HOLOMORPHIC POLAR COORDINATES AND SEGAL–BARGMANN SPACE

Abstract
by
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In Real Euclidean Space, polar coordinates allow mathematicians to calculate the norm of higher dimensional $SO$-invariant functions with relative ease by reducing the problem to a 1-dimensional integral. In this dissertation I look at the Complex Segal–Bargmann Space using the $C_t$ transform. I find there is a “holomorphic” version of polar coordinates that allows us to do the same in the odd dimensional cases. A geometric approach for this was done by Areerak Kaewthep and Wicharn Lewkeeratiyutkul using the $B_t$ transform in [9], but this method is not easily generalized to non-Euclidean Spaces. Motived by the works of Gestur Ólafsson and Henrik Schlichtkrull in [10], I use shift operators to find this “holomorphic” version of polar coordinates in $C_t$ version of the Segal–Bargmann transform.
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CHAPTER 1

INTRODUCTION

1.1 History

At the end of the 19th century, the understanding of physics began to change. Many people had been asking the question, “What is electricity?”, and the answers varied. Originally thought to be a fluid, J.J. Thomson showed through his cathode ray experiment that electricity consisted of particles called electrons. However, earlier experiments by James Clerk Maxwell, building off the work of Thomas Young, had shown how electricity, and thus electrons, behaved in a manner consistent with waves. This phenomena flew in the face of classical mechanics and so a new theory had to be developed which accounted for the wave-particle duality. Thus we have the study of quantum mechanics.

Among the first questions were how to determine the position and velocity of a particle. A simple question in classical mechanics, but already physicists hit a roadblock. It was proven that the exact position and momentum of the electron could not be determined. This was postulated originally by Werner Heisenberg in what is now referred to as the Heisenberg Uncertainty Principle. Mathematicians showed that it was better to encode the data of the position of the particle in a probability distribution, which we will call $\psi$, and the data of the momentum of the particle in a different probability distribution, which we will call $\phi$.

David Hilbert revolutionized quantum mechanics with his mathematical theories, viewing $\psi$ and $\phi$ as functions living in different Hilbert spaces. It was later shown that $\phi$ could be calculated, modulo some constants, by taking the Fourier transform of
Thus by our understanding of Hilbert spaces, we can equate “momentum space” to being the dual of “position space.” This was a breakthrough at the time, but physicists wanted to look at a space which represents both states at the same time, which we call phase space. However due to the uncertainty principle, the position space and momentum space cannot just be “glued” together. So the question had to be asked, “Is there a good way to have a phase space probability distribution that does not compromise the Uncertainty Principle?” This brings us to the work of Irving Segal and Valentine Bargmann in the 1960s.

They created a transformation, aptly named the Segal–Bargmann transform, that transformed $\psi \in L^2(\mathbb{R}^n)$ into a holomorphic function in square integrable space, $HL^2(\mathbb{C}^n)$. Using this transformation, quantum mechanics could take the original position function $\psi$, which lived in “position space”, to phase space. Phase space essentially illustrates the position of an electron on the real axis and the momentum on the imaginary axis. The functions in Segal–Bargmann space are holomorphic to preserve the Heisenberg Uncertainty Principle, since holomorphic functions cannot be localized. This is a consequence of Liouville’s Theorem. After all of this was discovered, techniques to understand phase space began to be developed. An important aspect of understanding functions in phase space is to be able to calculate the norm, because the norm can relay many different types of information.

1.2 The Segal–Bargmann Transform

The Segal–Bargmann transform is crucial to developments in quantum mechanics. There are a couple different forms the transform can take, but for this paper we are going to use the “$C_t$” transform used in [5]. For all $t > 0$, we define the Segal–Bargmann transform of functions in Euclidean space, $C_t(f) : L^2(\mathbb{R}^n) \rightarrow HL^2(\mathbb{C}^n)$ by

$$C_t(f) = (e^{\Delta t/2} f)_{\mathbb{C}}.$$

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where \( e^{t\Delta/2} \) is the heat operator on \( \mathbb{R}^n \) and \((\cdot)_C\) denotes the holomorphic extension.

An important note is the heat operator can be formally expressed as the power series expansion of powers of the Laplacian. This will become more relevant later in the section.

From the work of Segal and Bargmann, they calculated the norm of function in Segal–Bargmann space in the following way,

\[
||F||^2_{H^2(C^{2n+1},\nu_{t}^{2n+1})} = (2\pi \hbar)^{-n} \int_{\mathbb{C}^{2n+1}} |F(\omega)|^2 \ d\nu_t,
\]

where if \( \omega = (z_1, \ldots, z_{2n+1}) \),

\[
d\nu_t^{2n+1} := \frac{e^{-|\text{Im}(\omega)|^2/t}}{(\pi t)^{2n+1}} (dz_1 \wedge d\bar{z}_1) \cdots (dz_{2n+1} \wedge d\bar{z}_{2n+1}). \tag{1.1}
\]

The Segal-Bargmann transform is application of the heat operator and as such the norm is calculated with the heat kernel, which in the \( 2n+1 \)-dimensional case is \( \nu_t^{2n+1} \).

Another note is SO-invariant functions, where SO is the special orthogonal matrix group, on the real side, will still be SO-invariant when taken to Segal–Bargmann space. This is trivial to see since the Laplacian is rotationally invariant.

### 1.3 The Goal

Recall that in order to find the \( L^2 \) norm of a function, we try to integrate over the space of the magnitude of the function squared with respect to some measure. Thus the \( L^2 \) norm of \( \phi : M \to N \) is given by

\[
||\phi||^2_{L^2(M,\mu)} := \int_M |\phi|^2 \ d\mu,
\]

where \((M, \mu)\) is a measure space and \( N \) is a normed vector space.

Even in the Euclidean case, this becomes complicated as we extend into higher
dimensions and requires us to calculate several iterated integrals to evaluate the norm.

In certain cases, however, the norm of a function with a higher dimensional domain can be evaluated with ease, by reducing the integration to a one dimensional space. The classic example of this is functions in $\mathbb{R}^n$ that are rotationally invariant, meaning $f(Ax) = f(x)$ if $A$ is in the special orthogonal group, $SO(n, \mathbb{R})$. When we try evaluate the $L^2$ norm of these functions, we are integrate using “polar coordinates.” Thus the norm of a function

$$
||f||_{L^2(\mathbb{R}^{n+1})}^2 := \int_{\mathbb{R}^n} |f(x_1, x_2, \ldots, x_{n+1})|^2 \, dx,
$$

where $f \in L^2(\mathbb{R}^{n+1})^{SO}$, is now simplified to

$$
\frac{2\pi^{n+1}}{\Gamma\left(\frac{n+3}{2}\right)} \int_0^{\infty} |f(r)|^2 \, r^n dr.
$$

The above integral is quite easy to deal with, which makes finding the norm substantially easier. Other methods in different spaces were discussed by Camporesi in [2].

This also gives us hope for other such nice techniques to find the norm of $SO$ invariant functions in other types of spaces. We begin by examining the $SO(n, \mathbb{C})$—the complexification of $SO(n, \mathbb{R})$—invariant subspace of the holomorphic $L^2$ space of functions in $\mathbb{C}^n$. For sake of convention when referring to real space, $SO$ means $SO(n, \mathbb{R})$ and when referring to a complex space $SO$ means $SO(n, \mathbb{C})$. Due to the techniques we will be using, we are going to start by only looking at the higher odd dimensional spaces. Thus our goal is to find a measure, $\mu_t(z)$, such that we express the $L^2$ norm in the following way,

$$
||F||_{HL^2(\mathbb{C}^{2n+1}, \mu_t^{2n+1})}^2 = \int_{\mathbb{C}} |f(z, 0, \cdots, 0)|^2 \, d\mu_t(z),
$$

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recall $\nu_t^{2n+1}$ from equation \[1.1\] is a constant multiple of “bilinear-form Gaussian,” $e^{-|\text{Im}(\omega)|^2/t}$ with $\omega \in \mathbb{C}^{2n+1}$. This technique of “holomorphic polar coordinates” will allow us to calculate the norm of these $SO$ invariant functions of higher dimensional space using only one integration, which is much nicer than having to do iterated complex integrals.

1.4 Shift Operators

In order to better understand the space $HL^2(\mathbb{C}^{2n+1})$, we will take functions from the $SO$ invariant subspace and map them to functions in $HL^2(\mathbb{C})$ by an operator which we will call a shift operator. To find this shift operator, it is better to look at the real side and then work our way back to holomorphic space. A real shift operator, $\Lambda$, which takes functions in $L^2(\mathbb{R}^{2n+1})$ to functions in $L^2(\mathbb{R}^1)$ is defined to be an operator that intertwines with the radial part of the Laplacian,

$$\Delta_1 \Lambda = \Lambda \Delta_{2n+1}.$$ 

Since the paper will only address radial functions, $\Delta_n$ will be the radial part of the $n$—dimensional Laplacian, where $\Delta_n^T$ will be the full Laplacian talked about by Heckman \[8\] and Opdam \[11\]. More intuitively, we call it a shift operator because it shifts the dimension of the space by $2n$ dimensions, hence why we start by looking at the higher odd dimensional spaces. Now since we are looking at this intertwining map, we will be able to nicely transition between using the real shift operator and the Segal–Bargmann transform at a cost of changing the dimension by 2. This will help us in the long run because we can move from higher dimensional space to 1—dimensional space anyway. After combining the real shift operator with the Segal–Bargmann transform we will find the shift operator we are looking for, the complex shift operator. Thus the newly found complex shift operator will map $SO$-invariant functions of $HL^2(\mathbb{C}^{2n+1})$
to even or odd functions in $H L^2(\mathbb{C})$ depending on if $n$ is even or odd. In particular, the $SO$ invariant subspace will be mapped to either the odd function subspace or the even subspace, dependent of the value of $n$.

And thus by finding a map,

$$
\Lambda_z : H L^2(\mathbb{C}^{2n+1}) \mapsto H L^2(\mathbb{C})^{E/O},
$$

where $E/O$ indicates the appropriate even or odd space, which preserves the properties we are looking for, we begin to find ways to connect the multidimensional complex space with just $\mathbb{C}$. Now a real shift operator where $\Delta_1 \Lambda = \Lambda \Delta_{2n+1}$ is not unique. But not only do we want to reduce the dimension, but we also wish to find the norm so the property of $\Lambda$ being unitary is also desired. Making sure that we have an operator that has both properties is a nontrivial task.

1.5 A First Look

In their paper [10] Ólafsson and Schlichtkrull begin in the more abstract case dealing with a connected semisimple Lie group, $G$ with the maximal compact subgroup, $K$. They examine $K$-invariant functions in $L^2(G/K)$ and are able to identify an operator $\Lambda_z$ with all of the properties we are looking for. However, they did not calculate an explicit form of $\Lambda_z$ except in the most trivial case. Adapting their paper to the Euclidean case, we are motivated to find the norm of $F(\omega) \in H L^2(\mathbb{C}^{2n+1})^{SO}$ as an integral over the complex plane. I will show the existence of a unitary shift operator $\Lambda_z : H L^2(\mathbb{C}^{2n+1}, \nu_{t}^{2n+1})^{SO} \rightarrow H L^2(\mathbb{C}, e^{-y^2/t} \, dz)^{E/O}$, so that

$$
||F||_{H L^2(\mathbb{C}^{2n+1}, \nu_{t}^{2n+1})^{SO}}^2 = \int_{\mathbb{C}} |(\Lambda_z F)(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi} t} \, dz \, d\bar{z}.
$$
Recall that from equation 1.1, \[ e^{-y^2/t} \frac{dz \, d\bar{z}}{\sqrt{\pi t}} = d\nu_t^1. \] We will also use the notation \( \nu_t^1 := \frac{e^{-y^2/t}}{\sqrt{\pi t}}. \) This first look at the norm is helpful, since existence has been proven and in theory could be calculated without having to do all the iterated integrations as long as we knew where \( \Lambda_z \) took \( F. \) Without the explicit form of \( \Lambda_z \) being known, this task is still out of reach. Another issue with their result is the norm is not in the \( L^2 \) structure which mathematicians desire, which brings us to this dissertation.

1.6 This Dissertation

Motivated by Ólafsson and Schlichtkrull, we began by looking at the Euclidean case and seeing if an explicit form of \( \Lambda_z \) could be found. Starting by looking at the real case, I find an operator \( \Lambda \) which is unitary and is an intertwining map. By using previous results on the isometry of the Segal–Bargmann transform, this leads to a first main result.

**Theorem 1.** There exists a surjective map

\[ \Lambda_z : HL^2(\mathbb{C}^{2n+1}, \nu_{t}^{2n+1})^{SO} \to HL^2(\mathbb{C}, e^{-y^2/t} \frac{dz \, d\bar{z}}{\sqrt{\pi t}})^{E/O}, \]

such that

\[ ||F||_{HL^2(\mathbb{C}^{2n+1}, \nu_{t}^{2n+1})^{SO}}^2 = \int_{\mathbb{C}} |(\Lambda_z F)(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dz \, d\bar{z}. \]

However after finding \( \Lambda \) explicitly and from there calculating \( \Lambda_z, \) our next goal is to calculate the norm of a function in the space \( HL^2(\mathbb{C}^{2n+1})^{SO}, \) but expressed as a standard \( L^2 \) norm. Now by how \( \Lambda_z \) is defined as a shift operator, it takes holomorphic functions to holomorphic functions. Thus we are going to from now on use the notation \( \Lambda_z \) as the holomorphic operator and then use the notation \( \Lambda_{\bar{z}} \) to be the antiholomorphic operator of \( \Lambda_z. \) This operator will be defined in the following way.
\[ \Lambda_\bar{z} F(\bar{z}) := \overline{\Lambda_z F(z)}. \] Therefore we will show we can express the isometry motivated by Ólafsson and Schlichtkrull’s results by

\[
\int_\mathbb{C} |\Lambda_\bar{z} F(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dz \, d\bar{z} = \int_\mathbb{C} \Lambda_\bar{z} F(z) \Lambda_\bar{z} \overline{F(\bar{z})} \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dz \, d\bar{z} = \int_\mathbb{C} \Lambda_\bar{z} \Lambda_\bar{z} |F(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dz \, d\bar{z}.
\]

We are able to commute the \( \Lambda_\bar{z} \) with \( F(z) \) since an antiholomorphic operator does not interact with a holomorphic function. Now since the measure is some \( e^{-y^2/2t} \, dzd\bar{z} \), we express the norm with inner product notation,

\[
\left\langle |F|^2 \frac{e^{-\text{Im}(\omega,\text{Im}(\omega))/t}}{(t\pi)^{2n+1}2} \right\rangle_{\text{HL}^2(\mathbb{C}^{2n+1})^{SO}} = \left\langle \Lambda_\bar{z} \Lambda_\bar{z} |F|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right\rangle_{\text{HL}^2(\mathbb{C})^{E/O}}.
\]

Thus in order to get to the \( L^2 \) norm, we are going to take the adjoint of \( \Lambda_\bar{z} \) and \( \Lambda_\bar{z} \) and get

\[
\left\langle \Lambda_\bar{z} \Lambda_\bar{z} |F|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right\rangle_{\text{HL}^2(\mathbb{C})^{E/O}} = \left\langle |F|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right\rangle_{\text{HL}^2(\mathbb{C})^{E/O}}
\]

and thus by taking the adjoint we will have

\[
\left\langle |F|^2 \frac{\Lambda_\bar{z}^* \Lambda_\bar{z}^* e^{-y^2/t}}{\sqrt{\pi t}} \right\rangle_{\text{HL}^2(\mathbb{C}^{2n+1})^{SO}} = \int_\mathbb{C} |F(z)|^2 \Lambda_\bar{z}^* \Lambda_\bar{z}^* \left( \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right) \, dzd\bar{z}.
\]

Now by evaluating the new measure in the case of \( n = 1 \) and \( n = 2 \), we get the following results.

**Theorem 2.** If \( F \in \text{HL}^2 (\mathbb{C}^3, \nu_1^3)^{SO} \), then

\[
||F||^2_{\text{HL}^2(\mathbb{C}^3, \nu_1)^{SO}} := \int_{\mathbb{C}^3} |F(z)|^2 \, d\nu_1^3 = \frac{4\pi}{3} \int_{\mathbb{C}} |F(z, 0, \cdots, 0)|^2 \, |z|^2 \, d\nu_1^1.
\]

The result for \( \mathbb{C}^3 \) gives the naive assumption that “holomorphic” polar coordinates
might be the same as real polar coordinates. However, when we look at the result from $\mathbb{C}^5$, we see that new measure becomes significantly more difficult.

Theorem 3. If $F \in H^2 L^2(\mathbb{C}^5, \nu^5_t)_{\text{SO}}$, then

$$\|F\|^2_{H^2 L^2(\mathbb{C}^5, \nu^5_t)_{\text{SO}}} := \int_{\mathbb{C}^5} |F(z)|^2 \, d\nu^5_t = \frac{8\pi^2}{15} \int_{\mathbb{C}} |F(z, 0, \cdots, 0)|^2 (|z|^4 + 2t |z|^2) \, dv^1_t.$$ 

Something to note, however, is that the limit as $t$ goes to 0 of $|z|^4 + 2t |z|^2$ is equal to $|z|^4$. This is what we would expect from real polar coordinates. This leads us to the Theorem of the generalized case.

Theorem 4. If $F \in H^2 L^2(\mathbb{C}^{2n+1}, \nu^{2n+1}_t)_{\text{SO}}$, then

$$\|F\|^2_{H^2 L^2(\mathbb{C}^{2n+1}, \nu^{2n+1}_t)_{\text{SO}}} := \int_{\mathbb{C}^{2n+1}} |F(z)|^2 \, d\nu^{2n+1}_t = \int_{\mathbb{C}} |F(z, 0, \cdots, 0)|^2 P(|z|) \, dv^1_t,$$

where $P(|z|)$ is a polynomial of $|z|$ with the following properties:

1. $P(|z|)$ is of order $2n$
2. $\lim_{t \to 0} P(|z|) = |z|^{2n}$
3. $P(|z|)$ is an even polynomial.

The difficulty in the Euclidean case comes down to a couple of problems. The first issue is identifying the unitary shift operator. Although Ólafsson and Schlichtkrull showed existence, it is quite a bit harder to find the explicit form of $\Lambda$ or its adjoint. This was my first task. The second difficulty is actually showing the isometry. It is significant that $\Lambda^*_z \Lambda^*_z \phi(z)$ is a polynomial multiplied by a Gaussian, because it shows rapid decay along the imaginary axis. However, once the explicit form of $\Lambda_z$ is shown, the naive assumption would be the new measure would not decay. Thus demonstrating this rapid decay is a difficult part of the dissertation and has not been fully developed yet. We are able to show this easily in $\mathbb{C}^3$, but the $\mathbb{C}^5$ case was far
more challenging. The techniques used did not give an obvious generalization and more detailed analysis of the adjoint was necessary to complete the proof.
CHAPTER 2

A REAL SHIFT OPERATOR

As was introduced in the first section 1.3, polar coordinates allows us to identify the \( SO^- \) invariant subspace of \( L^2(\mathbb{R}^{2n+1}) \) with the even subspace of \( L^2(\mathbb{R}) \) with respect to a certain measure, which we denote \( L^2(\mathbb{R})^E \). Namely

\[
L^2(\mathbb{R}^{2n+1}, d\mathbf{x})^{SO} \cong L^2 \left( \mathbb{R}, \frac{2\pi^{n+1}}{\Gamma \left( \frac{2n+3}{2} \right)} r^{2n} \, dr \right).
\]

The right-hand side of the isometry is what we define as \( L^2(\mathbb{R})^E \) for concise notation. Throughout this paper we will refer to functions in \( L^2(\mathbb{R}^{2n+1}, d\mathbf{x})^{SO} \) using the notation \( F(r) \). The \( r \) and \( F \) are the common polar coordinate counterparts to the functions in \( L^2(\mathbb{R}^{2n+1}, d\mathbf{x})^{SO} \).

Additionally, the other space we will most often deal with is a subspace of \( L^2(\mathbb{R}) \) called \( L^2(\mathbb{R})^{E/O} \).

**Definition 5.** If \( L^2(\mathbb{R}^{2n+1})^{SO} \) and \( L^2(\mathbb{R})^{E/O} \) are in the same diagram, then \( L^2(\mathbb{R})^{E/O} \) is the even subspace of \( L^2(\mathbb{R}) \) is \( n \) is even and \( L^2(\mathbb{R})^{E/O} \) is the odd subspace of \( L^2(\mathbb{R}) \) is \( n \) is odd. This definition extends to \( HL^2(\mathbb{C}, \nu_\ell^1)^{E/O} \).

2.1 The Commutative Diagram

Because of this, when we apply the Laplacian to a function in this subspace, we only need to consider how the radial part of the operator acts on the identification. Recall that the Laplacian preserves the \( SO^- \)-invariance of the space. Since the
Segal–Bargmann transform, $C_t$, is formally a power series expansion of Laplacians followed by a holomorphic extension, it will be necessary to investigate this relationship. Recall that a shift operator, $Λ$, is an operator that intertwines the radial part of the Laplacian, thus it should formally intertwine the heat operator, $e^{t\Delta/2}$. We then have the following commutative diagram if we can find such a shift operator.

\[
\begin{array}{c}
L^2(\mathbb{R}^{2n+1})^{SO} \xrightarrow{\frac{i\Delta_{2n+1}}{2}} L^2(\mathbb{R}^{2n+1})^{SO} \\
\Lambda \downarrow \quad \Lambda \\
L^2(\mathbb{R}^{1})^{E/O} \xrightarrow{e^{\frac{i\Delta_1}{2}}} L^2(\mathbb{R}^{1})^{E/O}
\end{array}
\]

Recall that the notation $E/O$ indicates the even or odd subspace depending on the value of $n$. If $n$ is odd, then it is the odd subspace. If $n$ is even, then it is the even subspace. Now ultimately, we want to find a map,

\[
Λ_z: HL^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO} \to HL^2(\mathbb{C}, \nu_t^{1})^{E/O}.
\]

So we construct this diagram.

\[
\begin{array}{c}
L^2(\mathbb{R}^{2n+1})^{SO} \xrightarrow{C} HL^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO} \\
\Lambda \downarrow \quad \Lambda_z \\
L^2(\mathbb{R}^{1})^{E/O} \xrightarrow{C} HL^2(\mathbb{C}, \nu_t^{1})^{E/O}
\end{array}
\]

If we can find this $Λ$ and make it an isometry, then holomorphic extensions are also isometries, so we have a commutative diagram. And $Λ_z$ should just be the holomorphic extension of the operator $Λ$. Now to have the Segal–Bargmann transform we defined in the introduction we need to put the two diagrams together. The top two arrows combined make up the Segal–Bargmann transform, which is a unitary
Once we find a unitary shift operator $\Lambda$ and show surjectivity, we can establish that an isometry $\Lambda_\zeta$ exists. Since the work of Segal and Bargmann on the Segal–Bargmann transform has already shown top and bottom lines are unitary maps. Then from our unitary commutative diagram we can establish the following isometry:

$$||F||_{HL^2(\mathbb{C}^{2n+1}, \nu_2^{2n+1})^SO}^2 = \int_\mathbb{C} |(\Lambda_\zeta F)(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} d\zeta d\bar{\zeta}.$$ 

Our next goal after we find such a unitary operator we will be to try and push it over to the other side, so have an explicit way to calculate the $HL^2$ norm on $SO$-invariant subspaces of higher odd dimensional Segal-Bargmann space.

In order to establish the unitary commutative diagram we will need some machinery. We will revisit this commutative diagram after the machinery has been fully developed.

### 2.2 Defining a Real Shift Operator

In order to find the $\Lambda_\zeta$ we desire, we first need to find $\Lambda$ on the real side. Hopefully, then $\Lambda_\zeta$ will just be a holomorphic extension of $\Lambda$. This next part takes us through first identifying an operator between $L^2(\mathbb{R}^{2n+1})^{SO}$ and $L^2(\mathbb{R}^1)^{E/O}$ that is more obvious with most of the properties we want and then seeing how we can adjust that operator to make it a unitary operator. Our goal in the this chapter is to prove our first main result, namely
Theorem 1. There exists a $\Lambda_z : H^2(\mathbb{C}^{2n+1}, n^{2n+1})^{SO} \to H^2(\mathbb{C}, \frac{e^{-y^2/t}}{\sqrt{\pi t}} \; dzd\bar{z})^{E/O}$, such that

$$||F||^2_{H^2(\mathbb{C}^{2n+1}, n^{2n+1})^{SO}} = \int_{\mathbb{C}} |(\Lambda_z F)(z)|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \; dzd\bar{z}$$

and $\Lambda_z$ is a surjective map.

Since we are looking at the $SO$-invariant subspaces, the first task to break the Laplacian down to just the radial components.

Definition 6. $\Delta_{2n+1}^T$ is the differential operator known as the Laplacian which acts on $L^2(\mathbb{R}^{2n+1}) \to L^2(\mathbb{R}^{2n+1})$, defined by

$$\Delta_{2n+1}^T := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{2n+1}^2}.$$

We use the superscript $T$ to denote the total Laplacian, but since we only are using the radial part of the Laplacian we reserve $\Delta$ with no superscript to mean the radial part of the Laplacian. Throughout this paper we are going to be looking at the $SO$-invariant subspaces, so we only need to figure out how the radial part of $\Delta$ interacts with functions. Thus we have the following lemma

Lemma 7. If $\Delta_{2n+1}^T$ is the operator defined above but acts on the subspace $L^2(\mathbb{R}^{2n+1})^{SO} \to L^2(\mathbb{R}^{2n+1})^{SO}$, then

$$\Delta_{2n+1}^T f(r) = \left( \frac{d^2}{dr^2} + \frac{2n}{r} \frac{d}{dr} \right) f(r),$$

where $r := \sqrt{x_1^2 + x_2^2 + \cdots + x_{2n+1}^2}$. We call $\left( \frac{d^2}{dr^2} + \frac{2n}{r} \frac{d}{dr} \right)$ the radial part of the Laplacian or $\Delta_{2n+1}$.

The easiest way to show this is to apply $\Delta_{2n+1}$ to a general $SO$-invariant function, $f(r)$. It is only a simple calculation to prove the above lemma.

The next goal would be to formally define and find a shift operator.
Definition 8. An operator \( \Lambda : L^2(\mathbb{R}^{2n+1})^{SO} \rightarrow L^2(\mathbb{R})^{E/O} \) is called a real shift operator if
\[
\Delta_1 \Lambda = \Lambda \Delta_{2n+1}
\]
or alternatively
\[
\Delta_{2n+1} \Lambda^* = \Lambda^* \Delta_1.
\]

It is an easy calculation to see that a choice
\[
\Lambda^* = \left( \frac{1}{r} \frac{d}{dr} \right)^n
\]
will fit the second condition, given a dimension of \(2n + 1\). In the \(n = 1\) case, the reader can verify
\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \left( \frac{1}{r} \frac{d}{dr} \right) f(r) = \left( \frac{1}{r} \frac{d}{dr} \right) \frac{d^2}{dr^2} f(r).
\]

From there we can just use induction to verify the result. Thus we have the adjoint of a shift operator. Since I had an adjoint of a shift operator to start with, it was easier to work with the adjoint in general. For the rest of the time, we will focus mostly on exploring \( \Lambda^* \) instead of \( \Lambda \). Although we now have the adjoint of a shift operator, it is not unitary.

2.3 The Unitary Real Shift Operator

After starting with the nonunitary shift operator \( \left( \frac{1}{r} \frac{d}{dr} \right)^n \) that was discussed in equation 2.1, we seek to alter this shift operator to be unitary.

Definition 9. Let \( \Xi(n) \) be defined in the following way. If \( n \) is odd, then \( \Xi(n) := (-1)^{\frac{n+3}{2}} \). If \( n \) is even, then \( \Xi(n) := (-1)^{\frac{n}{2}} \).

Definition 10. We begin by defining \( \Lambda_n^* \) as a map from \( L^2(\mathbb{R}^1)^{E/O} \) to \( L^2(\mathbb{R}^{2n+1})^{SO} \).
by

$$\Lambda_n^* f(r) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} \left( \left( \frac{1}{r} \frac{d}{dr} \right)^n \left( \frac{d}{dr} \right)^{-n} \right) f(r),$$

where on the $\Lambda^*$ side we define the inverse derivative to be

$$\Lambda_n^* f(r) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} \left( \frac{1}{r} \frac{d}{dr} \right)^n \int_0^r \int_0^{s_n} \cdots \int_0^{s_2} \int_0 f(s_1) ds_1 ds_2 \cdots ds_n,$$

where $\text{Vol}(S^{2n}) = \frac{\pi^{2n+1}}{\Gamma\left( \frac{2n+3}{2} \right)}$. This is the volume of the $2n$-sphere with Lebesgue measure.

As previously discussed in the introduction to this chapter, we are taking functions from $L^2(\mathbb{R}^1)^{E/O}$ to $L^2(\mathbb{R}^{2n+1})^{SO}$, but it seems to take $L^2(\mathbb{R}^1)^{E/O}$ to $L^2(\mathbb{R}^1)^E$. The common identification between $L^2(\mathbb{R}^1)^E$ and $L^2(\mathbb{R}^{2n+1})^{SO}$ is such that if $f \in L^2(\mathbb{R}^{2n+1})^{SO}$, then there exists a $g \in L^2(\mathbb{R}^1)^E$ such that $f(x) = g\left( \sqrt{x_1^2 + x_2^2 + \cdots + x_{2n+1}^2} \right)$, denoted $g(|x|)$. Thus our $\Lambda_n^*$ truly finds our function $g$ and then we use the common identification.

With that understanding, the main theorem we want to show is the following:

**Theorem 11.** $\Lambda_n^*$ is an isometry between $L^2(\mathbb{R}^1)^{E/O}$ and $L^2(\mathbb{R}^{2n+1})^{SO}$, where $E/O$ denotes the space of odd functions if $n$ is odd and the space of even functions if $n$ is even. Thus a unitary operator exists.

2.3.1 Fourier Analysis

In order to prove the isometry Theorem 11 we will need the results of Fourier Analysis. In our case, we have the following definition of the Fourier Transform.

**Definition 12.** Let $\mathcal{F}_m$ be the Fourier Transform of functions on $\mathbb{R}^m$ defined by the integral

$$\mathcal{F}_m(g)(k) := \int_{\mathbb{R}^m} g(r) e^{i(r|k)} dr.$$
In particular, if $m = 1$, we use the notation

$$\hat{f} := F_1(g) \text{ and } \check{f} := F_1^{-1}(g).$$

The use of $r$ is telling because we will soon be looking at exclusively radial functions. Given small adjustments to a proof in a paper [3] by Grafakos and Teschl we have the following theorem.

**Theorem 13.** Let $n \geq 1$. Suppose that $g$ is a function on the real line such that the functions $g(|\cdot|)$ are in $L^1(\mathbb{R}^{n+2})$ and also in $L^1(\mathbb{R}^n)$. Then we have

$$F_{n+2}^{-1}(g)(r) = \frac{-1}{r} \frac{d}{dr} F_{n+1}^{-1}(g)(r).$$

Consequently, if $g(|r|) \in L^2(\mathbb{R}^{2n+1})$ and $g(r) \in L^2(\mathbb{R})$, then

$$F_{2n+1}^{-1}(g)(r) = \left( -\frac{1}{r} \frac{d}{dr} \right)^n F_{1}^{-1}(g)(r)$$

Thus if $g$ is a radial function defined on $[0, \infty)$, we can conclude that

$$F_{2n+1}^{-1}(g)(r) = \left( -\frac{1}{r} \frac{d}{dr} \right)^n \int_0^\infty g(k) \cos(kr) dk.$$  

This theorem and consequence will allow us to establish two lemmas that will allow us to prove the isometry. In particular,

**Lemma 14.** If $\phi(kr)$ is equal to sin($kr$) when $n$ is odd and $\phi(kr)$ is cos($kr$) when $n$ is even then,

$$k^n \Lambda_n^* \phi(kr) = \frac{1}{\sqrt{Vol(S^{2n})}} \left( -\frac{1}{r} \frac{d}{dr} \right)^n \cos(kr).$$

as well as
Lemma 15. Suppose $\hat{f} \in L^1$ and $\hat{f}_{kn} \in L^1$, then

$$F_{2n+1}(\Lambda^*_nf)(k) = \frac{\hat{f}(k)}{\sqrt{\text{Vol}(S^{2n})}kn}.$$ 

Assuming for now the above lemmas, let us prove the isometry.

2.4 Proof of Isometry

We begin by showing the shift operator is in fact an isometry, thus proving Theorem [11]. Let $f$ be in $L^2(\mathbb{R}^1)^{E/O}$. Assuming that Lemma 15 and 14 have been established, we can complete the proof of isometry in the Euclidean case by using multiple Plancherel’s theorems.

$$||\Lambda^*_nf||_{L^2(\mathbb{R}^{2n+1})^{SO}}^2 = 2 \text{Vol}(S^{2n}) \int_{0}^{\infty} |(\Lambda^*_nf)(r)|^2 r^{2n} dr$$

$$= 2 \text{Vol}(S^{2n}) \int_{0}^{\infty} |(F_{2n+1}(\Lambda^*_nf))(k)|^2 k^{2n} dk$$

$$= 2 \int_{0}^{\infty} \left| \frac{\hat{f}(k)}{kn} \right|^2 k^{2n} dk$$

$$= 2 \int_{0}^{\infty} |\hat{f}(k)|^2 dk$$

$$= 2 \int_{0}^{\infty} |f(r)|^2 dr$$

$$= ||f||_{L^2(\mathbb{R})^{E/O}}^2$$

and the isometry can be seen. Finally we need to show surjectivity to prove the isometry is between $L^2(\mathbb{R}^1)^{E/O}$ and $L^2(\mathbb{R}^{2n+1})^{SO}$ rather than just into. If $g \in L^2(\mathbb{R}^{2n+1})^{SO}$ then observe how $\mathcal{F}_{2n+1}(g)$ is in the space $L^2(\mathbb{R}^{2n+1})^{SO}$ by construction, but can be associated with $L^2(\mathbb{R}, \text{Vol}(S^{2n})k^{2n}dk)^{Even}$ by polar coordinates and properties of $SO$ invariance. Thus $\sqrt{\text{Vol}(S^{2n})}\mathcal{F}_{2n+1}(g)k^n$ is in $L^2(\mathbb{R}^1)^{E/O}$ since if $n$ is odd the function will be odd and if $n$ is even then the function will remain even. Thus $\mathcal{F}^{-1}_1(\mathcal{F}_{2n+1}(g)k^n)$
is also in $L^2(\mathbb{R}^1)^{E/O}$. Thus

\[
\left(\Lambda^*_n\left(F_1^{-1}(F_{2n+1}(g)k^n)\right)\right)(r) = \int_0^\infty k^nF_{2n+1}(g)(k)\left(\Lambda^*_n\phi(kr)\right)dk \\
= \int_0^\infty F_{2n+1}(g)(k)\left(-1\frac{d}{dr}\right)^n\cos(kr)dk \\
= g(r).
\]

Therefore, we can conclude the map $\Lambda^*_n$ is surjective.

2.5 Proof of Supporting Lemmas

Now that Theorem 11 has been established, let's revisit the two lemmas that supported the result and verify them.

2.5.1 Proof of Lemma 14

The first to do is prove Lemma 14, which is broken up into two cases. First suppose $n$ is even, thus $\phi(kr) = \cos(kr)$. We will be first integrating $\cos(kr)$ with respect to $r$ $n$ times producing $(\frac{-1}{r}\frac{d}{dr})^n\cos(kr)$, where $p(r)$ is an even polynomial of degree $n - 2$. Now observe if $l \geq 2$ and $l$ is even, then

\[
\left(\frac{1}{r}\frac{d}{dr}\right)(c_n r^l) = c'_n r^{l-2}
\]

and if $l = 0$, then $c' = 0$. Because of this, $r^l$ will always be of positive degree or $c'$ will be 0.

Thus from the calculations above, it is clear that after less than $n$ iterations of
\( \left( \frac{1}{r} \frac{d}{dr} \right), p(r) \) will become 0. Thus one can clearly see

\[
k^n \Lambda_n^* \cos(kr) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} k^n \left( \frac{1}{r} \frac{d}{dr} \right)^n \left( \frac{\cos(kr)}{k^n} + p(n) \right)
\]

\[
= \frac{1}{\sqrt{\text{Vol}(S^{2n})}} k^n \left( \frac{-1}{r} \frac{d}{dr} \right)^n \cos(kr)
\]

by our construction of \( \Xi \). Similarly if \( n \) is odd then we will be first integrating \( \sin(kr) \) with respect to \( r \) \( n \) times which produces \( \frac{\cos(kr)}{k^n} + p(n) \) since \( n \) is odd. Thus the equalities above will also hold, but we will swap the \( \cos(kr) \) for the \( \sin(kr) \).

### 2.5.2 Proof of Lemma 15

Lastly we need to prove Lemma 15. We know by Theorem 13 that

\[
\mathcal{F}_{2n+1}^{-1} \left( \frac{\hat{f}(k)}{k^n} \right) (r) = \left( \frac{-1}{r} \frac{d}{dr} \right)^n \int_0^\infty \frac{\hat{f}(k)}{k^n} \cos(kr) dk
\]

\[
= \int_0^\infty \frac{\hat{f}(k)}{k^n} \left( \frac{-1}{r} \frac{d}{dr} \right)^n \cos(kr) dk
\]

\[
= \sqrt{\text{Vol}(S^{2n})} \int_0^\infty \frac{\hat{f}(k)}{k^n} k^n \Lambda_n^* (\phi(kr)) dk
\]

\[
= \sqrt{\text{Vol}(S^{2n})} \int_0^\infty \hat{f}(k) \Lambda_n^* (\phi(kr)) dk
\]

\[
= \sqrt{\text{Vol}(S^{2n})} \Lambda_n^* \left( \int_0^\infty \hat{f}(k) \phi(kr) dk \right)
\]

\[
= \sqrt{\text{Vol}(S^{2n})} \Lambda_n^* f(r).
\]

Thus if we apply the Fourier Transform to both sides of the equality, then by uniqueness we get the desired result.

Lemma 15 also assists us in getting rid of any worries we had about the integrating and taking derivatives. We can now assume throughout this entire section that our
functions are “nice enough” to take derivatives and pull the differential operators inside the integrals. Schwartz functions would be such a “nice enough” space. And since Schwartz functions are dense, we will be able to extend our Isometry Theorem \[11\] to the entire space of \( L^2(\mathbb{R}^{2n+1}) \). 

\[2.6\] Commutative Diagram Revisited

After establishing the fact that \( \Lambda_n^* \) is a unitary shift operator from \( \mathbb{R}^1 \) to \( \mathbb{R}^{2n+1} \), we begin to look at the diagram below that we began discussing in Chapter 2.

The \( \Delta_{2n+1} \) is the \( 2n+1 \)-dimensional Laplacian. In order to establish the left square is commutative, it is sufficient to show \( \Lambda_{2n+1}^* e^{t \Delta_{1/2}} = e^{t \Delta_{2n+1}/2} \Lambda_{2n+1}^* \).

We can do this by simply applying Lemma \[15\] if we let our \( f \) be \( e^{t \Delta_{1/2}} g \).

\[
\mathcal{F}_{2n+1} \left( \Lambda_{2n+1}^* e^{t \Delta_{1/2}} g \right) (k) = \frac{e^{t \Delta_{1/2} / 2} \hat{g} (k)}{k^{2n+1}}.
\]

By using the properties of the 1-dimensional Fourier Transform we can rewrite the right-hand side as

\[
\frac{e^{-tk^2/2} \hat{g} (k)}{k^{2n+1}} = \frac{e^{-tk^2/2 / 2} \hat{g} (k)}{k^{2n+1}}.
\]

We can use Lemma \[15\] again and show this is equal to

\[
e^{-tk^2/2} \mathcal{F}_{2n+1} \left( \Lambda_{2n+1}^* g \right) (k).
\]
But by the properties of the $2n + 1$-dimensional Fourier transform, we can pull $e^{-tk^2/2}$ into the transform, thus we have

$$F_{2n+1} (\Lambda^{*}_{2n+1} e^{t\Delta_1/2} f) (k) = F_{2n+1} (e^{t\Delta_{2n+1}/2} \Lambda^{*}_{2n+1} f) (k).$$

Thus by the uniqueness of the Fourier Transform, we have our intertwining map shown.

With the intertwining map shown and the isometry and surjective shown, then the proof of Theorem 1 is established.
CHAPTER 3

\( \mathbb{C}^3 \) AND \( \mathbb{C}^5 \) CASE

Now that a real unitary shift operator \( \Lambda \) has been found, we are going to look at Segal–Bargmann space to complete our commutative diagram. Now we looked at \( \Lambda^* \) with the inner product

\[
\int_{\mathbb{R}} (\Lambda f(r)) g(r) \, dr = \text{Vol}(S^n) \int_{\mathbb{R}} f(r) (\Lambda^* g(r)) r^n \, dr.
\]

When we are looking at holomorphic extension \( \Lambda_z \), we are going to find the adjoint \( \Lambda^*_z \) with the inner product analogous to the real case

\[
\int_{\mathbb{C}} |\Lambda_z F(z)|^2 \nu^1_t \, dz \, d\bar{z} = \text{Vol}^2(S^n) \int_{\mathbb{C}} |F(z)|^2 \left( (\Lambda^*_z \Lambda^*_z \nu^1_t) \right) z^n dz \, \bar{z}^n d\bar{z}.
\]

We first need to establish then what \( \Lambda^*_z \Lambda^*_z \) is in order to prove Theorem 4. We begin by looking at the dimension 3 case and the dimension 5 case. The dimension 3 case will be too trivial to help us out, but the dimension 5 case gives us quite a bit of insight.

In order to properly identify all of the parts we are going to need, we start in real space because it is easier to find the shift operators there. From that point we will use the Segal–Bargmann transform to move into holomorphic space. It is for this reason that the shift operator needs to intertwine with the Laplacian.
3.1 The $\mathbb{C}^3$ Case

The $\mathbb{C}^3$ case is only going to be addressed briefly since it makes the measure seem too convenient. Recall we are trying to express the norm of $HL^2(\mathbb{C}^3, \nu_3^t)^{SO}$ by an integral over just $\mathbb{C}$.

**Theorem 2.** If $F \in HL^2(\mathbb{C}^3, \nu_3^t)^{SO}$, then

$$||F||_{HL^2(\mathbb{C}^3, \nu_3^t)^{SO}}^2 := \int_{\mathbb{C}^3} |F(\omega)|^2 \, d\nu_3^t(\omega) = \frac{4\pi}{3} \int_{\mathbb{C}} |F(z, 0, \cdots, 0)|^2 |z|^2 \, d\nu_1^t.$$ 

As established above, the adjoint of the nonunitary shift operator is $\frac{1}{r} \frac{d}{dr}$. However, based on Definition 10, if the $\frac{d}{dr}$ was just removed and we added a factor of $\sqrt{2\pi}$, then we would have the operator

$$\Lambda^* := \frac{\sqrt{3}}{\sqrt{4\pi r}},$$

which recall takes $L^2(\mathbb{R}^1)^O \rightarrow L^2(\mathbb{R}^3)^{SO}$. This new operator would still have the intertwining property

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}\right) \left(\frac{\sqrt{3}}{\sqrt{4\pi r}}\right) f(r) = \left(\frac{\sqrt{3}}{\sqrt{4\pi r}}\right) \frac{d^2}{dr^2} f(r).$$

The above equality may easily be verified through a small calculation. Furthermore,

$$\left|\frac{\sqrt{3} f(r)}{\sqrt{4\pi r}}\right|^2_{L^2(\mathbb{R}^3)^{SO}} \leq \frac{8\pi}{3} \int_0^\infty \left|\frac{\sqrt{3} f(r)}{\sqrt{4\pi r}}\right|^2 \, r^2 \, dr = 2 \int_0^\infty |f(r)|^2 \, dr = ||f(r)||_{L^2(\mathbb{R})^O}^2.$$ 

This shows that the adjoint, $\Lambda^*$, of the unitary operator $\Lambda$ is just $\Lambda^* = \frac{\sqrt{3}}{\sqrt{4\pi r}}$. From knowing this $\Lambda^*$, we can easily find the unitary shift operator $\Lambda$. All we need to do is reverse the above isometry. Thus $\Lambda$ is $\frac{\sqrt{4\pi}}{\sqrt{3}} r$. This is consistent with our result in Theorem 11.
This would suggest the holomorphic extension of $\Lambda_z$ to be $\sqrt{\frac{4\pi}{\sqrt{3}}}z$. Therefore the isometry between $HL^2(\mathbb{C}^3,\nu^3_t)$ and $HL^2(\mathbb{C},\nu^1_t)$ would be

$$\|F\|^2_{HL^2(\mathbb{C}^3,\nu^3_t)_{SO}} = \frac{4\pi}{3} \int_\mathbb{C} |F(z)|^2 |z|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dx \, dy.$$  

This is what Theorem 2 claims. This formula looks remarkably similar to the norm of $SO$-invariant functions in $\mathbb{R}^3$ when using polar coordinates and upon seeing this $\mathbb{C}^3$ result, one might hope that the norm could be calculated in a similar fashion.

Just moving to the $\mathbb{C}^5$ case, however, we begin to see that the general formula is not just a multiplication of powers of $|z|^2$, but in fact a polynomial of the term $|z|^2$. So we begin by looking at the $\mathbb{C}^5$ case as our motivating example for the general case.

### 3.2 The $\mathbb{C}^5$ Case

The isometry in the commutative diagram state that

$$\|F\|^2_{HL^2(\mathbb{C}^{2n+1},\nu^2_t)_{SO}} = \int_\mathbb{C} |\Lambda_z F|^2 \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dx \, dy.$$

We follow a similar outline for the $\mathbb{C}^3$ case, but acknowledge each step will be looked at in more depth, since the measure is far more interesting than in the previous case. We seek to prove

**Theorem 3.** If $F \in HL^2(\mathbb{C}^5,\nu^5_t)_{SO}$, then

$$\|F\|^2_{HL^2(\mathbb{C}^5)_{SO}} := \int_\mathbb{C} |F(z)|^2 \, d\nu^5_t = \frac{8\pi^2}{15} \int_\mathbb{C} |F(z,0,\ldots,0)|^2 \left(|z|^4 + 2t|z|^2\right) \, d\nu^1_t.$$

We begin by identifying the unitary shift operator that makes the above equality true. After I find the shift operator, I will find the adjoint and use it to get the $L^2$ norm.
3.2.1 The Real Unitary Shift Operator

Specifying Definition 10 to the case of \( n = 2 \), we have the unitary shift operator for the 5-dimensional case.

**Example 16.** The adjoint of the unitary shift operator, \( \Lambda^* \) from \( L^2(\mathbb{R})^E \) to \( L^2(\mathbb{R}^5)^{SO} \) is given by

\[
\Lambda^* = \frac{\sqrt{15}}{2\sqrt{2\pi}} \frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} \right)^{-1}.
\]

where in this context \( \left( \frac{d}{dr} \right)^{-1} f(r) \) is defined to mean

\[
\int_0^r f(s) \, dr.
\]

From this lemma we can discover the operator we truly want to know, \( \Lambda^*_z \). And thus we have

\[
\int_{\mathbb{C}} |\Lambda_z F|^2 e^{-y^2/t} \, dx \, dy = \text{Vol}^2(S^{2n}) \int_{\mathbb{C}} |F|^2 \left( \Lambda_z^* \Lambda_z^* e^{-y^2/t} \sqrt{\pi t} \right) |z|^8 \, dx \, dy.
\]

We can assume this is an isometry by using Theorem 11 and then proceed to looking at the other parts of commutative diagram.

Thus we have the following lemma, which can be shown formally quite easily, assuming all the boundary terms work out.

**Lemma 17.** If \( \Lambda^* = \frac{\sqrt{15}}{2\sqrt{2\pi}} \frac{1}{r} \frac{d}{dr} \int_0^r f(s) \, ds \), then the following is true

\[
\Lambda^*_z = \frac{\sqrt{15}}{2\sqrt{2\pi}} \frac{1}{z} \frac{\partial}{\partial z} \left( \int_0^z f(\omega) \, d\omega \right) \quad (3.1)
\]

\[
\Lambda = \frac{2\sqrt{2\pi}}{\sqrt{15}} \int_r^\infty \left( \frac{1}{s} \frac{d}{ds} s^4 f(s) \right) \, ds \quad (3.2)
\]
\[ \Lambda_z = \frac{2\sqrt{2\pi}}{\sqrt{15}} \int_{-\infty}^{\infty} \left( \frac{1}{\omega} \frac{\partial}{\partial \omega} \omega^4 f(\omega) \right) d\omega \] (3.3)

where in the complex operators, then integrals are contour integrals and in the \( \Lambda_z \) case \( \infty \) is \( \infty + 0i \).

These equivalences can easily be shown formally. Since we are right now only dealing with holomorphic functions, this lemma will hold. However the difficulty comes in applying the adjoint to \( \nu_t^1 \) which is not a holomorphic function. This is where a significant amount of difficulty comes into play with the \( \mathbb{C}^5 \) case.

### 3.2.2 \( \Lambda^* \) and the Gaussian

The significance of this isometry revolves around what happens when the adjoints of the unitary shift operators are applied to the Gaussian \( \nu_t^1 \), which is, again, not holomorphic. It is important to observe here that if everything plays nicely and \( \Lambda_z \) is just a holomorphic extension of \( \Lambda \) and \( \Lambda_z^\ast \) is just the antiholomorphic extension then

\[ \Lambda_z^* \Lambda_z^\ast = \frac{15}{8\pi^2 |z|^2} \Delta \frac{1}{|z|^2} \Delta^{-1}. \]

The first thing to observe is we have not actually defined \( \Delta^{-1} \) yet. And since we only will be applying this to a function of \( y \) in our calculation, we have the following definition.

**Definition 18.** Thus when restricted to functions on \( y \), we define \( \Delta^{-1} \) as

\[ \Delta^{-1} f(y) := I_y^2 f(y) = \int_{-\infty}^{y} \int_{-\infty}^{y_1} f(y_2) \, dy_2 \, dy_1. \]

It is not at first obvious that something nice and integrable will happen when \( f(y) \) is a Gaussian. This is because \( I_y^2 \) of a Gaussian does not decay at \( \infty \). However, this definition will help with our calculations of \( \Lambda_z^* \Lambda_z^\ast \). This will be something we define
with more rigor in the next section. Additionally when we simplify the holomorphic and antiholomorphic extensions of the standard shift operator, we get the following:

\[
\frac{4}{z} \frac{\partial}{\partial z} \frac{1}{z} \frac{\partial}{\partial \bar{z}} \frac{1}{\bar{z}} = \frac{1}{|z|^6} \left( \frac{\partial}{\partial z} - 1 \right) \left( \frac{\partial}{\partial \bar{z}} - 1 \right) = \frac{4}{|z|^6} \left( |z|^2 \frac{\partial^2}{\partial z \partial \bar{z}} - z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} + 1 \right) = \frac{4}{|z|^6} \left( \frac{|z|^2}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + 1 \right)
\]

Now since the Gaussian is only a function with respect to \( y \), we can get rid of the terms the derivative with respect to \( x \) comes first. We then apply \( I_y^2 f(y) \) to what we just calculated and in this motivating example we get

\[
\Lambda^*_z \Lambda^*_z f(y) = \frac{15}{8\pi^2} \frac{1}{|z|^6} \left( |z|^2 f(y) - 4yI_y f(y) + 4I_y^2 f(y) \right).
\]

This does not seem to help us at first glance, however the constants are in such a way that integration by parts yields a miraculous relationship. Integration by parts says that

\[
I_y (yf(y)) = yI_y f(y) - I_y^2 f(y).
\]

Thus we can simplify our expression further to

\[
\Lambda^*_z \Lambda^*_z f(y) = \frac{15}{8\pi^2} \frac{1}{|z|^6} \left( |z|^2 f(y) - 4I_y yf(y) \right)
\]

and since \( f(y) \) is a Gaussian this operator will produce for us a polynomial of \( |z|^2 \) multiplied by a Gaussian. Thus we have the equality

\[
\int_\mathbb{C} |\Lambda_z F|^2 e^{-y^2/t} \, dx \, dy = \frac{8\pi^2}{15} \int_\mathbb{C} |F(z)|^2 \left( |z|^4 + 2 |z|^2 t \right) \frac{e^{-y^2/t}}{\sqrt{\pi t}} \, dx \, dy. \tag{3.4}
\]

Recall Vol(\( S^{2n} \)) is defined in definition 10 and is the volume of the unit sphere with
Lebesgue measure. This nice map of the Gaussian to a polynomial multiplied by a 
Gaussian motivates us to explore the more generalized Euclidean spaces. Thus we can 
see that fits all the conditions in our Theorem 4. And thus we have proved Theorem 2 and motivated Theorem 3, which would establishes Theorem 4 in the case of \( n = 1 \) and \( n = 2 \). However proving taking the adjoint of our \( \Lambda_z \) is a little more tricky. This part is necessary for the complete proof.

3.2.3 Details about the Adjoint

As previously mentioned the issue of verifying the adjoint needs to be discussed. This will prove Theorem 3 as long as we can show

\[
\int_C |\Lambda_z F(z)|^2 \nu_1^1 \, dx \, dy = \text{Vol}^2(S^4) \int_C |F(z)|^2 \left( \Lambda_z^* \Lambda_z^* \nu_1^1 \right) |z|^8 \, dx \, dy.
\]

The first thought to do this would be to use integration by parts several times to move 
the derivatives over to the other side. However the integral operators on either side 
present a problem. In fact, when you try to move anything over it is not even obvious 
that you will get convergent integrals. A more innovative approach is necessary 
to show this equality, we will prove this in a general way in a later section. This 
calculation however, shows how even moving up to the \( \mathbb{C}^5 \) case makes the problem far more interesting. The leading term is expected, but there are other interesting 
terms that we have to account for. Next we will approach the general case.
CHAPTER 4

THE GENERAL EUCLIDEAN CASE

Now that the $\mathbb{C}^3$ and $\mathbb{C}^5$ case has been explored, our goal is to generalize this to every odd dimensional space. Much of the work will be quite similar to the $\mathbb{C}^5$ case. After we have our shift operator, we will be able to apply it the Gaussian and find the appropriate measure on $HL^2(\mathbb{C}^{2n+1}, \nu^{2n+1})^{SO}$. Recall from Definition 10 that

$$\Lambda^*_n f(r) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} \left( \frac{1}{r^d} \right)^n \left( \frac{d}{dr} \right)^{-n} f(r).$$

Then at least formally, by taking the holomorphic extension,

$$\Lambda^*_z f(z) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} \left( \frac{1}{z \partial z} \right)^n \left( \frac{\partial}{\partial z} \right)^{-n} f(z).$$

We should then formally have an antiholomorphic extension as well

$$\Lambda^*_\bar{z} f(\bar{z}) = \frac{\Xi(n)}{\sqrt{\text{Vol}(S^{2n})}} \left( \frac{1}{\bar{z} \partial \bar{z}} \right)^n \left( \frac{\partial}{\partial \bar{z}} \right)^{-n} f(\bar{z}).$$

The parity between the holomorphic and antiholomorphic operator has some nice results. We are dealing with Laplacian due to the heat operator and together these operators have $\frac{\partial^2}{\partial z \partial \bar{z}}$. And we have the relationship

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta.$$
Thus formally at least on functions of \( y \),

\[
\left( \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{-1} f(y) = 4I_y^2 f(y).
\]

These observations lead us to the suggestive definition of \( \Lambda^*_z \Lambda^*_\bar{z} \) for the isometry

\[
\int_{\mathbb{C}} |\Lambda_z F(z)|^2 \nu_1^z \, dz \, d\bar{z} = \text{Vol}^2(S^n) \int_{\mathbb{C}} |F(z)|^2 \left( (\Lambda^*_z \Lambda^*_\bar{z}) (\nu_1^z) \right) z^{2n} \, dz \, \bar{z}^{2n}.
\]

4.1 Desired Shift Operator

**Definition 19.**

\[
\Lambda^*_z \Lambda^*_\bar{z} f(y) := \frac{4^{2n}}{\text{Vol}(S^{2n})} \left( \frac{1}{|z|^2 \Delta} \right)^{n+1} I_y^{2n+2} f(y),
\]

where \( I_y^{2n+2} := \int_{-\infty}^{y} \cdots \int_{-\infty}^{s_{2n+2}} f(s_1) \, ds_1 \cdots ds_{2n+2}. \)

Using this definition, we are able to show that when applied to a Gaussian, get back a polynomial multiplied against a Gaussian. This result is one of the significant calculations for proving Theorem 4. First we are going to need the definitions of what will eventually be the coefficients of our polynomial.

**Definition 20.** Let \( n \in \mathbb{N} \), then \( \alpha_{n-1} = 0, \alpha_{n+1} = 0, \alpha_0 = 1 \) and

\[
\alpha_{n+1} = -4(n+k+1)\alpha_n^k + \alpha_{k+1}^n
\]

The recursive definition will make the most sense since we will be using proof by induction from some of the following theorems and lemmas. With this definition, we get the main result of this chapter.

**Theorem 21.** If \( \alpha \) is the constants defined in Definition 20, then

\[
(\Lambda_{n+1})^*_z (\Lambda_{n+1})^*_\bar{z} \left( \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right) = \frac{1}{\text{Vol}(S^{2n+2})} \sum_{i=0}^{n} \frac{\alpha_i^n}{|z|^{2(n+i+1)}} (I_y^i) \left( \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right).
\]
This is an incredible result because $\Delta^{-1}e^{-y^2/t}$ does not have the nice decay at infinity, but when $\Lambda_z^*\Lambda_z^*$ is applied to a Gaussian it has Gaussian decay. This will happen since $I_yy$ applied to a Gaussian is just a constant times a Gaussian.

Now, like we discussed at the end of the $\mathbb{C}^5$ case, if Definition 19 can be shown to be equivalent to moving $\Lambda_z$ to the other side of the integral then we can establish Theorem 4. As of right now, we are going to calculate what Definition 19 does to a Gaussian and what we want the norm to look like.

4.1.1 Calculating the Measure

The proof of Theorem 4 uses some seemingly miraculous calculations. Ultimately the calculation becomes nice to evaluate because the set of constants allow for “nice” integration against a Gaussian. The definition of $\alpha$ can be quite technical, so here is the first construction of the first couple layers. It is similar to that of Pascals Triangle

Let $n = 0,$

$$\alpha_0^0 = 1 \quad \alpha_1^0 = 0.$$  

This was given in the definition, then we create the $n = 1$ layer using the recursive definition

$$\alpha_0^1 = 1 \quad \alpha_1^1 = -4(\alpha_0^0) + \alpha_0^1 = -4 \quad \alpha_2^1 = 0.$$  

These coefficients should look familiar as they were seen in the $\mathbb{C}^5$ case. We then proceed for one more example layer to $n = 2.$

$$\alpha_0^2 = 1 \quad \alpha_1^2 = -4(\alpha_0^1) + \alpha_1^1 = -8 \quad \alpha_2^2 = -4(\alpha_1^1) + \alpha_2^1 = 16 \quad \alpha_3^2 = 0.$$  

We work layer by layer just like Pascals triangle. This kind of construction will be suggestive of a future proof by induction, which is how will begin to prove the following theorem.
Theorem 22. Suppose $\dim(C^m) = 2n+3$ and $f(y)$ be a function where $y$ is imaginary part, then

$$4^n \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^n \frac{1}{|z|^2} I_y^{2n} f(y) = \sum_{i=0}^{n} \frac{\alpha_i^n}{2^{(n+i+1)}} (I_y y)^i f(y).$$ (4.1)

The general structure of the proof follows from showing that if you take

$$\frac{\partial^2}{\partial z \partial \bar{z}} \sum_{i=0}^{n} \frac{\alpha_i^n}{|z|^{2(n+i+1)}} (I_y y)^i g(y)$$ (4.2)

and $g(y) = 4I_y^2 f(y)$, then the above expression can be simplified to

$$\sum_{i=0}^{n+1} \frac{\alpha_i^{n+1}}{|z|^{2(n+i+1)}} (I_y y)^i f(y).$$ (4.3)

A few lemmas are necessary to complete the proof.

Lemma 23. If $f(y)$ is a function in $y$, then for $k \geq 0$

$$(I_y y)^k f(y) = y^2 (I_y y)^{k-2} I_y^2 f(y) - (2k - 1) (I_y y)^{k-1} I_y^2 f(y),$$

where $(I_y y)^{-1} := \frac{1}{y} \frac{\partial}{\partial y}$.

We define $(I_y y)^{-1}$ this way because there is not a true inverse of the operator $I_y y$. However, the expression is an inverse up to a constant. If you observe the lemma, the only times this will occur is when $k = 0$ or $k = 1$. Both times, the inverse operator is applied after integration. The only time an unaccounted for constant would appear is when you apply integration after the inverse operator. This makes it so our lemma can still be well defined and we do not need to worry about stray constants floating about.
Lemma 24. Let $-1 \leq k \leq n$ then
\[ \alpha_{k+1}^{n+1} = \frac{-1}{2k+1} (4(n+k+1)^2 \alpha_k^n + \alpha_{k+1}^n). \]

Lemma 25. Let $k \geq 0$. The expression
\[ \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\alpha_k^n}{|z|^{2(n+k+1)}} (I_y y)^k g(y) \right) \]
can be expanded to the form
\[ \frac{\alpha_k^n}{4 |z|^{2(n+k+1)}} \left[ (I_y y)^{k-1} + y^2 (I_y y)^{k-2} \right] g(y) \]
\[ - \frac{\alpha_k^n}{|z|^{2(n+k+2)}} \left[ (n+k+1)y^2 (I_y y)^{k-1} + (n+k+1)^2 (I_y y)^k \right] g(y). \]

Looking at Lemma 25 we see that if we group by order of $|z|^2$, we see that we have $I_y y$ and $y^2 (I_y y)$ terms of the form necessary for Lemma 23. However the constants are not sufficient for Lemma 23 to be used. This unfortunately means that a singular term will not reduce nicely. However, given the form of Lemma 25 we will be able to combined the $k$ term with $k+1$ and find the constants needed for Lemma 23. We will begin by taking the expression of $k$ and $k+1$ given to us in Lemma 25 and grouping the similar $|z|^2$ terms. Eventually using Lemma 24 will we show the constants that appear are the ones we are looking for to use Lemma 23.

Proof. To begin the proof of Theorem 22 we start with our base case of $n = 0$. The left-hand side of the desired equality is
\[ 4^n \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^n \frac{1}{|z|^2} I_y^{2n} f(y) \]
which at $n = 0$ is just

$$4^0 \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^0 \frac{1}{|z|^2} I_y f(y) = \frac{1}{|z|^2} f(y).$$

However the right-hand side of the desired equality is

$$\sum_{i=0}^{n} \frac{\alpha_i^n}{|z|^{2(n+i+1)}} (I_y y^i f(y))$$

which at $n = 0$ is

$$\frac{\alpha_0^0}{|z|^2} f(y).$$

From the Definition 20 it is clear that the base case is true.

The base case is trivial since all the integrations and derivatives disappear.

Let Theorem 22 be true for a fixed $n$ as the induction hypothesis. We start with

the left-hand side of the desired equality in the $n + 1$ case and have

$$4^{n+1} \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n+1} \frac{1}{|z|^2} f^{2n+2} f(y)$$

$$= 4 \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right) \left( 4^n \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^n \frac{1}{|z|^2} f^{2n} f(y) \right)$$

$$= \frac{4}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \sum_{k=0}^{n} \frac{\alpha_k^n}{|z|^{2(n+k+1)}} (I_y y^k f(y))$$

$$= \sum_{k=-1}^{n} \frac{1}{|z|^{2(n+k+3)}} \left[ \alpha_{k+1}^n (I_y y)^k + \alpha_{k+1}^n y^2 (I_y y)^{k-1} \right.$$

$$- 4\alpha_k^n (n + k + 1)y^2 (I_y y)^{k-1} - 4\alpha_k^n (n + k + 1)^2 (I_y y)^k \right] 4I_y^2 f(y).$$

Let the inside of the bracket be $\beta_k^n$. Then we pair up the terms with the same $(I_y y)$

orders and the coefficients will combine nicely so we can use Lemma 23. Thus

$$\beta_k^n = (-4\alpha_k^n (n + k + 1) + \alpha_{k+1}^n) y^2 (I_y y)^{k-1} + (4\alpha_k^n (n + k + 1)^2 + \alpha_{k+1}^n) (I_y y)^k.$$
By Lemma 24 we can simplify $\beta_n^k$ by pulling out a factor of $\alpha_{k+1}^{n+1}$. Therefore

$$
\beta_n^k = \alpha_{k+1}^{n+1} \left[ y^2 (I_y y)^{k-1} + (2k + 1)(I_y y)^k \right]
$$

By applying $\beta_n^k$ to $I^2_y$ and using Lemma 23 we get the expression

$$
\alpha_{k+1}^{n+1} (I_y y)^{k+1} f(y).
$$

Thus

$$
4^{n+1} \frac{1}{|z|^2} \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n+1} \frac{1}{|z|^2} f_{2n+2} f(y) = \sum_{k=-1}^{n} \frac{\alpha_{k+1}^{n+1}}{|z|^{2(n+k+3)}} (I_y y)^{k+1} f(y),
$$

which could be reindexed as

$$
\sum_{k=0}^{n+1} \frac{\alpha_{k+1}^{n+1}}{|z|^{2(n+k+1)}} (I_y y)^{k} f(y).
$$

Thus given our lemmas, we have proven Theorem 22 which Theorem 21 is a special case.

4.2 Proof of Supporting Lemmas of the Main Result

Now that we have established our main theorem of this section, we have to go back and prove the supporting lemmas. Let’s start with the proof of Lemma 23.

4.2.1 Lemma 23

**Proof.** This will be a proof by induction. However the $k = 0$ case and $k = 1$ do not illuminate the induction proof, so we will show those cases independently, then use the $k = 2$ case as the base case.
For the $k = 0$ case, we can see

$$y^2(I_y y)^{-2} I_y^2 f(y) + (I_y y)^{-1} I_y^2 f(y) = y^2 \left( \frac{1}{y} \frac{\partial}{\partial y} \right)^2 I_y^2 f(y) + \frac{1}{y} \frac{\partial}{\partial y} I_y^2 f(y)$$

$$= y \frac{\partial}{\partial y} I_y f(y) + I_y f(y)$$

$$= y \left( \frac{y f(y) - I_y f(y)}{y^2} \right) + I_y f(y)$$

$$= f(y).$$

For the $k = 1$ case, we case see

$$y^2(I_y y)^{-1} I_y^2 f(y) - (I_y y)^{0} I_y^2 f(y) = y^2 \left( \frac{1}{y} \frac{\partial}{\partial y} \right) I_y^2 f(y) - I_y^2 f(y)$$

$$= y I_y f(y) - I_y^2 f(y).$$

Now if we look at $I_y y f(y)$ and let $u = y$ and $dv = f(y)$, then it is clear by integration by parts that

$$I_y y f(y) = y I_y f(y) - I_y^2 f(y).$$

This concludes the case for $k = 1$. However we will continue to use the above equation to prove this lemma. We start with the base case when $k = 2$ since this illuminates the induction proof. Observe how

$$I_y y I_y y f(y) = I_y y (y I_y f(y) - I_y^2 f(y))$$

$$= I_y y^2 I_y f(y) - I_y y I_y^2 f(y)$$

$$= (y^2 I_y^2 - 2 I_y y I_y^2) - I_y y I_y^2 f(y)$$

$$= y^2 I_y^2 - 3 I_y y I_y^2 f(y).$$

This all comes from the integration by parts done relationship shown in the $k = 1$ case and completes showing the base case.
Now we assume the induction hypothesis that
\[(I_y y)^k f(y) = y^2 (I_y y)^{k-2} I_y^2 f(y) - (2k - 1) (I_y y)^{k-1} I_y^2 f(y).\]

Thus
\[
(I_y y)^{k+1} f(y) = I_y y \left((I_y y)^k f(y)\right) = I_y y \left(y^2 (I_y y)^{k-2} I_y^2 f(y) - (2k - 1) (I_y y)^{k-1} I_y^2 f(y)\right) = I_y y^3 (I_y y)^{k-2} I_y^2 f(y) - (2k - 1) (I_y y)^k I_y^2 f(y).
\]

If you look at the first term and let \( u = y^2 \) and \( dv = y (I_y y)^{k-2} I_y^2 f(y) \), then it is clear that
\[
(I_y y)^{k+1} f(y) = y^2 (I_y y)^{k-1} I_y^2 f(y) - 2I_y y (I_y y)^{k-1} I_y^2 f(y) - (2k - 1) (I_y y)^k I_y^2 f(y) = y^2 (I_y y)^{k-1} I_y^2 f(y) - (2k + 1) (I_y y)^k I_y^2 f(y).
\]

This completes the induction on \( k \), proving Lemma 23.

4.2.2 Lemma 24

Proof. From the definition of \( \alpha_{k+1}^n \), we can show that proving Lemma 24 is equivalent to proving
\[
(k + 1)\alpha_{k+1}^n = 2\alpha_k^n \left( (k + 1)^2 - (k + 1) - n - n^2 \right).
\]

This is simply shown by putting the expression in a computer algebra system and commanding it to solve for \( \alpha_{k+1}^n \). We write the expression in this way, as opposed to dividing by \( k \), because if you look in Lemma 24, there is no issue with \( k = -1 \) since \( \alpha_{-1}^n = 0 \).

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The base case is when $n = 0$. Observe how

$$\alpha_{k+1}^1 := -4(0 + k + 1)\alpha_0^0 + \alpha_{k+1}^0.$$  

Thus if $k = -1$, $\alpha_0^1 = \alpha_0^0$, which we have defined to be 1. The definition of this makes sense by observing the statement of the theorem. Additionally if $k = 0$, then $\alpha_1^1 = -4\alpha_0^0 + \alpha_1^0 = -4$. Now if we look that the other side of Lemma 24, then for $k = -1$, $\alpha_0^1 = 1(4(0 + -1 + 1)^2\alpha_{-1}^0 + \alpha_0^0) = 1$. Now for $k = 0$, $\alpha_1^1 = -1(4(0 + 0 + 1)^2\alpha_0^0 + \alpha_0^0) = -4$. Each of the $\alpha$s are equivalent to one another, thus the base case is shown.

Let the above be true for all $k$ for a fixed $n$; this is the induction hypothesis, then by definition and the induction hypothesis we see that

$$\alpha_{k+1}^{n+1} = -4(n + k + 1)\alpha_k^n + \alpha_{k+1}^n$$

$$= -4(n + k + 1)\alpha_k^n + \frac{2\alpha_k^n((k + 1)^2 - (k + 1) - n - n^2)}{k + 1}$$

$$=\alpha_k^n\left(\frac{-2(2(n + k + 1)(k + 1) - (k + 1)^2 + (k + 1) + n + n^2)}{k + 1}\right)$$

Now if we approach from the other side, we can see by the induction hypothesis that

$$\alpha_k^{n+1}\left(\frac{2((k + 1)^2 - (k + 1) - (n + 1) - (n + 1)^2)}{k + 1}\right)$$

$$= (-4(n + k)\alpha_{k-1}^n + \alpha_k^n)\left(\frac{2((k + 1)^2 - (k + 1) - (n + 1) - (n + 1)^2)}{k + 1}\right)$$

$$=\alpha_k^n\left(-4(n + k)\frac{k}{2(k^2 - k - n - n^2) + 1}\right)$$

$$\times\left(\frac{2((k + 1)^2 - (k + 1) - (n + 1) - (n + 1)^2)}{k + 1}\right)$$

Thus this proof boils down to showing the coefficients of $\alpha_k^n$ of both expressions are equal, which is easily verifiable by any computer algebra system.
4.2.3 Lemma 25

Proof. Finally we conclude with the proof of Lemma 25, which is a series of calculations.

Let $k$ be fixed, then

$$h(y) = \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\alpha_k}{|z|^{2(n+k+1)}} (I_y y)^k \ g(y) \right)$$

$$= \alpha_k \frac{\partial}{\partial z} \frac{1}{z^{n+k+1}}$$

$$\times \left[ \frac{1}{\bar{z}^{2n+2k+2}} \left( z^{n+k+1} \frac{\partial}{\partial \bar{z}} (I_y y)^k \ g(y) - (n + k + 1) z^{n+k} (I_y y)^k \ g(y) \right) \right]$$

when we first apply the derivative with respect to $\bar{z}$. Then since $\bar{z}$ is just a constant when we take the derivative with respect to $z$ we show

$$h(y) = \alpha_k \left( \frac{1}{\bar{z}^{2n+2k+2}} \right) \left( \frac{1}{z^{2n+2k+2}} \right) \times$$

$$\left[ z^{n+k+1} \left( z^{n+k+1} \frac{\partial^2}{\partial z \partial \bar{z}} (I_y y)^k \ g(y) \right) - (n + k + 1) z^{n+k} \frac{\partial}{\partial z} (I_y y)^k \ g(y) \right]$$

$$- (n + k + 1) z^{n+k} \left( z^{n+k+1} \frac{\partial}{\partial \bar{z}} (I_y y)^k \ g(y) - (n + k + 1) z^{n+k} (I_y y)^k \ g(y) \right)$$

Afterwards we consolidate the $|z|^2$ and reduce to demonstrate

$$h(y) = \left( \frac{\alpha_k}{|z|^{2(n+k+1)}} \right) \times$$

$$\left[ \left| z \right|^2 \frac{\partial^2}{\partial z \partial \bar{z}} (I_y y)^k \ g(y) - (n + k + 1) \left( z \frac{\partial}{\partial z} + z \frac{\partial}{\partial \bar{z}} \right) (I_y y)^k \ g(y) \right.$$  

$$+ (n + k + 1)^2 (I_y y)^k \ g(y) \right]$$
And since \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \) and \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \), we can simplify the expression to

\[
\begin{align*}
    h(y) &= \frac{\alpha_k}{|z|^{2(n+k+2)}} \times \\
    &\left[ \frac{|z|^2}{4} \frac{\partial^2}{\partial y^2} (I_y y)^k \ g(y) - (n + k + 1) \left( y \frac{\partial}{\partial y} \right) (I_y y)^k \ g(y) \\
    &+ (n + k + 1)^2 (I_y y)^k \ g(y) \right]
\end{align*}
\]

Thus by applying our remaining derivatives we get the expression

\[
\begin{align*}
    h(y) &= \frac{\alpha_k}{|z|^{2(n+k+2)}} \left[ \frac{|z|^2}{4} (I_y y)^{k-1} \ g(y) + \frac{|z|^2}{4} y^2 (I_y y)^{k-2} \ g(y) \\
    &- (n + k + 1) y^2 (I_y y)^{k-1} \ g(y) + (n + k + 1)^2 (I_y y)^k \ g(y) \right] \\
    &= \frac{\alpha_k^n}{4 |z|^{2(n+k+1)}} \left[ (I_y y)^{k-1} + y^2 (I_y y)^{k-2} \right] g(y) \\
    &- \frac{\alpha_k^n}{|z|^{2(n+k+2)}} \left[ (n + k + 1) y^2 (I_y y)^{k-1} + (n + k + 1)^2 (I_y y)^k \right] g(y).
\end{align*}
\]

We group \( h(y) \) by order of \( |z|^2 \) for easier manipulations in the proof of Theorem \ref{thm:22}. \qed
CHAPTER 5

TIME ZERO

Ultimately the proof of all the theorems comes down to showing the adjoint of our shift operator moved over to the other side of the integral is what we defined in Definition 19. In order to do that, we first are going to need to find out what happens to both sides when \( t \) approaches 0. In order to do this, we need to establish a nice dense subspace for Segal–Bargmann Space.

If we start by looking at

\[
L = \text{Vol}^2(S^{2n+2}) \int |F(z)|^2 \left( \Lambda^*_z \Lambda^*_i \nu^1_\tau \right) |z|^{4n+4} \, dx \, dy.
\]

By using Theorem 21 we know

\[
L = \text{Vol}(S^{2n+2}) \int |F(z)|^2 \left( \sum_{i=0}^{n} \frac{\alpha^n_i}{|z|^{2(n+i+1)}} (I_y y)^i \left( \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right) \right) |z|^{4n+4} \, dx \, dy.
\]

The only situation where \( t \) is a positive power is when \( i = 0 \), thus we can say

\[
\lim_{t \to 0} L = \lim_{t \to 0} \text{Vol}(S^{2n+2}) \int |F(z)|^2 |z|^{2n+2} \left( \frac{e^{-y^2/t}}{\sqrt{\pi t}} \right) \, dx \, dy.
\]

We thus will have the following theorem

**Theorem 26.** If \( F \in HL^2(C^{2n+3}, \nu^{2n+3}_t) \), then

\[
\lim_{t \to 0} L = \text{Vol}(S^{2n+2}) \int |F(r)|^2 |r|^{2n+2} \, dr.
\]
Additionally we will prove the following theorem.

**Theorem 27.** If $F \in HL^2(\mathbb{C}^n, \nu^n_t)$, then $F \in HL^2(\mathbb{C}^n, \nu^n_s)$ for all $s \leq t$ and

$$\lim_{s \to 0} ||F||^2_{HL^2(\mathbb{C}^n, \nu^n_s)} = \int_{\mathbb{R}^n} |F(x)|^2 \, dx = \text{Vol}(S^{n-1}) \int_{\mathbb{R}} |F(r)|^2 r^{n-1} \, dr.$$ 

Having set up the first one, we actually use the same method to prove both theorems. Thus with both of these theorems, we can establish a result for the $t = 0$ case.

**Theorem 28.** If $F \in HL^2(\mathbb{C}^n, \nu^n_s)$

$$\lim_{t \to 0} \text{Vol}^2(S^{2n}) \int_{\mathbb{C}} |F(z)|^2 (\Lambda_x^* \Lambda_x^* \nu^1_t) |z|^{4n} \, dx \, dy = \lim_{t \to 0} \int_{\mathbb{C}} |\Lambda_x F|^2 \nu^1_t \, dx \, dy.$$ 

5.1 Proof of Theorem 26 and 27

Recall we are trying to show if $F \in HL^2(\mathbb{C}^n, \nu^n_t)$, then $F \in HL^2(\mathbb{C}^n, \nu^n_s)$ for all $s \leq t$ and

$$\lim_{s \to 0} ||F||^2_{HL^2(\mathbb{C}^n, \nu^n_s)} = \int_{\mathbb{R}^n} |F(x)|^2 \, dx.$$ 

For $y \in \mathbb{R}^n$, let $M_F(y) := \int_{\mathbb{R}^n} |F(x + iy)|^2 \, dx$, where $F$ is in $HL^2(\mathbb{C}^n, \nu^n_t)$. We have several goals with $M$ that will establish this proof. The first step is to show that $M$ is bounded, then we proceed with showing $M$ to be continuous. During this time, we will establish a nice dense subspace. Using this $M$, we can then do a manipulation to show that as we approach 0, our $\nu^1_s$ becomes a delta function. From this point it will not be hard to establish our result.

5.1.1 $M$ is bounded

We start with a lemma showing that $M$ is bounded for a fixed $y$ value.
Lemma 29. If \( M_F(y) = \int_{\mathbb{R}^n} |F(x + iy)|^2 \, dx \), where \( F \) is the \( HL^2(\mathbb{C}^n, \nu^n) \), then for all \( q > 0 \),

\[
M(y) \leq cn t^2 q^{-n} e^{(|y|+q)^2/4} ||F||^2_{HL^2(\mathbb{C}^n, \nu^n)},
\]

where \( c_n \) is a constant dependent on \( n \).

Proof. We know that

\[
F(z) = \int_{B_q(z)} \frac{F(v)}{\text{Vol}(B_q(z))} \, dv
\]

where \( B_q(z) = \{ v \in \mathbb{C}^n | |v - z| < q \} \).

Thus

\[
M_F(y) = \int_{\mathbb{R}^n} |F(z)|^2 \, dx \\
= \int_{\mathbb{R}^n} \left| \int_{B_q(z)} \frac{F(a + ib)}{\text{Vol}(B_q(z))} \, da \, db \right|^2 \, dx \\
\leq \int_{\mathbb{R}^n} \left( \int_{B_q(z)} |F(a + ib)| \frac{da \, db}{\text{Vol}(B_q(z))} \right)^2 \, dx.
\]

Therefore by Jensen’s Inequality, it is implied that

\[
M_F(y) \leq \int_{\mathbb{R}^n} \int_{B_q(z)} |F(a + ib)|^2 \frac{da \, db}{\text{Vol}(B_q(z))} \, dx.
\]

We then do a substitution of \( a = x + u \) and \( b = v + y \) to get

\[
M_F(y) \leq \frac{\Gamma(n + 1)}{\pi^n} s^{-2n} \int_{\mathbb{R}^n} \int_{B_q(0)} |F(x + u + iy + iv)|^2 \, du \, dv \, dx.
\]

By switching the order of integration, we get

\[
M_F(y) \leq \frac{\Gamma(n + 1)}{\pi^n} q^{-2n} \int_{B_q(0)} \int_{B_{\sqrt{q^{-n}|v|^2}(0)}} \int_{\mathbb{R}^n} |F(x + u + iy + iv)|^2 \, dx \, du \, dv.
\]

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Then since \( dx \) is translation invariant, we can express the integral as

\[
M_F(y) \leq \Gamma \left( \frac{n}{2} + 1 \right) q^{-n} \int_{B_q(0)} \int_{\mathbb{R}^n} |F(x + iy + iv)|^2 \, dx \, du \, dv.
\]

Now that the integrand is independent of \( u \), we can switch the order again and evaluate the integral with respect to \( u \) and get

\[
M_F(y) \leq \Gamma \left( \frac{n}{2} + 1 \right) q^{-n} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iu)|^2 \text{Vol} \left( B_{\sqrt{q^2 - |u|^2}}(0) \right) \, dx \, dv.
\]

We then perform another substitution \( w = y + v \) and get

\[
M_F(y) \leq \Gamma \left( \frac{n}{2} + 1 \right) \pi^{\frac{n}{2}} q^{-n} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iw)|^2 \text{Vol} \left( B_{\sqrt{q^2 - |w - y|^2}}(0) \right) \, dx \, dv.
\]

This can be equivalently seen as saying

\[
M_F(y) \leq \frac{\Gamma(n + 1) \pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right) \pi^{\frac{n}{2}}} q^{-n} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iw)|^2 \left( \sqrt{q^2 - |w - y|^2} \right)^n \, dx \, dv
\]

\[
\leq \frac{\Gamma(n + 1) \pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right) \pi^{\frac{n}{2}}} q^{-n} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iw)|^2 \, dx \, dv.
\]

We then perform the following series of manipulation to get our result:

\[
M_F(y) \leq \Gamma \left( \frac{n}{2} + 1 \right) \pi^{\frac{n}{2}} q^{-n} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iw)|^2 \text{Vol} \left( B_{\sqrt{q^2 - |w - y|^2}}(0) \right) \, dx \, dv
\]

\[
\leq \frac{\Gamma(n + 1) t^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right) t^{\frac{n}{2}}} q^{-n} e^{(|y| + q)^2/t} \int_{B_q(y)} \int_{\mathbb{R}^n} |F(x + iw)|^2 \text{Vol} \left( B_{\sqrt{q^2 - |w - y|^2}}(0) \right) \, dx \, dv
\]

\[
\leq \frac{\Gamma(n + 1) t^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right) t^{\frac{n}{2}}} q^{-n} e^{(|y| + q)^2/t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x + iw)|^2 \text{Vol} \left( B_{\sqrt{q^2 - |w - y|^2}}(0) \right) \, dx \, dv
\]

\[
\leq \frac{\Gamma(n + 1) t^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right) t^{\frac{n}{2}}} q^{-n} e^{(|y| + q)^2/t} ||F||_{H^{L^2}(\mathbb{C}^n, \nu_n^t)}^2.
\]

Thus \( M_F(y) \leq c_n t^{\frac{n}{2}} q^{-n} e^{(|y| + q)^2/t} ||F||_{H^{L^2}(\mathbb{C}^n, \nu_n^t)}^2 \).
5.1.2 $M$ is Continuous

Our next goal is to show that $M$ is continuous for all functions, $F$ in $HL^2(C^n, \nu^n_t)$. In particular, we want to show that it is continuous at 0.

**Lemma 30.** If $M_F(y) = \int_{\mathbb{R}^n} |F(x + iy)|^2 dx$, where $F$ is the $HL^2(C^n, \nu^n_t)$, then $M_F(y)$ is continuous at every value of $y$.

**Proof.** The strategy for proving $M_F$ is continuous will follow this outline

1. Find a dense subspace of $V$ of $HL^2(C^n, \nu^n_t)$, such that $M_F(y)$ is continuous for all $F$ in $V$.

2. We will let $F$ be arbitrary in $HL^2(C^n, \nu^n_t)$ and let $F_j$ be a sequence in $V$ such that $F = \lim_{j \to \infty} F_j = F$ and finally show $M_F(y)$ is continuous for all $F$ in $HL^2(C^n, \nu^n_t)$.

We start by looking at point (1).

**Lemma 31.** Let $b > 0$ and $t > 0$. If $\sigma = b + t$ and $V$ is the set of all functions of the form $F = z^m e^{-\frac{x^2}{2\sigma}}$, where $m$ is a multi-index, then $V$ is a dense subspace of $HL^2(C^n, \nu^n_t)$.

We take our dense subspace to be $F = z^m e^{-\frac{x^2}{2\sigma}}$.

Thus $|z^m e^{-\frac{x^2}{2\sigma}}|^2 = |(x + iy)^m e^{-\frac{(x+iy)^2}{2\sigma}}|^2 = (x^2 + y^2)^m e^{-\frac{x^2}{2\sigma}}$. It is clear that for a small ball around $y_0$, we could find a constant $\kappa$ such that

$$\int_{\mathbb{R}^n} (x^2 + y^2)^m e^{-\frac{x^2}{2\sigma}} e^{-\frac{y^2}{2\sigma}} \, dx < \kappa,$$

for all $y$ inside the ball. Then we will be able to use the Dominated Convergence Theorem. And since we have

$$|M_F(y) - M_F(y + \delta)| = \left| \int_{\mathbb{R}^n} |F(x + iy)|^2 - |F(x + iy + i\delta)|^2 \, dx \right|,$$
we pull the limit of $\delta$ approaching 0 into the integral and conclude that

$$\left| \int_{\mathbb{R}^n} |F(x + iy)|^2 - |F(x + iy + i\delta)|^2 \, dx \right|$$

must be small for small $\delta$ and as such $M$ is continuous on $V$ for a fixed $y$.

Now we move to part (2) of our outline. Since $V$ is dense in $HL^2(\mathbb{C}^n, \nu_t^n)$, if $F$ is an arbitrary function in $HL^2(\mathbb{C}^n, \nu_t^n)$ then there exists a sequence in $V$, $\{F_j\}$ where $\lim_{j \to \infty} F_j = F$ uniformly. Thus

$$\left| \left| F - F_j \right|_{HL^2(\mathbb{C}^n, \nu_t^n)} \right|$$

is small.

And we conclude with showing continuity for all $F$. Since $M_F(y) = \|F(x + iy)\|_{L^2(\mathbb{R}^n, dx)}^2$, then we will use the following definition.

**Definition 32.** $F|_y$ is a function $F(x + iy)$ restricted to a fixed $y$ value. Consequently,

$$\left| \left| F|_y \right|_{L^2(\mathbb{R}^n)} \right| = \|F(x + iy)\|_{L^2(\mathbb{R}^n, dx)}^2.$$ 

And we see that

$$\left| M_{F_j}(y) - M_F(y) \right|$$

$$= \left| \left| F_j|_y \right|_{L^2(\mathbb{R}^n)}^2 - \left| \left| F|_y \right|_{L^2(\mathbb{R}^n)}^2 \right|$$

$$= \left| \left| F_j|_y \right|_{L^2(\mathbb{R}^n)}^2 - \left| \left| F|_y \right|_{L^2(\mathbb{R}^n)}^2 \right| \cdot \left| \left| F_j|_y \right|_{L^2(\mathbb{R}^n)}^2 + \left| \left| F|_y \right|_{L^2(\mathbb{R}^n)}^2 \right|$$

$$\leq \left| \left| F_j|_y - F|_y \right|_{L^2(\mathbb{R}^n)} \cdot \left| \left| F_j|_y \right|_{L^2(\mathbb{R}^n)}^2 + \left| \left| F|_y \right|_{L^2(\mathbb{R}^n)}^2 \right|$$

$$\leq c_n q^{-n} e^{(|y| + q)^2/t} \left| F_j(z) - F(z) \right|_{HL^2(\mathbb{C}^n, \nu^n_t)} \cdot \left| \left| F_j|_y \right|_{L^2(\mathbb{R}^n)}^2 + \left| \left| F|_y \right|_{L^2(\mathbb{R}^n)}^2 \right|,$$

by Lemma 29. Thus $\left| M_{F_j}(y) - M_F(y) \right|$ is small and locally uniform. This implies
that $M_F$ converges locally uniformly to $M_F^j$ and therefore $M$ is continuous on all of $HL^2(C^n, \nu^*_n)$.

5.1.3 Proving Theorem \ref{thm:main}

Now we can give the proof of the Theorem \ref{thm:main} using our two established lemmas.

\textbf{Proof.} If we have
\[ \lim_{s \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(z)|^2 \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dx \, dy, \]
we can rewrite the limit to be
\[ \lim_{s \to 0} \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy, \]
since the Gaussian is independent of $x$. We are going to split up the integral into two components and have
\[ \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy = \int_{\mathbb{R}^n} M_F(0) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy + \int_{\mathbb{R}^n} (M_F(y) - M_F(0)) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy. \]
Since $M_F(0)$ is not dependent of $y$, we can pull it out of the integral, but what is left integrates to 1, thus we have
\[ \left| \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy - M_F(0) \right| = \left| \int_{\mathbb{R}^n} (M_F(y) - M_F(0)) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy \right| \leq \int_{\mathbb{R}^n} \left| M_F(y) - M_F(0) \right| \frac{e^{-|y|^2/s}}{\left(\pi s\right)^{n/2}} \, dy \]
Let $\epsilon > 0$. Now since $M(y)$ is continuous, in particular at 0, let’s pick a region $U$ dependant of $\epsilon$ such that
\[ \left| M_F(y) - M_F(0) \right| < \epsilon/2. \]
Therefore
\[ \int_U |M_F(y) - M_F(0)| \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy < \epsilon/2 \int_U \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy < \epsilon/2 \int_{\mathbb{R}^n} \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy = \epsilon/2. \]

Now we have
\[ \left| \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy - M_F(0) \right| \leq \int_U |M_F(y) - M_F(0)| \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy + \int_{U^c} |M_F(y) - M_F(0)| \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy < \epsilon/2 + \int_{U^c} |M_F(y) - M_F(0)| \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy. \]

But in our argument proving Lemma 30, we know \( M_F(y) - M_F(0) \) is bounded by
\[ k_n t^{n/2} q^{-n/2} e^{((|y|+q)^2)/2t} \]
for some constant \( k \) dependent on \( n \). Thus, since the Gaussian measure is normalized
\[ \left| \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy - M_F(0) \right| \leq \epsilon/2 + \int_{U^c} k_n t^{n/2} q^{-n/2} e^{((|y|+q)^2)/2t} \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy \]
\[ = \epsilon/2 + \int_{U^c} k_n t^{n/2} q^{-n/2} e^{((s-2t)|y|^2+2sq|y|+sq^2)/(2st)} \left(\pi s\right)^n/2 \, dy. \]

We can thus pick \( s \) to be sufficiently small for there to be rapid decay. The general idea is you make \( s \) small enough so the function is weighted inside on the region \( U \). This implies in particular we can pick \( s \) such that the integral of \( U^c \) is less than \( \epsilon/2 \). Therefore
\[ \left| \int_{\mathbb{R}^n} M_F(y) \frac{e^{-|y|^2/s}}{\left(\pi s\right)^n/2} \, dy - M_F(0) \right| \leq \epsilon. \]

This allows us to switch the limits and complete the proof. \( \square \)
5.2 Proof of Density

We used the a density argument for Theorem 27 as well as in proving the formula for the adjoint of $\Lambda$. Recall Lemma 31, which stated if $V$ is the set of all functions of the form $F = z^m e^{-\frac{x^2}{2b}}$, then $V$ is a dense subspace of $H^L_2(\mathbb{C}^n, \nu_n^t)$, where $m$ is a multi-index. In order to prove this statement we are going to take a dense subspace of $L^2(\mathbb{R}^n)$ and apply the Segal–Bargmann transform, which is an isometry. Using the Fourier Transform, it will be quite clear that we have this dense subspace.

**Proof.** Let $b > 0$. We start in the one dimensional case by applying the heat operator $e^{\Delta t/2}$ to $e^{-x^2/(2b)}$. In order to evaluate this expression, we are going to take the Fourier Transform and get

$$\mathcal{F} \left( e^{\Delta t/2} e^{-x^2/(2b)} \right) = e^{-k^2 t/2} \mathcal{F} \left( e^{-x^2/(2b)} \right)$$

$$= e^{-k^2 t/2} \int_{-\infty}^{\infty} e^{-x^2/(2b)} e^{ikx} \, dx$$

$$= e^{-k^2 t/2} \left( \sqrt{2\pi b} e^{-(b^2/2)} \right)$$

$$= \sqrt{2\pi b} e^{-(b+t)k^2/2}.$$ 

Thus

$$e^{\Delta t/2} e^{-x^2/(2b)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{2\pi b} e^{-(b+t)k^2/2} e^{ikx} \, dk = \sqrt{b} \frac{e^{-\frac{x^2}{2(b+t)}}}{\sqrt{(b+t)}}.$$ 

We then take a holomorphic extension and get

$$\sqrt{b} \frac{e^{\pi x^2/(b+t)}}{\sqrt{(b+t)}}.$$ 

From here it is quite clear that anything of the form $x^m e^{-x^2/(2b)}$ would be taken to a polynomial of $z$ multiplied by $e^{\pi x^2/(b+t)}$. This would be done by using the properties of the Fourier Transform and converting $x^m$ into derivatives of $k$. Since the Segal–Bargmann transform is an isometry and $x^m e^{-x^2/(2b)}$ forms a dense subspace
of $L^2(\mathbb{R}^n)$ according to Theorem 11.4 in Hall’s Quantum Theory for Mathematicians [5], then $z^m e^{-z^2} \frac{z^2}{2\sigma}$ will form a dense subspace in $HL^2(\mathbb{C}^n, \nu^n_t)$. Thus for all $\sigma > t$, the set $z^m e^{-z^2/(2\sigma)}$ is dense. The proof for the higher dimensional cases are trivially similar. \qed
CHAPTER 6

THE ADJOINT

So far we have moved our operators to the other side of the integral formally and not shown with rigor that we have an adjoint. The first attempt at doing this is quite tricky because of the integral operators. When you try to move each individual part of the Shift Operator over, you use integration by parts and the boundary terms diverge. There is not an obvious cancellation of the boundary terms and so proving that our adjoint is actually an adjoint requires a more interesting approach.

Theorem 33. Let $F \in H L^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO}$ and $(\Lambda_n)_z$ be our shift operator from $H L^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO}$ to $H L^2(\mathbb{C}^1, \nu_t^1)^{E/O}$ and $(\Lambda^*_n)_z (\Lambda^*_n)_\bar{z}$ be the operator defined in [19]. Then for all $n \in \mathbb{N}$

$$\int_{\mathbb{C}} |(\Lambda_n)_z F|^{2\nu_t^1} dy \, dx = \text{Vol}^2(S^n) \int_{\mathbb{C}} |F|^{2\nu_t^1} (|(\Lambda^*_n)_z (\Lambda^*_n)_\bar{z})| z|^{4n} dy \, dx.$$

In order to prove this theorem, we need to establish two essential lemmas. The first is the following the lemma.

Lemma 34. Let $F \in H L^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO}$ and $(\Lambda_n)_z$ be our shift operator from $H L^2(\mathbb{C}^{2n+1}, \nu_t^{2n+1})^{SO}$ to $H L^2(\mathbb{C}^1, \nu_t^1)^{E/O}$. Then for all $n \in \mathbb{N}$

$$\frac{d}{dt} \int_{\mathbb{C}} |(\Lambda_n)_z F|^{2\nu_t^1} dy \, dx = \int_{\mathbb{C}} |(\Lambda_{n-1})_z ((2n-1)F + zF')|^{2\nu_t^1} dy \, dx$$

And the second lemma we need is doing a similar thing on the other side.
Lemma 35. Let \( F \in H^2(\mathbb{C}^1, \nu^1_1)^{E/O} \) and \((\Lambda^*_n)_z\) be our shift operator from \( H^2(\mathbb{C}^1, \nu^1_1)^{E/O} \) to \( H^2(\mathbb{C}^{2n+1}, \nu^{2n+1}_t)^{SO} \). Then for all \( n \in \mathbb{N} \)

\[
\frac{d}{dt} \int_\mathbb{C} |F|^2 ((\Lambda^*_n)_z (\Lambda^*_n)_{\bar{z}} \nu^1_t) |z|^{4n} \, dy \, dx = \int_\mathbb{C} |((2n-1)F + zF')|^2 ((\Lambda^*_n-1)_z (\Lambda^*_n-1)_{\bar{z}} \nu^1_t) |z|^{4n-4} \, dy \, dx
\]

We are going to treat \( F \) as polynomial of \( z \) times a Gaussian. So we will need to establish what happens when we apply \( \Lambda_z \) to \( F \). The proof will be a consequence of our Fourier transform analysis and we have this lemma.

Lemma 36. If \( F \in H^2(\mathbb{C}^{2n+1}, \nu^{2n+1}_t)^{SO} \) and is of the form \( F(z) = z^m e^{-z^2/(2\sigma)} \), where \( \sigma > t \) and \( m \) is a multi-index, then

\[
\Lambda_z (F(z)) = p(z) e^{-z^2/\tau}.
\]

We have \( \tau < t \) and \( p(z) \) is a polynomial of \( z \). As a consequence of this, \( \Lambda_z (F(z)) \) has rapid decay at infinity as does it’s \( x \) derivatives.

Assuming these three lemmas, we are able to prove Theorem 33.

6.1 Proof of the Adjoint

Proof. We are going to use induction for this proof. The base case was shown rigorously in the trivial \( \mathbb{C}^3 \) case. Now assume the theorem of \( n - 1 \).

Let \( F(z) = z^m e^{-z^2/(2\sigma)} \), where \( \sigma > t \) and \( m \) is a multi-index. The set of all functions of this form is a dense subspace of \( H^2(\mathbb{C}^n, \nu^n_t) \) using Lemma 31. By using
Lemma 34, we are able to say

\[
\frac{d}{dt} \int_C |(\Lambda_n)_z z^m e^{-z^2/(2\sigma)}|^2 \nu_t^1 \, dy \, dx
\]

\[
= \int_C \left| (\Lambda_{n-1})_z \left( (2n-1)z^m e^{-z^2/(2\sigma)} + z \left( -\frac{z^{m+1}}{\sigma} e^{-z^2/(2\sigma)} + m z^{m-1} e^{-z^2/(2\sigma)} \right) \right) \right|^2 \, d\nu_t^1
\]

\[
= \int_C \left| (\Lambda_{n-1})_z \left( P(z)e^{-z^2/(2\sigma)} \right) \right|^2 \nu_t^1 \, dy \, dx,
\]

where \( P(z) \) is a polynomial of \( z \). \( P(z)e^{-z^2/(2\sigma)} \) can be considered to be a function in \( HL^2(C^{2n-1}, \nu_t^{2n-1})^{SO} \) and by our induction hypothesis we can now say

\[
\frac{d}{dt} \int_C |(\Lambda_n)_z z^m e^{-z^2/(2\sigma)}|^2 \nu_t^1 \, dy \, dx = \int_C \left| \left( P(z)e^{-z^2/(2\sigma)} \right) \right|^2 \left( \Lambda_{n-1}^* \right)_z \left( \Lambda_{n-1}^* \right)_z \, \nu_t^1 \, dy \, dx.
\]

But the \( P(z) \) is exactly the polynomial that allows us to use Lemma 35 and show

\[
\frac{d}{dt} \int_C |(\Lambda_n)_z z^m e^{-z^2/(2\sigma)}|^2 \nu_t^1 \, dy \, dx = \frac{d}{dt} \int_C \left| z^m e^{-z^2/(2\sigma)} \right|^2 \left( \Lambda_n^* \right)_z \left( \Lambda_n^* \right)_z \, \nu_t^1 \, dy \, dx.
\]

Since we did this for a dense subset, we can now conclude that

\[
\frac{d}{dt} \int_C |(\Lambda_n)_z F(z)|^2 \, dy \, dx = \frac{d}{dt} \int_C |F(z)|^2 \left( \left( \Lambda_n^* \right)_z \left( \Lambda_n^* \right)_z \nu_s^1 \right) \, dy \, dx
\]

for all \( F \in HL^2(C^{2n+1}, \nu_t^{2n+1})^{SO} \). Now by integrating both sides from 0 to \( s \) with respect to \( t \), we have

\[
\int_C |(\Lambda_n)_z F(z)|^2 \nu_s^1 \, dy \, dx - \lim_{t \to 0} \left( \int_C |(\Lambda_n)_z F(z)|^2 \nu_t^1 \, dy \, dx \right)
\]

\[
= \int_C |F(z)|^2 \left( \left( \Lambda_n^* \right)_z \left( \Lambda_n^* \right)_z \, \nu_s^1 \right) \, dy \, dx - \lim_{t \to 0} \left( \int_C |F(z)|^2 \left( \left( \Lambda_n^* \right)_z \left( \Lambda_n^* \right)_z \, \nu_t^1 \right) \, dy \, dx \right).
\]

Thus if we can conclude

\[
\lim_{t \to 0} \left( \int_C |(\Lambda_n)_z F(z)|^2 \nu_t^1 \, dy \, dx \right) = \lim_{t \to 0} \left( \int_C |F(z)|^2 \left( \left( \Lambda_n^* \right)_z \left( \Lambda_n^* \right)_z \, \nu_t^1 \right) \, dy \, dx \right)
\]

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then the proof is complete. Luckily this was already established in the general case with Theorem 28.

6.2 Proof of Lemma 34

Proof. To start the proof of Lemma 34, we are going to let $F(z) = z^m e^{-z^2/(2\sigma)}$, where $\sigma > t$ and $m$ is a multi-index. The set of all functions of this form is a dense subspace of $HL^2(\mathbb{C}^n, \nu^t)$ using Lemma 31. Thus

$$
\frac{d}{dt} \int_{\mathbb{C}} |(\Lambda_n)z F|^2 \nu_t^1 \, dy \, dx = \frac{1}{4} \int_{\mathbb{C}} |(\Lambda_n)z F|^2 \Delta \nu_t^1 \, dy \, dx
$$

$$= \frac{1}{4} \int_{\mathbb{C}} |(\Lambda_n)z F|^2 \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) \nu_t^1 \, dy \, dx
$$

We are then going to use integration by parts four times to get the relationship

$$
\frac{d}{dt} \int_{\mathbb{C}} |(\Lambda_n)z F|^2 \nu_t^1 \, dy \, dx = \frac{1}{4} \int_{\mathbb{C}} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) |(\Lambda_n)z F|^2 \nu_t^1 \, dy \, dx
$$

$$+ \frac{1}{4} \int_{-\infty}^{\infty} \left[ (\Lambda_n)z F \left( \frac{d}{dx} \right) \nu_t^1 \right]_{x=\infty} \, dy
$$

$$- \frac{1}{4} \int_{-\infty}^{\infty} \left[ (\Lambda_n)z F \left( \frac{d}{dx} \right) \nu_t^1 \right]_{x=-\infty} \, dy
$$

$$+ \frac{1}{4} \int_{-\infty}^{\infty} \left[ (\Lambda_n)z F \left( \frac{d}{dy} \right) \nu_t^1 \right]_{y=\infty} \, dx
$$

$$- \frac{1}{4} \int_{-\infty}^{\infty} \left[ (\Lambda_n)z F \left( \frac{d}{dy} \right) \nu_t^1 \right]_{y=-\infty} \, dx
$$

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All the boundary terms clearly go to 0 by Lemma 36, thus

\[
\begin{align*}
\frac{d}{dt} \int_C |(\Lambda_n)_z F|^2 \nu^1_t \, dy \, dx \\
= \frac{1}{4} \int_C \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) |(\Lambda_n)_z F|^2 \nu^1_t \, dy \, dx \\
= \int_C \left( \frac{\partial^2}{\partial z \partial \bar{z}} \right) |(\Lambda_n)_z F|^2 \nu^1_t \, dy \, dx \\
= \int_C \left( \frac{\partial^2}{\partial z \partial \bar{z}} \right) |I^n_{z-1} \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^{n-1} z^{2n-2} \left( \frac{1}{z^{n-2}} \frac{\partial}{\partial \bar{z}} \right) z^{2n} \right) F|^2 \nu^1_t \, dy \, dx \\
= \int_C |(\Lambda_{n-1})_z \left( \frac{1}{z^{2n-2}} \frac{\partial}{\partial z} \right) z^{2n} \right) F|^2 \nu^1_t \, dy \, dx \\
= \int_C |(\Lambda_{n-1})_z \left( (2n-1)F(z) + zF'(z) \right)|^2 \nu^1_t \, dy \, dx
\end{align*}
\]

This completes the proof of Lemma 34.

6.3 Proof of Lemma 35

Proof. To start the proof of Theorem 35, we are going to let \( F(z) = z^m e^{-z^2/(2\sigma)} \), where \( \sigma > t \) and \( m \) is a multi-index. The set of all functions of this form is a dense subspace of \( HL^2(\mathbb{C}^n, \nu^n_t) \) using Lemma 31. Thus

\[
\begin{align*}
\frac{d}{dt} \int_C |F|^2 \left( (\Lambda^*_n)_z (\Lambda^*_n)_z \nu^1_t \right) \, dy \, dx = \frac{d}{dt} \int_C |F|^2 \left( |z|^{4n} \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^n I_y^{2n} \right) \nu^1_t \, dy \, dx
\end{align*}
\]
Thus

\[
\frac{d}{dt} \int_C |F|^2 \left( (\Lambda^*_n)_z (\Lambda^*_n)_{\bar{z}} \nu^1_t \right) \, dy \, dx = \frac{1}{4} \int_C |z^{2n} F|^2 \left( \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^n 4^n I^{2n}_y \right) \frac{\partial^2}{\partial y^2} \nu^1_t \, dy \, dx
\]

But since \( \nu^1_t \) is a Gaussian we can simplify this expression. If we take two of the integral operators and look at the expression

\[
I^2_y \frac{\partial^2}{\partial y^2} \nu^1_t,
\]

then we get

\[
\int_{-\infty}^y \int_{-\infty}^{s_1} \frac{\partial^2}{\partial s_2} \nu^1_t(s_2) \, ds_2 \, ds_1 = \int_{-\infty}^y \frac{\partial}{\partial s_2} \nu^1_t(s_2) \bigg|_{s_2=-\infty}^{s_2=s_1} \, ds_1
\]

\[
= \int_{-\infty}^y \frac{\partial}{\partial s_1} \nu^1_t(s_1) \, ds_1
\]

\[
= \nu^1_t(s_1) \bigg|_{s_1=-\infty}^{s_1=y}
\]

\[
= \nu^1_t(y).
\]

Thus we can do write the equality

\[
\frac{d}{dt} \int_C |F|^2 \left( (\Lambda^*_n)_z (\Lambda^*_n)_{\bar{z}} \nu^1_t \right) \, dy \, dx = \int_C |z^{2n-1} F|^2 \left( \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n-1} 4^{n-1} I^{2n-2}_y \right) \nu^1_t \, dy \, dx
\]
Once again we use integration by parts four times and get

\[
\frac{d}{dt} \int_C |F|^2 \left( (\Lambda_n^*)_z (\Lambda_n^*)_{\bar{z}} \nu^1_t \right) \, dy \, dx
\]

\[
= \int_C \left( \frac{\partial^2}{\partial z \partial \bar{z}} \right) |z^{2n-1}F|^2 \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n-1} 4^{n-1} I^{2n-2}_y \nu^1_t \, dy \, dx
\]

\[
+ \int_{-\infty}^{\infty} \left[ |z^{2n-1}F|^2 \left( \frac{d}{dx} \right) \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n-1} 4^{n-1} I^{2n-2}_y \nu^1_t \right] \, dy \bigg|_{x=-\infty}^{x=\infty}
\]

\[
- \int_{-\infty}^{\infty} \left[ |z^{2n-1}F|^2 \left( \frac{d}{dy} \right) \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n-1} 4^{n-1} I^{2n-2}_y \nu^1_t \right] \, dx \bigg|_{x=-\infty}^{x=\infty}
\]

\[
+ \int_{-\infty}^{\infty} \left[ |z^{2n-1}F|^2 \left( \frac{d}{dy} \right) \left( \frac{1}{|z|^2} \frac{\partial^2}{\partial z \partial \bar{z}} \right)^{n-1} 4^{n-1} I^{2n-2}_y \nu^1_t \right] \, dx \bigg|_{y=-\infty}^{y=\infty}
\]

This completes the proof of Theorem 35.
6.3.1 Proof of Lemma 36

Proof. Lemma 15 states

\[ \Lambda_n^* f = F_{2n+1}^{-1} \left( c_n \frac{F_1 (f) (k)}{k^n} \right). \]

Now that we have shown \( \Lambda_n^* \) is unitary, we know

\[ \Lambda_n = (\Lambda_n^*)^{-1}, \]

thus by Lemma 15

\[ \Lambda_n f = F_{1}^{-1} (c_n k^n F_{2n+1} (f) (k)). \]

Suppose \( f(z) = z^m e^{-z^2/(2\sigma)} \) where \( \sigma > t \) and \( m \) is a multi-index, then

\[ F_{2n+1} f = F_{2n+1} z^m e^{-z^2/(2\sigma)} \]

which is a rotationally invariant polynomial times a Gaussian. However, we can identify that rotationally invariant polynomial with a polynomial of \( k \). Thus

\[ (c_n k^n F_{2n+1} (f) (k)) \]

is also a polynomial of \( k \) times a Gaussian and so by the properties of the 1-dimensional Fourier transform \( \Lambda_n f \) is also a polynomial of \( z \) times a Gaussian, which completes the proof.

\[ \square \]
CHAPTER 7

CONCLUDING REMARKS

As noted, these result hold for odd-dimensional Euclidean spaces. This comes from our shift operator moving functions by two dimensions. Some of the result hold however for all Euclidean space. Future work will involve looking at moving from 2—dimensional space to 1—dimensional space and then generalizing to all even-dimensional Euclidean spaces. We could check if our results are consistent with Kaewthep and Lewkeeratiyutkul in their paper [9], which used a more geometric approach.

Having completed the main result in the Euclidean Case, the next goal would be to see if something similar could happen in other spaces. A first generalization would be to see how this method could be used to tackle spherical space. If successful, this could indicate a nice method for approaching symmetric paces.

In the compact direction, we could look at dimension reduction to a maximal torus. The real case would be analogous to the Weyl Integral Formula with class functions. Previous results by Florentino, Mourao, and Nunes [1] have shown that there is a complex version of the Weyl Integral Formula which would give us motivation for this direction.

Another direction that research could take is applying this shift operator method to hyperbolic space and see if other noncompact spaces would become approachable. Hall and Mitchell in [7] discuss a hyperbolic shift operator

\[
\frac{1}{\sinh r} \frac{d}{dr}
\]
that is remarkable similar to the Euclidean case. Hopefully methods in the Euclidean case will generalize nicely to the shift operator in hyperbolic space.


