CONGRUENCES BETWEEN ORDINARY SYMPELLCTIC GALOIS REPRESENTATIONS

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Abstract

by

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We prove the existence of congruences between ordinary symplectic Galois representations in two different settings. First, we calculate lower bounds on the degree of the weight space map for the Hida families constructed in [23] given assumptions on the $p$-adic $\mathcal{L}$-invariant (or the adjoint $\mathcal{L}$-invariant) of a weight $(3, 3)$ automorphic representation $\pi$ on the Hida family when such an $\mathcal{L}$-invariant is defined using theorems of Rosso [21]. Second, we set up a Galois deformation problem for a fixed absolutely irreducible Galois representation $\bar{\rho} : G\mathbb{Q} \to \text{GSp}_4(\mathbb{F}_p)$ which is odd, ordinary and indecomposable at $p$, and unramified everywhere else. Under a mild local hypothesis, we prove the existence of at least two characteristic zero lifts of our fixed $\bar{\rho}$. 
In loving memory of my father

Timothy Allen Wawerczyk

(1960-2018)
## CONTENTS

Acknowledgments .................................................. vi

Chapter 1: Introduction ............................................. 1
  1.1 Number Theory Preliminaries ................................. 6

Chapter 2: Representation Theory of GSp\(_4\) ......................... 9
  2.1 Lie Theory .................................................. 9
  2.2 The Lie Algebra \(\mathfrak{gsp}_4\) .............................. 11
        2.2.1 The Adjoint Representation ........................... 13
  2.3 Representations of GSp\(_4(\mathbb{R})\) ......................... 14
        2.3.1 Discrete Series ....................................... 14
  2.4 Representations of GSp\(_4(\mathbb{Q}_p)\) ....................... 15
        2.4.1 Normalized Parabolic Induction ....................... 15
  2.5 \(L\)-Parameters ........................................... 17
        2.5.1 Local Langlands Correspondence for GSp\(_4(\mathbb{R})\) . 17
        2.5.2 Local Langlands for GSp\(_4(\mathbb{Q}_p)\) .............. 18
  2.6 Hecke Algebras ........................................... 19

Chapter 3: The Galois Representation .............................. 20
  3.1 Greenberg’s \(L\)-invariant ................................ 23
        3.1.1 Hypotheses (S), (U), (T) and the Exceptional Subquotient . 23
        3.1.2 The Balanced Selmer Groups .......................... 26
        3.1.3 Definition of the \(L\)-invariant ...................... 27

Chapter 4: Ordinary Hida Families ................................ 32
  4.1 Estimates of Analytic Hecke Operators ...................... 36
  4.2 Lower Bounds on the Degree of Weight Space map ............ 38

Chapter 5: Deformations of Galois Representations .................. 41
  5.1 Our Running Hypotheses on \(\hat{\rho}\) and its restrictions \(\hat{\rho}|_v\) . 42
        5.1.1 The adjoint representation of our \(\hat{\rho}\) ............ 44
  5.2 Tangent Spaces of Deformation Rings ........................ 45
        5.2.1 Local Deformations and our Deformation Conditions . 45
        5.2.2 \(H^1(\mathbb{R}, \text{Ad}^0(\hat{\rho}))\) ....................... 49
        5.2.3 \(H^1(\mathbb{Q}_p, \text{Ad}^0(\hat{\rho}))\) .................... 50
5.2.4  Local at $p$ deformation conditions ........................................ 55
5.3   Selmer Systems .............................................................................. 58
      5.3.1  Selmer Calculations ................................................................. 59
Bibliography ......................................................................................... 61
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CHAPTER 1

INTRODUCTION

Given a linear algebraic group $G$ and a Galois representation $\bar{\rho}: \mathbb{Q} \to G(\mathbb{F}_p)$, it has become a natural question to ask whether there exists characteristic 0 lifts of $\bar{\rho}$, i.e., is there a finite $\mathbb{Z}_p$-algebra $\mathcal{O}$ (with residue field $\mathbb{F}_p$) and a representation $\rho: \mathbb{Q} \to G(\mathcal{O})$ making the following diagram commute:

$$
\begin{array}{ccc}
G_{\mathbb{Q}} & \xrightarrow{\rho} & G(\mathcal{O}) \\
\downarrow{\bar{\rho}} & & \downarrow \\
G(\mathbb{F}_p) & & \\
\end{array}
$$

where the vertical map is induced by going modulo the maximal ideal of $\mathcal{O}$. Two such lifts, $\rho_1$ and $\rho_2$, are also said to be congruent mod $p$.

Questions like these have proven fruitful in many areas of modern algebraic number theory. For instance, one can ask that the lift $\rho$ be a Galois representation $\rho_f$ arising from a modular form $f$. In the case $G = \text{Gl}_2$, Khare-Wittenberg [13] proved the Serre conjectures about precisely which mod $p$ Galois representations $\bar{\rho}$ are modular (admit a characteristic zero lift $\rho_f$ such that $\bar{\rho}_f \cong \bar{\rho}$). Wiles’ proof of Fermat’s last theorem [24] relied on showing that the $p$-torsion of the rational points of an elliptic curve (which are $\text{Gl}_2(\mathbb{F}_p)$-valued Galois representations) were modular using the theory of universal deformation rings. Ramakrishna invented a general method in this area for producing “geometric” (in the sense of Fontaine-Mazur [6]) lifts of mod $p$ Galois representations which has been recently extended to orthogonal and symplectic groups by Patrikis [18] and Booher [3].
This work contains two different approaches to proving the existence of multiple characteristic zero lifts/proving congruences mod $p$ in the case that $G = \text{GSp}_4$, the symplectic similitude group. The first part of this work utilizes Hida families of ordinary cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ while the other uses the theory of universal deformation rings for symplectic Galois representations. Hida families, thought of as finite branched coverings of a $p$-adic unit disc (called the weight space), can be used to prove the existence of multiple characteristic zero lifts by determining the degree of the branching of the weight space map. By construction, the fibers of the weight space map (at a classical weight $(k_1, k_2; w)$) of a given Hida family correspond to ordinary irreducible cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ of weight $(k_1, k_2; w)$ whose Galois representations are all congruent mod $p$. The second part of this work uses universal deformation rings to conclude the existence of characteristic zero lifts of a fixed $\bar{\rho} : G_\mathbb{Q} \to \text{GSp}_4(\mathbb{F}_p)$ by calculating the tangent space of such universal deformation rings when it is known that the ring is smooth.

Let us provide some background for our first two results which use Hida families. Proposition 5.1 of Greenberg-Stevens [9] gives sufficient conditions for finding congruences for the mod $p$ Galois representation $\bar{\rho}_E$ an elliptic curve $E$ by estimating the $p$-adic valuation of an important and mysterious invariant coming from the theory of $p$-adic $L$-functions: the $p$-adic $L$-invariant. This number, denoted $L_E$, appears in the $p$-adic analogue of the Birch and Swinnerton-Dyer conjecture formulated by Mazur-Tate-Teitelbaum [15].

In their proof of the Mazur-Tate-Teitelbaum conjecture [9], Greenberg and Stevens use Hida families to provide a formula for the $p$-adic $L$-invariant of a split-multiplicative elliptic curve $E(\mathbb{Q}_p)$ in terms of logarithmic derivatives of $p$-adic analytic functions:
\[ \mathcal{L}_E = -2\frac{\alpha'_p}{\alpha_p}|_{f_E} \]  

(1.1)

where \( \alpha_p \) is the \( p \)-adic analytic function interpolating the \( U_p \)-eigenvalues of the modular forms in the Hida family and \( f_E \) is the modular form in the Hida family associated to the elliptic curve \( E \). They also prove in [9] that the \( \mathcal{L} \)-invariant of the elliptic curve can be (more intrinsically) defined by

\[ \mathcal{L}_E = \frac{\log_p(q_E)}{\text{ord}_p(q_E)} \]

where \( q_E \) is a choice of Tate period for \( E \) from the \( p \)-adic uniformization theorem: \( E(\overline{\mathbb{Q}_p}) \cong \overline{\mathbb{Q}_p}^\times / q_E^\mathbb{Z} \).

Along the way (see: Proposition 5.1 of [9]), Greenberg & Stevens are able to reprove a special case of Ribet’s level lowering theorem for the residual Galois representation \( \bar{\rho}_E \) of \( E \) under the assumption \( v_p(\mathcal{L}_E) < 1 \). The argument is as follows: if all of the weight 2 modular forms in the Hida family are new at \( p \), then \( v_p(\mathcal{L}_E) \geq 1 \). A consequence of this is that the weight space map of the Hida family passing through \( f_E \) is not an isomorphism (and hence, the existence of a characteristic zero lift of \( \bar{\rho}_E \) distinct from \( \rho_{f_E} \)).

In [8], Greenberg generalizes the construction of \( \mathcal{L}_E \) from [9] to any ordinary Galois representation \( V \) under several hypotheses explained in \( \S 3 \) of this work. More generally, one can now define a \( p \)-adic \( \mathcal{L} \)-invariant for any semistable \( G_{\mathbb{Q}_p} \) representation (following the work of Benois [1]) using the theory of \( (\varphi, \Gamma) \) modules over the Robba ring. In the case of an ordinary Galois representation, the definitions of Greenberg and Benois agree (remark 2.2.2 in [1]).

In [21], Rosso provides analogous logarithmic derivatives formulae for the \( \mathcal{L} \)-invariant of the Galois representation of an ordinary irreducible cuspidal automorphic
representation \( \pi \) of \( \text{GSp}_4(\mathbb{A}_\mathbb{Q}) \) of cohomological weight whose local component \( \pi_p \) is maximally Steinberg (type IV.a in [20]) as well as a formula for the adjoint \( \mathcal{L} \) invariant of such a \( \pi \) with no restriction on the local component at \( p \). By making appropriate assumptions on the Galois representations involved so that the \( \mathcal{L} \)-invariants are well defined, we may use the formulae of Rosso and the Hida families constructed by Tilouine [23] to prove analogous results on the degree of the weight space map for Hida families of irreducible cohomological cuspidal automorphic representations \( \pi \) of \( \text{GSp}_4(\mathbb{A}_\mathbb{Q}) \) with a hypothesis on the \( p \)-adic \( \mathcal{L} \)-invariant of a given weight \((3,3)\) point on the family.

Let \( V \) denote either the Galois representation associated to \( \pi \) (recalled in §3) or the adjoint representation \( \text{Ad}(V) \) of such a \( V \). Assume hypotheses (S), (U), and (T) of §3.1 and the vanishing of the Selmer group \( H^1_L(V) \) defined in §3.2. Moreover, assume that \( V \) is exceptional and critical at \( s = 1 \) in the sense of Deligne [5]. Under these hypothesis, the \( \mathcal{L} \)-invariant is defined as in §3.3. We prove the following two theorems:

**Theorem (1).** Let \( \mathbb{T}_{C_0} \) be a Hida family (as defined in §4) whose weight space map \( w \) is flat at the weight \((3,3)\) prime \( \mathfrak{p}_{(3,3)} \). Assume \( \mathbb{T}_{C_0} \) contains a weight \((3,3)\) point \( \pi \) whose local component \( \pi_p \) is maximal Steinberg (IV.a in [20]). Assume the hypothesis on \( \rho_{\pi,p} \) for this weight \((3,3)\) point \( \pi \) outlined above so that the \( p \)-adic \( \mathcal{L} \)-invariant \( \mathcal{L}_\pi \) is defined. If \( v_p(\mathcal{L}_\pi) < 1 \), then the Hida family \( \mathbb{T}_{C_0} \) must have at least one weight \((3,3)\) point which is not of Type IV.A at \( p \). In particular, the degree of the Hida family \( \mathbb{T}_{C_0} \) is at least 2 and the weight space map is not an isomorphism.

**Theorem (2).** Let \( \mathbb{T}_{C_0} \) be a Hida family (as defined in §4) whose weight space map \( w \) is flat at the weight \((3,3)\) prime \( \mathfrak{p}_{(3,3)} \). Assume \( \mathbb{T}_{C_0} \) contains a weight \((3,3)\) point \( \pi \). Assume the hypotheses outlined above so that the adjoint \( \mathcal{L} \)-invariant \( \mathcal{L}_{\pi,\text{AD}} \) is defined. If \( v_p(\mathcal{L}_{\pi,\text{AD}}) < 2 \), then the degree of the Hida family \( \mathbb{T}_{C_0} \) is at least 2 and the weight space map is not an isomorphism.
Chapter 5 of this work extends Corollary 16 of [19], utilizing Galois deformation theory, to give conditions under which a fixed residual representation \( \bar{\rho} : G_\mathbb{Q} \to \text{GSp}_4(\mathbb{F}_p) \) has multiple characteristic zero lifts. We prove the following:

Fix a prime \( p \). Let \( \bar{\rho} : G_\mathbb{Q} \to \text{GSp}_4(\mathbb{F}_p) \) be a fixed absolutely irreducible Galois representation such that:

- \( \bar{\rho}|_\infty \) is odd
- \( \bar{\rho}|_l \) is unramified at all primes \( l \neq p \)
- \( \bar{\rho}|_p \) is ordinary and indecomposable

More explicitly, we assume that the restriction of \( \bar{\rho} \) to the decomposition group at \( p \) has the form:

\[
\bar{\rho}|_p = \begin{pmatrix}
\bar{\epsilon}^3 & * & * & * \\
\bar{\epsilon}^2 & * & * & \\
\bar{\epsilon} & * & & \\
1 & & & \\
\end{pmatrix}
\]

where \( \bar{\epsilon} \) is the mod \( p \) cyclotomic character. Let \( \text{Ad}^0(\bar{\rho}) \) denote the Lie algebra \( \mathfrak{sp}_4(\mathbb{F}_p) \) equipped with the adjoint Galois action defined by conjugation with \( \bar{\rho} \). The main result of Chapter 5 is the following:

**Theorem (3).** Let \( p \) be a prime. Let \( \bar{\rho} : G_\mathbb{Q} \to \text{GSp}_4(\mathbb{F}_p) \) be a fixed odd, absolutely irreducible Galois representation which is unramified away from \( p \) and ordinary, indecomposable at \( p \). Let \( H^2_L(\mathbb{Q}^\Sigma, \text{Ad}^0(\bar{\rho})) = 0 \) for \( \Sigma = \{\infty, p\} \). If there exists a non-trivial ordinary deformation of the local representation \( \bar{\rho}|_p \) which is not semistable \( (\mathcal{L}^\text{ord}_{p,3} \supseteq \mathcal{L}^\text{st}_{p,3}) \) then there exists more than one \( p \)-ordinary characteristic zero lift of \( \bar{\rho} \).

We remark that the hypothesis that \( H^2_L(\mathbb{Q}^\Sigma, \text{Ad}^0(\bar{\rho})) = 0 \) of Theorem 3 is assumed in order to apply Proposition 3.6 of [15] to conclude that the universal deformation ring under consideration is isomorphic to a power series ring and, hence, is smooth.
1.1 Number Theory Preliminaries

For any field, $K$, let $\bar{K}$ denote the algebraic closure of $K$. Let $G_K := \text{Gal}(K/K)$ denote the absolute Galois group of $K$. For a field extension $L/K$, we denote the Galois group of this extension $\text{Gal}(L/K)$. If $X$ is a continuous $G_K$-module (respectively, a $\text{Gal}(L/K)$-module), we denote the $i$-th Galois cohomology group of $X$ by $H^i(K, X)$ (respectively, $H^i(\text{Gal}(L/K), X)$). If $X$ is a finite set, we denote $\#X$ for the cardinality of $X$.

We will use $\Sigma$ to denote a finite set of places of $\mathbb{Q}$. Let $\infty$ denote the real place of $\mathbb{Q} \rightarrow \mathbb{R}$ given by the completion of $\mathbb{Q}$ with respect to the archimedean metric, denoted $|.|$. Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the $p$-adic metric, denoted $|.|_p$. Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers, whose residue field is denoted $\mathbb{F}_p$. The ring of finite Adeles, $\mathbb{A}_\mathbb{Q}^f$, is the restricted direct product of all $\mathbb{Q}_p$ with respect to their ring of integers, $\mathbb{Z}_p$. The ring of adeles, denoted $\mathbb{A}_\mathbb{Q}$, is defined as $\mathbb{A}_\mathbb{Q}^f \times \mathbb{R}$. Let $G_\Sigma^\mathbb{Q}$ denote the Galois group of the extension $\mathbb{Q}^\Sigma/\mathbb{Q}$, where $\mathbb{Q}^\Sigma$ denotes the maximal extension of $\mathbb{Q}$ which is unramified at all places $v \notin \Sigma$. For each prime $p$, let $\mathbb{Q}_p^{ur}$ denote the maximal unramified extension of $\mathbb{Q}_p$. The Galois group of $\mathbb{Q}_p^{ur}$ over $\mathbb{Q}_p$ is denoted $I_{\mathbb{Q}_p}$ and called the inertia subgroup of $G_{\mathbb{Q}_p}$.

Recall the following exact sequence:

$$0 \rightarrow I_{\mathbb{Q}_p} \rightarrow G_{\mathbb{Q}_p} \rightarrow G_{\mathbb{F}_p} \rightarrow 0$$

An isomorphism $G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ is determined by a choice of topological generator $\overline{\text{Frob}_{\mathbb{F}_p}} \in G_{\mathbb{F}_p}$. We will drop the bar and denote a lift of this topological generator as $\text{Frob}_{\mathbb{F}_p} \in G_{\mathbb{Q}_p}$. The Weil group of $\mathbb{Q}_p$ is denoted $W_{\mathbb{Q}_p}$ and is defined by the following diagram being exact, topologized such that $I_{\mathbb{Q}_p}$ is an open subgroup:
The Weil group of $\mathbb{R}$ is denoted $W_{\mathbb{R}}$ and is a non-split extension of the group $G_{\mathbb{R}} \cong \mathbb{Z}/2\mathbb{Z} \cong \langle \sigma \rangle$ (where $\sigma$ denotes complex conjugation) by the group $\mathbb{C}^\times$. $W_{\mathbb{R}}$ can be identified with the subset $\mathbb{C}^\times \cup \mathbb{C}^\times j$ of the non-zero quaternions.

Lastly, we, once and for all, fix maps making the following diagram commute:

$$
\begin{array}{cccc}
\mathbb{Z} & \longrightarrow & \mathbb{Q} \\
\downarrow & & \downarrow \\
\mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \\
\end{array}
$$

Fixing these maps gives us contravariant maps of Galois groups whose images (for any place $p$ or $\infty$) are called the decomposition groups at $p$ or $\infty$ respectively. We shall denote the image of these maps with the same notation.

$$
G_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}} \quad G_{\mathbb{R}} \longrightarrow G_{\mathbb{Q}}
$$

Given a commutative topological ring, $R$, and a free topological $R$-module, $M$, a Galois representation is a continuous group homomorphism

$$
\rho : G_{\mathbb{Q}} \longrightarrow \text{Aut}_R(M)
$$

Given a Galois representation, $\rho$, we can restrict this representation to each of

\[ 0 \longrightarrow I_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}_p} \longrightarrow \langle \text{Frob}_p \rangle \longrightarrow 0 \\
0 \longrightarrow I_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p} \longrightarrow 0 \]
the decomposition groups. These restrictions are denoted

\[ \rho|_p : G_{Q_p} \longrightarrow \text{Aut}_R(M) \quad \rho|_\infty : G_\mathbb{R} \longrightarrow \text{Aut}_R(M) \]

Let \( \rho : G_\mathbb{Q} \to \text{Gl}_n(\mathbb{Z}_p) \) be a \( p \)-adic representation. The residual representation \( \bar{\rho} \) is the representation obtained by composing \( \rho \) with the map \( \text{Gl}_n(\mathbb{Z}_p) \to \text{Gl}_n(\mathbb{F}_p) \) determined by the quotient map \( \mathbb{Z}_p \to \mathbb{F}_p \). Two \( p \)-adic representations, \( \rho_1 \) and \( \rho_2 \), are congruent \( \mod p \) if their residual representations are isomorphic: \( \bar{\rho}_1 \cong \bar{\rho}_2 \).
CHAPTER 2

REPRESENTATION THEORY OF GSp$_4$

This work will study Galois representations which are valued in the four dimensional symplectic similitude group, denoted GSp$_4$, over various commutative rings. The first part of this work will give sufficient conditions to prove congruences between the Galois representations associated to cohomological automorphic representations of GSp$_4(\mathbb{A}_\mathbb{Q})$ while the second part will focus on giving sufficient conditions for the existence of characteristic zero lifts of fixed $\mathbb{F}_p$-valued Galois representations. This chapter will recall the basic facts from the abstract Lie theory of GSp$_4$, the representation theory of GSp$_4(\mathbb{R})$ and GSp$_4(\mathbb{Q}_p)$ acting on complex vector spaces, the assignment of $L$-parameters to these representations, and the definitions of the relevant Hecke algebras.

2.1 Lie Theory

We begin by fixing the symplectic form:

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and define the algebraic group GSp$_4$ over a ring $R$ as:

$$\text{GSp}_4(R) = \{ g \in \text{Gl}_4(R) \mid {}^t g J g = \nu(g) J \}$$
where $\nu : \text{GSp}_4 \to \mathbb{G}_m$ denotes the similitude character. This choice of $J$ determines the Borel subgroup, $B \subset \text{GSp}_4$, of upper triangular matrices, the maximal torus, $T \cong \mathbb{G}_m^3$, of diagonal matrices and the unipotent subgroup $N$ satisfying the Levi decomposition: $B = TN$

\[
B = \begin{pmatrix}
* & * & * & * \\
* & * & * & \\
* & * & & \\
* & & & \\
\end{pmatrix}
\begin{pmatrix}
* \\
* \\
* \\
1 \\
\end{pmatrix}
\begin{pmatrix}
1 & * & * & * \\
1 & & & \\
1 & & & \\
1 & & & \\
\end{pmatrix}
\]

The maximal torus has rank 3 with $t \in T$ embedded by

\[
t(a, b, c) = \begin{pmatrix} a \\ b \\ cb^{-1} \\ ca^{-1} \end{pmatrix}
\]

The Weyl group, $W = N_{\text{GSp}_4}(T)/T$, is a finite group of order 8 isomorphic to $D_4$, the dihedral group on a square. The Weyl group acts on $T$ by conjugation and, as a group, is generated by the two matrices $s_1$ and $s_2$ below.

\[
s_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

The character group of our torus, $X^*(T) = \text{Hom}(T, \mathbb{G}_m)$ is a free abelian group
and we choose the basis $e_1, e_2, e_3 \in X^*(T)$:

$$e_1(t(a, b, c)) = a \quad e_2(t(a, b, c)) = b \quad e_3(t(a, b, c)) = c$$

giving us the isomorphism $X^*(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$.

The choice of $B$ and $T$ determines a root system, $\Phi$, whose set of positive roots $\Phi^+$ are denoted

$$\alpha_1 = e_1 - e_2 \quad \alpha_2 = 2e_2 - e_3$$

$$\alpha_1 + \alpha_2 = e_1 + e_2 - e_3 \quad 2\alpha_1 + \alpha_2 = 2e_1 - e_3$$

with simple roots $\Pi = \{\alpha_1, \alpha_2\}$.

The cocharacter group of our torus, $X_*(T) = \text{Hom}(\mathbb{G}_m, T)$, is also a free abelian group and we choose the basis $f_1, f_2, f_3 \in X_*(T)$:

$$f_1(x) = (x, 1, 1) \quad f_2(x) = (1, x, 1) \quad f_3(x) = (1, 1, x)$$

giving us the isomorphism $X_*(T) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_3$.

2.2 The Lie Algebra $\mathfrak{g}sp_4$

The Lie algebra $\mathfrak{g} = \mathfrak{g}sp_4$ is 11 dimensional and decomposes $\mathfrak{g} = \mathfrak{1} \oplus \mathfrak{sp}_4$. The simple Lie algebra $\mathfrak{sp}_4$ decomposes $\mathfrak{sp}_4 = \mathfrak{h} \oplus_{\alpha > 0} \Phi_\alpha \oplus_{\alpha < 0} \Phi_\alpha$ into the Cartan subalgebra and positive and negative eigenspaces of the roots.

The basis of the subspace $\mathfrak{1}$ is:

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
The basis elements of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}_4$ are denoted

\[
\begin{align*}
    h_1 &= \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \\
    h_2 &= \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ & & & 0 \end{pmatrix}
\end{align*}
\]

Then we have the eigenspaces for the positive roots spanned by:

\[
\begin{align*}
    r_1 &= \begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -1 \\ & & & 0 & \end{pmatrix} \\
    r_2 &= \begin{pmatrix} & & & 1 & 0 \\ & & 1 & 0 & \\ & 0 & & \\ & -1 & 0 & & \\ & & & 0 & 1 \end{pmatrix} \\
    r_3 &= \begin{pmatrix} & & 0 & 1 \\ & & 0 & 0 \\ & 0 & & \\ & 0 & & & \end{pmatrix} \\
    r_4 &= \begin{pmatrix} & 0 & & & \\ & & & 0 & 1 \\ & & 1 & 0 & \\ & & & 0 & 0 \end{pmatrix}
\end{align*}
\]

and, for the negative roots,

\[
\begin{align*}
    s_1 &= \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -1 & 0 \end{pmatrix} \\
    s_2 &= \begin{pmatrix} & & & 1 & 0 \\ & & 1 & 0 & \\ & 0 & & \\ & & & 0 & 1 \end{pmatrix} \\
    s_3 &= \begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ & 0 & & \\ & 1 & 0 & & \end{pmatrix} \\
    s_4 &= \begin{pmatrix} & & & 0 & 1 \\ & & 0 & 1 & \\ & 0 & & \\ & & & 0 & 0 \end{pmatrix}
\end{align*}
\]
Here we collect the notation of all our Lie subalgebras for future use.

**Definition 2.2.1.**

\[
\mathfrak{sp}_4 = \begin{pmatrix}
    h_1 & r_1 & r_2 & r_3 \\
    s_1 & h_2 & r_4 & r_2 \\
    s_2 & s_4 & -h_2 & -r_1 \\
    s_3 & s_2 & -s_1 & -h_1
\end{pmatrix}
\]

\[
b = \begin{pmatrix}
    h_1 & r_1 & r_2 & r_3 \\
    0 & h_2 & r_4 & r_2 \\
    0 & 0 & -h_2 & -r_1 \\
    0 & 0 & 0 & -h_1
\end{pmatrix}
\]

\[
r = \begin{pmatrix}
    0 & r_1 & r_2 & r_3 \\
    0 & 0 & r_4 & r_2 \\
    0 & 0 & 0 & -r_1 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

2.2.1 The Adjoint Representation

Let \( \bar{\rho} : G \rightarrow \text{GSp}_4(R) \) be a representation. Define a new representation

\[
\text{Ad}(\bar{\rho}) : G \rightarrow \text{Aut}_R(\mathfrak{gsp}_4)
\]

\[
\text{Ad}(\bar{\rho})(g) \circ X = \bar{\rho}(g)X\bar{\rho}(g)^{-1}
\]

This representation, called the adjoint representation of \( \bar{\rho} \), contains a one dimensional direct summand on which it acts trivially:

\[
\text{Ad}(\bar{\rho}) \cong 1 \oplus \text{Ad}^0(\bar{\rho})
\]
corresponding to the Lie algebra decomposition:

\[ \mathfrak{gsp}_4 = 1 \oplus \mathfrak{sp}_4 \]

where \( \text{Ad}^0(\bar{\rho}) \) denotes the representation given by restriction to \( \mathfrak{sp}_4 \)

\[ \text{Ad}^0(\bar{\rho}) : G \to \text{Aut}_R(\mathfrak{sp}_4) \]

**Definition 2.2.2.** For an \( \text{Ad}^0(\bar{\rho}) \)-stable subspace \( \mathfrak{a} \) of \( \mathfrak{sp}_4 \), let \( (\rho_\mathfrak{a}, \mathfrak{a}) \) denote the representation given by restricting the action to \( \mathfrak{a} \).

**Definition 2.2.3.** For an \( \text{Ad}(\bar{\rho}) \)-stable pair of subspaces \( \mathfrak{b} \subset \mathfrak{a} \), let \( (\rho_{\mathfrak{a}/\mathfrak{b}}, \mathfrak{a}/\mathfrak{b}) \) denote the quotient representation of the restricted representation \( (\rho_\mathfrak{a}, \mathfrak{a}) \).

**Remark 2.2.1.** In chapter 5 we will study the Galois cohomology of \( \text{Ad}(\bar{\rho}), \text{Ad}^0(\bar{\rho}), \rho_\mathfrak{b}, \rho_\mathfrak{t}, \rho_{\mathfrak{sp}_4/\mathfrak{b}}, \text{and } \rho_{\mathfrak{sp}_4/\mathfrak{t}} \).

### 2.3 Representations of \( \text{GSp}_4(\mathbb{R}) \)

The \( \mathbb{R} \) points of the group scheme \( \text{GSp}_4 \) form an 11 dimensional Lie group whose Lie algebra is denoted \( \mathfrak{gsp}_4(\mathbb{R}) \). The maximal compact subgroup, \( K \subset \text{GSp}_4(\mathbb{R}) \), is the unitary group \( \text{U}(2) \). For the purposes of this work, we will only need to recall information about the discrete series representations of \( \text{GSp}_4(\mathbb{R}) \) described below.

#### 2.3.1 Discrete Series

A Theorem of Harish-Chandra states that a real semisimple Lie group \( G \) has a representation \( \rho \) whose matrix coefficients \( \langle \rho(g)v, w \rangle \) are square integrable with respect to a Haar measure on \( G \) (the discrete series representations) if and only if the rank of \( G \) is equal to the rank of the maximal compact subgroup \( K \). For our group \( \text{GSp}_4(\mathbb{R}) \), the nonsingular elements \( \lambda \) of the co-weight lattice \( X_*(T) \) which lie in a
dominant Weyl chamber (with respect to the choice of positive and compact roots) are in one-to-one correspondence with a packet of two discrete series representations: the holomorphic discrete series and the generic discrete series. These representations are denoted $\mathbb{D}_\lambda$ and for our work will always denote the holomorphic discrete series element in the packet.

In the case of $\text{GSp}_4(\mathbb{R})$, there are different conventions for the indexing the coordinates of the weights $\lambda = (m_1, m_2; w)$ (where $m_1 \geq m_2$ and $w \equiv m_1 + m_2 \mod 2$). Let $\lambda^{HC} = (\nu_1, \nu_2; w)$ denote the Harish-Chandra parameter and let $\lambda^B = (k_1, k_2; k_0)$ denote the Blattner parameter. They are related via the formulas

\begin{align*}
m_1 &= k_1 - 3 = \nu_1 - 2 \\
m_2 &= k_2 - 3 = \nu_2 - 1
\end{align*}

The representation $\mathbb{D}_\lambda$ is called cohomological if $m_2 \geq 0$. We remark that this condition ensures that the representation appears in the middle-degree cohomology of a Shimura variety of principally polarized abelian surfaces with level structure with coefficients in a local system indexed by the Blattner parameter.

2.4 Representations of $\text{GSp}_4(\mathbb{Q}_p)$

For this section, our main reference is [20] which organizes the irreducible smooth representations of $\text{GSp}_4(\mathbb{Q}_p)$ obtained as subquotients from a parabolic induction and describes the $L$-packets associated to each.

2.4.1 Normalized Parabolic Induction

Given three continuous characters $\chi_1, \chi_2$, and $\sigma: \mathbb{Q}_p^\times \to \mathbb{C}^\times$ we define a character of the $\mathbb{Q}_p$ points of $T$:

$$\chi = \chi_1 \times \chi_2 \rtimes \sigma : T(\mathbb{Q}_p) \to \mathbb{C}^\times$$
\[ \chi(t(a, b, c)) = \chi_1(a)\chi_2(b)\sigma(c) \]

which can be pulled back via the projection \( B \to T \cong B/N \) to define a character of \( B \). The normalized parabolic induction is defined as

\[
I_{GSp}^B(\chi) := \{ f : GSp_4(\mathbb{Q}_p) \to \mathbb{C} : f(hg) = |a^2b||c|^{-\frac{3}{2}}\chi_1(a)\chi_2(b)\sigma(c)f(g) \}
\]

where \( f \) is locally constant and \( h \in B \) having diagonal entries \([a, b, cb^{-1}, ca^{-1}]\) with the group \( GSp_4 \) acting by

\[ g(f(x)) = f(xg) \]

We remark that this normalized induction factor \( |ab^2||c|^{-\frac{3}{2}} \) comes from our choice of root system having compact root \( \rho = (2, 1, -\frac{3}{2}) \)

If \( I_{GSp}^B(\chi) \) is an irreducible representation of \( GSp_4(\mathbb{Q}_p) \) then it is called a principal series representation; otherwise, it is reducible and has irreducible constituents classified by Sally-Tadić and summarized in [20]. The representations \( I_{GSp}^B(\chi) \) are irreducible if and only if

\[ \chi_1 \neq |.|_p^{\pm 1} ; \chi_2 \neq |.|_p^{\pm 1} ; \chi_1 \neq |.|_p^{\pm 1} \chi_2^{\pm 1} \]

Given a choice of character \( \sigma \), the induced representation:

\[
I_B^G(\mathbb{Q}_p^2 \times |.|_p \rtimes \sigma, |.|_p^{\frac{3}{2}}) \]

is not irreducible but has four irreducible constituents, one of which is the has monodromy operator \( N \) of maximal rank (Type IV.A), denoted \( St_{\sigma} \). This representation is also referred to as the maximal Steinberg representation.
2.5 \textit{L-Parameters}

Given a non-archimedean local field \( F \) of characteristic 0 with residue characteristic \( p \) and a connected reductive group \( G \) over \( F \), let \( \Pi(G) \) denote the set of isomorphism classes of smooth irreducible representations of \( G(F) \) and let \( \Phi(G) \) denote the set of conjugacy classes of \( L \)-parameters (discussed below) valued in the Langlands dual group \( G^\vee \) (see §3). A local Langlands correspondence for \( G \) is a surjective finite-to-one map:

\[
L : \Pi(G) \to \Phi(G)
\]

which preserves natural invariants of \( L \)-functions one would attach to each side: \( \gamma \)-factors, \( L \)-factors, and \( \epsilon \)-factors.

The local Langlands correspondence for \( \text{GSp}_4(\mathbb{Q}_p) \) is known by the work of Gan-Takeda [7]. To an isomorphism class of an irreducible smooth representation \( \pi_p \) of \( \text{GSp}_4(\mathbb{Q}_p) \), this correspondence assigns a conjugacy class of \( L \)-parameters, i.e. \((\phi, N)\) where \( \phi \) is a semi-simple representation of the Weil group \( W_{\mathbb{Q}_p} \) (valued in the Langlands dual group \( \text{GSp}_4^\vee(\mathbb{C}) = \text{GSpin}_5(\mathbb{C}) \)) and a nilpotent monodromy operator \( N \). In the case of \( \text{GSp}_4 \), we denote this local Langlands correspondence map:

\[
\text{rec}_{\text{GSp}_4} : \Pi(\text{GSp}_4) \to \Phi(\text{GSp}_4)
\]

and call it the reciprocity map for \( \text{GSp}_4 \).

2.5.1 Local Langlands Correspondence for \( \text{GSp}_4(\mathbb{R}) \)

Let \( \mathcal{D}_{(m_1, m_2; w)} \) denote the discrete series representation of \( \text{GSp}_4(\mathbb{R}) \) where \( m_1 \geq m_2 \geq 0 \) and \( m_1 + m_2 \equiv w \mod (2) \). The local Langlands correspondence for \( \text{GSp}_4(\mathbb{R}) \) is known by the work of Langlands. Let \( W_\mathbb{R} = \mathbb{C}^\times \cup j\mathbb{C}^\times \) be the Weil group of \( \mathbb{R} \). The
representations $\mathbb{D}_{(m_1,m_2;w)}$ are in one-to-two correspondence with the $L$-parameters $\phi_{(m_1,m_2;w)}$ defined by

$$\phi_{(m_1,m_2;w)}(z) = |z|^{-w} \begin{pmatrix}
\left(\frac{z}{\bar{z}}\right)^{\frac{m_1+m_2}{2}}
\left(\frac{z}{\bar{z}}\right)^{\frac{m_1-m_2}{2}}
\left(\frac{\bar{z}}{z}\right)^{\frac{m_1-m_2}{2}}
\left(\frac{\bar{z}}{z}\right)^{\frac{m_1+m_2}{2}}
\end{pmatrix}
$$

$$\phi_{(m_1,m_2;w)}(j) = \begin{pmatrix}
1
1
(-1)^{w+1}
(-1)^{w+1}
\end{pmatrix}
$$

2.5.2 Local Langlands for $\text{GSp}_4(\mathbb{Q}_p)$

When the smooth representation $\pi_p$ appears as an irreducible constituent of a normalized parabolic induction of a continuous character $\chi = \chi_1 \times \chi_2 \rtimes \sigma$, the $L$-parameters may be assigned in a uniform fashion and are catalogued in the appendix of [20]. Let $\pi_p$ be an irreducible constituent of some induced representation $I_{\text{GSp}}^G(\chi_1 \times \chi_2 \rtimes \sigma)$. The $L$-parameter of $\pi_p$ is defined as

$$L(\pi_p) := (\sigma \oplus \chi_1 \sigma \oplus \chi_2 \sigma \oplus \chi_1 \chi_2 \sigma, N)$$

where $N$ is one of the five monodromy operators described in [20] or the 0 operator.
2.6 Hecke Algebras

Definition 2.6.1. For a prime number $l$, define the local Hecke algebra, denoted $\mathcal{H}_l$, as

$$\mathbb{Z}[\text{GSp}_4(\mathbb{Q}_l)//\text{GSp}_4(\mathbb{Z}_l)]$$

the convolution algebra of functions with compact support which are bi-invariant under the action of $\text{GSp}_4(\mathbb{Z}_l)$

Definition 2.6.2. For a positive integer $N$, let $\mathcal{H}^N$ denote the tensor product of all the $\mathcal{H}_l$ where $l$ does not divide $N$.

For our Hecke operators at $p$, the Iwahori subgroup $I \subset \text{GSp}_4(\mathbb{Q}_p)$ is defined as

$$I = \{g \in \text{GSp}_4(\mathbb{Z}_p) : g \in B(\mathbb{Z}/p\mathbb{Z})\}$$

We define the $U_{p,i}$ operators following [23] as double cosets using the diagonal matrices:

$$\beta_1 = t(1, 1; p^{-1}) \quad \beta_2 = t(1, p; p^{-1})$$

$$U_{p,1} = [I \beta_1 I] \quad U_{p,2} = [I \beta_2 I]$$

Definition 2.6.3. The local Hecke algebra at $p$ is denoted $\mathcal{U}_p$, and is defined as the $\mathbb{Z}$-algebra generated by $U_{p,1}$. 

19
CHAPTER 3

THE GALOIS REPRESENTATION

Let $G$ be a connected, reductive, linear algebraic group over an algebraically closed field $\bar{K}$ with root datum $(X^*(T), \Delta, X_*(T), \Delta^\vee)$. The Langlands dual $G^\vee(\mathbb{C})$ is the $\mathbb{C}$-points of the algebraic group with root datum obtained by switching the characters/roots with the co-characters/coroots: i.e. $G^\vee$ has root datum given by $(X_*(T), \Delta^\vee, X^*(T), \Delta)$. One formulation of the Langlands philosophy is as follows: given a suitable representation of $G(K)$, to assign a semisimple representation of the Weil group, $W_K$, valued in the Langlands dual group.

The Langlands dual group of $GSp_{2n}$ is $GSpin_{2n+1}$. In the special case of $n = 2$ there is an accidental isomorphism $GSpin_5 \sim GSp_4$ which, when composed with the Spin representation $GSpin_{2n+1} \to GL_{2n}$, gives us a $GSp_4$-valued Galois representation associated to suitable automorphic representations of $GSp_4(\mathbb{A}_Q)$ which we recall in this section. Specifically, to an irreducible cuspidal automorphic representation, $\pi$, of $GSp_4(\mathbb{A}_Q)$ of cohomological weight $(m_1, m_2; w)$ there is, for every prime $p$, the spin Galois representation $\rho_{\pi,p}$ [16]. Recall that in the introduction we have fixed, for each $p$, an isomorphism $\iota : \bar{\mathbb{Q}}_p \to \mathbb{C}$. Let $rec_{GSp_4}$ denote the reciprocity map discussed defined in [7] which was discussed in §2.5.

**Theorem 3.0.1.** Let $\pi$ be an irreducible cuspidal automorphic representation of $GSp_4(\mathbb{A}_Q)$ whose infinite component $\pi_\infty \cong D_{(m_1, m_2; w)}$ with $m_1 \geq m_2 \geq 0$. Then, for every prime number $p$, there exists a continuous Galois representation:

$$\rho_{\pi,p} : G_{\bar{\mathbb{Q}}} \to GSp_4(\bar{\mathbb{Q}}_p)$$
with the following properties:

1) If $\ell \nmid \infty$ with $\pi_\ell$-unramified we have local-global compatibility:

$$iWD(\rho_{\pi,p}|_{G_{Q_\ell}})^{Fr-ss} \cong \text{rec}_{\text{GSp}(4)}(\pi_\ell \otimes |_{\ell}^{-\frac{3}{2}})$$

2) If $\ell = p$ and $\pi_p$ is unramified, then $\rho_{\pi,p}|_{G_{Q_p}}$ is a crystalline representation with Hodge-Tate weights:

$$HT(\rho_{\pi,p}) = \frac{w - m_1 - m_2}{2} + \{0, m_2 + 1, m_1 + 2, m_1 + m_2 + 3\}$$

and if $\pi_p$ is induced from the Borel character $\chi = \chi_1 \times \chi_2 \ltimes \sigma$, then the eigenvalues of the crystalline Frobenius $\varphi_{\text{crys}}$ are

$$\varphi_{\text{crys}} = p^{-\frac{3}{2}}\{\sigma(p), \chi_1\sigma(p), \chi_2\sigma(p), \chi_1\chi_2\sigma(p)\}$$

Proof. This result is Theorem 3.5 from [16].

\[\square\]

Remark 3.0.2. Let $\pi$ be as above with cohomological weight $(m_1, m_2; w) = (0, 0, -\frac{3}{2})$ and $\rho_{\pi,p}$ its Galois representation. We will be assuming that $\rho_{\pi,p}|_p$ is ordinary, which tells us there exists unramified characters $\alpha$ and $\beta$ such that:

$$\rho_{\pi,p}|_p : G_{Q_p} \to \text{GSp}_4(\bar{Q}_p)$$

$$\rho_{\pi,p}|_p(g) = \begin{pmatrix} \alpha^{-1} \epsilon^3 & * & * & * \\ 0 & \beta^{-1} \epsilon^2 & * & * \\ 0 & 0 & \beta \epsilon & * \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

The following proposition is a main ingredient in the proof of our first main theorem.
Proposition 3.0.3. Let $\pi$ be as above and $\rho_{\pi,p}$ its Galois representation for the prime $p$. Assume $\rho_{\pi,p}$ is ordinary at $p$. Then, if $\pi_p$ is Type IV.A in [20], we have

\[\alpha(Frob_p) = \beta(Frob_p)\]

\[\alpha^2(Frob_p) = \beta^2(Frob_p) = 1\]

Proof. Using local-global compatibility, we need to twist $\pi_p$ by $\nu^{-\frac{3}{2}}$. The twist of a representation induced from the Borel twists only the multiplier character:

\[(\nu^2 \times \nu \times \nu^{-\frac{3}{2}}) \otimes \nu^{-\frac{3}{2}} = (\nu^2 \times \nu \times \nu^{-3}\sigma)\]

Thus, the $L$-parameter assigned to $\pi_p$ is the $L$-parameter assigned to this twist:

\[
\rho(Frob_p) = \begin{pmatrix}
(\sigma)(Frob_p) \\
(\nu^{-1}\sigma)(Frob_p) \\
(\nu^{-2}\sigma)(Frob_p) \\
(\nu^{-3}\sigma)(Frob_p)
\end{pmatrix}
\]

Meanwhile, on the right hand side, the Local Langlands correspondence applied to $\rho_{\pi,p}$ gives us the $(\varphi, N)$ module $D_{st}(\rho_{\pi,p})$ where $\varphi$ acts as:

\[\varphi = \begin{pmatrix}
\alpha^{-1}(Frob_p)p^3 \\
\beta^{-1}(Frob_p)p^2 \\
\beta(Frob_p)p \\
\alpha(Frob_p)
\end{pmatrix}\]

Matching these up we see that $\alpha = \beta = \sigma$ and since we assume $\sigma$ is unramified:
\[ v_p(\alpha(Frob_p)) = v_p(\beta(Frob_p)) = v_p(\sigma(Frob_p)) = 0 \]

For the last part of the lemma, we compute the multiplier character of both sides of the local Langlands correspondence to see that:

\[ p^3 \sigma^2(Frob_p) = p^3 \]
\[ \sigma^2(Frob_p) = 1 \]

\[ \square \]

3.1 Greenberg’s \( \mathcal{L} \)-invariant

In [8], Greenberg gives a definition for the \( p \)-adic \( \mathcal{L} \)-invariant of an ordinary Galois representation (under certain technical hypothesis we explain below). Another good reference for this definition is Harron’s Ph.D. thesis [10] whose exposition we will follow closely in this section.

For our two main results in §4, we remark that Rosso [21] uses the \( \mathcal{L} \)-invariant of a semistable Galois representation defined by Benois [1] which coincides with the Greenberg \( \mathcal{L} \)-invariant in the case of an ordinary Galois representation (see: Remark 2.2.2 in [1]).

3.1.1 Hypotheses (S), (U), (T) and the Exceptional Subquotient

Fix \((\rho, V)\), a continuous Galois representation on a \( \mathbb{Q}_p \) vector space \( V \)

\[ \rho : G_\mathbb{Q} \to \text{Aut}_{\mathbb{Q}_p}(V) \]

The representation \((\rho, V)\) is called ordinary at \( p \) if there exists a separated, exhaustive descending filtration \( \{F^iV\}_{i \in \mathbb{Z}} \) of \( V \) into \( G_{\mathbb{Q}_p} \)-stable \( \mathbb{Q}_p \)-subspaces such that,
on the $i$th graded piece $\text{gr}^i V = F^i V / F^{i+1} V$ the inertia subgroup $I_{Q_p}$ acts via multiplication by $\epsilon^i$ (where $\epsilon$ is the $p$-adic cyclotomic character). From now on, we will assume our representation $(\rho, V)$ satisfies the following hypotheses:

(S) : for all $i$, $G_{Q_p}$ acts semisimply on $\text{gr}^i V$;

(U) : $V$ has no $G_{Q_p}$ subquotient $U$ isomorphic to a crystalline extension of $Q_p$ by $Q_p(1)$

We remark that a crystalline extension arises from a cohomology class in the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}_p, Q_p(1))$ defined in [2] in terms of $p$-adic Hodge theory.

We now recall the definition of Greenberg’s exceptional subquotient $W$ of $V$ to formulate hypothesis (T). This involves a refinement of the ordinary filtration between the filtered pieces $F^2 V$ and $F^0 V$:

$$F^2(V) \subseteq F^{11} V \subseteq F^{1} V \subseteq F^{00} V \subseteq F^{0} V$$

**Definition 3.1.1.** 1) Let $F^{00} V$ be the maximal $G_{Q_p}$-stable subspace of $F^0 V$ such that $G_{Q_p}$ acts trivially on the quotient $F^{00} V / F^1 V$. The dimension of this quotient $F^{00} V / F^1 V$ is denoted $m_0$.

2) Let $F^{11} V$ be the minimal $G_{Q_p}$-stable subspace of $F^1 V$ such that $G_{Q_p}$ acts by multiplication by $\epsilon$ on the quotient $F^1 V / F^{11} V$. The dimension of this quotient $F^1 V / F^{11} V$ is denoted $m_1$.

3) Let $W$, called the exceptional subquotient of $V$, be defined

$$W := F^{00} V / F^{11} V$$

We remark that, by definition, the dimension of $W$ is $m_0 + m_1$ and $W$ is an ordinary $G_{Q_p}$ representation with the following two-step filtration:
\[ F^0W = W = F^{00}V/F^{11}V \]

\[ F^1W = F^1V/F^{11}V \]

\[ F^2W = 0 \]

Define \( t_0 := \dim_{\mathbb{Q}_p} W^{G_{\mathbb{Q}_p}} \) and \( t_1 := \dim_{\mathbb{Q}_p}(W^*(1))^{G_{\mathbb{Q}_p}} \). These subspaces define two \( G_{\mathbb{Q}_p} \)-linear maps which are injective and surjective, respectively:

\[ \phi_0 : \mathbb{Q}_p^{t_0} \to W \]

\[ \phi_1 : W \to \mathbb{Q}_p(1)^{t_1} \]

Letting \( M := \ker(\phi_1)/\im(\phi_0) \), we obtain an isomorphism of \( G_{\mathbb{Q}_p} \)-representations:

\[ W \cong M \oplus \mathbb{Q}_p^{t_0} \oplus \mathbb{Q}_p(1)^{t_1} \]

The subspace \( M \) is a non-split extension of \( \mathbb{Q}_p^{t_0} \) by \( \mathbb{Q}_p(1)^{t_1} \) where \( 2t = \dim_{\mathbb{Q}_p} W - t_0 - t_1 \) and we fix \( t \) to denote this number.

We can now state hypothesis \((T)\) from [8] which we note is conjectured to be true:

\[(T) : \text{At least one of } t_0 \text{ or } t_1 \text{ is zero}\]

**Definition 3.1.2.** Let \((\rho, V)\) be an ordinary \( \mathbb{Q}_p \) representation satisfying hypotheses \((S), (U), \) and \((T)\). We say \((\rho, V)\) is exceptional if the exceptional subquotient \( W \) is non-zero. In such a case, define the integer

\[ e := t + t_0 + t_1 \]

With this notation, we can equivalently define \((\rho, V)\) to be exceptional if, and only if, \( e \neq 0 \).
Remark 3.1.1. This $e$ defined above is conjectured (combining conjectures 1 and 2 in section 2 of [8]) to be the difference between the order of vanishing of the $p$-adic $L$ function of $V$ at $s = 1$ with the order of vanishing of the archimedean $L$-function of $V$.

3.1.2 The Balanced Selmer Groups

In order to define the $\mathcal{L}$-invariant of an ordinary $p$-adic Galois representation, we will need another hypothesis on the vanishing of a certain Selmer group of $V$ called the balanced Selmer group which we describe in this section. We will denote the local conditions of our Selmer systems in this section with $L_v(V)$ so as not to confuse the reader with our notation for $\mathcal{L}$-invariant.

Let $(\rho, V)$ be an ordinary continuous $G_{\mathbb{Q}}$-representation on a $\mathbb{Q}_p$ vectorspace $V$ satisfying properties (S), (U), and (T) from the last section. Recall that a Selmer system for a Galois representation $(\rho, V)$ is a collection of subspaces $L_v(V) \subset H^1(\mathbb{Q}_p, V)$ for each place $v$ of $\mathbb{Q}$. For places $v \neq p$, we define:

$$L_v(V) := H^1_{ur}(\mathbb{Q}_v, V) := \ker(H^1(\mathbb{Q}_v, V) \to H^1(I_{\mathbb{Q}_v}, V))$$

For $v = p$, we define the local condition $L_p(V)$ as the set of all cohomology classes $c \in H^1(\mathbb{Q}_p, V)$ such that:

$$\text{(Bal1)} : c \in \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V/F^{00}V))$$

$$\text{(Bal2)} : c \mod F^{11}V \text{ is in the image of}$$

$$H^1_f(\mathbb{Q}_p, F^1V/F^{11}V) \oplus H^1_{ur}(\mathbb{Q}_p, (F^{00}V/F^{11}V)^{G_{\mathbb{Q}_p}})$$

$$\text{in } H^1(\mathbb{Q}_p, V/F^{11}V)$$

Given these local conditions at all places $v$ of $\mathbb{Q}$, we define the balanced Selmer group of our $(\rho, V)$ which Greenberg denotes $\overline{\text{Sel}}_{\mathbb{Q}}(V)$.
Definition 3.1.3. The balanced Selmer group $H^1_\mathcal{L}(V)$ of our $(\rho, V)$ is defined as

$$H^1_\mathcal{L}(V) := \ker \left( H^1(\mathbb{Q}, V) \to H^1(\mathbb{Q}_p, V)/L_p(V) \bigoplus_{v \neq p} H^1(\mathbb{Q}_v, V)/L_v(V) \right)$$

with local conditions $L_p(V)$ and $L_v(V)$ defined above.

The term balanced in this definition is due to the following result of Greenberg. In it, he uses the hypothesis that $V$ be critical at $s = 1$, a condition defined by Deligne in terms of the archimedean $L$-function of $V$.

Proposition 3.1.2. (Proposition 2 of [8]) If $V$ is critical at $s = 1$, then

$$\dim H^1_\mathcal{L}(V) = \dim H^1_\mathcal{L}(V^*(1))$$

We will need this proposition to define our $\mathcal{L}$-invariants and so we now add the condition that $V$ be critical at $s = 1$ in the sense of Deligne.

3.1.3 Definition of the $\mathcal{L}$-invariant

Let $(\rho, V)$ be an ordinary continuous $G_{\mathbb{Q}}$-representation on a $\mathbb{Q}_p$ vector space $V$ which satisfies hypotheses (S), (U), and (T); assume $V$ is exceptional and that $V$ is critical at $s = 1$. We now add the hypothesis that $H^1_\mathcal{L}(V) = 0$.

Recall that $(\rho, V)$ has a filtration $\{F^i V\}_{i \in \mathbb{Z}}$ and the exceptional filtration pieces $F^{00} V$ and $F^{11} V$ which define the exceptional quotient $W$ which has a filtration as well. Define

$$F^{00} H^1(\mathbb{Q}_p, V) := \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V/F^{00} V))$$

Greenberg proved the following lemma on page 161 of [8].

Lemma 3.1.3. The local condition $L_p(V)$ from the definition of the balanced Selmer
group $H^1_L(V)$ has codimension $e$ in $F^{00}H^1(Q_p, V)$

Let $\Sigma$ be defined as the finite set of places where $V$ is ramified together with $p$ and $\infty$. By definition of the local conditions in the balanced Selmer group, we see that $H^1_L(V) \subseteq H^1(G_Q, V)$. Recall the first four terms of the Poitou-Tate exact sequence associated to our local conditions (see: 8.6.13 in $[17]$):

$$
0 \longrightarrow H^1_L(V) \longrightarrow H^1(G_Q, V) \longrightarrow \bigoplus_{v \in \Sigma} H^1(Q_v, V)/L_v(V) \longrightarrow H^1_L(V^*(1))^*
$$

By assuming $V$ is critical at $s = 1$, the vanishing of the balanced Selmer group implies the vanishing of the balanced Selmer group of $V^*(1)$ by Proposition 3.1.2 of the last section. Thus, the above exact sequence becomes an isomorphism of the middle terms:

$$
H^1(G_Q, V) \cong \bigoplus_{v \in \Sigma} H^1(Q_v, V)/L_v(V)
$$

We can then define a subspace on the left hand side by defining a subspace on the right hand side and composing with the above isomorphism.

**Definition 3.1.4.** Define $H^{exc}_{glob}(V)$ to be the $e$-dimensional subspace of $H^1(G_Q, V)$ corresponding to the $e$-dimensional quotient $F^{00}H^1(Q_p)/L_p(V)$ which sits inside

$$
\bigoplus_{v \in \Sigma} H^1(Q_v, V)/L_v(V)
$$

through the above isomorphism given by Poitou-Tate.

We now begin working under the assumption that $t_1 = 0$, as we are under hypothesis (T). Consider the image of our subspace $H^{exc}_{glob}(V)$ under the composition of maps

$$
\lambda : H^1(G_Q, V) \to H^1(Q_p, V) \to H^1(Q_p, V/F^1V)
$$

28
as well as the inclusion (since \( H^0(Q_p, V/F^{00}V) = 0 \))

\[
H^1(Q_p, W/F^1W) \to H^1(Q_p, V/F^1V)
\]

The following proposition-definition compares the images of these two maps and can be found in section 3 of \([8]\)

**Proposition-Definition 1.** The image of \( \text{H}^{\text{exc}}_{\text{glob}}(V) \) under the map \( \lambda \) above satisfies:

1) \( \lambda(\text{H}^{\text{exc}}_{\text{glob}}(V)) \subseteq H^1(Q_p, W/F^1W) \)

2) \( \dim \lambda(\text{H}^{\text{exc}}_{\text{glob}}(V)) = e \)

3) \( \lambda(\text{H}^{\text{exc}}_{\text{glob}}(V) \cap H^1_{\text{ur}}(Q_p, W/F^1W) = 0 \)

Based on the above, we define the new space \( \text{H}^{\text{exc}}_{\text{loc}}(V) \) to be the image \( \lambda(\text{H}^{\text{exc}}_{\text{glob}}(V)) \) viewed as a subspace of \( H^1(Q_p, W/F^1W) \).

The \( \mathcal{L} \)-invariant can be thought of as measuring how skewed \( \text{H}^{\text{exc}}_{\text{loc}}(V) \) sits as a subspace of \( H^1(Q_p, W/F^1W) \) with respect to certain coordinate systems which we describe below.

Since \( W/F^1W \) is a trivial \( G_{Q_p} \)-module, we have

\[
H^1(Q_p, W/F^1W) \cong \text{Hom}(G_{Q_p}, W/F^1W)
\]

A homomorphism out of \( G_{Q_p} \) to such a \( p \)-adic vectorspace must factor through the maximal pro-\( p \) quotient of \( G_{Q_p}^{\text{ab}} \). By local class field theory for \( Q_p \), this group is realized explicitly as the Galois group of the extension \( F_\infty \) which is the compositum of two extensions of \( Q_p \): the cyclotomic \( \mathbb{Z}_p \)-extension, denoted \( Q_\infty,p \), and the maximal unramified abelian extension which we denote \( Q_p^{\text{ur}} \). Let

\[
\Gamma_\infty := \text{Gal}(Q_\infty,p/Q_p) \cong \text{Gal}(F_\infty/Q_p^{\text{ur}})
\]

\[
\Gamma_{\text{ur}} := \text{Gal}(Q_p^{\text{ur}}/Q_p) \cong \text{Gal}(F_\infty/Q_\infty,p)
\]
denote the Galois groups of these two $\mathbb{Z}_p$ extensions which we remark are disjoint and give us the following isomorphism:

$$\text{Gal}(F_\infty/Q_p) \cong \Gamma_\infty \times \Gamma_{ur}$$

which allows us to write

$$H^1(Q_p, W/F^1W) = \text{Hom}(\Gamma_\infty, W/F^1W) \times \text{Hom}(\Gamma_{ur}, W/F^1W)$$

with projection maps to each factor denoted $pr_\infty$ and $pr_{ur}$. By proposition-definition 1, we know $H^\text{exc}_{\text{loc}}(V)$ is a subspace and we denote the restrictions of the projection maps to this subspace as $pr'_\infty$ (an invertible map, by proposition-definition 1) and $pr'_{ur}$ respectively.

Recall that the spaces $\text{Hom}(\Gamma_*, Q_p)$ for $* \in \{\infty, \text{ur}\}$ are one dimensional $\mathbb{Q}_p$ vector spaces with the $p$-adic logarithm of the cyclotomic character $\log \epsilon$ as a basis for $* = \infty$ and the ord function (which sends the Frobenius element to 1) for the case $* = \text{ur}$. These two chosen basis, and the isomorphisms

$$\text{Hom}(\Gamma_*, W/F^1W) \cong \text{Hom}(\Gamma_*, Q_p) \otimes W/F^1W$$

allow us to define the following isomorphisms

$$\iota_\infty : \text{Hom}(\Gamma_\infty, W/F^1W) \to W/F^1W$$

$$\log \epsilon \otimes w \to w$$

$$\iota_{ur} : \text{Hom}(\Gamma_{ur}, W/F^1W) \to W/F^1W$$

$$\text{ord} \otimes w \to w$$
Finally, we can define a new map from $W/F^{1}W$ to itself via the composition:

\[
\begin{array}{c}
\Hom(\Gamma_{\infty}, W/F^{1}W) \xrightarrow{pr_{\infty}^{-1}} H_{\text{loc}}^{\text{exc}}(V) \xrightarrow{pr_{ur}' \circ \iota_{\infty}^{-1}} \Hom(\Gamma_{ur}, W/F^{1}W) \\
W/F^{1}W & \text{W/F}^{1}W
\end{array}
\]

**Definition 3.1.5.** The $\mathcal{L}$-invariant of the representation $V$ is defined as

\[\mathcal{L}_V := \det(\iota_{ur} \circ pr_{ur}' \circ pr_{\infty}^{-1} \circ \iota_{\infty}^{-1})\]
CHAPTER 4

ORDINARY HIDA FAMILIES

The seminal work on $p$-adic families of ordinary modular forms $f \in S_k(\Gamma_0(Np^j))$ was undertaken by Hida in [11] & [12] and is a powerful tool for proving congruences. These ideas were quickly formalized into the theory of $\Lambda$-adic modular forms and $\Lambda$-adic Galois representations by Wiles [26] and were an integral part of his proof of the Iwasawa main conjecture for totally real number fields [25]. We quickly recall some basic facts about Hida’s original work to motivate the Hida families for ordinary irreducible cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ constructed by Tilouine [23] and used in this work.

Given a modular cusp form $f(z) = \sum_{n=1}^{\infty} a_n(f)q^n$ (where $q = e^{2\pi i z}$) whose $p$-th Fourier coefficient $a_p(f)$ has $p$-adic valuation equal to zero (the $p$-ordinary condition), Hida’s work gives a finite flat $\mathbb{Z}_p[[X]]$-algebra $\mathbb{T}$ called the Hida family of $f$ whose spectrum corresponds to $p$-adic ordinary modular forms with $p$-adic weight $k \in \mathbb{Z}_p^\times$. When $2 \leq k \in \mathbb{Z}$ is an integer, these $p$-adic ordinary modular forms are shown to be classical and, by Hida’s control theorem, the number $d$ of weight $k$ classical ordinary modular forms $f_1, \ldots, f_d$ on the Hida family is independent of the weight $k$. This invariant $d$ is called the degree of the Hida family. By construction, for each integer weight $k \geq 2$, the weight $k$ modular forms are all congruent mod $p$ in the sense that, for all primes $l \nmid Np$, the Fourier coefficients of the $f_i$ are all congruent mod $p$:

$$a_i(f_1) \equiv \ldots \equiv a_i(f_d) \mod (p)$$
Moreover, Hida constructed analytic Galois representations $\rho_T : G_\mathbb{Q} \to \text{Gl}_2(\mathbb{T})$ which interpolate the Galois representations $\rho_{f_i}$ of all the classical $p$-ordinary modular forms in the Hida family for all weights $k \geq 2$. The above congruences can be made even stronger as this analytic Galois representation proves the congruence of the Galois representations of these weight $k$ forms $f_i$ as well:

$$\bar{\rho}_{f_1} \cong \cdots \cong \bar{\rho}_{f_d} \mod p$$

One recovers the congruence of the Fourier coefficients by noting that $\rho_{f_i}(\text{Frob}_l) = a_l(f_i)$. Thus, computing the degree of a given Hida family $\mathbb{T}$ can be used to prove the existence of multiple (modular) characteristic zero lifts of the mod $p$ Galois representations of the modular forms of weight $k$ in the Hida family.

In [23], Tilouine constructs Hida families for nearly-ordinary irreducible cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_K)$ (with $K/\mathbb{Q}$ a totally real number field) of cohomological weight. We fix ourselves in the case $K = \mathbb{Q}$. The general strategy Tilouine uses is to construct cohomology modules $V_{(a,b,c)}$ for regular dominant weights $(a, b; c)$ and a large cohomology module $V$ (see §2.2 of [23]) which is the Pontryagin dual of an inverse limit of the localized middle degree cohomology of adelic Shimura varieties with coefficients in a fixed algebraic local system. Theorem 2 of [23] shows that this construction is independent of the fixed local system (the control theorem). In §2.3, Tilouine defines his big Hecke algebra $\mathbf{T}$ as the $\mathbb{Z}_p[[X, Y, Z]]$-algebra generated by the image of the Hecke algebra $\mathcal{H}^{Np}$ in $\text{End}_{\mathbb{Z}_p}(V)$ and similarly the Hecke algebras $\mathbf{T}_{(a,b,c)}$ as the $\mathbb{Z}_p$-algebra generated by the image of the Hecke algebra in $\text{End}_{\mathbb{Z}_p}(V_{(a,b,c)})$.

Let $C$ be an irreducible component of the spectrum of his big Hecke algebra $\mathbf{T}$ with ring of functions $\mathbb{T}_C$. The ring $\mathbb{T}_C$ is an algebra over the Iwasawa algebra in 3 variables $\Lambda_3 = \mathbb{Z}_p[[X, Y, Z]]$. Let $\Lambda_2 = \mathbb{Z}_p[[X, Y]]$ and consider the morphism:
\( \phi : \Lambda_3 \to \Lambda_2 \) sending \( Z \) to 0. Denote \( \mathcal{W}_i = \text{Spec}(\Lambda_i) \) for \( i = 2, 3 \). We will abuse notation and use \((\mathbb{T}_{C_0}, C_0)\) to refer to the base change of such a \((\mathbb{T}_C, C)\) given by the morphism

\[
\begin{array}{ccc}
C_0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
\mathcal{W}_2 & \longrightarrow & \mathcal{W}_3 \\
\text{Spec}(\phi) & & \\
\end{array}
\]

**Proposition 4.0.1.** For \( C_0 \) as above, the morphism

\[ w : C_0 \to \mathcal{W}_2 \]

is finite and surjective

This map \( w \) in the above proposition shall henceforth be called the weight space map. For positive integers \( k_1 \geq k_2 \geq 3 \) let \( \mathcal{P}_{(k_1,k_2)} \) denote the prime ideal of \( \Lambda_2 \) generated by the polynomials

\[
\mathcal{P}_{(k_1,k_2)} := (X + 1 - (1 + p)^{k_1-2}, Y + 1 - (1 + p)^{k_2-1})
\]

We will need the following lemma to control the Hecke eigenvalues of the arithmetic points in a given Hida Family \( \mathbb{T}_{C_0} \)

**Proposition 4.0.2.** Assume that the weight space map is flat at the prime \( \mathcal{P}_{(3,3)} \). Let \( Q_i \) denote the primes of \( \mathbb{T}_{C_0} \) lying above \( \mathcal{P}_{(3,3)} \). Then:

\[
\bigcap_i Q_i = \mathcal{P}_{(3,3)}
\]

**Proof.** We have \( \mathcal{P}_{(3,3)} \subset \bigcap_i Q_i \) since each of the kernels is a prime lying above \( \mathcal{P}_{(3,3)} \).
Consider the surjective morphism of finite flat rings over $\mathbb{Z}_p$:

$$\mathbb{T}_{C_0}/\mathcal{P}_{(3,3)} \to \mathbb{T}_{C_0}/\bigcap_i Q_i$$

Since cusp forms and Hecke algebras are dual, this surjection is between free modules of equal rank and hence an isomorphism. The ideals are thus equal, as was to be shown.

We also recall the existence of analytic Galois representations for these Hida families (Theorem 3 combined with Theorem 1 of [23])

**Proposition 4.0.3.** Let $C_0$ be a Hida Family with ring of functions $\mathbb{T}_{C_0}$. Then there exists a continuous Galois Representation:

$$\rho_{\mathbb{T}_{C_0}} : G_Q \to GSp_4(\mathbb{T}_{C_0})$$

which is ordinary at $p$ and the restriction to the decomposition group, specialized at a weight $(k_1, k_2)$-point has the form :

$$\rho_{\mathbb{T}_{C_0}}|_p = \begin{pmatrix}
    a^{-1}e^{k_1+k_2-3} & * & * & * \\
    0 & b^{-1}e^{k_1-2} & * & * \\
    0 & 0 & b e^{k_2-1} & * \\
    0 & 0 & 0 & a
\end{pmatrix}$$

where $a(Frob_p) := a_{p,1} \in \mathbb{T}_{C_0}$ interpolates the eigenvalue of $U_{p,1}$ and $ab(Frob_p) := a_{p,2} \in \mathbb{T}_{C_0}$ interpolates the eigenvalue of $U_{p,2}$ for all classical points of $\mathbb{T}_{C_0}$ suitably normalized so that $v_p(a(Frob_p)|_{Q_i}) = v_p(b(Frob_p)|_{Q_i}) = 0$ for any classical weight $(k_1, k_2)$ points $Q_i$.  


4.1 Estimates of Analytic Hecke Operators

This section contains all of the calculations used to prove the main theorem of the next section. Rosso’s formulas for the $L$-invariant and adjoint $L$-invariant in [21] are in terms of logarithmic derivatives of the analytic hecke operators $a_{p,1}, a_{p,2} \in \mathbb{T}_{C_0}$ described in Proposition 4.0.3 of the last section. In the first case, Theorem 4.18 of [21], calculates the $L$-invariant which we denote $L_{\pi}$ in the case that the local component $\pi_p$ is Type IV.A in [20] and the derivative is calculated in the parralell weight direction of the weight space. We denote this derivative as $\nabla_{(1,1)}$.

To streamline the proof of our first two main theorems we prove the following lemma:

**Lemma 4.1.1.** Assume that all of the weight $(3,3)$ points of $\mathbb{T}_{C_0}$ have local components Type IV.A from [20] with $\sigma$ an unramified character. Alternatively, assume that all weight $(3,3)$ points of $\mathbb{T}_{C_0}$ have the same $U_{p,i}$-eigenvalues.

Then, at all weight $(3,3)$-points $Q_i$:

\[ v_p(\nabla_{(1,1)}(\log_p(a_{p,1}))|_{Q_i}) \geq 1 - v_p(a(Frob_p)|_{Q_i}) \]

\[ v_p(\nabla_{(1,1)}(\log_p(a_{p,2}))|_{Q_i}) \geq 1 - v_p(ab(Frob_p)|_{Q_i}) \]

**Proof.** To calculate the derivatives in the weight direction, we first solve for the weights as functions of $X$ and $Y$:

\[ X + 1 = (1 + p)^{k_1 - 2} \quad \quad Y + 1 = (1 + p)^{k_2 - 1} \]

\[ k_1 = \frac{\log_p(X + 1)}{\log_p(1 + p)} + 2 \quad \quad k_2 = \frac{\log_p(Y + 1)}{\log_p(1 + p)} + 1 \]

\[ \frac{dk_1}{dX} = \frac{1}{\log_p(p + 1)(X + 1)} \quad \quad \frac{dk_2}{dY} = \frac{1}{\log_p(1 + p)(Y + 1)} \]
Plugging in a weight \((3, 3)\) point (going modulo the prime ideal \(Q_i\)) gives us:

\[
\frac{dk_1}{dX}|_{Q_i} = \frac{1}{\log_p(1+p)(1+p)} \quad \frac{dk_2}{dY}|_{Q_i} = \frac{1}{\log_p(1+p)(1+p)^2}
\]

Since \(v_p(\log_p(1+p)) = 1\) (by the power series expansion of \(\log_p\)) we have:

\[
v_p\left(\frac{dk_1}{dX}|_{Q_i}\right) = v_p\left(\frac{dk_2}{dY}|_{Q_i}\right) = -1
\]

We now solve for \(a_{p,i}\) in terms of \(X\) and \(Y\). Under our first assumption (Type IV.A) on the weight \((3, 3)\) points, \(Q_i\) of \(\mathbb{T}_{c_0}\), our proposition 3.0.3 shows \(a_{p,1}|_{Q_i} = a_{p,2}|_{Q_i} = \pm 1\). Let us organize the proof for both cases of hypothesis in this lemma.

Let \(a_{p,1}|_{Q_i} = \lambda_1\) and \(a_{p,2}|_{Q_i} = \lambda_2\)

With \(Q_i\) denoting the weight \((3, 3)\) primes in \(\mathbb{T}_C\) lemma 3.1 shows:

\[
a_{p,1} - \lambda_1, a_{p,2} - \lambda_2 \in \bigcap_i Q_i = \mathcal{P}_{(3,3)}
\]

With the generators of \(\mathcal{P}_{(3,3)}\) given, there exist \(F_i, G_i \in \mathbb{T}_C\):

\[
a_{p,1} - \lambda_1 = F_1(X + 1 - (1 + p)) + G_1(Y + 1 - (1 + p)^2)
\]

\[
a_{p,2} - \lambda_2 = F_2(X + 1 - (1 + p)) + G_2(Y + 1 - (1 + p)^2)
\]

Taking derivatives of both sides and specializing at a weight \((3, 3)\) point \(Q_i\) yeilds

\[
\frac{da_{p,1}}{dX}|_{Q_i} = F_1|_{Q_i} \quad \frac{da_{p,1}}{dY}|_{Q_i} = G_1|_{Q_i}
\]

\[
\frac{da_{p,2}}{dX}|_{Q_i} = F_2|_{Q_i} \quad \frac{da_{p,2}}{dY}|_{Q_i} = G_2|_{Q_i}
\]

Since \(F_i|_{Q_i}, G_i|_{Q_i}\) are integral, we know \(v_p(F_i|_{Q_i}) \geq 0\) and \(v_p(G_i|_{Q_i}) \geq 0\).
Now we can, in the direction \( \hat{u} = (1, 1) \), estimate \( v_p(\nabla_{(1,1)} \log_p(a_{p,j})|_{Q_i}) \), for \( j = 1, 2 \).

\[
\nabla_{(1,1)} \log_p(a_{p,1}) = \left( \frac{d a_{p,1}}{dk_1} + \frac{d a_{p,1}}{dk_2} \right) \frac{1}{a_{p,1}}
\]

\[
\nabla_{(1,1)} \log_p(a_{p,2}) = \left( \frac{d a_{p,2}}{dk_1} + \frac{d a_{p,2}}{dk_2} \right) \frac{1}{a_{p,2}}
\]

Using the chain rule, we calculate the logarithmic derivatives of \( a_{p,j} \) with respect to \( X \) and \( Y \).

\[
v_p \left( \left( \frac{d a_{p,1}}{dk_1} + \frac{d a_{p,1}}{dk_2} \right) |_{Q_i} \right) = v_p \left( \left( \frac{d a_{p,1}}{dX} \left( \frac{d k_1}{dX} \right)^{-1} + \frac{d a_{p,1}}{dY} \left( \frac{d k_2}{dY} \right)^{-1} \right) |_{Q_i} \right)
\]

\[
\geq \min \left\{ v_p \left( \frac{d a_{p,1}}{dX} |_{Q_i} \right) - v_p \left( \frac{d k_1}{dX} |_{Q_i} \right) , v_p \left( \frac{d a_{p,1}}{dY} |_{Q_i} \right) - v_p \left( \frac{d k_2}{dY} |_{Q_i} \right) \right\}
\]

\[
= \min \{0 - (-1), 0 - (-1)\} = 1
\]

Repeating the exact same calculation with \( a_{p,2} \) shows:

\[
v_p(\nabla_{(1,1)} \log_p(a_{p,1})|_{Q_i}) \geq 1 - v_p(a(Frob_p)|_{Q_i})
\]

\[
v_p(\nabla_{(1,1)} \log_p(a_{p,2})|_{Q_i}) \geq 1 - v_p(ab(Frob_p)|_{Q_i})
\]

\[
\square
\]

4.2 Lower Bounds on the Degree of Weight Space map

In [21], Rosso calculated the \( p \)-adic \( \mathcal{L} \) invariant (of Benois [1]) of the representation \( \rho_{\pi,p} \) of an irreducible cuspidal automorphic representation, \( \pi \), of \( \text{GSp}_4(A_{\mathbb{Q}}) \) of parallel weight \( \bar{k} = (k,k) \) whose local component \( \pi_p \) is Type IV.A from [20]. We remark
that since the monodromy of this representation has maximal rank, there is only one choice of the regular submodule $D$ in Benois’ definition of the $p$-adic $L$-invariant. Assume $\rho_{\pi,p}$ satisfies conditions (S), (U), and (T) from §3.1, assume $\text{Sel}_\mathbb{Q}(\rho_{\pi,p}) = 0$, and assume $\rho_{\pi,p}$ is exceptional and that it is critical at $s = 1$ in the sense of Deligne. Under these hypothesis, the $L$-invariant of $\pi$ is defined and we denote it with $L_\pi$. Rosso’s Theorem 4.18 of [21] states (using the notation of this paper):

$$L_\pi = \frac{d \log p(a_{p,1})}{dk} \bigg|_{k=(k,k)}$$

He also calculated a similar formula for the $p$-adic $L$ invariant of the adjoint of the spin representation $\text{Ad}(\rho_{\pi,p})$ without any hypothesis on the local component $\pi_p$. We remark that a choice of $p$-stabilization of the spin representation induces a choice of $p$-stabilization of $\text{Ad}(\rho_{\pi,p})$. Again, assume that $\text{Ad}(\rho_{\pi,p})$ satisfies the running hypothesis from §3 so that the $L$-invariant is defined. We denote the adjoint $L$-invariant as $L_{\pi,\text{Ad}}$. Rosso’s Theorem 5.2 of [21] says, in our notation:

$$L_{\pi,\text{Ad}} = 2 \det \begin{pmatrix} \frac{d \log p(a_{p,1})}{dk_1} & \frac{d \log p(a_{p,2})}{dk_1} \\ \frac{d \log p(a_{p,1})}{dk_2} & \frac{d \log p(a_{p,2})}{dk_2} \end{pmatrix} \bigg|_{k=(k_1,k_2)}$$

We can now state and prove the following theorems:

**Theorem 4.2.1.** Let $\mathcal{T}_{C_0}$ be a Hida family whose weight space map $w$ is flat at the weight $(3,3)$ prime $\mathcal{P}_{(3,3)}$. Assume $\mathcal{T}_{C_0}$ contains a weight $(3,3)$ point $\pi$ whose local component $\pi_p$ is maximal Steinberg (IV.a in [20]). Assume the hypothesis on $\rho_{\pi,p}$ for this weight $(3,3)$ point $\pi$ outlined in §3 so that the $p$-adic $L$-invariant $L_\pi$ is defined. If $v_p(L_\pi) < 1$, then the Hida family $\mathcal{T}_{C_0}$ must have at least one weight $(3,3)$ point which is not of Type IV.A at $p$. In particular, the degree of the Hida family $\mathcal{T}_{C_0}$ is at least 2 and the weight space map is not an isomorphism.

**Proof.** If the Hida family $\mathcal{T}_{C_0}$ only contains weight $(3,3)$ points $Q_i$ which are unram-
ified twists of Steinberg at $p$ (Type IV.A), we apply Lemma 4.1.1 to see

$$1 > v_p(\mathcal{L}_{\pi,p}) = v_p \left( \frac{d \log_p(a_{p,1})}{dk} \right) \geq 1 - v_p(a_{p,1}|_{\pi}) = 1 - 0 = 1$$

which is a contradiction. \qed

**Theorem 4.2.2.** Let $\mathbb{T}_{C_0}$ be a Hida family whose weight space map $w$ is flat at the weight $(3,3)$ prime $\mathfrak{p}_{(3,3)}$. Assume $\mathbb{T}_{C_0}$ contains a weight $(3,3)$ point $\pi$. Assume the hypotheses outlined in §3 so that the adjoint $\mathcal{L}$-invariant $\mathcal{L}_{\pi,AD}$ is defined. If $v_p(\mathcal{L}_{\pi,AD}) < 2$, then the degree of the Hida family $\mathbb{T}_{C_0}$ is at least 2 and the weight space map is not an isomorphism.

**Proof.** Using Rosso’s formula:

$$2 > v_p(\mathcal{L}_{\pi,Ad})$$

$$\geq \min \left\{ v_p \left( \frac{d \log_p(a_{p,1})}{dk_1} \frac{d \log_p(a_{p,2})}{dk_2} \right), v_p \left( \frac{d \log_p(a_{p,1})}{dk_2} \frac{d \log_p(a_{p,2})}{dk_1} \right) \right\}$$

$$\geq 2 - v_p(b(Frob_p)) = 2$$

a contradiction. \qed
CHAPTER 5
DEFORMATIONS OF GALOIS REPRESENTATIONS

In this chapter we will prove result about characteristic zero lifts of mod \( p \) symplectic Galois representations. The standard reference for the theory of deformations of Galois representations valued in \( \text{Gl}_n \) is by Mazur \cite{Mazur}. Patrikis \cite{Patrikis} & Booher \cite{Booher} have developed the theory of Galois deformations for symplectic groups which we take advantage of in this work.

Let \( \bar{\rho} : G_{\mathbb{Q}} \to \text{GSp}_4(\mathbb{F}_p) \) be absolutely irreducible. Let \( \Sigma \) be a finite set of places of \( \mathbb{Q} \) which includes \( p, \infty \), and all primes \( q \) at which \( \bar{\rho}|_q \) is ramified. Then, by definition of \( \Sigma \), the representation \( \bar{\rho} \) factors through the Galois group \( G_{\mathbb{Q}}^\Sigma \) of the field \( \mathbb{Q}^\Sigma \) which denotes the maximal \( \mathbb{Q} \) extension unramified outside at all places outside of \( \Sigma \):

\[
\bar{\rho} : G_{\mathbb{Q}}^\Sigma \to \text{GSp}_4(\mathbb{F}_p)
\]

The assumption that \( \bar{\rho} \) is absolutely irreducible guarantees the existence of a universal deformation ring equipped with a universal deformation, denoted \((\mathcal{R}_{\bar{\rho}}, \rho_{\bar{\rho}})\), pro-representing the functor \( D_{\bar{\rho}}(\_\_ \_ \_ \_ \_) \) of strict equivalence classes of \( \text{GSp}_4(R) \)-valued lifts of the \( G_{\mathbb{Q}}^\Sigma \)-representation \( \bar{\rho} \) where \( R \) is a complete local Noetherian ring with residue field \( \mathbb{F}_p \). Let \( t_{\mathcal{R}_{\bar{\rho}}} \) denote the Zariski tangent space of \( \mathcal{R}_{\bar{\rho}} \). Recall the following isomorphisms of \( \mathbb{F}_p \) vector spaces:

\[
t_{\mathcal{R}_{\bar{\rho}}} \cong D_{\bar{\rho}}(\mathbb{F}_p[\varepsilon]/(\varepsilon^2)) \cong H^1(\mathbb{Q}^\Sigma, \rho_{\text{gsp}_4})
\]

where \( \rho_{\text{gsp}_4} \) is the Adjoint representation of \( \bar{\rho} \) defined in chapter 2 of this work as
Ad(\bar{\rho}). In this work, we will be considering lifts of \bar{\rho} which have a fixed similitude character \nu = \epsilon^3 where \epsilon denotes the p-adic cyclotomic character. These are also representable and are given by cocyles in \rho_{\text{sp}_4}, defined as Ad^0(\bar{\rho}) in chapter 2 of this work.

The purpose of this chapter is to set up a global Galois deformation problem by choosing, for each place v of \mathbb{Q}, a subspace

$$\mathcal{L}_v \subset H^1(\mathbb{Q}_v, \text{Ad}^0(\bar{\rho}|_v))$$

which represents a smooth subfunctor of \textbf{D}_{\bar{\rho}|_v}(\_). The subspaces \mathcal{L} = \{\mathcal{L}_v\}_v are referred to as deformation conditions. For all \(v \notin \Sigma\) we choose \(\mathcal{L}_v = H^1_{\text{ur}}(\mathbb{Q}_v, \text{Ad}^0(\bar{\rho}))\). A suitable choice of \(\mathcal{L} = \{\mathcal{L}_v\}_v\) give rise to a global deformation problem (which is also representable by some pair \((R^\xi, \rho^\xi)\)) and has Zariski tangent space defined below.

$$H^1_{\mathcal{L}}(\text{Ad}^0(\bar{\rho})) = \ker(H^1(\mathbb{Q}^\Sigma, \text{Ad}^0(\bar{\rho})) \to \bigoplus_v (H^1(\mathbb{Q}_v, \text{Ad}^0(\bar{\rho}|_v))/\mathcal{L}_v))$$

5.1 Our Running Hypotheses on \(\bar{\rho}\) and its restrictions \(\bar{\rho}|_v\)

Fix a prime \(p > 5\) which is congruent to 2 mod 3. Let \(\bar{\rho} : G_{\mathbb{Q}} \to \text{GSp}_4(\mathbb{F}_p)\) be a fixed absolutely irreducible odd Galois representation which, when restricted to the decomposition group \(G_{\mathbb{Q}_p}\), is indecomposable and ordinary. In particular, we assume that \(\bar{\rho}|_p\) has the form

$$\bar{\rho}|_p := \bar{\rho}|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} \epsilon^3 & * & * \\ * & \epsilon^2 & * \\ * & * & \epsilon \end{pmatrix}$$
where $\bar{\epsilon}$ is the mod $p$ cyclotomic character.

To guarantee the smoothness of our global deformation ring $R^c_\rho$ it is enough, by Proposition 3.6 of [18], to assume the following hypothesis on the adjoint of our fixed Galois representation:

$$H^2_L(Q^\Sigma, \text{Ad}^0(\bar{\rho})) = 0$$

(5.1)

We remark that Proposition 3.6 of [18] gives us that the global deformation ring is a power series ring, and hence smooth.

For our local deformations at $p$, we prove the regularity assumptions in section 4 of Patrikis [18] (this is proved in Lemma 5.2.5) which will guarantee that our local deformation ring at $p$ is formally smooth by his Proposition 4.4 in the same work. These assumptions are, following our notation from 2.2.1 from this thesis:

$$\text{(REG)} \quad H^0(Q_p, \rho_{sp_4/b}) = 0$$

(5.2)

$$\text{(REG*)} \quad H^0(Q_p, \rho^*_{sp_4/b}(1)) = 0$$

(5.3)

We remark that the condition (REG) guarantees (by Lemma 4.2 and 4.3 of [18]) that the ordinary lifting condition is well defined and has the correct tangent space. The condition (REG*) guarantees that the local universal deformation at $p$ is a power series ring and hence formally smooth.

**Remark 5.1.1.** *The following equations shall be used throughout the computations*
to simplify convoluted formulas. Let:

\[ \tilde{\rho}\big|_p = \begin{pmatrix}
\bar{\epsilon}^3 & c_1 & c_2 & c_3 \\
\bar{\epsilon}^2 & c_4 & c_5 \\
\bar{\epsilon} & c_6 \\
1
\end{pmatrix} \]

That \( \tilde{\rho} \) preserves the symplectic form (up to similitude), tells us:

\[ ^t \tilde{\rho}(g) \circ J \circ \tilde{\rho}(g) = \nu(g)J \]

which, when applied to a general \( g \) and \( g^{-1} \) gives us the equations:

\[ \nu = \bar{\epsilon}^3 \quad (5.4) \]
\[ c_1 = -\bar{\epsilon}^2 c_6 \quad (5.5) \]
\[ c_4 c_6 + c_2 = \bar{\epsilon} c_5 \quad (5.6) \]

5.1.1 The adjoint representation of our \( \tilde{\rho} \)

For the proof of the lemmas in §5.2 we will make frequent use of the explicit adjoint action \( \text{Ad}^0(\tilde{\rho})\big|_p \) on \( \mathfrak{sp}_4(\mathbb{F}_p) \) with respect to a fixed basis which we fix in this section. This allows us to compute \( H^0 \) and \( H^2 \) (by Tate duality) of many subrepresentations and quotient representations.

We fix the ordered basis of \( \mathfrak{sp}_4 \) (from §2) once and for all:

\[ \mathfrak{sp}_4(\mathbb{F}_p) = \text{span}_{\mathbb{F}_p} < s_3, s_2, s_4, s_1, h_1, h_2, r_1, r_4, r_2, r_3 > \]

The adjoint action of the restriction to \( G_{\mathbb{Q}_p} \) on \( \mathfrak{gsp}_4(\mathbb{F}_p) \) with respect to this ordered basis is:
Ad(¯\(\rho\))|_p(g) =

\[
\begin{pmatrix}
\bar{\epsilon}^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_6 & \bar{\epsilon}^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_5 c_6 & \frac{2c_5 c_6}{\bar{\epsilon}^2} & \bar{\epsilon}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{c_5 c_6}{\bar{\epsilon}^3} & \frac{c_3}{\bar{\epsilon}^3} & 0 & \bar{\epsilon}^{-1} & 0 & 0 & 0 & 0 & 0 \\
\frac{c_3}{\bar{\epsilon}^3} & \frac{c_2}{\bar{\epsilon}^3} & 0 & -\frac{c_6}{\bar{\epsilon}} & 1 & 0 & 0 & 0 & 0 \\
c_5 c_6 & \frac{c_5 c_6 + c_4 c_5}{\bar{\epsilon}^3} & \frac{c_4}{\bar{\epsilon}^2} & \frac{c_6}{\bar{\epsilon}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{c_5}{\bar{\epsilon}^3} & -\frac{2c_4 c_5}{\bar{\epsilon}^3} & -\frac{c_2}{\bar{\epsilon}^2} & -\frac{2c_3}{\bar{\epsilon}^2} & 0 & -\frac{2c_3}{\bar{\epsilon}} & 0 & 0 & 0 \\
-\frac{c_5 c_3}{\bar{\epsilon}^3} - \frac{c_5 c_3 + c_2 c_5}{\bar{\epsilon}^3} - \frac{c_2 c_4}{\bar{\epsilon}^3} - \frac{c_3 c_5 - c_3}{\bar{\epsilon}^3} - c_5 \frac{c_4 c_6 - c_2}{\bar{\epsilon}} & -c_4 \frac{c_4 c_6 - c_2}{\bar{\epsilon}} & -c_4 & -c_6 \bar{\epsilon} & \bar{\epsilon}^2 & 0 \\
-\frac{c_5}{\bar{\epsilon}^3} - \frac{2c_2 c_5}{\bar{\epsilon}^3} - \frac{c_2}{\bar{\epsilon}^2} & -\frac{c_2}{\bar{\epsilon}^2} - 2c_3 & \frac{2c_2 c_6}{\bar{\epsilon}} & -2c_2 & \frac{c_2 c_6}{\bar{\epsilon}} & -c_6 \bar{\epsilon}^2 & \bar{\epsilon}^3
\end{pmatrix}
\]

By writing this in the right-multiplication representation, we can compute the Tate duals of our sub and quotient representations using the following formula:

\[
Ad(\bar{\rho})^*(1)(g) = Ad(\bar{\rho})(g^{-1})^t \cdot (\bar{\epsilon}(g) Id_{10})
\]

where \(Id_{10}\) is the \(10 \times 10\) identity matrix.

5.2 Tangent Spaces of Deformation Rings

In this section we define and compute the dimensions of our various local deformation conditions \(\mathcal{L}_v\). First, we recall the basic theory of local deformations to the dual numbers \(\mathbb{F}_p[\bar{\epsilon}]\) (where \(\bar{\epsilon}^2 = 0\)) to set up notation for our calculations.

5.2.1 Local Deformations and our Deformation Conditions

Let \(\bar{\rho}: G_{Q_v} \rightarrow GSp_4(\mathbb{F}_p)\) be a Galois representation for a non-archimedean place \(v\). Let \(M\) be a free rank 4 module over \(\mathbb{F}_p[\bar{\epsilon}] := \mathbb{F}_p[\bar{\epsilon}]/(\bar{\epsilon}^2)\). \(M\) is then a rank 8 vector
space over $\mathbb{F}_p$, equipped with an endomorphism $\varepsilon : M \to M$ whose square is the zero morphism ($\varepsilon^2 = 0$) giving the exact sequence:

$$0 \to \varepsilon M \to M \to M/\varepsilon M \to 0$$

Given a $G_{Q_v}$-action on $M$ by $\rho : G_{Q_v} \to \text{Aut}_{\mathbb{F}_p[\varepsilon]}(M)$, we say that $(M, \rho)$ is a lift of $\bar{\rho}$ if the above exact sequence of $\mathbb{F}_p[\varepsilon]$ modules are also $G_{Q_v}$-equivariant morphisms, such that the subrepresentation $\rho|_{\varepsilon M} \cong \bar{\rho}$ and the quotient $\rho/\rho|_{\varepsilon M} \cong \bar{\rho}$. This means that $M$ sits in a $G_{Q_v}$-exact sequence:

$$0 \to \bar{\rho} \to M \to \bar{\rho} \to 0$$

We can understand lifts of $\bar{\rho}$ to $M$ by viewing them with respect to a $\mathbb{F}_p[\varepsilon]$ basis

$$\rho : G_{Q_v} \to \text{Gl}_4(\mathbb{F}_p[\varepsilon])$$

or with respect to a $\mathbb{F}_p$ basis

$$\rho : G_{Q_v} \to \text{Gl}_8(\mathbb{F}_p)$$

and our calculations will make use of switching back and forth between these two different bases.

Choose a four dimensional $\mathbb{F}_p$-basis for the sub object $\varepsilon M \cong \bar{\rho}$ and extending it to an $\mathbb{F}_p$ basis of $M$ by lifting an $\mathbb{F}_p$ basis of the quotient object $M/\varepsilon M \cong \bar{\rho}$ (from the above exact sequence) so that we may now view the lift $(\rho, M)$ as an 8 dimensional representation.

$$\rho : G_{Q_v} \to \text{Gl}_8(\mathbb{F}_p)$$
where $\lambda$ is a function $\lambda : G_{Q_v} \to \mathfrak{gl}_4(F_p)$. The definition of $\rho$ being a group homomorphism is equivalent to factoring this function $\lambda$ as a product of two functions

$$\lambda(g) = \gamma(g)\bar{\rho}(g)$$

where $\gamma : G_{Q_v} \to \mathfrak{gl}_4(F_p)$ is a cocycle in $H^1(Q_v, \mathfrak{gl}_4(F_p))$ and $\bar{\rho}$ is our fixed representation. Thus, a lift $\rho$ is determined by a cocycle $\gamma$ and vice-versa.

Let $\rho^\gamma$ to denote the lift defined by the choice of cocycle $\gamma$. We will want to study lifts $\rho^\gamma$ which land in the symplectic group,

$$\rho^\gamma : G_{Q_v} \to \text{GSp}_4(F_p[\epsilon])$$

which means the cocycle $\gamma$ has its image valued in the Lie algebra $\mathfrak{sp}_4(F_p)$ and that $\gamma$ is a cocycle in $H^1(Q_v, \text{Ad}(\bar{\rho}))$. We will also only consider lifts to $\text{GSp}_4$ which have the fixed similitude character $\nu = \epsilon^3$, where $\epsilon$ is the $p$-adic cyclotomic character. This condition is a representable deformation condition by Corollary 4.5 of [IS] and, at the level of matrices, determines that cocycle $\gamma$ has its image in Lie algebra $\mathfrak{sp}_4(F_p)$ and thus is a cohomology class in $H^1(Q_p, \text{Ad}^0(\bar{\rho}))$. Such a lift has matrix coordinates:
\[ \rho^\gamma(g) = \begin{pmatrix} \bar{\rho}\big|_v(g) \\ h_1 & r_1 & r_2 & r_3 \\ 0 & 0 & 0 & 0 \\ s_1 & h_2 & r_4 & r_2 \\ 0 & 0 & 0 & 0 \\ s_2 & s_4 & -h_2 & -r_1 \\ 0 & 0 & 0 & 0 \\ s_3 & s_2 & -s_1 & -h_1 \\ 0 & 0 & 0 & 0 \
\end{pmatrix} \]

In §5.2.4 we are going to restrict ourselves to lifts \( \rho^\gamma \) which are ordinary and semistable. This means that the cohomology class \( \gamma \) lies in a linear subspace \( \mathcal{L}_{p,3}^* \subset H^1(\mathbb{Q}_v, \text{Ad}^0(\bar{\rho})) \) where \(* \in \{\text{ord, st, crys}\} \) which is the Zariski tangent space of the functor representing all lifts \( \bar{\rho} \) with each of these properties.

If \( \rho^\gamma \) is ordinary, meaning \( \gamma \in \mathcal{L}_{p,3}^{\text{ord}} \) then, at the level of matrices:

\[ \rho^\gamma(g) = \begin{pmatrix} \bar{\rho}\big|_v(g) \\ h_1 & r_1 & r_2 & r_3 \\ 0 & h_2 & r_4 & r_2 \\ 0 & 0 & -h_2 & -r_1 \\ 0 & 0 & 0 & -h_1 \\ 0 & 0 & 0 & 0 \
\end{pmatrix} \]

where the \( h_1, h_2 : G_{\mathbb{Q}_v} \to \mathbb{F}_p \) are unramified group homomorphisms.

If \( \rho^\gamma \) is semistable, meaning \( \gamma \in \mathcal{L}_{p,3}^{\text{st}} \) then, at the level of matrices:
\( \rho^\gamma(g) = \begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ \rho|_v(g) & 0 & 0 & r_4 \\ 0 & 0 & 0 & -r_2 \\ \rho|_v(g) & 0 & 0 & 0 \end{pmatrix} \)

5.2.2 \( H^1(\mathbb{R}, \text{Ad}^0(\bar{\rho})) \)

Let \( \bar{c} \in G_\mathbb{R} \) denote complex conjugation. Our hypothesis that \( \bar{\rho} \) be odd means, up to a choice of basis:

\[ \bar{\rho}|_{G_\mathbb{R}}(c) = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \]

The following proposition is Theorem 7.3.5 in [17].

**Proposition 5.2.1.** Let \( M \) be a finite \( G_\mathbb{R} \) module.

\[ \# M = \frac{\# H^0(\mathbb{R}, M) \cdot \# H^0(\mathbb{R}, \text{Hom}_{G_\mathbb{R}}(M, \mathbb{C}^\times))}{\# H^1(\mathbb{R}, M)} \]

where \( G_\mathbb{R} \) acts on \( \mathbb{C}^\times \) via conjugation.

**Proposition 5.2.2.**

\[ \dim H^0(\mathbb{R}, \text{Ad}^0(\bar{\rho})) = 4 \]

\[ \dim H^1(\mathbb{R}, \text{Ad}^0(\bar{\rho})) = 0 \]
Proof. There are only two possible representations of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$: either $c$ acts trivially or as $-1$. Applying $\text{Hom}(-, \mathbb{C}^\times)$ switches between the two of them. Direct calculation yields $H^0(\mathbb{R}, \text{Ad}^0(\bar{\rho}))$ is four dimensional and $H^0(\mathbb{R}, \text{Hom(Ad}^0(\bar{\rho}), \mathbb{C}^\times))$ is six dimensional. Applying the previous proposition to the result follows. \hfill \Box

5.2.3 $H^1(Q_p, \text{Ad}^0(\bar{\rho}))$

In this section we define and prove several key facts about certain subspaces of $H^1(Q_p, \text{Ad}^0(\bar{\rho}))$ classifying deformations with specific properties. For our computations we will use the local Euler-Poincaré characteristic formula (Theorem 7.3.1 in [17]):

**Proposition 5.2.3.** Let $V$ be a finite dimensional $\mathbb{F}_p$ vector space which is a $G_{Q_p}$ module. Then:

$$-\dim(V) = \sum_{i=0}^{2} (-1)^i \dim(H^i(Q_p, V))$$

First, we gather some lemmas on the cohomology of the terms in the exact sequence

$$0 \to \rho_b \to \text{Ad}^0(\bar{\rho}) \to \rho_{sp_4/b} \to 0$$

**Lemma 5.2.4.** Let $c_6$ and $c_4\bar{\epsilon}^{-1}$ be nontrivial cocyles in $H^1(Q_p, \mathbb{F}_p(\bar{\epsilon}))$.

$$\dim(H^0(Q_p, \rho_b)) = 0$$

$$\dim(H^1(Q_p, \rho_b)) = 8$$

$$\dim(H^2(Q_p, \rho_b)) = 2$$

**Proof.** Direct computation shows that for $H^0$ to vanish, it is equivalent to finding non-trivial solutions to the equations

$$c_6(h_2 - h_1) = (\bar{\epsilon} - 1)r_1$$
\[ 2c_4 h_2 = (\bar{c}^2 - \bar{c}) r_4 \]

There are no solutions by the assumption that \( c_b \) and \( c_4 \bar{c}^{-1} \) were non-trivial cocycles, and so \( H^0 \) vanishes.

Direct computation of the second cohomology group (using Tate Duality) shows that the dimension of \( H^2 \) is at least 2. To make it exactly equal to two requires the same equations as above (in the case of \( H^0 \)) and so is exactly 2 by our assumptions. By the Euler characteristic formula, the dimension of \( H^1(\mathbb{Q}_p, \rho_b) \) must then be 8.

**Lemma 5.2.5.**

\[
\begin{align*}
\dim(H^0(\mathbb{Q}_p, \rho_{sp_4/b})) &= 0 \\
\dim(H^1(\mathbb{Q}_p, \rho_{sp_4/b})) &= 4 \\
\dim(H^2(\mathbb{Q}_p, \rho_{sp_4/b})) &= 0
\end{align*}
\]

*Proof.* The adjoint action on the four dimensional space \( sp_4/b \) with respect to the ordered basis of the residue classes of \( s_3, s_2, s_4, s_1 \) is given by the following matrix.

\[
\begin{pmatrix}
\bar{c}^{-3} & 0 & 0 & 0 \\
\bar{c}^3 & \bar{c}^{-2} & 0 & 0 \\
\bar{c}^2 & 2c_6 & \bar{c}^{-1} & 0 \\
\bar{c} & \bar{c}^3 & \bar{c}^4 & \bar{c}^{-1}
\end{pmatrix}
\]

Solving for vectors fixed by the Galois action of \( \rho_{sp_4/b} \) tells us \( s_3 = 0 \), which then implies the rest are zero. So, by direct computation, \( H^0 \) vanishes.

Computing the dimension of \( H^2 \) by local Tate duality, we compute the action of \( \rho_{sp_4/b}^*(1) \) is the following matrix.
Solving for fixed vectors, we see the only fixed vector is the zero vector, and so $H^2$ vanishes. By the Euler characteristic formula, $H^1$ is 4 dimensional.

Now we collect information about the Galois cohomology of the terms in the exact sequence:

$$0 \to \rho \to \text{Ad}^0(\bar{\rho}) \to \rho_{sp_4} \to 0$$

**Lemma 5.2.6.**

$$\dim(H^0(Q_p, \rho_t)) = 0$$

$$\dim(H^1(Q_p, \rho_t)) = 6$$

$$\dim(H^2(Q_p, \rho_t)) = 2$$

**Proof.** The adjoint action on $\tau$ is given by the matrix:

$$\begin{pmatrix}
\bar{\epsilon} & 0 & 0 & 0 \\
0 & \bar{\epsilon} & 0 & 0 \\
-c_4 & -c_6 \bar{\epsilon} & \bar{\epsilon}^2 & 0 \\
-2c_2 & c_6 \bar{\epsilon} & -c_6 \epsilon^2 & \epsilon^3
\end{pmatrix}$$

which has no fixed vectors, so $H^0$ vanishes. To compute $H^2$ we use Tate duality to compute $H^0$ of the Tate twist of the dual, which has matrix:
Solving for fixed vectors, the last two basis elements must be zero, giving us the identity matrix in the top left corner which gives us 2 dimensional fixed vectors, hence $H^2$ is two dimensional.

\[ \begin{pmatrix} 1 & 0 & \frac{c_4}{\epsilon^2} & \frac{c_5\epsilon + c_3}{\epsilon^3} \\ 0 & 1 & \frac{c_6}{\epsilon} & 0 \\ 0 & 0 & \bar{\epsilon}^{-1} & \frac{c_6}{\epsilon^2} \\ 0 & 0 & 0 & \bar{\epsilon}^{-2} \end{pmatrix} \]

Lemma 5.2.7.

\[
\dim(H^0(\mathbb{Q}_p, \rho_{\text{sp}_4/\mathfrak{r}})) = 2 \\
\dim(H^1(\mathbb{Q}_p, \rho_{\text{sp}_4/\mathfrak{r}})) = 8 \\
\dim(H^2(\mathbb{Q}_p, \rho_{\text{sp}_4/\mathfrak{r}})) = 0
\]

Proof. The Galois action on $\text{sp}_4/\mathfrak{r}$ is given by the $6 \times 6$ matrix:

\[
\begin{pmatrix}
\bar{\epsilon}^{-3} & 0 & 0 & 0 & 0 & 0 \\
\frac{c_6}{\epsilon^3} & \bar{\epsilon}^{-2} & 0 & 0 & 0 & 0 \\
\frac{c_2}{\epsilon^2} & \frac{2c_6}{\epsilon^2} & \bar{\epsilon}^{-1} & 0 & 0 & 0 \\
\frac{c_5}{\epsilon^3} & \frac{c_6}{\epsilon^2} & 0 & \bar{\epsilon}^{-1} & 0 & 0 \\
\frac{c_3}{\epsilon^3} & \frac{c_6}{\epsilon^2} & 0 & -\frac{c_5}{\epsilon} & 1 & 0 \\
\frac{c_5c_6}{\epsilon^3} & \frac{c_4c_6}{\epsilon^2} & \frac{c_4}{\epsilon^2} & \frac{c_6}{\epsilon} & 0 & 1
\end{pmatrix}
\]

Calculating fixed vectors makes the first 4 basis elements zero, leaving us with the bottom right $2 \times 2$ identity matrix, which has 2 dimensional fixed vectors. So $H^0$ is two dimensional.

To compute $H^2$ we use Tate duality to compute $H^0$ of the Tate twist of the dual, which has the matrix:
which has no fixed vectors, and hence $H^2$ vanishes. The dimension of $H^1$ follows by Euler characteristic formula.

**Proposition 5.2.8.**

\[
\dim(H^0(Q_p, \text{Ad}^0(\bar{\rho}))) = 0 \\
\dim(H^1(Q_p, \text{Ad}^0(\bar{\rho}))) = 12 \\
\dim(H^1(Q_p, \text{Ad}^0(\bar{\rho}))) = 2
\]

**Proof.** Using the short exact sequence:

\[
b \to \mathfrak{sp}_4 \to \mathfrak{sp}/b
\]

the long exact sequence of cohomology groups associated to this short exact sequence gives us:

\[
0 \to H^0(Q_p, \rho) \to H^0(Q_p, \text{Ad}^0(\bar{\rho})) \to H^0(Q_p, \rho_{\mathfrak{sp}/b}) \to \\
\to H^1(Q_p, \rho) \to H^1(Q_p, \text{Ad}^0(\bar{\rho})) \to H^1(Q_p, \rho_{\mathfrak{sp}/b}) \to \\
\to H^2(Q_p, \rho) \to H^2(Q_p, \text{Ad}^0(\bar{\rho})) \to H^2(Q_p, \rho_{\mathfrak{sp}/b}) \to 0
\]
which, combining lemmas 5.2.6 and 5.2.7, is the exact sequence

$$0 \to 0 \to H^0(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \to 0 \to$$

$$\mathbb{F}_p^8 \to H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \to \mathbb{F}_p^6 \to$$

$$\mathbb{F}_p^2 \to H^2(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \to 0 \to 0$$

which tells us $H^0$ vanishes and that the dimension of our $H^2$ is bounded above by 2. We now argue that this upper bound is an equality. Direct computation of the Tate dual of the adjoint representation given in §5.1.1 shows that the basis vectors $h_1$ and $h_2$ are invariant under the $\text{Ad}^0(\bar{\rho})^*(1)$ action and so the dimension of $H^2$ is at least two. Thus it is equal to two and the dimension of $H^1$ is 12 by the local Euler-Poincaré characteristic formula.

\[\square\]

5.2.4 Local at $p$ deformation conditions

In this section we will define and compute the dimensions of the three different subspaces

$$\mathcal{L}_{p,3}^* \subset H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho}))$$

where $* \in \{\text{ord, st, crys}\}$ for our local deformations at $p$.

**Definition 5.2.1.** Define $\mathcal{L}_{p,3}^*$ for $* \in \{\text{st, ord}\}$ as follows:

$$\mathcal{L}_{p,3}^{\text{ord}} = \ker(H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \to H^1(I_{\mathbb{Q}_p}, \text{Ad}^0(\bar{\rho})/\mathfrak{r}))$$

$$\mathcal{L}_{p,3}^{\text{st}} = \ker(H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \to H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})/\mathfrak{r}))$$

For $* = \text{crys}$ define $\mathcal{L}_{p,3}^{\text{crys}}$ as in Theorem 5.2 [4]
Define \( L_{p,3}^b \) as

\[
L_{p,3}^b = \text{Im}(H^1(\mathbb{Q}_p, \rho_b) \rightarrow H^1(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})))
\]

**Remark 5.2.9.** Note that because \( H^0(\mathbb{Q}_p, \rho_{sp,4/b}) = 0 \), the map defining \( L_{p,3}^b \) is injective and thus the classes \( \gamma \in L_{p,3}^b \) correspond to deformations \( \rho \) whose cocyle lands in \( b \), and therefore are upper-triangular in an \( \mathbb{F}_{p}[\varepsilon] \) basis. Since the map is injective, the dimension of \( L_{p,3}^b \) is the dimension of \( H^1(\mathbb{Q}_p, \rho_b) \), which is 8 by Lemma 5.2.4.

**Proposition 5.2.10.**

\[
dim(L_{p,3}^{\text{ord}}) \leq 8
\]

\[
dim(L_{p,3}^{\text{st}}) = 4
\]

\[
dim(L_{p,3}^{\text{crys}}) = 4
\]

**Proof.** The dimension of \( L_{p,3}^{\text{crys}} \) is \( \frac{2^2}{4} = 4 \), as was calculated in Stefan Patrikis’ bachelors Thesis (Proposition 5.1) and were reproved in greater generality in Booher’s work (Theorem 5.2 of [4]).

First, we argue:

\[
L_{p,3}^{\text{ord}} \subset L_{p,3}^b
\]

where the right hand side has dimension 8. First, note that ordinary deformations are upper-triangular and unramified along the diagonal. For the inclusion, we use the short exact sequence:

\[
b \rightarrow \text{sp}_4 \rightarrow \text{sp}_4/b
\]

whose long exact sequence of cohomology groups contains the sequence

\[
0 \rightarrow H^0(\mathbb{Q}_p, \rho_b) \rightarrow H^0(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho})) \rightarrow H^0(\mathbb{Q}_p, \rho_{sp,4/b}) \rightarrow
\]
→ $H^1(Q_p, \rho_b) \rightarrow H^1(Q_p, \operatorname{Ad}^0(\bar{\rho})) \rightarrow H^1(Q_p, \rho_{sp4/b}) \rightarrow$

Recall that the unramified cohomology groups $H^1_{ur}$ are defined as the image of the inflation map induced by $I_{Q_p} \rightarrow G_{Q_p} \rightarrow G_{F_p}$. We have, for any finite $G_{Q_p}$ module $M$,

$$\dim(H^1_{ur}(Q_p, M)) = \dim(H^0(Q_p, M))$$

If we fix a class $\gamma \in \mathcal{L}^{ord}_{p,3}$, then it’s image in $H^1(Q_p, \rho_{sp4/b})$ must lie in $H^1_{ur}(Q_p, \rho_{sp4/b})$ whose dimension is equal to $\dim H^0(Q_p, \rho_{sp4/b}) = 0$ by Lemma 5.2.5. Such a $\gamma$ is then an element of $\mathcal{L}^b_{p,3}$ by definition so the inclusion follows. Thus the dimension of $\mathcal{L}^{ord}_{p,3}$ is bounded by $\dim \mathcal{L}^b_{p,3} = 8$ (Remark 5.2.9).

For the dimension of the semistable deformations, $\mathcal{L}^{st}_{p,3}$, consider the short exact sequence of $G_{Q_p}$ representations:

$$0 \rightarrow \rho_t \rightarrow \operatorname{Ad}^0(\bar{\rho}) \rightarrow \rho_{sp4/t} \rightarrow 0$$

giving the long exact sequence in cohomology:

$$0 \rightarrow H^0(Q_p, \rho_t) \rightarrow H^0(Q_p, \operatorname{Ad}^0(\bar{\rho})) \rightarrow H^0(Q_p, \rho_{sp4/t}) \rightarrow$$

$$\rightarrow H^1(Q_p, \rho_t) \rightarrow H^1(Q_p, \operatorname{Ad}^0(\bar{\rho})) \rightarrow H^1(Q_p, \rho_{sp4/t}) \rightarrow$$

$$\rightarrow H^2(Q_p, \rho_t) \rightarrow H^2(Q_p, \operatorname{Ad}^0(\bar{\rho})) \rightarrow H^2(Q_p, \rho_{sp4/t}) \rightarrow 0$$

By exactness of this diagram, $\mathcal{L}^{st}_{p,3}$ defined as a kernel of a map is equal to the image of the map preceeding it:

$$\mathcal{L}^{st}_{p,3} \cong \operatorname{im}(H^1(Q_p, \rho_t) \rightarrow H^1(Q_p, \operatorname{Ad}^0(\bar{\rho})))$$

This map is from a 6 dimensional space (Lemma 5.2.8) with a 2-dimensional kernel.
(Lemma 5.2.9) which tells us \( \dim(\mathcal{L}_{p,3}^{st}) = 4 \)

**Proposition 5.2.11.**

\[
\mathcal{L}_{p,3}^{st} \subset \mathcal{L}_{p,3}^{ord}
\]

**Proof.** The inclusion \( \mathcal{L}_{p,3}^{st} \subset \mathcal{L}_{p,3}^{ord} \) is by definition.

5.3 Selmer Systems

Let \( \Sigma \) be a finite set of primes including \( p \) and \( \infty \). Given a \( G_{\mathbb{Q}}^\Sigma \) module, \( X \), a Selmer system \( \mathcal{L} = \{ \mathcal{L}_v \}_v \) is a collection of subspaces

\[
\mathcal{L}_v \subset H^1(\mathbb{Q}_v, X)
\]

such that, for almost all \( v \notin \Sigma \) we have \( \mathcal{L}_v = H^1_{ur}(\mathbb{Q}_v, N) \).

Given a Selmer system \( \mathcal{L} \) for \( X \), define the dual Selmer system \( \mathcal{L}^\perp = \{ \mathcal{L}^\perp_v \} \) such that

\[
\mathcal{L}^\perp_v \subset H^1(\mathbb{Q}_v, X^* (1))
\]

is the annihilator of \( \mathcal{L}_v \) under the Tate pairing.

Define the Selmer group and dual Selmer group, respectively, as follows:

\[
H^1_\mathcal{L}(X) = \ker(H^1(G_{\mathbb{Q}}^\Sigma, X) \rightarrow \bigoplus_v H^1(\mathbb{Q}_v, X)/\mathcal{L}_v)
\]

\[
H^1_{\mathcal{L}^\perp}(X^* (1)) = \ker(H^1(G_{\mathbb{Q}}^\Sigma, X^* (1)) \rightarrow \bigoplus_v H^1(\mathbb{Q}_v, X^* (1))/\mathcal{L}^\perp_v)
\]

where the maps given above are the restriction maps composed with the quotient maps.

While Selmer groups are difficult to compute in general, using the Poitou-Tate
exact sequence (see: 8.6.13 of [17]), one can use the following Proposition 1.6 of Wiles [23] to gain some information.

**Proposition 5.3.1.** (Wiles) Let $X$ be a $G_{Q,S}$ module of $p$-power order. Then

$$\frac{\# H^1_L(X)}{\# H^1_L(X^*(1))} = \frac{\# H^0(G^\Sigma_{Q,S}, X)}{\# H^0(G^\Sigma_{Q,S}, X^*(1))} \prod_v \frac{\# L_v}{\# H^0(Q_v, X)}$$

We remark that the places $v \notin \Sigma$ are chosen as $L_v = H^1_{ur}(Q_v, X)$ which has the same dimension as $H^0(Q_v, X)$, making the product over all places in Wiles’ proposition above a finite product.

5.3.1 Selmer Calculations

We now consider a specific Selmer system:

**Definition 5.3.1.** Define a selmer system $L = \{L_v\}_v$ for the module $\text{Ad}^0(\bar{\rho})$ where

- $v = \infty \quad L_v = \{0\}$
- $v = p \quad L_p = L_{p,3}^{ord}$
- $v \neq p, \infty \quad L_p = H^1_{ur}(Q_v, \text{Ad}^0(\bar{\rho}))$

**Theorem 5.3.2.** Let $p$ be a prime. Let $\bar{\rho} : G_Q \to \text{GSp}_4(\mathbb{F}_p)$ be absolutely irreducible, odd, unramified at all primes $l \neq p$, and at $p$ assume $\bar{\rho}|_p$ is ordinary and indecomposable. Assume $H^2_L(Q^\Sigma, \text{Ad}^0(\bar{\rho})) = 0$ so the global deformation ring is formally smooth. Assume that there exists a non-trivial ordinary extension of $\bar{\rho}|_p$ which is not semistable ($L_{p,3}^{ord} \supset L_{p,3}^{st}$). Then there exists more than one $p$-ordinary lift of $\bar{\rho}$ to $\hat{\mathbb{Z}}_p$.

**Proof.** Using the Selmer system $L$ defined above and taking the log$_p$ of both sides of Wiles’ formula tells us:
\[
\dim(H^1_L(\text{Ad}^0(\bar{\rho})) - \dim(H^1_{L^\perp}(\text{Ad}^0(\bar{\rho})^*(1)))
\]
\[
= \dim(H^0(\mathbb{Q}^\Sigma/\mathbb{Q}, \text{Ad}^0(\bar{\rho}))) - \dim(H^0(\mathbb{Q}^\Sigma/\mathbb{Q}, \text{Ad}^0(\bar{\rho})^*(1))) + \mathcal{L}_\infty - \dim(H^0(\mathbb{R}, \text{Ad}^0(\bar{\rho})))
\]
\[
+ \mathcal{L}_{p,3}^\text{ord} - H^0(\mathbb{Q}_p, \text{Ad}^0(\bar{\rho}))
\]
\[
= 0 - 0 + 0 - 4 + \dim \mathcal{L}_{p,3}^\text{ord} - 0
\]
\[
\geq 1
\]

by the assumption that there is a non-semistable ordinary class and our calculation that the semistable classes have dimension 4. If there was no other lift, this number would have to be 0 since our hypothesis that \( H^2_L(\mathbb{Q}^\Sigma, \text{Ad}^0(\bar{\rho})) = 0 \) guarantees smoothness of the global deformation ring and we verified hypothesis (\( \text{REG} \)) and (\( \text{REG}^* \)) for the smoothness of the local deformation ring at \( p \) by Lemma 5.2.5 of this dissertation. \( \square \)


