CONTROL OF DISTRIBUTED SYSTEMS WITH INTERCONNECTION CONSTRAINTS

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Abstract
by
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As components shrink in size and cost, analysis and synthesis of distributed coordination and control algorithms for networked control systems with communication constraints has become an active area of control theory research. From quantization systems for networked control systems (NCS) to fundamental limitations of control systems under information constraints, and from consensus problems to formation control and sensing and coverage problems, researchers have been interested in distributed control algorithms that achieve global objectives with minimum communication requirements.

The main objective of this dissertation is to report novel results and techniques for controlling interconnected systems under communication constraints. We study the dynamics and classical control approaches (such as LQR, $H_\infty$) in complex interconnected systems under various communication and interconnection constraints. Of particular interest are theoretical questions regarding the influence of a complex system’s structure and limited communication on its stability and robustness, and the design of distributed algorithms and protocols to access and modify the global behavior of the system.

The first problem considered deals in this dissertation with the classical linear systems with bit-rate constraints in the feedback channel. Based on the bit-rate
constraints, an optimization problem is formulated to maximize the convergence rate of quantization error for these systems. The quantization error goes to zero if and only if the bit-rate is above some threshold determined by the unstable modes of the plants. This known result is shown here via an alternative proof which leads to the minimum bit-rate required to stabilize linear systems. We further show that a dynamic quantization algorithm will approach the analytic results of our optimization problem. Furthermore, we consider the performance of standard LQR problems under bit-rate constraint. Our results show that the performance of standard LQR problems under bit-rate constrain consists of two parts: the perfect information part and the cost introduced by stepwise quantization error. In this way, we can quantify the penalty of limited communication on LQR problems.

In the remainder of this dissertation, several novel results toward a distributed control theory for spatially invariant and distributed systems are introduced. For the spatially invariant systems, a multidimensional KYP-like lemma exists, such that, the $H_{\infty}$ performance can be characterized by a set of LMIs. These results are later utilized in distributed controller synthesis.

For distributed systems, a distributed stability condition to guarantee global quadratic performance under various communication constraints is presented. The idea is based on a theory of separation of graphs from an operator’s point of view, where the plant and the interconnections as modelled as two interconnected operators in proper signal space. The essential tools we are using here are two versions of S-procedures which are similar to Lagrange relaxation techniques, initially developed by researchers in the former Soviet Union. It is later shown that these distributed stability conditions can be utilized in distributed controller synthesis in a manner similar to those gain-scheduling controllers in Linear Parameter
Varying (LPV) systems. The synthesis is based on the elimination lemma, and convex conditions are derived for the existence of distributed controllers.

Comparisons of distributed controllers, centralized and decentralized controllers are also briefly discussed in the introduction and summary.
To my parents,

Fang Wenyuan and Yan Guizhen.
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PREFACE

Completing a PhD is truly a marathon event, and I would not have been able to complete this journey without the aid and support of countless people over the past six years.

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CHAPTER 1

INTRODUCTION

Over the past few years, there has been a rapidly growing interest in the system and control community in the study of coordination, communication and distributed control for networked dynamic systems of many components acting locally and interacting with each other under various interconnection constraints. In fact, most complex machinery can be described more concisely as an interconnected system than as aggregate whole. At this time in history, interconnected systems are commanding unprecedented consideration because advances in Micro-Electra-Mechanical System (MEMS) make feasible the idea of microscopic control with actuating, sensing, computing and communication capacities.

Analysis and synthesis of distributed coordination and control algorithms for networked control systems with communication constraints has become an active area of control theory research. From quantized networked control systems to limitations of control system under communication constraints, from consensus problem to formation control and sensing and coverage problem, researchers have been interested in control algorithms that are distributed in nature and locally implemented with minimum communication requirement to achieve global objectives.

Important questions that arise are the following:
• (Q1) What is the communication requirement (such as channel capacity/bit-rate) needed for control of distributed systems?

• (Q2) How to design controllers for these physically distributed systems with regard to global objectives under interconnection limitations?

• (Q3) How can classical control algorithms be implemented in a distributed way?

Before any attempts to answer the above questions, we need first understand the concept of “information” in control theory, which is not the general concept used in communication society. Historically, communication and control theory have traditionally been areas with little common ground. For the most part, communication theory has been concerned with reliable transmission of information, with no interest in its specific purpose. Control theory, in contrast, has been concerned primarily with using information, normally provided via a feedback loop, to achieve some performance objectives, with the assumption that data transmission can be performed with infinite precision. In recent years emerging applications such as micro-electro-mechanical systems (MEMS) and networked industrial control systems have begun to cross the boundary between these disciplines: bandwidth communication constraints are often major obstacles to control system design by means of classical theory. All these emerging applications have motivated the development of new research directions in control theory that deal with networked systems and combine control and communication issues, taking into account the communication constraints among sensors actuators and their physically distributed nature.

A key concept in control theory is “feedback”, i.e, observers “feed” the available information into the controller for design purpose, then inject the designed
command "back" into the original plant to close the feedback loop for the purpose of control. Through this loop, "information" is time-stamped and control is non-anticipating, in other word, we can not use unavailable information for control design. To formalize these ideas, the concept of "information structure" is introduced to model the information process. This idea which corresponds to the concept of "filtration" for stochastic processes has been well-known in the stochastic control community. The filtration denoted by $\mathcal{F}_t$ can be treated as a non-decreasing family of $\sigma$-field generated by the available observations of the underlying stochastic processes. We will not give a formal mathematical definition of filtration in this dissertation. Interested readers can consult textbooks on stochastic control [23],[24],[25]. More rigid definitions can be found in books on stochastic calculus and stochastic differential equations [39].

Following [103], the concept of "information structure" is introduced to define a simpler version of the "filtration" of the information processes in this dissertation. We will take an engineering approach to the definition of "information structure" to model what is known, when it is known, and to whom it is known.

1.1 Information Structure

Consider a distributed discrete-system operating for $T$ ($T \leq \infty$) time steps. Observations are made at each step from $M$ observations for all $N$ subsystems. Control inputs are applied at each step at $K$ control stations. Here $T > 0$, $M$, $N > 0$, $K \geq 0$ are given integers.

The operation of this system can be described chronologically as follows. Here the symbols $x^n_t$ denote state information of subsystem $n$, $n = 1, \ldots, N$, control input $u^k_t$ at the $k$-th control station and observation $y^m_t$ at the $m$-th observation.
station, $k = 1, \ldots, K$ and $m = 1, \ldots, M$ and $0 \leq t \leq T$.

The dynamics of the system can be modelled as some Markov processes,

$$
(x_t, u_t, v_t) \rightarrow x_{t+1} \tag{1.1}
$$

where $x_t = [(x_1^T, \ldots, x_N^T)^T, u_t = [(u_1^T, \ldots, u_K^T)^T, v_t = [(v_1^T, \ldots, v_M^T)^T]$. Notice here $x_t, u_t$ and $v_t$ denote the state, control and disturbance for the global system.

The uncertainties concerning operation of the system are modelled by the following set of independent random vectors with given probability distributions, called the "primitive random variables"

$$
(x_n^0, v_n^n, w_k^m); \quad t = 1, \ldots, T, n = 1, \ldots, N, k = 1, \ldots, K \tag{1.2}
$$

where $x_n^0$ is the initial state of subsystem $n$, $v_n^n$ is the system noise at subsystem $n$, and $w_k^m$ is the observation noise observation station $m$.

The variables are related by the state transition equations for the global system

$$
x_t = f_t(x_{t-1}, u_t, v_t) \tag{1.3}
$$

or equivalently the local state transition equations for subsystem $i$ can be modelled by

$$
x_t^i = f_t^i(x_{t-1}^i, v_t^i, u_t, \phi_1^i(t), \ldots, \phi_j^i(t), \ldots, \phi_N^i(t)), \quad i, j = 1, \ldots, N. \tag{1.4}
$$

where $\phi_j^i(t)$ denote the effect coming from subsystem $j$ for subsystem $i$, when they are connected with each other.
The observation equations can be represented by

\[ y^m_t = g^m_t(x_{t-1}, u_t, v_t, \phi^i_1(t), \ldots, \phi^i_j(t), \ldots, \phi^i_N(t), w^m_t), \quad (1.5) \]

for all \( i, j = 1, \ldots, N, m = 1, \ldots, M, \) and \( t = 1, \ldots, T. \)

Before we do any controller design, we need to specify the data available for control design at local controller \( u^k_t. \) The specification of the data available as arguments is the information pattern of the system. Because of its great importance, a special notation is introduced for this purpose.

Define the following sets of pairs of indices. For \( t = 1, \ldots, T, \)

\[ Y_t = \{(\tau, m) | \tau = 1, \ldots, T; m = 1, \ldots, M\} \quad (1.6) \]

for \( t = 2, \ldots, T + 1, \)

\[ U_t = \{(\tau, k) | \tau = 1, \ldots, T; k = 1, \ldots, K\} \quad (1.7) \]

Also \( U_1 = \emptyset. \)

A data basis at time \( t \) is the pair \((G, H), \) where \( G \) is a subset of \( Y_t, \) and \( H \) is a subset of \( U_t. \) The pair \((Y_t, U_t)\) is the maximal data basis at time \( t. \) The array of vectors designated by \((G, H)\) is denoted by \( y_G, u_H \) where \( y_G = \{y^i_t : (t, i) \in G\} \) and \( u_H = \{u^k_t : (t, k) \in H\}. \) This construction is similar to the construction of the \( \sigma-\)algebra generated by the observation and control up to time \( t. \) The pair \((Y_t, U_t)\) is the "filtration" up to time \( t \) for the global system.

An information pattern is the assignment to each index pair \((t, k)\) in \( U_{T+1} \) of a data basis at time \( t, \) denoted by \((Y_{t,k}, U_{t,k}). \) This is interpreted to mean that the control applied by station \( k \) at time \( t \) is based on argument \( y^k_t \) with \((\tau, \mu) \in Y_{t,k}\).
and \( u^k \in U_{t,k} \). In other words, there are the observations and controls that are available to station \( k \) at time \( t \) for controller design.

The control equations are

\[
\begin{align*}
  u^k_t &= \gamma^k_t(\ldots, y^\mu_{t_r}, \ldots; u^\theta_{t_r}, \ldots) \quad (1.8) \\
  &= \gamma^k_t(y_{Y_{t,k}}, u_{U_{t,k}}) \quad (1.9)
\end{align*}
\]

where \( \gamma^k_t \) is the control law for the index pair specified by the information pattern.

In general, the range of \( \gamma^k_t \) is assumed unrestricted. As a control law, function \( \gamma^k_t \) need only be measurable. The arguments \( (y_{Y_{t,k}}, u_{U_{t,k}}) \) of \( \gamma^k_t \) specify the available information generated via \( (Y_t, U_t) \) for control design. In this way, the information pattern regarding the questions on what is available, when it is available and where it is available can be unambiguously answered in this framework.

With all above definition, we can take a graph theory approach to model the subsystems as nodes and interconnections as edges. By specifying the index set at time \( t \) for the underlying subsystem, we can then define the available information pattern for these subsystems and give mathematical definitions of centralized control, distributed control and decentralized control via the specification of different information patterns for \( (Y_{t,k}, U_{t,k}) \) for each subsystem.

1.2 Model of Distributed Systems Via Graphs

Here we use graph-theoretic terminology to model the distributed systems with \( N \) subsystems. The global system can be modelled as an directed graph \( G = (V, E) \) where \( V = \{V_1, \ldots, V_N\} = \{G_1, \ldots, G_N\} \). Each vertex \( V_i \) of denotes a subsystem \( G_i \), where each \( G_i \) is a finite-dimensional system, and it can be described by an
ordinary differential equation as (1.4). The set of directed edges \( \mathcal{E} = \{(G_i, G_j), 1 \leq i, j \leq N\} \) can be used to model the interacting from subsystem \( G_i \) to subsystem \( G_j \). Signal \( v_{ij}(t) \) is used to denote the output signal from subsystem \( G_i \) to \( G_j \). It is denoted as \( w_{ji} \) as the input signal to subsystem \( G_j \) from \( G_i \) after it is transmitted over the directed edge \( (V_i, V_j) \) to subsystem \( G_j \). An operator \( \Delta_{ij} \) is used to model the channel condition on the edge \( (V_i, V_j) \) between subsystems \( G_i \) and \( G_j \), such that

\[
 w_{ji} = \Delta_{ij}v_{ij}. \tag{1.10}
\]

Here we use \( \Delta_{ij} \) to model the input-output relationship of \( v_{ij}, w_{ji} \) over the channel \( (V_i, V_j) \). The exact meaning of the "\( \Delta_{ij} \)" will be defined in later chapters after we have defined the signal spaces. The property of the interconnection is fully characterized by the property of \( \Delta_{ij} \).

In this graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), the model of the dynamics is defined by \( G_i \) corresponding to the vertex \( V_i \); the interconnection of the subsystems is defined by \( \mathcal{E} \). Note that the channel conditions are characterized by \( \Delta_{i,j}, 1 \leq i, j \leq N, \) since \( \Delta_{ij} \) is used here to model the input-output relationship of the channel from \( V_i \) to \( V_j \) in this graph.

For example, for the perfect interconnection case, \( \Delta_{ij} = I \), the bit-rate constraint of channel \( (V_i, V_j) \) can similarly modeled as \( C(\Delta_{ij}) \leq R \), where \( C \) denotes the channel capacity of the link \( \Delta_{ij} \). Other channel conditions, such as drop-out and delay, can be similarly modelled in this framework; for example, drop-out can be modelled as follows:
Example 1 \((\text{IID Drop-out with failure rate } p)\)

\[
P(\Delta(t)) = \begin{cases} 
1 & 1 - p \\
0 & p 
\end{cases}
\]  \hspace{1cm} (1.11)

1.3 Bit-rate Constraint in Simple Feedback Loop

The previous sections provide formal definitions on information structure on where, when and what is available for distributed systems modelled via graphs. In this way information considered here is time-stamped, location-dependent. Because of its distributed nature, limited bandwidth and transmission delay between subsystems, state information for the global system normally is not directly available for controller design. To address this problem, we want to start from the simplest case of one plant with bit-rate constraint for the feedback channel.

We would like to understand:

- The minimum bit-rate in the feedback channel to stabilize a linear stabilizable system.

- The performance degeneration of LQR controllers with limited bandwidth in the feedback channel.

To answer the first question (Q1), we will consider the problem of control under bit-rate constraints. The first step is to analyze the simplest possible network topology, consisting of one controller and a dynamical system connected via a feedback loop with a given data rate. In view of the limited communication resources, a natural questions we shall try to answer is the following:

What is minimum bit-rate \(R_{\text{min}}\) for the feedback channel to stabilize these systems? This can be treated as a feasible problem for stabilization. The feasibility
problem in terms of limited communication rate is analogous to Shannon’s source coding theory, which seeks to determine the smallest date rate, above which there exists a coding and control law to stabilize the system. However, Shannon’s information theory is not directly applicable here.

For communication theorists, Shannon’s classical channel capacity theorems are not just beautiful mathematical results, they are also useful as well. Broadly speaking, there are two sides of information theory. The better known side deals with the transmission of bit over channels. This part tells us in a precise way the fundamental limitation to reliable communication over a noisy channel. The crowning achievement of this theory is the Noisy Channel Coding Theorem, which identifies the channel in terms of the invariant quantity, called capacity of the channel, and reliable communication can take place if it occurs at a rate below capacity and cannot if it occurs at a rate above capacity. This theorem links the input side of the communication problem via the notion of capacity with the output side, namely, the ability of decode with arbitrarily small probability of error. However, there is another side to information theory. It relates the encoding of sources with respect to some criterion of fidelity. The rate distortion theorem establishes fundamental trade-offs between fidelity and the length of the encoding in bits. These two sides of information theory are joined together by the information transmission theorems relating to source/channel separation. Philosophically speaking, these theorems establish that the notion of ”reliable transmission” used for discussion channels is compatible with the notion of ”bits” used for encoding sources.

It is natural to ask whether Shannon’s classical capacity is also the right characterization for feedback channels used in control systems. However, there are
still many issues in modern communication systems for which information theoretic understanding eludes us. Networked control systems in particular have a whole host of such issues because of the delay introduced by the network and the key idea of feedback: using observed information to "control" the behavior of the plant. Unlike Shannon’s source coding theorem, the source can be treated as some probability distribution, where the information is fixed in the temporal domain. In the feedback control system, the information structure is influenced by the "control" actions, in other words, the information (such as state, observations) is time-stamped. Besides, normal feedback control is non-anticipating, i.e., we can not using unavailable future information for control design. Because of these properties, the information structure can be viewed as the filtration associated with a stochastic process. Because of this temporal dependence introduced by control, classical source coding theorem for a fixed source can not be directly used here. Notice here the uncertainty comes from the uncertainty of the initial state, which can be viewed as the fixed source in classical source coding theorem; however, long block coding techniques will not here for the simple reason the extra delay introduced by long block code.

In control, a particularly important problem is to stabilize the system. To do this, communicating the values of the underlying unstable process from one place to another is an important issue, since the physical location of the plant and the controllers is generally separated. In practice, people are already using quantizers and digital implementations to control unstable physical systems, and there has been some theoretical results on trying to extend source coding theorem to unstable processes.

Here we formulate this problem as an optimization problem to minimize the
maximum eigenvalue of the quantized system; somewhat counter-intuitively, the minimum rate depends only on the unstable dynamics of the plant, and it is independent of the communication channel. We have shown that a dynamic quantization strategy will converge to our optimal solution. The amount of attention in terms of communication resources associated with a specific unstable mode of the dynamical system is totally determined by its relative magnitude.

Even so, a major gap remains. Till now, there have been no information transmission theorems to tie the source coding of unstable processes to communication over noisy channels.

Once we have solved the feasibility problem in terms of the minimum bit-rate $R_{\text{min}}$ for stabilization, a natural problem to ask is the following:

*Given a bit-rate $R > R_{\text{min}}$, what are the "best" control and coding strategies according to some performance criteria?*

This is a control design problem related to the inherent trade-offs between control and communication costs. Because of the non-anticipating requirements of the controller design, networked control problems generally consist of designing the channel encoder and decoder as well as the controller to satisfy some given objectives; thus, independent design of communication and control for these problems generally does not work since communication systems can tolerate delay with long block coding schemes while delay is normally a main factor for instability in control.

For the classical LQG problem it has been shown in [93], [95], under some mild conditions, the classical separation property between estimation and control holds and certain equivalent control law is optimal. In particular, the optimal LQG cost decomposes into two terms: a full knowledge cost and a sequential rate distortion
cost introduced by the limited bit-rate of the feedback channel. These results justify our intuition that imperfect state/observation information will deteriorate the LQG performance.

Similarly, in this dissertation we try to analyze the performance degeneration of the LQR problem with limited bit-rate. It is well-known that under mild conditions on system controllability the LQR controller can be derived via the solution of an Algebraic Ricatti Equations (ARE), and the cost function is determined by the weighting matrix $Q, R$ and the system's initial state. For an LQR problem where the feedback channel only has limited bit-rate, the controller gets more and more accurate estimation of the initial state with time. By modelling the quantization error as decreasing white noise, we have derived a similar result as the LQG case with bit-rate constraints, i.e., the optimal LQR cost decomposes into two terms: a full knowledge cost and a quantization error cost introduced by the limited bit-rate of the feedback channel. The addition cost due to quantization error can be analytically derived as a function of the bit-rate and quantization sequences. Based on this result, an optimization problem can be formulated to solve for optimal bit-rate assignment policy to minimize this additional cost due to bit-rate constraint. However, due to the complexity of this problem (Integer Programming, higher dimension), we have not solve this optimization problem. For this reason, we have not come up with a quantization policy to approach this minimum.

1.4 Centralized, Decentralized and Distributed Control

To answer the second (Q2) and third (Q3) problem, we put emphasis on understanding how interconnection constraints create special challenges for designing
distributed controllers to guarantee performance of the global system. Here we concentrate on the control design part instead of the communication requirements emphasized in the previous section. Since system analysis and controller synthesis are highly coupled, we try to understand these two issues altogether.

One might initially attempt to control these systems using standard control design techniques, but several limitations will quickly be encountered with practical deployments of these controllers as dimension of the global system increases. In general, there are the following three main reasons why one might want to avoid centralized controllers:

- **Deployment Cost.** Centralized controllers typically need all state information of the global system even for controller design, which requires modifying the underlying interconnection to implement such controllers. This may be undesirable for applications such as communication networks, when introducing new channels between subsystems can normally be less practical and more expensive than increasing transmission rates over the pre-existing ones.

- **Communication Cost.** By the same reason of the information-structure requirement of centralized controllers, the centralized controller need all subsystem to communicate their own state to one central station, then the centralized controller generate appropriate control signal for each subsystems. Because of the communication limitations, some stringent specification of Quality of Service (OS) are needed for the network. Practical networks used in networked control systems sometimes cannot provide this service; for example, in wireless sensor networks it is impractical for controllers to subscribe such services because of the fading issues. Besides, communication leads to longer delay, which is a notorious situation we want to avoid as a
Computational Cost. By its own nature, centralized controllers are bound to have a large number of states, inputs and outputs, which is precisely the situation classical optimal control design algorithms can not handle. Besides, great computation power is normally required to implement these design, not to mention that convergence of computation of larger dimension matrices with great condition number is always a difficult numerical problem.

Because of all these limitations of centralized control for high dimension distributed systems, it has been taken for granted that the natural way to avoid the above limitations is to adopt a full decentralized architecture, where a local controller is attached to each subsystem, where the local control action is based on local measurements. From the communication point view, this approach greatly reduced the communication burden, and from the implementation point of view, these decentralized controller are easily to deploy. But the performance is normally conservative because of information structure constraints with these decentralized controllers, i.e, we are searching for a subset of all admissible controllers with the property that decentralized control can only use local information.

To balance between these two extreme control philosophies, it is natural to introduce an approach which respects the underlying interconnection, adopts a distributed architecture and scales well with the addition of new subsystems. We consider the problem of assessing the induced gain of the global system composed of non-identical interconnected subsystems, when the topology of the underlying graph is arbitrary. The problem taken here builds on recent papers [?], [33], where the idea of distributed control was first introduced.

Our research in this area is first to characterize the stability conditions for
distributed systems with IQC constraints, and then to introduce a distributed controller design method using topological structures and the property of the interconnections. The main contribution of our approach is to unify the stability results that first appeared in [14] in one general framework of IQC analysis. This analysis is similar to what was proposed in [1], [87]. However, here there is much more emphasis on the communication constraints between subsystems and the decoupled stability conditions. Although stability conditions are available for the global system when it is seen as a single system, these results are not directly applicable to distributed control design. In this dissertation, we show that distributed stability conditions exist for some IQCs modeled by a set of multipliers. These distributed stability conditions can be later utilized for distributed controller synthesis.

Another important contribution of our approach is to relate distributed control under communication constraints to the well-developed results in the literature of gain-scheduling techniques for Linear Parameter Varying (LPV) systems [87], [1]. Establishing and explaining the relations between distributed control and LPV is important as it opens the way for using in distributed control problems the results from the well-established field of LPV control. Geometrically, the stability results can be interpreted from a graph separation point view ([83], [61] [40]) following a similar proof as in [87]. As for synthesis, based on a recently extended elimination lemma in [36], the synthesis inequalities turn out to be convex in all variables, including the scalings [85]. It is also worth mentioning that in [14] the dimension of the controller $n_{ij}^K = 3n_{ij}$ for the synthesis condition, while here we show that if the dimension of the distributed controller is greater than or equal to the associated interconnected signals for the plants, i.e, $n_{ij}^K \geq n_{ij}$, there exist
distributed controllers to guarantee global control performance.

The ultimate objective of this dissertation is to summarize and report our current results and techniques for controlling interconnected systems under communication constraints. We try to understand the dynamics and control of complex interconnected systems. Of particular interest are theoretical questions regarding the influence of a complex system’s structure on its stability and robustness, and the design of distributed algorithms and protocols to access and modify the global behavior of the system.

1.5 Literature Review

Many challenging control problems in networking and communication systems can be cast in the framework of controlling complex system with a large number of decision-makers distributed over a widely distributed area. Although the application nature of these systems may vary, they all share three intrinsic aspects:

- Distributed controller design with global objective.
- Interconnection of information and control.
- Controller design subjected to parameter/model uncertainty.

which in our view make them also interesting from a theoretic point of view.

In the following introduction, we will briefly review three interesting distributed control problems in an information rich world related to the above three issues, namely, we concentrate on the ‘information pattern’ requirement for control design.
1.5.1 Information Structure Requirements–Consensus Problem

The distributed nature of these systems implies that the observation is highly distributed and the decision-makers have to determine their control in a distributed manner, based on limited, partial information to achieve some global objectives. One typical objective recently considered in the control society is consensus. The consensus problem has been recently address in [42], [78], [19], [82], [59], [91], [92], [108], [107] after the groundbreaking work [42], where a theoretical explanation is provided for the observed behavior of Viscek’s model [90]. A good review on current research on consensus can be found in the survey paper [77].

Consensus Protocol. Let $x_i$ be the information state of the $i$-th agent, the information state represents information that needs to be coordinated between agents. The information state may be agent position, velocity, oscillation phase, decision variable, and so on. The autonomous version of the linear systems with local state space description for each subsystem is given by

$$x_i^+ = A_{ii}x_i(\tau_i^i(t)) + \sum_{(j,i) \in \mathcal{E}} A_{ij}(t)x_j(\tau_j^i(t)) \quad (1.12)$$

and $x_j(\tau_j^i(t))$ can be the outdated state information, where $\tau_j^i(t) \in [0,t]$, and $t - \tau_j^i(t)$ represents communication and possible other type of delay.

The global autonomous system is given by:

$$x^+ = Ax \quad (1.13)$$

here $x^+$ can be either $\dot{x}(t)$ or $x[t + 1]$ for the continuous-time or discrete time systems respectively.
Consensus is said to be achieved for a group of agents if

\[ \|x_i(t) - x_j(t)\| \to 0 \quad \text{as} \quad t \to \infty, \forall \ i \neq j. \]  \hfill (1.14)

A directed graph \( G \) will be used to model the interaction topology among these agents. Let \( G = (V, \mathcal{E}) \), where \( V = \{A_1, A_2, \ldots, A_N\} = \{1, 2, \ldots, N\} \), and the edge set \( \mathcal{E} = \{(i, j)|i, j \in V, \text{there is a communication link from } A_i \text{ to } A_j\} \), we assume if there is communication link from \( A_i \) to \( A_j \), then \( A_j \) can access \( A_i \)'s state \( x_i \). \( G \) is called \textit{undirected} if \( (i, j) \in \mathcal{E} \), then \( (j, i) \in \mathcal{E} \). A graph is \textit{connected} if for any two vertexes \( i \) and \( j \), there exists a sequence of directed edges \( \{(i, k_1), (k_1, k_2), \ldots (k_{s-1}, k_s), (k_s, j)\} \subset \mathcal{E} \). A spanning-tree of a directed graph is a directed tree formed by graph edges that connected all nodes of the graph, we say that the graph has a spanning-tree if there is a spanning-tree formed by edges from \( \mathcal{E} \).

The time-varying communication graph between agents can be described by \( G(t) = (V, \mathcal{E}(t)) \), where \( \mathcal{E}(t) \) is the set of communication link at time \( t \). The sequence of communication can be either deterministic or stochastic. Let \( G_i = (V, \mathcal{E}_i), i = 1, \ldots, r \), denote a finite collection of graphs with common vertex \( V \). Their \textit{union} is graph \( G = \{V, \bigcup_{i=1}^{r} \mathcal{E}_i\} \). The set of graphs \( \{G_1, G_2, \ldots, G_r\} \) with common vertex is called \textit{jointed connected} if their union is a connected graph.

The 'control' part of the consensus problem comes from the design of the positive weighting factor \( A_{ij}(t) \) associated with the time-varying communication between agents. In other words, the information state of each agent is driven toward the states of its (possibly time-varying) neighbors at each time via proper \( A_{ij}(t) \). Note that some agents may not have any information exchange with other agents during some time intervals.
With a centralized control mechanism, this problem can be trivially solved. However, if each agent can only use its neighbor’s state information for control design, it is not even always feasible to design a control law for such simple global control objective as consensus. This problem can be viewed as an initial attempt to understand the information structure requirement for control design.

For the simple case of time-invariant information exchange topology without delay, it is proved that protocols achieve consensus asymptotically if and only if the information exchange topology has a spanning tree \([78]\) when the communication is uni-directional. It is later discovered that the consensus problem has been well studied in Tsitsiklis’s PhD work on ”agreement algorithm” \([96]\), \([97]\), and a simplified version is presented in the text \([43]\).

For the linear case, the crux of the consensus problem is essentially related to the algebraic multiplicity of the eigen-vector \([1, \ldots, 1]^T\) associated with the updating matrix \(A(t)\) formed by \(A_{ij}(t)\), which spans the consensus subspace. For the general case, the consensus is achieved if the intersection of all the para-contraction map associated with each updating mapping is \([1, \ldots, 1]^T\). The consensus problem reveals the importance of the information structure requirements for global control design.

Currently, there is significant research effort in the area of the asynchronous consensus problem. The main difficulty lies in the updating mechanism to achieve consensus: for a synchronous system, it is relatively easy to make the updating matrix para-contracting, and the dimension of consensus space spanned by \([1, \ldots, 1]^T\) is of dimension 1. However, for asynchronous systems, it is relatively difficult to coordinate the agent’s state with different time-index. We expect that the rich resources on asynchronous convergence analysis (for example, \([30]\), \([73]\))
from asynchronous iterative process will provide us some insight into this problem.

1.5.2 Information Bandwidth Requirement for Control Design

For a distributed control system, the observers, decision-makers, and physical controllers are not necessarily co-located, measurement and control information thus need to be communicated over a data network. In order not to impose a heavy overhead demand, it is often desirable to design algorithms that require low communication data rate. Since communication is an important component of these distributed and networked control systems, there is a need to understand the communication bandwidth requirement for control design of the distributed system.

The first step toward understanding these issues of communication bandwidth requirement is to analyze the simplest possible network topology, consisting of one controller and a dynamical system connected via a feedback loop with a given data rate. There are recent results reported in the literature that address some aspects of this problem. Following [21] and continuing with [10], [29], various schemes have been proposed to consider the stability of quantized system under communication rate constraints. After the first result on minimum bit rate [104] for a simple discrete scalar plant, similar tight bounds were subsequently obtained using different formulation and techniques [93], [66], [37], [52], [52]. Specifically, the necessary and sufficient condition on the rate for asymptotically stabilization in a linear, discrete time, system is

$$\sum_{\lambda(A)} \max \{0, \log(\|\lambda(A)\|)\}.$$  

This limitation is independent of the information pattern and totally determined by the unstable eigenvalues of the plant and relates the speed of dynamics of the plant to the channel capacity. Stabilization of nonlinear systems has also been studied by
[65], and a Slepian-Wolf coding scheme for stabilizing decentralized linear systems under rate constraints has been considered in [68]. A comprehensive review on quantization in networked control systems can be found in Nair’s paper [69].

Besides stability, there is emerging research on the performance of control systems under communication constraints. A first step toward a methodology for the design of controllers, in the presence of communication constraints, has been addressed in [7]. In a recent paper by Tatikonda [95], the classical Linear Quadratic Gaussian (LQG) problem is reconsidered in this framework. Under some mild assumption of no ‘dual-effect’, it is shown that the optimal LQG cost decomposes into two term: a full knowledge cost and a sequential rate distortion cost introduced by the communication constraints. Another recent area of investigation is stability analysis in the presence of disturbances and operator uncertainty [67]. The work by [51], has shown that the extra rate $C - \sum_{\lambda(A)} \max\{0, \log(\|\lambda(A)\|)\}$ is critical for performance, as measured by the expected power of the state of the plant. The work by [28] has used the integral of the log-sensitive, as seen by the noise in an additive channel, to establish that encoding/decoding schemes can be constructed using standard control theory on ”expensive control”. Researchers in this area are now considering some joint optimization problems of control and communication design, since no general separation theory seems to exist for control and communication.

1.5.3 Controller Design Subject to Parameter/Model Uncertainty

Finally, in many of these systems, system parameters and even the structure of the underlying dynamics may not be known or are at least only partially known. As a result, the control algorithms are required to be robust, so that
errors in knowledge about system parameters/structures do not seriously affect performance, that is to say, the controller must be robust with respect to imperfect information on system parameters and model uncertainties.

Motivated by the gain scheduling control methodology, the study of linear parameter varying (LPV) systems provides a systematic control design framework. LPV control theory is advantageous because it provides stability and performance guarantees over wide range of changing parameters, and has found wide application in various industrial problems. A thorough review on gain-scheduling and LPV research can be found in [80].

An LPV plant with LFT parameter dependence can be written in the following state space form

\[
\begin{bmatrix}
\dot{x} \\
z_{\sigma} \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A & B_{1} & B_{2} \\
C_{\sigma} & D_{\sigma1} & D_{\sigma2} \\
C_{1} & D_{11} & D_{12} \\
C_{2} & D_{21} & 0
\end{bmatrix} \begin{bmatrix}
x \\
v_{\sigma} \\
v \\
u
\end{bmatrix}.
\]

Here the \(z_{\sigma}, v_{\sigma}\) are 'artificial' signals related by

\[v_{\sigma} = \Delta(\sigma)z_{\sigma},\]

and \(\Delta\) takes the following diagonal form

\[\Delta(\sigma) = \text{Diag}(\sigma_{1}I_{n_{1}}, \ldots, \sigma_{p}I_{n_{p}}),\]

where the \(n_{k}\) are suitable positive integers and \(I_{n_{k}}\) denotes the \(n_{k} \times n_{k}\) identity matrix.
In LPV control, it is known that the parameter is contained in an a priori given set, whereas the actual parameter curve is unknown but can be measured on-line. The design procedure consists of determining an LPV controller which mimics the LFT parameter dependency of the plant, namely

\[
\begin{bmatrix}
\dot{x}^c \\
u \\
u_\sigma
\end{bmatrix} =
\begin{bmatrix}
F & G_1 & G_\sigma \\
H & E_{11} & E_{1\sigma} \\
H_\sigma & E_{\sigma 1} & E_{\sigma \sigma}
\end{bmatrix}
\begin{bmatrix}
\dot{x}^c \\
y \\
y_\sigma
\end{bmatrix},
\]

where \( u_\sigma, y_\sigma \) are related by

\[ u_\sigma = \Delta(\sigma)y_\sigma. \]

It is important to notice that in the above formulation, all \( \sigma \) dependence is contained in \( \Delta(\sigma) \), while other matrices are constant.
A Linear Fractional Transformation (LFT) gain scheduling technique was developed by Packard in [1] and [4]. Using the LFT representation for the system, conditions for the existence of a parameter-dependent controller that guarantees stability and $H_\infty$ performance for the closed-loop system are given in the form of linear matrix inequalities; however, there is one additional non-convex coupling condition involving the inverse of some design parameters.

Recently, on the basis of idealization lemma, a generalization formulation in geometric terms of what has proposed in [40], and an elimination lemma for quadratic inequalities extending the projection lemma in [4] was introduced in [85]. It is now possible to derive synthesis inequalities that turn out to be convex in all variables, including the scaling matrices [85]. Still, the technique could only be applied under certain inertia hypothesis on the multipliers which are unduly conservative [36].

Alternative LPV analysis and synthesis approaches for LTV systems can be found in [35], [41], [101] and [106].

Figure 1.1 shows the original block diagram. The blocks $P_{LTI}$ and $K_{LTI}$ represent the Linear Time-Invariant portion. The linear parameter-varying plant $P_{LPV}$ and controller $K_{LPV}$ have the following LFT representations:

$$P_{LTV} = \mathcal{F}_u(P_{LTI}, \Delta), \quad K_{LTV} = \mathcal{F}_l(K_{LTI}, \Delta).$$

Figure 1.2 shows the modified block diagram for control design. The block $P_{Aug}$ represents the augmented plant used for control design. The repeated block $\Delta$ represents parameter time variations which can be viewed as structure unknown perturbations.

There have been various recent works on robust stability and performance
analysis in the case of linear time varying perturbations. Generally speaking, the perturbation is modeled as set of contractive linear time-varying operators of some fixed block diagonal structure we assume that $\Delta : \mathcal{L}_2^n \rightarrow \mathcal{L}_2^n$ and $\Delta$ is of the form

$$\Delta = \text{Diag}(\tilde{\delta} I_{\tilde{n}_1}, \ldots, \tilde{\delta} I_{\tilde{m}_c}, \delta_1 I_{n_1}, \ldots, \delta_{m_c}, \Delta_1, \ldots, \Delta_{m_F}), \quad (1.15)$$

with

$$\begin{align*}
\tilde{\delta}_i : & \mathcal{L}_2 \rightarrow \mathcal{L}_2 \text{ LTV, self-adjointed, } \|\tilde{\delta}_i\| \leq 1 \\
\delta_i : & \mathcal{L}_2 \rightarrow \mathcal{L}_2 \text{ LTV and } \|\delta_i\| \leq 1 \\
\Delta_i : & \mathcal{L}_2 \rightarrow \mathcal{L}_2 \text{ LTV and } \|\Delta_i\| \leq 1
\end{align*} \quad (1.16)$$

The dimensions of the various identity matrix and $\Delta_i$ are fixed.

A crucial discovery concerning the robust stability/performance analysis was
the exactness of the constant \((D,G)\)-scaling test for arbitrary LTV perturbations [46], [88], [71], [56]. These results provide perfect justification for using the constant \((D,G)\)-scaling test for analysis against arbitrary LTV permutations, whereas on the other hand it was well-known from Structured Singular Value (SSV) analysis that the frequency-dependent \(D\)-scaling test is not generally exact in robust performance analysis for linear time-invariant (LTI) perturbations.

Combining the foundations of the methods of stability analysis that emerged during the robust control era with the classic input/output and absolute stability theory, a unifying approach for stability analysis was developed by [54] for robust stability analysis. Within this approach, the information on the structure and nature of the perturbations is modeled by an Integral Quadratic Constrain (IQC) that is to be satisfied by the uncertainties (by employing suitable relaxation if unavoidable) and the stability test is provided in terms of LMIs that is to be satisfied by the plant. This method admits a minor extension to robust performance analysis by a suitable application of the so-called \(S\)-procedure [110], [55]. All these results can be reformulated in the general topological separation framework. The robust analysis problem, in essence, can be interpreted as one of establishing a topological separation between the graph of a linear-time invariant nominal plant and the set of inverse graphs of all possible structured uncertainty [83]. That is, topological separation of the feedback system can be established by searching for an appropriate symmetric matrices which can be called ”quadratic separator” used for the quadratic form in the IQC conditions. While the existence of a quadratic separator is sufficient to conclude stability, its necessity is not clear for the general class of uncertainty; for specific LTV contractive operators, the \((D,G)\)-scaling theorem has been proven to be both necessary and sufficient [56].
The results from [40] have provided more insight into the exact analysis problem.

The numerical implementations of the tests developed within the IQC framework can be realized by using the Kalman-Yakubovich-Popov (KYP) lemma [75] to transfer the IQC expressed in the frequency domain inequalities to Linear Matrix Inequalities (LMI) and then employing the dedicated solver. In this fashion, robust stability/performance analysis or optimization problems are typically transformed into semi-definite programming problems. In fact, it has been a crucial observation that a variety of robust analysis and synthesis problems can be re-expressed as robust LMI problems. Among various other benefits, this also allowed for some extensions to analyze the conservatism of the relevant analysis/optimization methods, via the elegant Lagrange duality theory for optimization and in particular for semi-definite programming.

1.6 Dissertation Organization

The dissertation is organized as follows.

Chapter 2 In this chapter, we consider the problem of control of a closed-loop feedback system under bit-rate constraints in the feedback channel. It is called the quantized system in the recent literature in networked control systems. Because of the bit-rate constraints, the state information $x(t)$ is "quantized" and transmitted over the feedback channel to the controller. At each step, the controller has only access to the "quantized" state $\hat{x}(t)$. We assume that the controller knows the plant; however, the initial state $x_0$ is unknown. With the quantized information and its knowledge of the past control inputs, the controller’s estimation error
\[ e(t) = x(t) - \hat{x}(t) \] is decreasing given the bit-rate

\[ R > \sum_{\lambda(A)} \max\{0, \log \lambda_u(A)\} \]  

(1.17)

where \( \lambda_u(A) \) is the unstable eigenvalue of the state matrix \( A \) of the original system. We have formulated an optimization problem to maximize the convergence rate of the quantization error. It turns out that the feasibility problem of this optimization leads us to this bounds, and the solution of this optimization problem can lead us to design an dynamic quantization algorithm that approaches this maximal convergence rate. Notice here, under our formulation, the problem is an optimization continuous in the design variables; however, for any quantization policy, the bits are discrete integers, which means the practical bit-assignment policy is an integer programming problem. The existence of such dynamic policy justified our relaxation. Actually, most of the standard quantization policy reflects the intuition from our results, i.e. try to balance the allocation of communication resources between the unstable modes of the original system.

With some mild assumptions on the independence of quantization error at each step, we can model the quantization errors as white noise decaying in magnitude. Under these assumptions, the cost of the conventional LQR problem decomposes into two parts: the full-information part and the one introduced by quantization error due to bit-rate constraints. We can then try to solve the optimization problem to minimize the cost introduced by the network via orchestrating the bit-assignments. We have derived the analytical form of the cost, but at the current stage we have not come up with an analytical minimizer of this problem or a quantizer to approach this bound. Even so, these results are among the first few results to characterize performance of the quantized system.
Chapter 3  In this chapter, stability of spatially invariant systems is discussed in detail. To completely describe these systems, both the temporal and spatial index are introduced. The signal space needs to be first redefined to analyze such systems. Because of its spatial invariance, the standard approach is to take the spatial Fourier transform of these systems, such that the so-called spatially invariant system can be reduced to parameter-dependent time-invariant system, where the parameter denotes the spatial frequency. In this way, we can analyze these systems across each frequency in terms of well-posedness, stability and performance. We want to mention here that, unlike in the temporal domain, the control signal is non-causal in the temporal domain, i.e, the convolution kernel is unbounded in both sides. It can also be shown that the convolution kernel decays at an exponential rate, which justifies this distributed control design in practices.

Although we can use the conventional approach to analyze the system after taking the spatial Fourier transformation, the transformed system is still infinite dimension. Thanks to the KYP lemma, it is shown that these problem can be reduced to a sets of linear matrix inequalities, a convex optimization problem which can be numerically solved. Also, these stability conditions can be utilized for distributed controller synthesis.

Chapter 4  In this chapter, based on an extended verion of S-lemma for matrices, we have derived various distributed stability conditions for the global system to achieve various quadratic performance. When the communication is perfect, the interconnection signals can be viewed as algebraic constraints at each time instance. Performance such as stability conditions can be formulated as a set of LMIs with equality constraints coming from the interconnections. By the application of the extended S-procedure, the above constrained LMI can be reformulated
as a set of unconstrained LMIs. Mathematically, they are just Lagrange relaxation techniques. What is really interesting is the derived conditions can eventually be used for distributed controller synthesis, where these distributed controllers share the same interconnection as the underlying subsystems. The synthesis techniques are well-known in the literature of gain-scheduling control, which will presented in Chapter 5.

Chapter 5 In this chapter, we try to model more practical interconnections where the communications are not ideal. For simplicity, we model the channel as an operator in its input-output relationship in $\ell_2$. For some special operators, they can be equivalently characterized by Integral Quadratic Constraints. Similarly to the previous chapter, we use a lossless S-procedure for $\ell_2$ signals to introduce the multipliers into the original constrained problem to derive similar set of LMIs to guarantee performances. These multipliers reflect the constraints for the interconnection. The two version of $S$-procedure used in Chapter 4 and Chapter 5 differs in the signal space, but the idea are the same. Then we use the elimination lemma for controllers synthesis, as mentioned above, these synthesis conditions have a distributed structure, such that, controllers can be deployed in a distributed fashion.

Chapter 6 In this chapter, we will first summarize our research in this joint area of communication and distributed control, then point out some the unsolved problems for future research.
CHAPTER 2

STABILITY AND PERFORMANCE OF QUANTIZATION SYSTEM

2.1 Introduction and Literature Review

There is strong ongoing interest in the problem of control under communication constraints, attempting to bring together classical control theory and practical communication theoretical issues in the design of control system. There have been several results on stability [21], [10]. A number of these results have focused on the minimum bit rate required for stability, using different formulations and techniques [104], [66], [93], [37], [74], [53], [94], [58], [45]. Specifically, the necessary and sufficient condition on the rate for asymptotic stabilization in a linear, discrete time system is \( R > R_{\text{min}} \), where \( R_{\text{min}} = \sum_{\lambda(A)} \max\{0, \log |\lambda(A)|\} \) is the bit rate, and \( \lambda(A) \) denotes the eigenvalues of the state matrix \( A \). This result is independent of the particular quantization strategy, and the minimum \( R \) for stability is totally determined by the magnitude of the unstable eigenvalues of the plant. Stabilization of nonlinear systems has been studied by [64], [49] and a Slepian-Wolf coding scheme for stabilizing decentralized linear systems under rate constraints has also been considered in [68].

Besides stability, there is some research on the performance of control systems under communication constraints. The initial steps toward a methodology for the design of controllers, in the presence of communication constraints, has been
addressed in [7]. In a recent paper by Tatikonda [95], the classical linear quadratic Gaussian problem is reconsidered in this framework; under a mild assumption of no "dual-effect", it is shown that the optimal LQG cost decomposes into two terms: a full knowledge cost and a sequential rate distortion cost introduced by the communication constraints.

Another recent area of investigation is the analysis in the presence of disturbance and uncertainty. In [51], stability in the presence of disturbances and operator theoretic uncertainty is considered. For a particular class of channels, the work by [67] has shown that the extra rate $C - \sum_{\lambda(A)} \max\{0, \log(|\lambda(A)|)\}$ is critical for performance, as measured by the expected power of the state of the plant. The work by [29] has shown that the coarsest quantizer is logarithmic, and can be computed by solving a special linear quadratic regulator (LQR) problem. Research in this area currently focuses on understanding the fundamental limitations of control performance under communication constraints.

In this chapter, instead of viewing the channel capacity as a communication constraint, we regard $R$ as an available resource to minimize the quantization error. We investigate the optimization problem of maximizing the convergence rate of the quantization error $\|e[k]\|$, which is equivalent to minimizing the following asymptotic convergence factor:

$$r_{\text{asy}} = \sup_{\|e\|\neq 0} \lim_{k \to \infty} \left( \frac{\|e[k]\|}{\|e[0]\|} \right)^{1/k}.$$  \hspace{1cm} (2.1)

We formulate this problem as a min-max eigenvalue problem, and from the solution of this optimization problem, we find the following:

- the optimal communication rate allocation strategy to maximize the quantization error convergence rate of the quantized systems.
• a dynamic quantization policy that stabilizes the feedback system and the convergence rate of the quantization system approaches the solution of the optimal strategy.

It is shown that the error asymptotic convergence factor for the quantized system is determined by the ratio between $|\det A|$ and $R$. For the optimal strategy, the communication resources should be appropriately distributed among the different unstable modes of the system [37], [74]. It is further proven that a quantization policy presented in [74] approaches this optimal solution. Related problems on the design of a "communication sequence" to stabilize multiple systems whose feedback loops are closed over one common shared medium, can be found in [38].

Furthermore, we analyze the classical LQR problem with limited bit-rate in the feedback channel. The key idea is to introduce a probability distribution of the step-wide quantization error, so that the classical LQR problem can be reformulated as a stochastic LQR problem with uncorrelated, zero-mean decreasing white noise due to quantization error. In this way, the performance degeneration can be quantified. Our results show that the performance of the quantized system decomposes into two parts: a full-information case and a penalty from the step-wide quantization error.

2.2 Problem Formulation

This chapter studies discrete-time LTI systems of the following form:

\begin{align*}
x[k + 1] & = Ax[k] + Bu[k] \\
u[k] & = K x^q[k],
\end{align*}

(2.2) (2.3)
where the state $x[k] \in \mathbb{R}^n$, the control signal $u[k] \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $K \in \mathbb{R}^{m \times n}$. The state $x[k]$ is quantized and encoded into a symbol $s[k]$. The symbol $s[k]$ is transmitted to the decoder over a perfect communication channel with fixed rate $R$ bps. The symbol $s[k]$ belongs to the integer set $\Sigma$, $\Sigma = \{1, 2, \ldots, Q\}$, $Q = 2^R$. The decoder uses the received symbols to compute an estimate, $x^q[k]$ of the plant’s true state $x[k]$. The controller uses this estimate $x^q[k]$ to compute the control signal $u[k] = K x^q[k]$. We assume that the feedback gain matrix $K$ is properly designed to satisfy performance specifications and that the closed loop state matrix $(A + BK)$ is stable when there are no communication constraints. Here we can specify the information pattern of each of three components: encoder $E_t$, decoder $D_t$ and the controller $C_t$. Let $X^t = (X_0, \ldots, X_t)$.

- Encoder: The encoder at time $t$ is a map

$$E_t : \mathbb{R}^{l(t+1)} \times \Sigma^t \times \mathbb{R}^{mt} \to \Sigma$$

(2.4)

taking

$$(X^t, s^t, U^{t-1}) \to \sigma_t.$$  

(2.5)

The available inputs to the encoder is the information pattern of the encoder.

- Decoder: The decoder at time $t$ is a map

$$D_t : \Sigma^{t+1} \times \mathbb{R}^{mt} \to \mathbb{R}^d$$

(2.6)

taking

$$(s^t, U^{t-1}) \to \hat{X}_t.$$  

(2.7)

The output of decoder is an estimate of the state of the plant.
Controller: The controller at time $t$ is a map

$$C_t : \mathbb{R}^d \to \mathbb{R}^d$$

(2.8)

taking

$$\dot{X}_t \to U_t.$$ (2.9)

Note here that the decoder and controller both know the dynamics of the system. They also agree upon the initial uncertainty set, i.e. $P[0]$, such that $x[0] \in P[0]$.

We are interested in the stability of the system which guarantees

$$\lim_{k \to \infty} \|x[k]\| = 0$$

for any $x[0] \in \mathbb{R}^n$, where $\| \cdot \|$ denotes Euclidean 2-norm.

We study stability under the following assumptions:

1. $(A, B)$ is controllable. $A = \text{diag}(J_1, J_2, \ldots, J_p)$, where $J_i$ is an $n_i \times n_i$ Jordan block. We assume throughout this chapter that $A$ has only unstable real eigenvalues, i.e. $|\lambda_i| \geq 1$.

2. The initial condition $x[0]$ lies in a "known" parallelogram $P[0]$.

3. Both the encoder and decoder know the matrices $A, B$, the coding-decoding policy and control law. They also agree upon the initial uncertainty set, i.e. $P[0]$.

Assumption (2) requires the initial state to lie within a known parallelogram $P[0]$, which can be written as

$$P[0] = x^q[0] + U[0]$$
where \( x^q[0] \in \mathbb{R}^n \) is the centroid of \( P[0] \), and \( U[0] \) is a parallelogram centered at the origin. Similarly, the state \( x[k] \) at time \( k \) is quantized with respect to a parallelogram \( P[k] \) representing the quantization "uncertainty", i.e,

\[
P[k] = x^q[k] + U[k]
\]

(2.10)

where \( x^q[k] \in \mathbb{R}^n \) is the centroid of the \( P[k] \), and \( U[k] \) is a parallelogram with center at the origin. At time instance \( k \), we know that

\[
x[k] \in x^q[k] + U[k],
\]

where \( x^q[k] \) denote the current estimate of the state \( x[k] \), and \( U[k] \) is used to represent the uncertainty associated with such an estimate.

The quantization error \( e[k] \) is defined as the difference between the actual state \( x[k] \) and the estimated state \( x^q[k] \)

\[
e[k] = x[k] - x^q[k],
\]

(2.11)

and from our assumption, \( e[k] \in U[k] \).

The parallelogram \( U[k] \) is used to model the quantization error, which can be formally characterized by a set of vectors \( v_{i,j}[k] \in \mathbb{R}^{n_i} \) where \( i = 1, \ldots, p \) and \( j = 1, \ldots, n_i \). The parallelogram associated with the \( i \)-th Jordan block in \( A \) is denoted as the convex hull,

\[
S_i[k] = \text{Co} \left\{ v : v = \sum_{j=1}^{n_i} \frac{1}{2} v_{i,j}[k] \right\}.
\]

(2.12)
The entire parallelogram $U[k]$ may therefore be expressed as the Cartesian product of the sides, $S_i[k]$, i.e.:

$$U[k] = \prod_{i=1}^{p} S_i[k] \quad (2.13)$$

The volume of $U$ is defined as $\text{vol}(U) = \int_{x \in U} 1 \cdot dx$. The "size" of $U[k]$ is measured by its diameter, which is defined as

$$d_{\text{max}}(U) = \sup_{x, y \in U} \|x - y\|$$

As proven in [93] by an argument of the triangular inequality, the convergence of the diameter of $U[k]$ is equivalent to the asymptotic stability of the quantized system if the closed-loop state matrix $(A + BK)$ is stable, which implies that if the diameter of $U[k]$ converges to zero, then the quantized system is asymptotically stable.

The change of $\text{vol}(U[k])$ involves two parts: one is associated with quantization when $P[k]$ is partitioned into $2^R$ small uncertainty sets, described by:

$$\text{vol}(U^q[k]) = \frac{\text{vol}(U[k])}{2^R} \quad (2.14)$$

while the other is due to the dynamics of the underlying plant, namely:

$$\text{vol}(U[k]) = |\text{det}A| \cdot \text{vol}(U^q[k] - 1); \quad (2.15)$$

here we use the notation $U^q[k]$ to denote the uncertainty set $U[k]$ right after quantization. Equations (2.14), (2.15) provide the insight on the necessary low bound needed for stabilization since we have the following iteration relationship in terms of the "measure" of the uncertainty set $U[k]$; otherwise, the volume
(measure) of the uncertainty set \( U[k] \) goes to infinity.

\[
\text{vol}(U[k]) = \frac{|\text{det}A|}{2^R} \cdot \text{vol}(U[k-1])
\]  

(2.16)

Without going into details (see [94]), we can conclude from (2.16) a necessary condition for stabilization is

\[
R > \sum_{\lambda(A)} \max\{0, \log |\lambda(A)|\}.
\]

(2.17)

However, when the volume of the uncertainty set \( U[k] \) converges to zero, it does not necessarily mean that the diameter of \( U[k] \) converges to zero, because the edges of \( U[k] \) can be quite ”irregular” (such that some length of the supporting edge of \( \|v_{i,j}[k]\| \) never converges). Various techniques have been used to ”segment” the edge of \( U[k] \) and prove that the bound is sharp [37], [93], [74].

In this chapter, given bit rate \( R \), we want to find the best policy to maximize the convergence rate of quantization error. This can be posed as minimizing the asymptotic convergence factor. Throughout this chapter, the terms ”maximum convergence rate” and ”minimum asymptotic convergence factor” are used interchangeably.

2.3 Minimum Asymptotic Convergence Factor of Quantization Error

As we have mentioned, \( U[k] \) can be represented by the Cartesian product of convex hull of \( v_{i,j}[k] \). If \( q_{i,j} \) level is used to ”quantize” \( v_{i,j}[k] \) (2.14), then after quantization, \( v_{i,j}[k] \) is divided by \( q_{i,j} \), and \( v^q_{i,j}[k] \) is used to denote the vector \( v_{ij}[k] \) right after quantization,

\[
v^q_{i,j}[k] = \frac{1}{q_{i,j}} v_{i,j}[k].
\]

(2.18)
According to the dynamics of the system (2.15), \( v_{i,j}[k] \) will evolve as follows,

\[
v_{i,j}[k] = J_i v_{i,j}[k - 1].
\]

The new uncertainty convex hull \( S_i[k+1] \) associated with \( J_i \) can be represented as

\[
S_i[k] = \text{Co} \left\{ v : v = \sum_{j=1}^{n_i} \frac{1}{2} \frac{J_i}{q_{i,j}} v_{i,j}^q[k - 1] \pm \frac{1}{2} \right\},
\]

where \( \text{Co} \) denote the convex hull generate by the associated vectors.

Let us introduce the augmented state vector for \( v^a = [v^a_1, \ldots, v^a_p] \in \mathbb{R}^{\sum_p (n_i)^2} \), where

\[
v^a_i = [(v^a_{i,1})^T, \ldots, (v^a_{i,n_i})^T]^T.
\]

(2.19)

Assume that \( q_{i,j} \) is the quantization level associated with \( v_{i,j}[k] \), then

\[
\begin{bmatrix}
J_i \\
q_{i,1} \\
\vdots \\
\vdots \\
J_i \\
q_{i,n_i}
\end{bmatrix}
\begin{bmatrix}
\vdots \\
v^a_i[k]
\end{bmatrix}
= A_q^i v^a_i[k]
\]

Define

\[
A_q = \text{diag}\{A_1^q, \ldots, A_p^q\}.
\]

Then the system equation for the augmented vectors \( v^a \) to represent the uncertain set \( U[k] \) is

\[
v^a[k + 1] = A_q v^a[k].
\]

(2.20)
Here we allow $q_{i,j} \in \mathbb{R}^+$, with the constraint,

$$
\prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} \leq Q = 2^R
$$

(2.21)

Obviously, $\{\|v^a[k]\|\}$ converges to zero if and only if $\{\|v_{i,j}[k]\|\}$ converges to zero.

The convergence factor of $\{d_{max}(U[k])\}$ is determined by the maximum eigenvalue of $A_q$ [99], i.e.,

$$
r_{asm} = \lambda_{max}(A_q).
$$

(2.22)

In [111], a similar optimization problem for analyzing the minimum rate to achieve boundedness of the quantization error with a time-invariant quantizer is formulated with the quantization levels $q_{i,j}$ being integers. However, we allow $q_{i,j}$ to be positive reals in the following optimization. In this way we can overcome the difficulties associated with the integer programming problem. In the next section, we will prove that a dynamic quantization policy will use an average $q_{i,j}(q_{i,j} \in \mathbb{R}^+)$ quantization level per step if the time varying integer levels of $q_{i,j}$ are averaged over time. Before we begin any practical quantization design, let us first find the solution to the following minimization problem, which is equivalent to minimizing the maximum eigenvalue of $A_q$; the equivalence follows from the structure of $A_q$.

Min-Max Eigenvalue Problem (MMEP):

$$
\min \ t
$$

(2.23)
subject to

\[
\frac{\lambda_i}{q_{i,j}} \leq t \\
\prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} \leq Q
\]

**Theorem 1** The optimal solution to the above MMEP problem is

\[
t_{\text{min}} = t^* = \left[\frac{\det A}{Q}\right]^\frac{1}{n}
\]  

with

\[
q^*_{i,j} = |\lambda_i| \left[\frac{Q}{\det A}\right]^{\frac{1}{n_i}}.
\]

**Proof 1** This is a standard geometric programming [8] problem. To form the Lagrangian, we introduce the \( n + 1 \) multiplier \( \zeta_{i,j}, \nu \) for the equality constraints, then we obtain:

\[
L(t, q, \zeta, \nu) = t + \sum_i \sum_j \zeta_{i,j}(\frac{\lambda_i}{q_{i,j}} - t) + \nu(\prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} - Q),
\]

where \( q = (q_{i,j})^T, \zeta = (\zeta_{i,j}) \in \mathbb{R}^n, \nu \in \mathbb{R} \).
Apply the Kuhn-Tucker condition to $L(t, q, \zeta, \nu)$:

\[
0 = \frac{\partial L}{\partial t} = 1 - \sum_{i}^{p} \sum_{j}^{n_i} \zeta_{i,j}
\]

\[
0 = \frac{\partial L}{\partial q_{i,j}} = -\zeta_{i,j} \lambda_i q_{i,j}^2 + \nu \prod_{i=1}^{p} \prod_{j=1, j \neq i}^{n_i} q_{i,j}
\]

\[
0 = \frac{\partial L}{\partial \zeta_{i,j}} = \frac{\lambda_i}{q_{i,j}} - t
\]

\[
0 = \frac{\partial L}{\partial \nu} = \prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} - Q
\]

Combining these equations, we get

\[
t^* = \left[ \frac{\det A}{Q} \right]^{\frac{1}{n}}
\]

\[
q_{i,j}^* = \left| \lambda_i \right| \left[ \frac{Q}{\det A} \right]^{\frac{1}{n}}
\]

while the multiplier $\zeta_{i,j} = \frac{1}{n}$, $\nu = \frac{1}{nQ} \left[ \frac{\det A}{Q} \right]^{\frac{1}{n}}$.

It is straightforward to check that $t^*$ is the optimal solution of the above MMEP problem.

The above MMEP problem provides a clear justification for the different quantization policy [74], [94]. Intuitively, the relative magnitude of $|\det A|$ and $Q$ determines the convergence rate of quantization error, while the relative magnitude among $\lambda_i$ determines the fairness among the communication rate $\tau_{i,j} R (2.27)$.

Remark 1 It is interesting to notice that, in order to minimize the maximum eigenvalue of $A_q$, all the eigenvalues are set to be equal to $\left[ \frac{\det A}{Q} \right]^{\frac{1}{n}}$, which reflects the "balance" between the convergence rate of $\{||v_{i,j}[k]||\}$. This property can be used to guide our design of quantization policy. Furthermore, if we regard quantization
Q as another optimization parameter, the feasibility problem $t_{\min} < 1$ leads to the necessary condition needed for asymptotic stabilization.

**Remark 2** The portion of bit-rate $R$ associated with $q_{i,j}$ can be computed as follows,

$$
\tau^*_i,j = \log_Q q_{i,j} = \frac{1}{n} + \frac{1}{n} \log_Q \left| \frac{\lambda_i^n}{|\det A|} \right|
$$

(2.27)

Note that $\sum_{i=1}^p \sum_{j=1}^{n_i} \tau^*_{i,j} = 1$. $\tau^*_{i,j} R$ denotes the bit rate allocated to that particular mode. From a queuing-theoretical viewpoint, $\tau_{i,j}$ is used to capture the amount of communication resource needed to decrease the error associated with $v_{i,j}$ direction as well as the "attention" needed over the available channel [9].

**Remark 3** If we can arbitrarily assign $q_{ij}$ quantization levels to the vector $v_{i,j}[k]$, then, according to Lemma 2,

$$
\|e[k]\| \leq d_{\max}(U[k])
$$

(2.28)

$$
\leq nK \left[ \frac{|\det A|}{Q} \right] ^{k_{\text{max}} n_i - 1} d_{\max}(U[0]),
$$

(2.29)

and according to (2.1), (2.22), the minimum asymptotic convergence factor $r^{* \text{ asym}}$ is,

$$
r^{* \text{ asym}} = \frac{1}{n} \frac{|\det A|}{Q}.
$$

(2.30)

Generally speaking, the quantization level $q_{i,j}$ should be an positive integer instead of a positive real number. We will show that a novel dynamic quantization policy introduced in [74] will assign an "average" of $q^*_{ij}$ quantization level to the vector $v_{i,j}[k]$, and the convergence rate will approach $r^{* \text{ asym}}$. 

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2.4 A Policy to Achieve the Maximum Error Convergence Rate

This section introduces a theorem showing that the optimal convergence rate can be achieved by the dynamic bit assignment policy presented in [74]. On the average, the dynamic quantization policy from [74] will assign exact $\tau_{i,j}^* R$ to each of the unstable modes, which can be seen as a practical realization to achieve the theoretical bounds we have derived in the previous section. It is further shown that the $d_{\text{max}}(U[k])$ converges at a faster rate than the rate derived in [74], and this low bound is achieved by allocating an average of $\tau_{i,j}^*$ of the total resources $R$ to $v_{i,j}$ direction.

The dynamical quantizer is a three-tuple $(x^q[k], (I, J), U[k])$ where $x^q[k] \in \mathbb{R}^n$ represents the centroid, $U[k] = (S_1[k], \ldots, S_p[k]) \in \mathbb{R}^n$ represents the dynamical range, with $S_i[k]$ defined in (2.12), and $(I, J)$ represents the dynamical quantizer edge index. The dynamical quantizer uses a one-step prediction: at each step, choose $(I, J)$ according to (2.31) for quantization, and assign all $R$-bit to the vector $v_{I,J}$, thus partitioning the $U[k]$ along $v_{I,J}$ into boxes with side length $v_{I,J}/Q$. The mechanism of the dynamic bit assignment policy (DBAP) from [74] is described as follows.

Dynamic Bit Assignment Policy (DBAP):

Encoder/Decoder Initialization:
Initialize $X^q[0]$ and $\{v_{i,j}[0]\}$ so that $x[0] \in x^q[0] + U[0]$ and set $k = 0$.

Encoder Task:

1. Select the indices $(I, J)$ by

\[(I, J) = \arg \max_{i,j} \| J_i v_{i,j}[k] \|_2. \quad (2.31)\]
2. Quantize the state $x[k]$ by setting $s[k] = s$ if and only if

$$x[k] \in x^q[k] + x_{s}^{(I,J)} + U^{(I,J)}[k],$$

where

$$x_{s}^{(I,J)} = \begin{bmatrix} 0 & \cdots & 0 & v^T & 0 & \cdots & 0 \end{bmatrix}$$  \quad (2.32)$$

and $v = \frac{-Q + (2s-1)}{2Q} v_{I,J}[k]$ for $s \in 1, \ldots, Q$.

3. Transmit the quantized symbol $s[k]$ and wait for acknowledgment.

4. Update the variable

$$v_{i,j}[k + 1] = J_i v_{i,j}[k]$$

$$x^q[k + 1] = (A + BK)x^q[k].$$

5. If decode "ack" received:

$$v_{I,J}[k + 1] = \frac{1}{Q} v_{I,J}[k + 1]$$

$$x^q[k + 1] = x^q[k + 1] + Ax_s^{(I,J)}[k],$$

where $x_s^{(I,J)}$ is defined as in (2.32).

6. Update time, $k := k + 1$ and return to step 1.

Decode Task:
1. Update the variables

\[ v_{i,j}[k + 1] = J_i v_{i,j}[k] \]
\[ x^q[k + 1] = (A + B K)x^q[k] . \]

2. Wait for quantized data, \( s[k] \), from encoder.

3. If data received:

\[ v_{I,J}[k + 1] = \frac{1}{Q} v_{I,J}[k + 1] \]
\[ x^q[k + 1] = x^q[k + 1] + A x_{s[i,j]}^{(I,J)}[k] , \]

where \( x_{s[i,j]}^{(I,J)} \) is defined in (2.32). Then send ack back to the encoder.

4. Update the time index, \( k = k + 1 \), and return to step 1.

**Remark 4** The above algorithm assumes \( \{ v_{i,j}[k] \} \) and \( x^q[k] \) are synchronized at the beginning of the \( k \)-th time interval. The "ack" message is introduced to detect the possible packet-dropout from encoder to decoder on the communication channel.

The DBAP quantization policy, which seeks to "quantize the largest edge" can be viewed as the "contain the largest state" policy in [38] for switching between a group of unstable systems.

Let us define the average bit-rate associated with \( v_{i,j}[k] \) using the DBAP policy. Suppose from time instance 0 to \( k \), \( v_{i,j} \) has been quantized \( n_{i,j} \) times. The average bit-rate associated with \( v_{i,j} \) is defined as

\[ R_{i,j} = \lim_{k \to \infty} \frac{n_{i,j}}{k} R , \] (2.33)
assuming that the above limit exists.

In the following theorem, we show how the DBAP quantization policy relates to our MMEP problem as we investigate the average bit-rate assigned to the edge $v_{i,j}[k]$.

**Theorem 2** Let $R^*_{i,j}$ be the average bit rate for the DBAP policy. Then,

$$R^*_{i,j} = \tau^*_{i,j} R,$$

(2.34)

where $\tau^*_{i,j}$ is given in (2.27) through the optimal solution of the MMEP problem. Furthermore, the diameter of uncertainty set $U[k]$ converges at the following rate:

$$\|d_{\text{max}}(U[k])\| \sim 1/n\kappa \left[ \frac{|\det A|}{Q} \right]^{\frac{1}{n}} k^{\max n_i-1} d_{\text{max}}(U[0]),$$

(2.35)

where $\kappa$ is a time-independent constant. The asymptotic convergence factor for $\{\|e[k]\|\}$ is $r^*_{\text{asmp}}$ in (2.30).

The proof of Theorem 2 will need the following lemmas. Generally speaking, Lemma 1 is used to prove the fairness of the DBAP quantization policy [74]. A proof of Lemma 1 can be found in the appendix as Lemma 10. Lemma 2 is a matrix theory needed for our derivation.

**Lemma 1** For any $i_1, j_1, i_2, j_2$, there exists a finite constant $r$, such that

$$r^{-1} \leq \frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r.$$  

(2.36)

The following lemma is a standard result from matrix computation theory [99].

---

1By $f(m) = O(g(m))$ as $m \to \infty$, we mean that $0 \leq f(m) \leq \sigma g(m)$ for all $m$ sufficiently large, where $\sigma$ is a positive constant.
Lemma 2  Let $J$ be an upper bi-diagonal complex $p \times p$ matrix of the form

$$J = \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \ddots & \ddots \\
& \ddots & \ddots & \lambda \\
0 & 0 & \ddots & \ddots \\
\end{bmatrix}$$

(2.37)

$v = [v^1, \ldots, v^p]^T \in \mathbb{R}^p$, where $v^i \in \mathbb{R}$, then

$$\|J^m v\| \sim \left(\frac{m}{p-1}\right) \left(|\lambda|^{m-(p-1)p^2} + o\left(\frac{1}{m}\right)\right).$$

(2.38)

Proof 2

$$J^m = \begin{bmatrix}
\lambda^m & (m)\lambda^{m-1} & \ldots & (m)\lambda^{m-(p-1)} \\
0 & \lambda^m & \ldots & (m)\lambda^{m-(p-2)} \\
& \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \lambda^m \\
\end{bmatrix}$$

(2.39)
If we take the norm of both sides of the above equation, we get (2.38).

With the above two supporting lemmas, we are now ready to prove the Theorem 2.

**Proof 3 (Theorem 2)** Without loss of generality, let us assume that

\[ |v_{i,j}^{n_i}[0]| = v_c \neq 0. \quad (2.40) \]

For the time interval \([0, k]\), let us assume \(v_{i,j}\) is quantized exact \(n_{i,j}\) times, and
combine with (2.38) for sufficient large $k$,

$$
\|v_{i,j}[k]\| = \frac{1}{Q^{n_{i,j}}_i} \|J_i^k v_{i,j}[0]\|
\approx \frac{1}{Q^{n_{i,j}}_i} \left( \frac{k}{n_i - 1} \right) (|\lambda_i|^{k-(n_i-1)} v_{i,j}[0] + o\left(\frac{1}{k}\right)).
$$

from our previous lemma, $\|v_{i,j}[k]\|$ are balanced. Let $k$ go to infinity,

$$
r^{-1} \leq \frac{1}{Q^{n_{i,j}}_i} \left( \frac{k}{n_i - 1} \right) |\lambda_{i_{11}}|^{k-(n_{11}-1)} v_{i_{11},j_1}[0] \leq r
$$

i.e.,

$$
r^{-1} \leq \frac{Q^{n_{i,j}}_i}{Q^{n_{11},j_1}_{i_{11}}} \left( \frac{k}{n_i - 1} \right) |\lambda_{i_{11}}|^{k-(n_{11}-1)} \leq r
$$

$$
k^{-n_{11}+1} \left( \frac{\lambda_i}{\lambda_{i_{11}}} \right)^k \leq \frac{Q^{n_{i,j}}_i}{Q^{n_{11},j_1}_{i_{11}}} \leq k^{n_{11}-1} \left( \frac{\lambda_i}{\lambda_{i_{11}}} \right)^k
$$

We can multiply the above equations over all $(i_1, j_1) \neq (i, j)$, we have

$$
k^{-n_{\max n_i-1}} \frac{\lambda_i^{nk}}{|\det A|^k} \leq \frac{Q^{n_{i,j}}_i}{Q^k} \leq k^{n_{\max n_i-1}} \frac{\lambda_i^{nk}}{|\det A|^k}
$$

$$
\begin{align*}
n_{i,j} - k & \geq -n_{\max n_i-1} \log_Q k + k \log_Q \frac{\lambda_i}{|\det A|^k} \\
n_{i,j} - k & \leq n_{\max n_i-1} \log_Q k + k \log_Q \frac{\lambda_i}{|\det A|^k}
\end{align*}
$$

i.e.,

$$
n_{i,j} \sim \frac{k}{n} + \frac{k}{n} \log_Q \lambda_i^n |\det A| + O((\max n_i-1) \log_Q k)
$$

\footnote{By $f(m) = O(g(m))$ as $m \to \infty$, we mean that $0 \leq f(m) \leq \sigma g(m)$ for all $m$ sufficiently large, where $\sigma$ is a positive constant.}
if take the limit as \( k \) goes to infinity,

\[
\tau_{i,j} = \lim_{k \to \infty} \frac{n_{i,j}}{k} = \frac{1}{n} + \frac{1}{n} \log_q \left( \frac{\lambda_i^n}{|\det A|} \right) = \tau_{i,j}^* \tag{2.46}
\]

thus we have proven \( R_{i,j} = \tau_{i,j}^* R \).

\[
\|v_{i,j}[k]\| = \frac{1}{Q_{n_{i,j}}} \| J_i^k v_{i,j}[0]\| \sim \frac{1}{Q_{n_{i,j}}} \left( k \left| \lambda_i \right|^{k-(n_i-1)} v_{i,j}^{n_i}[0] + o\left( \frac{1}{k} \right) \right) \leq \frac{1}{Q_{n_{i,j}}} \left( \max_{n_i} k \left| \lambda_i \right|^{k-(n_i-1)} v_{i,j}^{n_i}[0] + o\left( \frac{1}{k} \right) \right) \leq \kappa \left[ \frac{|\det A|}{Q} \right]^{\frac{1}{n}} k^{n_{i}-1} \|v_{i,j}[0]\| \leq \kappa \left[ \frac{|\det A|}{Q} \right]^{\frac{1}{n}} k^{n_{i}-1} d_{\max}(U[0])
\]

where (a) follows from (2.45), and \( \kappa \) is some constant, and (b) follows since \( U[k] \) is a parallelogram with sides \( v_{i,j}[k] \).

The triangle inequality implies \( d_{\max}(U[k]) \)

\[
d_{\max}(U[k]) \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i} \|v_{i,j}[k]\| \leq n \kappa \left[ \frac{|\det A|}{Q} \right]^{\frac{1}{n}} k^{\max n_i-1} d_{\max}(U[0])
\]

Since \( \|e[k]\| \leq \|d_{\max}(U[k])\| \), and according to our definition (2.1), it is straight-
forward to prove that the asymptotic convergence rate for $\|e[k]\|$ is $r^*_\text{asmp}$.

**Remark 5** In [94], a similar bound in terms of the convergent rate of the quantization error is derived, namely

$$\|e[k]\| \leq \kappa k^{(\max_i n_i - 1)} 2^{-k(\min_i (R_i - \log |\lambda_i(A)|))}$$

Note that the DBAP policy will force the error to converge at a faster rate because of the dynamic and balanced property of the DBAP quantization policy. Besides, the convergence rate of $d_{\text{max}}(U[k])$ is actually tighter compared to the one found in [74] and approaches $r^*_\text{asym}$ from the MMEP problem.

### 2.5 LQR Performance Limitation of Quantization System

In this section, we try to characterize the optimal regulator problem of the quantized system with bit-rate constraints.

The classical linear discrete-time optimal regulator problem can be defined as follows. Consider the same linear system:

$$x[k+1] = Ax[k] + Bu[k], \quad k = 0, 1, \ldots, N - 1$$

and quadratic cost

$$x'[N]Q_Nx[N] + \sum_{k=0}^{N-1} (x'[k]Qx[k] + u'[k]Ru[k]).$$

We refer to this as the time-invariant discrete-time LQR problem. It is well-known
that the optimal input is given by

\[ u = L[k]x[k], \quad k = 0, \ldots, N - 1 \quad (2.48) \]

where

\[ L[k] = -(B'K_{k+1}B + R)^{-1}B'K_{k+1}A. \quad (2.49) \]

Here the inverse always exists and the symmetric positive matrices \( K_k \) are given recursively by the algorithm

\[ K_k = A'(K_{k+1} - K_{k+1}B(B'K_{k+1}B + R_k)^{-1}B'K_{k+1})A + Q_k \quad (2.50) \]
\[ K_N = Q_N \quad (2.51) \]

and the optimal cost is given by

\[ J_0(x_0) = x_0'K_0x_0. \quad (2.52) \]

For the quantized LQR problem, the exact state information is not available for the controller, since only the quantized version \( x^q[k] \) is available. As mentioned in the problem formulation, the quantized system can be described as follows,

\[ x[k + 1] = Ax[k] + Bu[k] \quad (2.53) \]
\[ u[k] = K[k]x^q[k], \quad (2.54) \]
such that

\[
x[k + 1] = Ax[k] + BK[k]x^q[k]
\]

\[
= Ax[k] + BK[k]x[k] + BK[k](x^q[k] - x[k])
\]

\[
= Ax[k] + Bu[k] + BK[k]e[k],
\]

where \(e[k]\) is the quantization error.

It is reasonable to assume that the quantization error is independent. From the DBAP quantization schemes, we know the supporting region of \(e[k]\) is fully bounded \(v_{i,j}^q[k]\). If we introduce the perturbation term \(w[k]\) defined as follows to the classical LQR problem, the perturbed LQR problem can be formulated as

\[
x[k + 1] = Ax[k] + Bu[k] + w[k] \quad (2.55)
\]

\[
w[k] = BK[k]e[k]. \quad (2.56)
\]

To characterize the LQR performance of the quantized systems, we need the following assumptions:

- \(e[k]\) is independent.
- \(e_{i,j}[k]\) has a normal distribution with variance \(v_{i,j}[k]\), i.e., \(e_{i,j}[k] \sim N(0, (v_{i,j}^q[k])^2)\).

In this way, the quantized system can be represented by the following linear model:

\[
\bar{x}[k + 1] = A[k]\bar{x}[k] + B[k]u(k) + w[k] \quad (2.57)
\]

\[
\bar{x}[0] = x_0. \quad (2.58)
\]
where $\bar{x}[k] \in \mathbb{R}^n$ is the $n$-dimensional state vector, $u[k] \in \mathbb{R}^m$ is the $m$-dimensional control vector, and $w[k]$ is the process noise vector with $n$-components.

From the assumptions on the distribution of $e[k]$, we know $w[k]$ has a zero-mean normal distribution,

$$
\mathbb{E}[w[k]w'[k]] = (BK_kv^q[k])(BK_kv^q[k])' = W[k]
$$

with initial condition

$$
\mathbb{E}[x[0]] = x_0
$$

$$
\mathbb{E}[e[0]e'[0]] = (v_0)^2,
$$

where $w[k], k = 0, \ldots, N - 1$, constitutes a sequence of uncorrelated, zero-mean stochastic variables with variance matrices $W[k], t = 0, \ldots, T - 1$. The original LQR problem can be formulated as a problem of minimizing the criteria

$$
\mathbb{E}\left\{\sum_{k=0}^{N-1} \{\bar{x}[k]Qx[k] + u^T[k]Ru[k] + \bar{x}[N]^TQ_N\bar{x}[N]\}\right\},
$$

where $Q > 0, R > 0$ and $Q_N > 0$. It is formulated as a stochastic discrete-time linear optimal regulator problem. If all the matrices in the problem formulation are constant, it is normally referred as the time-invariant stochastic discrete-time linear optimal regulator problem.

With the above formulation, we can quantify the performance of the original LQR problem subject to bit-rate constraint as the following theorem.

**Theorem 3** Consider the quantized system described by (2.53), (2.54). If we
assume that the step-wise quantization error $e_{i,j}[k]$ has IID zero-mean Gaussian distribution with variance $v_{i,j}^2[k]$, the criteria (2.63) is minimized by choosing the input according to the control law

$$u[k] = L_k x[k], \quad k = 0, \ldots, N - 1 \quad (2.64)$$

where the gain matrix $L_k$ are given by the backward equations (2.50) with terminal condition (2.51). Furthermore, the optimal value achieved with this control law is given by

$$\text{Tr}\{K_0X_0 + V_0 \sum_{k=0}^{N-1} K_{k+1}BK_k(BK_k)'(AA')^k \prod_{t=0}^{k} Q^2(I,J,t)\}$$

where $X_0 = x_0x_0'$ and $V_0 = v_0v_0'$.

**Proof 4** With our assumption on the Gaussian property of the quantization error, the original LQR problem is formulated as a stochastic optimal regulator problem. It is well-known that the classical LQR problem and the stochastic optimal regulator share the same feedback matrix $[23],[24]$; Besides, the optimal value is given by

$$J_0(x_0) = x_0'K_0x_0 + \sum_{k=0}^{N-1} \mathbb{E}[w'_kK_{k+1}w_k]. \quad (2.65)$$

Due to the "uncertainty" comes from initial state, our estimation error resulting from bit-rate constraints has a decreasing effect on our estimation of the current state $x[k]$, and here the term $e[k]$ represents our quantization/estimation error of
the true state $x[k]$. Notice here $w[k+1]$ has the following representation,

$$w[k+1] = BK_kQ(I,J)Av^q[k]$$

$$= BK_k \prod_{t=0}^{k} \{(Q(I,J,t)A)\}v^q[0],$$

where $Q(I,J)$ is diagonal as follows

$$Q(I,J,t) = Diag\{1,...,1\frac{1}{Q},...,1\},$$

and the $\frac{1}{Q}$ appears at the $i$-th Jordan block and $j$-th position at time $t$. We can reasonably assume that $w[k]$ are uncorrelated white noise and their variance are related by the recursive relationship bia (2.66).

With all this said, based on the results from linear quadratic control, we have the quadratic performance of the quantization problem as follows,

$$J_0(x_0) = x_0'K_0x_0 + \sum_{k=0}^{N-1} \mathbb{E}w_k'K_{k+1}w_k$$

$$= x_0'K_0x_0 + \sum_{k=0}^{N-1} \mathbb{E}\{\text{Tr}\{K_{k+1}w_kw'_k}\}\}$$

$$= x_0'K_0x_0 + \sum_{k=0}^{N-1} \text{Tr}\{K_{k+1}\mathbb{E}w_kw'_k\}\}$$

$$= x_0'K_0x_0 + \sum_{k=0}^{N-1} \text{Tr}\{K_{k+1}\mathbb{E}\{[BK_k\prod_{t=0}^{k} Q(I,J,t)Av_0][BK_k\prod_{t=0}^{k} Q(I,J,t)Av_0]''\}\}$$

$$= \text{Tr}\{K_0X_0 + V_0\sum_{k=0}^{N-1} K_{k+1}BK_k(BK_k)'(AA')^k \prod_{k=0}^{k} Q^2(I,J,t)\}$$

Clearly, the cost the performance of the quantization system has two parts; one part comes from the standard LQR problem denoted by $x_0K_0x_0'$, the other
part comes from the quantization error due to initial uncertainty and the step-wide quantization error. Notice here that the magnitude of $e[k]$ can be bounded by specific convergent geometric series given that $R > R_{\text{min}}$. In this way, we can quantify the performance degeneration as

$$\text{Tr}\{V_0 \sum_{k=0}^{N-1} K_{k+1}BK_k( BK_k)'( AA' )^k \prod_{k=0}^{k} Q^2(I, J, t) \},$$

since $K_i, i = 0, \ldots, N$ can be solved off-line given the weighting matrix $Q, R, Q_N$, and $V_0$ is the initial uncertainty set. We can formulate the following optimization problem to minimize the network issues,

$$\min_{Q(i, j, t); t=0, \ldots, N-1} \text{Tr}\{V_0 \sum_{k=0}^{N-1} K_{k+1}BK_k( BK_k)'( AA' )^k \prod_{k=0}^{k} Q^2(I, J, t) \}$$

where $Q(I, J, t)$ is defined by (2.68). At the current stage, we do not know how to solve the above optimization problem in terms of the optimal bit-assignment policy to minimize the quantization effect. The problem itself is an Integer Programming (IP) problem which has $n^N$ candidate solutions. To solve it, we might need some information on the $K_k$ matrices derived from the backward equations. We expect analytical solution of the above optimization problem will lead us to some smart, dynamical, optimal quantizers.

2.6 Conclusion

In this chapter, given bit rate $R$, we derived the best convergence rate for quantization error and the associated resource allocation strategy via the solution of a min-max eigenvalue problem associated with the quantized system. It is further proven that the result from our MMEP formulation is achievable through a dynamic bit assignment policy (DBAP) [74]. Our result provides the theoretical foundation for the dynamic quantization policy to increase its error convergence
rate.

The theoretical bit-rate low bound $R_{\text{min}}$ has been derived in the last decade. Currently, researchers are trying to find the inherent trade-offs between control and communication performance. Here we are interested in the degradation of control performance introduced by imperfect communication channels. Our result on the performance of LQR controllers with bit-rate constraints is among the first few results reported in this area to quantify the communication constraint on the performance of classical controllers. Because of the intrinsic difficulties with the optimization problems (IP programming), we have not come up with an analytical solution to guide our design of optimal quantizers. It is possible, for some carefully selected weighting matrix $Q, R$, that there might be some simple analytical solutions. We hope that our approach via optimizing the allocation of the communication resources might be a powerful approach to address the related performance problems.
3.1 Introduction

Spatially invariant systems have been an active topic of research in recent years [76], [79], [17], [50], [89], [5], [20], [44]. Such systems are composed by similar units which directly interact with their neighbors. These systems arise in many applications, such as the control of vehicle platoons [57], airplane formation flight control [31], cross-directional control in paper processing applications [89], and recent distributed control applications at a microscopic scale based on advances in micro electro-mechanical systems (MEMS) [6].

An important aspect of many of these systems is that sensing and actuation capabilities exist at each node. Although each node may be simple, their interdependence makes the resulting system display complex behavior. This brings new challenges to control theory, since standard methods cannot handle systems of such high dimension. Besides, it is not feasible to control such systems with centralized schemes since the centralized controller needs all the nodes’ state information, which makes such a controller impractical.

Since the spatially invariant systems can be diagonalized by a Fourier transform over the spatial domain, by the Plancherel’s theorem, the control design problem with quadratic criteria can be decoupled over spatial frequency, i.e, standard finite
dimensional theorems may be applied at each spatial frequency [5]. It is further shown in [5] that the optimal controller has an inherent degree of decentralization, which weighs the information coming from neighbors using a gain exponentially decaying with distance. Based on these properties, [20] presented a conservative method to impose localization in controller design for such systems, together with some sufficient condition for the $H_2$ problem which takes the form of Linear Matrix Inequalities (LMI) over each spatial frequency. In the one-dimensional case, by means of the Kalman-Yakubovich-Popov (KYP) lemma [75], these conditions can be further expressed as some LMIs independent of spatial frequency. In the first part of this chapter, we will use Cauchy estimation formulas for analytic complex functions to prove that bounded spatially invariant operators have exponentially decaying coefficients [27]. This result can be regarded as the discrete-time counterpart of the results appeared in [5] for continuous-time systems. A similar proof for the discrete-time case can be found in [63].

In another related line of work [76], analysis and synthesis results are developed for this class of systems using the $l_2$-induced norm as the performance criterion. With the introduction of a shift operator, a KYP-like lemma is obtained for the analysis and synthesis of controllers with rational spatial frequency dependence and $H_\infty$ norm guarantees. Methods of structured uncertainty analysis are extended to systems with dynamical and non-causal spatial coordinates in [17], and design techniques of robust spatially distributed controllers for paper machines are considered in [89]. The notion of loopshaping is extended to two-dimensional systems in [89]. In [48] the effect of structured uncertainty in terms of data dropouts for spatially invariant sensor-actuator networks is considered.

In the second part of this chapter, we will use a unified approach to address the
$H_{\infty}$ optimal control of spatially invariant systems. Since the underlying dynamics of these systems are spatially invariant, the interconnected system admits a linear fractional representation which can be represented by an interconnection of a generalized state-space realizations and a specific structural uncertainty which involving both the spatial and temporal variables. Stability and performance of such systems all reduce to one well-posedness problem, which can be efficiently solved by a method that involves searching for a quadratic separator. Based on paper of [40], we try to prove that both the internal stability condition derived in [?], and the $H_{\infty}$ analysis result in [76] can be derived through this approach, which yields conditions that can be expressed as linear matrix inequalities (LMIs), which can be solved easily using efficient interior-point methods.

The third part of this chapter considers the discrete time decentralized control of one-dimensional spatially invariant systems perturbed by white noise which is motivated by [20]. Note that examples of one dimensional spatially invariant systems includes platoons [57] and Cross Directional (CD) control in the chemical process industry [89]. Previous work ([76], [5] and [20]) has been more concerned with the induced gain for the overall system with the assumption that at fixed time the signal (noise, state) is square summable ($l_2$) in the spatial domain, as discussed in the second part of this chapter. We take a stochastic approach here with the assumption that the noise at each node is a white noise. A necessary and sufficient condition is obtained from the boundedness of the solution to a discrete-time Lyapunov equation across spatial frequency, which corresponds to the continuous-time stability condition in [5]. We further show how to convert those frequency-dependent stability conditions to finite dimensional LMIs using a result on non-negative pseudo-polynomial matrix form [34].
This chapter is organized as follows. Some preliminary concepts and the system model are first introduced in section 3.2; the exponentially decaying property of the bounded shift-invariant operators is also established in this part. In section 3.3, we use a unified approach to analyze the $H_\infty$ control of spatially invariant systems. In section 3.4, we study the stability of one-dimensional array perturbed by white noise. Stability of decentralized controller is also presented. Concluding remarks and future research are described in section 3.5.

3.2 Background

The notation in this chapter is fairly standard. Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ denote the set of real numbers, complex numbers and integers, and $\mathbb{R}_+$ be the set of non-negative real numbers. The space of $n$ by $m$ matrices in real and complex fields is denoted by $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$. The $n$ by $n$ identity matrix is denoted $I_n$. $\mathbb{R}_+^{n \times n}$ denotes the set of symmetric $n$ by $n$ real matrices, and $M > 0$ denotes a positive definite matrix, i.e, $x^* M x > 0, \forall x \neq 0$. The matrix singular value of $A \in \mathbb{C}^{n \times m}$ is denoted $\bar{\sigma}(A)$. $A^*$ denotes the complex conjugate transpose of matrix $A$.

For consistency, we utilize much of the same notation as in [76]. We are dealing with signals which are vector valued functions indexed by $L + 1$ independent variables, $d = (t, s_1, s_2, \ldots, s_L)$, where the $L$–tuple$(s_1, s_2, \ldots, s_L)$ is denoted by $s$. We restrict the temporal variable $t$ to be in $\mathbb{R}_+$, and $s_i \in \mathbb{Z}$, which captures infinite spatial extent in dimension $i$, i.e, the system is causal for temporal variable $t$, while non-causal with respect to the spatial variable $s_i$.

For fixed $t$, let the space $\ell_2$ be the set of functions for which:

$$\sum_{s_1 = -\infty}^{\infty} \sum_{s_L = -\infty}^{\infty} x^*(s)x(s) < \infty.$$  \hspace{1cm} (3.1)
Space $\mathcal{L}_2$ denotes the set of functions for which:

$$
\int_0^\infty \|u(t)\|_{\ell_2}^2 dt < \infty.
$$

(3.2)

The inner product on $\ell_2$ is given by

$$
<x, y>_{\ell_2} = \sum_{s_1=\infty}^{\infty} \cdots \sum_{s_L=\infty}^{\infty} x^*(s)x(s),
$$

(3.3)

where the inner product on $\mathcal{L}_2$ is given by

$$
<u, v>_{\mathcal{L}_2} = \int_0^\infty <u(t), v(t)>_{\ell_2} dt.
$$

(3.4)

The corresponding norms on $\ell_2$ and $\mathcal{L}_2$ are simply the square roots of their inner products, i.e, $\|a\| = \sqrt{<a,a>}$. Notice here, the signal space of $\ell_2$ and $\mathcal{L}_2$ are different: $\ell_2$ denotes the signal space with signals square-integrable in the spatial domain when $t$ is fixed; $\mathcal{L}_2$ denotes the signal space with signals square-integrable both in the spatial and temporal domains. By this definition $\mathcal{L}_2 \subset \ell_2$.

The induced gain of an operator $F$ on $\ell_2$ is given by:

$$
\|F\|_{\ell_2} = \sup_{x \in \ell_2, x \neq 0} \frac{\|Fx\|_{\ell_2}}{\|x\|_{\ell_2}}.
$$

(3.5)

Spatial shift operator $S_i$ on $\ell_2$ are given as

$$
(S_i u)(t,s) = u(t, s_1, \cdots, s_i + 1, \cdots, s_L), i = 1, \cdots, L.
$$

(3.6)

$S_i$ can be viewed as a complex variable, and $S_i^* S_i = 1$. One can verify that $\|S_i\|_{\ell_p \rightarrow \ell_p} = 1$. Higher order translation operators can be defined iteratively by
Definition 1  Operator $Q : \mathcal{D}(Q) \to \ell_p$ with domain $\mathcal{D}(Q) \subset \ell_p$ is translation invariant if it commutes with every translation operator $S^n : \mathcal{D}(Q) \to \mathcal{D}(Q)$, i.e., $S^n Q = QS^n$ for all $n \in \mathbb{Z}$.

It can be shown that all linear translation invariant operators on $\ell_p$ can be characterized by forming linear combinations of higher order translation operators of the form

$$Q(S) = \sum_{k \in \mathbb{Z}} Q_k S^k,$$

with $Q_k \in \mathbb{C}^{n \times n}$.

For every $x \in \ell_2$, the discrete Fourier transform is defined as

$$\hat{x}(z) = \sum_{k \in \mathbb{Z}} x_k z^{-k}.$$

where $z \in \mathbb{S}^1$. Using this definition, one can compute the discrete Fourier transform of a translation invariant operator in terms of $Q_k$ in (3.7).

According to [100], a translation invariant operator $Q$ is bounded on $\ell_2$ if and only if

$$\|Q\|_{\ell_2 \to \ell_2} = \sup_{z \in \mathbb{S}^1} \|\hat{Q}(z)\| < \infty.$$

(3.9)

The following decay result is similar to the Cauchy estimation formulas for analytic complex functions [27].

**Theorem 4** Let $Q$ be defined by (3.7) on $\mathbb{Z}$ with discrete Fourier transform $\hat{Q}(z)$. If $Q \in \mathcal{L}(\ell_2)$, then the coefficients of operator $Q$ decay exponentially in the spatial domain, i.e., for all $k \in \mathbb{Z}$

$$\|Q_k\| \leq \alpha e^{-\beta |k|},$$

(3.10)
for some $\alpha > 0$, and $0 < \beta < \ln(1 + r)$, where $r$ is the distance of the closest pole of $\hat{Q}(z)$ to $S^1$.

**Proof 5** According to (3.9), if $Q \in L(\ell_2)$, then $\hat{Q}(z)$ has no poles on $S^1$, and it is analytic on some annulus

$$\mathcal{A}(0, 1 - r, 1 + r) = \{z \in \mathbb{C} : 1 - r < |z| < 1 + r, r > 0\},$$  \hspace{1cm} (3.11)

where $r$ is the distance of the nearest pole of $\hat{Q}(z)$ to $S^1$. Then one can represent $Q$ as a Laurent series which normally converge in this annulus,

$$\hat{Q}(z) = \sum_{k \in \mathbb{Z}} a_k z^k \quad \text{for } z \in \mathcal{A}.$$  \hspace{1cm} (3.12)

Since Laurent representation is unique, we have $a_k = Q_{-k}$; moreover, $a_k$ is given by

$$a_n = \frac{1}{2\pi i} \oint_{|\varsigma| = \rho} \frac{\hat{Q}(\varsigma)}{(\varsigma - a)^{n+1}} d\varsigma, \quad n \in \mathbb{Z}, \quad 1 - r < \rho < 1 + r.$$  \hspace{1cm} (3.13)

Since $\hat{Q}$ is bounded, if we set $M_\rho(Q) = \sup\{|\hat{Q}(\varsigma)|; |\varsigma| = \rho\}$, then there hold the following Cauchy estimation formula:

$$|a_n| \leq \frac{M_\rho(Q)}{\rho^n}, \quad n \in \mathbb{Z}.$$  \hspace{1cm} (3.14)

If we choose $\rho = (1 + r)$ for $n \geq 0$ and $\rho = (1 - r)$ for $n < 0$ in the above Cauchy estimation formula, the decay result (3.10) can be obtained immediately.

Theorem 4 provides the theoretical foundation for the feasibility of those truncated controllers which only use neighboring information for controller designs. This approach can approximate the optimal controller only if the weighting for
further information has a fast decaying rate, which contributes little to the control effort.

3.3 $H_\infty$ Control of Spatially Invariant Systems

Consider the finite dimensional, linear time-invariant system governed by the following state-space equations:

$$
\begin{bmatrix}
\dot{x}(t,s) \\
w(t,s) \\
z(t,s)
\end{bmatrix} =
\begin{bmatrix}
A_{TT} & A_{TS} & B_T \\
A_{ST} & A_{SS} & B_S \\
C_T & C_S & D
\end{bmatrix}
\begin{bmatrix}
x(t,s) \\
v(t,s) \\
z(t,s)
\end{bmatrix}
$$

$$x(0,s) = x_0(s) \in \mathbb{R}^r , \tag{3.15}$$

where

$$v(t,s) = \begin{bmatrix} v_+(t,s) \\ v_-(t,s) \end{bmatrix}, \quad w(t,s) = \begin{bmatrix} w_+(t,s) \\ w_-(t,s) \end{bmatrix} \tag{3.16}$$

$$A_{TS} = \begin{bmatrix} A_{TS_1} & A_{TS_{-1}} \end{bmatrix}$$

$$A_{ST} = \begin{bmatrix} A_{ST_1} \\ A_{ST_{-1}} \end{bmatrix}$$

$$A_{SS} = \begin{bmatrix} A_{SS_{1,1}} & A_{SS_{1,-1}} \\ A_{SS_{-1,1}} & A_{SS_{-1,-1}} \end{bmatrix}$$

$$C_S = \begin{bmatrix} C_{S_1} & C_{S_{-1}} \end{bmatrix} \tag{3.17}$$

We assume that $v_+(t,s)$ and $w_+(t,s)$ are of the same size, and that $v_-(t,s)$ and $w_-(t,s)$ are of the same size. For simplicity, we restrict the spatial variable $s$ to be
of dimension 1; models of high dimensional interconnected systems can be found in [76].

Let us consider the following infinite interconnections, such that:

\[ v_+(s + 1) = w_+(s), \quad v_-(s - 1) = w_-(s), \quad \forall s \in \mathbb{Z} \]  

(3.18)

In (3.15), let \( m_0 \) denote the size of the subsystem states \( x(t, s) \), \( m_+ \) the size of interconnection variables \( v_+(t, s) \) and \( w_+(t, s) \), and \( m_- \) the size of interconnection variables \( v_-(t, s) \) and \( w_-(t, s) \).

Let \( \mathbf{m} = (m_0, m_+, m_-) \), and define the following structured operator on \( l_2 \):

\[
\Delta_{\mathbf{s}, \mathbf{m}} = \begin{bmatrix}
SI_{m_+} & 0 \\
0 & S^{-1}I_{m_-}
\end{bmatrix}, \quad (3.19)
\]

where \( \mathbf{S} \) denotes the spatial shift operator (3.6) for the one dimensional spatial variable \( s \), and the role of \( m_0 \) in \( \mathbf{m} \) will become apparent shortly. We further assume that

\[ m_+ = m_- \]  

(3.20)

in this chapter. Although generally we do not need this restriction, from (3.18), we know that

\[ w(t, s) = \Delta_{\mathbf{s}, \mathbf{m}} v(t, s). \]  

(3.21)

By eliminating the interconnected variable \( v \) and using the assumption of initial state \( x(0, s) = 0 \), we can further express the system as:

\[
\dot{x}(t, s) = Ax(t) + Bd(t) \]  

(3.22)

\[
z(t, s) = Cx(t) + Dd(t) \]  

(3.23)
where

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = 
\begin{bmatrix}
A_{TT} & B_T \\
C_T & D
\end{bmatrix} + 
\begin{bmatrix}
A_{TS} \\
C_S
\end{bmatrix}
\times (\Delta_{S,m} - A_SS)^{-1} 
\begin{bmatrix}
A_{ST} & B_S
\end{bmatrix},
\]

(3.24)

Here it is assumed that \((\Delta_{S,m} - A_SS)\) is invertible on \(l_2\), which is equivalent to assuming that the interconnection is well-posed.

Now let us consider \(m_0\) in \(m\) and introduce the following operator (3.26) and signal \(r\). Notice here the bold and normal font of \(A, B, C, D\) which denote two pairs of matrices. Define

\[
A = \begin{bmatrix}
A_{TT} & A_{TS} \\
A_{ST} & A_{SS}
\end{bmatrix},
B = \begin{bmatrix}
B_T \\
B_S
\end{bmatrix},
C = \begin{bmatrix}
C_T & C_S
\end{bmatrix}
\]

(3.25)

and

\[
\Delta_m = \text{diag}(\frac{d}{dt}I_{m_0}, SI_{m_+}, S^{-1}I_{m_-})
\]

(3.26)

where the shift operator \(S\) is extended to \(\mathcal{L}\).

Given constant matrices \(A, B, C\) and \(D\) of compatible dimension, and the three-tuple \(m\), we may thus express a system in the following succinct form:

\[
\begin{bmatrix}
(\Delta_m r)(t, s) \\
z(t, s)
\end{bmatrix} = 
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} 
\begin{bmatrix}
r(t, s) \\
d(t, s)
\end{bmatrix},
\]

(3.27)

where \(r(t, s) = (x(t, s), v(t, s))\). The system generated by \(A, B, C, D\) and \(m\) is defined by \(\mathcal{M} = \{A, B, C, D, m\}\). For simplicity, \(\mathcal{M}\) denote the LTI system described by (3.15).
The next two results are needed for our development and are included here for convenience. An important concept behind robust stability is the notion of topological separation; a system is stable if and only if the graph of $G$ and the inverse graph of uncertainty $\Delta$ are topologically separated [83]. Topological separation of the feedback system can be established by searching for an appropriate symmetric matrix. The following result from [40] deals with robust analysis with respect to generic uncertainties.

**Lemma 3** Given a set of matrices $\nabla \subset \mathbb{C}^{n \times m}$ and a matrix $M \subset \mathbb{C}^{m \times n}$, the following are equivalent:

- The inverse of $(I - M\Delta)$ exists for all $\Delta \in \nabla$.

- There exists a matrix $\Theta \in \mathbb{C}^{(m+n)\times(m+n)}$ such that the following inequalities are satisfied for all $\Delta \in \nabla$:

\[
\begin{bmatrix} M \\ I \end{bmatrix}^* \Theta \begin{bmatrix} M \\ I \end{bmatrix} < 0 \tag{3.28}
\]

\[
\begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Theta \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0. \tag{3.29}
\]

If $M$ is a real matrix, then the above equivalence still holds when $\Theta$ is restricted to a real matrix.

We present the celebrated Kalman-Yakubovich-Popov (KYP) lemma [75], through which frequency dependent matrix inequalities can be solved by first replacing the frequency dependence with a new matrix valued decision variable, then solving the resulting inequality numerically. The reason for this replacement is the de-
velopment of interior point algorithms for convex optimization with constraints defined by linear matrix inequality [70], [81].

**Lemma 4 (KYP lemma)** Given two matrices $A, B$ and a matrix $M = M^*$, with $\det(j\omega - A) \neq 0, \forall \omega \in \mathbb{R}$, the following two statements are equivalent:

- \[
\begin{pmatrix}
    (j\omega I - A)^{-1}B \\
    I
\end{pmatrix}^* M \begin{pmatrix}
    (j\omega I - A)^{-1}B \\
    I
\end{pmatrix} < 0, \forall \omega \in \mathbb{R}.
\]

- There exists a matrix $\mathcal{X} = \mathcal{X}^*$ such that $M + \begin{pmatrix}
    \mathcal{A}^* \mathcal{X} + \mathcal{X} A & \mathcal{X} B \\
    B^* \mathcal{X} & 0
\end{pmatrix} < 0$

### 3.3.1 Analysis of Internal Stability

Now let $G_{wv}(s)$ denote the transfer function from $v$ to $w$, that is:

$$G_{wv}(s) = A_{ST}(sI - A_{TT})^{-1}A_{TS} + A_{SS}. \quad (3.30)$$

Assume the system $G_{wv}(s)$ is stable. Then the interconnected system is well-posed if and only if the inverse

$$(I - G_{wv}(s)\Delta s_m)^{-1}$$

exists (in $RH_\infty$) for each $S$ on the unit circle, i.e,

$$\|G_{wv}\|_{l_2} = \sup_{|z|=1} \|G_{wv}(z)\| < \infty. \quad (3.31)$$

According to Lemma 3, $(I - G_{wv}(s)\Delta s_m)$ is invertible if and only if there exists
some separator $\Theta(\omega, S)$, such that

$$
\begin{bmatrix}
G_{wv}(j\omega) \\
I
\end{bmatrix}^* \Theta(\omega, S) \begin{bmatrix}
G_{wv}(j\omega) \\
I
\end{bmatrix} < 0 \quad (3.32)
$$

$$
\begin{bmatrix}
I \\
\Delta_{s,m}
\end{bmatrix}^* \Theta(\omega, S) \begin{bmatrix}
I \\
\Delta_{s,m}
\end{bmatrix} \geq 0. \quad (3.33)
$$

Consider the specific structure of $\Delta_{s,m}$, since

$$
S = \{zI : z \in \mathbb{C}, zz^* = 1\}, \quad (3.34)
$$

a quadratic separator between $\Delta_{s,m}$ and $G_{wv}(s)$ is

$$
\Theta_S(j\omega) = \begin{bmatrix}
X_S(j\omega) & 0 \\
0 & -X_S(j\omega)
\end{bmatrix}, \quad (3.35)
$$

where $X_S(j\omega) \in \mathbb{C}_s^{m \times m}$.

It is straightforward to verify that

$$
\begin{bmatrix}
I \\
\Delta_{s,m}
\end{bmatrix}^* \Theta_S(j\omega) \begin{bmatrix}
I \\
\Delta_{s,m}
\end{bmatrix} \geq 0. \quad (3.36)
$$

Using Lemma 3, we obtain a local stability analysis condition in the form of integral quadratic constraint (IQC).

**Proposition 1** The array system is well-posed if for every $\omega \in \mathcal{R}$, there exists
Hermitian matrix $X_S(j\omega) \in \mathbb{C}^{m \times m}$, such that the following IQC holds:

$$\begin{align*}
\begin{bmatrix}
    G_{ww}(j\omega)  \\
    I 
\end{bmatrix}^* \Theta_S(j\omega) 
\begin{bmatrix}
    G_{ww}(j\omega)  \\
    I 
\end{bmatrix} < 0 
\end{align*} \quad (3.37)$$

where $\Theta_S(j\omega)$ is defined as (3.35).

**Remark 6** Notice that $X_S(j\omega)$ only need to be Hermitian. Without loss of generality, we assume it can be partitioned $X_S(j\omega)$ as:

$$X_S(j\omega) = 
\begin{bmatrix}
    P(j\omega) & -Z^*(j\omega) \\
    -Z(j\omega) & -Q(j\omega) 
\end{bmatrix} \quad (3.38)$$

where $P(j\omega) > 0, Q(j\omega) > 0$. The quadratic separator $\psi(j\omega)$ derived in [18] is equivalent to $\Theta_S(j\omega)$, and they are related by

$$\psi(j\omega) = M \Theta_S(j\omega) M^T$$

where

$$M = 
\begin{bmatrix}
    I & 0 & 0 & 0 \\
    0 & 0 & 0 & I \\
    0 & 0 & I & 0 \\
    0 & I & 0 & 0 
\end{bmatrix} \quad (3.39)$$
By restricting $\Theta_S(j\omega)$ to be constant across all frequencies, we can get a sufficient condition in terms of LMI for the internal stability of such array systems. Similar ideas has been used in [20] to impose controller localization by restricting a Lyapunov-like matrix to be independent of spatial frequency.

Notice that since $G_{\text{uv}}(s)$ has the form (3.30), we can apply the KYP lemma to (3.37) to obtain the following Theorem 5, where $\Theta_S$ is defined as:

$$
\Theta_S = \begin{bmatrix}
X_S & 0 \\
0 & -X_S
\end{bmatrix}.
$$

(3.40)

It is relatively straightforward to derive the following theorem.

**Theorem 5** The array system is well-posed if there exists Hermitian matrix $X_S \in \mathbb{R}^{(m_++m_-)\times(m_++m_-)}$, $X_T \in \mathbb{R}^{m_0\times m_0}$, $X_T > 0$, such that the following LMI holds:

$$
J_{\Theta_S}(X_T, X_S) < 0
$$

(3.41)

where $J_{\Theta_S}(X_T, X_S)$ is defined in (3.42).

**Proof 6** From Proposition 1, the system is well-posed if there exists $\Theta_S$ as defined
in (3.40), such that \( J_{\Theta}(j\omega) < 0 \), where

\[
J_{\Theta_s}(j\omega) = \left[ \begin{array}{c} G_{ww}(j\omega) \\ I \end{array} \right]^* \Theta_s \left[ \begin{array}{c} G_{ww}(j\omega) \\ I \end{array} \right]
\]

\[
= \begin{bmatrix} A_{ST} & A_{SS} \\ 0 & I \end{bmatrix} \begin{bmatrix} (sI - A_{TT})^{-1}A_{TS} \\ I \end{bmatrix}
\]

\[
\cdot \begin{bmatrix} X_s \\ 0 \\ 0 \\ -X_s \end{bmatrix}
\]

\[
\cdot \begin{bmatrix} A_{ST} & A_{SS} \\ 0 & I \end{bmatrix} \begin{bmatrix} (sI - A_{TT})^{-1}A_{TS} \\ I \end{bmatrix}
\]

\[
= \begin{bmatrix} (sI - A_{TT})^{-1}A_{TS} \\ I \end{bmatrix}
\]

\[
\cdot \begin{bmatrix} A_{ST}^*X_sA_{ST} & A_{ST}^*X_sA_{SS} \\ A_{SS}X_sA_{ST} & A_{SS}X_sA_{SS} - X_s \end{bmatrix}
\]

\[
\cdot \begin{bmatrix} (sI - A_{TT})^{-1}A_{TS} \\ I \end{bmatrix}
\]

\[
< 0.
\]

Applying the KYP lemma to the frequency dependent inequality \( J_{\Theta_s}(j\omega) < 0 \), we get the equivalent matrix inequality:

\[
J_{\Theta_s}(X_T, X_S) = \begin{bmatrix} A_{ST}^*X_sA_{ST} & A_{ST}^*X_sA_{SS} \\ A_{SS}X_sA_{ST} & A_{SS}X_sA_{SS} - X_s \end{bmatrix}
\]

\[
+ \begin{bmatrix} A_{TT}^*X_T + X_T A_{TT} & X_T A_{TS} \\ A_{TS}^*X_T & 0 \end{bmatrix}
\]

\[
< 0.
\]
If we use the quadratic separator $\psi_S$ in (3.37), we obtain the following equivalent corollary.

**Proposition 2** The array system is well-posed if there exists matrices $X_S \in \mathbb{R}_s^{(m_+ + m_-) \times (m_+ + m_-)}$, $X_T \in \mathbb{R}_s^{m_0 \times m_0}$, $X_T > 0$, such that the following LMI holds:

$$J_{\psi_S} = \begin{bmatrix} A_{ST}^+ & A_{SS}^+ \\ A_{ST}^- & A_{SS}^- \end{bmatrix}^* \begin{bmatrix} X_S & 0 \\ 0 & -X_S \end{bmatrix} \begin{bmatrix} A_{ST}^+ & A_{SS}^+ \\ A_{ST}^- & A_{SS}^- \end{bmatrix}$$

$$+ \begin{bmatrix} A_{TT}^T X_T + X_T A_{TT} & X_T A_{TS} \\ (X_T A_{TS})^* & 0 \end{bmatrix} < 0,$$

where

$$A_{ST}^+ = \begin{bmatrix} A_{ST_1} \\ 0 \end{bmatrix},$$

$$A_{ST}^- = \begin{bmatrix} 0 \\ A_{ST_{-1}} \end{bmatrix},$$

$$A_{SS}^+ = \begin{bmatrix} A_{SS_{1,1}} & A_{SS_{1,-1}} \\ 0 & I \end{bmatrix},$$

$$A_{SS}^- = \begin{bmatrix} I & 0 \\ A_{SS_{-1,1}} & A_{SS_{-1,-1}} \end{bmatrix}.$$
Proof 7 Notice here

\[ J_{\psi_s}(j\omega) = \begin{bmatrix} G_{\psi_s}(j\omega) \\ I \end{bmatrix}^* M \begin{bmatrix} G_{\psi_s}(j\omega) \\ I \end{bmatrix} \]

\[ = \left\{ \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}^* \begin{bmatrix} X_S & 0 \\ 0 & -X_S \end{bmatrix} \right\} \]

\[ = \left\{ \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}^* \begin{bmatrix} X_S & 0 \\ 0 & -X_S \end{bmatrix} \right\} \]

\[ = \begin{bmatrix} A_{ST} & A_{SS_{1,1}} & A_{SS_{1,-1}} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \]

\[ = \begin{bmatrix} A_{ST}^T \\ 0 \\ A_{ST_{-1}} \end{bmatrix} \]

\[ = \begin{bmatrix} A_{ST}^T \\ 0 \\ A_{ST_{-1}} \end{bmatrix} \]

\[ < 0 \]

In a similar fashion to the proof of Theorem 1, by applying the KYP lemma to the above frequency dependent inequality, we get Proposition 2.

It is interesting to notice that \( J_{\psi_s}(2, 2) < 0 \) actually guarantees the well-posedness of the operator \((\Delta_S - A_{SS})^{-1}\) on \( \ell_2 \), which will be further analyzed in next section.
3.3.2 Robust Analysis of Spatially Invariant Systems

Now let us consider the realization (3.24). The interconnected system is well-posed if and only if \((\Delta_{S,m} - A_{SS})\) is invertible on \(\ell_2\). Using the quadratic separator \(\Theta_S\), it is straightforward to get the following sufficient condition for the well-posedness of \((\Delta_{S,m} - A_{SS})\).

**Proposition 3** \((\Delta_{S,m} - A_{SS})\) is invertible on \(\ell_2\) if there exists \(X_S \in \mathbb{R}^{(m_+ + m_-) \times (m_+ + m_-)}\), such that

\[
A_{SS}^* X_S A_{SS} - X_S < 0. \tag{3.48}
\]

**Proof 8**

\[(\Delta_{S,m} A_{SS}) = \Delta_{S,m} (I - \Delta_{S,m}^{-1} A_{SS}) \tag{3.49}\]

The inverse of \(\Delta_{S,m}\) is well defined:

\[
\Delta_{S,m}^{-1} = \begin{bmatrix}
S^{-1} I_{m_+} & 0 \\
0 & S I_{m_-}
\end{bmatrix} \tag{3.50}
\]

Thus, \((I - \Delta_{S,m} A_{SS})\) is invertible if and only if \((I - \Delta_{S,m}^{-1} A_{SS})\) is invertible, i.e., the feedback interconnection of the two operators \(\Delta_{S,m}^{-1}, A_{SS}\) is well-posed. Since

\[
\begin{bmatrix}
I \\
\Delta_{S,m}
\end{bmatrix}^* \begin{bmatrix}
X_S & 0 \\
0 & -X_S
\end{bmatrix} \begin{bmatrix}
I \\
\Delta_{S,m}
\end{bmatrix} \geq 0, \tag{3.51}
\]

applying Lemma 1, we know that a sufficient condition for the well-posedness of the feedback interconnection of \(\Delta_{S,m}^{-1}, A_{SS}\) is:

\[
\begin{bmatrix}
A_{SS} & 0 \\
I
\end{bmatrix}^* \begin{bmatrix}
X_S & 0 \\
0 & -X_S
\end{bmatrix} \begin{bmatrix}
A_{SS} \\
I
\end{bmatrix} < 0, \tag{3.52}
\]
i.e., (3.48) holds.

An equivalent well-posedness condition has been derived in [76] using a constructive approach, which can be similarly derived if we choose $\psi$ as the quadratic separator for the inverse image of $\Delta_{s,m}$ and $A_{SS}$.

**Corollary 1** $(\Delta_{s,m} - A_{SS})$ is well-posed if there exists $X_S \in \mathbb{R}_s^{(m_+ + m_-) \times (m_+ + m_-)}$, such that

$$(A_{SS}^+ X_SA_{SS}^+ - (A_{SS}^-)^* X_SA_{SS}^- < 0.)$$

(3.53)

Once the system is well-posed, we can study the exponential stability in the $\ell_2$ domain. This again reduces to one well-posedness problem in terms of the feedback interconnection of the operator $A$ (3.24) and $\Delta_m$ (3.26). In the frequency domain, when talking about the stability, we need the transfer function to be analytic in the right half plane (RHP), i.e., the partial operator $\frac{d}{dt}$ has the following equivalent representation:

$$\frac{d}{dt} = \{sI : s \in \mathbb{C}, s + s^* \geq 0\}.$$

(3.54)

For simplicity, we will denote $\Delta_m$ as

$$\Delta_m = \left\{ \begin{bmatrix} sI_{m_s} & 0 \\ 0 & \Delta_{S,m} \end{bmatrix} : s, S \in \mathbb{C}, s + s^* \geq 0 \right\}.$$

(3.55)

Consider the quadratic separator

$$P_S = \begin{bmatrix}
0 & 0 & X_T & 0 \\
0 & X_S & 0 & 0 \\
X_T & 0 & 0 & 0 \\
0 & 0 & 0 & -X_S
\end{bmatrix}.$$

(3.56)
It is easy to verify that

\[
J_{\Delta m}(P_S) = \begin{bmatrix}
I \\
\Delta_m \\
\end{bmatrix}^* \begin{bmatrix}
P_S \\
I \\
\end{bmatrix}
\begin{bmatrix}
I \\
\Delta_m \\
\end{bmatrix}
\begin{bmatrix}
s \\
0 \\
\end{bmatrix}^* \begin{bmatrix}
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
X_T \\
0 \\
\end{bmatrix} \begin{bmatrix}
I \\
0 \\
\end{bmatrix}
\begin{bmatrix}
I \\
0 \\
\end{bmatrix} \begin{bmatrix}
\Delta_{S,m} \\
I \\
0 \\
I \\
\end{bmatrix}
= \begin{bmatrix}
(s^* + s)X_T \\
0 \\
X_S - \Delta_{S,m}^*X_S\Delta_{S,m} \\
\end{bmatrix}
\geq 0.
\]

Thus we get the following condition for the stability of the system:

**Proposition 4** The interconnected system is stable if there exist symmetric matrices $X_S \in \mathbb{R}_{s}^{(m_+ + m_-) \times (m_+ + m_-)}$, $X_T \in \mathbb{R}_{s}^{m_0 \times m_0}$, $X_T > 0$, such that the following LMI holds:

\[
\begin{bmatrix}
\ast \\
\ast \\
\ast \\
\ast \\
\end{bmatrix}^* \begin{bmatrix}
0 \\
X_T \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
I \\
0 \\
A_{TT} \\
A_{TS} \\
\end{bmatrix} \begin{bmatrix}
I \\
0 \\
0 \\
I \\
\end{bmatrix} < 0.
\] (3.57)
**Proof 9** The stability problem can be again regarded as a well-posedness problem, (3.57) is equivalent to the following LMI,

\[
\begin{bmatrix}
  A \\
  I
\end{bmatrix}^* P_S \begin{bmatrix}
  A \\
  I
\end{bmatrix} < 0
\]  

(3.58)

According to lemma 1, stability thus follows.

However, we give an alternate proof here to emphasize the fact that a decentralized Lyapunov function \( V(x) \) exists to establish exponential stability of such system, let

\[
V(x) = \sum_{s=-\infty}^{\infty} x^T(t,s)X_Tx(t,s).
\]  

(3.59)

Multiplying (3.57) by \( \begin{bmatrix}
  x(t,s) \\
  v(t,s)
\end{bmatrix} \) from the left and by its transpose from the right-side, we obtain:

\[
V_t = \begin{bmatrix}
  * \\
  * \\
  * \\
  * \\
\end{bmatrix}^T \begin{bmatrix}
  0 & X_T & 0 & 0 \\
  X_T & 0 & 0 & 0 \\
  0 & 0 & X_S & 0 \\
  0 & 0 & 0 & -X_S \\
\end{bmatrix} \begin{bmatrix}
  x(t,s) \\
  \dot{x}(t,s) \\
  v(t,s) \\
  w(t,s)
\end{bmatrix}
\]

\[
= x^T(t,s)X_T\dot{x}(t,s) + v^T(t,s)X_Sv(t,s)
\]

\[
- w^T(t,s)X_Sw(t,s)
\]

\[
= \frac{d}{dt}V(x) + v^T(t,s)(X_S - \Delta^*_S\Delta_Sm)\Delta_Sm)v(t,s)
\]

\[
\leq \frac{d}{dt}V(x)
\]

\[
< 0.
\]
From (3.57), we have $\frac{d}{dt}V(x) < 0$, we thus conclude $x(t,s)$ is stable in $\ell_2$.

After establishing the stability of the interconnected systems, the following theorem provides a result on the $H_\infty$ performance of the spatially invariant system.

**Proposition 5** For the interconnected system, $\|z\|_{L_2} < \|d\|_{L_2}$ holds with $x(0,s) = 0$ if there exist symmetric matrices $X_S \in \mathbb{R}^{(m_+ + m_-) \times (m_+ + m_-)}$, $X_T \in \mathbb{R}^{m_0 \times m_0}$, $X_T > 0$, such that

$$\begin{bmatrix}
* & * & \ldots & * \\
0 & X_T & 0 & 0 \\
X_T & 0 & 0 & 0 \\
0 & 0 & X_S & 0 \\
0 & 0 & 0 & -X_S \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 \\
A_{TT} & A_{TS} & B_T & 0 \\
0 & I & 0 & 0 \\
A_{ST} & A_{SS} & B_S & 0 \\
0 & 0 & I & 0 \\
C_T & C_S & D & 0 \\
\end{bmatrix} < 0.$$ (3.60)

**Proof 10** Introduce the uncertainty set

$$\Delta_p = \left\{ \begin{bmatrix}
\Delta_m & 0 \\
0 & \Delta_p \\
\end{bmatrix} : \Delta_p \in \mathcal{L}(\mathcal{L}_2), \text{ with } \|\Delta_p\|_{\mathcal{L}_2} \leq 1 \right\},$$
where \( \Delta_p \) is the "performance block" inserted to accommodate the performance specification, and \( \Delta_m \) is defined in (3.55). According to the main loop theorem [26],

\[
\| F_u(M, \Delta_m) \|_{L^2} < 1 \tag{3.61}
\]

if and only if \((I - M \Delta_p)\) is well-posed for all \( \Delta_m, \Delta_p \), notice that:

\[
M = \begin{bmatrix}
A_{TT} & A_{TS} & B_T \\
A_{ST} & A_{SS} & B_S \\
C_T & C_S & D
\end{bmatrix}
\]

If we choose,

\[
P_p = \begin{bmatrix}
0 & 0 & 0 & X_T & 0 & 0 \\
0 & X_S & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
X_T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -X_S & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0
\end{bmatrix}
\tag{3.62}
\]

It is easy to show that

\[
J_{\Delta_p}(P_p) = \begin{bmatrix}
I \\
\Delta_p
\end{bmatrix}^* P_p \begin{bmatrix}
I \\
\Delta_p
\end{bmatrix}
\]

\[
\begin{bmatrix}
(s + s^*)X_T & 0 & 0 \\
0 & X_S - \Delta_{S,m}^* X_S \Delta_{S,m} & 0 \\
0 & 0 & (I - \Delta_p^* \Delta_p)
\end{bmatrix}
\]

Since

\[X_S - \Delta_{S,m}^* X_S \Delta_{S,m} \geq 0\]
\[(I - \Delta_p^* \Delta_p) > 0\]

\[X_T > 0, \text{ we conclude that} \]

\[J_{\Delta_p}(P_p) \geq 0\]

thus a sufficient condition for the well-posedness of \((I - M \Delta_p)^{-1}\) is that

\[
\begin{bmatrix}
\mathcal{M} \\
I
\end{bmatrix}^* P_p \begin{bmatrix}
\mathcal{M} \\
I
\end{bmatrix} < 0
\]

After some matrix manipulations, condition (3.60) is derived.

Alternatively, we give another proof similar to the techniques used in [105], [85], [76].

Multiplying (3.57) by

\[
\begin{bmatrix}
x(t, s) \\
v(t, s) \\
d(t, s)
\end{bmatrix}
\]

from the left and by its transpose from the
right-side, we obtain:

\[
V_t = \begin{bmatrix}
* \\
* \\
* \\
* \\
* \\
\end{bmatrix}^* \begin{bmatrix}
0 & X_T & 0 & 0 & 0 & 0 \\
X_T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X_S & 0 & 0 & 0 \\
0 & 0 & 0 & -X_S & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
x(t, s) \\
x(t, s) \\
v(t, s) \\
w(t, s) \\
d(t, s) \\
z(t, s) \\
\end{bmatrix}
\]

\[
= \dot{x}^T(t, s)X_Tx(t, s) + x^T(t, s)X_T\dot{x}(t, s)
+ v(t, s)^*X_Sv(t, s) - w(t, s)^*X_Sw(t, s)
+ d^T(t, s)d(t, s) - z^T(t, s)z(t, s)
= \frac{d}{dt}(x^T(t, s)X_Tx(t, s)) + d^T(t, s)d(t, s)
- z^T(t, s)z(t, s)
< 0
\]

Integrating \(V_t\) from 0 to \(+\infty\):

\[
\int_0^{+\infty} \left\{ d^T(t, s)d(t, s) - z^T(t, s)z(t, s) \right\} dt \\
+ x^T(\infty, s)X_Tx(\infty, s) - x^T(0, s)X_Tx(0, s) < 0
\]
\( x(\infty, s) = 0 \) because our system is exponentially stable from Proposition 3, and \( x(0, s) = 0 \) from our zero initial condition assumption, thus we get

\[
\int_{0}^{+\infty} \left\{ d^T(t, s)d(t, s) - z^T(t, s)z(t, s) \right\} dt < 0
\]

i.e

\[
\|z\|_{\mathcal{L}_2} < \|d\|_{\mathcal{L}_2}
\]

**Remark 7** The analysis result proved in [76] is equivalent to our result above, which can be derived if we choose the spatial separator \( \psi_s = M^T\Theta_s M \).

In this section, we transformed the \( H_\infty \) control problem into three well-posedness problems. The first well-posedness problem deals with the interconnection of the direct feed-through term \( A_{SS} \) of the internal signal \( v, w \) and the shift operator \( S \); here both \( A_{SS} \) and \( S \) are treated as operators in \( \ell_2 \). Then we reformulated the stability problem as another well-posedness problem in terms of \( \Delta_m \) and the original system without disturbance. The \( H_\infty \) performance problem, via the main-loop theorem, can also be formulated as a well-posedness problem. In this way, all the well-posedness problem can be solved via searching of some specific separators, so that \( H_\infty \) performance and stability conditions can be derived. The structure of these operators fully characterizes the shift invariant property of \( S \), the stability requirements and the \( H_\infty \) performance specification.

3.4 Stability of One-Dimensional Array Perturbed by White Noise

In this section, we will consider a one-dimensional spatially invariant system perturbed by white noise. When fixed \( t \), at each node \( x(t, s) \), there is a persistent white-noise. We want to study the steady-state properties of these systems.
Spatial Fourier transforms of these systems are first utilized for the one-dimensional spatially invariant array. A necessary and sufficient stability condition in terms of the Schur stability of a matrix over spatial frequency is obtained in this section. Based on the theorem on non-negative pseudo-polynomial matrices, the frequency-dependent stability condition is converted to a finite dimensional linear matrix inequality (LMI) problem, the solution of which is easy to compute.

Consider the following state-space representation for the plant (see [44], [48]):

$$x_{t+1,s} = Ax_{t,s} + A_{-1}x_{t,s-1} + A_1x_{t,s+1} + B_1w_{t,s} + B_2u_{t,s} \quad (3.63)$$

$$z_{t,s} = Cx_{t,s}, \quad (3.64)$$

where the state variable $x_{t,s} \in \mathbb{R}^n$, $w_{t,s} \in \mathbb{R}^p$, $u_{t,s} \in \mathbb{R}^m$, $z_{t,s} \in \mathbb{R}^q$, and $A$, $A_{-1}$, $A_1$, $B_1$, $B_2$, $C$ are real valued matrices of appropriate dimension.

The state of the plant is $x(t) = \{\ldots, x_{s-1,t}, x_{s,t}, x_{s+1,t}, \ldots\}$, $x(t)$ has infinite dimension, and $z_{t,s}$ is an output signal that can be used to characterize the overall system performance. The input disturbance $\{w_{t,s}\}$ is a two-index field of independent, identically distributed random vectors, of zero mean and unit covariance, i.e.,

$$E\{w_{t,s}w_{t,s}^T\} = I. \quad (3.65)$$

With the above model for such spatially invariant systems, each node’s dynamics directly depend on its neighbor’s state information. We assume that the nodes are synchronized in time and that the control input $u(t, s)$ for each node is a state-feedback control based only on all the state information about itself and
its direct neighbors, that is:

\[ u_{t,s} = Kx_{t,s} + K_{-1}x_{t,s-1} + K_1x_{t,s+1}. \] (3.66)

With this restriction on the decentralized controllers, we are searching within a subset of all possible feedback controllers which stabilize the above systems. Thus, the best performance achieved by the above decentralized controllers may be only suboptimal compared to the centralized controller. However, selecting the decentralized controllers (3.66) greatly decreases the complexity and the computation burden for each node, which makes such controllers practical and reliable. Additionally, from Theorem 4 in this chapter, the controller defined by (3.66) only use the first three dominant terms of all state information for control design since the feedback weighting matrix \( K_n \) of the optimal controller has an exponentially decaying rate.

We want to analyze the stability of the above system and minimize the induced power gain from the disturbance \( w_{t,s} \) to the state \( z_{t,s} \). [76], [20] have considered the setting in which for fixed \( t \), \( w_{t,x_t} \in l_2 \), which are square summable for all spatial index \( s \). The performance is evaluated in terms of the induced gain of the input signal to the output signal for the overall system \( x_{t,\cdot} \). In this section, we are more concerned with the signal power distributed at each node. The power of the output signal \( z_{t,s} \) is the output energy distributed in the system averaged both in time and space, which can be written as

\[ \|z(t,s)\|_p = \mathbb{E}\{z_{t,s}z_{t,s}^T\} \]

\[ = \text{Tr}\{CP_0^TC^T\} \] (3.67)
where $P^t_s$ is the covariance matrix $x_{t,s}$ for node $s$ at time $t$.

$$
P^t_{s,0} = \mathbb{E}\{x_{t,s}x_{t,s}^T\}. \quad (3.68)
$$

Similarly, we define $P^t_{s,d}$:

$$
P^t_{s,d} = \mathbb{E}\{x_{t,s}x_{t,s-d}^T\}. \quad (3.69)
$$

$P^t_{s,d}$ is simply denoted as $P^t_d$ because of its property of spatial invariance, and it has the following property.

$$
P^t_{-d} = P^t_d^T. \quad (3.70)
$$

In view of the above, we will present our main result in the next subsection.

### 3.4.1 Stability Analysis of Spatially Invariant System via Spatial Fourier Transformation

**Definition 2** Let $P^t_d, d \in \mathbb{Z}$ defined as above. The spatial power spectral density (spsd) $P^t(\omega)$ is the spatial Fourier transform of $P^t_d$, defined as

$$
P^t(\omega) = \sum_{d=-\infty}^{\infty} P^t_d e^{-j\omega d}. \quad (3.71)
$$

**Proposition 6** Consider the difference state equation (3.63) with $u_{t,s} = 0$, and $w_{t,s}$ i.i.d random vectors with zero mean and unit covariance in $t, s$, and $B_1B_1^T = R$. Then $P^t(\omega)$ satisfies the following difference equation:

$$
P^{t+1}(\omega) = A(\omega)P^t(\omega)A^*(\omega) + R, \quad (3.72)
$$
where

$$A(\omega) = A + A_1 e^{j\omega} + A_{-1} e^{-j\omega}. \hspace{1cm} (3.73)$$

**Proof 11** According to the definition,

$$P_0^{t+1} = E\{x_{t+1,s}x_{t+1,s}^T\}$$

$$= E\{(Ax_{t,s} + A_1 x(t, s + 1) + A_{-1} x(t, s - 1) + d(t, s)) \cdot (Ax_{t,s} + A_1 x(t, s + 1) + A_{-1} x(t, s - 1) + d(t, s))^T\}$$

$$= AP_0^t A^T + A_1 P_1^t A^T + A_{-1} P_{-1}^t A^T + 0 +$$

$$AP_{-1}^t A_1^T + A_1 P_0^t A_1^T + A_{-1} P_{-2}^t A_1^T + 0 +$$

$$AP_1^t A_{-1}^T + A_1 P_2^t A_{-1}^T + A_{-1} P_0^t A_{-1}^T + 0 +$$

$$0 + 0 + 0 + R. \hspace{1cm} (3.74)$$

For $P_k^t, k \neq 0, k \in Z$.

$$P_k^{t+1} = E\{x_{t+1,s}x_{t+1,s-k}^T\}$$

$$= E\{(Ax_{t,s} + A_1 x(t, s + 1) + A_{-1} x(t, s - 1) + d_{t,s}) \cdot (Ax_{t,s-k} + A_1 x(t, s - k + 1) +$$

$$A_{-1} x(t, s - k - 1) + d_{t,s-k})^T\}$$

$$= AP_k^t A^T + A_1 P_{k+1}^t A^T + A_{-1} P_{k-1}^t A^T +$$

$$AP_{k-1}^t A_1^T + A_1 P_k^t A_1^T + A_{-2} P_{k-2}^t A_1^T +$$

$$AP_{k+1}^t A_{-1}^T + A_1 P_{k+2}^t A_{-1}^T + A_{-1} P_k^t A_{-1}^T, \hspace{1cm} (3.75)$$

From (3.74), (3.75) and our definition, we can prove (3.72) by direct computation.

With the above definition, $P^t(\omega)$ represents the spatial power spectral density.
of signal \( x_{t,s} \) at time \( t \). The difference matrix equation (3.72) describes how the spatial power spectral density evolves with time. Note that a stable state for the \( P'(\omega) \) as \( t \) goes to infinity can be reached if and only if all the eigenvalues of \( A(\omega) \) lie inside the unit circle which is obtained in terms of the boundedness of the solution of the above difference Lyapunov matrix equation.

**Proposition 7** For the spatially invariant system described by (3.63) with \( u_{t,s} = 0 \), and \( w_{t,s} \) i.i.d random vectors with zero mean and unit covariance in \( t,s \), and \( B_1B_1^T = R \), the system is stable if and only if all the eigenvalues of \( A(\omega) \) are inside the unit circle. Furthermore, the stable state spatial power spectrum density \( P(\omega) \) satisfies the following algebraic Lyapunov matrix equation:

\[
P(\omega) = A(\omega)P(\omega)A^*(\omega) + R. \tag{3.76}
\]

**Remark 8** According to the discrete Lyapunov theorem [72], \( A(\omega) \) is Schur stable if and only if \( \forall \omega \in [-\pi, \pi], \exists \) a positive definite Hermitian matrix \( X(\omega) \), such that

\[
A(\omega)X(\omega)A^*(\omega) - X(\omega) < 0. \tag{3.77}
\]

**Remark 9** If we make \( X(\omega) \) to be a constant matrix over the spatial frequency \( \omega \), we can get a sufficient condition for \( X(\omega) \) to be Schur stable. The solution of (3.76) can be obtained by a simple recursion [32], which leads to the closed formula when all the eigenvalues of \( P(\omega) \) are inside the unit circle:

\[
P(\omega) = \sum_{i=0}^{\infty} A(\omega)^i R(A^*(\omega))^i. \tag{3.78}
\]

**Remark 10** Proposition 7 gives us a sufficient and necessary condition for the
system to reach a steady state, at which the energy distributed at each spatial frequency is bounded. Note that stability conditions for the continuous time case in terms of the boundedness of the solution of matrix Lyapunov equations were obtained in [5]. In view of the inverse Fourier transform, the above proposition provides a way to compute the steady state power at each node by the following integral:

\[
\| x(t,s) \|_p = \text{Trace} \{ P_0 \} = \text{Trace} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\omega) d\omega \right\} \quad (3.79)
\]

3.4.2 LMI Reformulation via Non-negative Pseudo-polynomial Matrix Theory

Before we convert the above stability condition to a \( \omega \) independent one, we shall introduce a result on the characterization of non-negative pseudo-polynomial matrix on the unit circle [34].

**Lemma 5** A pseudo-polynomial matrix

\[
\Upsilon(z) = \sum_{i=-k}^{k} P_i z^i \quad (3.80)
\]

whose coefficient matrices satisfy \( P_{-k} = P_k^* \), \( P_k \in C^{m \times m} \) is non-negative definite on the unit circle \( z = e^{j\theta}, \theta \in [0,2\pi] \) if and only if there exists a non-negative definite matrix \( Y \), such that

\[
Y = Y_0 + X - Z^T X Z, \quad (3.81)
\]
where $X, Y_0, Z$ are defined as follows:

$$
Y_0 = \begin{bmatrix}
P_0 & P_1 & \cdots & P_k \\
P_1^* & 0 & \vdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
P_n^* & 0 & \cdots & 0
\end{bmatrix}
$$

(3.82)

$$
Z = \begin{bmatrix}
0 & I_m \\
0 & \ddots \\
& \ddots & I_m \\
& & 0
\end{bmatrix}
$$

(3.83)

and $X$ is of the following form

$$
X = \begin{bmatrix}
X_h & 0 \\
0 & 0
\end{bmatrix}
$$

(3.84)

where $X_h$ is an $mk \times mk$ Hermitian matrix, i.e. $X_h = X_h^*$.

Generally speaking, it is difficult to find a positive definite Hermitian matrix $X(\omega)$ for (3.77) to prove the Schur stability of $A(\omega)$. However, if we restrict $X(\omega)$ to be a symmetric positive matrix $X_0 > 0$ which satisfies LMI (3.77), we can obtain a sufficient condition and later use the above lemma to convert this condition to its equivalent finite dimension LMI condition. The KYP lemma has been used to obtain finite dimensional LMIs independent of the spatial frequency $\omega$ in [20]. However, the application of KYP lemma in the discrete-time case is not that straightforward, especially when the degree of the state dependence is more
than one. With Lemma 5, we obtain the following sufficient finite dimensional LMI condition for stability.

**Proposition 8** \( A(\omega) \) is Schur stable if there exists a symmetric positive definite matrix \( X_0 \in \mathbb{R}^{n \times n} \) and a Hamilton matrix \( X \), such that

\[
Y_X + X - Z^T X Z < 0,
\]

where \( Y_X, X, Z \) are defined as follows:

\[
Y_X = \begin{bmatrix}
AX_0A^T + A_1X_0A_1^T + A_{-1}X_0A_{-1}^T - X_0 & (\bullet)^* & (\bullet)^* \\
A_{-1}X_0A^T + AX_0A_1^T & 0 & 0 \\
A_1X_0A_{-1}^T & 0 & 0
\end{bmatrix}
\]

(3.86)

\[
Z = \begin{bmatrix}
0 & I_n & 0 \\
0 & 0 & I_n \\
0 & 0 & 0
\end{bmatrix}
\]

(3.87)

\[
X = \begin{bmatrix}
X_{11} & X_{12} & 0 \\
X_{12}^* & X_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(3.88)

Note that \((\bullet)^*\) here denotes the transpose of \( T_X \)’s corresponding symmetric part, which is clear from context.

**Proof 12** \( A(\omega) \) is Schur stable if there exists a symmetric positive definite matrix \( X_0 \) such that

\[
\Gamma(\omega) = A(\omega)X_0A(j\omega)^* - X_0 < 0.
\]

(3.89)
Notice that $A(\omega) = A_0 + A_1 e^{j\omega} + A_{-1} e^{-j\omega}$, and let $e^{j\omega} = z$. Then from (3.89), we obtain

\[
\Gamma(\omega) = (A + A_1 z + A_{-1} z^{-1})X_0(A + A_1 z + A_{-1} z^{-1})^* - X_0
\]

\[
= (A + A_1 z + A_{-1} z^{-1})X_0(A^T + A_1^T z^{-1} + A_{-1}^T) - X_0
\]

\[
= AX_0A^T + A_1 X_0 A_1^T + A_{-1} X_0 A_{-1} - X_0
\]

\[
+ (A_{-1} X_0 A_1^T + A X_0 A_1^T) z^{-1}
\]

\[
+ (A X_0 A_{-1}^T + A_1 X_0 A^T) z
\]

\[
+ (A_1 X_0 A_{-1}^T) z^{-2}
\]

\[
+ (A_{-1} X_0 A_1^T) z^2. \quad (3.90)
\]

Now, according to Lemma 1, $\Gamma(\omega) \geq 0$ if and only if (3.85) holds, which guarantees that $A(\omega)$ is Schur stable.

In addition to the analysis of the spatially invariant systems, we consider the synthesis problem of the decentralized controller as defined in (3.66), and from proposition 2, we derive the following proposition:

**Proposition 9** System (3.63) is stabilizable by a local controller defined by (3.66), if and only if there exist a positive definite Hermitian matrix $X(\omega), \omega \in [-\pi, \pi]$, and $K, K_1, K_2$, such that

\[
(A(\omega) + B_2 K(\omega))X(\omega)(A(\omega) + B_2 K(\omega))^* - X(\omega) < 0 \quad (3.91)
\]
where

\[ K(\omega) = K + K_1 e^{j\omega} + K_{-1} e^{-j\omega}, \quad (3.92) \]

From proposition 3, by the same argument, if we choose \( X(\omega) \) to be independent of \( \omega \), we obtain sufficient conditions for stability of the closed loop system, i.e., the closed loop system is stable if there exists a symmetric positive definite matrix \( X_0 \), such that

\[
(A(\omega) + B_2 K(\omega))X_0(A(\omega) + B_2 K(\omega))^* - X_0 < 0. \quad (3.93)
\]

3.5 Conclusion

In this chapter, we considered stability/performance conditions for spatially invariant systems. We have provided the standard approaches used in this area for analysis of these systems. Because of their spatially invariant property, spatial Fourier transforms are used first to transform the original system into a classical LTI system with frequency-dependency, so that frequency-dependent stability/performance conditions can be derived. Then, based on extended version of the KYP lemma on pseudo-positive polynomial matrices, some equivalent LMI conditions are derived. We want to emphasize here again that Theorem 4 provides the theoretical foundation for the feasibility of localized controllers which only utilize local information for controller synthesis; information from subsystem further away has an exponentially decaying rate.
CHAPTER 4

DISTRIBUTED CONTROL FOR SYSTEMS WITH PERFECT INTERCONNECTIONS

The problem of performing distributed control over a network to implicitly solve a global optimization problem has been an active area of research over the past years. Many important control problem can be cast in the form of large-scale finite-dimensional or infinite-dimensional constraint optimization problems [81]. One of the fundamental problem in this area is to study the locality features of spatially distributed optimization problems which can be advantageous in the development of fast and well-conditioned distributed algorithms. There has been considerable progress in studying distributed $H_\infty$ control problems both in theory [14], [15], [16], [47], [22], [86], [62] and in computation [11]. A key feature of these problems is that the interconnection can be treated as algebraic constraints, so that distributed $H_\infty$ performance can be formulated as optimization problems with equality constraints on the interconnections.

In this chapter, we consider the distributed $H_\infty$ control problem with perfect interconnections formulated in [47]. The basic techniques we are using are block $S$-procedures from Scherer [85]. For this chapter, we require the interconnections to be perfect. The synthesis problem and interconnections modelled by Integral Quadratic Constraints (IQC) will be addressed in next chapter.
4.1 Systems Over Graphs

In this section, we deal with systems consisting of an assembly of \( L \) possibly different, arbitrarily interconnected linear time-invariant systems. In general, such underlying interconnected systems can be described by an undirected graph, each vertex representing a subsystem, with the edges denoting the coupling signals between subsystems. The system can be modelled by the state-space representation of the different subsystems and the interconnecting conditions.

Notation: Given matrices \( A_1, A_2, \ldots, A_L \), we define \( \text{diag}_{k \leq i \leq l} A_i \) by

\[
\text{diag}_{k \leq i \leq l} A_i = \begin{bmatrix}
A_k & 0 & \cdots & 0 \\
0 & A_{k+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_l
\end{bmatrix}.
\] (4.1)

Likewise, if \( e_1, \ldots, e_L \) are elements of sets \( E_1, \ldots, E_L \), \( \text{cat}_{k \leq i \leq l} e_i \) will denote the elements \((e_k, \ldots, e_l) \in E_k \times \cdots \times E_l\) when \( 1 \leq k \leq l \leq L \). We will sometimes write \( \text{diag}_i \) and \( \text{cat}_i \) instead of \( \text{diag}_{1 \leq i \leq L} \) and \( \text{cat}_{1 \leq i \leq L} \).

4.1.1 Subsystems and Interconnections

Each subsystem \( G_i \) is captured by the following state-space equations:

\[
\begin{bmatrix}
\dot{x}_i(t) \\
w_i(t) \\
z_i(t) \\
x_i(0)
\end{bmatrix} =
\begin{bmatrix}
A_{TT}^i & A_{TS}^i & B_{Td}^i \\
A_{ST}^i & A_{SS}^i & B_{Sd}^i \\
C_{Ts}^i & C_{Sz}^i & D_{zd}^i \\
x_0^i
\end{bmatrix}
\begin{bmatrix}
x_i(t) \\
v_i(t) \\
v_i(t) \\
d_i(t)
\end{bmatrix}
\] (4.2)

\[
x_i(0) = x_0^i
\] (4.3)
Here \( x_i(t) \in \mathbb{R}^{m_i}, d_i(t) \in \mathbb{R}^{p_i}, z_i(t) \in \mathbb{R}^{q_i}, v_i(t), w_i(t) \in \mathbb{R}^{n_i} \) for all \( t \geq 0 \). In (4.2), \( d_i \) is the disturbance and \( z_i \) is the performance associated with \( G_i \), while \( v_i \) and \( w_i \) are the overall interconnection signals used by \( G_i \). For each given \( i \), \( v_i, w_i \) is further partitioned into \( v_{ij}, w_{ij} \) respectively, i.e., the interconnection \( n_{ij} \)-dimension signal that is shared by \( G_i \) and \( G_j \). Note that \( v_{ij} \) and \( v \) denote a \( \mathbb{R}^{m_{ij}} \) and a \( \mathbb{R}^N \)-valued signal respectively, while \( v_{ij} \) is a component of the vector \( v \in \mathbb{R}^N \). A detailed description of the underlying graph structure of such systems has been presented in the Chapter 1.

4.1.2 Models of Interconnected Signals

Each \( v_i \) and \( w_i \) is further partitioned into \( v_{ij}, w_{ij} \) respectively, i.e. the interconnection \( n_{ij} \)-dimension signal that is shared by \( G_i \) and \( G_j \). We model the interconnection via a matrix operator \( \Delta_{ij} \), such that, in the general case

\[
    v_{ij}(t) = \Delta_{ji} w_{ji}(t), \quad \forall i, j, 1 \leq i, j \leq L.
\]

(4.4)

For example, the simplest case would be, \( w_{ij} = v_{ji} \) which is called ideal/perfect interconnection. In this chapter, we restrict the interconnection signal subspace as \( \mathcal{W}(\Delta_{ij}) \), such that

\[
    \mathcal{W}^R(\Delta_{ij}) = \left\{ \begin{bmatrix} v_{ij}(t) \\ w_{ji}(t) \end{bmatrix} \in \mathbb{R}^{2n_{ij}}_2 : v_{ij}(t) = \Delta_{ji} w_{ji}(t), v_{ij}, w_{ji} \in L^2_2 \right\}.
\]

(4.5)

We denote \( v = \text{cat}_i v_i \), where each \( v_i \) can be further partitioned as \( v_i = \text{cat}_j v_{ij} \). Note that the dimension of \( v_{ij}, v_i \) and \( v \) are \( n_{ij}, n_i \) and \( N \) where \( n_i = \sum_{j=1}^L n_{ij}, N = \sum_{i=1}^L n_i \). The global system signals, \( x = \text{cat}_i x_i, w = \text{cat}_i w_i, z = \text{cat}_i z_i, d = \)
\( \mathbf{c} \), \( \mathbf{d} \) are similarly defined.

Based on the state space representations of \( G_i \), the state space representation of the global system can be described by the following state-space representation

\[
\begin{pmatrix}
\dot{x}(t) \\
w(t) \\
z(t) \\
v(t)
\end{pmatrix} =
\begin{bmatrix}
A_{TT} & B_{TS} & B_{Td} \\
A_{ST} & A_{SS} & B_{Sd} \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
\begin{pmatrix}
\dot{x}(t) \\
v(t) \\
d(t)
\end{pmatrix}
\] (4.6)

where \( \Delta \) is a matrix operator from \( \mathbb{R}^N \) to \( \mathbb{R}^N \) generated via \( \Delta_{ij} \),

\[
\Delta = \text{diag}_i \text{diag}_j \Delta_{ji}
\] (4.8)

and the permutation matrix \( P_r \) is chosen such that

\[
\bar{w} = \mathbf{c} \mathbf{a}_i \mathbf{d}_j \mathbf{w}_{ji} = P_r w = P_r \mathbf{c} \mathbf{a}_i \mathbf{d}_j \mathbf{w}_{ij}
\]

matrix \( A_{TT} = \text{diag}_i A_{TT} \). Similarly, all other matrices for the global system in (4.6) are defined as diagonal matrices based on the underlying subsystem \( G_i \)'s representation.

The signals \( w(t) \) and \( v(t) \) are internal signals of the same dimension, \( w(t), v(t) \in \mathbb{R}^N \). The signal space for \( v, w \) can be described as

\[
W^R(\Delta) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{L}^{2N} : \begin{bmatrix} v_{ij} \\ w_{ji} \end{bmatrix} \in \mathbb{R}^{2n_j}, v_{ij} = \Delta_{ji} w_{ji} \right\}
\] (4.9)

For the state-space representation of the global system (4.6), we represent its
transfer function by
\[
G = \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\]
which has been partitioned to conform with the vector \((v, d)\). In this chapter, we define well-posed and stability of the interconnection system over signal space \(L_2\) if the system (4.6) is internal stable independent of the uncertainty of the interconnections \(\Delta_{ij}\) (4.9).

**Definition 3** The interconnected system consisting of subsystems (4.6) and the interconnection constraints (4.9) is said to be well-posed and stable if the map \((I - \Delta P r G_{11})\) has a bounded inverse on \(L_2\), for any \(\Delta\).

Finally, we will say that such a system (4.6) is contractive if it is stable and \(\|z\| < \|d\|\) for all \(d \in L_2\) and all interconnection \(\Delta_{ij}\) (4.9).

### 4.2 Full Block S-procedure

#### 4.2.1 The Extended Block S-procedure

To solve the problem, we need some technical results. The following lemmas will be of key importance in our derivation.

The following lemma is a generalization of the Finsler’s theorem proved independently by Scherer [85] and Iwasaki [41].

**Lemma 6** (Full block S-procedure) Suppose \(S\) is a subspace of \(\mathbb{R}^n\), \(T \in \mathbb{R}^{l \times n}\) is a full row rank matrix, and \(U \in \mathbb{R}^{k \times l}\) is a compact subset of matrices of full row rank. Define the family of subspaces

\[
S_U = S \cap \ker(UT) = \{x \in S : UTx = 0\}, U \in U.
\]
Suppose $N \in \mathbb{R}^{n \times n}$ is a fixed symmetric matrix. As a technical hypothesis, we require that all subspaces $S_U$ are complementary to a fixed subspace $S_0 \subset S$ that has the two properties

$$\dim(S_0) \geq k \text{ and } N \geq 0 \text{ on } S_0.$$  \hfill (4.11)

The two conditions:

$$\forall U \in U, S_U \cap S_0 = \{0\} \text{ and } N < 0 \text{ on } S_U$$  \hfill (4.12)

hold if and only if there exists a matrix $P$ that satisfies

$$\forall U \in U : N + T^T P T < 0 \text{ on } S \text{ and } P \geq 0 \text{ on } \ker(U).$$  \hfill (4.13)

In potential applications, one can think of $S$ as a nominal system, with $T$ specifying the interconnection of system with uncertainty. $U$ defines the uncertainty set, and therefore $S_U$ is the uncertainty system. The negativity of $N$ is a condition associated with the performance index. Then the above lemma renders the implicit condition based on the uncertain system data to an explicit expression through the full block multiplier $P$. Note that the aforementioned full block $S$-procedure will generally not be lossless when a general class of uncertain system is considered.
4.2.2 $H_\infty$ Performance

For the perfect interconnection case where the operator $\Delta_{ij} = I_{n_{ij}}$, we restrict the overall interconnection subspace to a subset of $\mathbb{R}^N \times \mathbb{R}^N$:

$$\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \in \mathcal{S}_I, \text{ for all } t \geq 0,$$

where the interconnection subspace $\mathcal{S}_I$ is defined as

$$\mathcal{S}_I = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^N \times \mathbb{R}^N : v_{ij} = I_{n_{ji}} w_{ji}, \text{ for all } i, j = 1, \ldots, L \right\}. \quad (4.14)$$

The construction of interconnected given above may not be always well-defined because the signals satisfying the interconnections maybe not exist. We are interested in the following definition of well-posedness for the perfect interconnection case:

**Definition 4** A prefect interconnected system defined by the adjacent matrix $A_G \in \mathbb{R}^{L \times L}$ and the state space representation (4.2) is well-posed if the two subspaces $\mathcal{S}_I$ and $\mathcal{S}_B$ satisfy

$$\mathcal{S}_I \cap \mathcal{S}_B = \{0\}$$

i.e $\mathcal{S}_I$ and $\mathcal{S}_B$ are topologically separated, where

$$\mathcal{S}_B = \left\{ (w, v) \in \mathbb{R}^{N \times N} : \begin{bmatrix} w_i \\ v_i \end{bmatrix} \in \mathcal{S}_B^i, \forall i = 1, \ldots, L \right\}. \quad (4.15)$$
and
\[
S^i_B = \text{Im} \begin{bmatrix}
A^i_{SS} \\
I
\end{bmatrix}
\] (4.16)

**Remark 11** The well-posedness defined above applies to interconnected signals in the real vector space \((w, v) \in \mathbb{R}^{2N}\). Matrices can be regarded as operators between vector spaces, and the above definition guarantees the uniqueness of the interconnected signals in \(\mathbb{R}^{2N}\).

When the interconnection is well-posed, initial condition \(x^0_i\) and disturbance \(d_i\) uniquely determine the signals \(x_i, v_i, w_i, z_i\). Moreover, if all disturbance signals belong to \(L_2\), then (4.2) has a unique solution in \(L_2\). We will call the global system stable if in the absence of disturbance \(d_i, \forall i = 1, \ldots, L\) and for any set of initial conditions \(x^0_i, i = 1, \ldots, L\), \(x_i\) is a continuous function of \(t\) and goes to 0 as \(t\) approaches infinity for all \(i = 1, \ldots, L\). Finally, it is easy to see that if an interconnected system is well-posed and stable, then \(x_i\) and \(z_i\) belong to \(L_2\) for all \(i\) when the disturbance input \(d_i\) do. Such a system then defines an input/output relationship from \(L_2\) to \(L_2\), and its induced gain characterizes the global performance. We will say that the global system is contractive if there exists \(\gamma \in [0, 1)\), such that

\[
\|z\|^2 = \sum_{i=1}^{L} \|z_i\|^2 \leq (1 - \epsilon) \sum_{i=1}^{L} \|d_i\|^2 = (1 - \epsilon)\|d\|^2.
\]

4.3 Performance Analysis via Block S-procedure

Before we go into details of the stability analysis of the distributed systems, let us first concentrate on the implicit defined Linear Parameter-Varying (LPV)
system as follows:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{w}(t) \\
\dot{z}(t) \\
v(t)
\end{bmatrix} =
\begin{bmatrix}
A_{TS} & B_{TS} & B_{Td} \\
A_{ST} & A_{SS} & B_{Sd} \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
v(t) \\
d(t)
\end{bmatrix}
\] (4.17)

where the admissible set of values that can be taken by the uncertainties is denoted by

\[
\Delta \subset \mathbb{R}^{k \times k}.
\]

We assume that \( \Delta(t) \) defines the algebraic constraints which are imposed on the implicitly defined signals \( v(t), w(t) \in \mathbb{R}^{k} \) through the parameter value \( \Delta(t) \). Technically, \( \Delta(t) \) is assumed to be compact and path-connected. Note that, \( \Delta \) captures both the size of the uncertainties as well as their structure. The main purpose of the full block \( S \)-procedure used in this chapter is to equivalently reformulate this test into a more explicit condition that makes use of multipliers. Here, the relevant set of multipliers is defined as

\[
\mathcal{P} = \left\{ P_m \in \mathbb{R}^{2k \times 2k} : P_m = P^T, \begin{bmatrix} \Delta \\ I \end{bmatrix}^T P \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \forall \Delta \in \Delta \right\}. \] (4.19)

Whenever required, we will tacitly assume that any such multiplier is partitioned as

\[
P_m = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \text{ conformable to } \begin{bmatrix} \Delta \\ I \end{bmatrix}.
\]
Equivalently, the LPV system (4.17) with interconnection (4.18) is well posed if

\[ I - \Delta A_{SS} \] is non-singular for every \( \Delta \in \Delta \quad (4.20) \]

By eliminating the interconnected signals \( v, w \) with the assumption on well-posedness, we can equivalently express the system (4.17), (4.18) as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bd(t) \quad (4.21) \\
z(t) &=Cx(t) + Dd(t), \quad (4.22)
\end{align*}
\]

where

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
A_{TT} & B_{Td} \\
C_{Tz} & D_{zd}
\end{bmatrix}
+ \begin{bmatrix}
B_{TS} \\
C_{Sz}
\end{bmatrix}(I - A_{SS}\Delta)^{-1}\begin{bmatrix}
A_{ST} & B_{Sd}
\end{bmatrix}.
\]

We are now ready to apply the block S-procedure lemma to obtain the following equivalent multiplier test for the well-posedness, stability and performance of the LPV system (4.17, 4.18).

**Proposition 10** The LPV system (4.17), (4.18) is well posed, stable and contractive if there exist \( X_T > 0 \) and a symmetric multiplier \( P_m \in \mathcal{P}(4.19) \), such
that

\[
\begin{bmatrix}
I & 0 & 0 \\
A_{TT} & B_{TS} & B_{Td} \\
0 & I & 0 \\
A_{ST} & A_{SS} & B_{Sd} \\
0 & 0 & I \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}^T
\begin{bmatrix}
0 & X_T & 0 & 0 & 0 & 0 \\
X_T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q & S & 0 & 0 \\
0 & 0 & S^T & R & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
A_{TT} & B_{TS} & B_{Td} \\
A_{ST} & A_{SS} & B_{Sd} \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
< 0.
\]

(4.23)

**Proof 13 Well-posedness:** We will show this via Lemma 3. From the above condition (4.23), we know that there exists a

\[
P_m = \begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix},
\]

for all \( \Delta(t) \in \Delta \), such that

\[
\begin{bmatrix}
I \\
A_{ss}
\end{bmatrix}^T
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix}
\begin{bmatrix}
I \\
A_{ss}
\end{bmatrix}
< 0
\]

Notice that \( P_m \in \mathcal{P} (4.19) \), i.e

\[
\begin{bmatrix}
\Delta \\
I
\end{bmatrix}^T
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix}
\begin{bmatrix}
\Delta \\
I
\end{bmatrix}
\geq 0.
\]

From Lemma 3, \((I - A_{ss}\Delta)^{-1}\) exists for all \( \Delta(t) \in \Delta \). Thus, we conclude that the interconnection (4.18) is well posed if condition (4.23) is satisfied.

**Stability and Performance:** Once we have established the well posedness of (4.18), we can equivalently represent the interconnected system (4.17),(4.18) as (4.21). It is obvious that system described by (4.21) is stable and contractive if
there exists $X_T > 0$ such that the following LMI is satisfied,

\[
\begin{bmatrix}
I & 0 \\
A & B \\
0 & I \\
C & D
\end{bmatrix}^T
\begin{bmatrix}
0 & X_T & 0 & 0 \\
X_T & 0 & 0 & 0 \\
0 & 0 & -I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
A & B \\
0 & I \\
C & D
\end{bmatrix} < 0. 
\tag{4.24}
\]
To apply the block S-procedure Lemma 6, we introduce

\[
N = \begin{bmatrix}
0 & X_T & 0 & 0 & 0 & 0 \\
X_T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}
\]  \tag{4.25}

\[
S_0 = \text{Im} \begin{bmatrix}
0 \\
B_{TS} \\
I \\
A_{SS} \\
0 \\
C_{Sz}
\end{bmatrix}
\]  \tag{4.26}

\[
S = \text{Im} \begin{bmatrix}
I & 0 & 0 \\
A_{TT} & B_{TS} & B_{Td} \\
0 & I & 0 \\
A_{ST} & A_{SS} & B_{Sd} \\
0 & 0 & I \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
\]  \tag{4.27}

and

\[
U = \begin{bmatrix}
I & -\Delta
\end{bmatrix}
\]  \tag{4.28}

\[
T = \begin{bmatrix}
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0
\end{bmatrix}
\]  \tag{4.29}
From the block $S$-procedure (6), condition (4.23) is a reformulation of (4.24), which finishes the proof for stability and performance.

With the above proposition, we are now ready to analyze the well-posedness, stability and performance of the distributed systems described via (4.6), (4.7).

Since $w_{ij}(t)$, and $v_{ji}(t)$ satisfy the following algebraic constraint $w_{ij} = v_{ji}$ under perfect interconnections, we are seeking a particular set of multipliers $X_{\text{ideal}}$ to model interconnections, such that for the interconnection signal associated with $v(t), w(t)$ in (4.6),

$$\begin{bmatrix} v \\ w \end{bmatrix} X_{\text{ideal}} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0,$$

Consider the quadratic term $P_{ij}$:

$$P_{ij}(w_{ij}, v_{ij}) = \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}^T \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ (X_{ij}^{12})^T & X_{ij}^{22} \end{bmatrix} \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}. \quad (4.30)$$

Then at any time $t$,

$$P_{ij}(w_{ij}, v_{ij}) + P_{ji}(w_{ji}, v_{ji}) = \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}^T \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ (X_{ij}^{12})^T & X_{ij}^{22} \end{bmatrix} \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix} + \begin{bmatrix} w_{ji} \\ v_{ji} \end{bmatrix}^T \begin{bmatrix} X_{ji}^{11} & X_{ji}^{12} \\ (X_{ji}^{12})^T & X_{ji}^{22} \end{bmatrix} \begin{bmatrix} w_{ji} \\ v_{ji} \end{bmatrix}$$

$$= \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}^T \{X_{ij} + E_{ij}X_{ji}E_{ij}\} \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix},$$
where

\[
E_{ij} = \begin{bmatrix}
0 & I_{n_{ij}} \\
I_{n_{ij}} & 0
\end{bmatrix}.
\]  

(4.31)

In this case, suppose we choose for all \(1 \leq i, j \leq L\)

\[
X_{ij}^{11} + X_{ji}^{22} = 0
\]

\[
X_{ij}^{12} + (X_{ji}^{12})^T = 0.
\]

Then

\[
\begin{bmatrix}
v \\
w
\end{bmatrix}^T X_{\text{ideal}} \begin{bmatrix}
v \\
w
\end{bmatrix} = \sum_{1 \leq i, j \leq L} \begin{bmatrix}
v_{ij} \\
w_{ij}
\end{bmatrix}^T X_{ij} \begin{bmatrix}
v_{ij} \\
w_{ij}
\end{bmatrix} = 0
\]

The family of multiplier \(X_{\text{ideal}}\) can thus be characterized by the two sets

\[
\left\{X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}} : i, j = 1, \ldots, L\right\}
\]

and

\[
\left\{X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}} : X_{ij}^{12}\text{ skew-symmetric}, 1 \leq j \leq i \leq L\right\}.
\]

We are now able to state our first analysis conditions. Note that the proof of proposition (11) follows from proposition (10) by utilizing the diagonal structure of the global system (4.6) and the above two set of multipliers.

**Proposition 11** The interconnected system (4.6), (4.7), is well-posed, stable and contractive for all \(\Delta_{ij} = I_{n_{ij}}\) if there exist symmetric matrices, \(X_T^i \in \mathbb{R}^{m_i \times m_i}\) and \(X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}}\) for all \(i, j = 1, \ldots, L\), and matrices \(X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}\) for all \(i \geq j\).
with $X_{1i}^{12}$ skew-symmetric, such that $X_T^i > 0$ and the LMIs

$$
\begin{bmatrix}
I & 0 & 0 \\
A_{TT}^i & A_{TS}^i & B_T^i \\
0 & I & 0 \\
A_{ST}^i & A_{SS}^i & B_S^i \\
0 & 0 & I \\
C_T^i & C_S^i & D_i
\end{bmatrix}^* \begin{bmatrix}
0 & X_T^i & 0 & 0 & 0 & 0 \\
X_T^i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{11}^i & P_{12}^i & 0 & 0 \\
0 & 0 & (P_{12}^i)^* & P_{22}^i & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix} \begin{bmatrix}
I & 0 & 0 \\
A_{TT}^i & A_{TS}^i & B_T^i \\
0 & I & 0 \\
A_{ST}^i & A_{SS}^i & B_S^i \\
0 & 0 & I \\
C_T^i & C_S^i & D_i
\end{bmatrix} < 0
$$

(4.32)

hold true for all $i = 1, \ldots, L$ with

$$
P_{11}^i = \text{diag}_{1 \leq j \leq L} X_{ij}^{11}
$$

$$
P_{22}^i = \text{diag}_{1 \leq j \leq L} - X_{ji}^{11}
$$

$$
P_{12}^i = \text{diag} \left( \text{diag}_{1 \leq j \leq i} X_{ij}^{12}, \text{diag}_{i \leq j \leq L} - (X_{ji}^{12})^T \right).
$$

4.4 Conclusion

In this chapter, we have discussed how to handle the distributed $H_\infty$ control problem with perfect interconnections. Here the perfect interconnections when $v_{ij}(t) = w_{ji}(t)$ are treated as algebraic constraints. It is well-known that the stability and $H_\infty$ performance of the global linear time-invariant system can be characterized as a set of Linear Matrix Inequalities when the interconnected signals are eliminated from the original state space representations. Here we introduced a general technique, the so-called full block $S$-procedure, which allows us to formally introduce full block multipliers to model the interconnected system. In this way, the constrained LMI conditions in terms of the simplified system (4.21, 4.22) are used to obtain the conditions to guarantee well-posedness, stability and per-
formance of the original interconnected systems. More general applications of the full block S-procedure can be found in [84], [85], [41]. The $H_{\infty}$ distributed control problem with Integral Quadratic Constraints (IQC) on interconnections will be analyzed in detail in the following chapter.
CHAPTER 5
DISTRIBUTED CONTROL FOR SYSTEMS WITH INTERCONNECTION
MODELLED VIA IQC

In this chapter, we present some stability conditions for distributed control problems under general Integral Quadratic Constraints (IQC) to achieve quadratic performance. These results take the form of coupled LMIs, and the multipliers are specified by the underlying IQCs to model interconnections between the subsystems. It is further shown that these stability results can be exploited for distributed controller synthesis in a manner similar to the gain-scheduling controller design in the LPV systems in Chapter 4. The present contribution of our approach is to unify all the stability results which first appeared in [14] in one general framework of IQC analysis and provide lower dimension controller synthesis conditions.

5.1 Introduction

Over the past few years, there has been renewed research interest in distributed control of large scale systems [14], [33], [47], [12], [98], [86], [15], [60]. Many of these systems are formed by the interconnection of multiple homogeneous or heterogeneous subsystems. These systems exhibit an overall complex dynamical behavior dictated by their distributed nature and the dynamical interactions between the subsystems.
A great challenge that the control community faces is to deal with these systems that are physically distributed and consist of a large number of interacting subsystems. The distributed nature of the systems implies that the observation information is highly distributed, which has motivated the development of new research directions in control theory to take into account the communication constraints. In particular, researchers have considered control problems with non-ideal communication links such as limited bandwidth [93], [65], delay, and packet dropout between sensors and actuator of these subsystems, and new results have been reported (see for example [2], [3]). Note that standard control design techniques often fail because of the high dimension of the overall system and the forbidding communication and computation burden to implement such centralized control algorithms. Therefore, it is often useful to utilize special structures of the underlying topology of the systems, such that decentralized control schemes are deployed for these large-scale applications. Successful synthesis methods have been proposed for the existence of a decentralized controller guaranteeing performance of the closed-loop systems. However, the techniques only apply to specific types of interconnections.

Recently, a distributed control theory was developed in for spatially-invariant distributed systems [14]. It was shown that the controllers have 'identical' structure as the underlying subsystems. A Linear Matrix Inequality (LMI) based control synthesis algorithms for this class of interconnected systems was developed in [14], [33] using multidimensional system theory. These results is further extended in [14], [12], [22] to distributed system over a arbitrary graph under various communication connections. Specifically, the results take the form of a set of coupled linear matrix inequalities, where the particular design variables for the LMIs are
shaped by the interconnections. As mentioned in [14], these stability results can be explained in the general framework of dissipative theory [102], and are well connected to the integral quadratic constraints [54] analysis methods since the interconnections are generally modeled as IQCs. This distributed control theory has recently been applied in the control of large-scale irrigation networks in Australia [13] and the performance is compared to those standard centralized and decentralized control techniques. It is shown in [13] that distributed controllers have achieved similar performance to the centralized controller and significant, better error propagation performance than the simple decentralized feedback controllers. Since the computational cost and the infrastructure cost for decentralized control do not scale well as the number of subsystem is increased, the distributed control thus appears to achieve an acceptable trade-off between performance and complexity.

In our view, this distributed control approach [14], [47] has been well-developed in the literature of gain-scheduling techniques for linear parameter varying (LPV) systems [87], [1]. The stability results follow from a application of the S-procedure developed in [109], [55], [110] when parameterizing the interconnections as a family of IQCs. Besides, the stability condition under perfect communication can be proved via the block S-procedure [85], [41] if a proper quadratic separator is chosen. Geometrically, all the stability results can be interpreted from a graph separation point of view ([83], [61] [40]) following a similar proof as in [87]. While the stability results can be easily derived via a graph separation argument, the necessity part for some specific interconnections follows from the lossless $(D,G)$ scaling theorem for LPV uncertainties [56]. As for synthesis, based on a recently extended elimination lemma in [36], the synthesis inequalities turn out to be convex in all variables,
including the scalings [85]. However, the technique could only be applied under certain inertia hypothesis on the multipliers which are unduly conservative.

The objective of this chapter is to characterize the stability condition for distributed systems with IQC constraints, and then to utilize the topological structures and intrinsic property of the interconnections for distributed controller design. Instead of presenting new stability results, the present contribution of our approach is to unify all the stability results first appeared in [14] in one general framework of IQC analysis. This analysis is similar to what proposed in [1], [87]. However, here there is much more emphasis on the communication constraints between subsystems and the decoupled stability condition. Another important contribution of the present approach is to relate distributed control under communication constraints to the well-developed results in the literature of gain-scheduling techniques for linear parameter varying (LPV) systems [87], [1]. For stability, this is achieved by an application of the S-procedure developed in [109], [55], [110] when parameterizing the interconnections as a family of IQCs.

Geometrically, the stability results can be interpreted from a graph separation point of view ([83], [61] [40]) following a similar proof as in [87]. As for synthesis, based on a recently extended elimination lemma in [36], the synthesis inequalities turn out to be convex in all variables, including the scalings [85]. It is also worth mentioning that instead of $n_{ij}^K = 3n_{ij}$ for the synthesis condition in [14], if the dimension of the interconnected signals for the controller is great than or equal to the associated interconnected signals for the plants, i.e., $n_{ij}^K \geq n_{ij}$, there exist distributed controllers to guarantee the global control performance.

This chapter is organized as follows: we begin with some mathematical preliminaries which outline the notation used throughout this chapter. In Section
5.2, the distributed system models are introduced. Section 5.3 is devoted to the
analysis of stability and performance of the distributed systems under various
interconnections. In Section 5.4, we move on to deriving distributed controller
synthesis results, and some concluding remarks are made in Section 5.5.

The notation is standard. The set of real number is denoted by \( \mathbb{R} \), the non-
negative reals by \( \mathbb{R}_+ \). \( \mathbb{R}^{n \times m} \) is the set of \( n \times m \) matrices. The transpose (com-
plex conjugate transpose) of matrix \( M \) is denoted by \( M^T(M^*) \). We use \( \mathbb{R}_S^n \) to
denote \( n \times n \) real symmetric matrices. If \( M \in \mathbb{R}_S^n \), then \( M > 0(M \geq 0) \) in-
dicate \( M \) is positive definite (positive semidefinite) matrix, and \( M < 0(M \leq) \)
denotes negative (negative semidefinite) matrix. For any matrix \( P \), \( \ker(P) \) stands
for the null space of the linear operator associated with \( P \). The inertia of a
symmetric matrix \( A \) is the ordered triple \( \text{in}(A) = (i_+(A), i_0(A), i_-(A)) \) where
\( i_+(A), i_-(A), i_0(A) \) is the number of positive, negative and zero eigenvalues of
\( A \), all counting multiplicity. A block diagonal matrix with \( X_k, \ldots, X_l \) is de-
dnoted \( \text{diag}_{k \leq i \leq l} X_i = \text{diag} \{ X_k, \ldots, X_l \} \); likewise, if \( e_1, \ldots, e_L \) are elements of
sets \( E_1, \ldots, E_L \), \( \text{cat}_{k \leq i \leq l} E_i \) will designate the elements \( (e_k, \ldots, e_l) \in E_k \times \ldots E_l \)
when \( 1 \leq k \leq l \leq L \). We will sometimes write \( \text{diag}_i \) and \( \text{cat}_i \) instead of \( \text{diag}_{1 \leq i \leq L} \)
and \( \text{cat}_{1 \leq i \leq L} \).

The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( \| x \| = (x^T x)^{1/2} \). The
space of square integrable \( n \)-dimensional functions \( f : (0, \infty) \rightarrow \mathbb{R}^n \) is denoted by
\( \mathcal{L}_2^n \); this is abbreviated as \( \mathcal{L}_2 \) when \( n \) is clear from context or not relevant. The
Fourier transform of a \( \mathcal{L}_2 \) function \( f \) is denoted as \( \hat{f}(j\omega) \). The norm of an \( \mathcal{L}_2 \)
signal and the induced norm of an operator on \( \mathcal{L}_2 \) is denoted by \( \| \cdot \| \), so for an
operator \( F : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \), \( \| F \| = \sup_{u \in \mathcal{L}_2} \frac{\| Fu \|}{\| u \|} \). An operator \( \Delta : \mathcal{L}_2^n \rightarrow \mathcal{L}_2^n \) is said to
be contractive if \( \| \Delta v \| < \| v \|, \forall v \in \mathcal{L}_2^n \). Lower case \( \delta \)'s always denote operators
from $L^1_2$ to $L^1_2$, for $u, v \in L^n_2$. The express $v = \delta I_n u$ is defined to mean that $u_k$ of $u$ and $v_k$ of $v$ satisfy $u_k = \delta v_k$. An operator $\delta : L_2 \rightarrow L_2$ is called self-adjoint if $< u, \delta v > = < \delta u, v >, \forall u, v \in L_2$. Note that all real-valued static LTV operators are self-adjoint.

5.2 Problem Formulation

In this chapter, we will concern ourselves with systems formulated in [14]. The global system consists of an assembly of $L$ subsystem $G_i, i = 1, \ldots, L$ connected arbitrarily. Each subsystem $G_i$ is captured by the following state-space equations as defined in the previous chapter (4.2):

$$
\begin{bmatrix}
\dot{x}_i(t) \\
w_i(t) \\
z_i(t) \\
x_i(0)
\end{bmatrix}
= 
\begin{bmatrix}
A^i_{TT} & A^i_{TS} & B^i_T \\
A^i_{ST} & A^i_{SS} & B^i_S \\
C^i_T & C^i_S & D^i
\end{bmatrix}
\begin{bmatrix}
x_i(t) \\
v_i(t) \\
d_i(t)
\end{bmatrix}
$$

(5.1)

$$
x_i(0) = x^0_i.
$$

(5.2)

A detailed description of the above model can be found in the previous chapter.

In this chapter, we model the signal subspace defined via $\Delta_{ij}$ as $\mathcal{W}(\Delta_{ij})$, such that

$$
\mathcal{W}(\Delta_{ji}) = \left\{ \begin{bmatrix} v_{ij} \\ w_{ji} \end{bmatrix} \in L^2_{2n_{ji}} : v_{ij} = \Delta_{ji} w_{ji} \right\}.
$$

(5.3)

As we mentioned in the previous chapter that the signals $w(t)$ and $v(t)$ are internal signals of the same dimension. The signal space for $v, w$ can be described as

$$
\mathcal{W}(\Delta) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in L^2_{2N} : \begin{bmatrix} v_{ij} \\ w_{ji} \end{bmatrix} \in L^2_{2n_{ji}}, v_{ij} = \Delta_{ji} w_{ji} \right\}.
$$

(5.4)
Now, we define well-posedness and stability of the interconnection system over signal space if the system (4.6) is internal stable independent of the uncertainty of the interconnections $\Delta_{ij}$ (5.4).

**Definition 5** The interconnected system consisting of subsystems (4.6) and the interconnection constraints (5.4) is said to be well-posed and stable if the map $(I - \Delta PG_{11})$ has a bounded inverse on $L_2$, for any choice of $\Delta \in \Delta$.

Finally, we will say that such a system (4.6) is contractive if it is stable and $\|z\| < \|d\|$ for all $d \in L_2$ and all interconnection $\Delta_{ij}$ (5.4).

5.3 Stability analysis for Distributed Systems

The main idea here is to first use integral quadratic constraints to model the interconnection operator $\Delta_{ij}$. The performance under IQCs for the internal signal $v, w$ can then be cast as an unconstrained quadratic optimization problem [54]. For the LTI system, the stability results admit a LMI formulation. For this purpose, we need the following definition of IQC and dissipativity.

5.3.1 Stability Analysis

**Definition 6** [54] A signal class is a collection of vector signals $W = \{w : w \in L_2^n\}$. A class of signal $W$ is said to satisfy the IQC defined by $\sigma$ defined by $\Pi(\omega)$ (denoted by $W \in IQC(\sigma, \Pi(\omega))$) if

$$\sigma(\omega, \Pi(\omega)) = \int_{-\infty}^{\infty} \hat{w}(j\omega)^* \Pi(\omega) \hat{w}(j\omega) d\omega$$

(5.5)

$\hat{w}(j\omega)$ is the Fourier transform of $w$, and $\Pi(j\omega) = \Pi^*(j\omega)$ is a matrix function referred to as the multiplier of $\sigma$ and assumed to be bounded on the image axis.

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In the sequel, we will refer to condition $\sigma(w, \Pi(\omega)) \geq 0$ (5.5) as an IQC with multiplier $\Pi(\omega)$.

**Definition 7** Let $\mathcal{H} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be an operator, we say $\mathcal{H}$ is $\{X, Y, Z\}$-dissipative if there exist real matrices $X, Y, Z$ such that

$$
\Phi = \begin{bmatrix}
X & Y \\
Y^T & Z
\end{bmatrix}
$$

is a full rank matrix and with $p(t) = \mathcal{H}(q(t)), p, q \in \mathcal{L}_{2e}$

$$
\int_0^\infty \begin{bmatrix} p(t) \\
q(t)
\end{bmatrix}^T \begin{bmatrix}
X & Y \\
Y^T & Z
\end{bmatrix} \begin{bmatrix} p(t) \\
q(t)
\end{bmatrix} dt \geq 0.
$$

(5.6)

Note that condition (5.6) can be easily represented in the frequency domain as IQC (5.5) where $\Pi(\omega)$ is restricted to be constant matrix. We often call (5.6) an IQC in the time-domain form.

Many important properties of basic system interconnections used in stability analysis can be characterized by IQC’s with a proper multiplier $\Pi(\omega)$. A collection of common used IQC’s has been summarized in [54]. Based on results on $(D, G)$-scaling [56], the following linear time varying operators of fixed block and scalar operators can be equivalently represented by IQCs with proper constant multiplier $\Pi$’s.

**Lemma 7** Suppose $\tilde{\delta} : \mathcal{L}^2_n \to \mathcal{L}^2_n$, if the LTV operator $\tilde{\delta}$ is self-adjoint and contractive, then for any $D \in \mathbb{R}^{n\times n}_S, D \geq 0$ and $G = -G^T, \tilde{\delta}I_n$ is $(-D, G, D)$-dissipative.
• Suppose $\delta : L^2_n \to L^2_n$, if the LTV operator is contractive, then for any $D \in \mathbb{R}^{n \times n}$, $D \geq 0$, $\delta I_n$ is $(-D, 0, D)$-dissipative.

• There is a contractive LTV operator, $\Delta : L^2_n \to L^2_n$ such that $p = \Delta q$ if and only if $\Delta$ is $(-I, 0, I)$-dissipative.

**Definition 8** A quadratic performance is a quadratic functional $\sigma_z(z, d)$ defined as

$$\sigma_p(z, d) = \int_0^\infty \begin{bmatrix} d(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} \Pi_{p1} & \Pi_{p2} \\ \Pi_{p2}^T & \Pi_{p3} \end{bmatrix} \begin{bmatrix} d(t) \\ z(t) \end{bmatrix} dt.$$  \hspace{1cm} (5.7)

A system is said to satisfy the $\sigma_p$-performance criterion over a set of disturbance $W$ if the system is well-posed, internally stable and its performance measurement $z$ satisfies $\sigma_z(z, d) < 0$.

The following proposition gives a sufficient condition for the system that satisfies the performance criterion $\sigma_p < 0$ over a class of signals $W$ which can be characterized by IQCs.

**Proposition 12** [87] Suppose the operator $\Delta P$ in (4.7) is $\{X, Y, Z\}$-dissipative. Then the interconnected systems (4.6), (4.7) satisfies $\sigma_p(z, d)$ performance (5.7), if there exists a symmetric matrix $X_T \in \mathbb{R}^{m \times m}_S$, $X_T > 0$, such that the following
LMI holds true,

\[
\begin{bmatrix}
I & 0 & 0 \\
A_{TT} & B_{TS} & B_{Td} \\
0 & I & 0 \\
A_{ST} & A_{SS} & B_{Sd} \\
0 & 0 & I \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 \\
A_{TT} & B_{TS} & B_{Td} \\
0 & I & 0 \\
A_{ST} & A_{SS} & B_{Sd} \\
0 & 0 & I \\
C_{Tz} & C_{Sz} & D_{zd}
\end{bmatrix}
\begin{bmatrix}
0 & X_T & 0 & 0 & 0 & 0 \\
X_T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Y & 0 & 0 & 0 \\
0 & 0 & Y^T & Z & 0 & 0 \\
0 & 0 & 0 & 0 & \Pi_{p1} & \Pi_{p2} \\
0 & 0 & 0 & 0 & \Pi^T_{p2} & \Pi_{p3}
\end{bmatrix}
< 0
\]

(5.8)

**Remark 12** The operator $\Delta$ used to model the interconnection $v = \Delta w$ is characterized by several IQCs, $\sigma_{w1}, \sigma_{w2}, \ldots, \sigma_{wn}$. In this case the performance can be formulated as a convex feasibility problem over the set of IQCs via the lossless $S$-procedures,

\[
\sigma_p(z, d) + \sum_{i=1}^{n} \lambda_i \sigma_{wi}(w) < 0, \quad \forall \ w \in \mathcal{L}_2.
\]

(5.9)

Considering the state space representation of the global system, the above LMI condition (5.8) is simply a reformulation of (5.9) [87].

**Remark 13** From the lossless $(D, G)$ scaling theorem for linear time invariant (LTI) systems with LPV uncertainties, we know for the contractive operators ($\bar{\delta}, \delta$ and $\Delta$) considered in lemma 7, the above result is both necessary and sufficient [56] with proper multiplier $X, Y, Z$, which is called $(D, G)$-scalings for such LTV operators. Here, we only use the sufficient part of the lossless $S$-procedure for Proposition 12. Generally speaking, this sufficient part of proposition 12 can be easily proved via a separation of graph argument, and the inner matrix in equation 5.8 can be interpreted as a hyperplane to separate the graph of the linear time invariant system and the time-varying interconnections. The sufficient condition derived in
Proposition 12 is also necessary for the 3 types of operators mentioned in Lemma 7 according to [56]. The proof of necessity part follows the idea proposed in [88] to construct a causal destabilizing operator when strict separation of the two graphs is violated. The scalar case \( \delta, \tilde{\delta} \) has been proved in [55],[56] respectively. For the contractive operators list in Lemma 7, the above proposition is a LMI reformulation of the necessary, and sufficient condition presented in [56] via an application of the celebrated KYP lemma to the LTI system (4.6).

5.3.2 IQC for the Interconnections

We use IQCs to model the interconnections between subsystems. For each of the subsystems \( G_i, i = i, \ldots, L \), let us introduce the quadratic form on \( \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \), such that

\[
\mathcal{P}_{ij}(v_{ij}, w_{ij}) = \begin{bmatrix} v_{ij}^T & w_{ij} \end{bmatrix} \begin{bmatrix} X_{ij} \end{bmatrix} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}.
\]  

(5.10)

The scaling matrix \( X_{ij} \) is further partitioned into four \( n_{ij} \) by \( n_{ij} \) blocks as

\[
X_{ij} = \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ (X_{ij}^{12})^T & X_{ij}^{22} \end{bmatrix}.
\]  

(5.11)

We are now in a position to state our first analysis results.

**Theorem 6** The interconnected system is well-posed, stable and contractive if there exist symmetric matrices, \( X_i^T \in \mathbb{R}^{m_i \times m_i} \) and \( X_{ij} \in \mathbb{R}^{2n_{ij} \times 2n_{ij}} \), \( X_i^T > 0 \) such that

\[
\begin{bmatrix} I & 0 & 0 \\ A_i^T & A_i^T & B_i^T \\ 0 & I & 0 \\ A_{ij}^T & A_{ij}^T & B_{ij}^T \\ 0 & 0 & I \\ C_i & C_i & D^+ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X_i^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_i & C_i & D^+ \end{bmatrix} < 0 \quad (5.12)
\]

\[
\sigma(\mathcal{P}_X) > 0 \quad (5.13)
\]
for all \( i = 1, \ldots, L \). where

\[
P^{11}_i = \text{diag}_{1 \leq j \leq L} X_{ij}^{11} \quad (5.14)
\]

\[
P^{22}_i = \text{diag}_{1 \leq j \leq L} X_{ij}^{22} \quad (5.15)
\]

\[
P^{12}_i = \text{diag}_{1 \leq j \leq L} X_{ij}^{12} \quad (5.16)
\]

\[
\sigma(P_X) = \int_0^\infty \begin{bmatrix} v \\ w \end{bmatrix}^T P_X \begin{bmatrix} v \\ w \end{bmatrix} \, dt \quad (5.17)
\]

\[
= \sum_{1 \leq i, j \leq L} \int_0^\infty \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}^T \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ (X_{ij}^{12})^T & X_{ij}^{22} \end{bmatrix} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix} \, dt \quad (5.18)
\]

As applications of Theorem 6, we consider different IQCs to model interconnections \( \Delta_{ij} = I_{n_{ij}} \).

5.3.2.1 Perfect Interconnections

Here we assume that \( \Delta_{ij} = I_{n_{ij}}, \forall i, j \), i.e., at any time instance \( t \)

\[
v_{ij}(t) = w_{ji}(t), \quad \forall i, j, t \geq 0. \quad (5.19)
\]

In this case,

\[
\sigma(P_X) = \sum_{1 \leq i, j \leq L} \int_0^\infty \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}^T X_{ij} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix} \, dt \quad (5.20)
\]

\[
= \sum_{i, j \geq j} \int_0^\infty \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}^T \begin{bmatrix} X_{ij}^{11} + X_{ji}^{22} & X_{ij}^{12} + (X_{ji}^{12})^T \\ (X_{ij}^{12})^T + X_{ji}^{12} & X_{ij}^{22} + X_{ji}^{11} \end{bmatrix} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix} \quad (5.21)
\]
Suppose we choose for all $1 \leq i, j \leq L$

\[
X_{ij}^{11} + X_{ji}^{22} = 0 \quad (5.22)
\]

\[
X_{ij}^{12} + (X_{ji}^{12})^T = 0. \quad (5.23)
\]

Then

\[
\sigma(P_{X_{ideal}}) = 0. \quad (5.24)
\]

The family of multipliers $X_{ideal}$ can thus be characterized by the two sets

\[
\left\{ X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}}, i, j = 1, \ldots, L \right\}
\]

and

\[
\left\{ X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}} : X_{ii}^{12} \text{ skew-symmetric}, 1 \leq j \leq i \leq L \right\}. \quad (5.25)
\]

**Proposition 13** The interconnected system is well-posed, stable and contractive for all $\Delta_{ij} = I_{n_{ij}}$ if there exist symmetric matrices, $X_T^i \in \mathbb{R}^{n_i \times m_i}$ and $X_{ij} \in \mathbb{R}^{2n_{ij} \times 2n_{ij}}, X_T^i > 0$ such that

\[
\begin{bmatrix}
I & 0 & 0 \\
A_{TT}^i & A_{TS}^i & B_T^i \\
0 & I & 0 \\
A_{ST}^i & A_{SS}^i & B_S^i \\
0 & 0 & I \\
C_T^i & C_S^i & D^i
\end{bmatrix}^* \begin{bmatrix}
0 & X_T^i & 0 & 0 & 0 & 0 \\
X_T^i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{11}^i & P_{12}^i & 0 & 0 \\
0 & 0 & (P_{12}^i)^* & P_{22}^i & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix} \begin{bmatrix}
I & 0 & 0 \\
A_{TT}^i & A_{TS}^i & B_T^i \\
0 & I & 0 \\
A_{ST}^i & A_{SS}^i & B_S^i \\
0 & 0 & I \\
C_T^i & C_S^i & D^i
\end{bmatrix} < 0,
\]

(5.26)
for all \(i = 1, \ldots, L\). where

\[
P^{11}_i = \text{diag}_{1 \leq j \leq L} X^{11}_{ij}
\]

\[
P^{22}_i = \text{diag}_{1 \leq j \leq L} - X^{11}_{ji}
\]

\[
P^{12}_i = \text{diag}(\text{diag}_{1 \leq j \leq i} X^{12}_{ij}, \text{diag}_{i \leq j \leq L} - (X^{12}_{ji})^T).
\]

### 5.3.2.2 Directed Interconnection with \(\Delta_{ij} = \delta_{ij} I_{n_{ij}}\)

Let us now consider the new class of interconnected systems with \(\Delta_{ij} = \delta_{ij} I_{n_{ij}}\) and \(\|\delta_{ij}\| \leq 1\). We are seeking a new IQC to model such interconnections.

Following similar derivation, suppose we parametrize the multiplier \(X_{ij}\) by the following sets of matrix

\[
\{X^{11}_{ij} \in \mathbb{R}^{n_{ij} \times n_{ij}}, X^{11}_{ij} < 0, i, j = 1, \ldots, L\}
\]

and

\[
\{X^{22}_{ij} = X^{11}_{ji}, X^{12}_{ij} = 0\}.
\]

In this case, it is straightforward to verify

\[
\sigma(P_{X_s}) = \frac{1}{2} \sum_{1 \leq i,j \leq L} \left< \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}, \begin{bmatrix} X^{11}_{ij} & 0 \\ 0 & -X^{11}_{ji} \end{bmatrix} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix} \right> + \frac{1}{2} \sum_{1 \leq i,j \leq L} \left< \begin{bmatrix} v_{ji} \\ w_{ji} \end{bmatrix}, \begin{bmatrix} X^{11}_{ji} & 0 \\ 0 & -X^{11}_{ij} \end{bmatrix} \begin{bmatrix} v_{ji} \\ w_{ji} \end{bmatrix} \right> = \sum_{1 \leq j \leq i \leq L} \left< v_{ij}, X^{11}_{ij} v_{ij} \right> - \left< w_{ji}, X^{11}_{ij} w_{ji} \right> \geq 0.
\]
Following similar arguments, we have the following propositions (14, 16), the sufficient part can be similarly proved as Proposition 13, and the necessity part follows from the lossless-$(D,G)$-scaling theorem for LTV uncertainties. We omit the details for reason of space.

**Proposition 14** The interconnected system is well-posed, stable and contractive for all $\Delta_{ij} = I_{n_{ij}} \delta, \|\delta\| \leq 1$ if and only if there exist symmetric matrices, $X^i_T \in \mathbb{R}^{m_i \times m_i}, X^i_T > 0$ and for all $i, j = 1, \ldots, L$, $X_{ij} \in \mathbb{R}^{n_{ij} \times n_{ij}}, X_{ij}^{11} < 0$ and LMI (5.26) are satisfied for all $i = 1 \ldots, L$, with $P^{11}_i = \text{diag}_j(X^{11}_{ij}), P^{22}_i = \text{diag}_j(-X^{11}_{ji})$ and $P^{12}_i = 0$.

**Proposition 15** The interconnected system is well-posed, stable and contractive for all LTV $\Delta_{ij} = I_{n_{ij}} \tilde{\delta}, \|\tilde{\delta}\| \leq 1$, $\tilde{\delta}$ self-adjoint if and only if there exist symmetric matrices, $X^i_T \in \mathbb{R}^{m_i \times m_i}, X^i_T > 0$ and for all $i, j = 1, \ldots, L$, $X_{ij}^{11}, X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}, X_{ij}^{11} < 0$ and LMI (5.26) are satisfied for all $i = 1 \ldots, L$, with $P^{11}_i, P^{22}_i$ and $P^{12}_i$ defined as (5.27), (5.28), (5.29).

**Proposition 16** The interconnected system is well-posed, stable and contractive for all LTV $\Delta_{ij}, \|\Delta_{ij}\| \leq 1$, if and only if there exist symmetric matrices, $X^i_T \in \mathbb{R}^{m_i \times m_i}, X^i_T > 0$ and for all $i, j = 1, \ldots, L$, $d_{ij} < 0, X_{ij}^{11} = d_{ij}^{11} I_{n_{ij}}$ and LMI (5.26) are satisfied for all $i = 1 \ldots, L$, with $P^{11}_i = \text{diag}_j(X^{11}_{ij}), P^{22}_i = \text{diag}_j(-X^{11}_{ji})$ and $P^{12}_i = 0$.

**Remark 14** Before we apply the stability analysis results for synthesis, we would like to make a comment here. Theorem 6 unified the stability results for different interconnections which can be modeled as integral quadratic constraints. This theorem renders the performance specification based on the interconnected uncertain systems to an explicit expression through S-procedure, where the multipliers
$X_{ij}$ are shaped by the structure and properties of the interconnection operator $\Delta_{ij}$. Note that, although sufficient stability conditions can be derived in this framework easily, the necessity part is not trivial, and can be constructed by utilizing the special structure of the operator. Generally speaking, Theorem 6 reflects the simple idea of topological separation of the graph generated via the LTI plant and the LTV uncertainty.

5.4 Synthesis via Elimination lemma

The synthesis part of this chapter follows the same line of derivation presented in [14], which is based on the extended elimination lemma. We want to point out that for the synthesis condition corresponding to Theorem 2 in [14], $n^K_{ij} = n_{ij}$ is adequate.

Now let us consider each of subsystem $G_i$ with control input $u_i$ and a measured output $y_i$, in addition to the signals given in 5.1, such that

$$
\begin{bmatrix}
\dot{x}_i(t) \\
w_i(t) \\
\vdots \\
y_i(t)
\end{bmatrix}
= 
\begin{bmatrix}
A^i_{TT} & A^i_{TS} & B^i_{Td} & B^i_{Tu} \\
A^i_{ST} & A^i_{SS} & B^i_S & B^i_{Su} \\
C^i_T & C^i_S & D^i & D^i_{zu} \\
C^i_{Ty} & C^i_{Sy} & D^i_{yu} & D^i_{yu}
\end{bmatrix}
\begin{bmatrix}
x_i(t) \\
v_i(t) \\
\vdots \\
d_i(t)
\end{bmatrix}
$$

(5.30)

for all $t \geq 0$ and $i = 1, \ldots, L$. Here $\Delta_{ji}$ is an operator used to specify the interconnection. In the rest of this chapter, without loss of generality, we assume that $D^i_{yu} = 0, \forall i$. Similar to the controller considered in the LPV literature, we are seeking a controller with "similar" structure as the plant: another interconnected
system $K$ with subsystems $K_i, i = 1, \ldots, L$ given by

$$
\begin{bmatrix}
\dot{x}_i^K(t) \\
 w_i^K(t) \\
u_i(t)
\end{bmatrix} =
\begin{bmatrix}
(A_{TT}^i)_K & (A_{TS}^i)_K & (B_T^i)_K \\
(A_{ST}^i)_K & (A_{SS}^i)_K & (B_S^i)_K \\
(C_T^i)_K & (C_S^i)_K & D_K
\end{bmatrix}
\begin{bmatrix}
x_i^K(t) \\
v_i^K(t) \\
y_i(t)
\end{bmatrix} \quad (5.32)
$$

such that the closed loop system is well-posed, stable and contractive. In addition, we require $n_{ij}^K = 0$ whenever $n_{ij} = 0$, which means if subsystem $G_i$ does not communicate with subsystem $G_j$, neither will controller $K_i$ communicate with controller $K_j$.

Straightforward matrix manipulation shows that each of the closed-loop sys-
tems admits the state-space matrices,

\[
\begin{bmatrix}
  (A^i_{TT})_C & (A^i_{TS})_C & (B^i_T)_C \\
  (A^i_{ST})_C & (A^i_{SS})_C & (B^i_S)_C \\
  (C^i_T)_C & (C^i_S)_C & (D^i)_C
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  A^i_{TT} & 0 & A^i_{TS} & 0 & B^i_{Td} \\
  0 & 0 & 0 & 0 & 0 \\
  A^i_{ST} & 0 & A^i_{SS} & 0 & B^i_{Sd} \\
  0 & 0 & 0 & 0 & 0 \\
  C^i_{Tz} & 0 & C^i_{Sz} & 0 & D^i_{zd}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  0 & 0 & B^i_{Td} \\
  I & 0 & 0 \\
  0 & 0 & B^i_{Su} \\
  0 & I & 0 \\
  0 & 0 & D^i_{zu}
\end{bmatrix}
\Theta_i
\times
\begin{bmatrix}
  0 & I & 0 & 0 & 0 \\
  0 & 0 & I & 0 & 0 \\
  C^i_{Ty} & 0 & C^i_{Sy} & 0 & D^i_{yd}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  A^i & B^i \\
  C^i & D^i
\end{bmatrix}
+ \begin{bmatrix}
  B^i_{Tu} \\
  B^i_{Su} \\
  D^i_{zu}
\end{bmatrix}
\Theta_i
\begin{bmatrix}
  C^i_{Ty} & C^i_{Sy} & D^i_{yd}
\end{bmatrix}
\]
which is linear in the controller matrix denoted by $\Theta_i$,

$$\Theta_i = \begin{bmatrix}
(A_{IT})_K & (A_{TS})_K & (B_T)_K \\
(A_{ST})_K & (A_{SS})_K & (B_S)_K \\
(C_T)_K & (C_S)_K & D^i_K
\end{bmatrix}. \quad (5.33)$$

The state variable for the subsystem $x^i_c$ has size $m_i + m^i_K$,

$$x^i_c = \begin{bmatrix}
x_i \\
x^K_i
\end{bmatrix}. \quad (5.34)$$

The interconnection signal $w^C_{ij}, v^C_{ij}$ has size $n^C_{ij} = n_{ij} + n^K_{ij}$,

$$w^C_{ij} = \begin{bmatrix} w_{ij} \\ w^K_{ij} \end{bmatrix}, \quad (5.34)$$

$$v^C_{ij} = \begin{bmatrix} v_{ij} \\ v^K_{ij} \end{bmatrix}. \quad (5.35)$$

In addition, since the controller $K$ and the plant $G$ share the same communication channel between each subsystem, we further require

$$v^C_{ij} = \Delta_{ji} w^C_{ji}. \quad (5.36)$$

Before we apply the analysis results to the close-loop system, we would like to make the following convention that every symmetric matrix $T^i_C \in \mathbb{R}^{m^i_c \times m^i_c}$ is
partitioned according to the plant’s and controller’s state

\[ T^i_C = \begin{bmatrix} T^i_G & T^i_{GK} \\ (T^i_{GK})^T & T^i_K \end{bmatrix}, \]

with \( T^i_G \in \mathbb{R}^{m_i \times m_i} \), and \( T^i_K \in \mathbb{R}^{m^K_i \times m^K_i} \), where \( m^K_i \) is the dimension of the controller’s state. Similarly, if \((T_{ij})_C \) belongs to \( \mathbb{R}^{n_{ij}^C \times n_{ij}^C} \), we partition it according to the plant’s and controller’s interconnection signals as

\[ (T_{ij})_C = \begin{bmatrix} (T_{ij})_G & (T_{ij})_{GK} \\ (T_{ij})_{GK}^* & (T_{ij})_K \end{bmatrix}, \]

with \((T_{ij})_G \in \mathbb{R}^{n_{ij} \times n_{ij}}\) and \((T_{ij})_K \in \mathbb{R}^{n^K_{ij} \times n^K_{ij}}\). We are now ready to apply the analysis result to the closed-loop systems.

**Proposition 17** The closed-loop system is well-posed, stable and contractive if there exist symmetric matrices \((X^i_T)_C \in R^{m^C_i \times m^C_i}_S\) and \(X^{11}_{ij} \in R^{n^C_{ij} \times n^C_{ij}}_S\) for all \(i, j = 1, \ldots, L\), and \((X^{12}_{ij})_C \in R^{n^C_{ij} \times n^C_{ij}}_S\) for all \(i \geq j\), with \((X^{12}_{ii})_C \) skew symmetric, such that \((X^i_T)_C > 0\) and

\[(M^C_i)^T P^C_i M^C_i < 0 \quad (5.37)\]
with

\[
M_i^C = \begin{bmatrix}
I & 0 & 0 \\
(A^i_{TT})_C & (A^i_{TS})_C & (B^i_T)_C \\
0 & I & 0 \\
(A^i_{ST})_C & (A^i_{SS})_C & (B^i_S)_C \\
0 & 0 & I \\
(C^i_T)_C & (C^i_S)_C & (D^i)_C
\end{bmatrix}
\]

(5.38)

\[
P_i^C = \begin{bmatrix}
0 & (X^i_T)_C & 0 & 0 & 0 & 0 \\
(X^i_T)_C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (Z^{11}_i)_C & (Z^{12}_i)_C & 0 & 0 \\
0 & 0 & (Z^{12}_i)_C^* & (Z^{22}_i)_C & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix}
\]

(5.39)

for all \( i = 1, \ldots, L \) and

\[
(Z^{11}_i)_C = \text{diag}_j(X^{11}_{ij})_C
\]

\[
(Z^{22}_i)_C = \text{diag}_j -(X^{11}_{ij})_C
\]

\[
(Z^{12}_i)_C = \text{diag}(\text{diag}_{1 \leq j \leq i}(X^{12}_{ij})_C, \text{diag}_{i < j \leq L}((X^{12}_{ji})^T)_C).
\]

The derivation of (5.37) is a reformulation of the analysis result of Proposition 13. But since the controller matrix \( \Theta_i \) is generally unknown at this stage, the above condition is a Bilinear Matrix Inequality (BMI) condition for the set of matrices \( \Theta_i \) and the scaling matrices \( (X^i_T)_C, (X_{ij})_C \), which means if we fixed either of the sets, the above condition is a LMI condition for the remaining sets.
Using the elimination lemma, the following equivalent LMI conditions can be obtained. Notice here the following conditions do not depend on the controller matrices \( \Theta_i \) to be designed.

**Proposition 18** There exist distributed controllers with state representation (5.32) with \( n^K_{ij} = n_{ij} \) and interconnection \( \Delta_{ij} = I \) such that the closed-loop system conditions (5.37) are satisfied if and only if for all \( i = 1, \ldots, L \), there exist symmetric matrices \( (X^i_T)_G \), \( (Y^i_T)_G \in \mathbb{R}^{T_{ij}^T} \) for all \( i, j = 1, \ldots, L \), and \( (X^i_{11})_G \), \( (Y^i_{11})_G \in \mathbb{R}^{T_{ij}^T} \) for all \( i, j = 1, \ldots, L \) and \( (X^i_{12})_G \), \( (Y^i_{12})_G \in \mathbb{R}^{n_{ij} \times n_{ij}} \) for \( i \geq j \), with \( (X^i_{12})_G \), \( (Y^i_{12})_G \) skew-symmetric such that \( (X^i_T) > 0, (Y^i_T) > 0 \) and LMIs (5.48), (5.49), (5.50) are satisfied, and \( \Psi^i, \Phi^i \) are defined as (5.40), (5.41) respectively.

\[
\begin{align*}
\Psi^i &= \ker \begin{bmatrix} C^i_{Ty} & C^i_{Sy} & D^i_{yd} \end{bmatrix} \quad (5.40) \\
\Phi^i &= \ker \begin{bmatrix} (B^i_{Tu})^T & (B^i_{Su})^T & (D^i_{zu})^T \end{bmatrix} \quad (5.41)
\end{align*}
\]

and

\[
\begin{align*}
(Z^i_{11}) &= \text{diag}_{1 \leq j \leq L}(X^i_{11}^j)_G \\
(Z^i_{22}) &= -\text{diag}_{1 \leq j \leq L}(X^i_{11}^j)_G \\
(Z^i_{22}) &= \text{diag} \left\{ \text{diag}_{1 \leq j \leq L}(X^i_{12}^j)_G, -\text{diag}_{1 \leq j \leq L}(X^i_{12}^j)^T \right\} \quad (5.44) \\
(\tilde{Z}^i_{11}) &= \text{diag}_{1 \leq j \leq L}(Y^i_{11}^j)_G \\
(\tilde{Z}^i_{22}) &= -\text{diag}_{1 \leq j \leq L}(Y^i_{11}^j)_G \\
(\tilde{Z}^i_{22}) &= \text{diag} \left\{ \text{diag}_{1 \leq j \leq L}(Y^i_{12}^j)_G, -\text{diag}_{1 \leq j \leq L}(Y^i_{12}^j)^T \right\} \quad (5.47)
\end{align*}
\]
controller described by (5.32) is linear in the controller’s parameter \( \Theta_i \) with

\[
\Theta_i = \begin{bmatrix}
(A_{TT}^i)_{K} & (A_{TS}^i)_{K} & (B_T^i)_{K} \\
(A_{ST}^i)_{K} & (A_{SS}^i)_{K} & (B_S^i)_{K} \\
(C_T^i)_{K} & (C_S^i)_{K} & D_K^i
\end{bmatrix}
\]

then apply the elimination lemma from [36] (see appendix) to each individual stability condition derived in Proposition 13 (5.26) for the closed-loop system. The

Proof 14 Notice that the closed loop system for the individual subsystem with the controller described by (5.32) is linear in the controller’s parameter \( \Theta_i \) with

\[
\begin{bmatrix}
(X_T^i)_{G} & I \\
I & (Y_T^i)_{G}
\end{bmatrix} > 0
\]

(Necessity) Suppose there exist symmetric matrices \((X_T^i)^C \in R_{S^{m_i} \times m_i}^C\) and \((X_{ij}^{12})^C \in R_{S^j \times n_{ij}^C}^C\) for all \( i, j = 1, \ldots, L, \) and \((X_{ij}^{12})^C \in R_{S^j \times n_{ij}^C}^C\) for all \( i \geq j, \) with \((X_{ij}^{12})^C\) skew symmetric, such that \((X_T^i)_{C} > 0 \) and (5.26) is satisfied, then try to
show that (5.48), (5.49) are satisfied. Let

\[
P_C^i = \begin{bmatrix}
0 & (X^i_T) & 0 & 0 \\
(X^i_T) & 0 & 0 & 0 \\
0 & 0 & (Z^{11}_i) & (Z^{12}_i) \\
0 & 0 & (Z^{12}_i)^* & (Z^{22}_i)
\end{bmatrix}
\]  \quad (5.52)

and

\[
\tilde{P}_C^i = (P_C^i)^{-1} = \begin{bmatrix}
0 & (\tilde{X}^i_T) & 0 & 0 \\
(\tilde{X}^i_T) & 0 & 0 & 0 \\
0 & 0 & (\tilde{Z}^{11}_i) & (\tilde{Z}^{12}_i) \\
0 & 0 & (\tilde{Z}^{12}_i)^* & (\tilde{Z}^{22}_i)
\end{bmatrix}.
\]  \quad (5.54)

Suppose $P_C^i$ is non-singular. Applying the elimination lemma 11 to the stability condition (5.26) of the closed-loop system, we have

\[
\begin{bmatrix}
\Phi_x
\end{bmatrix}^T < 0
\]
Due to the zero block of $\Phi_e$, it is obvious that this is the same as

$$
\begin{pmatrix}
\begin{array}{cccc}
0 & (X^i_C) & 0 & 0 \\
0 & (X^i_C) & 0 & 0 \\
0 & 0 & (Z^{11}_i)^T C & (Z^{12}_i)^T C \\
0 & 0 & (Z^{12}_i)^T C & (Z^{22}_i)^T C \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0
\end{array}
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
0 & A^i_{TT} & A^i_{TS} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
C^i_T, C^i_S, B^i_d
\end{pmatrix}
< 0
$$

The four zero block rows in the outer factors allow to simplify this inequality to (5.48). (5.49) can be derived via similar argument with respect to $\bar{P}^i_C$.

(Sufficiency) If the LMIs (5.48), (5.48), (5.50) are feasible, we can find the block multiplier $(X^i_C), (Z_i)^C$, such that proposition (5.26) can be satisfied. We see that it is enough to show that

$$
\begin{pmatrix}
(X^i_T)G & (X^i_T)GK \\
((X^i_T)G)^* & (X^i_T)K
\end{pmatrix}
= \begin{pmatrix}
(Y^i_T)G & (Y^i_T)GK \\
((Y^i_T)G)^* & (Y^i_T)K
\end{pmatrix} > 0
$$

This can be proved via Lemma 12.

$$
(X^{-1}_{ij}C) = \begin{pmatrix}
(X_{ij}^{11})C & (X_{ij}^{12})C \\
(X_{ij}^{12})^* & -(X_{ji}^{11})C
\end{pmatrix}
$$

for all $i \geq j$. Notice that for $i = j$, $(X_{ij})_C \in \kappa^C_{ij} \times 2n^C_{ij}$, which can be proved via lemma 14, and for the general case $i > j$, this can be proved via Lemma
Corresponding to Proposition 14, Proposition 15 and Proposition 16, the following synthesis condition can be proved similarly to Proposition 18.

**Proposition 19** There exist distributed controllers with state representation (5.32) with $n^K_{ij} = n_{ij}$ and interconnection $\Delta_{ij} = \delta I, \|\delta\| \leq 1$ such that the closed-loop system is well-posed, stable and contractive if and only if for all $i = 1, \ldots, L$, there exist symmetric matrices $(X^i_T)G, (Y^i_T)G \in R^{m_i \times m_i} (X^i_T)G, (Y^i_T)G \in R^{n_{ij} \times n_{ij}}$ for all $i, j = 1, \ldots, L$, and $(X^{11}_{ij})G, (Y^{11}_{ij})G \in R^{n_{ij} \times n_{ij}}, (X^{11}_{ij})G < 0, (Y^{11}_{ij})G < 0$ for all $i, j = 1, \ldots, L$ and $(X^{12}_{ij})G, (Y^{12}_{ij})G \in R^{n_{ij} \times n_{ij}}$ for $i, j = 1, \ldots, L$, with $(X^{12}_{ij})G = (Y^{12}_{ij})G = 0$ such that $(X^i_T) > 0, (Y^i_T) > 0$ and LMIs (5.48), (5.49), (5.50) are satisfied, and $\Psi^i, \Phi^i$ are defined as (5.40), (5.41) respectively.

**Proposition 20** There exist distributed controllers with state representation (5.32) with $n^K_{ij} = n_{ij}$ and interconnection $\Delta_{ij} = \tilde{\delta} I$ such that the closed-loop system is well-posed, stable and contractive if and only if for all $i = 1, \ldots, L$, there exist symmetric matrices $(X^i_T)G, (Y^i_T)G \in R^{m_i \times m_i} (X^i_T)G, (Y^i_T)G \in R^{n_{ij} \times n_{ij}}$ for all $i, j = 1, \ldots, L$, and $(X^{11}_{ij})G, (Y^{11}_{ij})G \in R^{n_{ij} \times n_{ij}}, (X^{11}_{ij})G < 0, (Y^{11}_{ij})G < 0$ for all $i, j = 1, \ldots, L$ and $(X^{12}_{ij})G, (Y^{12}_{ij})G \in R^{n_{ij} \times n_{ij}}$ for $i \geq j$, with $(X^{12}_{ij})G, (Y^{12}_{ij})G$ skew-symmetric such that $(X^i_T)G > 0, (Y^i_T)G > 0$ and LMIs (5.48), (5.49), (5.50) are satisfied, and $\Psi^i, \Phi^i$ are defined as (5.40), (5.41) respectively.

**Proposition 21** There exist distributed controllers with state representation (5.32) with $n^K_{ij} = n_{ij}$ and interconnection $\Delta_{ij}, \|\Delta_{ij}\| \leq 1$ such that the closed-loop system is well-posed, stable and contractive if and only if for all $i = 1, \ldots, L$, there exist symmetric matrices $(X^i_T)G, (Y^i_T)G \in R^{m_i \times m_i}$, and $x_{ij}, y_{ij} \in R$ such that $x_{ij} < 0, y_{ij} < 0, (X^{11}_{ij})G = x_{ij}I_{n_{ij}}, (Y^{11}_{ij})G = y_{ij}$ for all $i, j = 1, \ldots, L$ and
\((X_{ij}^{12})_G, (Y_{ij}^{12})_G \in \mathbb{R}^{n_{ij} \times n_{ij}}, (X_{ij}^{12})_G = (Y_{ij}^{12})_G = 0\) for \(i \geq j\) such that \((X_T^i)_G > 0, (Y_T^i)_G > 0\) and LMIs (5.48), (5.49), (5.50) are satisfied, and \(\Psi^i, \Phi^i\) are defined as (5.40), (5.41) respectively.

5.5 Conclusion

In this chapter, we derived stability conditions for distributed systems with IQC constraints for the internal interconnections. Technically, the sufficient stability results follow from the application of the S-procedure and can be proved via a graph separation argument. Our stability theorem reduces the global performance with implicit uncertainty interconnections to explicit conditions with design multipliers parametrized by the uncertainty. Specifically, our theorem generalizes the results presented in [14].
6.1 Summary of Dissertation

In this dissertation, we reported our current results and techniques for controlling interconnected systems under communication constraints. Of particular interest are theoretical questions regarding the limitations of the performance of distributed systems under bit-rate constraint and the influence of a complex system’s structure on its stability and robustness. Specifically, this work addressed three important problems and related issues: quantization, stability of spatially invariant systems, and distributed systems with interconnection constraints. The results obtained in this dissertation provide some fundamental ideas of the communication limitation and interconnection constraints on the performance of the global system.

We first studied the problem of control under bit-rate constraints in Chapter 2. We investigated the optimization problem of maximizing the convergence rate of the quantization error. Based on our formulation, the feasible problem of convergence of the quantization error leads to the minimum bit-rate $R_{\text{min}}$ for stabilization known in the literature. The solution of our optimization problem provides the tightest bound on the maximum convergence rate of the quantization error. Similar bounds on the convergence rate of the quantization error have been
derived in the literature. The bound derived in this dissertation is the tightest achievable one, which is obtained by transforming the original integer programming problem to a continuous optimization problem. We have also proved that this bound is achievable via a dynamic quantization policy. In this way, our results reveal the fundamental limitations of the quantized system under bit rate constraints in terms of the asymptotic behavior of the quantization error. Later in the same chapter, we studied the performance degeneration under communication constraints. Our novel contribution is to provide a closed-form characterization of the performance degeneration of the LQR controller under bit-rate constraints and quantization. It is one of the first few results reported to quantify the performance degeneration of classical control problems. With the assumption on the IID and Gaussian properties of the quantization error, it is shown that the stochastic LQR controller for the quantized system shares the same feedback matrices as the original problem, which greatly simplifies the controller design part of these systems; furthermore, since the performance degeneration is a function of quantization policy and bit-rate constraints, we can then concentrate on the quantization policy to minimize the effect from the bit-rate constraint. In this way, these results reflect a simple idea of separation theory for these quantized systems: the separate design of controllers and quantizers.

Next we take a robust control approach to address the stability problems of spatially invariant systems in Chapter 3 and control of distributed systems in Chapter 4 and Chapter 5. In these chapters, we have taken a LFT based uncertain interconnection modelling approach which is well-known in the literature of LPV control to model distributed systems and interconnections. In this framework, the rich results on structured singular values, KYP lemma and gain-scheduling
techniques are used to obtain the LMI results for computation.

In Chapter 3, we analyzed the stability of spatially invariant systems. Three important contributions are summarized:

- The exponentially decaying property of the bounded shift-invariant operators for spatially invariant systems is proved via the Cauchy estimation formulas.
- Stability analysis of spatially invariant systems via the main-loop theorem.
- The theorem on non-negative pseudo-polynomial matrix theory is introduced for LMI reformulation of the infinite-dimensional optimization problem.

Although some of these results have already been reported in the literature, in this dissertation we have provided more intuitive proofs and pointed out their relations to more fundamental mathematical theorems such as the Cauchy estimation formulas.

In Chapter 4, we considered the stability of distributed systems over graphs. In this framework, the interconnections are represented by algebraic constraints. It is well-known that stability as well as $H_\infty$ performance for the LTI global system can be formulated as a set of LMIs in terms of the state-space representation matrices from the input-output point view, with a set of equality constraints on the internal signals. With the simple idea of Lagrange multipliers on constrained optimization problems, based on an extended version of the block S-procedure, we have shown in this chapter that the so-called distributed stability conditions are simple applications of the S-procedure. This is a two-step approach. The first step is to model constraints by properly chosen multipliers; the second step is to introduce these multiplies into the stability/performance conditions to recover the
stability conditions in terms of all external signals. These insightful techniques such as the newly-developed block S-procedure and the optimization ideas are what we try to emphasize in this dissertation for our research.

Noticing the diagonal structure of the global system and the corresponding diagonal structure of the multipliers. These distributed conditions can be obtained easily. The derived stability results for distributed control systems are direct applications of the block S-procedure for the global system.

In Chapter 5, integral quadratic constraints (IQC) are used to model the interconnections for more general set of signals. Compared to the algebraic constraints to model interconnection operators in $\mathbb{R}^n$, IQCs are used to model operators in $\mathcal{L}_2^n$. For shift-invariant operators, the lossless S-procedure plays the same role as the block S-procedures in Chapter 4, i.e., introducing the multipliers into the constrained design problems in the general theory of Lagrange multipliers. In this way, similar stability conditions are derived for a set of structured operators. We have also shown that based on some of the newly-developed synthesis techniques and using the elimination lemma, the distributed closed-loop stability conditions are convex. Technically, all the synthesis results are essentially the same as those convex conditions in gain-scheduling control. As mentioned earlier in this chapter, all the stability conditions are equivalent to a centralized stability condition for the global system. It is the special structure of the interconnection operators that provides a diagonal structure to the centralized stability conditions. It might seem that these results are just reformulations of the known LMI stability techniques for some special IQCs. We want to emphasize here that for the special IQCs considered in the Chapter 5, a set of decoupled stability conditions not only exist but also may be used for distributed controller synthesis, i.e, they can be utilized for
distributed controller synthesis in a way identical to gain-scheduling controllers in LPV systems.

It is known that distributed control is not always feasible. In Chapter 4 and Chapter 5 of this dissertation, we have identified and characterized a few special structures that can be controlled by distributed controllers.

6.2 Future Directions

In this dissertation we summarize some of the latest theoretical progress in the area of distributed control under information constraints. Our work only takes preliminary steps toward understanding robust control under interconnection constraints for distributed systems. There are still a lot of difficult questions unresolved that need to be addressed. In particular, we need to address the following:

- A modified information theory for control purpose. As we have mentioned in previous chapters, the quantization problem is similar to a source coding problem. With information on observations of the system and control applied to the system, how can we design optimal coding schemes under different control objectives? We formulate a joint optimization for the code-design of control and quantization in Chapter 2, but we do not know how to solve it.

- Joint optimization of communication and control. The beauty of the LQR problem is that for different proper weighting matrices $Q, R$ on state and control, we can come up with an optimal controller to minimize the LQR cost. Although initial efforts have been made, we still do not know how
to add communication costs into the control costs to come up with some meaningful optimization problems.

- Quantifying performance degeneration of classical controllers under communication constraints.

- Quantifying performance degeneration of distributed controllers and its robustness. We know that centralized controllers are demanding in computation and communication, but we can come up with optimal centralized controllers with classical control theory. Distributed controllers are conservative since they are constrained optimization problems, but there are few tangible results to quantify their limitations. Researchers have begun to study distributed control problems since the 1970s, using Lyapunov approaches which normally treat interconnections as unwanted perturbations. These approaches normally end with conservative sufficient conditions. The robust control approaches as treated in this dissertation utilize more structural information on the interconnections and the analysis results can be directly used for controller synthesis. Even so, there is need to quantify the performance gap between distributed controllers and centralized controllers.
This appendix contains the supporting lemmas to prove the fairness of the DBAP quantization policy considered in Chapter 1.

Lemma 8 Let $J_i$ be defined as above, and $|\lambda_i| > 1$. For any non-zero $v_i \in \mathbb{R}^n$, we have

$$\lim_{k \to \infty} \frac{\|J_i^{k+1}v_i\|_2}{\|J_i^k v_i\|_2} = |\lambda_i|. \quad (A.1)$$

Since $v_{i,j}[k]$ is a scaled version of $J_i^k v_{i,j}[0]$. thus, $\forall \epsilon > 0$, $\exists K_\epsilon$ such that, $\forall k \geq K_\epsilon$,

$$(1 - \epsilon)|\lambda_i| \leq \frac{\|J_i^{k}v_{i,j}[k]\|_2}{\|v_{i,j}[k]\|_2} \leq (1 + \epsilon)|\lambda_i|. \quad (A.2)$$

In particular let $T_{i,j}$ denote the time instants when side $v_{i,j}$ was successfully quantized,

$$T_{i,j} = \{ k : I_k = i, J_k = j, \}. \quad (A.3)$$

The following lemma is from [74], which can be easily proved.

Lemma 9 If $v_{i,j}[0] \neq 0$, then $\text{card}(T_{i,j}) = \infty$.

Now we set up to prove the fairness of the algorithm in terms of the relatively length of $\|v_{i,j}[k]\|$.
Lemma 10 For any $i_1, j_1, i_2, j_2$, there exists a finite constant $r$, such that

$$r^{-1} \leq \frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r.$$  \hfill (A.4)

Proof 15 Suppose at time $K_0 > K_e$, all $v_{i,j}$ has been quantized for at least once. By Lemma 9, $K_0$ is finite.

For all $k \leq K_0$, suppose $v_{i,j}[0] \neq 0$, there exist a constant $r_0$, such that

$$r_0^{-1} \leq \frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r_0$$  \hfill (A.5)

Then for all $k \geq K_0$ choose $i_0, j_0$, such that

$$(i_0, j_0) = \arg \min_{i,j} \|J_iv_{i,j}[k]\|_2$$  \hfill (A.6)

with our assumption on $K_0$ and Lemma 9, we know there must be some finite $l$ such that $(k - l) \in T_{i_0,j_0}$. Choose $\bar{l} = \min l$, that is at time instant $k - \bar{l}$, $v_{i_0,j_0}$ is selected before $k$ for the last quantization. According to our selection policy, we have $\forall (i,j) \neq (i_0,j_0)$

$$\|J_iv_{i,j}[k - \bar{l}]\| \leq \|J_{i_0}v_{i_0,j_0}[k - \bar{l}]\|.$$  \hfill (A.7)

Let $l_{i,j}$ denote the number of quantization for $v_{i,j}$ from $k - \bar{l}$ to $k$, from the inequality (A.2)

$$\|J_iv_{i,j}[k]\| \leq \frac{(1 + \epsilon)^{\bar{l}}\lambda_{i,j}^{\bar{l}}}{Q_{i,j}} \|J_iv_{i,j}[k - \bar{l}]\|.$$  \hfill (A.8)
Applying (A.2) to $v_{i_0,j_0}$, we have

$$\frac{(1 - \epsilon)\lambda_{i_0}^l}{Q}\|J_{i_0}v_{i_0,j_0}[k - \bar{l}]\| \leq \|J_{i_0}v_{i_0,j_0}[k]\|.$$  \tag{A.9}$$

Combining all (A.6), (A.7), (A.8), (A.9), we have

$$\frac{(1 - \epsilon)\lambda_{i_0}^l}{Q} \leq \frac{(1 + \epsilon)\lambda_i^l}{Q^{|i,j|}}.$$  \tag{A.10}$$

Multiply over all $(i, j) \neq (i_0, j_0)$, and notice that $\sum_{i,j:(i,j)\neq(i_0,j_0)} l_{i,j} = \bar{l}$.

$$((1 + \epsilon)\lambda_{i_0})^{(n-1)\bar{l}} \geq \frac{(1 - \epsilon)^{(n-1)\bar{l}}|\det A|^{\bar{l}}}{Q^{|i_0|\lambda_{i_0}^l}}.$$  \tag{A.11}$$

Taking $\log_Q$ of the above equation, we get a upper bound of $\bar{l}$.

$$\bar{l} \leq \frac{n}{(n - 1) \log_Q \frac{1 - \epsilon}{1 + \epsilon} + n \log_Q \lambda_{i_0} + 1 - \log_Q |\det A|}$$

$$\leq \frac{n}{(n - 1) \log_Q \frac{1 - \epsilon}{1 + \epsilon} + 1 - \log_Q |\det A|} = l_c,$$  \tag{A.12}$$

where $l_c = \frac{n}{(n - 1) \log_Q \frac{1 - \epsilon}{1 + \epsilon} + 1 - \log_Q |\det A|}$.

For any $i_1, j_1, i_2$ and $j_2$, we have

$$\frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq \alpha \frac{\|J_{i_2}v_{i_2,j_2}[k]\|_2}{\|J_{i_2}v_{i_2,j_2}[k]\|_2},$$  \tag{A.13}$$
where $\alpha = \frac{1+\epsilon}{1-\epsilon} \max_{i_1,i_2} \frac{\lambda_{i_1}}{|\lambda_{i_2}|}$ following the argument, we know that

$$\frac{\|J_{i_1}v_{i_1,j_1}[k]\|}{\|J_{i_0}v_{i_0,j_0}[k]\|} \leq \frac{|\lambda_{i_1}|(1+\epsilon)^t \|J_{i_1}v_{i_1,j}[k'-\overline{l}]\|}{Q |\lambda_{i_0}|(1-\epsilon)^t \|J_{i_0}v_{i_0,j_0}[k'-\overline{l}]\|}$$ (A.14)

$$\leq Q \left( \frac{|\lambda_{i_1}|(1+\epsilon)}{|\lambda_{i_0}|(1-\epsilon)} \right)^t.$$ (A.15)

$$\leq r_1,$$ (A.16)

where

$$r_1 = Q \left( \max_{i_1,i_2} \frac{|\lambda_{i_1}|}{|\lambda_{i_2}|} \right)^t \left( \frac{1+\epsilon}{1-\epsilon} \right)^t.$$

Since we know $\|J_{i_0}v_{i_0,j_0}[k]\|$ is the smallest among $\|J_i v_{i,j}[k]\|$, then

$$\frac{\|J_{i_1}v_{i_1,j_1}[k]\|}{\|J_{i_2}v_{i_2,j_2}[k]\|} \leq r_1.$$ (A.17)

From (A.13), we know $\forall k \geq K_0$, $$(\alpha r_1)^{-1} \leq \frac{\|v_{i_1,j_1}[k]\|}{\|v_{i_2,j_2}[k]\|} \leq \alpha r_1.$$ (A.18)

Combining (A.5), (A.18), and choosing $r = \max(r_0, \alpha r_1)$, we have proved (A.4) for all $k$. 

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APPENDIX B

SUPPORTING LEMMAS FOR CONTROLLER SYNTHESIS

The following result is basically an extension of the well-known elimination lemma to a quadratic matrix inequality. It is convenient for elimination of controller parameters from the synthesis conditions.

Lemma 11 (Elimination Lemma [36]) Let $P$ be a symmetric matrix with inertia $\text{in}(P) = (m, 0, n)$ and $C \in \mathbb{R}^{n \times m}$. The quadratic matrix inequality,

$$
\begin{bmatrix}
I \\
A^T XB + C
\end{bmatrix}^T P
\begin{bmatrix}
I \\
A^T XB + C
\end{bmatrix} < 0
$$

in the unstructured unknown matrix variable $X$ has a solution if and only if

$$
B_\perp^T
\begin{bmatrix}
I \\
C
\end{bmatrix}^T P
\begin{bmatrix}
I \\
C
\end{bmatrix} B_\perp < 0
\quad (B.2)
$$

$$
A_\perp^T
\begin{bmatrix}
-C^T \\
I
\end{bmatrix}^T P^{-1}
\begin{bmatrix}
-C^T \\
I
\end{bmatrix} A_\perp > 0.
\quad (B.3)
$$

Lemma 12 ([1]) Suppose $S_1, R_1 \in \mathbb{R}_{n \times n}^{s \times s}$, with $S_1 > 0, R_1 > 0$. Let $m$ be a
positive integer. Then there exists matrices $S_2 \in \mathbb{R}^{n \times m}$, $S_3 \in \mathbb{R}^{m \times m}$, with

$$
\begin{bmatrix}
S_1 & S_2 \\
S_2^T & S_3
\end{bmatrix} > 0 \quad \text{and} \quad
\begin{bmatrix}
S_1 & S_2 \\
S_2^T & S_3
\end{bmatrix}^{-1} =
\begin{bmatrix}
R_1 & R_2 \\
R_2^T & R_3
\end{bmatrix}
$$

if and only if

$$
\begin{bmatrix}
S_1 & I_n \\
I_n & R_1
\end{bmatrix} \geq 0 \quad \text{and} \quad
\text{rank}
\begin{bmatrix}
S_1 & I_n \\
I_n & R_1
\end{bmatrix} \leq n + m.
$$

Let us define the following four types of matrices $\chi_G^{2n \times 2n}$, $\kappa_G^{2n \times 2n}$, $\chi_C^{2(n+m) \times 2(n+m)}$, $\kappa_C^{2(n+m) \times 2(n+m)}$.

$$
\chi_G^{2n \times 2n} = \left\{ \Theta : \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{bmatrix}, \Theta_{11}, \Theta_{22} \in \mathbb{R}^{n \times n}, \Theta_{12} \in \mathbb{R}^{n \times n} \right\}
$$

$$
\kappa_G^{2n \times 2n} = \left\{ \Theta : \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & -\Theta_{11} \end{bmatrix}, \Theta_{11} \in \mathbb{R}^{n \times n}, \Theta_{12} \in \mathbb{R}^{n \times n}, \Theta_{12} = -\Theta_{11} \right\}
$$

$$
\chi_C^{2(n+m) \times 2(n+m)} = \left\{ \Theta : \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{bmatrix}, \Theta_{11} \in \chi_G^{2n \times 2n}, \Theta_{12} \in \chi_G^{2n \times 2n} \right\}
$$

$$
\kappa_C^{2(n+m) \times 2(n+m)} = \left\{ \Theta : \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{bmatrix}, \Theta_{11} \in \kappa_G^{2n \times 2n}, \Theta_{12} \in \kappa_G^{2n \times 2n} \right\}
$$

**Lemma 13** Suppose $S_1, R_1 \in \chi_G^{2n \times 2n}$, then for some $m$, there exist matrices $S, R \in \chi_C^{2(n+m) \times 2(n+m)}$, such that,

$$
S =
\begin{bmatrix}
S_1 & S_2 \\
S_2^T & S_3
\end{bmatrix}^{-1} =
\begin{bmatrix}
R_1 & R_2 \\
R_2^T & R_3
\end{bmatrix} = R.
$$

**Proof 16** Suppose $N = (S_1 - R_1^{-1})^{-1}$ is nonsingular. For any $m \geq n$, choose $S_3 \in \mathbb{R}_S$ such that, $\text{in}_+(S_3) \geq \text{in}_-(N)$, $\text{in}_-(S_3) \geq \text{in}_-(N)$. Then there exist $T$, such that

$$
N^{-1} = TS_3^{-1}T^*.
$$

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Choose \( S_3 = T \), then (13) are satisfied with \( S_3, S_2 \) specified above, and \( R_2, R_3 \) can be easily determined.

**Lemma 14** Suppose \( S_1, R_1 \in \kappa^2_{2n \times 2n} \), then for some \( m \), there exist matrices \( S, R \in \kappa^2_{2(n+m) \times 2(n+m)} \), such that,

\[
S = \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}^{-1} = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix} = R. \tag{B.11}
\]

**Proof 17** Since for any matrix \( M \in \kappa^2_{2n \times 2n} \), \( \text{in}_+(M) = \text{in}_-(M) = n \), there exists a matrix \( E_{ii} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \), such that

\[
E_{ii}ME_{ii}^* = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}. \tag{B.12}
\]

besides,

\[
N = (S_1 - R_1^{-1})^{-1} \in \kappa^2_{2n \times 2n} \tag{B.13}
\]

By a similar argument, choose \( S_3 \in \kappa^2_{2n \times 2n} \), and \( \text{in}(S_3) = \text{in}(N) \). Then there exists \( S_2 \), such that

\[
N^{-1} = S_2S_3^{-1}S_2^*. \tag{B.14}
\]

and \( R_2, R_3 \) can be easily determined.
BIBLIOGRAPHY


