from lemmas 4.12 and 4.13.

69
We will now show that

\[ G_2 = \{ w_{1,i}w_{1,j}w_{1,k} \mid 3 \leq i \leq j \leq k \leq d \} \subset G. \]

We will break the proof into four lemmas. Lemma 4.14 will account for the case \( w_{1,i}w_{1,j}w_{1,k} \) with \( i, j, k \) all distinct values as well as the case where \( i = j \) and \( k \) is something else. Lemma 4.15 will account for the case \( w_{1,i}w_{1,j}w_{1,k} \) where \( j = k \) and \( i \) is something else. Lemma 4.16 will account for the case \( w_{1,i}w_{1,j}w_{1,k} \) where \( i = j = k \) and \( i \neq 3 \). Finally, Lemma 4.17 will account for the case \( w_{1,i}w_{1,j}w_{1,k} \) where \( i = j = k = 3 \).

**Lemma 4.14.** Adopt Data 4.5. The elements \( w_{1,i}w_{1,j}w_{1,k} \) for \( 3 \leq i \leq j < k \leq d \) can be obtained as the leading monomial of an element of a minimal Gröbner basis of \( \Lambda \). That is

\[ \{ w_{1,i}w_{1,j}w_{1,k} \mid 3 \leq i \leq j < k \leq d \} \subset G. \]

**Proof.** To begin, we list two generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \).

Let \( f \) be the element of \( \Lambda_0 \) obtained by selecting rows 1 and \( k \) and columns 2 and \( i \). More precisely,

\[
    f = \det \begin{bmatrix} w_{1,2} & w_{1,i} \\ w_{2,k} & w_{i,k} \end{bmatrix} = w_{1,2}w_{i,k} - w_{1,i}w_{2,k}
\]

Let \( g \) be the element of \( \Lambda_1 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,j,k} \) and then choosing the complementary subminors in red and orange as follows.
More precisely,

\[ g = w_{1,1}w_{j,k} - w_{1,j}w_{1,k} + w_{2,j}w_{2,k} - w_{2,2}w_{j,k}. \]

**Claim:** The monomial \( w_{1,i}w_{1,j}w_{1,k} \) is in \( G \).

The monomial \( w_{1,i}w_{1,j}w_{1,k} \) is the leading monomial of the S-polynomial between \( f \) and \( g \). Indeed,

\[ S(f, g) = w_{2,j}f + w_{1,i}g = w_{1,2}w_{2,j}w_{i,k} + w_{1,1}w_{1,i}w_{j,k} - w_{1,i}w_{1,j}w_{1,k} - w_{2,2}w_{1,i}w_{j,k}. \]

To show \( w_{1,i}w_{1,j}w_{1,k} \in G \), we must show \( w_{1,i}w_{1,j}w_{1,k} \notin K \). That is, no degree two divisor is in \( L \).

According to Remark 4.4, if \( w_{1,i}w_{1,k} \in L \) it must be in \( L_1 \). However, any time \( w_{1,i}w_{1,k} \) is a term of a generator of \( \Lambda_1 \) it appears together with the larger monomial \( w_{l,i}w_{l,k} \) where \( l \geq 2 \). Thus \( w_{1,i}w_{1,k} \notin L \). By the same reasoning \( w_{1,j}w_{1,k} \notin L \) if \( i \neq j \), this also shows \( w_{1,i}w_{1,j} \notin L \).

If \( i = j \), again by Remark 4.4, if \( w_{1,i}^2 \in L \) it must be in \( L_2 \). However, any time \( w_{1,i}^2 \) is a term of a generator of \( \Lambda_2 \) it appears together with the larger monomial \( w_{l,i}^2 \) where \( l > 1 \). Thus \( w_{1,i}^2 \notin L \). Hence \( w_{1,i}w_{1,i}w_{1,k} \notin K \). Therefore \( w_{1,i}w_{1,j}w_{1,k} \in G \), as claimed. \( \square \)

**Lemma 4.15.** Adopt Data 4.5. The elements \( w_{1,i}w_{1,j}^2 \) for \( 3 \leq i < j \leq d \) can be obtained as the leading monomial of an element of a minimal Gröbner basis of \( \Lambda \). That is

\[ \{ w_{1,i}w_{1,j}^2 \mid 3 \leq i < j \leq d \} \subset G. \]

**Proof.** To begin, we list two generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \). Notice that if \( j = d \) the proof is the same except for setting \( w_{j,j} = 0 \).

Let \( f \) be the element of \( \Lambda_0 \) obtained by selecting rows 1 and \( j \) and columns 2 and \( i \).
More precisely,

\[ f = \det \begin{bmatrix} w_{1,2} & w_{1,i} \\ w_{2,j} & w_{i,j} \end{bmatrix} = w_{1,2}w_{i,j} - w_{1,i}w_{2,j} \]

Let \( h \) be the element of \( \Lambda_2 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,i,j} \) and then choosing the pair of complementary minors in red, orange, green and purple as follows.

\[
\begin{bmatrix}
  w_{1,1} & w_{1,2} & w_{1,i} & w_{1,j} \\
  w_{1,2} & w_{2,2} & w_{2,i} & w_{2,j} \\
  w_{1,i} & w_{2,i} & w_{i,i} & w_{i,j} \\
  w_{1,j} & w_{2,j} & w_{i,j} & w_{j,j}
\end{bmatrix}
\]

More precisely,

\[ h = w_{2,j}^2 + w_{1,i}^2 - w_{2,i}^2 - w_{1,1}w_{i,i} + w_{1,1}w_{j,j} + w_{2,2}w_{i,i} - w_{2,2}w_{j,j}. \]

**Claim:** The monomial \( w_{1,i}w_{1,j}^2 \) is in \( G \).

The monomial \( w_{1,i}w_{1,j}^2 \) is the leading monomial of the S-polynomial between \( f \) and \( h \). Indeed,

\[
S(f,h) = w_{2,j}f + w_{1,i}h = w_{1,2}w_{2,j}w_{i,j} + w_{1,i}^3 - w_{1,i}w_{1,j}^2 - w_{1,1}w_{2,i} - w_{1,1}w_{1,j}w_{i,i} + w_{1,1}w_{1,j}w_{i,j} + w_{2,2}w_{i,i} - w_{2,2}w_{1,i}w_{j,j}.
\]

To show \( w_{1,i}w_{1,j}^2 \in G \), we must show \( w_{1,i}w_{1,j}^2 \notin K \).

According to Remark 4.4, if \( w_{1,i}w_{1,j} \in L \) it must be in \( L_1 \). However, any time \( w_{1,i}w_{1,j} \) is a term of a generator of \( \Lambda_1 \) it appears together with the larger monomial \( w_{l,i}w_{l,j} \) where \( l \geq 2 \). Thus \( w_{1,i}w_{1,j} \notin L \).

Again by Remark 4.4, if \( w_{1,j}^2 \in L \) it must be in \( L_2 \). However, any time \( w_{1,j}^2 \) is a term
of a generator of $\Lambda_2$ it appears together with the larger monomial $w^2_{k,j}$ where $k > 1$. Thus $w^2_{1,j} \notin L$. Hence $w_{1,i}w^2_{1,j} \notin K$. Therefore $w_{1,i}w^2_{1,j} \in G$, as claimed.

\[ \square \]

**Lemma 4.16.** Adopt Data 4.5. The elements $w^3_{i,j}$ for $3 < i \leq d$ can be obtained as the leading monomial of an element of a minimal Gröbner Basis of $\Lambda$. That is,

\[ \{w^3_{i,j} \mid 3 < i \leq d\} \subseteq G. \]

**Proof.** To begin, we list two generators of $\Lambda$ that will be involved in the S-polynomials used to show the desired monomials are elements of $G$. Notice that if $i = d$ the proof is the same except for setting $w_{i,i} = 0$.

Let $g$ be the element of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,i}$ and then choosing the complementary minors in red and orange as follows.

\[
\begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,3} & w_{1,i} \\
    w_{1,2} & w_{2,2} & w_{2,3} & w_{2,i} \\
    w_{1,3} & w_{2,3} & w_{3,3} & w_{3,i} \\
    w_{1,i} & w_{2,i} & w_{3,i} & w_{i,i}
\end{bmatrix}
\]

More precisely,

\[ g = w_{1,2}w_{3,3} - w_{1,3}w_{2,3} + w_{1,i}w_{2,i} - w_{1,2}w_{i,i}. \]

Let $h$ be the element of $\Lambda_2$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,i}$ and then choosing the pair of complementary minors in red, orange, green and purple as
follows.

\[
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,i} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,i} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,i} \\
w_{1,i} & w_{2,i} & w_{3,i} & w_{i,i}
\end{bmatrix}
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,i} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,i} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,i} \\
w_{1,i} & w_{2,i} & w_{3,i} & w_{i,i}
\end{bmatrix}
\]

More precisely, \( h = w_{2,2,i}^2 + w_{1,3,i}^2 - w_{1,1,i}^2 - w_{2,3,i}^2 + w_{1,1}w_{3,3,i} + w_{1,1}w_{i,i} + w_{2,2}w_{3,3,i} - w_{2,2}w_{i,i} \).

**Claim:** The monomial \( w_{3,1,i} \) is in \( G \).

The monomial \( w_{3,1,i} \) is the leading monomial of the S-polynomial between \( g \) and \( h \).

Indeed,

\[
S(g, h) = w_{2,i}g - w_{1,i}h \\
= w_{1,2}w_{3,3}w_{2,i} - w_{1,3}w_{2,3}w_{2,i} - w_{1,2}w_{2,i}w_{i,i} - w_{1,1}w_{3,3}w_{1,i} + w_{1,i}^3 \\
+ w_{2,3}w_{1,i} + w_{1,1}w_{3,3}w_{1,i} - w_{1,1}w_{1,i}w_{i,i} - w_{2,2}w_{3,3}w_{1,i} + w_{2,2}w_{1,i}w_{i,i}.
\]

To show \( w_{3,1,i} \in G \), we must show \( w_{3,1,i} \notin K \).

According to Remark 4.4, if \( w_{1,1,i} \in L \) it must be in \( L_2 \). However, any time \( w_{1,1,i} \) is a term of a generator of \( L_2 \) it appears together with the larger monomial \( w_{k,i}^2 \) where \( k > 1 \). Thus \( w_{1,1,i}^2 \notin L \). Hence \( w_{3,1,i} \notin K \). Therefore \( w_{3,1,i} \in G \), as claimed.

**Lemma 4.17.** Adopt Data 4.5. The element \( w_{3,1,3} \) can be obtained as the leading monomial of a minimal Gröbner Basis of \( \Lambda \). That is

\[
\{w_{3,1,3}\} \subset G.
\]

**Proof.** To begin, we list four generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \). Additionally, let \( F \) be the degree 3
element from Lemma 4.12 from which we concluded $w_{1,3}w_{2,3}^2 \in G$. That is,

$$F = w_{1,2}w_{2,d}w_{3,d} + w_{1,2}w_{2,3}w_{3,3} - w_{1,3}w_{2,3}^2.$$ 

Let $f_1$ be the element of $\Lambda_0$ obtained by selecting rows 1 and $d$ and columns 2 and 3. Let $f_2$ be the element of $\Lambda_0$ obtained by selecting rows 1 and 3 and columns 2 and $d$. More precisely,

$$f_1 = \text{det} \begin{bmatrix} w_{1,2} & w_{1,3} \\ w_{2,d} & w_{3,d} \end{bmatrix} = w_{1,2}w_{3,d} - w_{1,3}w_{2,d}$$

$$f_2 = \text{det} \begin{bmatrix} w_{1,2} & w_{1,d} \\ w_{2,3} & w_{3,d} \end{bmatrix} = w_{1,2}w_{3,d} - w_{2,3}w_{1,d}.$$ 

Let $g$ be the element of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,d}$ and then choosing the complementary minors in red and orange as follows.

$$g = w_{1,1}w_{3,d} - w_{1,3}w_{1,d} + w_{2,3}w_{2,d} - w_{2,2}w_{3,d}.$$ 

Let $h$ be the element of $\Lambda_2$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,d}$ and then choosing the pair of complementary minors in red, orange, green and purple as
follows.

\[
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\]

More precisely,

\[h = w_{2,d}^2 + w_{1,3}^2 - w_{1,1}^2 - w_{2,3}^2 - w_{1,1}w_{3,3} + w_{2,2}w_{3,3} \]

**Claim:** The monomial \(w_{1,3}^3\) is in \(G\).

The monomial \(w_{1,3}^3\) arises from the S-polynomial between \(F_1 = S(f_1, h)\) and \(F_2 = S(f_2, g)\) and then further reduction with \(F\). Indeed,

\[F_1 = S(f_1, h)
= w_{2,d}f_1 + w_{1,3}h
= w_{1,2}w_{2,d}w_{3,d} + w_{1,3}^2 - w_{1,1}w_{3,3} + w_{2,2}w_{1,3}w_{3,3}
\]

\[F_2 = S(f_2, g)
= w_{2,d}f_2 + w_{1,d}g
= w_{1,2}w_{2,d}w_{3,d} + w_{1,1}w_{1,d}w_{3,d} - w_{1,3}w_{1,d}^2 - w_{2,2}w_{1,d}w_{3,d}
\]

Notice \(F_1\) and \(F_2\) have the same leading monomial and let \(F_3\) be their S-polynomial. Indeed,

\[F_3 = S(F_1, F_2) = F_1 - F_2 = w_{1,2}w_{2,d}w_{3,d} + w_{1,3}^3 - w_{1,3}w_{2,3}^2 - w_{1,1}w_{1,3}w_{3,3} + w_{2,2}w_{1,3}w_{3,3} - w_{1,2}w_{2,d}w_{3,d} - w_{1,1}w_{1,d}w_{3,d} + w_{2,2}w_{1,d}w_{3,d}
\]
Now $F_3$ has the same leading monomial as $F$, again take the S-polynomial.

\[
S(F_3, F) = F_3 - F = w_{1,2}w_{2,d}w_{3,d} + w_{1,3}^3 - w_{1,1}w_{1,3}w_{3,3} + w_{2,2}w_{1,3}w_{3,3} - w_{1,2}w_{2,d}w_{3,d} - w_{1,1}w_{1,d}w_{3,d} + w_{2,2}w_{1,d}w_{3,d} - w_{1,2}w_{2,d}w_{3,d} - w_{1,2}w_{2,3}w_{3,3}.
\]

To show $w_{1,3}^3 \in G$, we must show $w_{1,3}^3 \notin K$.

According to Remark 4.4, if $w_{1,3}^2 \in L$ it must be in $L_2$. However, any time $w_{1,3}^2$ is a term of a generator of $\Lambda_1$ it appears together with the larger monomial $w_{k,3}^2$ where $k > 1$. Thus $w_{1,3}^2 \notin L$. Hence $w_{1,3}^3 \notin K$. Therefore $w_{1,3}^3 \in G$, as claimed.

We are now ready to conclude $G_2 \subset G$ and count its size.

**Proposition 4.6.** Adopt Data 4.5. There is an inclusion of sets

\[
G_2 = \{w_{1,i}w_{1,j}w_{1,k} | 3 \leq i \leq j \leq k \leq d\} \subset G.
\]

Further $|G_2| = (d-2) + (d-2)(d-3) + \binom{d-2}{3}$.

**Proof.** The inclusion $G_2 \subset G$ follows directly from lemmas 4.14, 4.15, 4.16 and 4.17.

Regarding the size of $G_2$, notice that either $i = j = k$, two are equal, or they are all distinct. If $i = j = k$, there are $d-2$ elements since $i = j = k$ can range from 3 to $d$. If two are equal, there are $d-2$ ways to select the index for the squared term and $d-3$ ways to select the other index. This gives a total of $(d-2)(d-3)$ elements. Finally, if all indices are distinct, there are $\binom{d-2}{3}$ ways to select them. The sum gives the size of $G_2$ as claimed.

We will now show that

\[
G_3 = \{w_{1,3}w_{2,3}w_{i,j} | 3 \leq i \leq j \leq d\} \subset G.
\]

We will break the proof into three lemmas. Lemma 4.18 will account for the case $w_{1,3}w_{2,3}w_{i,j}$.
Lemma 4.19 will account for the case \( w_{1,3}w_{2,3}w_{3,j} \) with \( 3 < j \). Finally, Lemma 4.20 will account for the case \( w_{1,3}w_{2,3}w_{i,j} \) with \( 3 \leq i = j \).

Lemma 4.18. Adopt Data 4.5. The elements \( w_{1,3}w_{2,3}w_{i,j} \) for \( 3 < i < j \) can be obtained as the leading monomial of an element of a minimal Gröbner Basis of \( \Lambda \). That is

\[
\{ w_{1,3}w_{2,3}w_{i,j} \mid 3 < i < j \} \subset G.
\]

Proof. To begin, we list two generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \).

Let \( f_1 \) be the element of \( \Lambda_0 \) obtained by selecting rows 1 and \( i \) and columns 2 and 3. Let \( f_2 \) be the element of \( \Lambda_0 \) obtained by selecting rows 3 and \( i \) and columns 2 and \( j \). More precisely,

\[
f_1 = \begin{vmatrix}
  w_{1,2} & w_{1,3} \\
  w_{2,i} & w_{3,i}
\end{vmatrix} = w_{1,2}w_{3,i} - w_{1,3}w_{2,i}
\]

\[
f_2 = \begin{vmatrix}
  w_{2,3} & w_{3,j} \\
  w_{2,i} & w_{i,j}
\end{vmatrix} = w_{2,3}w_{i,j} - w_{2,i}w_{3,j}.
\]

Claim: The monomial \( w_{1,3}w_{2,3}w_{i,j} \) is in \( G \).

The monomial \( w_{1,3}w_{2,3}w_{i,j} \) is the leading monomial of the S-polynomial between \( f_1 \) and \( f_2 \). Indeed,

\[
S(f_1, f_2) = w_{3,j}f_1 - w_{1,3}f_2 = w_{1,2}w_{3,i}w_{3,j} - w_{1,3}w_{2,3}w_{i,j}.
\]

To show \( w_{1,3}w_{2,3}w_{i,j} \in G \), we must show \( w_{1,3}w_{2,3}w_{i,j} \notin K \). That is, no degree two divisor is in \( L \). Since \( w_{1,3}w_{i,j} \) and \( w_{2,3}w_{i,j} \) are not antidiagonals they are not in \( L \).

According to Remark 4.4, if \( w_{1,3}w_{2,3} \in L \) it must be in \( L_1 \). However, any time \( w_{1,3}w_{2,3} \) is a term of a generator of \( \Lambda_1 \) it appears together with the larger monomial \( w_{1,k}w_{2,k} \) with \( k > 3 \).
Thus $w_{1,3}w_{2,3} \notin L$. Hence $w_{1,3}w_{2,3}w_{i,j} \notin K$. Therefore $w_{1,3}w_{2,3}w_{i,j} \in G$, as claimed.

**Lemma 4.19.** Adopt Data 4.5. The elements $w_{1,3}w_{2,3}w_{3,j}$ for $3 < j$ can be obtained as the leading monomial of an element of a minimal Gröbner Basis of $\Lambda$. That is

$$\{w_{1,3}w_{2,3}w_{3,j} \mid 3 < j\} \subset G.$$  

**Proof.** To begin, we list two generators of $\Lambda$ that will be involved in the S-polynomials used to show the desired monomials are elements of $G$. Notice that if $j = d$ the proof is the same except for setting $w_{j,j} = 0$.

Let $f$ be the element of $\Lambda_0$ obtained by selecting rows 1 and $j$ and columns 2 and 3. More precisely,

$$f = \det \begin{bmatrix} w_{1,2} & w_{1,3} \\ w_{2,j} & w_{3,j} \end{bmatrix} = w_{1,2}w_{3,j} - w_{1,3}w_{2,j}.$$  

Let $g$ be the element of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,j}$ and then choosing the complementary minors in red and orange as follows.

$$g = w_{1,1}w_{2,j} - w_{1,2}w_{1,j} - w_{2,3}w_{3,j} - w_{3,3}w_{2,j}.$$  

**Claim:** The monomial $w_{1,3}w_{2,3}w_{3,j}$ is in $G$. 

79
The monomial $w_{1,3}w_{2,3}w_{3,j}$ arises from the S-polynomial between $f$ and $g$. Indeed,

$$S(f, g) = w_{3,3}f - w_{1,3}g = w_{1,2}w_{3,3}w_{3,j} - w_{1,1}w_{1,3}w_{2,j} - w_{1,2}w_{1,3}w_{2,j} + w_{1,3}w_{2,3}w_{3,j}$$

To show $w_{1,3}w_{2,3}w_{3,j} \in G$, we must show $w_{1,3}w_{2,3}w_{3,j} \notin K$. Since $w_{1,3}w_{3,j}$ and $w_{2,3}w_{3,j}$ are not antidiagonals they are not in $L$.

According to Remark 4.4, if $w_{1,3}w_{2,3} \in L$ it must be in $L_1$. However, any time $w_{1,3}w_{2,3}$ is a term of a generator of $L$ it appears together with the larger monomial $w_{1,k}w_{2,k}$ with $k > 3$. Thus $w_{1,3}w_{2,3} \notin L$. Hence $w_{1,3}w_{2,3}w_{3,j} \notin K$. Therefore $w_{1,3}w_{2,3}w_{3,j} \in G$, as claimed. □

**Lemma 4.20.** Adopt Data 4.3. The elements $w_{1,3}w_{2,3}w_{i,i} \in G$ for $3 \leq i < d$ can be obtained as the leading monomial of an element of a minimal Gröbner Basis of $\Lambda$. That is,

$$\{w_{1,3}w_{2,3}w_{i,i} \in G \mid 3 \leq i < d\} \subset G.$$ 

**Proof.** To begin, we list three generators of $\Lambda$ that will be involved in the S-polynomials used to show the desired monomials are elements of $G$.

Let $g_1$ be the element of $\Lambda_1$ obtained by selecting $4 \times 4$ principal submatrix $M_{1,2,3,d}$ and then pairing according to the colors red and orange as follows. Additionally, let $g_2, g_3$ be the elements of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,i,d}$ and then pairing according to the colors red and orange as follows.

$$
\begin{aligned}
g_1 & : \\
& \begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix} \\
g_2 & : \\
& \begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix} \\
g_3 & : \\
& \begin{bmatrix}
w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix}
\end{aligned}
$$
More precisely,

\[ g_1 = w_{1,d}w_{2,d} + w_{1,2}w_{3,3} - w_{1,3}w_{2,3} \]
\[ g_2 = w_{1,i}w_{i,d} - w_{i,i}w_{1,d} - w_{1,2}w_{2,d} + w_{2,2}w_{1,d} \]
\[ g_3 = w_{2,d}w_{i,d} + w_{1,1}w_{2,i} - w_{1,2}w_{1,i} \]

**Claim:** The monomial \( w_{1,3}w_{2,3}w_{i,i} \) is in \( G \).

The monomial \( w_{1,3}w_{2,3}w_{i,i} \) arises from S-polynomial between \( g_1 \) and \( g_2 \) and then further reduction with \( g_3 \). Indeed,

\[ F_1 = S(g_1, g_2) \]
\[ = w_{i,i}g_1 + w_{2,d}g_2 \]
\[ = w_{1,2}w_{3,3}w_{i,i} - w_{1,3}w_{2,3}w_{i,i} + w_{1,i}w_{2,d}w_{i,d} - w_{1,2}w_{2,d}^2 + w_{2,2}w_{1,d}w_{2,d} \]

Notice that the leading monomial of \( F_1 \) is \( w_{1,i}w_{2,d}w_{i,d} \) and the leading monomial of \( g_3 \) is \( w_{2,d}w_{i,d} \), so we will reduce \( F_2 \) using \( g_3 \). Indeed,

\[ F_2 = S(F_1, g_3) \]
\[ = F_1 - w_{1,3}g_3 \]
\[ = w_{1,2}w_{3,3}w_{i,i} - w_{1,3}w_{2,3}w_{i,i} - w_{1,2}w_{2,d}^2 + w_{2,2}w_{1,d}w_{2,d} - w_{1,1}w_{1,i}w_{2,i} + w_{1,2}w_{1,i}^2 \]

To show \( w_{1,3}w_{2,3}w_{i,i} \in G \), we must show \( w_{1,3}w_{2,3}w_{i,i} \notin K \). Since \( w_{1,3}w_{i,i} \) and \( w_{2,3}w_{i,i} \) are not antidiagonals, they are not in \( L \).

According to Remark 4.4, if \( w_{1,3}w_{2,3} \in L \) it must be in \( L_1 \). However, any time \( w_{1,3}w_{2,3} \) is a term of a generator of \( \Lambda_1 \) it appears together with the larger monomial \( w_{1,k}w_{2,k} \) where \( k > 3 \). Thus \( w_{1,3}w_{2,3} \notin L \). Hence \( w_{1,3}w_{2,3}w_{i,i} \notin K \). Therefore \( w_{1,3}w_{2,3}w_{i,i} \in G \), as claimed.

We are now ready to conclude \( G_3 \subset G \) and count its size.
Proposition 4.7. Adopt Data 4.5 There is an inclusion of sets

\[ G_3 = \{ w_{1,3}w_{2,3}w_{i,j} \mid 3 \leq i \leq j \leq d \} \subset G. \]

Further \(|G_3| = \frac{d(d-3)}{2} \).

Proof. The inclusion \(G_3 \subset G\) follows directly from lemmas 4.18, 4.19, and 4.20.

Regarding the size of \(G_3\), notice that when \(i = j\), there are \(d - 3\) possibilities since both cannot be \(d\). When \(i < j\), for each \(i\) there are \(d - i\) possibilities for \(j\). Therefore we have the claimed size of \(G_3\). That is,

\[ |G_3| = (d - 3) + [(d - 3) + (d - 2) + \cdots + 2 + 1] \]
\[ = (d - 3) + \frac{(d-2)(d-3)}{2} = \frac{(d-3)(2+d-2)}{2} \]
\[ = \frac{d(d-3)}{2}. \]

We will now show that \(G_4 = \{ w_{2,3}^2w_{i,j} \mid 3 \leq i \leq j \leq d \} \subset G.\)

We will break the proof into three lemmas. Lemma 4.21 will account for the case \(w_{2,3}^2w_{i,j}\) with \(3 < i < j\). Lemma 4.22 will account for the case \(w_{2,3}^2w_{3,j}\) with \(3 < j\). Finally, Lemma 4.23 will account for the case \(w_{2,3}^2w_{i,j}\) with \(i = j\).

Lemma 4.21. Adopt Data 4.5 The elements \(w_{2,3}^2w_{i,j} \in G\) for \(3 < i < j\) can be obtained as the leading monomial of an element of a minimal Gröbner Basis of \(\Lambda\). That is,

\[ \{ w_{2,3}^2w_{i,j} \in G \mid 3 < i < j \} \subset G. \]
Proof. To begin, we list two generators of $\Lambda$ that will be involved in the S-polynomials used to show the desired monomials are elements of $G$.

Let $f$ be the element of $\Lambda_0$ obtained by selecting rows 2 and $i$ and columns 3 and $j$. More precisely,

$$f = \det \begin{bmatrix} w_{2,3} & w_{2,j} \\ w_{3,i} & w_{i,j} \end{bmatrix} = w_{2,3}w_{i,j} - w_{3,i}w_{2,j}.$$

Let $g$ be the element of $\Lambda_1$ obtained by selecting the $4 \times 4$ principal submatrix $M_{1,2,3,j}$ and then choosing the complementary minors in red and orange as follows.

$$g = w_{2,3}w_{2,j} - w_{2,2}w_{3,j} - w_{1,3}w_{1,j} + w_{1,1}w_{3,j}.$$

Claim: The monomial $w_{2,3}^2w_{i,j}$ is in $G$.

The monomial $w_{2,3}^2w_{i,j}$ arises from the S-polynomial between $f$ and $g$. Indeed,

$$S(f, g) = w_{2,3}f + w_{3,i}g = w_{2,3}^2w_{i,j} - w_{2,2}w_{3,j}w_{3,3,j} - w_{1,3}w_{3,3,j}w_{1,1} + w_{1,1}w_{3,3,j}.$$

To show $w_{2,3}^2w_{i,j} \in G$, we must show $w_{2,3}^2w_{i,j} \notin K$. That is, no degree two divisor is in $L$. Since $w_{2,3}^2w_{i,j}$ is not an antidiagonal it is not in $L$.

According to Remark 4.4 if $w_{2,3}^2 \in L$ it must be in $L_2$. However, any time $w_{2,3}^2$ is a term of a generator of $\Lambda_2$ it appears together with the larger monomial $w_{k,l}^2$ with $l > 3$. Thus
\[ w_{2,3}^2 \notin L. \text{ Hence } w_{2,3}^2 w_{i,j} \notin K. \text{ Therefore } w_{2,3}^2 w_{i,j} \in G, \text{ as claimed.} \]

**Lemma 4.22.** Adopt Data 4.5. The elements \( w_{2,3}^2 w_{3,j} \) for \( 3 < j \leq d \) can be obtained as the leading monomial of an element of a minimal Gröbner Basis of \( \Lambda \). That is,

\[ \{ w_{2,3}^2 w_{3,j} \mid 3 < j \leq d \} \subset G. \]

**Proof.** To begin, we list two generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \).

Let \( g_1, g_2 \) be elements of \( \Lambda_1 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,3,j} \) and then choosing the complementary minors in red and orange as follows.

\[
g_1: \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,j} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,j} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,j} \\ w_{1,j} & w_{2,j} & w_{3,j} & w_{j,j} \end{bmatrix} \quad \quad g_2: \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,j} \\ w_{1,2} & w_{2,2} & w_{2,3} & w_{2,j} \\ w_{1,3} & w_{2,3} & w_{3,3} & w_{3,j} \\ w_{1,j} & w_{2,j} & w_{3,j} & w_{j,j} \end{bmatrix}
\]

More precisely,

\[
g_1 = w_{2,3}w_{2,j} - w_{2,2}w_{3,j} - w_{1,3}w_{1,j} + w_{1,1}w_{3,j} \\
g_2 = -w_{2,j}w_{3,3} + w_{2,3}w_{3,j} + w_{1,1}w_{2,j} - w_{1,2}w_{1,j}.
\]

**Claim:** The monomial \( w_{2,3}^2 w_{3,j} \) is in \( G \).

The monomial \( w_{2,3}^2 w_{3,j} \) arises from the S-polynomial between \( g_1 \) and \( g_2 \). Indeed,

\[
S(g_1, g_2) = w_{3,3}g_1 + w_{2,3}g_2 = -w_{2,2}w_{3,j} - w_{1,3}w_{3,j} + w_{1,1}w_{3,j} + w_{2,3}^2 w_{3,j} \\
+ w_{1,1}w_{2,2}w_{1,j} - w_{1,2}w_{2,3}w_{1,j}.
\]

To show \( w_{2,3}^2 w_{3,j} \in G \), we must show \( w_{2,3}^2 w_{3,j} \notin K \). Since \( w_{2,3} w_{3,j} \) is not an antidiagonal it
is not in \( L \).

According to Remark 4.4 if \( w_{2,3}^2 \in L \) it must be in \( L_2 \). However, any time \( w_{2,3}^2 \) is a term of a generator of \( \Lambda_2 \) it appears together with the larger monomial \( w_{k,l}^2 \) with \( l > 3 \). Thus \( w_{2,3}^2 \notin L \). Hence \( w_{2,3}^2 w_{3,j} \notin K \). Therefore \( w_{2,3} w_{3,j} \in G \), as claimed.

\[ \square \]

**Lemma 4.23.** Adopt Data 4.5. The elements \( w_{2,3}^2 w_{i,i} \) for \( 3 \leq i < d \) can be obtained as the leading monomial of an element of a minimal Gröbner Basis of \( \Lambda \). That is

\[ \{ w_{2,3}^2 w_{i,i} \mid 3 \leq i < d \} \subset G. \]

**Proof.** To begin, we list five generators of \( \Lambda \) that will be involved in the S-polynomials used to show the desired monomials are elements of \( G \).

Let \( g_1, g_2, g_3, g_4 \) be elements of \( \Lambda_1 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,i,d} \) and then choosing the complementary minors in red and orange as follows.

\[
\begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
    w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
    w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
    w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
    w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
    w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
    w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
    w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
    w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
    w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    w_{1,1} & w_{1,2} & w_{1,i} & w_{1,d} \\
    w_{1,2} & w_{2,2} & w_{2,i} & w_{2,d} \\
    w_{1,i} & w_{2,i} & w_{i,i} & w_{i,d} \\
    w_{1,d} & w_{2,d} & w_{i,d} & 0
\end{bmatrix}
\]
More precisely,

\[ g_1 = -w_{2,d}w_{i,i} + w_{2,i}w_{i,d} + w_{1,1}w_{2,d} - w_{1,2}w_{1,d} \]
\[ g_2 = w_{1,i}w_{i,d} - w_{i,i}w_{1,d} - w_{1,2}w_{2,d} + w_{2,2}w_{1,d} \]
\[ g_3 = w_{2,d}w_{i,d} + w_{1,1}w_{2,i} - w_{1,2}w_{1,i} \]
\[ g_4 = -w_{1,d}w_{i,d} + w_{1,2}w_{2,i} + w_{2,2}w_{1,i} \]

Let \( h \) be the element of \( \Lambda_2 \) obtained by selecting the \( 4 \times 4 \) principal submatrix \( M_{1,2,3,d} \) and then choosing the pair of complementary minors in \textbf{red}, \textbf{orange}, \textbf{green} and \textbf{purple} as follows.

\[
\begin{bmatrix}
  w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
  w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
  w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
  w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  w_{1,1} & w_{1,2} & w_{1,3} & w_{1,d} \\
  w_{1,2} & w_{2,2} & w_{2,3} & w_{2,d} \\
  w_{1,3} & w_{2,3} & w_{3,3} & w_{3,d} \\
  w_{1,d} & w_{2,d} & w_{3,d} & 0
\end{bmatrix}
\]

More precisely,

\[ h = w_{2,2}^2 + w_{1,3}^2 - w_{1,1}^2 - w_{2,3}^2 - w_{1,1}w_{3,3} + w_{2,2}w_{3,3}. \]

**Claim:** The monomial \( w_{2,3}^2w_{i,i} \) is in \( G \).

The monomial \( w_{2,3}^2w_{i,i} \) arises from S-polynomial between \( g_1 \) and \( h \) and then further reductions with the other defined generators. Let

\[ F_1 = S(g_1, h) \]
\[ = w_{2,d}g_1 + w_{i,i}h \]
\[ = w_{2,i}w_{2,d}w_{i,i} + w_{1,1}w_{2,d} - w_{1,2}w_{1,1}w_{2,d} + w_{2,2}w_{1,3}w_{i,i} - w_{i,i}w_{1,1}w_{2,d} \]
\[ - w_{2,3}^2w_{i,i} - w_{1,1}w_{3,3}w_{i,i} + w_{2,2}w_{3,3}w_{i,i}. \]
Notice that the leading monomial of $F_1$ is $w_i w_{1,d}^2$ and the leading monomial of $g_2$ is $w_{i,d} w_{1,d}$, so we will reduce $F_1$ using $g_2$. Let

$$F_2 = S(F_1, g_2)$$

$$= -F_1 + w_{1,d} g_2$$

$$= -w_2,i w_{2,d} w_{1,d} - w_{1,1} w_{2,d}^2 + w_{1,2} w_{1,d} w_{2,d} - w_{1,3} w_i + w_{2,3} w_{i,i} + w_{1,1} w_{3,3} w_i$$

Notice that the leading monomial of $F_2$ is $w_2,i w_{2,d} w_{1,d}$ and the leading monomial of $g_3$ is $w_{2,d} w_{1,d}$, so we will reduce $F_2$ using $g_3$. Let

$$F_3 = S(F_2, g_3)$$

$$= F_2 + w_{2,i} g_3$$

$$= -w_{1,1} w_{2,d}^2 + w_{1,2} w_{1,d} w_{2,d} - w_{1,3} w_i + w_{2,3} w_{i,i} + w_{1,1} w_{3,3} w_i$$

Notice now that we have two possible leading monomials, depending on if $i = 3$ or $i \neq 3$.

**Claim:** If $i = 3$, the monomial $w_{2,3}^2 w_{3,3}$ is in $G$.

If $i = 3$, then the leading monomial of $F_3$ is $w_{2,3}^2 w_{i,i} = w_{2,3}^2 w_{3,3}$. To show $w_{2,3}^2 w_{3,3} \in G$, we must show $w_{2,3}^2 w_{3,3} \notin K$. Since $w_{2,3} w_{3,3}$ is not an antidiagonal, it is not in $L$.

According to Remark 4.4, if $w_{2,3}^2 \in L$ it must be in $L_2$. However, any time $w_{2,3}^2$ is a term of a generator of $A_2$ it appears together with the larger monomial $w_{k,l}^2$ with $l > 3$. Thus $w_{2,3}^2 \notin L$. Hence $w_{2,3}^2 w_{3,3} \notin K$. Therefore $w_{2,3}^2 w_{3,3} \in G$, as claimed.

**Claim:** If $i > 3$, the monomial $w_{2,3}^2 w_{i,i}$ is in $G$.

If $i > 3$, then $w_{1,i} w_{1,d} w_{i,d}$ is the leading monomial of $F_3$ and the leading monomial of
$g_4$ is $w_{1,d}w_{i,d}$, so we will reduce $F_3$ using $g_4$. Let

$$F_4 = S(F_3, g_4)$$

$$= F_3 + w_{1,i}g_4$$

$$= w_{1,1}w_{2,d}^2 - w_{1,2}w_{1,d}w_{2,d} - w_{1,3}w_{i,i} + w_{2,3}^2w_{i,i} + w_{1,1}w_{3,3}w_{i,i} - w_{2,2}w_{3,3}w_{i,i}$$

$$- w_{1,2}w_{1,d}w_{2,d} + w_{2,2}w_{1,d}^2 + w_{1,1}w_{2,2}^2 - w_{1,2}w_{1,i}w_{2,i} + w_{1,2}w_{1,i}w_{2,i} + w_{2,2}w_{1,1}^2.$$

To show $w_{2,3}^2w_{i,j} \in G$, we must show $w_{2,3}^2w_{i,i} \notin K$. Since $w_{2,3}w_{i,i}$ is not an antidiagonal, it is not in $L$. We saw above that $w_{2,3}^2 \notin L$. Hence $w_{2,3}^2w_{i,i} \notin K$. Therefore $w_{2,3}^2w_{i,i} \in G$, as claimed.

We are now ready to conclude $G_4 \subset G$ and count its size.

**Proposition 4.8.** Adopt Data 4.5. There is an inclusion of sets

$$G_4 = \{w_{2,3}^2w_{i,j} \mid 3 \leq i \leq j \leq d\} \subset G.$$

Further $|G_4| = \frac{d(d-3)}{2}$.

**Proof.** The inclusion $G_4 \subset G$ follows directly from lemmas 4.21, 4.22, and 4.23.

Regarding the size of $G_4$, notice that when $i = j$, there are $d - 3$ possibilities since both cannot be $d$. When $i < j$, for each fixed $i$, there are $d - i$ possibilities for $j$. Therefore we have the claimed size of $G_4$. That is

$$|G_4| = (d - 3) + [(d - 3) + (d - 2) + \cdots + 2 + 1]$$

$$= (d - 3) + \frac{(d - 2)(d - 3)}{2}$$

$$= \frac{(d - 3)(2 + d - 2)}{2} = \frac{d(d - 3)}{2}.$$
At last, we are able to compute the Hilbert function of $\Lambda$ in degree 3.

**Theorem 4.5.** Adopt Data 4.5. The Hilbert Functions of $\Lambda$ and $I(X)$ are equivalent in degree 3, that is

$$HF_\Lambda(3) = HF_{I(X)}(3).$$

**Proof.** Recall that $\Lambda \subset I(X)$, so we have $HF_\Lambda(3) \leq HF_{I(X)}(3)$.

Notice that $G_i$ are disjoint subsets of $G$. Now, by propositions 4.5, 4.6, 4.7, and 4.8, we have

$$|G| \geq |G_1| + |G_2| + |G_3| + |G_4|$$

$$= 6 + (d - 2) + (d - 2)(d - 3) + \binom{d - 2}{3} + d(d - 3)$$

$$= \frac{120d^3 + 360d^2 - 1920d + 4320}{6!}.$$

Additionally, by Theorem 4.4 we have

$$|K| = \frac{14d^6 + 30d^5 - 40d^4 - 330d^3 - 694d^2 + 1740d - 4320}{6!}$$

By definition of $G$ and $K$, we have

$$HF_\Lambda(3) = |H| = |G| + |K|$$

$$\geq \frac{120d^3 + 360d^2 - 1920d + 4320}{6!} + \frac{14d^6 + 30d^5 - 40d^4 - 330d^3 - 694d^2 + 1740d - 4320}{6!}$$

$$= \frac{14d^6 + 30d^5 - 40d^4 - 210d^3 - 334d^2 - 180d}{6!}$$

$$= HF_{I(X)}(3).$$

Therefore the Hilbert functions of $\Lambda$ and $I(X)$ are equal in degree 3. \qed
4.5 Main Theorem 1

We are finally able to conclude that $\Lambda$ is the defining ideal of $\mathcal{F}(I)$.

**Theorem 4.6** (Main Theorem 1). Adopt Data 4.5. The ideal $\Lambda$ is the defining ideal of $\mathcal{F}(I)$, that is

$$\Lambda = I(X).$$

Further

$$\mathcal{J} = \mathcal{L} + \Lambda S$$

is the defining ideal of $\mathcal{R}(I)$.

**Proof.** By Theorem 4.3 the Hilbert functions satisfy $HF_{\Lambda}(2) = HF_{I(X)}(2)$. In particular, since $\Lambda \subseteq I(X)$ by Theorem 4.1 the degree two graded components of $I(X)$ and $\Lambda$ are equal. By Theorem 4.5 the Hilbert functions satisfy $HF_{\Lambda}(3) = HF_{I(X)}(3)$. In particular, since $\Lambda \subseteq I(X)$ by Theorem 4.1 the degree three graded components of $I(X)$ and $\Lambda$ are equal. By Theorem 4.2 we have $I(X)$ is generated in degrees 2 and 3. In particular, $I(X)$ and $\Lambda$ must have the same generating set and $I(X) = \Lambda$. Since the Rees algebra of $I$ is fiber type, we have $\mathcal{J} = \mathcal{L} + \Lambda S$. \qed
CHAPTER 5

JACOBIAN DUAL

The goal of this chapter is once again to study the defining ideal of the Rees algebra for ideals which are \( m \)-primary, Gorenstein, and have a Gorenstein linear resolution. The focus, in particular, will be to study the defining ideal via the Jacobian dual. Some theorems in this chapter are joint work with Claudia Polini and Bernd Ulrich. In the previous chapter, we found explicit polynomial generators for the defining ideal for the Rees algebra when \( \delta = 2 \). This chapter has a broader focus and some of theorems do not require \( \delta = 2 \). We will show that for all \( \delta \), the defining ideal of the Rees algebra is of expected form up to radical. Further, for \( \delta = 2 \), the \( I(X) \) is a saturation of the ideal of maximal minors of the Jacobian dual.

**Data 5.1.** Let \( R = k[x_1, \ldots, x_d] \) be a polynomial ring over a field \( k \).

- Let \( m = (x_1, \ldots, x_d) \) be the homogeneous maximal ideal of \( R \).
- Let \( I \) be an \( m \)-primary Gorenstein ideal having a Gorenstein linear resolution.
- Let \( n = \mu(I) \) denote the minimal number of generators of \( I \).
- Let \( g_1, \ldots, g_n \) be forms in degree \( \delta \) which minimally generate \( I \).
- Let 
  \[
  \Psi = [g_1 : \ldots : g_n] : \mathbb{P}^{d-1}_k \dashrightarrow \mathbb{P}^{n-1}_k .
  \]
  be the rational map defined by the forms \( g_i \).
- Let \( X = \text{Im}(\Psi) \) be the variety parametrized by \( \Psi \).
- Let \( A \) be the \( k \)-subalgebra \( k[\ g_1, \ldots, g_n \] of \( R \).
- Let \( \varphi \) be a minimal presentation matrix of \( g_1, \ldots, g_n \).
• Let \( S = R[w_1, \ldots, w_n] \) and \( W = k[w_1, \ldots, w_n] \) be polynomial rings.

**Theorem 5.1** (Liske, Polini, Ulrich). Adopt Data 5.1. The map \( \Psi \) is biregular onto its image. In particular, \( A \) has multiplicity \( \delta^{d-1} \).

**Proof.** Notice since \( I \) is \( m \)-primary the map \( \Psi \) is a regular map, defined everywhere. Let \( P \in \mathbb{P}^{n-1} \) be a point in \( X \). According to Proposition 2.2 in [14], the ideal \( I_1(P \cdot \varphi) : m^\infty \) defines the fiber \( \Psi^{-1}(P) \) scheme theoretically. Since the ideal \( I \) is \( m \)-primary, the fiber consists of finitely many points. On the other hand, the ideal \( I_1(P \cdot \varphi) \) is generated by linear forms in the \( x_i \) because \( \varphi \) has linear entries. Hence \( I_1(P \cdot \varphi) \) is a prime ideal of \( R \). Thus the fiber \( \Psi^{-1}(P) \) consists of a single reduced point and \( \Psi \) is biregular onto its image.

Corollary 5.8 in [20] states that \( \Psi \) is birational onto its image if and only if the multiplicity of \( A \) is \( \delta^{d-1} \). \qed

**Data 5.2.** Adopt Data 5.1

• Let \( \underline{x} = [x_1, \ldots, x_d] \) be the \( 1 \times d \) row vector of variables from \( R \).

• Let \( \underline{w} = [w_1, \ldots, w_n] \) be the \( 1 \times n \) row vector of variables from \( W \).

• Let \( B \) be the Jacobian dual of \( \varphi \), that is the matrix with linear entries in \( W \) satisfying
\[
\underline{w} \cdot \varphi = \underline{x} \cdot B.
\]

• Let \( \mathcal{J} \) be the defining ideal of the Rees algebra \( \mathcal{R}(I) \).

• Let \( \mathcal{L} = I_1(\underline{w} \cdot \varphi) \) be the defining ideal of the symmetric algebra \( S(I) \).

• Let \( I(X) \) be the defining ideal of the special fiber ring \( \mathcal{F}(I) \).

Recall that the defining ideal of the Rees algebra is said to be of **expected form** if \( \mathcal{J} = (\mathcal{L}, I_d(B)) \).

**Theorem 5.2** (Liske, Polini, Ulrich). Adopt Data 5.2. The defining ideal of the special fiber ring is
\[
I(X) = \sqrt{I_d(B)}.
\]
In particular, the defining ideal of $R(I)$ is of expected form up to radical. That is,

$$J = \sqrt{(L, I_d(B))} = (L, \sqrt{I_d(B)}) .$$

Proof. By Theorem 9.1 (c) in [12], we have $J = \mathcal{L} : S \mathfrak{m}$. In other words, $J$ is the annihilator of the $S$-module $M = \mathfrak{m}S / \mathcal{L}$. Since we always have the inclusion $\text{ann}_S M \subset \sqrt{\text{Fitt}_0(M)}$, we are interested in computing the zero-th fitting ideal of $M$. We will now construct a presentation matrix $\phi$ of $M$. Let $\theta$ be a $d \times \left( \genfrac{[}{]}{0pt}{}{d}{2} \right)$ presentation matrix of $x$. Since $\mathfrak{m}S$ is generated by the entries of $x$, each column of $\theta$ will be a column of $\phi$. Since $\mathcal{L}$ is generated by the entries $x \cdot B$ and $M = \mathfrak{m}S / \mathcal{L}$, each column of $B$ is also a column of $\phi$. More precisely, $M$ has a presentation matrix $\phi = \left[ \begin{array}{c|c} \theta & B \end{array} \right]$. Since $\text{Fitt}_0(M) = I_d\left( \left[ \begin{array}{c|c} \theta & B \end{array} \right] \right)$, we have

$$J = \text{ann}_S M \subset \sqrt{\text{Fitt}_0(M)} \subset \sqrt{(\mathfrak{m}, I_d(B))} .$$

Think of $S$, temporarily, as $S = W[x_1, \ldots, x_d]$. Then $W$ is the degree zero graded component of $S$ and $I(X)$ is the degree zero graded component of $J$. Therefore,

$$I(X) = [J]_0 \subset [\sqrt{(\mathfrak{m}, I_d(B))}]_0 = \sqrt{I_d(B)} .$$

Since we always have the containment $I_d(B) \subset I(X)$, we have

$$I(X) = \sqrt{I_d(B)} .$$

In particular, since the Rees algebra of $I$ is of fiber type, we have $J = (\mathcal{L}, \sqrt{I_d(B)})$. That is, the defining ideal of the Rees algebra is expected form up to radical.

We will now return to the case studied in Chapter 4. That is, when the ideal $I$ is generated in degree $\delta = 2$. We will also return to the notation for this case.
Data 5.3. Adopt Data 3.2 and let $B$ be the Jacobian dual of $\phi$. Let $n = (w_{1,1}, \ldots, w_{d-1,d})$ be the homogeneous maximal ideal of $W$.

The goal is to show $J = (L, I_d(B) : n^\infty)$ for $m$-primary, Gorenstein ideals, with a Gorenstein linear resolution, and generated in degree $\delta = 2$. Since the Rees algebra is fiber type, we will again focus on the defining ideal of the special fiber ring and show $I(X) = I_d(B) : n^\infty$. The inclusion $I_d(B) : n^\infty \subseteq I(X)$ is trivial since $I_d(B) \subset I(X)$ and $I(X)$ is prime, and hence saturated. We will begin with two lemmas which study the structure of the Jacobian dual after localization at the variables $w_{i,j}$ of $W$. These lemmas will aid in the proof of the other inclusion.

To help understand the Jacobian dual $B$, recall that the map $\zeta : S \to S(I)$ which presents the symmetric algebra satisfies

$$
\zeta(w_{i,j}) = \begin{cases} 
  x_i x_j & \text{for } i \neq j \\
  x_i^2 - x_d^2 & \text{for } i = j
\end{cases}
$$

and $\zeta(x_k) = x_k$.

Further, the kernel is $L = I_1(w \cdot \phi) = I_1(x \cdot B)$.

Lemma 5.1. Adopt Data 5.3. Let $1 \leq i < j \leq d$. After row and column operations, there is a $d - 1 \times d - 1$ upper (or lower triangular) submatrix of $B$ whose diagonal entries are $w_{i,j}$.

Proof. Let $k \in \{1, 2, \ldots, d\} \setminus \{i, j\}$. Let $C_k$ be the column vector with $-w_{k,j}$ in the $i$th row, $w_{i,j}$ in the $k$th row, and zeros everywhere else. Notice that $x_k(w_{i,j}) - x_i(w_{k,j}) \in L$. Hence, after possible row and column operations, this equation will contribute the column $C_k$ to the Jacobian dual. Indeed, we can construct $d - 2$ such columns of $B$ in this way.

Now let $l = \min\{1, 2, \ldots, d\} \setminus \{i, j\}$. Let $C_j$ be the column vector with $-w_{i,j}$ in the $l$th row, $w_{i,j}$ in the $j$th row, $w_{l,j} - w_{j,j}$ in the $i$th row, and zeros everywhere else. Notice that $-x_l(w_{i,j}) + x_j(w_{i,j}) + x_i(w_{l,j} - w_{j,j}) \in L$. Hence, after possible column operations, this equation will contribute the column $C_j$ to the Jacobian dual.
Let $C$ be the $d \times d - 1$ submatrix

$$C = [C_1|\cdots|\hat{C}_i|\cdots|C_d].$$

Let $C'$ be the $d - 1 \times d - 1$ submatrix of $C$ consisting of all rows except $i$. Notice that the only nonzero entry in $C'$ that does not lie on the diagonal is $-w_{i,l}$ in the $l$th row of $C_j$. Further, the diagonal entries of $C'$ are all $w_{i,j}$. Therefore $C'$ is the desired $d - 1 \times d - 1$ upper (or lower) triangular submatrix of $B$.

Lemma 5.2. Adopt Data 5.3. Let $1 \leq i < d$. After row and column operations, there is a $d - 1 \times d - 1$ lower triangular submatrix of $B$ whose diagonal entries are $w_{i,i}$.

Proof. Let $l = \min\{1, 2, \ldots, d - 1\} \setminus \{i\}$. Let $C_d$ be the column vector with $-w_{d,l}$ in the $i$th row, $w_{i,l}$ in the $d$th row, and zeros everywhere else. Notice that $x_d(w_{i,l}) - x_l(w_{d,l}) \in \mathcal{L}$. Hence, after possible row and column operations, this equation will contribute the column $C_d$ to the Jacobian dual.

Let $k \in \{1, \ldots, d - 1\} \setminus \{i\}$. Let $C_k$ be the column vector with $-w_{i,k}$ in the $i$th row, $w_{k,d}$ in the $d$th row, $w_{i,i}$ in the $k$th row and zeros everywhere else. Notice that $-x_l(w_{i,k}) + x_d(w_{k,d}) + x_k(w_{i,i}) \in \mathcal{L}$. Hence, after possible row and column operations, this equation will contribute the column $C_k$ to the Jacobian dual. Indeed, we can construct $d - 2$ such columns of $B$ in this way.

Let $C$ be the $d \times d - 1$ submatrix

$$C = [C_1|\cdots|\hat{C}_i|\cdots|C_d].$$

Let $C'$ be the $d - 1 \times d - 1$ submatrix of $C$ consisting of all rows except $i$. Notice that the only nonzero entries in $C'$ that do not lie on the diagonal are in the last row. Further, the diagonal entries of $C'$ are all $w_{i,i}$. Therefore $C'$ is the desired $d - 1 \times d - 1$ lower triangular submatrix of $B$.  \qed
Theorem 5.3 (Main Theorem 2). Adopt Data \textcolor{red}{5.3} The defining ideal of the special fiber ring is

\[ I(X) = I_d(B) : n^\infty. \]

In particular, the defining ideal of the Rees algebra is

\[ J = (\mathcal{L}, I_d(B) : n^\infty). \]

Proof. By Theorem \textcolor{red}{5.2} we have \( I(X) = \sqrt{I_d(B)} \). Since \( I(X) \) is saturated, we have the inclusion \( I_d(B) : n^\infty \subseteq I(X) \). It remains to show \( I(X) \subseteq I_d(B) : n^\infty \). It is enough to show equality after localization at the associated primes of \( I_d(B) : n^\infty \). Let \( p \in \text{Ass}(I_d(B) : n^\infty) \).

We will show that \( I_d(B)_p \) is prime of the correct height. Since \( I(X) = \sqrt{I_d(B)} \) they have the same height. Further, since localization preserves height \( I(X)_p \) and \( \sqrt{I_d(B)}_p \) also have the same height.

Now we will show that \( I_d(B)_p \) is prime. By the definition of saturation, there is some \( w_{i,j} \notin p \). In particular, localization at \( p \) is a further localization of \( w_{i,j} \). That is, \( J_p = (J_{w_{i,j}})_p \).

So it is enough to show \( I_d(B)_{w_{i,j}} \) is prime for every variable of \( W \). Let \( w_{i,j} \) be a variable of \( W \). By Lemmas \textcolor{red}{5.1} and \textcolor{red}{5.2} there is a \( (d-1) \times (d-1) \) upper (or lower) triangular submatrix of the Jacobian dual with \( w_{i,j} \) in the diagonal. After row and column operations, the localization of the Jacobian dual at the variable \( w_{i,j} \) is

\[
B_{w_{i,j}} = \begin{bmatrix}
Id_{d-1} & 0 \\
0 & B'
\end{bmatrix},
\]

where \( Id_{d-1} \) is the \( (d-1) \times (d-1) \) identity matrix and \( B' \) consists of exactly one row. Since localization commutes with taking minors, we have

\[ I_d(B)_{w_{i,j}} = I_d(B_{w_{i,j}}) = I_1(B'). \]
The ideal $I_1(B')$ is a linear ideal, and hence prime. Therefore $I(X)_{w_i,j} = I_d(B)_{w_i,j}$. In particular, $I_d(B) : n^\infty = I(X)$ and since $R(I)$ is of fiber type, $\mathcal{J} = (\mathcal{L}, I_d(B) : n^\infty)$. $\square$
CHAPTER 6

OPEN QUESTIONS

The purpose of this chapter is to survey some open questions relating to this thesis.

In this thesis, we have described the defining ideal of $F(I)$ in two ways. Now that we
have an explicit polynomial generating set for $I(X)$, it would now be interesting to study its
resolution. In the proof of Theorem 4.2, we studied the long exact sequence of Tor which
is a graded sequence. The degree two part of the sequence is

$$0 \rightarrow \text{Tor}^W_{1}(F(I), k) \rightarrow \text{Tor}^W_{1}(F(m^2), k) \rightarrow \text{Tor}^W_{1}(k(-1), k) \rightarrow 0$$

and in particular, this short exact sequence splits off. I would like to study this long exact
sequence further to see if the higher Tor’s also split into short exact sequences. If so, this
could aid in understanding the syzygies of $I(X)$ since $\text{Tor}^W_{1}(k(-1), k)$ is well understood
and $\text{Tor}^W_{1}(F(m^2), k)$ is closely related to $\text{Tor}^U_{1}(F(m^2), k)$ which is also well understood.

Now that we have described $I(X)$ and $J$ it would also be interesting to study the al-
gerbraic properties of the Rees algebra. Some properties of the Rees algebra such as the
regularity and the relation type were studied in [18], but some are still unknown. By study-
ing depth on the short exact sequence

$$0 \rightarrow F(I) \rightarrow F(m^2) \rightarrow k(-1) \rightarrow 0$$

we see that the depth of $F(I)$ must be 1. One might ask, is what is the depth of the
Rees algebra and is it normal? Since $\Lambda$ can be described in terms of minors of a symmetric

98
matrix of variables, it would be interesting to see if there is a way to view $I(X)$ alternatively using representation theory via the symmetric group action.

In Chapter 5, we were able to show that $I(X) = I_d(B) : n^\infty$. It would be of particular interest to calculate the degree of the saturation. In fact, it is very likely that the index of saturation is $d - 2$. Since $\Lambda$ is generated in degree 2, the index of saturation must be at least $d - 2$ due to degree. Computation in Macaulay 2 has confirmed that the index of saturation is indeed $d - 2$ in low dimension. Further, for each generator $f \in \Lambda$, there exists an element $b \in I_d(B)$ and a form $g$ of degree $d - 2$ such that $b = f \cdot g$. This fact is not strong enough to show the desired inequality, but hopefully will lead to some insight after further examination. Additionally, Polini and Ulrich are able to show that for $m$-primary ideals with a Gorenstein linear resolution $I(X) = I_d(B) : n^\infty$ using information about heights of fitting ideals. We would also be interesting in computing the degree of saturation in this more general setting.

The $m$-primary Gorenstein ideals having a Gorenstein linear resolution which were generated in degree $\delta = 2$ correspond to projections of the Veronese from a general point. A natural line of investigation would be to consider ideals which correspond to projections of the Veronese from a special point. These are the Cohen-Macaulay ideals submaximally generated by quadrics studied in [18], the same paper which gave the structure theorem for the Gorenstein case. They also have a structure theorem for the Cohen-Macaulay case. The structure depends on the Cohen-Macaulay type of $I$, and is likely to be fruitful in understanding the defining ideal of the special fiber ring. Indeed, by current research I have obtained a candidate for $I(X)$. The Rees algebra of these ideals is not usually of fiber type, so understanding $I(X)$ will not give the full picture of the Rees algebra. Regardless, it would interesting to at least study special fiber ring, its defining ideal, and the corresponding resolution.

Chapter 4 only focused on the $m$-primary Gorenstein ideals having a Gorenstein linear resolution which were generated in degree $\delta = 2$. It would be interesting see if one could
also describe explicit polynomial generators for the $\delta = 3$ case. The Rees algebra in this case will be of fiber type. One obstacle to computing the defining ideal, is that we would no longer have the structure theorem coming from [13]. However, there is a series of papers by El Khoury and Kustin which describes the resolution of the resolution of $m$-primary Gorenstein ideals having a Gorenstein linear resolution for general $\delta$. We will no longer have a single structure controlling all ideals in the $\delta = 3$ case. Instead we will have a handful of classes each with their own associated structure theorems. It is worth investigating if any or all of these classes resemble $m^3$ closely enough to compute an explicit polynomial generating set for the special fiber ring using methods similar to Chapter [4]. In this case, I also hope to show that $I(X) = I_d(B) : n^\infty$ and compute the degree of saturation.


