COST DENSITY-SHAPING
FOR STOCHASTIC OPTIMAL CONTROL

A Dissertation

Submitted to the Graduate School
of the University of Notre Dame
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

by

Matthew J. Zyskowski

Panos Antsaklis, Co-Director

Ronald Diersing, Co-Director

Graduate Program in Electrical Engineering
Notre Dame, Indiana
December 2010
© Copyright by
Matthew J. Zyskowski
2010
All Rights Reserved
COST DENSITY-SHAPING
FOR STOCHASTIC OPTIMAL CONTROL

Abstract
by
Matthew J. Zyskowski

In stochastic optimal control theory, the complete specification of the probability density function of the random cost functional might be considered the most a designer can do when formulating an optimal control law. The field of cost cumulant controls has made considerable advances towards this capability in the past few decades, largely because of the advantage gained from controlling the cost cumulants instead of the cost moments. However, current cost cumulant control paradigms have left the deliberate specification of the probability density function for the random cost outside the designer’s direct influence. The ability to design control laws upon the desired shape and location of the cost density would be highly valuable to control engineers, since there is evidence that the shape of the cost density under high-performance controllers directly corresponds to the resulting closed-loop system behavior.

This dissertation proposes a Multiple-Cumulant Cost Density-Shaping (MC-CDS) optimization problem for the LQG framework. The control solution to the MCCDS optimization is derived using dynamic programming techniques; it is the finite-horizon, linear state-feedback control that minimizes a smooth, convex, scalar function of arbitrarily-many initial cost cumulants and target initial cost
cumulants. The MCCDS theory is shown to generalize the Linear Quadratic Gaussian (LQG), $k$ Cost Cumulant ($kCC$), and Risk-Sensitive (RS) control paradigms for zero targets and certain linear performance indices. Additionally, the MCCDS framework enables the minimization of well-known distance functions between the cost density and a target cost density, such as the Kullback-Leibler Divergence, Bhattacharyya Distance, and the Hellinger Distance.

The finite-horizon MCCDS control is extended to the infinite-horizon in this dissertation. Other areas of investigation include MCCDS performance index construction, cost density-shaping minimax and Nash games, and a Statistical Target Selection (STS) iterative procedure for control design. Together MCCDS and STS enable the design of control laws with optimality among a family of target cost densities, which might be regarded as a new approach to robust LQG control design. For structures excited by seismic disturbances, numerical experiments show that the MCCDS controls identified thru STS can achieve greater vibration suppression than nominal $kCC$ controllers, without any compromise to robust stability.
Dedication

In the memory of Michael K. Sain
CONTENTS

FIGURES ................................................................. vii

TABLES ............................................................... ix

SYMBOLS ............................................................. x

ACKNOWLEDGMENTS ................................................ xiv

CHAPTER 1: INTRODUCTION .......................................... 1
  1.1 Background ...................................................... 1
  1.2 A Brief History of Cost Density-Shaping for the LQG Framework . 5
  1.3 Summary of Work ............................................... 7

CHAPTER 2: COST CUMULANT-GENERATING EQUATIONS
  OF THE LQG FRAMEWORK ...................................... 9
  2.1 A Derivation of the Cumulant-Generating Equations .............. 9
  2.2 Some Properties of the Cost Cumulant-Generating Equations .... 19
  2.3 Mean-Value Stationary Assumption for the Markov Process ...... 24
  2.4 Some Less Well-Known Properties of Cumulants ................. 25

CHAPTER 3: MULTIPLE-CUMULANT COST DENSITY-SHAPING ....... 32
  3.1 Problem Statement and Solution ................................ 33
    3.1.1 Process and Cost .......................................... 33
    3.1.2 Cost Cumulants ............................................ 34
    3.1.3 Notation ................................................... 36
    3.1.4 Target Cumulants .......................................... 37
    3.1.5 Perturbations to Terminal Conditions ...................... 39
    3.1.6 MCCDS Problem Formulation .............................. 48
    3.1.7 MCCDS Solution via Dynamic Programming ................. 51
  3.2 Derivation of Classical Controllers using MCCDS Theory ....... 80
  3.3 Weighted Least-Squares Cost Density-Shaping Optimal Control ... 82
### 5.4.3 Main Result

### 5.5 MCCDS Infinite-Horizon Controller Computation

5.5.1 Algorithm

5.5.2 Validation

### CHAPTER 6: CUMULANT-REPRESENTATIONS OF HYBRID VARIANTS TO PROBABILITY DISTANCE MEASURES

6.1 Introduction

6.2 Foundational Results

6.2.1 An Alternative Representation for the Random Cost

6.2.2 Gram-Charlier Series, Edgeworth Series, and a Central Limit Theorem

6.2.3 Common Material, MCCDS Performance Index Construction

6.2.4 Overview of MCCDS Performance Index Construction

6.3 Hybrid Variants of Probability Distance Measures

6.3.1 Hybrid Variant, Kullback-Leibler Divergence

6.3.2 Hybrid Variant, Bhattacharyya Coefficient

6.3.3 Hybrid Variants, Bhattacharyya and Hellinger Distance

6.4 Application of MCCDS, Four-Cumulant Cost Density-Shaping

### CHAPTER 7: COST DENSITY-SHAPING GAMES

7.1 Introduction

7.2 Cost Cumulants of the LQG Framework, and Games

7.3 Zero-Sum Cost Density-Shaping Games

7.3.1 Process and Cost

7.3.2 Cost Cumulants (ZS-CDS)

7.3.3 Notation (ZS-CDS)

7.3.4 Target Cost Statistics (ZS-CDS)

7.3.5 Problem Formulation (ZS-CDS)

7.3.6 A Dynamic Programming Framework (ZS-CDS)

7.3.7 Problem Solution (ZS-CDS)

7.4 Non-Zero-Sum Cost Density-Shaping Games

7.4.1 Process and Cost (NZS-CDS)

7.4.2 Cost Cumulants, NZS-CDS Optimization

7.4.3 Notation (NZS-CDS)

7.4.4 Target Cost Statistics (NZS-CDS)

7.4.5 Problem Formulation (NZS-CDS)

7.4.6 A Dynamic Programming Framework (NZS-CDS)

7.4.7 Problem Solution (NZS-CDS)

7.4.8 Cost Density-Shaping, N-Player Game

7.5 Simulation Results
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>STATISTICAL TARGET SELECTION</td>
<td>349</td>
</tr>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>349</td>
</tr>
<tr>
<td>8.2</td>
<td>Overview</td>
<td>350</td>
</tr>
<tr>
<td>8.2.1</td>
<td>Cost Cumulants</td>
<td>351</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Notation</td>
<td>353</td>
</tr>
<tr>
<td>8.2.3</td>
<td>Target Cost Cumulants</td>
<td>354</td>
</tr>
<tr>
<td>8.3</td>
<td>Theory of Statistical Target Selection</td>
<td>358</td>
</tr>
<tr>
<td>8.4</td>
<td>Numerical Experiments</td>
<td>363</td>
</tr>
<tr>
<td>8.4.1</td>
<td>Statistical Target Selection and MHD²-CDS</td>
<td>363</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Statistical Target Selection using Four Cost Cumulants</td>
<td>374</td>
</tr>
<tr>
<td>9</td>
<td>CONCLUSIONS</td>
<td>387</td>
</tr>
<tr>
<td>9.1</td>
<td>Summary</td>
<td>387</td>
</tr>
<tr>
<td>9.2</td>
<td>Future Work</td>
<td>390</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>392</td>
</tr>
</tbody>
</table>
FIGURES

1.1 Key Developments for MCCDS Control Theory ........................................ 7
3.1 MCCDS Controller Computation .............................................................. 79
3.2 WLS-CDS Cost Mean ................................................................................. 90
3.3 Normalized Error, Cost Mean ................................................................. 90
3.4 WLS-CDS Cost Variance ......................................................................... 91
3.5 Normalized Error, Cost Variance ............................................................ 91
3.6 WLS-CDS Cost Skew .............................................................................. 92
3.7 Normalized Error, Cost Skew ................................................................. 92
4.1 Performance and Control Implementation Costs, MKLD-CDS Control .......... 119
4.2 First-Generation Benchmark, Finite-Horizon MKLD-CDS Control, Application Results ................................................................. 120
4.3 First-Generation Benchmark, Per-Story Displacements and Per-Story Accelerations, MKLD-CDS vs. 2CC ....................................................... 121
4.4 Performance and Control Implementation Costs, MBC-CDS Control 135
4.5 First-Generation Benchmark, Per-Story Displacements and Per-Story Accelerations MBC-CDS vs. 2CC ....................................................... 136
4.6 First-Generation Benchmark, Finite-Horizon MBC-CDS Control, Application Results ................................................................. 137
4.7 Single Degree-of-Freedom Structure ....................................................... 148
4.8 Single Degree-of-Freedom Problem, MBC-CDS Cost-Cumulant Tracking and Error ................................................................. 152
4.9 Single Degree-of-Freedom Problem, MBC-CDS Cost Densities ............... 153
4.10 Single Degree-of-Freedom Problem, MBC-CDS Cost Densities ............. 154
4.11 Single Degree-of-Freedom Problem, MKLD-CDS Cost-Cumulant Tracking and Error ................................................................. 155
4.12 Single Degree-of-Freedom Problem, MKLD-CDS Cost Densities ........... 156
4.13 Single Degree-of-Freedom Problem, MKLD-CDS Cost Densities .......... 157
TABLES

4.1 CONTROL LIMITS, LQG VS. 2CC VS. MKLD-CDS .............. 117
4.2 CONTROL LIMITS, LQG VS. 2CC VS. MBC-CDS .............. 135
8.1 CONTROL LIMITS, TOP-FIVE MCCDS ($\theta_1, \theta_2, \theta_3, \theta_4$) CONTROLS 380
8.2 CONTROL PERF./EFFORT, TOP-FIVE MCCDS ($\theta_1, \theta_2, \theta_3, \theta_4$) CONTROLS ................................................................. 381
**SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0$</td>
<td>initial time</td>
</tr>
<tr>
<td>$t_f$</td>
<td>terminal time</td>
</tr>
<tr>
<td>$A(t)$</td>
<td>time-varying system matrix for state</td>
</tr>
<tr>
<td>$B(t)$</td>
<td>time-varying system matrix for control</td>
</tr>
<tr>
<td>$G(t)$</td>
<td>time-varying system matrix for noise</td>
</tr>
<tr>
<td>$Q(t)$</td>
<td>time-varying cost weighting matrix for state</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>time-varying cost weighting matrix for control</td>
</tr>
<tr>
<td>$Q_f$</td>
<td>constant cost weighting matrix for terminal-state penalty</td>
</tr>
<tr>
<td>$F(t)$</td>
<td>closed-loop system matrix</td>
</tr>
<tr>
<td>$N(t)$</td>
<td>closed-loop cost weighting matrix</td>
</tr>
<tr>
<td>$w(t)$</td>
<td>Wiener process at time $t$</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>target set</td>
</tr>
<tr>
<td>$\mathcal{K}$</td>
<td>admissible control space</td>
</tr>
<tr>
<td>$W$</td>
<td>constant matrix for second-order statistics of $w(t)$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>backwards-time variable</td>
</tr>
<tr>
<td>$K(\alpha)$</td>
<td>time-varying gain of linear control input to system</td>
</tr>
<tr>
<td>$\tilde{K}(\alpha)$</td>
<td>time-varying gain driving the target variables</td>
</tr>
<tr>
<td>$J(\alpha, x)$</td>
<td>cost-to-go at $\alpha$</td>
</tr>
<tr>
<td>$\phi(\theta, \alpha, x)$</td>
<td>moment-generating function of $J(\alpha, x)$</td>
</tr>
</tbody>
</table>
ψ(θ, α, x) \quad \text{cumulant-generating function of } J(α, x)

m_r \quad r\text{th moment of general variate}

\mathfrak{M} \quad \text{space of all moment sequences } \{m_i\}_{i=1}^\infty

\kappa_r \quad r\text{th cumulant of general variate}

\mathfrak{K} \quad \text{space of all cumulant sequences } \{\kappa_i\}_{i=1}^\infty

H_r(α) \quad r\text{th order } H \text{ variable for } J(α, x)

D_r(α) \quad r\text{th order } D \text{ variable for } J(α, x)

κ_r(α) \quad r\text{th order cost cumulant for } J(α, x)

\bar{H}(α) \quad H \text{ aggregate for } J(α, x)

\bar{D}(α) \quad D \text{ aggregate for } J(α, x)

κ(α) \quad \text{cost cumulant aggregate for } J(α, x)

\hat{H}_r(α) \quad r\text{th order target } H \text{ variable for } J(α, x)

\hat{D}_r(α) \quad r\text{th order target } D \text{ variable for } J(α, x)

\hat{κ}_r(α) \quad r\text{th order target cost cumulant for } J(α, x)

\hat{H}(α) \quad \text{target } H \text{ aggregate for } J(α, x)

\hat{D}(α) \quad \text{target } D \text{ aggregate for } J(α, x)

\hat{κ}(α) \quad \text{target cost cumulant aggregate for } J(α, x)

ε \quad \text{displaced terminal time } t_f \to ε \text{ for dynamic programming}

\mathcal{Y}_i(ε) \quad \text{displaced terminal condition for } H_i(t_f)

\mathcal{Z}_i(ε) \quad \text{displaced terminal condition for } D_i(t_f)

\hat{\mathcal{Y}}_i(ε) \quad \text{displaced terminal condition for } \hat{H}_i(t_f)

\hat{\mathcal{Z}}_i(ε) \quad \text{displaced terminal condition for } \hat{D}_i(t_f)

\mathcal{Y}(ε) \quad \text{displaced terminal condition aggregate for } \mathcal{H}(t_f)

\mathcal{Z}(ε) \quad \text{displaced terminal condition aggregate for } \mathcal{D}(t_f)

\hat{\mathcal{Y}}(ε) \quad \text{displaced terminal condition aggregate for } \hat{\mathcal{H}}(t_f)
\( \tilde{Z}(\varepsilon) \) displaced terminal condition aggregate for \( \dot{D} (t_f) \)

\( \mathcal{Q} \) reachable set

\( \mathcal{V}(\cdot) \) value function

\( \mathcal{W}(\cdot) \) candidate value function

\( F_\infty \) limiting form, closed-loop system matrix

\( N_\infty \) limiting form, closed-loop cost-weighting matrix

\( G_\infty \) limiting form, noise system matrix

\( \mathcal{H}_l \) \( l \)th order cumulant variable for infinite-horizon

\( \mathcal{H} \) infinite-horizon aggregate of cumulant variables

\( \tilde{\mathcal{H}}_l \) \( l \)th order target cumulant variable for infinite-horizon

\( \dot{\mathcal{H}} \) infinite-horizon aggregate of target cumulant variables

\( \kappa_{\infty,l} \) \( l \)th order infinite-horizon cost cumulant

\( \kappa_\infty \) infinite-horizon cost cumulant aggregate

\( \tilde{\kappa}_{\infty,l} \) \( l \)th order infinite-horizon target cost cumulant

\( \kappa_\infty \) infinite-horizon target cost cumulant aggregates

\( \mathcal{B}_{n,k}(\cdot) \) \( n \)th Bell polynomial

\( \mathcal{B}_n(\cdot) \) \( n \)th complete Bell polynomial

\( \phi_i(t) \) \( i \)th orthonormal basis function

\( \mathcal{H}_k(z) \) \( k \)th Hermite polynomial

\( \kappa^\xi(\omega, \alpha) \) \( i \)th random variable for STS control design

\( \kappa(\omega, \alpha) \) random vector for STS control design

\( \theta_j \) probability that \( \kappa^\xi(\omega_j, \alpha) \) occurs

\( \theta \) vector of frequencies \( \{\theta_i\}_{i=1}^N \)

\( E[\cdot] \) expectation operator

\( \mathcal{F}_t \) \( \sigma \)-algebra generated by \( w(t) \), the Wiener process at time \( t \)
\[ L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \] square-integrable, \( \mathbb{R}^n \)-valued, \( \mathcal{F}_t \)-adapted processes on \([t_0, t_f]\)

\[ L^2_{\mathcal{F}_t}(\Omega; C([t_0, \infty); \mathbb{R}^n)) \] square-integrable, \( \mathbb{R}^n \)-valued, \( \mathcal{F}_t \)-adapted processes on \([t_0, \infty)\)

\[ \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n}) \] class of continuous, \( n \times n \) real functions defined on \([t_0, t_f]\)

\[ \mathcal{C}^p([t_0, t_f]; \mathbb{R}^{n \times n}) \] class of \( p \)-differentiable continuous, \( n \times n \) real functions

\( \mathbb{R} \) real numbers

\( \mathbb{R}^+ \) positive reals

\( \mathbb{R}^{n \times n} \) real-valued, \( n \times n \) matrices

\( \mathbb{S}^n_+ \) real-valued, symmetric, positive semi-definite \( n \times n \) matrices

\( \mathbb{S}^{n++} \) real-valued, symmetric, positive definite \( n \times n \) matrices

\( X \succeq Y \) means the difference \( X - Y \in \mathbb{S}^n_+ \)

\( X \succ Y \) means the difference \( X - Y \in \mathbb{S}^{n++} \)
ACKNOWLEDGMENTS

First and foremost, I want to acknowledge a good friend and colleague, Dr. Michael Sain, who was the original director for this dissertation. Without his important recommendations and guidance, this work would be markedly different than the final product. I count myself privileged to have worked under his direction, and I enjoyed the fellowship we had. I also would like to thank Dr. Panos Antsaklis and Dr. Ron Diersing, who graciously agreed to supervise this project when I was without an advisor. If not for their immense helpfulness the completion of this dissertation would not be possible. I also thank the readers of this dissertation - Dr. Vijay Gupta, Dr. Peter Bauer, and Dr. Yih-Fang Huang - for their valuable inputs to my work.

Being an employee at Hamilton Sundstrand during part of my Ph.D. career, my progress has definitely been a result of my employer allowing me to dedicate time for research meetings with my advisors, presenting research results at technical conferences, and so on. I am grateful to Jim Durand, Jack Waters, Michael Krenz, and Nolan Wanner, among others, for the many accommodations that have made my pursuit of the Ph.D. successful while balancing a full-time job.

Lastly, I would like to acknowledge my friends and family, who were a source of constant encouragement during my time at Notre Dame. I am thankful for their kindness, support, and understanding.
CHAPTER 1

INTRODUCTION

1.1 Background

In stochastic optimal control theory, the complete specification of the probability density function of the random cost functional might be considered the most a designer can do when formulating an optimal control law. The field of cost cumulant controls has made considerable advances towards this capability in the past few decades, largely because of the advantage gained from controlling the cost cumulants instead of the cost moments. Sain and Liberty [41] found that changing cost cumulants beyond the fourth has little or no effect on the cost density function. In agreement with this observation, Hald [2] states that most distributions can be almost completely constrained by four cumulants in his treatise on the history of cumulants and their general role in probability theory and statistics. While this conjecture has yet to be formally established in the literature [16-17], it has been the case that cost cumulants are of more use than the cost moments for constraining the cost density due to a number of reasons.

Higher-order cost moments are not known to correspond with any feature of the cost density’s appearance where on the other hand, the first four cost cumulants correspond directly to the shape and rough location of the cost density. Cost moments also give rise to problems with accurate cost density approximation, as
it remains unknown how many cost moments are needed to tightly constrain the cost density. Indeed, higher-order cost moments might have more of an effect on the cost density than the lower-order cost moments chosen for control. This is a general challenge with using cost moments for cost density-shaping. Conversely, as stated in the prior paragraph, it seems that four cost cumulants suffice to constrain the cost density function.

If the foregoing remarks were not enough to make cost moment-based controllers prohibitive, it has been shown that cost moment optimizations for linear systems and quadratic costs can lead to non-linear control laws [42], which may pose challenges to the implementation of cost moment-based controls for the Linear Quadratic Gaussian (LQG) framework. Conversely, cost cumulant optimizations have led to linear control laws, which are amenable to implementation. Another advantage of using cost cumulants for cost density-shaping is the cost cumulant-generating equations of the LQG framework, which currently have no known counterpart for the cost moments.

Cost density-shaping with cost cumulants is by no means a new idea, as cost cumulant controls with limited cost density-shaping capability have been the subject of investigation for decades. The earliest such control is Kalman’s well-known and widely-used LQG controller, which minimizes the expectation of an integral-quadratic cost involving a process with dynamics subject to additive white Gaussian noise. Since the expected cost is minimized, this resulting optimal control constrains the mean of the cost density to lie as close to the origin as possible. Other features of the cost density are without the designer’s specification under this control strategy. Through research post-dating that of Kalman’s, a generalization has been made of the LQG control, which seeks to minimize a finite linear
combination of $k$ cost cumulants, and thus provides more influence over the shape and location of the cost density.

For the LQG framework, Pham’s $k$ Cost Cumulant ($kCC$) controller has been developed to constrain a linear combination of cost cumulants, and some might regard this control strategy as providing the most complete approximation of the cost density for $k \geq 4$. Pham’s simulation results support that under the $kCC$ control selection, the cost density is indeed shaped in accordance with the weightings of the cost cumulants in the $kCC$ performance index. However no direct association between weights and pre-specified target cost densities has been proposed as of yet. These static time-invariant weights of the cost cumulants in the $kCC$ problem formulation have been left as parametric design freedom that may be used for ad hoc density-shaping and possibly to achieve desired closed-loop properties of the system. To date a control strategy that can steer cost cumulants to some pre-selected, achievable values that will in effect shape the cost density has yet to be developed.

The goal of this research is to develop a novel cumulant-based linear state-feedback control paradigm that is capable of achieving cost density-shaping objectives by steering cost cumulants towards pre-specified target cost cumulant values. The research concerns the well-known and widely-used LQG framework for systems and costs, where the process dynamics are linear (under linear state-feedback control) with additive Gaussian white noise, and the cost functional is integral quadratic in the system state and control input. The respective control solutions of the Multiple-Cumulant Cost Density-Shaping (MCCDS) optimizations formulated in this dissertation provide a time-varying, multi-dimensional degree of freedom that has a direct connection with target initial cost cumulants. This
control solution generalizes the LQG, $kCC$, and Risk-Sensitive control paradigms for zero target cost cumulants and certain linear performance indices, which shows that MCCDS supports existing theory. The derived MCCDS control is illustrated through a variety of applications to structural control, where it is shown to track target cost cumulants while converging to a stabilizing controller with modest performance gains over other control paradigms.

It is important to separately address two important pieces of existing theory that are built upon with the development of the MCCDS control paradigm. First is the well-accepted cost cumulant-generating equations of the LQG framework, which were originally developed by Liberty and Hartwig [61-62]. Secondly is the notation of Pham in [26], and the isomorphic vectorization operation between the matrix cumulant-generating equations and their vectorized equivalents which enabled the derivation of an HJB Verification Lemma for Mayer optimizations involving the functions that satisfy the cost cumulant-generating equations. Together, these contributions are crucial to formulating the MCCDS optimization in the space of cost cumulant variables, which can be thought of as the “building blocks” for cost cumulants of the LQG framework.

A note of distinction on the MCCDS theory from other cumulant control theories should be made before continuing. The MCCDS theory introduces “target cost cumulants” into the new kind of cost cumulant optimization, which seeks to minimize a general smooth, convex function of cost cumulants and target cost cumulants. The introduction of target statistics into the optimization represents a new direction for the stochastic optimal control problem, and allows for the minimization of distance functions between a cost density and a target density when expressed in terms of the cumulants of both densities. Despite the novelty
of these results, the MCCDS paradigm is in the family of statistical controls that have stemmed from [36].

1.2 A Brief History of Cost Density-Shaping for the LQG Framework

Cost density-shaping has its historical beginnings with the classical work of Kalman involving the widely-known LQG control, which is the linear state-feedback control that minimizes the expectation of an integral-quadratic cost on the control input and the process state. Here the assumption is that the process has dynamics subject to additive white Gaussian noise. This formulation is easily adaptable to a large class of problems, and as a result the LQG solution has been extremely useful in applications.

Being a random variable, the cost will have a distribution of possible values for any or no control input. As such, it is natural to give a statistical interpretation to the resultant effect of the LQG control on the aforementioned non-negative integral-quadratic cost. Minimizing the cost expectation will evidently move the cost density closer to the origin so in some sense this is a cost density-shaping objective, albeit one that does not constrain the dispersion of cost values about the mean, the skewness in the cost density, the weight in the tails of the cost density, and so forth. In this way, LQG control has laid the foundation for further research in cost density-shaping for the LQG framework.

The initial work of Sain [36-39] extended traditional LQG stochastic control beyond the cost’s mean to the cost variance for the open-loop problem. This contribution represents the inception of cost cumulant control, the birth of a new “Statistical Control Theory”. This milestone in stochastic control was followed by the seminal work of Liberty and Hartwig [61-62] that showed all higher-order
cost cumulants for the LQG framework are affine quadratic in the known mean of the initial state under linear state-feedback control and together satisfy a system of coupled differential Lyapunov equations.

Cost cumulant control involving the cost variance was extended by Won [7] to the closed-loop case thru the Minimum Cost Variance (MCV) control problem, which minimized the cost variance subject to a constraint that the cost mean remain beneath a certain upper bound. The problem was posed and solved in the more general framework of non-quadratic costs and non-linear systems, and the case of an integral-quadratic cost involving a process with dynamics subject to additive noise was later considered as a special case of the theory. The results stemming from this consideration gave rise to a MCV control law that is composed of two matrix functions that satisfy a system of two coupled Riccati equations. Both cost mean and variance were shown to have a quadratic structure, and also to be a function of these matrices. The MCV problem for the LQG framework achieves a cost density-shaping objective where it is sought to minimize the dispersion of cost values about the cost mean while keeping the cost density suitably close to the origin.

The form of MCV control for the LQG case by Sain and Won was generalized by Pham with his $k$CC control which minimizes a linear combination of the first $k$ cost cumulants. In his work, Pham builds on the previous results of Liberty and Hartwig [61-62] pertaining to cost cumulants having quadratic structure and satisfying a system of differential Lyapunov equations. Given the foregoing remarks, it might be said that to date, the $k$CC control provides the most complete specification of the cost density for $k \geq 4$.

From a cost density-shaping perspective, the previous paragraphs represent a
Figure 1.1. Key Developments for MCCDS Control Theory

brief historical account of the developments of cost cumulant control for continuous Markov process and integral-quadratic cost of the LQG framework. The objective of statistical control might be naturally regarded as selecting a control input that influences the cost cumulants to achieve desired qualities in the distribution of the random cost functional.

1.3 Summary of Work

This dissertation is organized as follows. Chapter 2 contains a derivation of the cost-cumulant generating equations of the LQG framework, and sets forth some of the previously established properties of these equations from [26]. Also, a few interesting properties are presented for cumulants of a general variate (e.g. not necessarily the random cost functional of the LQG framework), as discussed in Port [9]. In Chapter 3, the MCCDS optimization is formalized, and then solved using dynamic programming techniques. The Weighted-Least-Squares (WLS) MCCDS optimization are presented, and validated through a simulation involving a Single-Degree-of-Freedom (SDOF) problem. Chapter 4 presents many concrete examples of the MCCDS theory for the important case of the first two cost cumulants, the

<table>
<thead>
<tr>
<th>Year(s)</th>
<th>Event/Inventor(s)</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>Kalman</td>
<td>LQG Control</td>
</tr>
<tr>
<td>1965</td>
<td>Sain</td>
<td>MCV Control (open-loop)</td>
</tr>
<tr>
<td>1978</td>
<td>Liberty &amp; Hartwig</td>
<td>MCC Control (closed-loop)</td>
</tr>
<tr>
<td>1994</td>
<td>Won &amp; Sain</td>
<td>MCV Control</td>
</tr>
<tr>
<td>2004</td>
<td>Pham &amp; Sain</td>
<td>kCC Control</td>
</tr>
</tbody>
</table>
mean and the variance. In particular, the Hellinger Distance, Kullback-Leibler Divergence, Bhattacharyya Distance, and Bhattacharyya Coefficient between densities associated with normalized cost and target cost variates are optimized using the MCCDS framework. Simulation results are presented for validation purposes. Sufficient conditions for so-called delta-mean zero-crossings are developed.

In Chapter 5, the infinite-horizon MCCDS optimization problem is formalized and solved. An algorithm to compute the infinite-horizon MCCDS optimal control is presented that maintains well-defined convergence properties of the Pham’s Lyapunov iterate solutions technique. The results of a numerical experiment are provided to validate the algorithm and MCCDS infinite-horizon theory. Chapter 6 uses Edgeworth approximations to the densities associated with cost and target cost variates in order to derive cumulant representations of hybrid variants to the Bhattacharyya Coefficient and the Kullback-Leibler Divergence that facilitate cost density-shaping with up to four cost cumulants. The first-generation benchmark problem for seismically-excited buildings is used to validate the theory for four cost cumulants, and shows that these cumulant representations of distance functions can indeed be used in the framework of MCCDS optimal control. Chapter 7 formalizes cost density-shaping games on the finite-horizon and presents the minimax and Nash solutions. In Chapter 8, some material is dedicated to numerical experiments concerning the selection of target statistics for the cost functional. Design by Statistical Target Selection (STS) is formally established and illustrated through simulation. Finally, Chapter 9 concludes this work with a summary and directions for future research.
CHAPTER 2
COST CUMULANT-GENERATING EQUATIONS
OF THE LQG FRAMEWORK

2.1 A Derivation of the Cumulant-Generating Equations

The cost cumulant-generating equations of the LQG framework are crucial to the cost density-shaping methods developed in this dissertation. The original derivation of these equations in [61-62] relies on the representation of the process in a Karhunen-Loeve expansion and using iterated-integral kernels to derive a set of backwards-in-time differential equations, whereby the cumulants of any order can be calculated exactly for the “cost-to-go”. This development is rather ingenious, but because of its length these results are not presented here. For a full development, consult Chapter 2 of [26].

To provide some background for the cost cumulant-generating equations, the more concise and rather clever development of Mou is presented. A supporting lemma is first given, and then the main result is presented. The proof of the main result is sketched along the lines provided in [33]. The essential starting point for this discussion will be the cumulant-generating and moment-generating functions for the random cost,

\[
J(t_0, x_0) = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_x(t_f) \tag{2.1}
\]
where $Q_f \in \mathbb{R}^{n \times n}$ and $Q_f \succeq 0^{n \times n}$, and $N \in C([t_0, t_f]; \mathbb{S}_+^n)$ and $x(t)$ is the stochastic process
\[
    dx(t) = A(t)x(t)dt + B(t)u(t)dt + G(t)dw(t), \quad t \in [t_0, t_f]
\]
\[\tag{2.2}\]
\[x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n.\]

where $A \in C([t_0, t_f]; \mathbb{R}^{n \times n})$, $B \in C([t_0, t_f]; \mathbb{R}^{n \times m})$, $G \in C([t_0, t_f]; \mathbb{R}^{n \times p})$ and $w(t)$ is a $p$-dimensional stationary Wiener process having correlation of increments
\[E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}.\]  \[\tag{2.3}\]

The following will refer to the “cost-to-go” $J(\alpha, x)$ given below,
\[
    J(\alpha, x) = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_f x(t_f)
\]
\[\tag{2.4}\]

where $\alpha \in [t_0, t_f]$ will replace $t_0$ and $x(\alpha) = x$ will replace $x_0$. Consider the case when $u(t) = K(t)x(t)$ is a linear, state-feedback control input to the system (2.2) and in the following denote the closed-loop matrix as $F \triangleq (A + BK)$.

It is a known result that the moment-generation function $\phi(\theta, \alpha, x) = E\{e^{\theta J(\alpha, x)}\}$ and the cumulant-generating function $\psi(\theta, \alpha, x) = \ln \phi(\theta, \alpha, x)$ for the random cost $J(\alpha, x)$ under linear state-feedback control take the form given in the following theorem. Note that $\phi(j\omega, \alpha, x)$ and $\psi(j\omega, \alpha, x)$ are the first and second characteristic functions for $J(\alpha, x)$.

**Theorem 2.1.1 (Representation, Cost Cumulant-Generating Function)**

For fixed $\theta$ let $S(\theta, \alpha)$ and $\rho(\theta, \alpha)$ denote functions that satisfy the partial differential equations,
\[
\frac{\partial S(\theta, \alpha)}{\partial \alpha} = -F^T(\alpha)S(\theta, \alpha) - S(\theta, \alpha)F(t) - 2S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) - \theta N(\alpha)
\]
\[ S(\theta, t_f) = \theta Q_f \]
\[ \frac{\partial \rho(\theta, \alpha)}{\partial \alpha} = -\rho(\theta, \alpha) \text{Tr}(S(\theta, \alpha)G(\alpha)WG^T(\alpha)) \]
\[ \rho(\theta, t_f) = 1 \]

where \( F \triangleq (A + BK) \) is the closed-loop matrix of the system (2.2) under linear state-feedback control \( u = Kx \). Then the moment-generating function \( \phi(\theta, \alpha, x) \) and cumulant-generating function \( \psi(\theta, \alpha, x) \) of \( J(\alpha, x) \) have the representations,

\[ \phi(\theta, \alpha, x) = \rho(\theta, \alpha) \cdot \exp(x^T S(\theta, \alpha)x) \]
\[ \psi(\theta, \alpha, x) = d(\theta, \alpha) + x^T S(\theta, \alpha)x \]

where \( d(\theta, \alpha) = \ln \rho(\theta, \alpha) \) is the function that satisfies

\[ \frac{\partial d(\theta, \alpha)}{\partial \alpha} = -\text{Tr}(S(\theta, \alpha)G(\alpha)WG^T(\alpha)), \quad d(\theta, t_f) = 0. \]

**Proof** See [1].

Before proceeding, it is noted that the cumulant-generating function of any variate has the MacLaurin series expansion,

\[ \psi(\theta, t) = \sum_{j=1}^{\infty} \kappa_j \frac{\theta^j}{j!} \]

where \( \kappa_j \) is the \( j \)th order cumulant of the variate, here \( J(\alpha, x) \). More precisely, for every pair \((\alpha, x)\), this variate will have cumulants with an explicit designation of this dependence \( \kappa_j(\alpha, x) \). The main result is now presented.

**Theorem 2.1.2** (Cost Cumulant-Generating Equations)

When the process having dynamics (2.2) is subjected to linear state-feedback control \( u = Kx \), the cumulants of (2.4) have the representation,

\[ \kappa_j(\alpha, x) = x^T H_j(\alpha)x + D_j(\alpha) \quad (2.5) \]

where the functions \( D_i(\alpha) \) and \( H_i(\alpha) \) satisfy the family of differential equations,

\[ \frac{dH_1(\alpha)}{d\alpha} = -F^T(\alpha)H_1(\alpha) - H_1(\alpha)F(\alpha) - N(\alpha) \]
\[
\frac{dH_i(\alpha)}{d\alpha} = -F^T(\alpha)H_i(\alpha) - H_i(\alpha)F(\alpha) \\
-2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha), \ i \geq 2
\]
\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}(H_j(\alpha)G(\alpha)WG^T(\alpha)), \ \alpha \in [t_0, t_f], \ j \geq 1
\]
\[
H_i(t_f) = Q_f, \ H_i(t_f) = 0^{n \times n}, \ D_j(t_f) = 0, \ i \geq 2, \ j \geq 1.
\]

**Proof** Begin by using the functions \(S(\theta, \alpha)\) and \(d(\theta, \alpha)\) from Theorem 2.1.1 and define,

\[
H_i(\alpha) \triangleq \frac{\partial^i}{\partial \theta^i} S(0, \alpha), \ D_i(\alpha) \triangleq \frac{\partial^i}{\partial \theta^i} d(0, \alpha).
\]

Write the MacLaurin series for the cumulant-generating function for \(J(\alpha, x)\),

\[
\psi(\theta, \alpha, x) = \sum_{j=1}^{\infty} \kappa_j(\alpha, x) \frac{\theta^j}{j!} = d(\theta, \alpha) + x^T S(\theta, \alpha)x.
\]

Now consider the MacLaurin series representations of \(x^T S(\theta, \alpha)x\) and \(d(\theta, \alpha)\) as below,

\[
d(\theta, t) = \sum_{j=1}^{\infty} \frac{\partial^j}{\partial \theta^j} d(0, \alpha) \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} D_j(\alpha) \frac{\theta^j}{j!}
\]
\[
x^T S(\theta, t)x = \sum_{j=1}^{\infty} x^T \left( \frac{\partial^j}{\partial \theta^j} S(0, \alpha) \right) x \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} x^T H_j(\alpha) x \frac{\theta^j}{j!}
\]

Inserting these representations into the MacLaurin series for \(\psi(\theta, \alpha, x)\) yields,

\[
\psi(\theta, \alpha, x) = \sum_{j=1}^{\infty} \kappa_j(\alpha, x) \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} (D_j(\alpha) + x^T H_j(\alpha)x) \frac{\theta^j}{j!}
\]
from which a comparison of coefficients yields the result,

$$
\kappa_j(\alpha, x) = x^T H_j(\alpha) x + D_j(\alpha).
$$

Now all that is left to show is that the functions $H_i(\alpha)$ and $D_i(\alpha)$ satisfy the given family of differential equations. Recall that $S(\theta, \alpha)$ satisfies the partial differential equation

$$
\frac{\partial S(\theta, \alpha)}{\partial \alpha} = -F^T(\alpha)S(\theta, \alpha) - S(\theta, \alpha)F(\alpha) - 2S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) - \theta N(\alpha)
$$

$$
S(\theta, t_f) = \theta Q_f.
$$

For $\theta = 0$, this becomes

$$
\frac{\partial S(0, \alpha)}{\partial \alpha} = -F^T(\alpha)S(0, \alpha) - S(0, \alpha)F(\alpha) - 2S(0, \alpha)G(\alpha)WG^T(\alpha)S(0, \alpha)
$$

$$
S(0, t_f) = 0^{n \times n}.
$$

The unique solution to this equation is $S(0, \alpha) = 0^{n \times n}$, $\forall \alpha \in [t_0, t_f]$. The consequence of this for $\rho(0, \alpha)$ can be understood by considering the equation

$$
\frac{\partial \rho(0, \alpha)}{\partial \alpha} = -\rho(0, \alpha) \text{Tr}(S(0, \alpha)G(\alpha)WG^T(\alpha)) = 0, \ \rho(0, t_f) = 1
$$

from which it is clear that $\rho(0, \alpha) = 1$, $\forall \alpha \in [t_0, t_f]$. Now differentiate the equation giving the time evolution of $S(\theta, \alpha)$ with respect to $\theta$,

$$
\frac{\partial}{\partial \alpha} \left( \frac{\partial S(\theta, \alpha)}{\partial \theta} \right) = -F^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} - \frac{\partial S(\theta, \alpha)}{\partial \theta} F(\alpha) - N(\alpha)
$$

$$
- 2 \frac{\partial}{\partial \theta} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)), \ \frac{\partial S(\theta, t_f)}{\partial \theta} = Q_f. \tag{2.6}
$$
Let $\theta = 0$ and it is readily observed that this gives,

$$
\frac{dH_1(\alpha)}{d\alpha} = - F^T(\alpha)H_1(\alpha) - H_1(\alpha)F(\alpha) - N(\alpha), \quad H_1(t_f) = Q_f
$$

accounting for the definition of $H_1(\alpha)$ and also noting that

$$
\frac{\partial}{\partial \theta} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right) = \frac{\partial S(\theta, \alpha)}{\partial \theta}G(\alpha)WG^T(\alpha)S(\theta, \alpha) + S(\theta, \alpha)G(\alpha)WG^T(\alpha)\frac{\partial S(\theta, \alpha)}{\partial \theta}
$$

for which it is apparent that $\theta = 0$ renders this expression equal to a $n \times n$ zero matrix. Now differentiate (2.6) with respect to $\theta$,

$$
\frac{\partial}{\partial \alpha} \left( \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} \right) = - F^T(\alpha)\frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} - \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2}F(\alpha)
$$

$$
-2\frac{\partial^2}{\partial \theta^2} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right), \quad \frac{\partial^2 S(\theta, t_f)}{\partial \theta^2} = 0^{n \times n}.
$$

where

$$
\frac{\partial^2}{\partial \theta^2} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right) = 2\frac{\partial S(\theta, \alpha)}{\partial \theta}G(\alpha)WG^T(\alpha)\frac{\partial S(\theta, \alpha)}{\partial \theta}
$$

$$
+ \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2}G(\alpha)WG^T(\alpha)S(\theta, \alpha)
$$

$$
+ S(\theta, \alpha)G(\alpha)WG^T(\alpha)\frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2}.
$$

(2.7)

so that when $\theta = 0$ in (2.7) this yields,

$$
\frac{dH_2(\alpha)}{d\alpha} = - F^T(\alpha)H_2(\alpha) - H_2(\alpha)F(\alpha)
$$

$$
- 4H_1(\alpha)G(\alpha)WG^T(\alpha)H_1(\alpha), \quad H_2(t_f) = 0^{n \times n}.
$$
By similar reasoning,

\[
\frac{\partial}{\partial \alpha} \left( \frac{\partial^i S(\theta, \alpha)}{\partial \theta^i} \right) = -F^T(\alpha) \frac{\partial^i S(\theta, \alpha)}{\partial \theta^i} F(\alpha) - 2 \frac{\partial^i}{\partial \theta^i} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)), \quad \frac{\partial^i S(\theta, t_f)}{\partial \theta^i} = 0^{n \times n}.
\]

and it is easily shown that

\[
\frac{\partial^i}{\partial \theta^i} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)) \bigg|_{\theta = 0} = \sum_{j=1}^{i-1} \binom{i}{j} \frac{\partial^i S(0, \alpha)}{\partial \theta^i} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(0, \alpha)}{\partial \theta^{i-j}}.
\]

so that

\[
\frac{dH_i(\alpha)}{d\alpha} = -F^T(\alpha)H_i(\alpha) - H_i(\alpha)F(\alpha) - 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha), \quad \alpha \in [t_0, t_f], \quad H_i(t_f) = 0^{n \times n}, \quad i \geq 2.
\]

The above relation can be established with a simple inductive proof. Use (2.7) as the base case, and suppose that the following holds for \(i - 1\),

\[
\frac{\partial^{i-1}}{\partial \theta^{i-1}} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha))
\]

\[
= \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-1-j} S(\theta, \alpha)}{\partial \theta^{i-1-j}} + \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha)WG^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha)G(\alpha)WG^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}}.
\]

The above form is entirely motivated by the expression (2.7) where \(i = 3\). Consider
the derivative for $i$ and use the result above,

$$
\frac{\partial^i}{\partial \theta^i} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right)
$$

$$
= \frac{\partial}{\partial \theta} \left( \frac{\partial^{i-1}}{\partial \theta^{i-1}} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right) \right)
$$

$$
= \frac{\partial}{\partial \theta} \left( \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha) W G^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}} \right)
$$

$$
+ \frac{\partial}{\partial \theta} \left( \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} \right).
$$

From the above expression, consider the first term

$$
\frac{\partial}{\partial \theta} \left( \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha) W G^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}} \right)
$$

$$
= \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^{j+1} S(\theta, \alpha)}{\partial \theta^{j+1}} G(\alpha) W G^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}}
$$

$$
+ \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^{j+1} S(\theta, \alpha)}{\partial \theta^{j+1}} G(\alpha) W G^T(\alpha) \frac{\partial^{(i-1)-j+1} S(\theta, \alpha)}{\partial \theta^{(i-1)-j+1}}
$$

$$
= \sum_{j=2}^{i-1} \binom{i-1}{j-1} \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}} G(\alpha) W G^T(\alpha)
$$

$$
+ \sum_{j=1}^{i-2} \binom{i-1}{j} \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}} G(\alpha) W G^T(\alpha)
$$

$$
= \sum_{j=2}^{i-2} \left( \binom{i-1}{j-1} + \binom{i-1}{j} \right) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}} G(\alpha) W G^T(\alpha)
$$

$$
+ \binom{i-1}{1} \frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha)
$$
\[
\frac{(i-1)!}{(j-1)! \cdot (i-j)!} + \frac{(i-1)!}{j! \cdot (i-j)!} = \frac{j \cdot (i-1)!}{j! \cdot (i-j)!} + \frac{(i-j) \cdot (i-1)!}{j! \cdot (i-j)!} = \frac{(j + (i-j)) \cdot (i-1)!}{j! \cdot (i-j)!} = \frac{i \cdot (i-1)!}{j! \cdot (i-j)!} = \binom{i}{j}
\]

The relations below are not difficult to establish,

\[
\begin{align*}
\binom{i-2}{j-1} + \binom{i-1}{j} &= \binom{i-1}{j-1} - 1, \\
\binom{i-1}{1} + \binom{i-2}{i-1} &= \binom{i}{i-1} - 1.
\end{align*}
\]

Using the above results, expression (2.9) becomes

\[
\sum_{j=2}^{i-2} \left( \binom{i-1}{j-1} + \binom{i-1}{j} \right) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} + \binom{i-1}{1} \frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta} \]

\[
+ \binom{i-1}{i-2} \frac{\partial^{i-1} S(\theta, \alpha)}{\partial^{i-1} \theta} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} = \sum_{j=1}^{i-1} \frac{\partial^{i} S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}}
\]
\[-\frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} - \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta}\]

Expanding the derivative in the last line of (2.8) yields

\[
\frac{\partial}{\partial \theta} \left( \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} \right) 
= \frac{\partial^i S(\theta, \alpha)}{\partial \theta^i} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta}. \tag{2.11}\]

Using the results (2.10) and (2.11) in (2.8), leads to

\[
\frac{\partial^i}{\partial \theta^i} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right) 
= \sum_{j=1}^{i-1} \binom{i}{j} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha) W G^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}} + \frac{\partial^{i} S(\theta, \alpha)}{\partial \theta^i} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial^i S(\theta, \alpha)}{\partial \theta^i}.
\]

which completes the inductive proof. Now, use the fact that \( S(0, \alpha) = 0 \), \( \forall \alpha \) from which it follows that

\[
\frac{\partial^i}{\partial \theta^i} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right) \bigg|_{\theta=0} 
= \sum_{j=1}^{i-1} \binom{i}{j} \frac{\partial^j S(0, \alpha)}{\partial \theta^j} G(\alpha) W G^T(\alpha) \frac{\partial^{i-j} S(0, \alpha)}{\partial \theta^{i-j}} + \frac{\partial^{i} S(0, \alpha)}{\partial \theta^i} G(\alpha) W G^T(\alpha) S(0, \alpha) + S(0, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial^i S(0, \alpha)}{\partial \theta^i} 
= \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha).
\]

18
Consider the equation

$$\frac{\partial}{\partial \alpha} \left( \frac{\partial^i d(\theta, \alpha)}{\partial \theta^i} \right) = -\text{Tr} \left( \frac{\partial^i S(\theta, \alpha)}{\partial \theta^i} G(\alpha) W G^T(\alpha) \right), \frac{\partial^i d(\theta, t_f)}{\partial \theta^i} = 0$$

which makes it immediate that

$$\frac{dD_i(\alpha)}{d\alpha} = -\text{Tr} (H_i(\alpha) G(\alpha) W G^T(\alpha)), D_i(\alpha) = 0, i \geq 1.$$

when $\theta = 0$. The proof is complete.

\[\square\]

2.2 Some Properties of the Cost Cumulant-Generating Equations

The cost-cumulant generating equations are a family of coupled Lyapunov equations, which have many desirable properties. These properties are established in section 2.3 of Pham’s thesis [26], and this section of the dissertation is dedicated to those results. In particular, properties of existence and uniqueness, symmetry, and positive-definiteness of solutions to the cost cumulant-generating equations are presented, and also the monotonicity of solutions backwards-in-time on $[t_0, t_f]$. The proofs have been omitted, but they can be referenced in [26].

The results in [26] make use of a different, though equivalent, definition of the cost cumulants that carries an impact to the cumulant-generating equations of the LQG framework. Given this difference, it is important to establish the equivalency of the two forms of cost cumulants, and show formally how changes are manifest in the cumulant-generating equations. A lemma is given here before the properties of cost cumulants are shown.
Lemma 2.2.1 (Equivalency of Cost Cumulants Forms)
When the process having dynamics (2.2) is subject to a linear state-feedback control input, Theorem 2.1.2 states that the cumulants of (2.4) have the following representation,

$$\kappa_j(\alpha, x) = x^T H_j(\alpha) x + D_j(\alpha)$$

where the functions $D_i(\alpha)$ and $H_i(\alpha)$ satisfy the family of differential equations,

$$\frac{dH_1(\alpha)}{d\alpha} = - F^T(\alpha) H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha)$$

$$\frac{dH_i(\alpha)}{d\alpha} = - F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha)$$

$$- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \; i \geq 2$$

$$\frac{dD_j(\alpha)}{d\alpha} = - \text{Tr}(H_j(\alpha) G(\alpha) W G^T(\alpha)), \; \alpha \in [t_0, \; t_f], \; j \geq 1$$

$$H_1(t_f) = Q_f, \; H_i(t_f) = 0^{n \times n}, \; D_j(t_f) = 0, \; i \geq 2, \; j \geq 1.$$

which equivalently can be written as

$$\kappa_j(\alpha, x) = x^T \left( 2^{j-1} \cdot j! H_j(\alpha) \right) x + (2^{j-1} \cdot j! D_j(\alpha))$$

where the functions $D_i(\alpha)$ and $H_i(\alpha)$ satisfy the family of differential equations,

$$\frac{dH_1(\alpha)}{d\alpha} = - F^T(\alpha) H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha)$$

$$\frac{dH_i(\alpha)}{d\alpha} = - F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha)$$

$$- \sum_{j=1}^{i-1} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \; i \geq 2$$

$$\frac{dD_j(\alpha)}{d\alpha} = - \text{Tr}(H_j(\alpha) G(\alpha) W G^T(\alpha)), \; \alpha \in [t_0, \; t_f], \; j \geq 1$$

$$H_1(t_f) = Q_f, \; H_i(t_f) = 0^{n \times n}, \; D_j(t_f) = 0, \; i \geq 2, \; j \geq 1.$$

Proof It is proposed that the following relation hold

$$H_i(\alpha) = \frac{H_i(\alpha)}{2^{i-1} \cdot i!}, \; D_i(\alpha) = \frac{D_i(\alpha)}{2^{i-1} \cdot i!}.$$
which can be established by showing that

$$\frac{dH_i(\alpha)}{d\alpha} = \left(\frac{1}{2^{i-1} \cdot i!}\right) \frac{dH_i(\alpha)}{d\alpha}, \quad \frac{dD_i(\alpha)}{d\alpha} = \left(\frac{1}{2^{i-1} \cdot i!}\right) \frac{dD_i(\alpha)}{d\alpha}$$

It is evident that

$$\frac{dH_1(\alpha)}{d\alpha} = \frac{dH_1(\alpha)}{d\alpha}, \quad \mathcal{H}_1(t_f) = H_1(t_f) = Q_f$$

so that $\mathcal{H}_1(\alpha) = H_1(\alpha)$ from the statement of the lemma. Consider that

$$\frac{dH_i(\alpha)}{d\alpha} = - F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha)$$

$$- \sum_{j=1}^{i-1} \mathcal{H}_j(\alpha) G(\alpha) W^T(\alpha) H_{i-j}(\alpha)$$

$$= - F^T(\alpha) \left( \frac{H_i(\alpha)}{2^{i-1} \cdot i!} \right) - \left( \frac{H_i(\alpha)}{2^{i-1} \cdot i!} \right) F(\alpha)$$

$$- \sum_{j=1}^{i-1} \left( \frac{H_j(\alpha)}{2^{j-1} \cdot j!} \right) G(\alpha) W^T(\alpha) \left( \frac{H_{i-j}(\alpha)}{2^{i-j-1} \cdot (i-j)!} \right)$$

from which it follows that

$$\frac{dH_i(\alpha)}{d\alpha} = - F^T(\alpha) \left( \frac{H_i(\alpha)}{2^{i-1} \cdot i!} \right) - \left( \frac{H_i(\alpha)}{2^{i-1} \cdot i!} \right) F(\alpha)$$

$$- \sum_{j=1}^{i-1} \left( \frac{H_j(\alpha)}{2^{j-1} \cdot j!} \right) G(\alpha) W^T(\alpha) \left( \frac{H_{i-j}(\alpha)}{2^{i-j-1} \cdot (i-j)!} \right)$$

$$= \left( \frac{1}{2^{i-1} \cdot i!} \right) \left( - F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha) \right)$$

$$- \sum_{j=1}^{i-1} \frac{i!}{j! \cdot (i-j)!} \left( \frac{1}{2^{j-1} \cdot 2^{i-j-1} \cdot j!} \right) H_j(\alpha) G(\alpha) W^T(\alpha) H_{i-j}(\alpha)$$

$$= \left( \frac{1}{2^{i-1} \cdot i!} \right) \left( - F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha) \right)$$

$$+ \left( \frac{1}{2^{i-1} \cdot i!} \right) \left( - 2 \sum_{j=1}^{i-1} \left( \begin{array}{c} i \\ j \end{array} \right) H_j(\alpha) G(\alpha) W^T(\alpha) H_{i-j}(\alpha) \right)$$

21
\[
= \left(\frac{1}{2i-1 \cdot i!}\right) \cdot \frac{dH_i(\alpha)}{d\alpha}.
\]

The terminal conditions are

\[H_i(t_f) = \frac{H_i(t_f)}{2i-1 \cdot i!} = 0^{n \times n} = H_i(t_f).\]

Also, it readily is established that

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}(H_j(\alpha)G(\alpha)WG^T(\alpha))
= \left(\frac{1}{2i-1 \cdot i!}\right) \cdot (-\text{Tr}(H_j(\alpha)G(\alpha)WG^T(\alpha)))
= \left(\frac{1}{2i-1 \cdot i!}\right) \cdot \frac{dD(\alpha)}{d\alpha}
\]

and analogously

\[
D_i(t_f) = \frac{D_i(t_f)}{2i-1 \cdot i!} = 0 = D_i(t_f).
\]

Now the properties of the cost cumulant-generating equations are presented.

**Theorem 2.2.2** *(Existence and Uniqueness)*

Let \( r \in \mathbb{N} \) be a fixed, positive integer, and let \( F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p}) \), \( N \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_{++}) \), and \( W \in \mathbb{S}^p_{++} \). Then the solutions \( \{H_i(\alpha)\}_{i=1}^r \) to the following equations exist on \([t_0, t_f]\), and these solutions are unique.

\[
\frac{dH_1(\alpha)}{d\alpha} = -F(\alpha)^T H_1(\alpha) - H_1(\alpha)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_i(\alpha)}{d\alpha} = -F(\alpha)^T H_2(\alpha) - H_2(\alpha)F(\alpha)
- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha), \quad 2 \leq i \leq r.
\]

**Proof** See [26].
Theorem 2.2.3 (Symmetry)
Let $r \in \mathbb{N}$ be a fixed, positive integer, and let $F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$, $N \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_+)$, and $W \in \mathbb{S}^p_+$. Suppose that solutions $\{H_i(\alpha)\}_{i=1}^r$ to the following equations exist on $[t_0, t_f]$, 

$$
\frac{dH_1(\alpha)}{d\alpha} = -F(\alpha)^T H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha) \\
\frac{dH_i(\alpha)}{d\alpha} = -F(\alpha)^T H_2(\alpha) - H_2(\alpha) F(\alpha) \\
- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \ 2 \leq i \leq r.
$$

Under these conditions, the unique solutions $\{H_i(\alpha)\}_{i=1}^r$ are symmetric on $[t_0, t_f]$ so that

$$H_i(\alpha) = H_i^T(\alpha), \ 1 \leq i \leq r.$$

Proof See [26].

Theorem 2.2.4 (Positive-Semidefiniteness)
Let $r \in \mathbb{N}$ be a fixed, positive integer, and let $F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$, $N \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_+)$, and $W \in \mathbb{S}^p_+$. Suppose that solutions $\{H_i(\alpha)\}_{i=1}^r$ to the following equations exist on $[t_0, t_f]$, 

$$
\frac{dH_1(\alpha)}{d\alpha} = -F(\alpha)^T H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha) \\
\frac{dH_i(\alpha)}{d\alpha} = -F(\alpha)^T H_2(\alpha) - H_2(\alpha) F(\alpha) \\
- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \ 2 \leq i \leq r.
$$

Under these conditions, the unique, symmetric solutions $\{H_i(\alpha)\}_{i=1}^r$ are positive-semidefinite on $[t_0, t_f]$ so that

$$H_i(\alpha) \succeq 0^{n \times n}, \ 1 \leq i \leq r.$$

Proof See [26].

Theorem 2.2.5 (Monotonicity)
Let $r \in \mathbb{N}$ be a fixed, positive integer, and let $F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$, $N \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_+)$, and $W \in \mathbb{S}^p_+$. Suppose that solutions $\{H_i(\alpha)\}_{i=1}^r$ to the following
different equations exist on \([t_0, t_f]\),
\[
\frac{dH_1(\alpha)}{d\alpha} = -F(\alpha)^T H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha)
\]
\[
\frac{dH_2(\alpha)}{d\alpha} = -F(\alpha)^T H_2(\alpha) - H_2(\alpha) F(\alpha)
\]
\[
- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \ 2 \leq i \leq r.
\]

Under these conditions, the unique, symmetric, positive-semidefinite solutions \(\{H_i(\alpha)\}_{i=1}^r\) are monotonic-increasing backwards-in-time, from \(t_f\) to \(t_0\).

**Proof** See [26].

The theorems above represent a selection from those presented in [26], and the text can be consulted for the results, which have not been shown here.

### 2.3 Mean-Value Stationary Assumption for the Markov Process

The assumption that is made repeatedly in this work is that \(x_0 = E\{x(t_0)\}\). The cost cumulants (2.5) are written hereafter with the assumption that \(x_\alpha = E\{x(\alpha)\} = x_0, \forall \alpha \in [t_0, t_f]\). This implies that \(x(t)\) is mean-value stationary. Briefly some conditions that ensure this property are considered here. Let \(m_x(t) \triangleq E\{x(t)\}\), and observe that it is determined by the following equations with \(m_x(t_0) = x_0\)
\[
dm_x(t) = E\{dx(t)\} = \left( A(t) + B(t)K(t) \right) E\{x(t)\} dt = \left( A(t) + B(t)K(t) \right) m_x(t) dt.
\]

It follows that
\[
\frac{dm_x(t)}{dt} = \left( A(t) + B(t)K(t) \right) m_x(t), \ m_x(t) = \Phi(t,t_0)x_0, t \in [t_0, t_f].
\]

If \(x_0 = 0^r\), then clearly \(m_x(t) = x_0, \forall t \in [t_0, t_f]\). Conversely, suppose that \(x_0 \neq 0^r\) and that the system (2.2) with \(w(t) = 0^p\) under influence of \(u = Kx\) is asymptotically,
exponentially stabilized. Then $t^* \in [t_0, t_f)$ can be found when $t_f$ is sufficiently large such that $\forall t \geq t^*$, $m_x(t) \approx 0^n$, and this approximates the first case considered (i.e. $x_0 = 0^n$). Thus under the conditions of $x_0 = 0^n$ or exponential stability rendered by $u = Kx$, the process having dynamics (2.2) will be approximately mean-value stationary.

2.4 Some Less Well-Known Properties of Cumulants

This section is dedicated to a discussion and presentation of some less well-known properties of cumulants that appear in Port [9], which certainly pertain to the cost cumulants of the LQG framework. The theorems and corollaries which follow stem from the bijective relationship between sequences of moments and cumulants via the exponential Bell polynomials, which are well-known in combinatorial mathematics for their natural occurrence in a host of counting problems. See, for instance, [30] and [14]. Port shows additional examples of the bijection between moments and cumulants through moments of sums of independent, identically-distributed random variables, recurrent events on a boolean lattice, and compound Poisson processes.

The $n$th exponential Bell polynomial $\mathcal{B}_n(b_1, \ldots, b_n)$ is also referred to as the $n$th complete Bell polynomial, and these are defined as the sum of $n$ Bell polynomials $\mathcal{B}_{n,k}(b_1, \ldots, b_{n-k+1})$ as below,

$$\mathcal{B}_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \mathcal{B}_{n,k}(b_1, \ldots, b_{n-k+1}).$$ \hspace{1cm} (2.12)

The function $\mathcal{B}_{n,k}(b_1, \ldots, b_{n-k+1})$ is defined in the following way. Consider a sort of convolution operation $\star$ on sequences $a = \{a_n\}_{n=1}^{\infty}$ and $b = \{b_n\}_{n=1}^{\infty}$ to form a new
sequence given by \((a \star b)_{n=1}^{\infty}\), the \(n\)th term of which is

\[
(a \star b)_n = \sum_{j=1}^{n-1} \binom{n}{j} a_j b_{n-j}.
\]

When \(a = b\) and the \(\star\) convolution operation is completed repeatedly on \(b\), and subsequently \(B_{n,k}(b_1, \ldots, b_{n-k+1})\) is formed. More explicitly, this is

\[
B_{n,k}(b_1, \ldots, b_{n-k+1}) = \frac{(b \star \cdots \star b)_n}{k!}
\]

The interpretation of the polynomial \(B_{n,k}(b_1, \ldots, b_{n-k+1})\) in combinatorics is that each coefficient in this expression represents the total distinguishable number of ways of partitioning an integer \(n\) into \(k\) integers \(\{c_j\}_{j=1}^{k}\), where the notion of partitioning means that \(c_1 + \cdots + c_k = n\). The properties of Bell polynomials and complete Bell polynomials are many and fascinating, for instance \(B_{n,k}(1, \ldots, 1)\) and \(B_n(1, \ldots, 1)\) are

\[
B_{n,k}(1, \ldots, 1) = \binom{n}{j}, \quad B_n(1, \ldots, 1) = \sum_{j=1}^{n-1} \binom{n}{j}
\]

where it might be clear that \(B_{n,k}(1, \ldots, 1)\) is a Stirling number of the second kind and \(B_n(1, \ldots, 1)\) represents the total number of unique partitions of a set of \(n\).

How these interesting expressions \(B_n(b_1, \ldots, b_n)\) and \(B_{n,k}(b_1, \ldots, b_{n-k+1})\) connect the moments and cumulants is now discussed. Consider analytic functions \(F(z)\) and \(G(z)\) that satisfy the formal differential equation,

\[
\frac{1}{G(z)} \cdot \frac{dG(z)}{dz} = \frac{dF(z)}{dz}
\]

26
that clearly satisfy the relation, $G(z) = \exp(F(z))$.

Given the functions described above, expand the function $F(z)$ in formal power series along with $\exp(z)$,

$$F(z) = \sum_{i=0}^{\infty} x_i \frac{z^i}{i!}, \quad \exp(z) = 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!}.$$ 

Use Faà di Bruno’s formula [67] for power series in terms of Bell polynomials, which states that for functions $g(z)$ and $f(z)$ with

$$f(z) = \sum_{i=1}^{\infty} a_i \frac{z^i}{i!}, \quad g(z) = \sum_{i=1}^{\infty} b_i \frac{z^i}{i!}$$

that the composition $g \cdot f$ is given by

$$g(f(z)) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{i} \frac{z^j}{j!} B_{i,j}(a_1, \ldots, a_{i-j+1}) \right) \frac{z^i}{i!}.$$ 

To utilize the form above, take $a_i = x_i$ in $f(z)$ for $F(z)$ and note

$$g(z) = \exp(z) = 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!}$$

so that $b_j = 1, \forall j$ and using the relation (2.12) the following holds

$$G(z) = g(f(z)) = 1 + \sum_{i=1}^{\infty} \left( \sum_{j=1}^{i} \frac{(1)B_{i,j}(x_1, \ldots, x_{i-j+1})}{j!} \right) \frac{z^i}{i!}$$

$$= 1 + \sum_{i=1}^{\infty} B_i(x_1, \ldots, x_i) \frac{z^i}{i!}$$

$$= B_0(x_0) \frac{z^0}{0!} + \sum_{i=1}^{\infty} B_i(x_1, \ldots, x_i) \frac{z^i}{i!}.$$
\[= \sum_{i=0}^{\infty} B_i(x_1, \ldots, x_i) \frac{z^i}{i!}.\]

Now bear this relationship in mind in the following. Consider a random variable \(X\) that assumes values on \(\mathbb{R}\) with probability distribution function \(P(x)\) and density \(p(x)\). The \(n\)th moment is defined by

\[m_n = \int_{-\infty}^{\infty} x^n dP(x).\]

Let the moment sequence for \(X\) be defined as \(\{m_i\}_{i=1}^{\infty}\). Denote the set of all such moment sequences by \(\mathfrak{m}\) and consider the Laplace transform of \(p(x)\),

\[\varphi(s) = \int_{-\infty}^{\infty} p(x) \exp(-sx) dx = E[\exp(-sx)].\]

It is easy to see that the moment-generating function has the form,

\[\varphi(-z) = E[\exp(zx)] = E \left[ \sum_{i=0}^{\infty} \frac{(xz)^i}{i!} \right] = \sum_{i=0}^{\infty} E\left[ x^i \right] \frac{z^i}{i!} = \sum_{i=0}^{\infty} m_i \frac{z^i}{i!} = 1 + \sum_{i=1}^{\infty} m_i \frac{z^i}{i!}.\]

The definition of the cumulant-generating function is defined by \(\psi(z) = -\log(\varphi(-z))\), and has the series expansion,

\[\psi(z) = \sum_{i=1}^{\infty} \kappa_i \frac{z^i}{i!}.\]

so it must be true that,

\[\varphi(-z) = 1 + \sum_{i=1}^{\infty} m_i \frac{z^i}{i!} = \exp(\sum_{i=1}^{\infty} \kappa_i \frac{z^i}{i!}) = \exp(\psi(z))\]
and

\[ m_i = B_i(\kappa_1, \ldots, \kappa_i), \ i \geq 0. \]

It is discussed in Port [9] that \( \mathfrak{K} \), the cumulant space, is the inverse image of the moment space \( \mathfrak{M} \) under the exponential Bell map (E.B.M.) formed by the complete Bell polynomials. This is written concisely as \( \mathfrak{K} \xrightarrow{\text{E.B.M.}} \mathfrak{M} \). It can be shown \( \mathfrak{M} \) is closed in the weak topology of the sequence space on \( \mathbb{R} \), and so is \( \mathfrak{K} \) by this bijective relationship.

A few moments are written in terms of the cumulants, and vice versa, to illustrate the above fact. First, the moments via the exponential Bell polynomials

\[
\begin{align*}
m_1 &= \kappa_1 = B_1(\kappa_1) \\
m_2 &= \kappa_2 + \kappa_1^2 = B_2(\kappa_1, \kappa_2) \\
m_3 &= \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3 = B_3(\kappa_1, \kappa_2, \kappa_3) \\
m_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4 = B_4(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \\
&\quad\vdots
\end{align*}
\]

and the inversion,

\[
\begin{align*}
\kappa_1 &= m_1 \\
\kappa_2 &= m_2 - \kappa_1^2 \\
\kappa_3 &= m_3 - 3m_1m_2 + 2m_1^3 \\
\kappa_4 &= m_4 - 4m_3m_1 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4 \\
&\quad\vdots
\end{align*}
\]
The map \( \mathfrak{R} \xrightarrow{E.B.M.} \mathfrak{M} \) gives rise to an extension of the classical solution to the problem of moments to the problem of cumulants, which is defined here as characterizing the set \( \mathfrak{R} \). That is, what are necessary and sufficient conditions for a given sequence of reals \( \{\kappa_i\}_{i=1}^{\infty} \) to belong to \( \mathfrak{R} \)? The question can be answered by first examining the result of Hamburger’s theorem, which gives the necessary and sufficient conditions for a set of reals \( \{m_i\}_{i=1}^{\infty} \) to belong to \( \mathfrak{M} \).

**Theorem 2.4.1 (Hamburger’s Theorem)**

A sequence \( \{m_i\}_{i=1}^{\infty} \) is a sequence of moments in \( \mathfrak{M} \) if and only if the determinants below are non-negative,\
\[
\begin{vmatrix}
    m_0 & m_1 & m_2 & \ldots & m_n \\
    m_1 & m_2 & m_3 & \ldots & m_{n+1} \\
    m_2 & m_3 & m_4 & \ldots & m_{n+2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    m_n & m_{n+1} & m_{n+2} & \ldots & m_{2n}
\end{vmatrix} \geq 0, \ n \geq 0.
\]

**Proof** See [20].

**Corollary 2.4.2 (Problem of Cumulants)**

A sequence \( \{\kappa_i\}_{i=1}^{\infty} \) is a sequence of cumulants in \( \mathfrak{R} \) if and only if the determinants below are non-negative for \( n \geq 0 \),\
\[
\begin{vmatrix}
    1 & B_1(\kappa_1) & B_2(\kappa_1, \kappa_2) & \ldots & B_n(\kappa_1, \ldots, \kappa_n) \\
    B_1(\kappa_1) & B_2(\kappa_1, \kappa_2) & B_3(\kappa_1, \kappa_2, \kappa_3) & \ldots & B_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) \\
    B_2(\kappa_1, \kappa_2) & B_3(\kappa_1, \kappa_2, \kappa_3) & B_4(\kappa_1, \kappa_2, \kappa_3, \kappa_4) & \ldots & B_{n+2}(\kappa_1, \ldots, \kappa_{n+2}) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    B_n(\kappa_1, \ldots, \kappa_n) & B_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) & B_{n+2}(\kappa_1, \ldots, \kappa_{n+2}) & \ldots & B_{2n}(\kappa_1, \ldots, \kappa_{2n})
\end{vmatrix}.
\]

**Proof** The corollary is presented as a direct result in [9].

**Remark(s)** Hamburger’s theorem is for random variables \( X \) with support on \( \mathbb{R} \). Since the random cost has support only on \( \mathbb{R}^+ \), the positive reals, the necessary and sufficient conditions of the Stieltjes moment problem are more suited for the development in this work. The analogous conditions for the cumulants result from a direct replacement of the moments \( m_i \) for the expressions \( B_i(\kappa_1, \ldots, \kappa_i) \), as done above.
A few more properties of the cumulants stem from the relation $\mathfrak{R} \xrightarrow{E.B.M.} \mathfrak{M}$, and these will now be presented. Since these are a consequence of *Hankel Mean Independence*, they are given as corollaries of the result established in Port.

**Corollary 2.4.3 (Mean Independence)**
Suppose that $\{\kappa_i\}_{i=1}^{\infty}$ is a sequence of cumulants in $\mathfrak{R}$. Then for any constant $c$, $\{c, \{\kappa_i\}_{i=2}^{\infty}\}$ is also in $\mathfrak{R}$.

**Corollary 2.4.4 (Cumulants of Translated Variate)**
Suppose that $\{\kappa_i\}_{i=1}^{\infty}$ is a sequence of cumulants in $\mathfrak{R}$ for a random variable $X$. Then for any constant $c$, $\{\kappa_1 + c, \{\kappa_i\}_{i=2}^{\infty}\}$ is the sequence of cumulants in $\mathfrak{R}$ for the translated random variable $X + c$.

**Remark(s)** The above property that the first cumulant is *shift-equivariant* and the higher-order cumulants are *shift-invariant* is partly why cumulants are sometimes referred to as the *semi-invariants*. See [2] for a complete account of this historical nomenclature.

A concluding remark should be made as to what motivates this presentation of less well-known cumulant properties. It will become apparent in this work that for the cost (2.1), it is desired to identify a linear control input that optimizes some distance measure expressed in terms of the cost cumulants $\{\kappa_i\}_{i=1}^{r}$ and pre-specified target cost cumulants $\{\tilde{\kappa}_i\}_{i=1}^{r}$, where $r$ has been chosen such that his number of cumulants provides a reasonable approximation to the cost density and a target density. To this end, the conditions that characterize valid target cumulants for the random cost are very important and deserve some attention.
CHAPTER 3

MULTIPLE-CUMULANT COST DENSITY-SHAPING

The Multiple-Cumulant Cost Density-Shaping (MCCDS) optimization is posed in this chapter. Its solution is the linear control that minimizes a smooth, convex function of arbitrarily-many initial cost cumulants and target initial cost cumulants. The solution is obtained via a dynamic programming approach that is a natural extension of the theory in [66], but is adapted to the cost cumulant-generating equations of the LQG framework along the same lines as [26]. After the MCCDS problem is formalized and solved, it is shown that the MCCDS control solution generalizes the LQG, $k$CC, Risk-Sensitive (RS) controllers when certain linear MCCDS performance indices are chosen with zero targets. This material embellishes the work [48].

Towards the end of the chapter, the MCCDS control solution is derived for a Weighted Least-Squares, Cost Density-Shaping (WLS-CDS) optimization. After this, the WLS-CDS controller computation is validated through simulation, where it is shown that a family of target cost cumulants resultant from $3CC$ compensations can be tracked with high accuracy. This WLS-CDS framework does not directly involve a distance function between the cost density and the target density, and presently lacks a precise probabilistic meaning. It is shown later in Chapter 4 and Chapter 6 that distance functions can be expressed in terms
of the cumulants of both densities, and the MCCDS control directly gives the optimization solution.

3.1 Problem Statement and Solution

3.1.1 Process and Cost

Consider the process with dynamics subject to additive white Gaussian noise,

\[ dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t), \quad t \in [t_0, t_f] \]  

\[ x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n \]

where \( A \in C([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( B \in C([t_0, t_f]; \mathbb{R}^{n \times m}) \), \( G \in C([t_0, t_f]; \mathbb{R}^{n \times p}) \) and \( w(t) \) is a \( p \)-dimensional Wiener process having a correlation of increments defined by

\[ E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}. \]

Control inputs are assumed to be \( \mathbb{R}^m \)-valued, square-integrable, non-anticipating processes \( u \in \mathcal{U} \subset L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^m)) \) adapted to the filtration \( \mathcal{F}_t \) generated by \( w(t) \). Under this assumption, \( x \in L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \). The control input \( u \) is chosen with respect to the cost \( J[x, u; t_0, x_0] \) that is integral-quadratic and defined by

\[ J = \int_{t_0}^{t_f} \left( x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau) \right) d\tau + x^T(t_f)Q_f x(t_f) \]

where \( Q \in C([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( R \in C([t_0, t_f]; \mathbb{R}^{m \times m}) \), and \( Q_f \in \mathbb{R}^{n \times n} \). For the well-posedness of the associated stochastic optimal control problem, it is imposed further that \( Q, Q_f \succeq 0^{n \times n}, \ R \succ 0^{m \times m} \).
3.1.2 Cost Cumulants

When the process \( x(t) \) is subjected to linear state-feedback control \( u = Kx \) where \( K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n}) \) with \( K \subset \mathbb{R}^{m \times n} \) and \( K \) compact, the process dynamics become

\[
\begin{align*}
  dx(t) &= \left( A(t) + B(t)K(t) \right)x(t)dt + G(t)dw(t), \ t \in [t_0, t_f] \\
  x_0 &= E\{x(t_0)\}, \ x_0 \in \mathbb{R}^n
\end{align*}
\]

and the cost can be written as

\[
J = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_fx(t_f)
\]

where \( N(\tau) = K(\tau)^TR(\tau)K(\tau) + Q(\tau) \). The development assumes that the controlled Markov process determined by (3.3) is mean-value stationary. Sufficient conditions for this are described in Section 2.3.

The work of Liberty [61-62] established that the under the foregoing assumptions of the cost being integral-quadratic and the control being linear state-feedback for the system (3.1), the initial cost cumulants have the following quadratic-affine form

\[
\kappa_i(\alpha) = x_0^TH_i(\alpha)x_0 + D_i(\alpha), \ 1 \leq i \leq r
\]

with \( \alpha = t_0 \). The functions \( H_i(\alpha) \) satisfy the following system of backwards-in-time, matrix differential equations. The dynamics of the functions \( D_i(\alpha) \) depend on the \( H_i(\alpha) \) functions.
\[
\frac{dH_1(\alpha)}{d\alpha} = -\left( (A(\alpha) + B(\alpha)K(\alpha))^T H_1(\alpha) - H_1(\alpha) A(\alpha) + B(\alpha)K(\alpha) \right) - K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha) \triangleq F_1(H(\alpha), K(\alpha))
\]

\[
\frac{dH_i(\alpha)}{d\alpha} = -\left( (A(\alpha) + B(\alpha)K(\alpha))^T H_i(\alpha) - H_i(\alpha) A(\alpha) + B(\alpha)K(\alpha) \right) - 2 \sum_{j=1}^{i-1} \binom{i-1}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha) \triangleq F_i(H(\alpha), K(\alpha)), \quad 2 \leq i \leq r
\]

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}\left( H_j(\alpha)G(\alpha)WG^T(\alpha) \right) \triangleq G_j(H(\alpha)), \quad \alpha \in [t_0, t_f], \quad 1 \leq j \leq r.
\]

These functions satisfy the terminal conditions

\[
H_1(t_f) = Q_f, \quad H_i(t_f) = 0^{n \times n}, \quad i \geq 2 \tag{3.5}
\]

\[
D_1(t_f) = 0, \quad D_2(t_f) = 1, \quad D_j(t_f) = 0, \quad j \geq 3.
\]

For linear state-feedback control inputs, it is shown in [27] that \( J \) is a finite \( \chi^2 \) random variable on a probability space \((\Omega, \mathcal{F}, \mathcal{P})\). The finiteness of \( J \) stems from the fact that for linear state-feedback controls, the “running cost” and “terminal cost” functions of (3.2) always satisfy the suitable polynomial growth conditions necessary for boundedness of the expectation of the cost functional [66]. So under this class of control inputs, a finite number of \( r \) cumulants exist for \( J \).

The initial cumulants of (3.2) are given explicitly by the following well-known recursive relationship,

\[
\kappa_1(t_0) \triangleq E\{J\}, \quad \kappa_r(t_0) \triangleq E\{J^r\} - \sum_{i=1}^{r-1} \binom{r-1}{i-1} \kappa_i(t_0)E\{J^{r-i}\}, \quad r \geq 2.
\]
3.1.3 Notation

Some notation is introduced to make restatements of the above equations more concise in the development. This notation was originally proposed by Pham in [26], and is heavily used in the MCCDS problem formulation. Begin by defining the state variables $H(\alpha) \in \mathbb{R}^{r \times r \times n}$ and $D(\alpha) \in \mathbb{R}^r$ as below

\[
H(\alpha) \triangleq \begin{bmatrix}
H_1(\alpha) \\
\vdots \\
H_r(\alpha)
\end{bmatrix}, \\
D(\alpha) \triangleq \begin{bmatrix}
D_1(\alpha) \\
\vdots \\
D_r(\alpha)
\end{bmatrix}.
\]

Using these state variables, define the functions

\[
\mathcal{F}(H(\alpha), K(\alpha)) \triangleq \begin{bmatrix}
\mathcal{F}_1(H(\alpha), K(\alpha)) \\
\vdots \\
\mathcal{F}_r(H(\alpha), K(\alpha))
\end{bmatrix}, \\
\mathcal{G}(H(\alpha)) \triangleq \begin{bmatrix}
\mathcal{G}_1(H(\alpha)) \\
\vdots \\
\mathcal{G}_r(H(\alpha))
\end{bmatrix}.
\]

Let $\mathcal{F}_i(\cdot)$ and $\mathcal{G}_i(\cdot)$ in the above definitions be defined as beforehand in (3.4). Also to be introduced is a condensed form for the terminal conditions

\[
H_f \triangleq \begin{bmatrix}
Q_f \\
0^{n \times n} \\
\vdots \\
0^{n \times n}
\end{bmatrix}, \\
D_f \triangleq \begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]
Finally, denote the vector of cost cumulants $\kappa(\alpha) \in \mathbb{R}^r$ as

$$
\kappa(\alpha) \triangleq \begin{bmatrix}
\kappa_1(\alpha) \\
\vdots \\
\kappa_r(\alpha)
\end{bmatrix}.
$$

### 3.1.4 Target Cumulants

Given matrices for a system characterization $(A, B, G)$, an integral-quadratic cost characterization $(Q, R, Q_f)$, and the second-order statistics of the noise $(W)$, consider the cost cumulants as a result of the alternative (and unknown) linear state-feedback control $\hat{u}(t) = \hat{K}(t)\hat{x}(t)$, where $\hat{K} \in C([t_0, t_f]; \mathbb{R}^{m \times n})$. The initial cost cumulants are given by

$$
\hat{\kappa}_i(\alpha) = x_0^T \hat{H}_i(\alpha)x_0 + \hat{D}_i(\alpha), \ 1 \leq i \leq r
$$

(3.6)

with $\alpha = t_0$. Let this set of numbers be regarded as target initial cost cumulants. Here the functions $\hat{H}_i(\alpha)$ are determined by the same system of backwards-in-time, matrix differential equations as (3.4). The dynamics of $\hat{D}_i(\alpha)$ will also be as before.

$$
\frac{d\hat{H}(\alpha)}{d\alpha} = \mathcal{F}(\hat{H}(\alpha), \hat{K}(\alpha)), \quad \frac{d\hat{D}(\alpha)}{d\alpha} = \mathcal{G}(\hat{H}(\alpha))
$$

(3.7)

$$
\hat{H}(t_f) = H_f, \hat{D}(t_f) = D_f, \quad \alpha \in [t_0, t_f]
$$

Above the terminal conditions are

$$
\hat{H}_1(t_f) = Q_f + \mathcal{E}, \quad \hat{H}_i(t_f) = 0^{n \times n}, \ i \geq 2
$$

(3.8)

$$
\hat{D}_1(t_f) = \mathcal{E}, \quad \hat{D}_2(t_f) = 1, \hat{D}_j(t_f) = 0, \ j \geq 3
$$

(3.9)
and the short-hand notation is used,

\[
\mathbf{H}_{f,\varepsilon^*} \triangleq \begin{bmatrix}
Q_f + \mathcal{E}^* \\
0_{n \times n} \\
\vdots \\
0_{n \times n}
\end{bmatrix}, \quad \mathbf{D}_{f,\varepsilon^*} \triangleq \begin{bmatrix}
\varepsilon^* \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

Here \( \varepsilon^* > 0 \) is a small perturbation constant, and \( \mathcal{E}^* > 0_{n \times n} \) is a positive-definite perturbation matrix. As with the cost cumulants, compose a vector of target cost cumulants \( \bar{\kappa}(\alpha) \in \mathbb{R}^r \) defined as below,

\[
\bar{\kappa}(\alpha) \triangleq \begin{bmatrix}
\bar{\kappa}_1(\alpha) \\
\vdots \\
\bar{\kappa}_r(\alpha)
\end{bmatrix}.
\]

The equations (3.7) must be integrable under \( \bar{K}(\alpha) \) for \( \bar{H}(\alpha) \) and \( \bar{D}(\alpha) \) to exist, and hence for \( \bar{\kappa}(\alpha) \) to also exist. This requirement means that a \( r \)-cumulant approximation for the target cost density exists, and that subsequently the MCCDS problem is well-posed in this respect. The important assumption of integrability of (3.7) is made for some \( \bar{K}(\alpha) \).

Additionally it is assumed that this control asymptotically and exponentially stabilizes the system (3.1) when \( w(t) = 0^p \). The purpose for this assumption will become clear in Section 3.1.5. See also Section 2.3 for another motivation of assuming exponentially, stabilizing control for the generation of target cost cumulants.
Remark(s)

- It should be noted that the target cumulants \( \tilde{\kappa} \) will appear in the formulation of a novel dynamic optimization as uncontrolled modes of an augmented system of initial cost cumulants and target initial cost cumulants. The target initial cost cumulants have dynamics achieved by some fixed, though unknown linear control input with gain \( \tilde{K} \). The formulation is such that for a general control gain \( K \), the dynamics of the controlled modes (i.e. the initial cost cumulants) are uncoupled from the dynamics of the uncontrolled modes (i.e. the target initial cost cumulants).

- For purposes of simulation, it is assumed the target initial cost cumulants are computed \emph{a priori} and full information on the target cumulants \( \tilde{\kappa}(\alpha) \) is available during the MCCDS controller computation.

3.1.5 Perturbations to Terminal Conditions

Recall the generating equations for the target cost cumulants,

\[
\frac{d\tilde{H}_1(\alpha)}{d\alpha} = - \left( A(\alpha) + B(\alpha)\tilde{K}(\alpha) \right)^T \tilde{H}_1(\alpha) - \tilde{H}_1(\alpha) \left( A(\alpha) + B(\alpha)\tilde{K}(\alpha) \right) - K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha) \tag{3.10}
\]

\[
\frac{d\tilde{H}_i(\alpha)}{d\alpha} = - \left( A(\alpha) + B(\alpha)\tilde{K}(\alpha) \right)^T \tilde{H}_i(\alpha) - \tilde{H}_i(\alpha) \left( A(\alpha) + B(\alpha)\tilde{K}(\alpha) \right) - 2 \sum_{j=1}^{i-1} \begin{pmatrix} i \ j \end{pmatrix} \tilde{H}_j(\alpha)G(\alpha)WG^T(\alpha)\tilde{H}_{i-j}(\alpha), \ 2 \leq i \leq r
\]

\[
\frac{d\tilde{D}_j(\alpha)}{d\alpha} = - \text{Tr}\left( \tilde{H}_j(\alpha)G(\alpha)WG^T(\alpha) \right), \ \alpha \in [t_0, t_f], \ 1 \leq j \leq r. \tag{3.11}
\]

In particular, alternative terminal conditions for (3.10) and (3.11) appear:

\[
\tilde{H}_1(t_f) = Q_f + E^*, \ \tilde{H}_i(t_f) = 0^{n \times n}, \ i \geq 2
\]

\[
\tilde{D}_1(t_f) = e^*, \ \tilde{D}_2(t_f) = 1, \ \tilde{D}_j(t_f) = 0, \ j \geq 3 \tag{3.12}
\]
These differ from the terminal conditions established in [61-62],

\[ \begin{align*}
\tilde{H}_1(t_f) &= Q_f, \quad \tilde{H}_i(t_f) = 0^{n \times n}, \quad i \geq 2, \\
\tilde{D}_j(t_f) &= 0, \quad 1 \leq j \leq r.
\end{align*} \tag{3.13} \]

Perturbation to the terminal conditions of (3.4) and (3.7) makes it possible to integrate the equations of motion using traditional fixed-step numerical integration schemes. This is because without perturbations, a singularity is present at the start of the integration for many cost density-shaping controls, and it is well-known that Runge-Kutta and multi-step methods cannot integrate through singularities [25]. The so-called Möbius scheme discussed in the referenced work may be used to integrate through singularities in the solutions of (3.4) and (3.7) under MCCDS control, as it is easily seen that these equations become differential matrix Riccati equations. This possibility, while plausible, is not explored.

Because of the challenges with numerical integration mentioned above, it is worthwhile to consider what effects perturbing the terminal conditions might render to the solutions of (3.4) and (3.7). One consequence of perturbing the terminal conditions is that the singularities in the solution of the evolution equations will be changed. A large class of differential equations is characterized by fixed singularities that are associated with the operator itself, and also moveable singularities that depend on the initial (or terminal) conditions in the boundary-value problem [5]. See the reference for an example that illustrates the singularities of a Riccati equation.

In the following, unity perturbations and matrix perturbations are examined separately. In the context of perturbations appearing in (3.4) and (3.7), theorems are provided on the effects of each kind of perturbation.
Unity Perturbation, $D_1(t_f)$ and $\tilde{D}_1(t_f)$:

The following theorems are presented to detail the effects of unity perturbation to the terminal conditions $D_2(t_f)$ and $\tilde{D}_2(t_f)$, as well as the $\epsilon^*$ perturbation to $\tilde{D}_1(t_f)$. The proof supporting the following theorem exploits the fact that the time evolution of $H_i(\alpha)$ for $K(\alpha)$ constrains the time evolution of $D_i(\alpha)$. Analogously, the time evolution of $\tilde{H}_i(\alpha)$ for $\tilde{K}(\alpha)$ constrains the time evolution of $\tilde{D}_i(\alpha)$.

**Theorem 3.1.1 (Effects of Scalar Perturbation to $\tilde{D}(t_f)$)**

Define the following systems of equations,

\[
\frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha))
\]

$\tilde{H}(t_f) = H_f$, $\tilde{D}(t_f) = D_{f^*}, \alpha \in [t_0, t_f]$.

and

\[
\frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha))
\]

$\tilde{H}(t_f) = H_f$, $\tilde{D}(t_f) = 0^*, \alpha \in [t_0, t_f]$.

Then it is true that

\[
\tilde{D}(\alpha) = \mathcal{D}(\alpha) + D_{f^*}.
\]

**Proof** Observe that the following holds,

\[
\tilde{H}(t_f) = \tilde{H}(t_f) = H_f.
\]

By the uniqueness property in [26] for the solution $\tilde{H}(\alpha)$ with fixed terminal conditions, it is true that

\[
\tilde{H}(\alpha) = \tilde{H}(\alpha), \quad \forall \alpha \in [t_0, t_f].
\]
From this result it follows that,

\[
\frac{d\tilde{D}(\alpha)}{d\alpha} = g(H(\alpha)) = g(\tilde{H}(\alpha)) = \frac{d\tilde{D}(\alpha)}{d\alpha}.
\]

Consider the function \( \hat{D}(\alpha) = \tilde{D}(\alpha) - \tilde{D}(\alpha) \), where evidently from above,

\[
\frac{d\hat{D}(\alpha)}{d\alpha} - \frac{d\hat{D}(\alpha)}{d\alpha} = 0, \quad \hat{D}(t_f) = \tilde{D}(t_f) - \tilde{D}(t_f) = D_{f,\epsilon}.
\]

From these relations, it must be that \( \hat{D}(\alpha) = c, \forall \alpha \in [t_0, t_f] \) where \( c \in \mathbb{R} \). To satisfy the terminal condition \( \hat{D}(t_f) \), it must be that

\[
\hat{D}(\alpha) = \hat{D}(\alpha) + D_{f,\epsilon}, \forall \alpha \in [t_0, t_f].
\]

\[\square\]

**Theorem 3.1.2 (Effects of Unity Perturbation to \( D(t_f) \))**

Define the following systems of equations,

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{dD(\alpha)}{d\alpha} = G(H(\alpha))
\]

\[H(t_f) = H_f, \quad D(t_f) = D_f, \alpha \in [t_0, t_f]\]

and

\[
\frac{d\hat{H}(\alpha)}{d\alpha} = F(\hat{H}(\alpha), K(\alpha)), \quad \frac{d\hat{D}(\alpha)}{d\alpha} = G(\hat{H}(\alpha))
\]

\[\hat{H}(t_f) = H_f, \quad \hat{D}(t_f) = 0, \alpha \in [t_0, t_f].\]

Then it is true that

\[D(\alpha) = D(\alpha) + D_f.\]

**Proof** The proof is identical to that of Theorem 3.1.1.
The next result follows immediately from the above development.

**Theorem 3.1.3 (Effects of Scalar Perturbation)**

Suppose on the finite-horizon \([t_0, t_f]\) the following is satisfied for solutions \(D_2(\alpha)\), \(\tilde{D}_2(\alpha)\), and \(\tilde{D}_1(\alpha)\) of equations (3.4) and (3.7),

\[
\kappa_2(\alpha) \geq D_2(t_0) >> 1, \quad \tilde{\kappa}_2(\alpha) \geq \tilde{D}_2(t_0) >> 1, \quad \tilde{\kappa}_1(\alpha) \geq \tilde{D}_1(t_0) >> \epsilon^*.
\]

When these conditions are satisfied, the unity perturbations to \(D_2(t_f)\) and \(\tilde{D}_2(t_f)\) are of negligible effect. Furthermore, under the above conditions, the \(\epsilon^*\) perturbation to \(\tilde{D}_1(t_f)\) is also of negligible effect.

**Proof** The relations below are immediate from Theorem 3.1.1.

\[
\tilde{D}_1(\alpha) = \tilde{D}_1(\alpha) + \epsilon^*, \quad \tilde{D}_2(\alpha) = \tilde{D}_2(\alpha) + 1, \forall \alpha \in [t_0, t_f].
\]

Also Theorem 3.1.2 guarantees that the following relationship holds

\[
D_2(\alpha) = D_2(\alpha) + 1, \forall \alpha \in [t_0, t_f].
\]

Since \(x_0^T H_2(t_0)x_0 \geq 0\) and \(x_0^T \tilde{H}_i(t_0)x_0 \geq 0, 1 \leq i \leq 2\), it follows that conditions under which scalar perturbations are negligible are

\[
\kappa_2(t_0) \geq D_2(t_0) >> 1, \quad \tilde{\kappa}_2(t_0) \geq \tilde{D}_2(t_0) >> 1, \quad \tilde{\kappa}_1(t_0) \geq \tilde{D}_1(t_0) >> \epsilon^*.
\]

**Matrix Perturbation, \(\tilde{H}_1(t_f)\):**

The *Generalized Partitioned Riccati Solutions* attributed to Lainiotis [10] can be used in order to better understand the effect of perturbing \(\tilde{H}_1(t_f)\). Essentially, generalized partition solutions of a Riccati equation are when the equation
solutions are expressed in terms of a solution to another Riccati equation. In particular, this enables the solution of a Riccati equations with perturbed terminal condition to be expressed in terms of solution to that same Riccati equation, but with unperturbed terminal conditions. The main results are summarized below.

**Theorem 3.1.4 (Generalized Partitioned Riccati Solutions)**

Consider now the symmetric matrix-Ricatti differential equation,

\[
\frac{\partial X_n(t, t_0)}{\partial t} = Q(t) + X_n(t, t_0)F(t) + F^T(t)X_n(t, t_0) - X_n(t, t_0)R(t)X_n(t, t_0) \tag{3.14}
\]

and also

\[
\frac{\partial X(t, t_0)}{\partial t} = Q(t) + X(t, t_0)F(t) + F^T(t)X(t, t_0) - X(t, t_0)R(t)X(t, t_0) \tag{3.15}
\]

Then the solution \( X(t, t_0) \) can be expressed as,

\[
X(t, t_0) = X_n(t, t_0) + \Phi_n(t, t_0) \left( O_n(t, t_0) + X_r^{-1} \right)^{-1} \Phi_n^T(t, t_0)
\]

where the functions \( \Phi_n(t, t_0) \) and \( O_n(t, t_0) \) satisfy the following differential equations,

\[
\frac{\partial O_n(t, t_0)}{\partial t} = \Phi_n(t, t_0)R(t)\Phi_n(t, t_0), \quad O_n(t_0, t_0) = 0^{n\times n}
\]

\[
\frac{\partial \Phi_n(t, t_0)}{\partial t} = (F^T(t) - X_n(t, t_0)R(t))\Phi_n(t, t_0), \quad \Phi_n(t_0, t_0) = I^{n\times n} \tag{3.16}
\]

**Proof** See [10] and [6].

In order to adapt this theory to the cost cumulant-generating equations of the LQG framework, the equations of motion must be written in forward-time variables \( t \) instead of the backwards-time variable \( \alpha \triangleq t_f + t_0 - t \). Make the following definitions.

\[
\dot{A}(t) = A(t_f + t_0 - t), \quad \dot{B}(t) = B(t_f + t_0 - t), \quad \dot{G}(t) = G(t_f + t_0 - t)
\]

44
\[
\dot{K}(t) = K(t_f + t_0 - t), \quad \dot{Q}(t) = Q(t_f + t_0 - t), \quad \dot{H}_l(t) = H_l(t_f + t_0 - t)
\]

It is not hard to see that

\[
\frac{d\dot{H}_l(t)}{dt} = \frac{dH_l(\alpha)}{d\alpha} \cdot \frac{d\alpha}{dt} = -\frac{dH_l(\alpha)}{d\alpha}, \quad 1 \leq l \leq r.
\]

The equations above gives rise to the forward-in-time equations

\[
\frac{d\dot{H}_1(t)}{dt} = \left( \dot{A}(t) + \dot{B}(t)\dot{K}(t) \right)^T \dot{H}_1(t) + \dot{H}_1(t) \left( \dot{A}(t) + \dot{B}(t)\dot{K}(t) \right) + \dot{K}^T(t)\dot{R}(t)\dot{K}(t) + \dot{Q}(t)
\]

\[
\frac{d\dot{H}_l(t)}{dt} = \left( \dot{A}(t) + \dot{B}(t)\dot{K}(t) \right)^T \dot{H}_l(t) + \dot{H}_l(t) \left( \dot{A}(t) + \dot{B}(t)\dot{K}(t) \right) + \sum_{j=1}^{l-1} \left( \begin{array}{c} l \\ j \end{array} \right) \dot{H}_l(t)\dot{G}(t)W\dot{G}^T(t)\dot{H}_{l-j}(t), \quad 2 \leq l \leq r
\]

which satisfy the initial conditions

\[
\dot{H}_1(t_0) = Q_f, \quad \dot{H}_l(t_0) = 0^{n \times n}, \quad 2 \leq l \leq r.
\]

Analogously the equations that \( \dot{\hat{H}}_l(t) \) must satisfy are given by

\[
\frac{d\dot{\hat{H}}_1(t)}{dt} = \left( \dot{\hat{A}}(t) + \dot{\hat{B}}(t)\dot{\hat{K}}(t) \right)^T \dot{\hat{H}}_1(t) + \dot{\hat{H}}_1(t) \left( \dot{\hat{A}}(t) + \dot{\hat{B}}(t)\dot{\hat{K}}(t) \right) + \dot{\hat{K}}^T(t)\dot{\hat{R}}(t)\dot{\hat{K}}(t) + \dot{\hat{Q}}(t)
\]

\[
\frac{d\dot{\hat{H}}_l(t)}{dt} = \left( \dot{\hat{A}}(t) + \dot{\hat{B}}(t)\dot{\hat{K}}(t) \right)^T \dot{\hat{H}}_l(t) + \dot{\hat{H}}_l(t) \left( \dot{\hat{A}}(t) + \dot{\hat{B}}(t)\dot{\hat{K}}(t) \right) + \sum_{j=1}^{l-1} \left( \begin{array}{c} l \\ j \end{array} \right) \dot{\hat{H}}_l(t)\dot{\hat{G}}(t)W\dot{\hat{G}}^T(t)\dot{\hat{H}}_{l-j}(t), \quad 2 \leq l \leq r
\]
which satisfy the initial conditions

\[
\hat{H}_1(t_0) = Q_f + \mathcal{E}^*, \quad \hat{H}_l(t_0) = 0^{n \times n}, \quad 2 \leq l \leq r. \tag{3.20}
\]

The forward-time representations of the cost cumulant-generating \( H_i(\cdot) \) equations (3.4) are consistent with the form of the equations used in the generalized partition Riccati equations results with \( R(t) = 0^{n \times n} \). A new result is now presented which indicates that when the equations (3.17) and (3.19) are defined on \([t_0, \infty)\), the effects of perturbation to \( \hat{H}_l(t) \) asymptotically and exponentially diminish when the control input uniformly, exponentially stabilizes the process having dynamics (3.1), though in the absence of noise \( w(t) = 0^p, \forall t \).

**Theorem 3.1.5 (Effects of Perturbation to \( \hat{H}_1(t_f) \))**

Suppose that the system below is uniformly, exponentially stable,

\[
\frac{dx(t)}{dt} = \left( \hat{A}(t) + \hat{B}(t) \hat{K}(t) \right)^T x(t), \quad x(t_0) = x_0. \tag{3.21}
\]

Under this condition, and when \( \hat{H}_1(t) \) and \( \hat{H}_1(t) \) exist on \( t \in [t_0, \infty) \), the difference between the solution \( \hat{H}_1(t) \) of the first forward-in-time cost cumulant-generating equations (3.17) and the solution \( \hat{H}_1(t) \) of the first cumulant-generating equation (3.19) asymptotically and exponentially vanishes,

\[
\lim_{t \to \infty} ||\hat{H}_1(t) - \hat{H}_1(t)|| = 0.
\]

**Proof** By the result of Lainiotis the following relation holds,

\[
\hat{H}_1(t) - \hat{H}_1(t) = \Phi_n(t, t_0) \mathcal{E}^* \Phi_n^T(t, t_0), \quad \mathcal{E}^* > 0^{n \times n}.
\]

46
where the function $\Phi_n(t, t_0)$ satisfies the following differential equation,

$$\frac{\partial \Phi_n(t, t_0)}{\partial t} = \left( \hat{A}(t) + \hat{B}(t) \hat{K}(t) \right)^T \Phi_n(t, t_0), \quad \Phi_n(t_0, t_0) = I_{n \times n}. $$

Notice that $\Phi_n(t, t_0)$ is the transition matrix for the system (3.21). Since the system (3.21) is uniformly, exponentially stable, it is true that

$$||\Phi_n(t, t_0)|| \leq \gamma e^{-\lambda (t-t_0)}, \quad \lambda > 0, \quad \gamma > 0.$$ 

Immediately it follows that,

$$||\hat{H}_1(t) - \hat{\tilde{H}}_1(t)|| = ||\Phi_n(t, t_0) E^* \Phi_n^T(t, t_0)||$$

$$\leq ||\Phi_n(t, t_0)|| \cdot ||E^*|| \cdot ||\Phi_n^T(t, t_0)||$$

$$\leq \left( \gamma e^{-\lambda (t-t_0)} \right) \cdot ||E^*|| \cdot \left( \gamma e^{-\lambda (t-t_0)} \right)$$

so that

$$\lim_{t \to \infty} ||\hat{H}_1(t) - \hat{\tilde{H}}_1(t)|| \leq \lim_{t \to \infty} \left\{ \left( \gamma e^{-\lambda (t-t_0)} \right) \cdot ||E^*|| \cdot \left( \gamma e^{-\lambda (t-t_0)} \right) \right\} = 0.$$ 

The following extends the above result to the target equations (3.7).

**Theorem 3.1.6 (Effects of Matrix Perturbation)**

Assume that the function $\hat{H}_1(\alpha)$ determined by (3.7) for $i = 1$ with $E^* > 0_{n \times n}$ is defined on $[t_0, \infty)$. Similarly, consider the function $\hat{\tilde{H}}_1(\alpha)$ determined by (3.7) for $i = 1$, but instead with $E^* = 0_{n \times n}$. Suppose this solution is defined on $[t_0, \infty)$. Let the deterministic system below be uniformly, exponentially stable

$$\frac{d\hat{x}(t)}{dt} = \left( A(\alpha) + B(\alpha) \hat{K}(\alpha) \right)^T \hat{x}(t), \quad \hat{x}(t_0) = x_0, \quad t \in [t_0, \infty).$$
Under these assumptions, the effect of perturbation $\mathcal{E}^*$ to the terminal condition $\tilde{H}_1(t_f)$ vanishes asymptotically and exponentially,

$$\lim_{t_f \to \infty} ||\tilde{H}_1(t_f + t_0 - t) - \tilde{H}_1(t_f + t_0 - t)|| = 0.$$  

**Remark(s)** Theorem 3.1.6 makes a very important statement. That is, when the controller $\tilde{K}(\alpha)$ underlying the target cost cumulants $\tilde{\kappa}_i(\alpha), 1 \leq i \leq r$ is stabilizing, the effects of perturbing $\tilde{H}_1(t_f)$ are negligible as $t_f \to \infty$. Hence for the MCCDS optimization, as will be posed with a perturbation to $\tilde{H}_1(t_f)$, it is clear the target cost cumulants resultant from stabilizing compensations are ideal for tracking in the sense that perturbations to $\tilde{H}_1(t_f)$ should not have a noticeable effect in the solution $\tilde{H}_1(\alpha)$.

3.1.6 MCCDS Problem Formulation

The goal of this section is to pose a novel cost density-shaping optimization, termed as the MCCDS problem. Before proceeding to the problem statement, two important definitions must be introduced - the target set and admissible feedback gains. Namely, the target set is a space that the end value of the state trajectories must lie in. Put somewhat loosely, the set of admissible control gains are those gains that can steer the state variables into the target set. The following definitions formalize these ideas.

**Definition 3.1.7 (Target Set)**  
Let $(t_0, \mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0)) \in \mathcal{M}$, where $\mathcal{M}$ denotes the target set which is a closed subset of

$$[t_0, t_f] \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times \mathbb{R}^r \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times \mathbb{R}^r.$$  

**Remark(s)** Note that the goal of tracking target cumulants suggests that predetermined trajectories of $\tilde{\mathbf{H}}(\alpha)$ and $\tilde{\mathbf{D}}(\alpha)$ will inherently have initial values in $\mathcal{M}$ by their definition.

For given terminal conditions, the set of admissible feedback gains is the set
of matrices $K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})$ such that values

$$(t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}$$

are obtained at the end of the trajectories for the state equations (3.4) and (3.7).

This is formally stated in the following definition.

**Definition 3.1.8 (Admissible Feedback Gains)**

Denote the allowable set of control gain values by $\bar{K} \subset \mathbb{R}^{m \times n}$ and let this set be compact. For fixed $\epsilon \in \mathbb{N}$ let $K_{t_f}, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)$ characterize a class of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})$ such that for any $K \in K_{t_f}, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)$ the solutions to

$$\begin{align*}
\frac{dH(\alpha)}{d\alpha} &= \mathcal{F}(H(\alpha), K(\alpha)), \\
\frac{d\tilde{H}(\alpha)}{d\alpha} &= \mathcal{F}(\tilde{H}(\alpha), \tilde{K}(\alpha)) \\
\frac{dD(\alpha)}{d\alpha} &= \mathcal{G}(H(\alpha)), \\
\frac{d\tilde{D}(\alpha)}{d\alpha} &= \mathcal{G}(\tilde{H}(\alpha)), \\
H(t_f) &= H_f, \\
\tilde{H}(t_f) &= \tilde{H}_f, \\
D(t_f) &= D_f, \\
\tilde{D}(t_f) &= \tilde{D}_f
\end{align*}$$

exist and the initial values of the state trajectories satisfy

$$(t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}.$$
**Theorem 3.1.9** (Positivity of the Performance Index)
*If the function $g_{\tilde{x}}$ is analytic and convex in $x$, and such that $g_{\tilde{x}}(\tilde{x}) = 0$ and also $\nabla_x g_{\tilde{x}}(\tilde{x}) = 0^{1 \times r}$, then $g_{\tilde{x}}(x) \geq 0$ on some neighborhood of $\tilde{x}$.***

**Proof** Because the function $g_{\tilde{x}}(x)$ is analytic on its domain, the function’s Taylor series expansion about the target vector $\tilde{x}$ exists,

$$
g_{\tilde{x}}(x) = g_{\tilde{x}}(\tilde{x}) + \nabla_x g_{\tilde{x}}(\tilde{x})(x - \tilde{x}) + \frac{1}{2}(x - \tilde{x})^T \nabla^2_x g_{\tilde{x}}(\tilde{x})(x - \tilde{x}) + \ldots
$$

Since the Taylor series exists about $\tilde{x}$ and the function $g_{\tilde{x}}(x)$ is convex, the first-order condition of convexity can be written.

$$
g_{\tilde{x}}(x) \geq g_{\tilde{x}}(\tilde{x}) + \nabla_x g_{\tilde{x}}(\tilde{x})(x - \tilde{x})
$$

Using the conditions $g_{\tilde{x}}(\tilde{x}) = 0$ and $\nabla_x g_{\tilde{x}}(\tilde{x}) = 0^{1 \times r}$ gives

$$
ge_{\tilde{x}}(x) \geq g_{\tilde{x}}(\tilde{x}) + \nabla_x g_{\tilde{x}}(\tilde{x})(x - \tilde{x}) = 0^{1 \times r} = 0
$$

and the proof is complete.

\[\square\]

**Definition 3.1.10** (MCCDS Performance Index)
*Let the MCCDS Performance index be defined as the function

$$
\phi(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) = g(\kappa(t_0), \tilde{\kappa}(t_0)).
$$

Consider the following optimization problem of minimizing the function $g(\kappa(t_0), \tilde{\kappa}(t_0))$ of the initial cost cumulants for a given set of target initial cost cumulants, over the admissible space of controls. Ideally, the function $g(\kappa(t_0), \tilde{\kappa}(t_0))$ will be some positive measure between initial cost cumulants and target initial cost cumulants.
cumulants, so that deriving a control input which minimizes this function will drive initial cost cumulants closer to the targets, and in effect will drive a \( r \)-cumulant approximation to the initial cost density closer to a \( r \)-cumulant approximation to the target initial cost density.

**Definition 3.1.11 (MCCDS Optimization)**

For every \( \tilde{\kappa}(t_0) \), let \( g(\kappa(t_0), \tilde{\kappa}(t_0)) \) be an analytic function, convex in \( \kappa(t_0) \), defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of \( \tilde{\kappa}(t_0) \). Let \( r \in \mathbb{N} \) be a fixed positive integer, where \( \kappa(t_0), \tilde{\kappa}(t_0) \in \mathbb{R}^r \) are the vectors of initial cost cumulants and target initial cost cumulants, respectively. Then the MCCDS optimization can be formulated as,

\[
\min_{K \in K, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)} \phi(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0))
\]

subject to:

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha))
\]

\[
\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\( H(t_f) = H_f, \quad \tilde{H}(t_f) = H_{f, \epsilon^*}, \quad D(t_f) = D_f, \quad \tilde{D}(t_f) = D_{f, \epsilon^*} \)

where the initial values of the state trajectories satisfy

\( (t_0, H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \in M \).

**Remark(s)** It is assumed in the following development that conditions on the system characterization can be imposed to assure \( \frac{\partial g(\kappa(\alpha))}{\partial \kappa(\alpha)} \neq 0, \forall \alpha \in [t_0, t_f] \). For many ideal choices of \( \phi(\cdot) \), it will be seen that this amounts to no delta-mean zero-crossings.

### 3.1.7 MCCDS Solution via Dynamic Programming

A solution to the Multiple-Cumulant Cost Density-Shaping (MCCDS) Optimization is now derived by employing the traditional techniques of dynamic programming for Mayer Form problems as described in Fleming and Rishel [66]. This
development will specifically adapt the Verification Theorem to the formulation of the MCCDS problem.

The following notations will be used in the solution derivation. Define the block matrices $\mathbf{Y}(\tau), \mathbf{\hat{Y}}(\tau) \in \mathbb{R}^{r \times n}$ and the vectors $\mathbf{Z}(\tau), \mathbf{\hat{Z}}(\tau) \in \mathbb{R}^r$ as below:

$$\mathbf{Y}(\tau) = \begin{bmatrix} Y_1(\tau) \\ Y_2(\tau) \\ \vdots \\ Y_r(\tau) \end{bmatrix}, \quad \mathbf{\hat{Y}}(\tau) = \begin{bmatrix} \hat{Y}_1(\tau) \\ \hat{Y}_2(\tau) \\ \vdots \\ \hat{Y}_r(\tau) \end{bmatrix}, \quad \mathbf{Z}(\tau) = \begin{bmatrix} Z_1(\tau) \\ Z_2(\tau) \\ \vdots \\ Z_r(\tau) \end{bmatrix}, \quad \mathbf{\hat{Z}}(\tau) = \begin{bmatrix} \hat{Z}_1(\tau) \\ \hat{Z}_2(\tau) \\ \vdots \\ \hat{Z}_r(\tau) \end{bmatrix}.$$

$$Y_i(\tau) = H_i(\tau), \quad Z_i(\tau) = D_i(\tau), \quad \hat{Y}_i(\tau) = \hat{H}_i(\tau), \quad \hat{Z}_i(\tau) = \hat{D}_i(\tau), \quad 1 \leq i \leq r$$

References to states $\mathbf{Y}(\tau), \mathbf{\hat{Y}}(\tau), \mathbf{Z}(\tau),$ and $\mathbf{\hat{Z}}(\tau)$ will be made, and the above definitions of block matrices and vectors apply $\forall \tau \in [t_0, \tau)$. Before continuing, note that in this development it is assumed that the terminal time $t_f$ and the associated boundary conditions have been displaced to corresponding ones at some time $\tau$, and the nature of the control that drives the state from $t_f$ to $\tau$ must be regarded.

In cases where the control is assumed optimal, a $\ast$ designation will appear with the control denotation, for example $K^\ast$. The state variables under this control will receive a similar distinction. For instance, $(\tau, \mathbf{Y}^\ast(\tau), \mathbf{Z}^\ast(\tau), \mathbf{\hat{Y}}(\tau), \mathbf{\hat{Z}}(\tau))$ will refer to “displaced” terminal conditions arrived at from the optimal control $K^\ast$ on $(\tau, t_f]$, and $(\tau, \mathbf{Y}(\tau), \mathbf{Z}(\tau), \mathbf{\hat{Y}}(\tau), \mathbf{\hat{Z}}(\tau))$ will refer to terminal conditions arrived at from any control $K$ on $(\tau, t_f]$.

It is time to define the reachable set $\mathcal{Q}$ and the value function $\mathcal{V}(\cdot)$, which are core constructs to this dynamic programming formulation.
Definition 3.1.12 (Reachable Set)
Define the reachable set as the set of initial values from which there exists a control that can take the system to the target set. More formally, this is

\[ Q = \{ (\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) \mid K_{\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) \neq \emptyset \}. \]

Definition 3.1.13 (Value Function)
Let \( (\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) \in [t_0, t_f] \times (\mathbb{S}^n)^r \times \mathbb{R}^r \times (\mathbb{S}^n)^r \times \mathbb{R}^r \) and let \( V(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) \) be a scalar function

\[ V : [t_0, t_f] \times (\mathbb{S}^n)^r \times \mathbb{R}^r \times (\mathbb{S}^n)^r \times \mathbb{R}^r \to \mathbb{R} \]

such that

\[ V(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) = \begin{cases} \inf_{K \in K(\epsilon)} \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)), & K(\epsilon) \neq \emptyset \\ \infty, & K(\epsilon) = \emptyset \end{cases} \]

where \( K(\epsilon) \triangleq K_{\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)} \).

The foundational theorems of dynamic programming are now presented, which lead naturally to the celebrated verification theorem. First, two necessary conditions are established that describe the nature of the value function when evaluated along trajectories corresponding to general and optimal controls. These conditions are then shown to be sufficient as well as necessary. The sufficient conditions can be equated to finding a scalar function that satisfies the HJB equation of Dynamic Programming, which essentially is the provision of the verification theorem. This line of thinking will be expanded in the following.

Lemma 3.1.14 (Value Function is Non-Increasing Function)
The value function is a non-increasing function of time. Thus for \( t_0 \leq \tau_1 \leq \tau_2 \leq \epsilon \leq t_f \) the value function \( V(\tau_1, \mathbf{y}(\tau_1), \mathbf{z}(\tau_1), \mathbf{\dot{y}}(\tau_1), \mathbf{\dot{z}}(\tau_1)) \) satisfies the property

\[ V(\tau_1, \mathbf{y}(\tau_1), \mathbf{z}(\tau_1), \mathbf{\dot{y}}(\tau_1), \mathbf{\dot{z}}(\tau_1)) \geq V(\tau_2, \mathbf{y}(\tau_2), \mathbf{z}(\tau_2), \mathbf{\dot{y}}(\tau_2), \mathbf{\dot{z}}(\tau_2)). \]

Proof Let \( K \in K_{\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)} \) be such that

\[ K = \arg \min_{K \in K(\epsilon)} \left\{ g(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \right\} \]

53
where $K(\epsilon)$ again denotes $\mathcal{K}_{\epsilon,Y(\epsilon),Z(\epsilon),\tilde{Y}(\epsilon),\tilde{Z}(\epsilon)}$'. Given arbitrary terminal conditions at $\epsilon$, $K$ is considered the optimal control on $[t_0, \epsilon]$. Consider the restriction of $K$ to the interval $[t_0, \tau_2]$, denoted here as $K_2$. Clearly $K_2 \in \mathcal{K}_{\tau_2,Y(\tau_2),Z(\tau_2),\tilde{Y}(\tau_2),\tilde{Z}(\tau_2)}$. Now consider the control gain

$$K^*(\tau) = \begin{cases} K_2(\tau), & \tau_1 < \tau \leq \tau_2 \\ K_1(\tau), & t_0 \leq \tau \leq \tau_1 \end{cases}$$

where $K_1$ is any feasible control gain on $[t_0, \tau_1]$, that is $K_1 \in \mathcal{K}_{\tau_1,Y(\tau_1),Z(\tau_1),\tilde{Y}(\tau_1),\tilde{Z}(\tau_1)}$. It is evident that $K^* \in \mathcal{K}_{\tau_2,Y(\tau_2),Z(\tau_2),\tilde{Y}(\tau_2),\tilde{Z}(\tau_2)}$. Use the control $K^*$ to write the apparent inequality

$$\phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \geq V(\tau_2, Y(\tau_2), Z(\tau_2), \tilde{Y}(\tau_2), \tilde{Z}(\tau_2))$$

from which the infimum of both sides with respect to $K(\tau_1)$ can be taken, since by construction $K^*$ is optimal over $(\tau_1, \tau_2]$,

$$V(\tau_1, Y(\tau_1), Z(\tau_1), \tilde{Y}(\tau_1), \tilde{Z}(\tau_1)) = \min_{K \in \mathcal{K}(\tau_1)} \phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \geq \min_{K \in \mathcal{K}(\tau_1)} V(\tau_2, Y(\tau_2), Z(\tau_2), \tilde{Y}(\tau_2), \tilde{Z}(\tau_2)) = V(\tau_2, Y(\tau_2), Z(\tau_2), \tilde{Y}(\tau_2), \tilde{Z}(\tau_2))$$

\[\square\]

**Remark(s)** The traditional value function characteristic is that it be a non-increasing function, forward in time. The fact that this optimization has an initial cost formulation leads to this fundamental difference.

**Lemma 3.1.15 (Value Function is Constant, Along Optimal Trajectories)**

The value function $V(\epsilon, Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))$ is constant when evaluated along an optimal trajectory and $\frac{dV(\epsilon, Y^*(\epsilon), Z^*(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} = 0$.  

54
Proof Let $K^* \in \mathcal{K}_{t_f, \mathcal{Y}(t_f), \mathcal{Z}(t_f), \tilde{\mathcal{Y}}(t_f), \tilde{\mathcal{Z}}(t_f)}$ be such that

$$K^* = \arg \min_{K \in \mathcal{K}(t_f)} \left\{ \phi(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \right\}.$$ 

Under the optimal control $K^*$ it must be that

$$\mathcal{V}(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \tilde{\mathcal{Y}}(\epsilon), \tilde{\mathcal{Z}}(\epsilon)) \leq \phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0)).$$

since $K^*$ restricted to $[t_0, \epsilon]$ is in $\mathcal{K}(\epsilon)$. And also, it is true that

$$\mathcal{V}(t_f, \mathcal{Y}(t_f), \mathcal{Z}(t_f), \tilde{\mathcal{Y}}(t_f), \tilde{\mathcal{Z}}(t_f)) = \phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0)).$$

from which it follows

$$\mathcal{V}(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \tilde{\mathcal{Y}}(\epsilon), \tilde{\mathcal{Z}}(\epsilon)) \leq \mathcal{V}(t_f, \mathcal{Y}(t_f), \mathcal{Z}(t_f), \tilde{\mathcal{Y}}(t_f), \tilde{\mathcal{Z}}(t_f)).$$

which is a contradiction since the value function is non-increasing. It follows that

$$\mathcal{V}(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \tilde{\mathcal{Y}}(\epsilon), \tilde{\mathcal{Z}}(\epsilon)) = \phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0)), \ \epsilon \in [t_0, t_f].$$

and

$$\frac{d\mathcal{V}(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \tilde{\mathcal{Y}}(\epsilon), \tilde{\mathcal{Z}}(\epsilon))}{d\epsilon} = 0, \ \epsilon \in [t_0, t_f].$$

The above two results are necessary conditions that characterize optimal controls and the value function. It is natural to wonder if these conditions are suf-
sufficient to verify that a given function is the value function and also that a given control is optimal. The following theorem shows that this conjecture is indeed the case.

**Theorem 3.1.16 (Sufficient Conditions)**

Assume a function $W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon))$ is known that is non-increasing for $\epsilon \in [t_0, tf]$. This function has this property when evaluated along the trajectories of $\mathbf{y}(\tau)$ and $\mathbf{z}(\tau)$ corresponding to any control gain $K$. Assume further that $W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon))$ satisfies the boundary condition

$$W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon)) = \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0))$$  \hspace{1cm} (3.22)

Assume lastly that for a control gain $K^*$ that

$$W(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{\dot{y}}(t_f), \mathbf{\dot{z}}(t_f)) = \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)).$$  \hspace{1cm} (3.23)

Under these conditions the control gain $K^*$ is optimal for the problem on $[t_0, tf]$ and

$$W(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{\dot{y}}(t_f), \mathbf{\dot{z}}(t_f)) = V(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{\dot{y}}(t_f), \mathbf{\dot{z}}(t_f)).$$

**Proof** Since $W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\dot{y}}(\epsilon), \mathbf{\dot{z}}(\epsilon))$ is non-increasing, it is true that

$$W(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{\dot{y}}(t_f), \mathbf{\dot{z}}(t_f)) \leq W(t_0, \mathbf{y}(t_0), \mathbf{z}(t_0), \mathbf{\dot{y}}(t_0), \mathbf{\dot{z}}(t_0))$$
$$= \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)), \; \forall K \in \mathcal{K}(\epsilon).$$  \hspace{1cm} (3.24)

Above the boundary condition (3.22) is used on the right-hand side of the inequality. For some control $K^*$ it is assumed that

$$W(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{\dot{y}}(t_f), \mathbf{\dot{z}}(t_f)) = \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0))$$

from which it follows that

$$\phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) \leq \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)), \; \forall K \in \mathcal{K}(\epsilon).$$
Here (3.23) has been inserted into the left-hand side of (3.24). It is immediate from this relation that $K^*$ is optimal and also

$$W(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{z}(t_f), \mathbf{t}(t_f)) = \phi(\mathbf{H}^*(t_0), \mathbf{D}^*(t_0), \mathbf{H}(t_0), \tilde{\mathbf{D}}(t_0))$$

$$= \min_{K \in K(t_f)} \phi(\mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0))$$

$$= \mathcal{V}(t_f, \mathbf{y}(t_f), \mathbf{z}(t_f), \mathbf{z}(t_f), \mathbf{t}(t_f)).$$

\[ \square \]

**Remark(s)** The argument above can be easily extended to show that the restriction of the control $K^*$ to $[t_0, \epsilon]$ satisfies the above conditions and is therefore optimal on $[t_0, \epsilon]$, and

$$W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon)) = \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon)).$$

A sufficient condition is now presented for differentiability of the value function.

**Theorem 3.1.17 (Differentiability of Value Function)**

Suppose that controls $(K_1(\alpha), \mathbf{H}(\alpha), \mathbf{D}(\alpha), \tilde{\mathbf{H}}(\alpha), \tilde{\mathbf{D}}(\alpha))$, $K_2(\alpha)$, $\mathbf{H}(\alpha)$, $\mathbf{D}(\alpha)$, $\tilde{\mathbf{H}}(\alpha)$, $\tilde{\mathbf{D}}(\alpha))$ are given, and let the initial time $t_0(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon))$ and

$$\mathbf{H}(t_0(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon))), \mathbf{D}(t_0(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon)))$$

denote initial states of the trajectories

$$\frac{d\mathbf{H}(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{d\tilde{\mathbf{H}}(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha))$$

$$\frac{d\mathbf{D}(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}(\alpha)), \quad \frac{d\tilde{\mathbf{D}}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{\mathbf{H}}(\alpha)), \quad \alpha \in [t_0, \epsilon]$$

$$\mathbf{H}(\epsilon) = \mathcal{V}(\epsilon), \quad \tilde{\mathbf{H}}(\epsilon) = \tilde{\mathbf{V}}(\epsilon), \quad \mathbf{D}(\epsilon) = \mathbf{Z}(\epsilon), \quad \tilde{\mathbf{D}}(\epsilon) = \tilde{\mathbf{Z}}(\epsilon).$$

The value function $\mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{z}(\epsilon), \mathbf{t}(\epsilon))$ is differentiable in $(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon))$ when the terminal states and initial time are also differentiable in those variables.

**Proof** Since $\text{vec}: \mathbb{R}^{m \times n} \to \mathbb{R}^mn$ is an isomorphic map, it is true that the map

$$\frac{d\mathbf{H}(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}(\alpha), K(\alpha))$$

57
If there exists an optimal control $K$ increasing along trajectories $Y$ sense. Thus Theorem 4.3 on pg. 83 of [66] can be invoked. The proof is complete.

\[\square\]

It is quite a challenging task to verify directly if a given function $W(\cdot)$ is non-increasing along trajectories $Y(\epsilon), Z(\epsilon), \forall K \in K(\epsilon), \forall \epsilon \in [t_0, t_f]$. Fortunately, if a function satisfies the \textit{HJB equation of Dynamic Programming}, then the above sufficient conditions are met. Not only is this easier to verify in practice, but sometimes candidate value functions can be constructed using a particular control, and then this function can be used to establish that the selected control is optimal once the candidate value function can be shown to satisfy the HJB equation. These ideas are explored in the following development.

\textbf{Theorem 3.1.18 (HJB Equation for MCCDS Optimization)}

Let $(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))$ be an interior point of the reachable set $Q$. Then the scalar function $V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))$ satisfies the HJB equation of dynamic programming,

\[-\frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \epsilon} \geq \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \text{vec}(Y_i(\epsilon))} \text{vec}(F_i(Y(\epsilon), K(\epsilon)))
+ \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \text{vec}(Y_i(\epsilon))} \text{vec}(F_i(\dot{Y}(\epsilon), \dot{K}(\epsilon)))
+ \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial Z_i(\epsilon)} \mathcal{G}_i(Y(\epsilon))
+ \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial Z_i(\epsilon)} \mathcal{G}_i(\dot{Y}(\epsilon)), \forall K \in K(\epsilon)\]

If there exists an optimal control $K^* \in K(\epsilon)$, then the following equation is satisfied,

\[-\frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \epsilon} = \min_{K \in K} \left( \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \text{vec}(Y_i(\epsilon))} \text{vec}(F_i(Y(\epsilon), K(\epsilon)))
+ \sum_{i=1}^{r} \frac{\partial V(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \text{vec}(Y_i(\epsilon))} \text{vec}(F_i(\dot{Y}(\epsilon), \dot{K}(\epsilon))) \right)\]

58
From the previous result, \( V_0 \geq \) the left derivative, where on the interval, which is denoted as \( D[\tau] \).

**Remark(s)** The minimum in the second equation is obtained by the limit \( \lim_{\tau \to \epsilon^-} K^*(\tau) \).

**Proof** Let \((e, Y(e;K), Z(e;K), \dot{Y}(e), \dot{Z}(e)) \in Q\) for some control \(K\) and consider a constant control \(K_{\Delta e}\) such that the state equations \((\tau, Y(\tau;K_{\Delta e}), Z(\tau;K_{\Delta e}), \dot{Y}(\tau), \dot{Z}(\tau)) \in Q\) for \([\epsilon - \Delta e \leq \tau \leq \epsilon]\). Consider the restriction of \(K\) to \([t_0, \epsilon - \Delta e]\), denoted here as \(K_{\Delta e} \in \mathcal{C}(\epsilon - \Delta e)\). Using this, define the control \(K_{\Delta} \in \mathcal{K}(e)\) by

\[
K_{\Delta}(\tau) = \begin{cases}
K_{\Delta e}(\tau), & t_0 \leq \tau \leq \epsilon - \Delta e \\
K_{\Delta e}(\tau), & \epsilon - \Delta e \leq \tau \leq \epsilon
\end{cases}
\]

Write the equations of motion under the control \(K_{\Delta}\) as

\[
\frac{dH_{\Delta}}{d\tau} = \mathcal{F}(H_{\Delta}(\tau), K_{\Delta}(\tau)), \quad \frac{dD_{\Delta}}{d\tau} = \mathcal{G}(H_{\Delta}(\tau), \tau \in [t_0, \epsilon])
\]

\[
H_{\Delta}(e) = Y(e), \quad D_{\Delta}(e) = Z(e)
\]

From the previous result, \(\mathcal{V}(e, Y(e), Z(e), \dot{Y}(e), \dot{Z}(e))\) is non-increasing and finite on \([t_0, \epsilon]\). Hence \(\mathcal{V}(e, Y_{\Delta}(e), Z_{\Delta}(e), \dot{Y}(e), \dot{Z}(e))\) has a non-positive derivative almost everywhere on the interval, which is denoted as \(D\mathcal{V}(e, Y_{\Delta}(e), Z_{\Delta}(e), \dot{Y}(e), \dot{Z}(e))\). Consider the left derivative,

\[
0 \geq D^-\mathcal{V}(e, Y_{\Delta}(e), Z_{\Delta}(e), \dot{Y}(e), \dot{Z}(e)) = \\
\lim_{\Delta \tau \to 0} \frac{\mathcal{V}(\tau, Y_{\Delta}(\tau), Z_{\Delta}(\tau), \dot{Y}(\tau), \dot{Z}(\tau)) - \mathcal{V}(\tau - \Delta \tau, Y_{\Delta}(\tau - \Delta \tau), Z_{\Delta}(\tau - \Delta \tau), \dot{Y}(\tau - \Delta \tau), \dot{Z}(\tau - \Delta \tau))}{\Delta \tau}
\]
Use the chain rule of differentiation at $\tau = \epsilon$, which yields

$$
0 \geq D^- \mathcal{V}(\epsilon, \mathbf{y}_0(\epsilon), \mathbf{z}_0(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon)) = \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \epsilon} = \\
= \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} \text{vec} \left( \frac{d\mathbf{y}_i}{d\epsilon} \right) + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} \text{vec} \left( \frac{d\mathbf{\hat{y}}_i}{d\epsilon} \right) + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \mathbf{z}_i(\epsilon)} \frac{d\mathbf{\hat{z}}_i}{d\epsilon} + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \mathbf{z}_i(\epsilon)} \frac{d\mathbf{\hat{z}}_i}{d\epsilon} \quad K \text{ arbitrary.}
$$

If $K^*$ is an optimal control then

$$
D^- \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon)) = 0
$$

which can be expressed as

$$
D^- \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon)) = \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \epsilon} = \\
= \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i^*(\epsilon))} \text{vec} \left( \frac{d\mathbf{y}_i^*}{d\epsilon} \right) + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i^*(\epsilon))} \text{vec} \left( \frac{d\mathbf{\hat{y}}_i}{d\epsilon} \right) + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \mathbf{z}_i(\epsilon)} \frac{d\mathbf{\hat{z}}_i}{d\epsilon} + \\
+ \sum_{i=1}^{r} \frac{\partial \mathcal{V}(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \mathbf{\hat{y}}(\epsilon), \mathbf{\hat{z}}(\epsilon))}{\partial \mathbf{z}_i(\epsilon)} \frac{d\mathbf{\hat{z}}_i}{d\epsilon} \quad K^* \text{ arbitrary.}
$$
\[
\begin{align*}
+ \sum_{i=1}^{r} & \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{z}_i(e)} \mathbf{d} \mathbf{z}_i \quad \text{de} \\
+ \sum_{i=1}^{r} & \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{\hat{z}}_i(e)} \mathbf{d} \mathbf{\hat{z}}_i \quad \text{de} \\
\end{align*}
\]

The proof is complete. \(\square\)

**Lemma 3.1.19 (HJB Verification)**

Let \((e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))\) be an interior point of the reachable set \(\mathcal{Q}\). Assume the scalar function \(W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))\) satisfies the boundary condition

\[W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e)) = \phi(H(t_0), D(t_0), H(t_0), \dot{H}(t_0))\]

\((e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e)) \in \mathcal{M}\) \hspace{1cm} (3.25)

and also the HJB equation of dynamic programming,

\[-\frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial e} \geq \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{y}_i(e)} \text{vec}(\mathbf{F}_i(\mathbf{y}(e), \mathbf{K}(e)))
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{\hat{y}}_i(e)} \text{vec}(\mathbf{F}_i(\mathbf{\hat{y}}(e), \mathbf{\hat{K}}(e)))
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{z}_i(e)} \mathbf{g}_i(\mathbf{y}(e))
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{\hat{z}}_i(e)} \mathbf{g}_i(\mathbf{\hat{y}}(e)), \quad \forall \mathbf{K} \in \mathcal{K}(e)\]

(3.26)

If further for a given control \(\mathbf{K}^* \in \mathcal{K}(e)\) this function satisfies the following equation

\[-\frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial e} = \min_{\mathbf{K} \in \mathcal{K}(e)} \left\{ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{y}_i(e)} \text{vec}(\mathbf{F}_i(\mathbf{y}(e), \mathbf{K}(e))) \right\}
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{\hat{y}}_i(e)} \text{vec}(\mathbf{F}_i(\mathbf{\hat{y}}(e), \mathbf{\hat{K}}(e)))
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{z}_i(e)} \mathbf{g}_i(\mathbf{y}(e))
\]

\[+ \sum_{i=1}^{r} \frac{\partial W(e, \mathbf{y}(e), \mathbf{z}(e), \mathbf{\hat{y}}(e), \mathbf{\hat{z}}(e))}{\partial \mathbf{\hat{z}}_i(e)} \mathbf{g}_i(\mathbf{\hat{y}}(e)) \right\} \quad (3.27)
\]
then the control $K^*$ is optimal and

$$W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon)) = V(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon)).$$

**Proof** The second relation means that

$$\frac{dW(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))}{d\epsilon} \leq 0$$

which means $W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))$ is non-increasing on $[t_0, t_f]$. The third relation means that

$$\frac{dW(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))}{d\epsilon} = 0, \text{ } K^* \text{ optimal.}$$

which implies $W(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))$ is constant. By the previous result, it is clear that these conditions are sufficient for the control gain $K^*$ to be optimal and

$$W(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon)) = V(\epsilon, \mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon)).$$

\[\square\]

**Definition 3.1.20 (Candidate Value Function)**

Consider a solution of the form

$$W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon)) = \eta(\epsilon) - \phi(\mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon)) \tag{3.28}$$

where the function $\eta(\tau) \in C^1([t_0, t_f], \mathbb{R})$ is to be determined.

For any point $(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))$ in the reachable set $Q$ where the candidate value function is differentiable, it may differentiated directly. The following will establish the form of the total time derivative of $W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))$, which essentially is constrained by the HJB equations of dynamic programming.
Lemma 3.1.21 (Derivative of Candidate Value Function)

Let \( r \in \mathbb{N} \) be fixed and let \((\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon)) \in \mathcal{Q}\) be an interior point of the set at which the candidate value function

\[
W(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon)) = \eta(\epsilon) - \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))
\]
is differentiable. Then its total time derivative takes the form

\[
\frac{dW(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{d\epsilon} = -\sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T \mathcal{F}_i(\mathbf{y}(\epsilon), K(\epsilon)) x_0
- \sum_{i=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T \mathcal{F}_i(\hat{\mathbf{y}}(\epsilon), \hat{K}(\epsilon)) x_0
- \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\mathbf{y}(\epsilon))
- \sum_{i=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\hat{\mathbf{y}}(\epsilon)) + \frac{d\eta(\epsilon)}{d\epsilon}.
\]

Proof Proceed by directly differentiating (3.28) to obtain the following formal expression which follows directly from definition which holds \( \forall K \in \mathcal{K}(\epsilon) \).

\[
\frac{dW(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{d\epsilon} = -\sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} \text{vec}(\mathcal{F}_i(\mathbf{y}(\epsilon), K(\epsilon)))
- \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{\partial \text{vec}(\hat{\mathbf{y}}_i(\epsilon))} \text{vec}(\mathcal{F}_i(\hat{\mathbf{y}}(\epsilon), \hat{K}(\epsilon)))
- \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{\partial \hat{\mathbf{z}}_i(\epsilon)} \mathcal{G}_i(\mathbf{y}(\epsilon))
- \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))}{\partial \hat{\mathbf{z}}_i(\epsilon)} \mathcal{G}_i(\hat{\mathbf{y}}(\epsilon)) + \frac{d\eta(\epsilon)}{d\epsilon}.
\]

Note the cumulant and target cost cumulant forms when the cost is integral quadratic and the control input is of the linear state feedback type. In particular,

\[
\kappa_i(\epsilon) = x_0^T \mathbf{y}_i(\epsilon) x_0 + \mathbf{z}_i(\epsilon), \quad \tilde{\kappa}_i(\epsilon) = x_0^T \hat{\mathbf{y}}_i(\epsilon) x_0 + \hat{\mathbf{z}}_i(\epsilon)
\]

Note also the the properties below of the \text{vec}(\cdot) and \text{Tr}(\cdot) operators for \( x \in \mathbb{R}^r \) and
\[ A, B \in \mathbb{R}^{n \times n}, \]

\[ x^T Ax = \text{Tr}(Ax^T) = \text{Tr}(xx^T A), \quad \text{vec}(B)^T \text{vec}(A) = \text{Tr}(AB) = \text{Tr}(BA). \]

Recall that the trace operator \( \text{Tr}(\cdot) \) is defined for square matrices \( U \in \mathbb{R}^{n \times n} \) with components \([u_{ij}]_{i,j=1}^n\) as

\[ \text{Tr}(U) = \sum_{i=1}^n \sum_{j=1}^n u_{ij} \delta_{ij} = \sum_{i=1}^n u_{ii} \]

The \( \text{vec}(\cdot) \) operator is defined for the matrix

\[
U = \begin{bmatrix}
    u_{11} & u_{12} & \cdots & u_{1n} \\
    u_{21} & u_{22} & \cdots & u_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{n1} & u_{n2} & \cdots & u_{nn}
\end{bmatrix}
\]

as the \( n^2 \times 1 \) column vector formed by stacking the columns of \( U \),

\[ \text{vec}(U) = \begin{bmatrix}
    u_{11} & u_{21} & \cdots & u_{n1} & u_{1n} & u_{2n} & \cdots & u_{nn}
\end{bmatrix}^T \]

Using the cumulant forms, the chain rule of differentiation, and the properties above, write

\[
\frac{\partial \phi(Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))}{\partial \text{vec}(Y_i(\epsilon))} = \frac{\partial g(\kappa(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \frac{\partial \kappa_i(\epsilon)}{\partial \text{vec}(Y_i(\epsilon))} \text{vec}(F_i(Y(\epsilon), K(\epsilon))) \\
= \frac{\partial g(\kappa(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \text{vec}(x_0 x_0^T) \text{vec}(F_i(Y(\epsilon), K(\epsilon))) \\
= \frac{\partial g(\kappa(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} x_0^T F_i(Y(\epsilon), K(\epsilon)) x_0 
\]

(3.31)
\[
\frac{\partial \phi(Y(\epsilon), Z(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))}{\partial \text{vec}(\hat{Y}_i(\epsilon))} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \bar{\kappa}_i(\epsilon)} \frac{\partial \bar{\kappa}_i(\epsilon)}{\partial \text{vec}(\hat{Y}_i(\epsilon))} \text{vec}(F_i(\hat{Y}(\epsilon), \bar{K}(\epsilon)))
\]

\[
= \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \text{vec}(x_0 x_0^T) \text{vec}(F_i(\hat{Y}(\epsilon), \bar{K}(\epsilon)))
\]

\[
= \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \bar{\kappa}_i(\epsilon)} x_0^T F_i(\hat{Y}(\epsilon), \bar{K}(\epsilon)) x_0
\]

\[
\frac{\partial \phi(Y(\epsilon), \mathbf{Z}(\epsilon), \hat{Y}(\epsilon), \hat{Z}(\epsilon))}{\partial \mathbf{Z}_i(\epsilon)} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \bar{\kappa}_i(\epsilon)} \frac{\partial \bar{\kappa}_i(\epsilon)}{\partial \mathbf{Z}_i(\epsilon)} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \frac{\partial \kappa_i(\epsilon)}{\partial \mathbf{Z}_i(\epsilon)}.
\]

Inserting the relations (3.31), (3.32), and (3.33) into the expression (3.30) gives the desired form (3.29), and the proof is complete.

\[\square\]

In the dynamic programming formulation, the variables that embed this problem into a larger space of varying terminal conditions are always independent. Hence differentiation can be performed directly on the dynamic programming variables without any consideration of inter-relationships between them, even though these variables compose the cumulants, which are coupled. As a result, differentiation is done with respect to the cumulants without explicitly accounting for the inherent inter-dependence between them. However, an essential consideration here is that the feasible space is restricted by the cumulant-generating equations, which are sufficient for the solutions to be cumulants.

An inspection of the cumulant-generating equations reveals that the computa-
tion of the cumulants is influenced by the system and cost structure, the second-order statistics of the noise, *but not the moments*. So no new information is brought into the optimization by imposing relationships between cumulants, such as the Bell Polynomials which commonly are employed to relate the cumulants to each other and also to the moments. In this regard, the cumulant-generating equations ensure the inter-relationships between the cumulants are maintained, and introducing additional constraints between the cumulants will over-constrain the problem needlessly. Given these remarks, partial derivatives of the objective function with respect to the cumulants are taken directly without reservation.

As a final remark, it is mentioned that partial derivatives of a function of cumulants associated with a variate have been considered in the literature. The infinitesimal perturbation of one cumulant in such a function, while holding the other cumulants fixed, is an action that under the appropriate regulatory conditions will result in a set of cumulants pertaining to distribution within the set of *rapidly decreasing functions*, denoted as $O'$. This normal space of distributions contains the set $\Pi$ of distributions with well-defined cumulant-generating functions and therefore well-defined cumulants of all orders. The set $\Pi$ contains the generalized $\chi^2$ distribution so this fact has particular relevance to the MCCDS problem. For further details on these topics, consult [50]. Background theory pertinent to the development can be found in [19], [69].

The solution of the MCCDS optimization is captured in the following theorem, proceeded by its proof.

**Theorem 3.1.22 (State-Feedback Solution to MCCDS Optimization)**

Consider the LQG stochastic optimal control problem involving the process having dynamics (3.1) and the cost (3.2). Then the linear state-feedback, finite-horizon, optimal control solution to the MCCDS optimization is characterized by the opti-
mal gain

\[ K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( H_1^*(\alpha) + \sum_{i=2}^r \left( \frac{\partial g(\kappa^*(\alpha),\tilde{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \right) H_i^*(\alpha) \right) \]

where the optimal cost cumulants and target cost cumulants are defined by

\[ \kappa_i^*(\alpha) = x_0^T H_i^*(\alpha)x_0 + D_i^*(\alpha) \text{ and } \tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha), \quad 1 \leq i \leq r. \]

The optimal state variables \( \mathbf{H}^*(\alpha), \mathbf{D}^*(\alpha) \) and \( \hat{\mathbf{H}}(\alpha), \hat{\mathbf{D}}(\alpha) \) follow the equations of motion

\[
\frac{d\mathbf{H}^*(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}^*(\alpha), K^*(\alpha)), \quad \frac{d\hat{\mathbf{H}}(\alpha)}{d\alpha} = \mathcal{F}(\hat{\mathbf{H}}(\alpha), \hat{K}(\alpha))
\]

\[
\frac{d\mathbf{D}^*(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}^*(\alpha)), \quad \frac{d\hat{\mathbf{D}}(\alpha)}{d\alpha} = \mathcal{G}(\hat{\mathbf{H}}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
\mathbf{H}^*(t_f) = \mathbf{H}_f, \quad \hat{\mathbf{H}}(t_f) = \hat{\mathbf{H}}_{f, \epsilon^*}, \quad \mathbf{D}^*(t_f) = \mathbf{D}_f, \quad \hat{\mathbf{D}}(t_f) = \hat{\mathbf{D}}_{f, \epsilon^*}
\]

exist and also the initial values of the state trajectories satisfy

\[
(t_0, \mathbf{H}(t_0), \mathbf{D}(t_0), \hat{\mathbf{H}}(t_0), \hat{\mathbf{D}}(t_0)) \in \mathcal{M}.
\]

**Proof** The objective is to identify a control gain \( K^* \) and a function \( \eta(\epsilon) \) such that

\[
\frac{d\eta(\epsilon)}{d\epsilon} = \min_{\kappa \in \mathcal{K}} \left\{ \sum_{i=1}^r \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T \mathcal{F}_i(\mathcal{Y}(\epsilon), K(\epsilon)) x_0 + \sum_{i=1}^r \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\mathcal{Y}(\epsilon)) \right. \\
+ \left. \sum_{i=1}^r \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T \mathcal{F}_i(\tilde{\mathcal{Y}}(\epsilon), \tilde{K}(\epsilon)) x_0 + \sum_{i=1}^r \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\tilde{\mathcal{Y}}(\epsilon)) \right\}.
\]

(3.34)

Begin with the minimization on the right-hand side of the equation above. In the following differentiation, denote the partial derivatives of \( \phi(\mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \tilde{\mathcal{Y}}(\epsilon), \tilde{\mathcal{Z}}(\epsilon)) \) as shown below to simplify the notation,

\[
e_i(\epsilon) = \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)), \quad 1 \leq i \leq r.
\]

67
The expression in braces in (3.34) is differentiated with respect to $K$, and the resulting form is set equal to a zero matrix with the appropriate dimension. This is the necessary condition for the expression to take an extremal value on the interior of its domain.

$$-2B^T(\epsilon) \sum_{i=1}^{r} c_i(\epsilon)Y_i(\epsilon)(x_0x_0^T) - 2c_1(\epsilon)R(\epsilon)K(\epsilon)(x_0x_0^T) = 0^{m \times n}$$

Assume that $c_1(\epsilon) \neq 0$, $\forall \epsilon \in [t_0, t_f]$. Since $x_0x_0^T$ is a fixed rank-one matrix, it must be that

$$K^*(\epsilon) = -R^{-1}(\epsilon)B^T(\epsilon) \left( Y^*_1(\epsilon) + \sum_{i=2}^{r} \frac{c_i^*(\epsilon)}{c_1^*(\epsilon)} Y_i^*(\epsilon) \right) \tag{3.35}$$

Let $Y^*(\tau)$ and $Z^*(\tau)$ denote the solutions of the equations of motion under this control selection:

$$dY^*(\tau) = F(Y^*(\tau), K^*(\tau)), \quad dZ^*(\tau) = G(Y^*(\tau)) \tag{3.36}$$

$$Y_i^*(t_f) = H_i(t_f), \quad Z_i^*(t_f) = D_i(t_f), \quad \tau \in [\epsilon, \ t_f], \ 1 \leq i \leq r.$$ 

It is assumed that the solutions to the above equations exist. In addition, let $\kappa^*(\epsilon)$ be the vector of cost cumulants determined using $Y^*(\epsilon)$ and $Z^*(\epsilon)$ determined above according to the relations

$$\kappa_i^*(\epsilon) = x_0^T Y_i^*(\epsilon)x_0 + Z_i^*(\epsilon), \ 1 \leq i \leq r.$$

Return again to the minimized expression on the right-hand side of (3.34),
inserting the controller (3.35),

\[
\frac{d\eta(e)}{d\epsilon} = \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(e)} g(\kappa^*(e), \tilde{\kappa}(e))x_0^T F_i(Y^*(e), K^*(e)) + \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(e)} g(\kappa^*(e), \tilde{\kappa}(e))G_i(Y^*(e)) + \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(e)} g(\kappa^*(e), \tilde{\kappa}(e))F_i(Y(e))
\]

Note that \((e, Y(e), Z(e), \tilde{Y}(e), \tilde{Z}(e)) \in Q\), for some \(K \in \mathcal{K}(e)\). Recall by assumption, \(K^*\) is admissible on \([t_0, \epsilon]\). Thus \((e, Y^*(e), Z^*(e), \tilde{Y}(e), \tilde{Z}(e)) \in Q\) since \(K^* \in \mathcal{K}(e)\). Consider now that any displaced terminal conditions at \(\tau < \epsilon\) along the respective state trajectories resultant from \(K^*\) will also be in \(Q\) due to the fact that the restrictions of the control \(K^*\) to \([t_0, \tau]\) will be in \(\mathcal{K}(\tau)\). Clearly then, it can be similarly argued that \((\tau, Y^*(\tau), Z^*(\tau), \tilde{Y}(\tau), \tilde{Z}(\tau)) \in Q, \forall \tau \in [t_0, \epsilon]\). It is desired to determine \(\eta(e)\) in order to enforce the above relationship for all possible displaced terminal times \(\tau < \epsilon\) and the associated terminal conditions, obtained from anywhere along the optimal trajectory of the state equations.

\[
\frac{d\phi(Y^*(\tau), Z^*(\tau), \tilde{Y}(\tau), \tilde{Z}(\tau))}{d\tau} = \frac{d\eta(\tau)}{d\tau}, \tau \in [t_0, \epsilon]
\]

This differential equation can be integrated over a reduced time-horizon since the state equations \(Y_i^*(\tau), Z_i^*(\tau), \tilde{Y}_i(\tau), \tilde{Z}_i(\tau)\) for \(1 \leq i \leq r\) are continuously
differentiable. By the Fundamental Theorem,

\[
\phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) - \phi(Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) \\
= -\int_{t_0}^{\epsilon} \frac{d}{d\tau} \left( \phi(Y^*(\tau), Z^*(\tau), \dot{Y}(\tau), \dot{Z}(\tau) \right) \\
\eta(\epsilon) - \eta(t_0) = -\int_{t_0}^{\epsilon} \frac{d}{d\tau} (\eta(\tau))
\]

from which it is immediate that

\[
\eta(\epsilon) = \eta(t_0) - \phi(Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)).
\]

Since \((t_0, Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) \in \mathcal{M},\) the target set, we can re-write the above equation as

\[
\eta(\epsilon) = \eta(t_0) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)).
\]

The function \(\eta(\epsilon)\) is determined up to its initial condition, which remains to be constrained for this problem. To achieve this end, the Mayer-form boundary condition (3.25) of the verification lemma is used, which is

\[
\mathcal{W}(t_0, Y(t_0), Z(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) = \eta(t_0) - \phi(Y(t_0), Z(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) \\
= \eta(t_0) - \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \\
= \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)).
\]

This condition requires \(\eta(t_0) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0))\) and thus \(\eta(\epsilon)\) is deter-
mined completely as

\[ \eta(\epsilon) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) \]

\[ + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)). \]  

(3.38)

The candidate value function becomes

\[ W(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) \]

\[ + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) - \phi(Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)). \]  

(3.39)

The selection of \( K^* \) per (3.35) and selecting \( \eta(\epsilon) \) per (3.38) make \( W(\cdot) \) satisfy the requirements of the verification lemma, which are verified briefly in the following.

First recall that the selection of \( \eta(t_0) \) in (3.37) achieves (3.25). To verify (3.26) and (3.27), consider the following expansion of the total time derivative of \( W(\cdot) \),

\[ \frac{dW}{d\epsilon} = \frac{\partial W}{\partial \epsilon} + \frac{\partial W}{\partial \text{vec}(Y(\epsilon))} \text{vec}(F(Y(\epsilon), K(\epsilon))) + \frac{\partial W}{\partial Z(\epsilon)} G(Y(\epsilon)) \]

\[ + \frac{\partial W}{\partial \text{vec}(\dot{Y}(\epsilon))} \text{vec}(F(\dot{Y}(\epsilon), \dot{K}(\epsilon))) + \frac{\partial W}{\partial \dot{Z}(\epsilon)} G(\dot{Y}(\epsilon)). \]  

(3.40)

A slight manipulation of the right-hand side of (3.40) makes it easier to see that (3.41) below is equivalent to (3.26), and (3.42) below is equivalent to (3.27).

\[ \frac{dW}{d\epsilon} \leq 0, \ K \neq K^* \]  

(3.41)

\[ \frac{dW}{d\epsilon} = 0, \ K = K^* \]  

(3.42)

The condition (3.41) requires \( W(\cdot) \) be non-increasing along non-optimal state trajectories, and (3.42) requires \( W(\cdot) \) be constant along optimal state trajectories.
Now to confirm that both of the above conditions hold for the determined candidate value function (3.39). A brief check reveals that (3.39) is non-increasing when evaluated along terminal conditions from trajectories resultant from any and all $K \neq K^*$,

$$
\frac{dW(\epsilon, Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} = \frac{d\phi(Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} - \frac{d\phi(Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} \\
= \min_{k \in K} \left\{ \frac{d\phi(Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} \right\} - \frac{d\phi(Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} \\
\leq 0.
$$

Further, using the displaced terminal condition $(Y^*(\epsilon), Z^*(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))$ and then choosing $K = K^*$ over $[t_0, \epsilon]$ makes $H(t_0) = H^*(t_0)$, $D(t_0) = D^*(t_0)$ and also $Y(\epsilon) = Y^*(\epsilon)$, $Z(\epsilon) = Z^*(\epsilon)$ so that (3.39) assumes the constant value

$$
W(\epsilon, Y^*(\epsilon), Z^*(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon)) = \phi(H^*(t_0), D^*(t_0), \tilde{H}(t_0), \tilde{D}(t_0))
$$

and clearly $\frac{dW(\epsilon, Y^*(\epsilon), Z^*(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))}{d\epsilon} = 0$.

By the verification lemma, $W(\epsilon, Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon)) = V(\epsilon, Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))$ and the derived control (3.35) is optimal.

Parameter Optimality of MCCDS Solution

A general control gain $K$ was sought to solve the MCCDS optimization. The peculiar structure of the resulting solution resembles Pham’s parametric $kCC$ controller, but instead with dynamic weights. This motivates the MCCDS optimization be completed over the parameter space $\Gamma_{t_f, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)}$, where a controller having the $kCC$ form is assumed. As opposed to optimizing over a space
of control gains, it is reiterated that the optimization is done with respect the free parameters in the kCC controller gain.

This study confirms that the weightings on the terms in the controller appearing in Theorem 3.1.22 are indeed the optimal ones. This problem formulation is free from the requirement that the weighting selections be positive constants as they do not appear explicitly in the MCCDS performance index \( \phi(\cdot) = \dot{g}(\cdot) \) and impact the problem’s well-posedness. More precisely, for the kCC performance index to be bounded below, the parameters \( \mu_i \geq 0, 1 \leq i \leq k \) where as the MCCDS performance index is always positive by selection of \( g(\cdot) \). Such enables a parameter selection that can depend on state feedback and is not restricted to positive values.

It will be seen that the solution to this problem yields the same solution as previously derived. This suggests that any initial cost cumulant optimization where the performance index has the character of \( g(\cdot) \) or when it is a strictly positive function of the cost cumulants (though not involving target cost cumulants, e.g. \( \kappa_0(\alpha) = 0^{r \times 1}, \forall \alpha \)), admits a control solution bearing a unique dependence on the cost cumulant-generating equations of the LQG framework, as manifest in the LQG, kCC, RS, and MCCDS controllers.

**Theorem 3.1.23 (Parameter Optimality of MCCDS State Feedback Solution)**

Consider the LQG stochastic optimal control problem involving the process having dynamics (3.1) and the cost (3.2). Then for a chosen analytic and convex function \( g(\kappa, \tilde{\kappa}) \), with non-negative rate of change and value at \( \tilde{\kappa} \), and any dynamically-weighted controller of the form

\[
K(\alpha, H(\alpha), D(\alpha), \tilde{H}(\alpha), \tilde{D}(\alpha)) = -R^{-1}(\alpha)B^T(\alpha) \left( \sum_{i=1}^{r} \gamma_i H_i(\alpha) \right)
\]

the optimal weighting of the \( \{ H_i(\alpha) \}_{i=2}^{r} \) matrices to minimize the MCCDS objective
is the same as in the MCCDS optimal controller. That is,

\[ \gamma^*_1 = 1, \quad \gamma^*_i = \begin{pmatrix} \frac{\partial g(\kappa^*(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa^*_i(\alpha)} \\ \frac{\partial \kappa^*_i(\alpha)}{\partial g(\kappa^*(\alpha), \tilde{\kappa}(\alpha))} \end{pmatrix}, \quad 2 \leq i \leq r \]

and

\[ K^*(\alpha, H^*(\alpha), D^*(\alpha), \tilde{H}(\alpha), \tilde{D}(\alpha)) = -R^{-1}(\alpha)B^T(\alpha) \sum_{i=1}^{r} \begin{pmatrix} \frac{\partial g(\kappa^*(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa^*_i(\alpha)} \\ \frac{\partial \kappa^*_i(\alpha)}{\partial g(\kappa^*(\alpha), \tilde{\kappa}(\alpha))} \end{pmatrix} H^*_i(\alpha). \]

where the cumulants are defined by \( \kappa^*_i(\alpha) = x_0^T H^*_i(\alpha) x_0 + D^*_i(\alpha) \) and \( \tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha) x_0 + \tilde{D}_i(\alpha) \) and \( H^*(\alpha), D^*(\alpha) \) follow the equations of motion

\[
\frac{dH^*(\alpha)}{d\alpha} = F(\bar{H}(\alpha), H^*(\alpha)), \quad \frac{d\bar{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \bar{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[ H^*(t_f) = H_f, \quad \tilde{H}(t_f) = \tilde{H}_{f, \bar{\alpha}}, \quad D^*(t_f) = D_f, \quad \tilde{D}(t_f) = \tilde{D}_{f, \bar{\alpha}}. \]

**Proof** Begin by minimizing the right-hand side of the equation of dynamic programming with respect to the parameter vector,

\[
\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{bmatrix}
\]

\[
\min_{\gamma \in \Gamma, \gamma, \bar{\gamma}, \bar{\gamma}, \bar{\gamma}} \left\{ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\alpha)} g(\kappa(\alpha), \tilde{\kappa}(\alpha)) x_0^T F_i(\bar{\gamma}(\alpha), K(\bar{\gamma}(\alpha), \gamma)) x_0 \right. \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\alpha)} g(\kappa(\alpha), \tilde{\kappa}(\alpha)) x_0^T F_i(\bar{\gamma}(\alpha), \tilde{K}) x_0 \right. \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\alpha)} g(\kappa(\alpha), \tilde{\kappa}(\alpha)) \bar{F}_i(\bar{\gamma}) + \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\alpha)} g(\kappa(\alpha), \tilde{\kappa}(\alpha)) \bar{F}_i(\bar{\gamma}) \right\}
\]
Use the $c_i(e)$ notation for the partial derivatives, and consider the sum inside the product involving $x_0$,

$$
\sum_{i=1}^{r} c_i(e) F_i(Y(e), K(Y(e), \gamma))
$$

$$
= - c_1(e)((A(e) + B(e)K(Y(e), \gamma))^T Y_1 + Y_1(A(e) + B(e)K(Y(e), \gamma))
+ K(Y(e), \gamma)^T R(e)K(Y(e), \gamma) + Q(e))
- \sum_{i=2}^{r} c_i(e)((A(e) + B(e)K(Y(e), \gamma))^T Y_i(e) + Y_i(e)(A(e) + B(e)K(Y(e), \gamma)))
+ 2 \sum_{i=2}^{r} \sum_{j=1}^{i-1} \binom{i}{j} Y_j(e) G(e)WG(e)^T Y_{i-j}(e)
$$

$$
= -(B(e)K(Y(e), \gamma))^T \sum_{i=1}^{r} c_i(e) Y_i(e) - \left( \sum_{i=1}^{r} c_i(e) Y_i(e) \right) B(e)K(Y(e), \gamma)
- c_1(e)K(Y(e), \gamma)^T R(e)K(Y(e), \gamma)
- \sum_{i=1}^{r} c_i(e)(A(e)^T Y_i(e) + Y_i(e)A(e)) - c_1(e)Q(e)
- 2 \sum_{i=2}^{r} \sum_{j=1}^{i-1} \binom{i}{j} c_i(e) Y_j(e) B(e) R^{-1}(e) B^T(e) Y_{i-j}(e)
$$

The last three lines above vanish when the gradient is taken with respect to $\gamma$ because there is no dependence on any component of the parameter vector.

The three terms in the lines above the third from the bottom can be expressed
individually. The first term is,

\[-(B(e)K(\mathcal{Y}(e), \gamma))^T \sum_{j=1}^{r} c_j(e) \mathcal{Y}_j(e)\]

\[= -(-R^{-1}(e)B^T(e) \sum_{i=1}^{r} \frac{\gamma_i}{\gamma_1} \mathcal{Y}_i(e))B^T(e) \sum_{j=1}^{r} c_j(e) \mathcal{Y}_j(e)\]

\[= \sum_{i=1}^{r} \sum_{j=1}^{r} c_j(e) \frac{\gamma_i}{\gamma_1} \mathcal{Y}_i(e) B(e) R^{-1}(e) B^T(e) \mathcal{Y}_j(e).\]

The second term is,

\[-\left( \sum_{i=1}^{r} c_i(e) \mathcal{Y}_i(e) \right) B(e) K(\mathcal{Y}(e), \gamma)\]

\[= -\left( \sum_{i=1}^{r} c_i(e) \mathcal{Y}_i(e) \right) B(e) (-R^{-1}(e)B^T(e) \sum_{j=1}^{r} \frac{\gamma_j}{\gamma_1} \mathcal{Y}_j(e))\]

\[= \sum_{i=1}^{r} \sum_{j=1}^{r} c_i(e) \frac{\gamma_j}{\gamma_1} \mathcal{Y}_i(e) B(e) R^{-1}(e) B^T(e) \mathcal{Y}_j(e).\]

And next, the third term is given by,

\[-c_1(e)K(\mathcal{Y}(e), \gamma)^T R(e) K(\mathcal{Y}(e), \gamma)\]

\[= -c_1(e) (-R^{-1}(e)B^T(e) \sum_{i=1}^{r} \frac{\gamma_i}{\gamma_1} \mathcal{Y}_i(e))^T R(e) (-R^{-1}(e)B^T(e) \sum_{j=1}^{r} \frac{\gamma_j}{\gamma_1} \mathcal{Y}_j(e))\]

\[= -\sum_{i=1}^{r} \sum_{j=1}^{r} c_1(e) \frac{\gamma_i \gamma_j}{\gamma_1^2} \mathcal{Y}_i(e) B(e) R^{-1}(e) B^T(e) \mathcal{Y}_j(e).\]

Combining terms leads to

\[-(B(e)K(\mathcal{Y}(e), \gamma))^T \sum_{i=1}^{r} c_i(e) \mathcal{Y}_i(e)\]

\[-\left( \sum_{i=1}^{r} c_i(e) \mathcal{Y}_i(e) \right) B(e) K(\mathcal{Y}(e), \gamma) - c_1(e) K(\mathcal{Y}(e), \gamma)^T R(e) K(\mathcal{Y}(e), \gamma)\]

\[= \sum_{i=1}^{r} \sum_{j=1}^{r} \left( c_i(e) \frac{\gamma_j}{\gamma_1} + c_j(e) \frac{\gamma_i}{\gamma_1} - c_1(e) \frac{\gamma_i \gamma_j}{\gamma_1^2} \right) \mathcal{Y}_i(e) B(e) R^{-1}(e) B^T(e) \mathcal{Y}_j(e).\]
Using this form, write

\[
\nabla \gamma \left( \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_{i}(\epsilon)} g(\kappa(\epsilon), \bar{\kappa}(\epsilon)) x_{0}^{T} F_{i}(\mathbf{Y}(\epsilon), K(\mathbf{Y}(\epsilon), \gamma)) x_{0} \right) = \nabla \gamma \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \left( c_{i}(\epsilon) \frac{\gamma_{j}}{\gamma_{1}} + c_{j}(\epsilon) \frac{\gamma_{i}}{\gamma_{1}} - c_{1}(\epsilon) \frac{\gamma_{i} \gamma_{j}}{\gamma_{1}^{2}} x_{0}^{T} Y_{i}(\epsilon) B(\epsilon) R^{-1}(\epsilon) B^{T}(\epsilon) Y_{j}(\epsilon) x_{0} \right) \right)
\]

Taking the derivative of the coefficients with respect to \( \gamma_{i} \) yields one equation that must be satisfied in order for all the coefficients of the terms \( Y_{i}(\epsilon) B(\epsilon) R^{-1}(\epsilon) B^{T}(\epsilon) Y_{j}(\epsilon) \) to vanish for \( 1 \leq i \leq r \), which effectively makes the gradient vanish.

\[
\frac{\partial}{\partial \gamma_{i}} \left( c_{i}(\epsilon) \frac{\gamma_{j}}{\gamma_{1}} + c_{j}(\epsilon) \frac{\gamma_{i}}{\gamma_{1}} - c_{1}(\epsilon) \frac{\gamma_{i} \gamma_{j}}{\gamma_{1}^{2}} \right) = 0
\]

\[
\frac{c_{j}(\epsilon)}{\gamma_{1}} - \gamma_{j} \frac{c_{1}(\epsilon)}{\gamma_{1}^{2}} = 0, \quad 1 \leq i \leq r
\]

The solution of this relation is \( \gamma_{j}^{*} = \frac{c_{j}(\epsilon)}{c_{1}(\epsilon)} \gamma_{1}, \quad 1 \leq j \leq r \). The value \( \gamma_{1} \) remains free in these relations, but will fall out in the controller form. It follows readily that,

\[
K(\epsilon) = -R^{-1}(\epsilon) B^{T}(\epsilon) \left( H_{1}(\epsilon) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_{i}(\epsilon)} \right) \frac{\partial \kappa_{i}(\epsilon)}{\partial \kappa_{1}(\epsilon)} H_{i}(\epsilon) \right)
\]

The solution proceeds as before, and given that control gain is identical here, it is clear that the end result is the same as before.

Concerning Theorem 3.1.22 and Theorem 3.1.23, a few remarks are made about the nature of the solution.
Remark(s)

- The optimal control uses state information for both controlled (i.e. the cumulant variables) and uncontrolled states (i.e. the target variables) of the augmented system. This is because the MCCDS performance index is a function of both controlled and uncontrolled states.

- The optimal control also reflects the structure of the controlled system, the process and control weighting matrices in $J$, the second order statistics of the noise process and its weighting in the process dynamics. Once all of these are specified, the optimal control can be implemented by a linear feedback gain that is calculated by solving the differential matrix Riccati equations $\{H_i(\tau)\}_{i=1}^r$ and the scalar $\{D_i(\tau)\}_{i=1}^r$ equations and iteratively updating the controller to drive the controlled system states. The Riccati equations are coupled under the optimal control $K^*$, and full information on the $r$ cost cumulants and the $r$ target cost cumulants is needed to compute the solutions to the controller design equations.

- The formulation of the problem involves an unknown control gain $\tilde{K}$ and target cost cumulant trajectories characterized by the equations $\{\tilde{H}_i(\alpha)\}_{i=1}$ and $\{\tilde{D}_i(\alpha)\}_{i=1}$. All of these items are unknown in practice. However, the only item that is needed for cost-cumulant tracking is the target cost cumulant trajectory $\hat{\kappa}(\alpha)$. While this too is often unknown in practice, nominal functions of time can sometimes be used to approximate $\hat{\kappa}(\alpha)$.

The procedure for the MCCDS control computation might be better understood through studying Figure 3.1. It is shown here that the target cost statistics are produced in an isolated computation, which is not affected by the control designer’s input. The equations of motion for $\tilde{H}(\alpha)$, $\tilde{D}(\alpha)$ are influenced by some fixed, but unknown, linear control input with gain $\tilde{K}(\alpha)$. These constructs represent the ideal cost statistics that are to be achieved through the MCCDS control $K^*(\alpha)$. The equations produce $\tilde{H}(\alpha)$, $\tilde{D}(\alpha)$, which enter the optimal MCCDS control computation as the target cost cumulants $\hat{\kappa}(\alpha)$ through the optimal parameters $\gamma^*$ discussed above. The trajectories of the target cost cumulants $\hat{\kappa}(\alpha)$ will reflect a perturbation $(E^*, \epsilon^*)$ to the terminal conditions $H_f$, $D_f$ cost cumulant-generating equations subjected to the MCCDS optimal control.
Figure 3.1: MCCDS Controller Computation
3.2 Derivation of Classical Controllers using MCCDS Theory

When a performance index is considered that is a linear combination of cost cumulants with positive weights, and all target statistics are set to zero, it is directly verified that the performance index is non-negative, convex, and analytic. Hence, such selections lead to optimizations that are well-suited for the MCCDS optimization formulation and we can apply the form of the MCCDS control solution directly. Three particular cases are considered, and in each it can be observed that the MCCDS control matches a classical control solution.

**Linear Quadratic Gaussian** Assume a performance index of the form

\[ g_{LQG}(\kappa_1(t_0), \bar{\kappa}_1(t_0)) = \kappa_1(t_0) - \bar{\kappa}_1(t_0). \]  

Assume the target \( \bar{\kappa}_1(\alpha) = 0, \alpha \in [t_0, t_f] \). We apply the form of the MCCDS control solution directly. It is immediate that the MCCDS controller form yields the desired LQG controller, that is

\[ K^{MCCDS}_{LQG}(\alpha) = -R^{-1}(\alpha)B^T(\alpha)H_1(\alpha) - R^{-1}(\alpha)B^T(\alpha) \sum_{i=2}^{r-1} \left( \frac{\partial g_{LQG}(\kappa_1(\alpha), 0)}{\partial \kappa_i(\alpha)} \right) \left( \frac{\partial \kappa_i(\alpha)}{\partial \kappa_1(\alpha)} \right) H_i(\alpha) \]

\[ = -R^{-1}(\alpha)B^T(\alpha)H_1(\alpha). \]

**k Cost Cumulants** Now assume a performance index that is a weighted sum of the first \( k \) cost cumulants,

\[ g_{kCC}(\kappa(t_0), \bar{\kappa}(t_0)) = \sum_{i=1}^{k} \mu_i(\kappa_i(t_0) - \bar{\kappa}_i(t_0)), \mu_1 > 0, \mu_i \geq 0, 2 \leq i \leq k, k \in \mathbb{N}. \]  

(3.44)
Assume the targets $\tilde{\kappa}_i(\alpha) = 0, \alpha \in [t_0, t_f], 1 \leq i \leq k$. We again apply the form of the MCCDS control solution. It is immediate that the MCCDS controller form yields the desired $k$CC controller, that is

$$K_{kCC}^{MCCDS}(\alpha) = -R^{-1}(\alpha)B^T(\alpha)H_1(\alpha) - R^{-1}(\alpha)B^T(\alpha) \sum_{i=2}^{k} \left( \frac{\partial g_{kCC}(\kappa(\alpha), \theta^{k+1})}{\partial \kappa_i(\alpha)} \right) H_i(\alpha)$$

$$= -R^{-1}(\alpha)B^T(\alpha) \left( H_1(\alpha) + \sum_{i=2}^{k} \frac{\mu_i}{\mu_1} H_i(\alpha) \right)$$

$$= -R^{-1}(\alpha)B^T(\alpha) \left( \sum_{i=1}^{k} \hat{\mu}_i H_i(\alpha) \right), \quad \hat{\mu}_i = \frac{\mu_i}{\mu_1}, \ 1 \leq i \leq k.$$

**Risk-Sensitive Control** Assume $k \to \infty$ presents no problems with the convergence of the performance index given in (3.44) when the weights are chosen to be $\mu_i = \frac{\theta^i}{i!}$ and the targets are zero. That is, we assume the function below is well-defined,

$$g_{RS}(\kappa(t_0), \tilde{\kappa}(t_0)) = \sum_{i=1}^{\infty} \frac{\theta^i}{i!} (\kappa_i(t_0) - \tilde{\kappa}_i(t_0)), \quad \theta > 0. \quad (3.45)$$

Assume the targets $\tilde{\kappa}_i(\alpha) = 0, \alpha \in [t_0, t_f], i \in \mathbb{N}$. Consider a solution fitting the MCCDS form, assuming that differentiation of the series (3.45) can be completed term-by-term. The RS control shown in [33] follows,

$$K_{RS}^{MCCDS}(\alpha) = -R^{-1}(\alpha)B^T(\alpha)H_1(\alpha) - R^{-1}(\alpha)B^T(\alpha) \sum_{i=2}^{\infty} \left( \frac{\partial g_{RS}(\kappa(\alpha), 0, 0, \ldots)}{\partial \kappa_i(\alpha)} \right) H_i(\alpha)$$

$$= -R^{-1}(\alpha)B^T(\alpha) \left( \frac{\theta}{\theta} H_1(\alpha) + \frac{\theta}{\theta} \sum_{i=2}^{\infty} \frac{\theta^{i-1}}{i!} H_i(\alpha) \right)$$

81
\[
R^{-1}(\alpha)B^T(\alpha) \left( \frac{1}{\theta} \sum_{i=1}^{\infty} \theta^i H_i(\alpha) \right) \\
= -\theta R(\alpha)^{-1} B^T(\alpha) \left( \sum_{i=1}^{\infty} \frac{\theta^i}{i!} H_i(\alpha) \right).
\]

3.3 Weighted Least-Squares Cost Density-Shaping Optimal Control

Here a novel Weighted Least-Squares, Cost Density-Shaping (WLS-CDS) optimization problem of minimizing a weighted sum of squared differences between initial cost cumulants and target initial cost cumulants is posed, which considers arbitrarily-many terms. The problem is solved using the dynamic programming techniques developed previously, which represent a classical theory adapted to the cost cumulant-generating equations of the Linear Quadratic Gaussian (LQG) framework. The WLS-CDS optimal controller results as the solution to the aforementioned optimization, and is applied in a building protection problem. It is shown that WLS-CDS controls can achieve target cost cumulants resultant from a family of 3CC controls to within a 0.5% margin of normalized error. This is a full development of the theory presented in [47].

3.3.1 WLS-CDS Problem Formulation

The optimization problem can be formulated by defining a control space over which the WLS-CDS performance index can be minimized. The definitions for target set and the admissible control space presented earlier are referred to below.

**Definition 3.3.1 (WLS-CDS Performance Index)**

Let the WLS-CDS Performance index be defined as the function

\[
\phi_{\text{WLS}}(\mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0)) = \sum_{i=1}^{r} w_i \left( x_0^T H_i(t_0) x_0 + D_i(t_0) - x_0^T \tilde{H}_i(t_0) x_0 - \tilde{D}_i(t_0) \right)^2 \\
= \sum_{i=1}^{r} w_i (\kappa_i(t_0) - \tilde{\kappa}(t_0))^2
\]

82
defined for positive weights $w_1 > 0$, $w_j \geq 0$, $2 \leq j \leq r$.

The WLS-CDS performance index is non-negative, and convex in the initial cost cumulants. As such, it is well-suited for the performance index in the following WLS-CDS optimization problem.

**Definition 3.3.2 (WLS-CDS Optimization)**

Let $r \in \mathbb{N}$ be a fixed positive integer. Then the WLS optimization can be formulated as,

$$\min_{K \in K(t_f)} \phi_{WLS}(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0))$$

subject to:

$$\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha))$$

$$\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]$$

$$H(t_f) = H_f, \quad \tilde{H}(t_f) = H_{f,e}^*, \quad D(t_f) = D_f, \quad \tilde{D}(t_f) = D_{f,e}^*.$$  

### 3.3.2  WLS-CDS Solution via Dynamic Programming

If a candidate value function can be proposed that satisfies a Mayer-form boundary condition, in addition to the HJB equations of dynamic programming (shown previously), then that candidate value function agrees with the value function along trajectories of the state variables corresponding to any linear control. Using the HJB verification lemma, the WLS-CDS optimal control is derived below. The proof follows.

**Theorem 3.3.3 (State-Feedback Solution, WLS-CDS)**

Consider the LQG stochastic optimal control problem involving the process having dynamics (3.1) and the cost (3.2). Then the linear state-feedback, finite-horizon,
optimal control solution to the WLS-CDS optimization is characterized by the optimal gain

\[ K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( H^*_i(\alpha) + \sum_{i=2}^{r} \frac{w_i}{w_1} \left( \frac{\kappa^*_i(\alpha) - \hat{\kappa}_i(\alpha)}{\kappa^*_1(\alpha) - \hat{\kappa}_1(\alpha)} \right) H^*_i(\alpha) \right) \]  

(3.46)

where the optimal cost cumulants and target cost cumulants are defined by

\[ \kappa^*_i(\alpha) = x_0^T H^*_i(\alpha)x_0 + D^*_i(\alpha) \]

and

\[ \hat{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha), \quad 1 \leq i \leq r. \]

The optimal state variables \( H^*(\alpha), D^*(\alpha) \) and \( \tilde{H}(\alpha), \tilde{D}(\alpha) \) follow the equations of motion

\[ \frac{dH^*(\alpha)}{d\alpha} = F(H^*(\alpha), K^*(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha)) \]

\[ \frac{dD^*(\alpha)}{d\alpha} = G(H^*(\alpha), \frac{dD(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f] \]

\( H^*(t_f) = H_f, \quad \tilde{H}(t_f) = \tilde{H}_f, \quad D^*(t_f) = D_f, \quad \tilde{D}(t_f) = \tilde{D}_f. \)

**Proof** In the following, let \( \mathbf{Y} \) and \( \mathbf{Z} \) denote

\[ \mathbf{Y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_r \end{bmatrix} \in \mathbb{R}^{rn \times n}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_r \end{bmatrix} \in \mathbb{R}^r \]

where the functions \( \mathbf{y}_i = H_i(\epsilon) \) and \( \mathbf{z}_i = D_i(\epsilon), \quad 1 \leq i \leq r. \) Analogously, \( \tilde{\mathbf{Y}} \) and \( \tilde{\mathbf{Z}} \) are referred to, which denote

\[ \tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{y}}_1 \\ \vdots \\ \tilde{\mathbf{y}}_r \end{bmatrix} \in \mathbb{R}^{rn \times n}, \quad \tilde{\mathbf{Z}} = \begin{bmatrix} \tilde{\mathbf{z}}_1 \\ \vdots \\ \tilde{\mathbf{z}}_r \end{bmatrix} \in \mathbb{R}^r \]

where the functions \( \tilde{\mathbf{y}}_i = \tilde{H}_i(\epsilon) \) and \( \tilde{\mathbf{z}}_i = \tilde{D}_i(\epsilon), \quad 1 \leq i \leq r. \) Now determine \( K^* \) such
that it is the minimum,

\[ K^* = \arg \min_{K \in \bar{K}} \left\{ \frac{d}{d\epsilon} \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}) \right\}. \]

Consider that the total time derivative above has the representation,

\[
\frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}) \right) = \frac{\partial \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \text{vec}((\mathbf{Y}))} \text{vec}(\mathcal{F}(\mathbf{Y}, K)) + \frac{\partial \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \text{vec}(\tilde{\mathbf{Y}})} \text{vec}(\mathcal{F}(\tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})) + \frac{\partial \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \tilde{\mathbf{Y}}} \mathcal{G}(\mathbf{Y}) + \frac{\partial \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \tilde{\mathbf{Z}}} \mathcal{G}(\tilde{\mathbf{Y}}). \tag{3.47}
\]

The partial derivatives below have the alternative form via the chain rule,

\[
\frac{\partial \phi_{WLS}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \text{vec}(\mathbf{Y}_i)} = 2w_i(\kappa_i(\mathbf{Y}_i, \mathbf{Z}_i) - \tilde{\kappa}_i(\tilde{\mathbf{Y}}_i, \tilde{\mathbf{Z}}_i)) \cdot \text{vec}(x_0 x_0^T).
\]

Insert the derivatives into (3.47), then differentiate (3.47) with respect to \( K \), and set the resulting expression equal to a zero matrix with the appropriate dimension. This is the necessary condition for the expression to take an extremal value on the interior of its domain.

\[
-2 \sum_{i=1}^{r} (2w_i(\kappa_i(\mathbf{Y}_i, \mathbf{Z}_i) - \tilde{\kappa}_i(\tilde{\mathbf{Y}}_i, \tilde{\mathbf{Z}}_i)))B^T(\epsilon)\mathcal{Y}_i(x_0 x_0^T)
- \frac{1}{2}(2w_1(\kappa_1(\mathbf{Y}_1, \mathbf{Z}_1) - \tilde{\kappa}_1(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Z}}_1)))R(\epsilon)K(\epsilon)(x_0 x_0^T) = 0
\]
Since $x_0x_0^T$ is a fixed rank-one matrix, the control below is found when $\kappa_1(\epsilon) \neq \tilde{\kappa}_1(\epsilon)$,

\[
K^*(\epsilon, \mathbf{Y}, \mathbf{Z}, \check{\mathbf{Y}}, \check{\mathbf{Z}}) = -R^{-1}(\epsilon)B^T(\epsilon)\mathbf{Y}_1 - R^{-1}(\epsilon)B^T(\epsilon) \sum_{i=2}^{r} \frac{w_i}{w_1} \left( \frac{\kappa_i(\mathbf{Y}_i, \mathbf{Z}_i) - \tilde{\kappa}_i(\check{\mathbf{Y}}_i, \check{\mathbf{Z}}_i)}{\kappa_1(\mathbf{Y}_1, \mathbf{Z}_1) - \tilde{\kappa}_1(\check{\mathbf{Y}}_1, \check{\mathbf{Z}}_1)} \right) \mathbf{Y}_i.
\]

Let $\mathbf{Y}^*$ and $\mathbf{Z}^*$ denote

\[
\mathbf{Y}^* = \begin{bmatrix} \mathbf{Y}_1^* \\ \vdots \\ \mathbf{Y}_r^* \end{bmatrix}, \quad \mathbf{Z}^* = \begin{bmatrix} \mathbf{Z}_1^* \\ \vdots \\ \mathbf{Z}_r^* \end{bmatrix}
\]

where the functions $H_i^*(\epsilon) = \mathbf{Y}_i^*$ and $D_i^*(\epsilon) = \mathbf{Z}_i^*$ satisfy the equations of motion under the control selection $K^*$,

\[
\frac{dH_i^*(\tau)}{d\tau} = -(A(\tau) + B(\tau)K^*(\tau))^T H_i^*(\tau) - H_i^*(\tau)(A(\tau) + B(\tau)K^*(\tau)) - K^*T(\tau)R(\tau)K^*(\tau) - Q(\tau)
\]

\[
\frac{dH_1^*(\tau)}{d\tau} = -(A(\tau) + B(\tau)K^*(\tau))^T H_1^*(\tau) - H_1^*(\tau)(A(\tau) + B(\tau)K^*(\tau)) - K^*T(\tau)R(\tau)K^*(\tau) - Q(\tau)
\]

\[
\frac{dD_1^*(\tau)}{d\tau} = -\text{Tr}(H_i^*(\tau)G(\tau)WG^T(\tau)), \quad \tau \in [\epsilon, t_f]
\]

\[
H_i^*(t_f) = Q_f, \quad H_i^*(t_f) = 0^{n \times n}, \quad i \geq 2
\]

\[
D_i^*(t_f) = 0, \quad D_j^*(t_f) = 1, \quad j \geq 3
\]

It is assumed that $\mathbf{Y}^*$ and $\mathbf{Z}^*$ exist in the following. Using these dynamic
programming variables under $K^*$, the candidate value function is now proposed, 

$$W(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) = 2\phi_{WLS}(\mathbf{H}(t_0), \mathbf{D}(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0))$$

$$- \phi_{WLS}(\mathbf{H}^*(t_0), \mathbf{D}^*(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0))$$

$$+ \phi_{WLS}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\dot{y}}, \mathbf{\dot{z}}) - \phi_{WLS}(\mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}).$$

(3.50)

Note first that the Mayer-form boundary condition is satisfied, by observing that when $(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \in \mathcal{M}$, it is true that $\mathbf{y}^* = \mathbf{H}^*(t_0)$, $\mathbf{z}^* = \mathbf{D}^*(t_0)$, $\mathbf{\dot{y}} = \mathbf{\dot{H}}(t_0)$, $\mathbf{\dot{z}} = \mathbf{\dot{D}}(t_0)$, $\mathbf{y} = \mathbf{H}(t_0)$, and $\mathbf{z} = \mathbf{D}(t_0)$. Using these relations in (3.50) yields the expected result (3.25).

The function $W(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\dot{y}}, \mathbf{\dot{z}})$ is constant for displaced terminal conditions obtained from anywhere along the trajectory of the state variables under the influence of $K^*$. This can be verified by observing that when $\mathbf{y} = \mathbf{y}^*$ and $\mathbf{z} = \mathbf{z}^*$, it must be true that $\mathbf{H}(t_0) = \mathbf{H}^*(t_0)$ and $\mathbf{D}(t_0) = \mathbf{D}^*(t_0)$ so that

$$W(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\dot{y}}, \mathbf{\dot{z}}) = \phi_{WLS}(\mathbf{H}^*(t_0), \mathbf{D}^*(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0)).$$

Furthermore, (3.26) is satisfied since $W(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}})$ is non-increasing along $\mathbf{y} \neq \mathbf{y}^*$ and $\mathbf{z} \neq \mathbf{z}^*$,

$$\frac{dW(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}})}{d\epsilon} = \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right) - \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right)$$

$$= \min_{\mathbf{K} \in \mathcal{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right) \right\} - \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right) \leq 0.$$

Finally, equation (3.27) is satisfied since the total time derivative of $W(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}})$ vanishes,

$$\min_{\mathbf{K} \in \mathcal{K}} \left\{ \frac{dW(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}})}{d\epsilon} \right\} = \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right) - \min_{\mathbf{K} \in \mathcal{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{WLS}(\mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) \right) \right\} = 0.$$
Thus by the HJB Verification Lemma,

$$\mathcal{W}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}) = \mathcal{V}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\dot{y}}, \mathbf{\dot{z}}).$$

and $K^*$ is optimal.

3.3.3 Simulation Results

The goal of this section is to conduct a numerical experiment in order to apply and validate the WLS-CDS theory. In particular, a single-degree of freedom (SDOF) problem is considered involving a one-story structure, which is described fully in [32]. The system dynamics and cost are given by,

$$dx(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) dt + \begin{bmatrix} 0 \\ -4k_c \cdot \cos(\alpha)/m \end{bmatrix} u(t) dt + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dw(t)$$

$$J[x, u] = \int_{t_0}^{t_f} \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(\tau) dt + \begin{bmatrix} 0 \\ -4k_c \cdot \cos(\alpha)/m \end{bmatrix} u(\tau) dt + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dw(\tau)$$

The system constants are $m = 16.69$ lb-s$^2$/in, $c = 9.020$ lb-s/in, $k = 7934$ lb/in, $k_c = 2124$ lb/in, $\alpha = 36^\circ$, $W = 2\pi$ in$^2$/s$^3$, and $x_0 = \begin{bmatrix} .00176 \\ .039 \end{bmatrix}$. The time horizon is between $t_0 = 0$ and $t_f = 10$ sec. The system state $x_1(t)$ is the floor displacement and $x_2(t)$ is the floor velocity. Now consider the WLS-CDS performance index,

$$\phi_{WLS}(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) = \sum_{i=1}^{3} w_i (\kappa_i(t_0) - \dot{\kappa}(t_0))^2$$

where arbitrary weighting constants $w_1, w_2 = 1$ and $w_3 = 0.2$ are chosen, and the
functions \( \{ \tilde{\kappa}_i(\alpha) \}_{i=1}^3 \) are determined for arbitrary perturbation constants \( \epsilon^* = 0.0075 \)
and \( \mathcal{E}^* = 0^{2\times2} \) by the equations (3.7) with \( r = 3 \), under the control selection

\[
\hat{K}(\alpha) = - R(\alpha)^T B(\alpha) \left( \hat{H}_1(\alpha) + \mu_2 \hat{H}_2(\alpha) + \mu_3 \hat{H}_3(\alpha) \right).
\] (3.52)

Here the parameters are \( \mu_2 \in [0,9] \) and \( \mu_3 \in [0,9] \). That is, let each parameter pair
\( (\mu_2, \mu_3) \in [0,9] \times [0,9] \) characterize a control gain (3.52) under which the equations
(3.7) with \( r = 3 \) can be solved to yield \( \{ \tilde{\kappa}_i(\alpha) \}_{i=1}^3 \). It should be reiterated that
the control gain (3.52) will ordinarily be unknown, and only the target statistics
\( \{ \tilde{\kappa}_i(\alpha) \}_{i=1}^3 \) are available to the designer. Nevertheless, to inspect the capability
of the WLS-CDS control, the aforementioned linear controls are used directly to
realize the targets so that with the cost cumulant information \( \{ \tilde{\kappa}_i(\alpha) \}_{i=1}^3 \) alone, the
optimal linear control solution to the minimization of (3.51) may be computed.

The cost cumulants \( \{ \kappa_i(\alpha) \}_{i=1}^3 \) under the optimal control (3.46) are displayed in
Figures 3.2, 3.4, and 3.6. The plots correspond with scaled-down versions of those
computed by Pham in [26] under 3CC controls. This shows that a controller
has been constructed that approximately achieves the targets, only using the
information \( \{ \tilde{\kappa}_i(\alpha) \}_{i=1}^3 \). The normalized errors are shown in Figures 3.3, 3.5, and
3.7 between the initial cost cumulants and target initial cost cumulants, which
enables a quantitative assessment of how close the cost cumulants are to the
targets. Notice that there is less than 0.5% normalized error for each of the three
cost cumulants. Clearly, a three-cumulant approximation to the cost density will
approximately align with that of the target cost density.
Figure 3.2. WLS-CDS Cost Mean

Figure 3.3. Normalized Error, Cost Mean
Figure 3.4. WLS-CDS Cost Variance

Figure 3.5. Normalized Error, Cost Variance
Figure 3.6. WLS-CDS Cost Skew

Figure 3.7. Normalized Error, Cost Skew
Chapter 3 has provided a very general framework for developing cost density-shaping controls that can steer finitely-many cost cumulants towards nominal target values. However, the framework is theoretical and thus it is beneficial to apply the theory to some specific performance indices of the appropriate character. One instance where this has been done is the Weighted Least-Squares, Cost Density-Shaping (WLS-CDS) optimal control, which was presented in Chapter 3 with a simulation to illustrate the controller computation. However, the WLS-CDS performance index does not have an apparent probabilistic meaning in terms of the distance between the cost density and a nominal target cost density. This makes way for considering well-known probabilistic measures between density functions such as the Kullback-Leibler Divergence (KLD), the Hellinger Distance (HD), and the Bhattacharyya Distance (BD). Cumulant representations of the aforementioned measures are quite complicated and the general $k$th cost cumulant case requires the methodology presented in Chapter 6.

Nevertheless, the mean-variance representations of these probabilistic measures are more wieldy, facilitating a number of numerical experiments in this chapter to examine the cost-density shaping ability gained through a family of Mean-Variance Cost Density-Shaping (MVCDS) controls. The importance of mean-
variance cost density-shaping should make it clear that this restriction on the number of cumulants in the expressions for distance functions is not done out of convenience. Indeed, for many design problems, controlling the mean and the variance of the cost has been yielded significant performance gains to LQG control [45], and the it can be seen in the simulation results of this chapter that important performance and stability characteristics of the controller producing the target density are actually embedded in the mean-variance approximation to that density.

Each subsection in this chapter is meant to be self-contained. Thus, the development is such that each optimization can be considered on its own; simulation results for validation are provided per MVCDS optimal control. Afterwards, the “delta-means” appearing in the MVCDS controls are studied so to develop a variety of sufficient conditions such that the difference of means does not vanish on the finite horizon. This event is termed generally herein as a delta-mean zero-crossing.

Before delving into the results, some common material is introduced that pertains to all derived MVCDS control laws.

4.1 Common Concepts

A variety of common concepts are pertinent to the MVCDS controls presented in this chapter. The class of process, cost, and noise, the cost and target cost cumulants, the normalized cost and target cost variates, the target set and the notion of admissible feedback gains, along with the HJB verification lemma are all essential to deriving the control solutions that solve the optimizations to be posed. In addition to these ideas, the formal definition of the \( f \)-divergence is presented and some of its properties; this formalism is important because each MVCDS
optimization involves a different instance of f-divergence. The aforementioned common concepts are provided here in one central location; these definitions will be referred to for each and every problem studied in this chapter.

4.1.1 Problem Class

This work pertains exclusively to the process \( x(t) \) with dynamics subject to additive white Gaussian noise,

\[
dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t),
\]

\[x_0 = E\{x(t_0)\}, \ x_0 \in \mathbb{R}^n, \ t \in [t_0, t_f]\]

where \( A \in C([t_0, t_f]; \mathbb{R}^{n \times n}), \ B \in C([t_0, t_f]; \mathbb{R}^{n \times m}), \ G \in C([t_0, t_f]; \mathbb{R}^{n \times p}) \) and \( w(t) \) is a \( p \)-dimensional stationary Wiener process having a correlation of increments defined by

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}.
\]

Control inputs are assumed to be \( \mathbb{R}^m \)-valued, square-integrable, non-anticipating processes \( u \in U \subset L^2_{\mathcal{F}}(\Omega; C([t_0, t_f]; \mathbb{R}^m)) \) adapted to the filtration \( \mathcal{F}_t \) generated by \( w(t) \). Under this assumption, \( x \in L^2_{\mathcal{F}}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \). Control inputs \( u \) are chosen with respect to the cost \( J[x, u; t_0, x_0] \), which is an integral-quadratic form and is defined by

\[
J = \int_{t_0}^{t_f} \left( x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau) \right) dt + x^T(t_f)Q_f x(t_f)
\]

where \( Q = Q^T \in C([t_0, t_f]; S^n_+) \), \( R = R^T \in C([t_0, t_f]; S^m_+) \), and \( Q_f = Q_f^T \in S^n_+ \). Under this membership, \( Q_f, Q \succ 0^{n \times n} \) and \( R \succ 0^{m \times m} \); these are typical conditions that ensure
well-posedness for the associated stochastic optimal control problem.

For linear state-feedback control inputs, it is shown in [27] that $J$ is a finite $\chi^2$ random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The finiteness of $J$ stems from the fact that for linear state-feedback controls, the “running cost” and “terminal cost” functions of (4.3) always satisfy the suitable polynomial growth conditions necessary for boundedness of the expectation of the cost functional [66]. Under this class of control inputs, a finite number of $r$ cumulants exist for $J$.

This material deals with the mean and variance of $J$, which are defined as

$$
\kappa_1(t_0) \triangleq E\{J\}, \quad \kappa_2(t_0) \triangleq E\{J^2\} - (E\{J\})^2.
$$

4.1.2 Cost Cumulants

The work of Liberty and Hartwig [61-62] established that when the process and cost take the LQG form, and when the control is a linear state-feedback input, the $r$ cumulants of (4.3) are quadratic in the known initial state and are composed of functions that satisfy a family of coupled, backwards-in-time, differential equations.

In particular, the first two initial cost cumulants are given by

$$
\kappa_i(\alpha) = x_0^T H_i(\alpha)x_0 + D_i(\alpha), \quad 1 \leq i \leq 2 \quad (4.4)
$$

when $\alpha = t_0$. The functions $H_i(\alpha)$ satisfy the following system of backwards-in-time, matrix differential equations. The dynamics of the functions $D_i(\alpha)$ depend
on the $H_i(\alpha)$ functions.

$$\frac{dH_1(\alpha)}{d\alpha} = -\left( A(\alpha) + B(\alpha)K(\alpha) \right)^T H_1(\alpha) - H_1(\alpha) \left( A(\alpha) + B(\alpha)K(\alpha) \right)$$

$$- K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha) \triangleq \mathcal{F}_1(\mathbf{H}(\alpha), K(\alpha))$$

$$\frac{dH_2(\alpha)}{d\alpha} = -\left( A(\alpha) + B(\alpha)K(\alpha) \right)^T H_2(\alpha) - H_2(\alpha) \left( A(\alpha) + B(\alpha)K(\alpha) \right)$$

$$- 4H_1(\alpha)G(\alpha)WG^T(\alpha)H_1(\alpha) \triangleq \mathcal{F}_2(\mathbf{H}(\alpha), K(\alpha))$$

$$\frac{dD_i(\alpha)}{d\alpha} = -\text{Tr} \left( H_i(\alpha)G(\alpha)WG^T(\alpha) \right) \triangleq \mathcal{G}_i(\mathbf{H}(\alpha)), \quad \alpha \in [t_0, t_f], 1 \leq i \leq 2$$

These functions satisfy the terminal conditions

$$H_1(t_f) = Q_f, \quad H_2(t_f) = 0^{n \times n}, \quad D_1(t_f) = 0, \quad D_2(t_f) = 1. \quad (4.6)$$

### 4.1.3 Notation

Some notation is introduced to make restatements of the above equations more concise in the development. Begin by defining the state variables $\mathbf{H}(\alpha) \in \mathbb{R}^{2n \times n}$ and $\mathbf{D}(\alpha) \in \mathbb{R}^2$ as below

$$\mathbf{H}(\alpha) \triangleq \begin{bmatrix} H_1(\alpha) \\ H_2(\alpha) \end{bmatrix}, \quad \mathbf{D}(\alpha) \triangleq \begin{bmatrix} D_1(\alpha) \\ D_2(\alpha) \end{bmatrix}.$$  

Using these state variables, define the functions

$$\mathcal{F}(\mathbf{H}(\alpha), K(\alpha)) \triangleq \begin{bmatrix} \mathcal{F}_1(\mathbf{H}(\alpha), K(\alpha)) \\ \mathcal{F}_2(\mathbf{H}(\alpha), K(\alpha)) \end{bmatrix}, \quad \mathcal{G}(\mathbf{H}(\alpha)) \triangleq \begin{bmatrix} \mathcal{G}_1(\mathbf{H}(\alpha)) \\ \mathcal{G}_2(\mathbf{H}(\alpha)) \end{bmatrix}.$$  

Let $\mathcal{F}_i(\cdot)$ and $\mathcal{G}_i(\cdot)$ in the above definitions be defined as beforehand in (4.5). A
condensed form for the terminal conditions is also introduced as below,

\[
\begin{bmatrix}
Q_f \\
0^{n \times n}
\end{bmatrix}, \quad D_f \triangleq \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

Finally, denote the vector of cost cumulants \( \kappa(\alpha) \in \mathbb{R}^2 \) as

\[
\kappa(\alpha) \triangleq \begin{bmatrix}
\kappa_1(\alpha) \\
\kappa_2(\alpha)
\end{bmatrix}.
\]

### 4.1.4 Target Cost Statistics

Given matrices for a system characterization \((A, B, G)\), an integral-quadratic cost characterization \((Q, R, Q_f)\), and the second-order statistics of the noise \((W)\), consider the cost cumulants as a result of the alternative (and unknown) linear state-feedback control \( \tilde{u}(t) = \tilde{K}(t)\tilde{x}(t) \), where \( \tilde{K} \in C([t_0, t_f]; \mathbb{R}^{m \times n}) \).

The initial cost mean and variance are given by

\[
\tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha), \quad 1 \leq i \leq 2
\]  

when \( \alpha = t_0 \). Let this set of numbers be regarded as target cost mean and variance. Here the functions \( \tilde{H}_i(\alpha) \) are determined by the same system of backwards-in-time, matrix differential equations as (4.5). The dynamics of \( \tilde{D}_i(\alpha) \) will also be as before.

\[
\frac{d\tilde{H}(\alpha)}{d\alpha} = \mathcal{F}(\tilde{H}(\alpha), \tilde{K}(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{H}(\alpha))
\]

\[
\tilde{H}(t_f) = H_{f, \xi^*}, \quad \tilde{D}(t_f) = D_{f, \xi^*}, \quad \alpha \in [t_0, t_f]
\]
The terminal conditions for these equations are

\[
\hat{H}_1(t_f) = Q_f + \mathcal{E}^*, \quad \hat{H}_2(t_f) = 0^{n \times n},
\]
\[
\hat{D}_1(t_f) = \epsilon^*, \quad \hat{D}_2(t_f) = 1.
\] (4.9)

Above, the short-hand notation is used

\[
\mathbf{H}_{f;\epsilon^*} \triangleq \begin{bmatrix} Q_f + \mathcal{E}^* \\ 0^{n \times n} \end{bmatrix}, \quad \mathbf{D}_{f;\epsilon^*} \triangleq \begin{bmatrix} \epsilon^* \\ 1 \end{bmatrix}.
\]

Here $\epsilon^* > 0$ is a small perturbation constant, and $\mathcal{E}^* 

\geq 0^{n \times n}$ is a positive-definite perturbation matrix. As with the cost cumulants, compose a vector of target initial cost cumulants $\tilde{\kappa}(\alpha) \in \mathbb{R}^2$ defined as below,

\[
\tilde{\kappa}(\alpha) \triangleq \begin{bmatrix} \tilde{\kappa}_1(\alpha) \\ \tilde{\kappa}_2(\alpha) \end{bmatrix}.
\]

When the Markov process determined by (4.1) is mean-value stationary for $u$ (respectively, $\tilde{u}$), the function (4.4) (respectively, (4.7)) for $\alpha > t_0$ gives a family of initial cost means and variances for the new cost (4.3) under $K$ (respectively, $\tilde{K}$) from $\alpha > t_0$ to the fixed terminal time $t_f$, for $\alpha \in (t_0, t_f)$. Even if the process having dynamics (4.1) is not mean-value stationary (e.g. approximately mean-value stationary), the functions (4.7) are special and it will be observed empirically that tracking these quantities throughout the finite-horizon $[t_0, t_f]$ is necessary to achieve a nominal pair $(\tilde{\kappa}_1(t_0), \tilde{\kappa}_2(t_0))$. It is assumed in this work that the Markov process with dynamics (4.1) and input $u(t) = K(t)x(t)$ is mean-value stationary per the sufficient conditions presented in Section 2.3.
4.1.5 Cost and Target Cost Variates

This work considers linear optimal MVCDS controls that optimize probability distance measures between best Gaussian density approximations to normalized cost and target cost variates. Hence it is worthwhile to define the aforementioned approximations, as they will be referred to repeatedly in the development.

**Definition 4.1.1 (Gaussian Density Approximations)**
Consider Gaussian Density Approximations (GDA) for the true cost density which is $\chi^2$, and consider the appropriate standardized variates,

$$Z = \frac{J - \kappa_1(t_0)}{\kappa_2(t_0)^{1/2}}$$

$$\tilde{Z} = \frac{J - \tilde{\kappa}_1(t_0)}{\tilde{\kappa}_2(t_0)^{1/2}} = \frac{J - \kappa_1(t_0) + \kappa_1(t_0) - \tilde{\kappa}_1(t_0)}{\kappa_2(t_0)^{1/2}}$$

where evidently

$$\int_{-\infty}^{\infty} p_Z(z) dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp \left( \frac{-z^2}{2} \right) dz = 1$$

$$\int_{-\infty}^{\infty} p_{\tilde{Z}}(\tilde{z}) d\tilde{z} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp \left( \frac{-(az + b)^2}{2} \right) d(az) = 1.$$

4.1.6 Problem Formulation

The optimizations in this chapter will either minimize a nominal performance index over a space of linear controller gains defined by the target set and the notion of admissible feedback gains defined below. The two concepts parallel the
definitions given in Chapter 3.

**Definition 4.1.2 (Target Set)**
Let \((t_0, H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \in \mathcal{M}\), where \(\mathcal{M}\) denotes the target set which is a closed subset of

\[ [t_0, t_f] \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}) \times \mathbb{R}^2 \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}) \times \mathbb{R}^2. \]

**Remark(s)** Note that the goal of tracking target cumulants suggests that predetermined trajectories of \(\tilde{H}(\alpha)\) and \(\tilde{D}(\alpha)\) will inherently have initial values in \(\mathcal{M}\) by their definition.

For given terminal conditions, the set of admissible feedback gains

\[ K_{t_f, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)} \]

is the set of matrices \(K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})\) such that values \((t_0, H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \in \mathcal{M}\) are obtained at the end of the trajectories for the state equations (4.5) and (4.8). This is formally stated in the following definition.

**Definition 4.1.3 (Admissible Feedback Gains)**
Denote the allowable set of control gain values by \(\bar{K} \subset \mathbb{R}^{m \times n}\) and let this set be compact. For fixed \(r \in \mathbb{N}\) let \(K_{t_f, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)} \)
characterize a class of \(\mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})\) such that for any \(K \in K_{t_f, H(t_f), D(t_f), \tilde{H}(t_f), \tilde{D}(t_f)}\) the solutions to

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = \tilde{F}(\tilde{H}(\alpha), \tilde{K}(\alpha))
\]

\[
\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = \tilde{G}(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[ H(t_f) = H_f, \quad \tilde{H}(t_f) = \tilde{H}_f, \quad D(t_f) = D_f, \quad \tilde{D}(t_f) = \tilde{D}_f. \]

exist and also the initial values of the state trajectories satisfy

\((t_0, H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \in \mathcal{M}\).

### 4.1.7 Dynamic Programming

The solution approach for all of the optimizations presented in the chapter will be the same as in Chapter 3. In particular, the dynamic programming concepts in Fleming and Rishel [66] are adapted to the cumulant-generating equations of
the LQG framework, where the cost cumulants considered are limited to the first two, the mean and the variance. Three concepts are seminal to the dynamic programming solution, those being the reachable set $Q$, the value function $V(\cdot)$, and the HJB verification lemma. The definitions and lemma, as presented in Chapter 3, are presented here once more. Chapter 3 contains a more complete development, and all of those results apply to this formalism.

**Definition 4.1.4 (Reachable Set)**
The reachable set is the set of terminal values from which there exists a control that can take the system to the target set. More formally, this is

$$Q = \{(\epsilon, \mathbf{Z}, \mathbf{\hat{z}}) \mid K_{\epsilon, \mathbf{Z}, \mathbf{\hat{z}}} \neq \emptyset\}.$$ 

**Definition 4.1.5 (Value Function)**
Let $(\epsilon, \mathbf{Y}, \mathbf{Z}, \mathbf{\hat{Y}}, \mathbf{\hat{Z}}) \in [t_0, t_f] \times (\mathbb{S}^n)^2 \times \mathbb{R}^2 \times (\mathbb{S}^n)^2 \times \mathbb{R}^2$ and let $V(\epsilon, \mathbf{Y}, \mathbf{Z}, \mathbf{\hat{Y}}, \mathbf{\hat{Z}})$ be a scalar function

$$V : [t_0, t_f] \times (\mathbb{S}^n)^2 \times \mathbb{R}^2 \times (\mathbb{S}^n)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$V(\epsilon, \mathbf{Y}, \mathbf{Z}, \mathbf{\hat{Y}}, \mathbf{\hat{Z}}) = \inf_{K \in K_{\epsilon, \mathbf{Y}, \mathbf{Z}, \mathbf{\hat{Y}}, \mathbf{\hat{Z}}}} \left\{ GDA^{GDA}(\mathbf{H}(t_0), \mathbf{D}(t_0), \mathbf{H}(t_0), \mathbf{D}(t_0)) \right\}$$

when $K_{\epsilon, \mathbf{Y}, \mathbf{Z}, \mathbf{\hat{Y}}, \mathbf{\hat{Z}}} \neq \emptyset$.

**Remark(s)** In each case, $\chi$ denotes the optimization under consideration, though in this development the convention is implicit in the value function. The values $\chi = \{KLD, HD, BD, BC\}$ refer to the Kullback-Leibler Divergence, the Hellinger Distance, the Bhattacharyya Distance, and the Bhattacharyya Coefficient, respectively.

The value function $V(\epsilon, \mathbf{Y}(\epsilon), \mathbf{Z}(\epsilon), \mathbf{\hat{Y}}(\epsilon), \mathbf{\hat{Z}}(\epsilon))$ is a function of the “displaced” terminal conditions, and it possesses two inherent properties. First, is that the value function is non-increasing when evaluated at terminal conditions from anywhere along a trajectory of the state equations resultant from a non-optimal control
$K \neq K^*$. Second, is that the value function is constant when evaluated at terminal conditions from anywhere along the optimal trajectory of the state equations resultant from the control $K^*$. These necessary conditions characterize the value function, and are sufficient.

The HJB verification lemma encapsulates these two innate features of the value function, and provides a means by which a control input can be shown to be optimal. That is, if a candidate value function $W(\epsilon, Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))$ can be found that satisfies the partial differential equation of dynamic programming and a Mayer-form boundary condition for a nominal control, then that control is optimal and the candidate value function is the value function. Formally, the result is as follows.

**Lemma 4.1.6 (HJB Verification)**

Let $(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})$ be an interior point of the reachable set $Q$. Assume the scalar function $W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})$ satisfies the boundary condition

$$W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) = \phi^{GDA}_\chi(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \quad (4.11)$$

when $(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) \in M$.

Let $W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})$ also satisfy the HJB equation of dynamic programming,

$$-\frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \epsilon} \geq \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y)} g(Y) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y)} g(Y) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(\tilde{Y})} g(\tilde{Y}) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(\tilde{Y})} g(\tilde{Y}) \quad (4.12)$$

If further for a given control $K^* \in \mathcal{K}(\epsilon)$ this function satisfies the following equation

$$-\frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \epsilon} = \min_{K \in \mathcal{K}} \left\{ \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y)} g(Y) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y)} g(Y) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(\tilde{Y})} g(\tilde{Y}) + \frac{\partial W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(\tilde{Y})} g(\tilde{Y}) \right\} \quad (4.13)$$

then the control $K^*$ is optimal and

$$W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) = V(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}).$$
Remark(s) In each case, \( \chi \) denotes the optimization under consideration when the HJB verification lemma is referred to. The values \( \chi = \{KLD, HD, BD, BC\} \) refer to the Kullback-Leibler Divergence, the Hellinger Distance, the Bhattacharyya Distance, and the Bhattacharyya Coefficient, respectively. What this means is that while the HJB partial differential equations of dynamic programming remain independent of the optimization under consideration, the Mayer-form boundary condition must reflect the appropriate performance index.

4.1.8 Background: \( f \)-Divergence

Probabilistic measures, or distance functions, are used to assess the degree of correlation between a number of statistical models or between a model and measurement data. Such measures have become prevalent in the domains of signal processing, information theory, pattern recognition, estimation and detection theory, and coding. The usefulness of distance measures in these fields is apparent from the literature. The utility of probability distance measures is furthered with this work, though not for just any distance measure. In particular, the discussion will be concerned only with distance measures that meet a variety of properties that make them suitable for the MVCDS framework.

As a first requirement, only distance measures are considered that pertain to measuring the distance between information models, or in other words pre-specified probability density functions. Second, the distance measure must meet the criteria of a MCCDS performance index, that is it’s non-negative, smooth, and convex (or concave) in its arguments. Thirdly, though perhaps a natural requirement of MCCDS performance indices, is that the distance measure between densities be a function of both the cost cumulants and the target cost cumulants.

The fundamental definition of the \( f \)-divergence has given rise to many useful distance measures, some of which meet the aforementioned requirements and hence are considered in this work. The definition is first stated and then the properties
of the $f$-divergence are examined.

**Definition 4.1.7 (f-Divergence)**

The $f$-divergence is defined for a convex, continuous function $f(\cdot)$ such that $f(1) = 0$ on $\mathbb{R}^+$ and an increasing function $g(\cdot)$ on $\mathbb{R}$ between the probability density functions $p_1(x)$ and $p_2(x)$ with support on $\mathbb{R}$ having respective distribution functions $P_1(x)$ and $P_2(x)$ as,

$$d(p_1(x), p_2(x)) = g \left( \int_{-\infty}^{\infty} f \left( \frac{dP_2(x)}{dP_1(x)} \right) dP_1(x) \right)$$

$$= g \left( \int_{-\infty}^{\infty} f \left( \frac{p_2(x)}{p_1(x)} \right) p_1(x) dx \right)$$

$$= g \left( E \left( f \left( \frac{p_2(x)}{p_1(x)} \right) \right) \right).$$

**Remark(s)** The derivatives above are taken as the Radon-Nikodym kind, which possibly can be generalized in the case where $P_2(x)$ has a singular component with respect to $P_1(x)$ [34]; the expectation above is with respect to $p_1(x)$.

Given this general definition, it is readily seen that the $f$-divergence gives the measure between densities $p_1(x)$ and $p_2(x)$ by considering dispersion in the likelihood ratio $\frac{p_2(x)}{p_1(x)}$. A natural question is whether or not the $f$-divergence yields a true metric for probability measures. Recall that a metric on a space $X$ is a function $d : X \times X \rightarrow \mathbb{R}^+$, which satisfies the following properties,

1. $d(x, y) \geq 0$, $\forall x, y \in X$

2. $d(x, y) = 0$, if and only if $x = y$

3. $d(x, y) = d(y, x)$, $\forall x, y \in X$

4. $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$

It turns out that many of the distance measures attainable from the $f$-divergence do not satisfy all the conditions above, and hence these may not be metrics on a functional space in the true sense. However, any $f$-divergence distance function
is a premetric, the definition of which relaxes the triangle inequality and symmetry conditions above. In particular, for a space $X$ a premetric is defined as some function $\overline{d}: X \times X \rightarrow \mathbb{R}^+$ such that,

1. $\overline{d}(x, y) \geq 0$, $\forall x, y \in X$

2. $\overline{d}(x, y) = 0$, if and only if $x = y$

This characteristic of $f$-divergence distance function is easily established. Write $\phi(x) = \frac{p_2(x)}{p_1(x)}$, and consider that since the function $f(\cdot)$ is convex, it follows by Jensen’s inequality that

$$E\left\{ f(\phi(x)) \right\} \geq f \left( E\left\{ \phi(x) \right\} \right) = f \left( \int_{-\infty}^{\infty} \frac{p_2(x)}{p_1(x)} \cdot p_1(x) dx \right) = f(1) = 0.$$  

Since $g(\cdot)$ is increasing on $\mathbb{R}$, it must be true that

$$\overline{d}(p_1(x), p_2(x)) = g \left( E\left\{ f(\phi(x)) \right\} \right) \geq g \left( f \left( E\left\{ \phi(x) \right\} \right) \right) \geq 0.$$  

When $p_1(x) = p_2(x)$, $\phi(x) = 1$ and thus $f(\phi(x)) = f(1) = 0$, from which it is obvious that $\overline{d}(p_1(x), p_2(x)) = 0$.

4.2 Minimum Kullback-Leibler Divergence Cost-Density Shaping

4.2.1 Kullback-Leibler Divergence and Properties

In this section the definition of the KLD is presented, and then derive the special form of the KLD between Gaussian approximations to the densities of normalized cost and target cost variates. The function will ultimately be proposed as the MKLD-CDS performance index in the ensuing optimization.
Definition 4.2.1 (Kullback-Leibler Divergence)
The Kullback-Leibler Divergence (KLD) between probability densities $p_1(x)$ and $p_2(x)$ with support on $\mathbb{R}$ is defined as

$$KLD\{p_1(x), p_2(x)\} = \int_{-\infty}^{\infty} \log\left(\frac{p_1(x)}{p_2(x)}\right) p_1(x)dx.$$  

The KLD fits the f-divergence form for the functions $g(x) = x$ and $f(x) = -\log(x)$.

Remark(s) Note that $f(\phi(x)) = -\log\left(\frac{p_2(x)}{p_1(x)}\right) = \log\left(\frac{p_1(x)}{p_2(x)}\right)$ in the KLD. The function $\log(.)$ refers to the natural logarithm $\ln(.)$; the two are used interchangeably.

Theorem 4.2.2 (Derivation of KLD-GDA)
The Kullback-Leibler Divergence (KLD) between $p_Z(z)$ and $\tilde{p}_Z(\tilde{z})$ can be written in terms of $\kappa_1(t_0)$, $\kappa_2(t_0)$, $\tilde{\kappa}_1(t_0)$, and $\tilde{\kappa}_2(t_0)$ as below,

$$KLD\{p_Z(z), \tilde{p}_Z(\tilde{z})\} = \frac{1}{2} \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} - 1 - \log\left(\frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)}\right) + \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{\tilde{\kappa}_2(t_0)} \right). \quad (4.14)$$

Proof Consider the definition of KLD and the definitions of the Gaussian approximations to the densities of normalized cost and target cost variates directly,

$$KLD\{p_Z(z), \tilde{p}_Z(\tilde{z})\} = \int_{-\infty}^{\infty} p_Z(z) \log\left(\frac{p_Z(z)}{\tilde{p}_Z(\tilde{z})}\right) dz$$

$$= \int_{-\infty}^{\infty} p_Z(z) \log\left(\frac{p_Z(z)}{ap_Z(az+b)}\right) dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \log\left(a^{-1} e^{-\frac{(az+b)^2}{2}}\right) dz$$

$$= -\log(a) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$+ \int_{-\infty}^{\infty} \frac{-(az+b)^2}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= -\log(a)(1) + \int_{-\infty}^{\infty} \left(\frac{(a^2-1)z^2 + 2abz + b^2}{2}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= -\frac{1}{2} \log(a^2) + \left(\frac{(a^2-1)+b^2}{2}\right)$$

$$= \frac{1}{2} \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} - 1 - \log\left(\frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)}\right) + \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{\tilde{\kappa}_2(t_0)} \right).$$

The proof is complete.
Before proceeding to the MKLD-CDS problem formulation, a few properties of the KLD shown above are considered to ensure well-posedness of the ensuing optimization problem. The function shown above is denoted as $g^{GDA}_{KLD}(\kappa(t_0), \tilde{\kappa}(t_0))$. It can be shown that the function $g^{GDA}_{KLD}(\kappa(t_0), \tilde{\kappa}(t_0))$ has many desirable properties that make it well-suited for the MKLD-CDS problem. Some of these properties are explored here, and a few associated theorems are presented. In the following, the cumulant vector designations $\kappa(t_0)$ and $\tilde{\kappa}(t_0)$ will be replaced with general vector arguments,

$$
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}, \quad
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix}.
$$

Before proceeding, it is imperative to note that for fixed $\tilde{x}_1 \geq 0$ and $\tilde{x}_2 > 0$, the value $g^{GDA}_{KLD}(x, \tilde{x})$ is real and exists for values $0 < x_2 < \infty$ and $0 \leq x_1 < \infty$. This region is an open set of contiguous values in $\mathbb{R}^2$. Accounting for these remarks, write

$$
\text{dom } g^{GDA}_{KLD} = [0, \infty) \times (0, \infty) \times [0, \infty) \times (0, \infty).
$$

**Theorem 4.2.3 (Non-Negativeness)**

The function $g^{GDA}_{KLD}(x, \tilde{x})$ is nonnegative on $\text{dom } g^{GDA}_{KLD}$.

**Proof** The first-order condition of convexity for a convex function $f(x)$ is

$$
f(y) \geq f(x) + \nabla_x f(x)(y - x).
$$
Since for a given $\tilde{x}_2 > 0$, the function $-\log \left( \frac{x_2}{\tilde{x}_2} \right)$ is convex for $x_2 > 0$ this leads to

$$-\log \left( \frac{x_2}{\tilde{x}_2} \right) \geq - \left( \frac{x_2}{\tilde{x}_2} - 1 \right)$$

and thus $(\frac{x_2}{\tilde{x}_2} - 1) - \log \left( \frac{x_2}{\tilde{x}_2} \right) \geq 0$. Hence it follows that

$$\left( \frac{x_2}{\tilde{x}_2} - 1 \right) - \log \left( \frac{x_2}{\tilde{x}_2} \right) + \frac{(x_1 - \tilde{x}_1)^2}{\tilde{x}_2} \geq \frac{(x_1 - \tilde{x}_1)^2}{\tilde{x}_2}$$

and

$$g_{GDA}^{KLD}(x, \tilde{x}) = \frac{1}{2} \left( \left( \frac{x_2}{\tilde{x}_2} - 1 \right) - \log \left( \frac{x_2}{\tilde{x}_2} \right) + \frac{(x_1 - \tilde{x}_1)^2}{\tilde{x}_2} \right)$$

$$\geq \frac{1}{2} \left( \frac{(x_1 - \tilde{x}_1)^2}{\tilde{x}_2} \right) \geq 0.$$

\[\square\]

**Theorem 4.2.4 (Convexity)**

*For fixed $\tilde{x}$, the function $g_{KLD}^{GDA}(x, \tilde{x})$ is strictly convex in $x$ on dom $g_{KLD}^{GDA}$.***

**Proof** Appeal to the second-order sufficient condition of convexity, that for fixed $\tilde{x}$ with $\tilde{x}_1 > 0$ and $\tilde{x}_2 > 0$ the Hessian of $g_{KLD}^{GDA}(x, \tilde{x})$ is positive semi-definite and its domain is convex. That is $\nabla^2 g_{KLD}^{GDA}(x, \tilde{x}) \succeq 0^{2 \times 2}$ on some contiguous square $R \in \mathbb{R}^2$. By definition,

$$\nabla^2 g_{KLD}^{GDA}(x, \tilde{x}) = \begin{bmatrix}
\frac{\partial^2 g_{KLD}^{GDA}(x, \tilde{x})}{\partial^2 x_1} & \frac{\partial^2 g_{KLD}^{GDA}(x, \tilde{x})}{\partial x_1 \partial x_2} \\
\frac{\partial^2 g_{KLD}^{GDA}(x, \tilde{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 g_{KLD}^{GDA}(x, \tilde{x})}{\partial^2 x_2}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{x_2} & 0 \\
0 & \frac{1}{2x_2^2}
\end{bmatrix} \succeq 0^{2 \times 2}, \forall x_2.
$$

It follows that the function $g_{KLD}^{GDA}(x, \tilde{x})$ is strictly convex in $x$.\[109\]
4.2.2 MKLD-CDS Problem Formulation

The optimization problem can be formulated by defining a control space over which the MKLD-CDS performance index can be minimized. This can be characterized with the target set and the space of admissible feedback gains as defined beforehand. The function (4.14) is used as the performance index for the MKLD-CDS optimization.

**Definition 4.2.5 (MKLD-CDS Performance Index)**

Let the MKLD-CDS Performance index be defined as the function

\[ \phi_{KLD}^{GDA}(\mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0)) = g_{KLD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0)). \]

The MKLD-CDS performance index has previously been shown to be non-negative, and strictly convex in \( \kappa(t_0) \). As such, it is well-suited for the objective function in the following MKLD-CDS optimization problem. The optimal MKLD-CDS controller is to be identified by minimizing (4.14) over the control space, as below.

**Definition 4.2.6 (MKLD-CDS Optimization)**

The MKLD-GDA optimization can be formulated as,

\[
\min_{\kappa \in \mathcal{K}(t_f)} \phi_{KLD}^{GDA}(\mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0))
\]

subject to:

\[
\frac{d\mathbf{H}(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}(\alpha), K(\alpha)), \quad \frac{d\tilde{\mathbf{H}}(\alpha)}{d\alpha} = \mathcal{F}(\tilde{\mathbf{H}}(\alpha), \tilde{K}(\alpha)),
\]

\[
\frac{d\mathbf{D}(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}(\alpha)), \quad \frac{d\tilde{\mathbf{D}}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{\mathbf{H}}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
\mathbf{H}(t_f) = \mathbf{H}_f, \quad \tilde{\mathbf{H}}(t_f) = \mathbf{H}_{f, \epsilon} \ast, \quad \mathbf{D}(t_f) = \mathbf{D}_f, \quad \tilde{\mathbf{D}}(t_f) = \mathbf{D}_{f, \epsilon} \ast.
\]
exist and also the initial values of the state trajectories satisfy

\[(t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}.
\]

4.2.3 MKLD-CDS Solution via Dynamic Programming

If a candidate value function can be proposed for a control \(K^*\) that satisfies three conditions, then the control \(K^*\) is optimal. In particular, the candidate value function must satisfy a Mayer-form boundary condition, in addition to satisfying the HJB equations of dynamic programming shown in the Lemma 4.1.6. Under these conditions, the candidate value function agrees with the value function along trajectories of the state variables corresponding to any linear control gain \(K\). This essentially is the provision of the HJB Verification lemma, which is used to establish the main result stated below. The proof follows.

**Theorem 4.2.7 (State-Feedback Solution, MKLD-CDS)**

Consider the MKLD-CDS optimization involving the process having dynamics (4.1) and the cost (4.3). Then the linear state-feedback, finite-horizon, optimal control solution to the MKLD-CDS optimization is characterized by the optimal gain

\[
K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\left(H_1^*(\alpha) + \frac{1}{2}\left(\frac{1 - \tilde{\kappa}_2(\alpha)}{\kappa_1^*(\alpha) - \tilde{\kappa}_1(\alpha)}\right)H_2^*(\alpha)\right) \tag{4.15}
\]

where the optimal cost mean \((i = 1)\) and variance \((i = 2)\) are defined by

\[
\kappa_i^*(\alpha) = x_0^T H_i^*(\alpha)x_0 + D_i^*(\alpha)
\]

and target cost mean \((i = 1)\) and the target cost variance \((i = 2)\) are given by

\[
\tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha).
\]

The optimal state variables \(H^*(\alpha), D^*(\alpha)\) and \(H(\alpha), D(\alpha)\) follow the equations of motion

\[
\frac{dH^*(\alpha)}{d\alpha} = \mathcal{F}(H^*(\alpha), K^*(\alpha)), \quad \frac{dH(\alpha)}{d\alpha} = \mathcal{F}(H(\alpha), \tilde{K}(\alpha))
\]
\[
\frac{d \mathbf{D}^*(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}^*(\alpha)), \quad \frac{d \tilde{\mathbf{D}}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{\mathbf{H}}(\alpha)), \quad \alpha \in [t_0, t_f]
\]
\[
\mathbf{H}^*(t_f) = \mathbf{H}_f, \quad \tilde{\mathbf{H}}(t_f) = \mathbf{H}_{f, \epsilon}, \quad \mathbf{D}^*(t_f) = \mathbf{D}_f, \quad \tilde{\mathbf{D}}(t_f) = \mathbf{D}_{f, \epsilon}.
\]

**Proof** In the following, let \( \mathbf{Y} \) and \( \mathbf{Z} \) denote
\[
\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \in \mathbb{R}^r
\]
where the functions \( \mathbf{Y}_i = \mathbf{H}_i(\epsilon) \) and \( \mathbf{Z}_i = \mathbf{D}_i(\epsilon), \quad 1 \leq i \leq 2 \). Analogously, \( \tilde{\mathbf{Y}} \) and \( \tilde{\mathbf{Z}} \) denote
\[
\tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}, \quad \tilde{\mathbf{Z}} = \begin{bmatrix} \tilde{\mathbf{Z}}_1 \\ \tilde{\mathbf{Z}}_2 \end{bmatrix} \in \mathbb{R}^2
\]
where the functions \( \tilde{\mathbf{Y}}_i = \tilde{\mathbf{H}}_i(\epsilon) \) and \( \tilde{\mathbf{Z}}_i = \tilde{\mathbf{D}}_i(\epsilon), \quad 1 \leq i \leq 2 \). Determine \( K^* \) such that it is the minimum,
\[
K^* = \arg \min_{K \in \bar{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}) \right) \right\}.
\]
Consider that the total time derivative above has the representation,
\[
\frac{d}{d\epsilon} \left( \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}) \right) = \frac{\partial \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \operatorname{vec}(\mathbf{Y})} \operatorname{vec}(\mathcal{F}(\mathbf{Y}, K))
+ \frac{\partial \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \operatorname{vec}(\tilde{\mathbf{Y}})} \operatorname{vec}(\mathcal{F}(\tilde{\mathbf{Y}}, \tilde{K}))
+ \frac{\partial \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \mathbf{Z}} \mathcal{G}(\mathbf{Y})
+ \frac{\partial \phi_{KLD}^{GDA}(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})}{\partial \mathbf{Z}} \mathcal{G}(\tilde{\mathbf{Y}})
\]
\[\tag{4.16}\]

112
The partial derivatives below have the alternative form via the chain rule,

\[
\frac{\partial \phi_{KLD}(Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y_1)} = \left( \frac{\kappa_1(Y_1, Z_1) - \tilde{\kappa}_1(\tilde{Y}_1, \tilde{Z}_1)}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} \right) \cdot \text{vec}(x_0 x_0^T)^T
\]

\[
\frac{\partial \phi_{KLD}(Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y_2)} = \frac{1}{2} \left( \frac{1}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} - \frac{1}{\kappa_2(Y_2, Z_2)} \right) \cdot \text{vec}(x_0 x_0^T)^T
\]

where the cost mean and variance, along with the target counterparts are

\[
\kappa_i(Y_i, Z_i) = x_0^T Y_i x_0 + Z_i, \quad \tilde{\kappa}_i(\tilde{Y}_i, \tilde{Z}_i) = x_0^T \tilde{Y}_i x_0 + \tilde{Z}_i, \quad 1 \leq i \leq 2.
\]

Insert (4.27) into (4.26), differentiate (4.26) with respect to \( K \), and set the resulting expression equal to a zero matrix with the appropriate dimension. This is the necessary condition for the expression to take an extremal value on the interior of its domain.

\[
-2 \left( \frac{\kappa_1(Y_1, Z_1) - \tilde{\kappa}_1(\tilde{Y}_1, \tilde{Z}_1)}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} \right) B^T(\epsilon) Y_1(x_0 x_0^T)
\]

\[
-2 \left( \frac{1}{2} \left( \frac{1}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} - \frac{1}{\kappa_2(Y_2, Z_2)} \right) \right) B^T(\epsilon) Y_2(x_0 x_0^T)
\]

\[
-2 \left( \frac{\kappa_1(Y_1, Z_1) - \tilde{\kappa}_1(\tilde{Y}_1, \tilde{Z}_1)}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} \right) R(\epsilon) K(x_0 x_0^T) = \mathbf{0}^{m \times n}
\]

Since \( x_0 x_0^T \) is a fixed rank-one matrix, the control below results when \( \kappa_1(Y_1, Z_1) \neq \tilde{\kappa}_1(\tilde{Y}_1, \tilde{Z}_1) \),

\[
K^*(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) = - R^{-1}(\epsilon) B^T(\epsilon) \left( Y_1 + \frac{1}{2} \left( \frac{1 - \tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)}{\kappa_2(Y_2, Z_2)} \right) \frac{\kappa_1(Y_1, Z_1) - \tilde{\kappa}_1(\tilde{Y}_1, \tilde{Z}_1)}{\tilde{\kappa}_2(\tilde{Y}_2, \tilde{Z}_2)} \right) Y_2. \quad (4.18)
\]
Let \( Y^\star \) and \( Z^\star \) denote

\[
Y^\star = \begin{bmatrix} Y^\star_1 \\ Y^\star_2 \end{bmatrix}, \quad Z^\star = \begin{bmatrix} Z^\star_1 \\ Z^\star_2 \end{bmatrix}
\]

\[
Y^\star_i = H^\star_i(\epsilon), \quad Z^\star_i = D^\star_i(\epsilon), \quad 1 \leq i \leq 2
\]

where the functions \( Y^\star_i \) and \( Z^\star_i \) satisfy the equations of motion under the control selection \( K^\star \),

\[
\frac{dH^\star_1(\tau)}{d\tau} = - (A(\tau) + B(\tau)K^\star(\tau))^T H^\star_1(\tau) - H^\star_1(\tau)(A(\tau) + B(\tau)K^\star(\tau)) - K^\star(\tau)R(\tau)K^\star(\tau) - Q(\tau)
\]

\[
\frac{dH^\star_2(\tau)}{d\tau} = - (A(\tau) + B(\tau)K^\star(\tau))^T H^\star_2(\tau) - H^\star_2(\tau)(A(\tau) + B(\tau)K^\star(\tau)) - 4H^\star_1(\tau)G(\tau)WG^T(\tau)H^\star_1(\tau)
\]

\[
\frac{dD^\star_i(\tau)}{d\tau} = - \text{Tr}(H^\star_i(\tau)G(\tau)WG^T(\tau)), \quad \tau \in [\epsilon, t_f]
\]

\[
H^\star_1(t_f) = Q_f, \quad H^\star_2(t_f) = 0, \quad D^\star_1(t_f) = 0, \quad D^\star_2(t_f) = 1
\]

It is assumed that \( Y^\star \) and \( Z^\star \) exist in the following. Using these dynamic programming variables under \( K^\star \), form the candidate value function,

\[
W(\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) = 2\phi^{GDA}_{KLD}(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) - \phi^{GDA}_{KLD}(H^\star(t_0), D^\star(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) + \phi^{GDA}_{KLD}(Y^\star, Z^\star, \tilde{Y}, \tilde{Z}) - \phi^{GDA}_{KLD}(Y, Z, \tilde{Y}, \tilde{Z}).
\]

The above function can be identified by integrating a determined differential equation over the space of displaced terminal conditions \((\epsilon, Y, Z, \tilde{Y}, \tilde{Z}) \in Q\), but the results have been omitted since this has already been shown in Chapter 3. Several
properties of this candidate value function are verified to show it is indeed the
value function.

First, observe that the Mayer-form boundary condition is satisfied when

\[(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \in \mathcal{M}\]

since \(\mathbf{y}^* = \mathbf{H}^*(t_0), \mathbf{z}^* = \mathbf{D}^*(t_0), \mathbf{\hat{y}} = \mathbf{\hat{H}}(t_0), \mathbf{\hat{z}} = \mathbf{\hat{D}}(t_0), \mathbf{y} = \mathbf{H}(t_0), \text{ and } \mathbf{z} = \mathbf{D}(t_0)\).

Using these relations in (4.30) yields the expected result (4.11).

The function \(\mathcal{W}(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) is constant for displaced terminal conditions
from anywhere along the trajectory of the state variables under the influence of
\(K^*\). This can be verified by observing that when \(\mathbf{y} = \mathbf{y}^*\) and \(\mathbf{z} = \mathbf{z}^*\), it follows
that \(\mathbf{H}(t_0) = \mathbf{H}^*(t_0)\) and \(\mathbf{D}(t_0) = \mathbf{D}^*(t_0)\) and thus

\[\mathcal{W}(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}}) = \phi^{GDA}_{KL}(\mathbf{H}^*(t_0), \mathbf{D}^*(t_0), \mathbf{\hat{H}}(t_0), \mathbf{\hat{D}}(t_0)).\]

Furthermore, (4.12) is satisfied since \(\mathcal{W}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) is non-increasing, forwards
in time, along \(\mathbf{y} \neq \mathbf{y}^*\) and \(\mathbf{z} \neq \mathbf{z}^*\),

\[\frac{d\mathcal{W}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})}{d\epsilon} = \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right) - \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right)
= \min_{K \in \bar{K}} \left\{ \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right) \right\} - \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right) \leq 0.\]

Finally, (4.13) is satisfied since the total time derivative of \(\mathcal{W}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\)
vanishes,

\[\min_{K \in \bar{K}} \left\{ \frac{d\mathcal{W}(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})}{d\epsilon} \right\} = \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right) - \min_{K \in \bar{K}} \left\{ \frac{d}{d\epsilon} \left(\phi^{GDA}_{KL}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\right) \right\} = 0.\]
Thus by the HJB Verification Lemma,

\[ \mathcal{W}(\epsilon, \mathbf{y}, \mathbf{Z}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}) = \mathcal{V}(\epsilon, \mathbf{y}, \mathbf{Z}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}). \]

and \( K^* \) is optimal.

\[ \square \]

4.2.4 Simulation Results

The AMD benchmark problem serves to validate the MKLD-CDS theory. The problem involves a 3-story test structure, that is subject to 1-dimensional ground motion in order to simulate the effects of seismic disturbances. Historical data from the El Centro and Hachinohe earthquake is used to excite the structure, along with a random process having the spectral density of the Kanai-Tajimi spectrum. For control purposes, a representative AMD has been deployed on the third story of the structure. It is comprised of a single hydraulic actuator with steel masses attached to the end of a piston rod. For further details on the experiment’s setup, system model, and controller evaluations, consult [27], [3].

The following computations use the system matrices for the reduced-order model for control design, and also the weighting matrices for the baseline LQG control. Using the perturbation constant \( \epsilon^* = 1.0 \times 10^{-9} \) and a perturbation matrix \( \mathbf{X}^* = 0^{10 \times 10} \), the 2CC control is computed

\[
\begin{align*}
\tilde{K}(\alpha) &= -R^{-1}(\alpha)B^T(\alpha) \left( \tilde{H}_1(\alpha) + \mu_2 \tilde{H}_2(\alpha) \right) \\
&= -R^{-1}(\alpha)B^T(\alpha) \left( \tilde{H}_1(\alpha) + \mu_2 \tilde{H}_2(\alpha) \right) \quad (4.21)
\end{align*}
\]

with \( \mu_2 = 1.0 \times 10^{-5} \) to compute a target cost mean \( \tilde{\kappa}_1(\alpha) \) and a target cost variance \( \tilde{\kappa}_2(\alpha) \). An attempt is made to realize these cost statistics using the MKLD-
TABLE 4.1

CONTROL LIMITS, LQG VS. 2CC VS. MKLD-CDS

<table>
<thead>
<tr>
<th>Limit</th>
<th>LQG</th>
<th>2CC</th>
<th>MKLD-CDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>max_u</td>
<td>0.5215</td>
<td>0.9946</td>
<td>1.0034</td>
</tr>
<tr>
<td>max_x</td>
<td>1.9963</td>
<td>3.5856</td>
<td>3.6164</td>
</tr>
<tr>
<td>max_a</td>
<td>4.6865</td>
<td>5.2860</td>
<td>5.3159</td>
</tr>
<tr>
<td>σ_u</td>
<td>0.1441</td>
<td>0.2336</td>
<td>0.2353</td>
</tr>
<tr>
<td>σ_x</td>
<td>1.0452</td>
<td>1.3073</td>
<td>1.3133</td>
</tr>
<tr>
<td>σ_a</td>
<td>0.6302</td>
<td>0.9433</td>
<td>0.9490</td>
</tr>
</tbody>
</table>

CDS paradigm. Thus numerical integration of (4.5) is performed using a fourth-order Runge-Kutta method, where these equations are under the influence of the control (4.15) with the aforementioned target statistics for the random cost functional.

The cost cumulants resultant from the MKLD-CDS control (4.15) using the target information indeed match the targets over the time horizon with less than 1% error as seen in Figure 4.2, (c). The plots of the time-varying optimal parameter

$$\gamma_2(\alpha) = \frac{1}{2} \left( \frac{1 - \tilde{\kappa}_2(\alpha)}{\kappa_2(\alpha)} \right)$$

in Figure 4.2, (b) show convergence to \(\mu_2\), which implies that there is a bijective relationship between the cost mean and variance and the 2CC linear state-feedback
controls used to drive the targets. The computed MKLD-CDS control produces the same performance in reducing the inter-story drifts and per-story accelerations, as seen in Figure 4.1 (\(J_1 - J_2\) and \(J_6 - J_7\)). This suggests that the performance of the controller is embedded in the target cost mean \(\tilde{\kappa}_1(\alpha)\) and the target cost variance \(\tilde{\kappa}_2(\alpha)\).

The determinant of the return difference matrix corresponding to the computed MKLD-CDS control has the same stability margins as seen for 2CC control, as seen from Figure 4.2, (a). Note also that the stability margins of the determinant of the return difference matrix for the MKLD-CDS control in Figure 4.2, (a) matches those of the return difference for the 2CC control. This suggests that robust stability properties of a controller are embedded within the cost cumulants it drives. The strong inferences of performance and stability properties of linear controllers being embedded in the resulting cost cumulants are made because the MKLD-CDS control is computed from the target cost mean \(\tilde{\kappa}_1(\alpha)\) and target cost variance \(\tilde{\kappa}_2(\alpha)\) alone, and it shown that the performance and stability features are preserved. Thus the statistical characterization of the cost via these lower-order cumulants is clearly linked to the performance and stability features of the 2CC controller.

Figure 4.3, (a) and (b), shows the structure displacements and accelerations respectively. It can be seen that the response reduction achieved by the MKLD-CDS control is greater than that achieved with the baseline LQG compensation. It further can be noted that the drifts for the MKLD-CDS and 2CC controls are approximately equal, and the same is evident for the per-story accelerations corresponding to each control. This confirms yet again that the statistical characterization of the random cost under the 2CC control, as described by \(\tilde{\kappa}_1(\alpha)\) and
Figure 4.1. Performance and Control Implementation Costs, MKLD-CDS Control

$\tilde{\kappa}_2(\alpha)$, contains enough information to re-construct the 2CC control, and reproduce its control performance and robust stability in the MKLD-CDS control.
(a) Return Difference, LQG vs. 2CC vs. MKLD-CDS

(b) Trajectory, Optimal Parameter

(c) Trajectories, Cost Cumulants

Figure 4.2. First-Generation Benchmark, Finite-Horizon MKLD-CDS Control, Application Results
Figure 4.3. First-Generation Benchmark, Per-Story Displacements and Per-Story Accelerations, MKLD-CDS vs. 2CC
4.3 Maximum Bhattacharyya Coefficient Cost Density-Shaping

4.3.1 Bhattacharyya Coefficient and Properties

**Definition 4.3.1 (Bhattacharyya Coefficient)**
The Bhattacharyya Coefficient (BC) between probability densities $p_1(x)$ and $p_2(x)$ with support on $\mathbb{R}$ is defined as

$$BC(p_1(x), p_2(x)) = \int_{-\infty}^{\infty} \sqrt{p_1(x)p_2(x)} dx.$$  \hfill (4.22)

**Theorem 4.3.2 (BC, between Gaussian Approximations)**
The BC between $p_Z(z)$ and $\tilde{p}_Z(\tilde{z})$, as defined previously, can be written in terms of $\kappa_1$, $\kappa_2$, $\tilde{\kappa}_1$, and $\tilde{\kappa}_2$ as below,

$$BC(p_Z(z), \tilde{p}_Z(\tilde{z})) = \sqrt{2} \cdot \frac{(\kappa_2(t_0)\tilde{\kappa}_2(t_0))^{\frac{1}{4}}}{\sqrt{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)}} \cdot \exp\left(-\frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{4(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))}\right).$$ \hfill (4.23)

**Proof** The BC is

$$BC(p_Z(z), \tilde{p}_Z(\tilde{z})) = \int_{-\infty}^{\infty} \sqrt{p_Z(z) \cdot \tilde{p}_Z(\tilde{z})} dz$$

$$= \int_{-\infty}^{\infty} \sqrt{p_Z(z) \cdot \alpha p_Z(\alpha z + \beta)} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \cdot \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{(\alpha z + \beta)^2}{2}\right) dz$$

$$= \int_{-\infty}^{\infty} \frac{\alpha}{2\pi} \exp\left(-\frac{z^2}{4}\right) \cdot \exp\left(-\frac{(\alpha z + \beta)^2}{4}\right) dz$$

$$= \int_{-\infty}^{\infty} \frac{\alpha}{2\pi} \exp\left(-\frac{(\alpha^2 + 1)z^2 + 2\alpha\beta z + \beta^2}{4}\right) dz$$

The argument of the exponential in the last line can be expressed in an alternative form by completing the square as follows.

$$-\frac{(\alpha^2 + 1)z^2 + 2\alpha\beta z + \beta^2}{4} = -\frac{(\sqrt{\alpha^2 + 1}z + \frac{\beta}{\sqrt{\alpha^2 + 1}})^2 - \frac{\beta^2}{\alpha^2 + 1}}{4}$$

$$= -\frac{(\alpha\sqrt{\frac{\alpha^2 + 1}{2}} + \frac{\beta}{\sqrt{2(\alpha^2 + 1)}})^2}{2} - \frac{\beta^2}{4(\alpha^2 + 1)}$$

122
This enables the exponential to be expressed in a more convenient way.

\[
\exp\left(-\frac{(\alpha^2 + 1)z^2 + 2\alpha\beta z + \beta^2}{4}\right) = \exp\left(-\left(\frac{\sqrt{\alpha^2 + 1}}{4}z + \frac{\alpha\beta}{\sqrt{\alpha^2 + 1}}\right)^2 - \frac{\beta^2}{\alpha^2 + 1}\right)
\]

\[
= \exp\left(-\left(\frac{\sqrt{\alpha^2 + 1}}{2}z + \frac{\alpha\beta}{\sqrt{2(\alpha^2 + 1)}}\right)^2 - \frac{\beta^2}{4(\alpha^2 + 1)}\right)
\]

Inserting the new expression into the integral makes it clear that,

\[
BC(p_Z(z), p_{\tilde{Z}}(\tilde{z})) = \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{(\alpha^2 + 1)z^2 + 2\alpha\beta z + \beta^2}{4}\right) dz
\]

\[
= \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\left(\frac{\sqrt{\alpha^2 + 1}}{2}z + \frac{\alpha\beta}{\sqrt{2(\alpha^2 + 1)}}\right)^2 - \frac{\beta^2}{4(\alpha^2 + 1)}\right) \cdot \exp\left(-\frac{\beta^2}{4(\alpha^2 + 1)}\right) dz
\]

\[
= \sqrt{\alpha} \cdot \exp\left(-\frac{\beta^2}{4(\alpha^2 + 1)}\right) \cdot \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\sqrt{\frac{\alpha^2 + 1}{2}}z + \frac{\alpha\beta}{\sqrt{2(\alpha^2 + 1)}})^2}{2}\right) dz\right)
\]

\[
= \sqrt{\frac{2\alpha}{\alpha^2 + 1}} \cdot \exp\left(-\frac{\beta^2}{4(\alpha^2 + 1)}\right) \cdot \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\sqrt{\frac{\alpha^2 + 1}{2}} z + \frac{\alpha\beta}{\sqrt{2(\alpha^2 + 1)}})^2}{2}\right) \left(\sqrt{\frac{\alpha^2 + 1}{2}}\right) dz\right)
\]

The integral in the last line above can be simplified using the following substitution,

\[
u = \left(\sqrt{\frac{\alpha^2 + 1}{2}} z + \frac{\alpha\beta}{\sqrt{2(\alpha^2 + 1)}}\right), \quad du = \left(\sqrt{\frac{\alpha^2 + 1}{2}}\right) dz
\]
which leads immediately to

\[
BC(p_Z(z), p_{\tilde{Z}}(\tilde{z})) = \sqrt{\frac{2\alpha}{\alpha^2 + 1}} \cdot \exp \left( \frac{-\beta^2}{4(\alpha^2 + 1)} \right) \cdot \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\alpha^2 z - \alpha \beta \sqrt{2(\alpha^2 + 1)}}{2} \right) \left( \sqrt{\frac{\alpha^2 + 1}{2}} \right) dz \right)
\]

\[
= \sqrt{\frac{2\alpha}{\alpha^2 + 1}} \cdot \exp \left( \frac{-\beta^2}{4(\alpha^2 + 1)} \right) \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2}{2} \right) du \right)
\]

\[
= \sqrt{\frac{2\alpha}{\alpha^2 + 1}} \cdot \exp \left( \frac{-\beta^2}{4(\alpha^2 + 1)} \right)
\]

\[
= \sqrt{2} \cdot \frac{(\kappa_2 \tilde{\kappa}_2)^{1/4}}{\sqrt{\kappa_2 + \tilde{\kappa}_2}} \cdot \exp \left( -\frac{(\kappa_1 - \tilde{\kappa}_1)^2}{4(\kappa_2 + \tilde{\kappa}_2)} \right).
\]

\[\square\]

Before proceeding to the MBC-CDS problem formulation, a few properties of the BC are established below that ensure well-posedness of the ensuing optimization problem.

The function derived above is denoted as \( g^{GDA}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)) \). It can be shown that the function \( g^{GDA}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)) \) has desirable properties that make this function well-suited for the MBC-CDS problem. Some of these properties are explored here, and a few associated theorems are presented. In the following, the cumulant vector designations \( \kappa(t_0) \) and \( \tilde{\kappa}(t_0) \) are replaced with general vector arguments are considered

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.
\]

For fixed \( \tilde{x}_1 \geq 0 \) and \( \tilde{x}_2 > 0 \), the value \( g^{GDA}_{BC}(\mathbf{x}, \tilde{\mathbf{x}}) \) is real and exists for values \( 0 < x_2 < \infty \) and \( 0 \leq x_1 < \infty \). This region is an open set of contiguous values in \( \mathbb{R}^2 \),

\[
\text{dom} \ g^{GDA}_{BC} = [0, \infty) \times (0, \infty) \times [0, \infty) \times (0, \infty).
\]
Theorem 4.3.3 (Non-Negativeness)
The function $g^{\text{GDA}}_{\text{BC}}(x, \bar{x})$ is nonnegative on $\text{dom} g^{\text{GDA}}_{\text{BC}}$ and is such that
\[ 0 \leq g^{\text{GDA}}_{\text{BC}}(x, \bar{x}) \leq 1. \]

Proof It is not difficult to establish upper and lower bounds for the BC. For measures $p_1(x)$ and $p_2(x)$ the lower bound is apparent,
\[ BC(p_1(x), p_2(x)) = \int_{-\infty}^{\infty} \sqrt{p_1(x)p_2(x)} \, dx \geq 0. \]

Use the general arithmetic-geometric mean inequality for $a, b \geq 0$ and $0 \leq \theta \leq 1$,
\[ a^\theta \cdot b^{1-\theta} \leq \theta a + (1-\theta)b \]
to write
\[ \int_{-\infty}^{\infty} p_1(x)^\theta \cdot p_2(x)^{1-\theta} \, dx \leq \theta \int_{-\infty}^{\infty} p_1(x) \, dx + (1-\theta) \int_{-\infty}^{\infty} p_2(x) \, dx = \theta + (1-\theta) = 1. \]

Choosing $\theta = \frac{1}{2}$ gives
\[ 0 \leq \int_{-\infty}^{\infty} \sqrt{p_1(x)p_2(x)} \, dx = BC(p_1(x), p_2(x)) \leq 1. \]

Theorem 4.3.4 (Maximum at Target)
The function $g^{\text{GDA}}_{\text{BC}}(x, \bar{x})$ is such that $g^{\text{GDA}}_{\text{BC}}(\bar{x}, \bar{x}) = 1$.

Proof Evaluate the $g^{\text{GDA}}_{\text{BC}}(x, \bar{x})$ for $x = \bar{x}$; it can easily be verified that
\[ g^{\text{GDA}}_{\text{BC}}(\bar{x}, \bar{x}) = 1. \]
4.3.2 MBC-CDS Problem Formulation

The optimization problem can be formulated by defining a control space over which the MBC-CDS performance index can be minimized. This can be characterized with the target set and also the space of admissible feedback gains as defined beforehand. The function (4.23) is now used as the performance index for the MBC-CDS optimization.

**Definition 4.3.5 (MBC-CDS Performance Index)**

Let the MBC-CDS Performance index be defined as the function

\[ \phi_{BC}^{GDA}(H(t_0), D(t_0), \hat{H}(t_0), \hat{D}(t_0)) = -g_{BC}^{GDA}(\kappa(t_0), \hat{\kappa}(t_0)). \]

The MBC-CDS performance index has previously been shown to be negative and bounded below, with a minimum at \( \hat{\alpha} \). As such, it is well-suited for the objective function in the following MBC-CDS optimization problem.

**Definition 4.3.6 (MBC-CDS Optimization)**

The MBC-GDA optimization can be formulated as,

\[
\min_{K \in K(t_f)} \phi_{BC}^{GDA}(H(t_0), D(t_0), \hat{H}(t_0), \hat{D}(t_0))
\]

subject to:

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{d\hat{H}(\alpha)}{d\alpha} = F(\hat{H}(\alpha), \hat{K}(\alpha))
\]

\[
\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\hat{D}(\alpha)}{d\alpha} = G(\hat{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
H(t_f) = H_f, \quad \hat{H}(t_f) = H_{f, \hat{\alpha}^*}, \quad D(t_f) = D_f, \quad \hat{D}(t_f) = D_{f, \hat{\alpha}^*}.
\]

Observing the optimization posed above, dynamic programming solution can be used to derive the MBC-CDS optimal control in the next section.
4.3.3 MBC-CDS Solution via Dynamic Programming

If a candidate value function can be proposed that satisfies a Mayer-form boundary condition, in addition to satisfying the HJB equations of dynamic programming shown in the lemma for some nominal control $K^*$, then that candidate value function agrees with the value function along trajectories of the state variables corresponding to any linear control and the control $K^*$ is optimal. This essentially is the provision of the HJB Verification lemma, which is used to establish the main result stated below. The proof follows.

**Theorem 4.3.7 (State-Feedback Solution, MBC-CDS)**

Consider the MBC-CDS optimization involving the process having dynamics (4.1) and the cost (4.3). Then the linear state-feedback, finite-horizon, optimal control solution to the MBC-CDS optimization is characterized by the optimal gain

$$
K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\left( H_1^*(\alpha) + \gamma_{BC}^{GDA}(\alpha)H_2^*(\alpha) \right)
$$

(4.24)

where the optimal parameter is

$$
\gamma_{BC}^{GDA}(\alpha) = \left( \frac{1}{2} \cdot \frac{(1 - \tilde{\kappa}_2(\alpha)) - \left( \frac{\tilde{\kappa}_1(\alpha) - \tilde{\kappa}_1(\alpha)}{\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha)} \right)^2}{\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha)} \right)
$$

(4.25)

the optimal cost mean ($i = 1$) and variance ($i = 2$) are defined by

$$
\kappa_i^*(\alpha) = x_0^T H_i^*(\alpha)x_0 + D_i^*(\alpha)
$$

and target cost mean ($i = 1$) and the target cost variance ($i = 2$) are given by

$$
\tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha).
$$

The optimal state variables $H^*(\alpha), D^*(\alpha)$ and $\tilde{H}(\alpha), \tilde{D}(\alpha)$ follow the equations of motion

$$
\frac{dH^*(\alpha)}{d\alpha} = F(H^*(\alpha), K^*(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha))
$$

$$
\frac{dD^*(\alpha)}{d\alpha} = G(H^*(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
$$
\(H^*(t_f) = H_f, \ \bar{H}(t_f) = H_{f,e}, \ D^*(t_f) = D_f, \ \bar{D}(t_f) = D_{f,e} \).

**Proof** In the following, let \(\mathbf{y}\) and \(\mathbf{z}\) denote

\[
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}, \ \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^r
\]

where the functions \(y_i = H_i(\epsilon)\) and \(z_i = D_i(\epsilon), 1 \leq i \leq 2\). Analogously, \(\hat{\mathbf{y}}\) and \(\hat{\mathbf{z}}\) denote

\[
\hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}, \ \hat{\mathbf{z}} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} \in \mathbb{R}^2
\]

where the functions \(\hat{y}_i = \bar{H}_i(\epsilon)\) and \(\hat{z}_i = \bar{D}_i(\epsilon), 1 \leq i \leq 2\). Determine \(K^*\) such that it is the minimum,

\[
K^* = \arg \min_{K \in \overline{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \right) \right\}.
\]

Consider that the total time derivative above has the representation,

\[
\frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \right) = \frac{\partial \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}})}{\partial \text{vec}(\mathbf{y})} \text{vec}(F(\mathbf{y}, K)) + \frac{\partial \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}})}{\partial \text{vec}(\mathbf{y})} \text{vec}(F(\hat{\mathbf{y}}, \hat{K})) + \frac{\partial \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}})}{\partial \mathbf{z}} g(\mathbf{y}) + \frac{\partial \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \hat{\mathbf{y}}, \hat{\mathbf{z}})}{\partial \mathbf{z}} g(\hat{\mathbf{y}})
\]

(4.26)
The partial derivatives below have the alternative form via the chain rule,

\[
\frac{\partial \phi^{GDA}_{BC}(Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y_1)} = \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_1(Y_1, Z_1)} \right) \left( \frac{\partial \kappa_1(Y_1, Z_1)}{\partial \text{vec}(Y_1)} \right) \\
= \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_1(Y_1, Z_1)} \right) \text{vec}(x_0x_0^T)^T
\]

\[
\text{(4.27)}
\]

\[
\frac{\partial \phi^{GDA}_{BC}(Y, Z, \tilde{Y}, \tilde{Z})}{\partial \text{vec}(Y_2)} = \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_2(Y_2, Z_2)} \right) \left( \frac{\partial \kappa_2(Y_2, Z_2)}{\partial \text{vec}(Y_2)} \right) \\
= \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_2(Y_2, Z_2)} \right) \text{vec}(x_0x_0^T)^T
\]

where the cost mean and variance, along with the target counterparts are

\[
\kappa_i(Y_i, Z_i) = x_0^T Y_i x_0 + Z_i
\]

\[
\tilde{\kappa}_i(\tilde{Y}_i, \tilde{Z}_i) = x_0^T \tilde{Y}_i x_0 + \tilde{Z}_i, \ 1 \leq i \leq 2.
\]

Insert (4.27) into (4.26), differentiate (4.26) with respect to \( K \), and set the resulting expression equal to a zero matrix with the appropriate dimension. This is the necessary condition for the expression to take an extremal value on the interior of its domain.

\[
-2 \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_1(Y_1, Z_1)} \right) B^T(\epsilon) Y_1(x_0x_0^T) - 2 \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_2(Y_2, Z_2)} \right) B^T(\epsilon) Y_2(x_0x_0^T)
\]

\[
-2 \left( \frac{\partial g^{GDA}_{BC}(\cdot)}{\partial \kappa_1(Y_1, Z_1)} \right) R(\epsilon) K(x_0x_0^T) = 0^{m \times n}
\]

Since \( x_0x_0^T \) is a fixed rank-one matrix, the control below results when \( \kappa_1(Y_1, Z_1) \neq \)
\[ K^*(\epsilon, Y, Z, \hat{Y}, \hat{Z}) = -R^{-1}(\epsilon)B^T(\epsilon) \begin{pmatrix} Y_1 + \frac{\partial \phi^GDA(\cdot)}{\partial \phi^GDA(\cdot)} Y_2 \end{pmatrix} \]  

(4.28)

Let \( Y^* \) and \( Z^* \) denote

\[ Y_i^* = H_i^*(\epsilon), \quad Z_i^* = D_i^*(\epsilon), \quad 1 \leq i \leq 2 \]

where the functions \( H_i^*(\epsilon) \) and \( D_i^*(\epsilon) \) satisfy the equations of motion under the control selection \( K^* \),

\[
\begin{align*}
\frac{dH_i^*(\tau)}{d\tau} &= - (A(\tau) + B(\tau)K^*(\tau))^T H_i^*(\tau) - H_i^*(\tau)(A(\tau) + B(\tau)K^*(\tau)) \\
&\quad - K^{*T}(\tau)R(\tau)K^*(\tau) - Q(\tau) \\
\frac{dH_2^*(\tau)}{d\tau} &= - (A(\tau) + B(\tau)K^*(\tau))^T H_2^*(\tau) - H_2^*(\tau)(A(\tau) + B(\tau)K^*(\tau)) \\
&\quad - 4H_1^*(\tau)G(\tau)WG^T(\tau)H_1^*(\tau) \\
\frac{dD_i^*(\tau)}{d\tau} &= - \text{Tr}(H_i^*(\tau)G(\tau)WG^T(\tau)), \quad \tau \in [\epsilon, t_f] \\
H_1^*(t_f) &= Q_f, \quad H_2^*(t_f) = 0^{n \times n}, \quad D_1^*(t_f) = 0, \quad D_2^*(t_f) = 1.
\end{align*}
\]

(4.29)

It is assumed that \( Y^* \) and \( Z^* \) exist in the following. Using these dynamic programming variables under \( K^* \), consider now the candidate value function,

\[
\mathcal{W}(\epsilon, Y, Z, \hat{Y}, \hat{Z}) = 2\phi^GDA(H(t_0), D(t_0), \hat{H}(t_0), \hat{D}(t_0)) \\
- \phi^GDA(H^*(t_0), D^*(t_0), \hat{H}(t_0), \hat{D}(t_0))
\]

(4.30)
\[ + \phi_{BC}^{GDA}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}}) - \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}). \]

Observe first that the Mayer-form boundary condition is satisfied when

\[(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \in \mathcal{M}\]

since it follows that \(\mathbf{y}^* = \mathbf{H}^*(t_0), \mathbf{z}^* = \mathbf{D}^*(t_0), \mathbf{\hat{y}} = \mathbf{\hat{H}}(t_0), \mathbf{\hat{z}} = \mathbf{\hat{D}}(t_0), \mathbf{y} = \mathbf{H}(t_0), \) and \(\mathbf{z} = \mathbf{D}(t_0).\) Using these relations in (4.30) yields the expected result (4.11).

The function \(W(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) is constant for displaced terminal conditions gotten from anywhere along the trajectory of the state variables under the influence of \(K^*.\) This can be verified by observing that when \(\mathbf{y} = \mathbf{y}^*\) and \(\mathbf{z} = \mathbf{z}^*\), it follows that \(\mathbf{H}(t_0) = \mathbf{H}^*(t_0)\) and \(\mathbf{D}(t_0) = \mathbf{D}^*(t_0)\) so that

\[W(\epsilon, \mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}}) = \phi_{BC}^{GDA}(\mathbf{H}^*(t_0), \mathbf{D}^*(t_0), \mathbf{\hat{H}}(t_0), \mathbf{\hat{D}}(t_0)).\]

Furthermore, (4.12) is satisfied since \(W(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) is non-increasing along \(\mathbf{y} \neq \mathbf{y}^*\) and \(\mathbf{z} \neq \mathbf{z}^*\),

\[
\frac{dW(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})}{d\epsilon} = \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) - \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) \\
= \min_{K \in \mathcal{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) \right\} - \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) \leq 0.
\]

Finally, (4.13) is satisfied since the total time derivative of \(W(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) vanishes,

\[
\min_{K \in \mathcal{K}} \left\{ \frac{dW(\epsilon, \mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}})}{d\epsilon} \right\} = \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}^*, \mathbf{z}^*, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) - \min_{K \in \mathcal{K}} \left\{ \frac{d}{d\epsilon} \left( \phi_{BC}^{GDA}(\mathbf{y}, \mathbf{z}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \right) \right\} = 0.
\]
Thus by the HJB Verification Lemma,

\[ W(\epsilon, y, z, \dot{y}, \dot{z}) = V(\epsilon, y, z, \dot{y}, \dot{z}). \]

and \( K^* \) is optimal.

\[ \square \]

4.3.3.1 MBC-CDS Validation, First-Generation Benchmark

4.3.4 Simulation Results

The AMD benchmark problem serves to validate the MBC-CDS theory. The problem involves a 3-story test structure, that is subject to 1-dimensional ground motion in order to simulate the effects of seismic disturbances. Historical data from the El Centro and Hachinohe earthquake is used to excite the structure, along with a random process having the spectral density of the Kanai-Tajimi spectrum. For control purposes, a representative AMD has been deployed on the third story of the structure. It is comprised of a single hydraulic actuator with steel masses attached to the end of a piston rod. For further details on the experiment’s setup, the system model, and controller evaluations, consult [27], [3].

The following computations use the system matrices for the reduced-order model for control design, and also the weighting matrices for the baseline LQG control. Using the perturbation constant \( \epsilon^* = 1.0 \times 10^{-9} \) and a perturbation matrix \( \mathcal{E}^* = 0^{10 \times 10} \), the 2CC control is computed

\[ \tilde{K}(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( \tilde{H}_1(\alpha) + \mu_2 \tilde{H}_2(\alpha) \right) \quad (4.31) \]
with $\mu_2 = 1.0 \times 10^{-5}$ to compute a target cost mean $\tilde{\kappa}_1(\alpha)$ and a target cost variance $\tilde{\kappa}_2(\alpha)$. An attempt is made to realize these cost statistics using the MBC-CDS paradigm. Thus numerical integration of (4.5) is performed using a fourth-order Runge-Kutta method, where these equations are under the influence of the control (4.24) with the aforementioned target statistics for the random cost functional.

The cost cumulants resultant from the MBC-CDS control (4.24) using the target information indeed match the targets over the time horizon with less than 1% error as seen in Figure 4.6, (c). The plots of the time-varying optimal parameter

$$
\gamma_2(\alpha) = \left( 1 + \frac{1}{2} \left( \frac{1 - \tilde{\kappa}_2(\alpha)}{\kappa_2(\alpha)} - \frac{(\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha))^2}{\kappa_2(\alpha) + \tilde{\kappa}_2(\alpha)} \right) \right)
$$

(4.32)
in Figure 4.6, (b) show convergence to $\mu_2$, which implies that there is a bijective relationship between the cost mean and variance and the 2CC linear state-feedback controls used to drive the targets. The computed MBC-CDS control produces the same performance in reducing the inter-story drifts and per-story accelerations, as seen in Figure 4.4 ($J_1 - J_2$ and $J_6 - J_7$). This suggests that the performance of the controller is embedded in the target cost mean $\tilde{\kappa}_1(\alpha)$ and the target cost variance $\tilde{\kappa}_2(\alpha)$.

The determinant of the return difference matrix corresponding to the computed MBC-CDS control has the same stability margins as seen for 2CC control, as seen from Figure 4.6, (a). Note also that the stability margins of the determinant of the return difference matrix for the MBC-CDS control in Figure 4.6, (a) matches those of the return difference for the 2CC control. This suggests that robust stability properties of a controller are embedded within the cost cumulants it drives. The
strong inferences of performance and stability properties of linear controllers being embedded in the resulting cost cumulants are made because the MBC-CDS control is computed from the target cost mean $\tilde{\kappa}_1(\alpha)$ and target cost variance $\tilde{\kappa}_2(\alpha)$ alone, and it shown that the performance and stability features are preserved. Thus the statistical characterization of the cost via these lower-order cumulants is clearly linked to the performance and stability features of the 2CC controller.

Figure 4.5, (a) and (b), shows the structure displacements and accelerations respectively. It can be seen that the response reduction achieved by the MBC-CDS control is greater than that achieved with the baseline LQG compensation. It further can be noted that the drifts for the MBC-CDS and 2CC controls are approximately equal, and the same is evident for the per-story accelerations corresponding to each control. This confirms yet again that the statistical characterization of the random cost under the 2CC control, as described by $\tilde{\kappa}_1(\alpha)$ and $\tilde{\kappa}_2(\alpha)$, contains enough information to re-construct the 2CC control, and reproduce its control performance and robust stability in the MBC-CDS control.
TABLE 4.2

CONTROL LIMITS, LQG VS. 2CC VS. MBC-CDS

<table>
<thead>
<tr>
<th>Limit</th>
<th>LQG</th>
<th>2CC</th>
<th>MBC-CDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>max_t u</td>
<td>0.5215</td>
<td>0.9946</td>
<td>1.0049</td>
</tr>
<tr>
<td>max_t x_m</td>
<td>1.9963</td>
<td>3.5856</td>
<td>3.6224</td>
</tr>
<tr>
<td>max_t a_m</td>
<td>4.6865</td>
<td>5.2860</td>
<td>5.3218</td>
</tr>
<tr>
<td>σ_u</td>
<td>0.1441</td>
<td>0.2336</td>
<td>0.2356</td>
</tr>
<tr>
<td>σ_x_m</td>
<td>1.0452</td>
<td>1.3073</td>
<td>1.3145</td>
</tr>
<tr>
<td>σ_a_m</td>
<td>0.6302</td>
<td>0.9433</td>
<td>0.9501</td>
</tr>
</tbody>
</table>

Figure 4.4. Performance and Control Implementation Costs, MBC-CDS Control
(a) Per-story Displacement (Hachinohe)

(b) Per-Story Acceleration (Hachinohe)

Figure 4.5. First-Generation Benchmark, Per-Story Displacements and Per-Story Accelerations MBC-CDS vs. 2CC
Figure 4.6. First-Generation Benchmark, Finite-Horizon MBC-CDS Control, Application Results
4.4 Hellinger and Bhattacharyya Distance Cost Density-Shaping

The (squared) Hellinger Distance (HD) and the Bhattacharyya Distance (BD) are two probability distance functions that actually lead to the same cost density-shaping controllers as derived when maximizing the Bhattacharyya Coefficient (BC). The reason for this is straightforward, given that both HD and BD are functions of the BC such that when their ratio of partial derivations (with respect to the cumulants) is taken, a cancellation occurs and produces the same time-varying optimal parameter as the MBC-CDS problem.

This section is dedicated to establishing the aforementioned fact. Before demonstrating the proof, it is important to first provide the definition of the HD and BD measures, and their cumulant representations when considering the normalized cost and target cost variates (4.10). Also, several properties of the HD and BD are discussed, and from them it follows immediately that the BC is concave and therefore the MBC-CDS maximization problem is well-posed.

4.4.1 Hellinger Distance and Properties

**Definition 4.4.1 (Hellinger Distance)**
The Hellinger-Distance (HD) between probability densities $p_1(x)$ and $p_2(x)$ with support on $\mathbb{R}$ is defined as,

$$HD(p_1(x), p_2(x)) = \sqrt{\frac{1}{2} \int_{-\infty}^{\infty} (\sqrt{p_1(x)} - \sqrt{p_2(x)})^2 dx}.$$  \hspace{1cm} (4.33)

**Theorem 4.4.2 (HD, between Gaussian Approximations)**
The HD between $p_Z(z)$ and $\tilde{p}_Z(\tilde{z})$, as defined previously, can be written in terms of $\kappa_1(t_0)$, $\kappa_2(t_0)$, $\tilde{\kappa}_1(t_0)$, and $\tilde{\kappa}_2(t_0)$ as below,

$$HD(p_Z(z), \tilde{p}_Z(\tilde{z})) = g_{HD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0)) = \sqrt{1 - \sqrt{2} \cdot \frac{(\kappa_2(t_0)\tilde{\kappa}_2(t_0))^\frac{3}{2}}{\sqrt{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)}} \cdot \exp \left( -\frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{4(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))} \right)}.$$  \hspace{1cm} (4.34)
Proof A direct expansion of the HD gives

$$HD(p_1(x), p_2(x)) = \sqrt{\frac{1}{2} \int_{-\infty}^{\infty} (\sqrt{p_1(x)} - \sqrt{p_2(x)})^2 dx}$$

$$= \sqrt{\frac{1}{2} \left( \int_{-\infty}^{\infty} p_1(x) dx - 2 \int_{-\infty}^{\infty} \sqrt{p_1(x)} p_2(x) dx + \int_{-\infty}^{\infty} p_2(x) dx \right)}$$

$$= \sqrt{\frac{1}{2} \left( 2 - 2 \int_{-\infty}^{\infty} \sqrt{p_1(x)} p_2(x) dx \right)}$$

$$= \sqrt{1 - \int_{-\infty}^{\infty} \sqrt{p_1(x)} p_2(x) dx}$$

$$= \sqrt{1 - BC(p_1(x), p_2(x))}.$$

Insert the previously-derived form of the Bhattacharyya Coefficient (BC), the function $g_{GDA}^{BC}(\kappa(t_0), \tilde{\kappa}(t_0))$, in the above expression. The proof is complete.

It can be shown that the function $g_{HD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0))$ has desirable properties that make it well-suited for optimization. Some of these properties are explored here, and a few associated theorems are presented. On occasion, the function $g_{HD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0)) = (g_{HD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0)))^2$ will be referred to. In the following, the cumulant vector designations $\kappa(t_0)$ and $\tilde{\kappa}(t_0)$ are dropped, and general vector arguments are considered

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

For fixed $\tilde{x}_1 \geq 0$ and $\tilde{x}_2 > 0$, the value $g_{HD}^{GDA}(x, \tilde{x})$ is real and exists for values $0 < x_2 < \infty$ and $0 \leq x_1 < \infty$. This region is an open set of contiguous values in $\mathbb{R}^2$. Accounting for these remarks, denote
Theorem 4.4.3 (Non-Negativeness)
The function $g_{\text{HD}}^{\text{GDA}}(x, \bar{x})$ is nonnegative on $\text{dom} \ g_{\text{HD}}^{\text{GDA}}$ and is such that

$$0 \leq g_{\text{HD}}^{\text{GDA}}(x, \bar{x}) \leq 1.$$ 

Proof

Use the fact that

$$0 \leq BC(p_1(x), p_2(x)) \leq 1.$$ (4.35)

The non-negativity result is immediate.

\[ \square \]

Theorem 4.4.4 (Minimum at Target)
The function $g_{\text{HD}}^{\text{GDA}}(x, \bar{x})$ is such that $g_{\text{HD}}^{\text{GDA}}(\bar{x}, \bar{x}) = 0$.

Proof

It is apparent that,

$$\text{HD}(p_1(x), p_1(x)) = 0.$$ (4.36)

It is not difficult to verify $x = \bar{x}$ means $p_1(x) = p_2(x)$ and so $g_{\text{HD}}^{\text{GDA}}(\bar{x}, \bar{x}) = 0$.

\[ \square \]

4.4.2 Bhattacharyya Distance and Properties

Definition 4.4.5 (Bhattacharyya Distance)
The Bhattacharyya-Distance (BD) between probability densities $p_1(x)$ and $p_2(x)$ with support on $\mathbb{R}$ is defined as

$$BD(p_1(x), p_2(x)) = -\ln \left( \int_{-\infty}^{\infty} \sqrt{p_1(x) \cdot p_2(x)} \, dx \right).$$ (4.37)

The BD fits the $f$-divergence form for the functions $g(x) = -\log(-x)$ and $f(x) = -\sqrt{x}$.
\textbf{Theorem 4.4.6} \textit{(BD, between Gaussian Approximations)}

The BD between $p_Z(z)$ and $p_{\tilde{Z}}(\tilde{z})$, as defined previously, can be written in terms of $\kappa_1$, $\kappa_2$, $\tilde{\kappa}_1$, and $\tilde{\kappa}_2$ as below:

$$
BD(p_Z(z), p_{\tilde{Z}}(\tilde{z})) = -\ln(\sqrt{2}) - \frac{1}{2} \ln \left( \frac{(\kappa_2(t_0)\tilde{\kappa}_2(t_0))}{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)} \right) + \frac{1}{4} \left( \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))} \right).
$$

(4.38)

\textbf{Proof} Negate the logarithm of $g_{BD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0))$, then expand it using the rule that the logarithm of products is equal to the product of logarithms for each product; the result is immediate.

\hfill \Box

A few properties of the BD are presented below that make it well-suited for optimization. The function derived above is denoted as $g_{BD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0))$. Some of these properties are explored here, and present a few associated theorems. In the following, the cumulant vector designations $\kappa$ and $\tilde{\kappa}$ are dropped and only general vector arguments are considered

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.
$$

For fixed $\tilde{x}_1 \geq 0$ and $\tilde{x}_2 > 0$, the value $g_{BD}^{GDA}(x, \tilde{x})$ is real and exists for values $0 < x_2 < \infty$ and $0 \leq x_1 < \infty$. This region is an open set of contiguous values in $\mathbb{R}^2$.

Accounting for these remarks, write

$$
\text{dom } g_{BD}^{GDA} = [0, \infty) \times (0, \infty) \times [0, \infty) \times (0, \infty).
$$

\textbf{Theorem 4.4.7} \textit{(Non-Negativeness)}

The function $g_{BD}^{GDA}(x, \tilde{x})$ is nonnegative on $\text{dom } g_{BD}^{GDA}$ and is such that

$$
0 \leq g_{BD}^{GDA}(x, \tilde{x}) \leq \infty
$$

141
Proof Write

\[ BD(p_1(x), p_2(x)) = \ln \left( \frac{1}{BC(p_1(x), p_2(x))} \right), \quad BC(p_1(x), p_2(x)) \neq 0 \]

and note that when \( p_1(x) = p_2(x) \), \( BC(p_1(x), p_2(x)) = 1 \) so that \( BD(p_1(x), p_2(x)) = 0 \). This is the lower bound. Note that if

\[ BC(p_1(x), p_2(x)) \to 0, \text{ it follows } BD(p_1(x), p_2(x)) \to \infty. \tag{4.39} \]

\[ \square \]

**Theorem 4.4.8 (Minimum at Target)**
The function \( g^{GDA}_{BD}(\tilde{x}, \tilde{x}) \) is such that \( g^{GDA}_{BD}(\tilde{x}, \tilde{x}) = 0 \).

Proof When \( p_1(x) = p_2(x) \), it is true that \( BD(p_1(x), p_2(x)) = 0 \) and \( x = \tilde{x} \) because the densities are Gaussian. Clearly then, \( g^{GDA}_{BD}(\tilde{x}, \tilde{x}) = 0 \).

\[ \square \]

### 4.4.3 Convexity and the Well-Posedness of MBC-CDS

Convexity is now established for the \( g^{GDA}_{BD}(\tilde{x}, \tilde{x}) \), \( g^{GDA}_{HD_2}(\tilde{x}, \tilde{x}) \) distance measures with the \( \alpha \)-divergence, which is a well-known generalization of probability distance functions.

**Definition 4.4.9 \( \alpha \)-divergence**
The \( \alpha \)-divergence between densities \( p_1(x) \) and \( p_2(x) \) on \( \Omega \) is defined as,

\[ D_\alpha(p_1(x), p_2(x)) = \frac{1}{\alpha(1-\alpha)} \int_{\Omega} \left( \alpha p_1(x) + (1-\alpha)p_2(x) - \alpha p_1^\alpha(x)p_2^{1-\alpha}(x) \right) dx, \quad \alpha \in (-\infty, \infty). \]

The following theorem shows convexity of these \( D_\alpha(\cdot) \) in its density arguments.
Theorem 4.4.10 (Convexity in Density, $\alpha$-divergence)

The $\alpha$-divergence $D_\alpha(p_1, p_2)$ is convex with respect to $p_1$ and $p_2$.

Proof Consider that

$$
\frac{\partial^2 D_\alpha(p_1, p_2)}{\partial p_1^2} = \frac{1}{\alpha(1-\alpha)} \int_\Omega -\alpha \cdot (\alpha - 1) \cdot p_1^{\alpha-2} p_2^{1-\alpha} dx
$$

$$
= \frac{\alpha(1-\alpha)}{\alpha(1-\alpha)} \int_\Omega p_1^{\alpha-2} p_2^{1-\alpha} dx
$$

$$
= \int_\Omega p_1^{\alpha-2} p_2^{1-\alpha} dx \geq 0
$$

and similarly that

$$
\frac{\partial^2 D_\alpha(p_1, p_2)}{\partial p_2^2} = \frac{1}{\alpha(1-\alpha)} \int_\Omega -(1-\alpha) \cdot (\alpha - 1) \cdot p_1^{\alpha-2} p_2^{1-\alpha} dx
$$

$$
= \frac{\alpha(1-\alpha)}{\alpha(1-\alpha)} \int_\Omega p_1^{\alpha-2} p_2^{1-\alpha} dx
$$

$$
= \int_\Omega p_1^{\alpha-2} p_2^{1-\alpha} dx \geq 0.
$$

The convexity result follows immediately.

It is a well-known result that for a variate $X$, the characteristic function $\phi_X(\omega)$ of $X$ uniquely characterizes its probability density function $p_X(x)$. If $X$ is a Gaussian random variable, then the isomorphism $\xi$ can be defined with its inverse $\xi^{-1}$:

$$
\xi : \{ \mathbb{R}^+ \times \mathbb{R}^+ - (0,0) \} \rightarrow L^1_x(\Omega, \mathcal{P})
$$

$$
\xi(\kappa_1(X), \kappa_2(X)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega x) \phi_X(\omega; \kappa_1(X), \kappa_2(X)) d\omega = p_X(x)
$$

$$
\xi^{-1} : L^1_x(\Omega, \mathcal{P}) \rightarrow \{ \mathbb{R}^+ \times \mathbb{R}^+ - (0,0) \}
$$

143
\[ \xi^{-1}(p_X(x)) = \left( j^{-1} \cdot \left( \frac{d}{d\omega} \int_{\Omega} \exp(j\omega x)p_X(x)dx \right) \left|_{\omega=0} \right. \right), j^{-2} \cdot \left( \frac{d^2}{d\omega^2} \int_{\Omega} \exp(j\omega x)p_X(x)dx \right) \left|_{\omega=0} \right. \right) = (\kappa_1(X), \kappa_2(X)) \]

The convexity result can now be stated, of which MBC-CDS’ well-posedness is a direct consequence.

**Theorem 4.4.11 (Convexity of Mean-Variance Probability Distance Functions)**
The functions \( g_{\text{GDA}}^{\text{BD}}(\tilde{x}, \tilde{x}) \) and \( g_{\text{HD}}^{\text{GDA}}(\tilde{x}, \tilde{x}) \) and convex, and as a result the function \( g_{\text{BC}}^{\text{GDA}}(\tilde{x}, \tilde{x}) \) is concave.

**Proof** Being isomorphic via \( \xi \), the Gaussian parameter space is algebraically and topologically equivalent to the Gaussian density space, and properties holding in one space must hold in the other. It must be that convexity of \( D_\alpha(p_1, p_2) \) in \( p_1 \) and \( p_2 \) means that for \( \alpha = \frac{1}{2} \) and

\[
D_{\frac{1}{2}}(p_Z(z), p_\tilde{Z}(z)) = 4 \cdot HD(p_Z(z), p_\tilde{Z}(z)) = 4 \cdot g_{\text{HD}}^{\text{GDA}}(\kappa(t_0), \tilde{\kappa}(t_0))
\]

this function is convex in \( (\kappa_1(t_0), \kappa_2(t_0)) \) and \( (\tilde{\kappa}_1(t_0), \tilde{\kappa}_2(t_0)) \). This implies that \( g_{\text{BC}}^{\text{GDA}}(\cdot) \) is concave in its cumulant arguments, and the MBC-CDS maximization is well-posed. Further, since \( -\ln(\cdot) \) is is convex, and \( g_{\text{BD}}^{\text{GDA}}(\kappa(t_0), \tilde{\kappa}(t_0)) \) is concave in \( (\kappa_1(t_0), \kappa_2(t_0)) \) and \( (\tilde{\kappa}_1(t_0), \tilde{\kappa}_2(t_0)) \), the function \( g_{\text{BD}}^{\text{GDA}}(\kappa(t_0), \tilde{\kappa}(t_0)) \) is convex by the composition rules in Boyd [59].

\[ \square \]

### 4.4.4 Equivalency of Optimal Cost Density-Shaping Controls

As stated in the beginning of this section, the optimal controls that result from minimizing the \( HD \) or \( BD \) are consistent with the optimal control solution that
results from maximizing the $BC$. The Multiple-Cumulant Cost Density-Shaping (MCCDS) theory states that the gain in an optimal control that minimizes a convex, smooth function of the first two cost cumulants contains a ratio of partial derivatives.

Consider the optimization below.

$$\min_k \left\{ g\left( \begin{bmatrix} \kappa_1(t_0) \\ \kappa_2(t_0) \end{bmatrix}, \begin{bmatrix} \tilde{\kappa}_1(t_0) \\ \tilde{\kappa}_2(t_0) \end{bmatrix} \right) \right\}$$

$$dx(t) = \left( A(t) + B(t)K(t) \right)x(t) + G(t)dw(t), \ x_0 = E\{x(t_0)\}, \ t \in [t_0, t_f]$$

Recall the MCCDS control solution takes the form,

$$K(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( H_1^*(\alpha) + \gamma(\alpha)H_2^*(\alpha) \right)$$

where the optimal time-varying parameter is

$$\gamma(\alpha) = \left( \begin{bmatrix} \frac{\partial g(\kappa(\alpha),\tilde{\kappa}(\alpha))}{\partial \kappa_2(\alpha)} \\ \frac{\partial g(\kappa(\alpha),\tilde{\kappa}(\alpha))}{\partial \kappa_1(\alpha)} \end{bmatrix} \right).$$

Define the functions $h_1(x)$ and $h_2(x)$ as below,

$$h_1(x) = 1 - x, \ h_2(x) = -\ln(x).$$

The proof of equivalency of the optimal controls mentioned above is now given.

**Theorem 4.4.12 (Equivalency of MVCDS Controls)**

*Define the Minimum Hellinger Distance, Cost Density-Shaping (MHD²-CDS)*
optimization as below,
\[
\min_k \left\{ 1 - \sqrt{2} \cdot \frac{(\kappa_2(t_0) \tilde{\kappa}_2(t_0))^\top}{\sqrt{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)}} \cdot \exp \left( -\frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{4(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))} \right) \right\}
\]
\[
dx(t) = \left( A(t) + B(t) K(t) \right) x(t) + G(t) dw(t), \ x_0 = E\{x(t_0)\}, \ t \in [t_0, t_f].
\]
Define the Minimum Bhattacharyya Distance, Cost Density-Shaping (MBD-CDS) optimization as below,
\[
\min_k \left\{ -\ln \left( \sqrt{2} \cdot \frac{(\kappa_2(t_0) \tilde{\kappa}_2(t_0))^\top}{\sqrt{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)}} \cdot \exp \left( -\frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{4(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))} \right) \right) \right\}
\]
\[
dx(t) = \left( A(t) + B(t) K(t) \right) x(t) + G(t) dw(t), \ x_0 = E\{x(t_0)\}, \ t \in [t_0, t_f].
\]
The optimal control solutions to the MHD²-CDS and MBD-CDS optimizations (with $\chi = BD$ and $\chi = HD^2$) are,
\[
K^*(\alpha) = -R^{-1}(\alpha) B^T(\alpha) \left( H^*_1(\alpha) + \gamma_{GDA}^G(\alpha) H^*_2(\alpha) \right)
\]
where the optimal parameters are
\[
\gamma_{GDA}^{BD}(\alpha) = \gamma_{GDA}^{HD}(\alpha) = \gamma_{BC}^{GDA}(\alpha) = \left( \frac{1}{2} \cdot \frac{(1 - \tilde{\kappa}_2(\alpha)) - (\kappa_1^*(\alpha) - \tilde{\kappa}_1(\alpha))^2}{\kappa_2^*(\alpha) \tilde{\kappa}_2(\alpha) + \tilde{\kappa}_1(\alpha)} \right)
\]
with the optimal cost mean ($i = 1$) and variance ($i = 2$) are defined by
\[
\kappa_i^*(\alpha) = x_0^T H^*_i(\alpha) x_0 + D_i^*(\alpha)
\]
and also with the target cost mean ($i = 1$) and the target cost variance ($i = 2$) are given by
\[
\tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha) x_0 + \tilde{D}_i(\alpha).
\]
The optimal state variables $H^*(\alpha), D^*(\alpha)$ and $\tilde{H}(\alpha), \tilde{D}(\alpha)$ follow the equations of motion
\[
\frac{dH^*(\alpha)}{d\alpha} = F(H^*(\alpha), K^*(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha))
\]
\[
\frac{dD^*(\alpha)}{d\alpha} = G(H^*(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]
\[
H^*(t_f) = H_f, \quad \tilde{H}(t_f) = \tilde{H}_{f,e^*}, \quad D^*(t_f) = D_f, \quad \tilde{D}(t_f) = D_{f,e^*}.
\]
**Proof** Observe that the following relations hold

\[ g^{\text{GDA}}_{HD^2}(\kappa(t_0), \tilde{\kappa}(t_0)) = h_1(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))) \]

\[ g^{\text{GDA}}_{BD}(\kappa(t_0), \tilde{\kappa}(t_0)) = h_2(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))). \]

Write the derivatives for \( i = 1 \) and \( 2 \),

\[
\frac{\partial g^{\text{GDA}}_{HD^2}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)} = \frac{\partial h_1(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)))}{\partial \kappa_i(\alpha)} = \frac{\partial h_1(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)))}{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))} \cdot \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)} = (-1) \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)}.
\]

\[
\frac{\partial g^{\text{GDA}}_{BD}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)} = \frac{\partial h_2(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)))}{\partial \kappa_i(\alpha)} = \frac{\partial h_2(g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)))}{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))} \cdot \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)} = \left( \frac{1}{g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))} \right) \cdot \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_i(\alpha)}.
\]

The assumption on the finite-horizon \( \kappa_1(\alpha) - \tilde{\kappa}_1(\alpha) \neq 0 \) implies in fact that

\[ g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0)) \neq 0. \]

Thus the ratio of derivatives can be considered,

\[ \gamma^{\text{GDA}}_{HD^2}(\alpha) = \frac{\partial g^{\text{GDA}}_{HD^2}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial g^{\text{GDA}}_{HD^2}(\kappa(t_0), \tilde{\kappa}(t_0))} \cdot \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_2(\alpha)} = \frac{\partial g^{\text{GDA}}_{BD}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))} \cdot \frac{\partial g^{\text{GDA}}_{BC}(\kappa(t_0), \tilde{\kappa}(t_0))}{\partial \kappa_1(\alpha)} = \gamma^{\text{GDA}}_{BC}(\alpha). \]
and

\[
\gamma_{BD}^{GDA}(\alpha) = \frac{\partial g_{BD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0))}{\kappa_2(\alpha)} \quad = \frac{\partial g_{BC}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0))}{\kappa_1(\alpha)} = \gamma_{BC}^{GDA}(\alpha).
\]

4.5 Comparison of MVCDS Methods

The single-degree of freedom problem discussed in [32] serves to demonstrate the performance of the MKLD-CDS, MHD-CDS, MBD-CDS, MBC-CDS, and WLS-CDS \((r = 2)\) controllers. The system dynamics are given by the perturbed linear system and the performance index

\[\text{Figure 4.7. Single Degree-of-Freedom Structure}\]
The system constants are \( m = 16.69 \text{ lb-s}^2/\text{in} \), \( c = 9.020 \text{ lb-s/\text{in}} \), \( k = 7934 \text{ lb/in} \), \( k_c = 2124 \text{ lb/in} \), \( \alpha = 36^\circ \), \( W = 2\pi \text{ in}^2/\text{s}^3 \), and \( x_0 = \begin{bmatrix} .00176 \\ .039 \end{bmatrix} \). The time horizon is between \( t_0 = 0 \) and \( t_f = 15 \) sec. The system state \( x_1(t) \) is the floor displacement and \( x_2(t) \) is the floor velocity. The setup is illustrated in Figure 4.7 above.

\[
dx(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) dt + \begin{bmatrix} 0 \\ -4k_c \cdot \cos(\alpha)/m \end{bmatrix} u(t) dt + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dw(t)
\]

\[
J[x, u] = \int_{t_0}^{t_f} \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} x(\tau) + u(\tau) Ru(\tau) d\tau
\]

In this example, several constant parameters in the range \( \mu_2 \in [0, 0.9] \) are used to characterize a \( k \)CC controller, which drives the the mean and variance of the cost to certain target initial values. When the trajectories of the target mean and variance are inserted into the MKLD-CDS, MBC-CDS, and WLS-CDS \((r = 2)\) controllers, excellent tracking performance is observed. It can be seen that \( \gamma_2^x \) values, where \( x = MKLDCDS, MBCCD, WLSCDS \) converge very quickly to the associated parameter \( \mu_2 \) in the \( 2 \)CC control, for each selection. This is shown in Figures 4.10, 4.13, and 4.16. As a result, the structure of the MKLD-CDS, MBC-CDS, and WLS-CDS \((r = 2)\) is consistent with that underlying the target cost statistics, in particular the \( 2 \)CC control. Given the approximate coincidence of initial conditions for the system, it’s likely that the state variables will be approximately equal under these near equal controls. The ultimate resultant effect is that the cost mean and variance track the target cost statistics.

Not shown is the Frobenius norm for matrices between the optimal control gains and the pre-selected \( 2 \)CC controller though convergence in this norm does occur, which is reasonable given the convergence of the optimal parameters. This...
convergence of control gains demonstrates how the cost mean and variance driven by the optimal MVCDS controllers are effectively tracking the target cost mean and variance by matching the control gain, which is reconstructed through information of the target cumulant trajectories. Another observation that can be made from Figures 4.10, 4.13, and 4.16 is that there seems to exist a bijective relationship between the 2CC controls and the initial mean and variance of the cost $J$. That is, the $\gamma_2^x(\tau)$ trajectories did not obtain the target cost mean and variance with parameter values different than $\mu_2$. It appears there is a one-to-one correspondence between the 2CC control inputs and the first two cost cumulants.

The cost-cumulant tracking of the MKLD-CDS, MBC-CDS, and WLS-CDS ($r = 2$) controls is shown as well. It can be seen that under each controller the cost mean and variance track the target mean and variance over the time horizon $[0, 15 \text{ sec}]$. The amount of error is shown in each case, which is minimal and decreasing in time. See Figures 4.8, 4.11, and 4.14, where $|\kappa_i(\alpha) - \tilde{\kappa}_i(\alpha)|$ is shown where $1 \leq i \leq 2$ and the different target statistics are chosen appropriately. It’s a natural expectation to expect the cost and target cost densities to reflect the achieved tracking of target cost cumulants. The cost and target cost densities are shown in Figures 4.9, 4.12, and 4.15 and it can be seen that they align almost exactly, which is not surprising since the controls become approximately equal and will drive higher-order cumulants to be approximately equal as well. This statement is confirmed in the nature of the cost density approximation, which is accurate up to the first four cumulants.

Figures 4.17 (a) and (b) show the error in the cost mean and variance, respectfully, for each of the MKLD-CDS, MBC-CDS, and WLS-CDS paradigms. Error is shown in both bar plots for the 5 levels of $\mu_2$. It can be seen that the error in
the cost mean for WLS-CDS is considerable compared to the other paradigms, whereas the error in cost variance is significantly reduced with WLS-CDS compared to that realized with MKLD-CDS and MBC-CDS methods. Nevertheless, the realized error is very small for all methods, and each of them appears to be capable of high-accuracy cost density-shaping.
Figure 4.8: Single Degree-of-Freedom Problem, MBC-CDS Cost-Cumulant Tracking and Error
Figure 4.9: Single Degree-of-Freedom Problem, MBC-CDS Cost Densities
Figure 4.10: Single Degree-of-Freedom Problem, MBC-CDS Cost Densities
Figure 4.11: Single Degree-of-Freedom Problem, MKLD-CDS Cost-Cumulant Tracking and Error
Figure 4.12: Single Degree-of-Freedom Problem, MKLD-CDS Cost Densities
Figure 4.13: Single Degree-of-Freedom Problem, MKLD-CDS Cost Densities
Figure 4.14: Single Degree-of-Freedom Problem, WLS-CDS Cost-Cumulant Tracking and Error
Figure 4.15: Single Degree-of-Freedom Problem, WLS-CDS Cost Densities
Figure 4.16: Single Degree-of-Freedom Problem, WLS-CDS Cost Densities
Figure 4.17. Error in Cost Mean and Variance
4.6 Delta-Mean Zero-Crossings

The material of this chapter and Chapter 3 develops Multiple-Cumulant Cost Density-Shaping (MCCDS) and Mean-Variance Cost Density-Shaping (MVCDS) optimal controls that are characterized by gains with optimal time-varying parameters. Such parameters are formed by a quotient of partial derivatives of the performance index for each problem with respect to different orders of the cost cumulants. For several of these problems, the denominator in the quotient has amounted to the difference between the cost mean and the target cost mean.

The investigation of sufficient conditions for where the difference of means does not vanish on \([t_0, t_f]\) is motivated by considering two important cases where this difference of cost means appears. For Weighted Least Squares, recall the performance index is

\[
g_{WLS}^{MC}(\kappa(t_0), \tilde{\kappa}(t_0)) = \sum_{i=1}^{r} w_i (\kappa_i(t_0) - \tilde{\kappa}_i(t_0))^2, \quad w_1 > 0, \quad w_i \geq 0, \quad i \geq 2
\]

so that in the associated MCCDS optimal control has parameters

\[
\left( \frac{\partial g_{WLS}^{MC}(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \right) = 2w_i (\kappa_i(\alpha) - \tilde{\kappa}_i(\alpha))
\]

and

\[
\gamma_i^*(\alpha) = \frac{\partial g_{WLS}^{MC}(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_1(\alpha)} = \frac{w_i}{w_1} \frac{\kappa_i(\alpha) - \tilde{\kappa}_i(\alpha)}{\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha)}.
\]

Refer to the Weighted-Least-Squares, Cost Density-Shaping (WLS-CDS) controller of Section 3.3 if needed.

Now for the Kullback-Liebler Divergence between the best Gaussian approxi-
motions to densities of normalized cost and target cost variates,

$$g_{KLD}^{GDA}(\kappa(t_0), \tilde{\kappa}(t_0)) = \frac{1}{2} \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} - 1 - \log \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} \right) \right) + \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{\tilde{\kappa}_2(t_0)}$$

so that

$$\frac{\partial g_{KLD}^{GDA}(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_1(\alpha)} = \frac{(\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha))}{\tilde{\kappa}_2(\alpha)}, \quad \frac{\partial g_{KLD}^{GDA}}{\partial \kappa_2(\alpha)} = \frac{1}{2} \left( \frac{1}{\tilde{\kappa}_2(\alpha)} - \frac{1}{\kappa_2(\alpha)} \right)$$

and

$$\gamma_2 = \left( \frac{\partial g_{KLD}^{GDA}(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \right) \left( \frac{\partial g_{KLD}^{GDA}(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \tilde{\kappa}_1(\alpha)} \right) = \frac{1}{2} \left( \frac{1 - \tilde{\kappa}_2(\alpha)}{\kappa_2(\alpha)} \right) \left( \frac{\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha)}{\kappa_2(\alpha)} \right)$$

is the optimal parameter selection for the Minimum Kullback-Leibler Divergence, Cost Density-Shaping (MKLD-CDS) controller.

Since the difference of the cost mean and target cost mean $\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha)$ is in the denominator for these important cases of MVCDS and MCCDS optimal controls, it is expedient to investigate sufficient conditions under which the quantity does not vanish. Several conditions are presented for control gains $K$ and $\tilde{K}$ under which the aforementioned MVCDS and MCCDS optimal controls are well-defined. Here $\tilde{K}$ is the control underlying the target cost statistics.

4.6.1 Delta Mean Zero-Crossings, $kCC$ Existence

Going forward, the discussion may on occasion refer to the closed-loop matrices $F \triangleq A + BK$ and $\tilde{F} \triangleq A + B\tilde{K}$. Also, the denotation $S \triangleq GWG^T$ may be used sparingly. Some non-restrictive requirements on the system characterization $(A, B, G)$ and also the control gains $K$ and $\tilde{K}$ are made, in order to ensure $\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha) \neq 0$ everywhere.
on the finite horizon $\alpha \in [t_0, t_f]$. In particular, it is proposed that the conditions (4.40) and (4.41) hold. These effectively impose a mild restriction on $K$ and $\tilde{K}$ such that an ordering can be established between solutions of the differential matrix Riccati equation. In particular, condition (4.40) ensures $\Delta H_1(\tau) = H_1(\tau) - \tilde{H}_1(\tau) \succeq 0_{n \times n}$ when $H_1(t_f) = \tilde{H}_1(t_f)$. If instead (4.41) holds in the opposite direction, this ensures $\Delta H_1(\tau) \preceq 0_{n \times n}$. See Freiling and Jank for a thorough treatment of comparison theorems for generalized differential matrix Riccati equations [15].

For general control gains $K$ and $\tilde{K}$ and fixed $R, A, B$, and $Q$ it is assumed that either

$$
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0_{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0_{n \times n}
\end{bmatrix}, \ \forall \tau \in [t, t_f] \quad (4.40)
$$

or

$$
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0_{n \times n}
\end{bmatrix} \succeq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0_{n \times n}
\end{bmatrix}, \ \forall \tau \in [t, t_f] \quad (4.41)
$$

The system dynamics are such that the additive noise has independent effects on the individual dimensions of the process. Formally, this requires that $Nul(G) = \emptyset$.

The first “no zero-crossings” condition is now presented, which essentially requires that (4.40) or (4.41) hold, and also the target cost mean generating equation for $\tilde{H}_1(\tau)$ has a positive-definite perturbation to its terminal condition system. The result appears in a theorem, which will be given after some preliminary results are introduced. In the following, let $\lambda_1(A)$ denote the smallest eigenvalue of the
matrix $A \in \mathbb{R}^n$ and $\lambda_n(A)$ its largest eigenvalue.

**Lemma 4.6.1** If $A, B \in \mathbb{R}^{n \times n}$ and $A, B \succeq 0$ then

$$\lambda_1(A) \text{Tr}(B) \leq \text{Tr}(AB) \leq \lambda_n(A) \text{Tr}(B). \quad (4.42)$$

If $A, B \in \mathbb{R}^{n \times n}$ and instead $A \succeq 0$, $B \preceq 0$ then $-B \succeq 0$ and

$$\lambda_n(A) \text{Tr}(B) \leq \text{Tr}(AB) \leq \lambda_1(A) \text{Tr}(B). \quad (4.43)$$

**Lemma 4.6.2** *(Integral Representation of Cost Cumulants)*

The cost cumulants $\kappa_i(\alpha) = x_0^T H_i(\alpha)x_0 + D_i(\alpha)$ have the alternative form

$$\kappa_i(\alpha) = \text{Tr}(x_0x_0^T H_i(\alpha)) - \int_{t_f}^\alpha \text{Tr}(G(\alpha^*)WG(\alpha^*)H_i(\alpha^*))d\alpha^*, \ \alpha \in [t_0, t_f].$$

**Proof** Write $\bar{D}_i(t) \triangleq D_i(t_f + t_0 - t)$ with $\alpha \triangleq t_f + t_0 - t$, for which it is clear that

$$\bar{D}_i(t) - \bar{D}_i(t_0) = \int_{t_0}^t -\text{Tr}(H_i(t_f + t_0 - \tau)G(t_f + t_0 - \tau)WG(t_f + t_0 - \tau))d\tau$$

Take $\alpha^* = t_f + t_0 - \tau$ and note that $d\alpha^* = -d\tau$ and so from the above relation it follows that

$$D_i(\alpha) - D_i(t_f) = D_i(\alpha) - 0 = -\int_{t_f}^\alpha \text{Tr}(H_i(\alpha^*)G(\alpha^*)WG(\alpha^*))d\alpha^*.$$

With the form for the $i$th cost cumulant,

$$\kappa_i(\alpha) = x_0^T H_i(\alpha)x_0 + D_i(\alpha)$$

and using the fact that

$$\text{Tr}(H_i(\alpha^*)G(\alpha^*)WG(\alpha^*)) = \text{Tr}(G(\alpha^*)WG(\alpha^*)H_i(\alpha^*))$$

165
it is apparent that,

\[ \kappa_i(\alpha) = \text{Tr}(x_0 x_0^T H_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) H_i(\alpha^*)) d\alpha^*. \]

\[ \kappa_i(\alpha) = \text{Tr}(x_0 x_0^T H_i(\alpha) - \tilde{H}_i(\alpha)) x_0 + (D_i(\alpha) - \tilde{D}_i(\alpha)) \]

have the alternative form

\[ \kappa_i(\alpha) - \tilde{\kappa}_i(\alpha) = \text{Tr}(x_0 x_0^T \Delta H_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) \Delta H_i(\alpha^*)) d\alpha^*, \quad t \in [t_0, t_f] \]

where \( \Delta H_i(\alpha) = H_i(\alpha) - \tilde{H}_i(\alpha) \).

**Proof** For \( \kappa_i(\alpha) \), write

\[ \kappa_i(\alpha) = \text{Tr}(x_0 x_0^T H_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) H_i(\alpha^*)) d\alpha^*. \]

For \( \tilde{\kappa}_i(\alpha) \), write

\[ \tilde{\kappa}_i(\alpha) = \text{Tr}(x_0 x_0^T \tilde{H}_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) \tilde{H}_i(\alpha^*)) d\alpha^*. \]

Taking the difference gives,

\[ \kappa_i(\alpha) - \tilde{\kappa}_i(\alpha) = \text{Tr}(x_0 x_0^T H_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) H_i(\alpha^*)) d\alpha^* \\
- \text{Tr}(x_0 x_0^T \tilde{H}_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) \tilde{H}_i(\alpha^*)) d\alpha^* \\
= \text{Tr}(x_0 x_0^T (H_i(\alpha) - \tilde{H}_i(\alpha))) \\
- \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) (H_i(\alpha^*) - \tilde{H}_i(\alpha^*))) d\alpha^* \\
= \text{Tr}(x_0 x_0^T \Delta H_i(\alpha)) - \int_{t_f}^t \text{Tr}(G(\alpha^*) W G^T(\alpha^*) \Delta H_i(\alpha^*)) d\alpha^*. \]
Consider a process with linear dynamics and additive noise defined by

\[ dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t), \quad t \in [t_0, t_f] \]

\[ x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n \]

where \( A \in C([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( B \in C([t_0, t_f]; \mathbb{R}^{n \times m}) \), \( G \in C([t_0, t_f]; \mathbb{R}^{n \times p}) \), \( w(t) \) is a \( p \)-dimensional stationary Wiener process having a correlation of increments defined by

\[ E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}. \]

Consider a cost \( J[x, u; t_0, x_0] \) that is an integral-quadratic form and defined by

\[ J[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left( x(\tau)^T Q(x(\tau) + u(\tau)^T R(u(\tau)) \right) d\tau + x(t_f)^T Q_f x(t_f). \]

Subject the process to linear state-feedback control, \( u = Kx \), and assume also that the process may be alternatively controlled by \( u = \bar{K}x \). Let \( \text{Null}(G) = \emptyset \) and assume \( K \) and \( \bar{K} \) are two control gains satisfying either

\[ \begin{bmatrix} \bar{K}^T R \bar{K} + Q & (A + B \bar{K})^T \\ A + B \bar{K} & 0_{n \times n} \end{bmatrix} \preceq \begin{bmatrix} K^T R K + Q & (A + B K)^T \\ A + B K & 0_{n \times n} \end{bmatrix} \]

or

\[ \begin{bmatrix} \bar{K}^T R \bar{K} + Q & (A + B \bar{K})^T \\ A + B \bar{K} & 0_{n \times n} \end{bmatrix} \succeq \begin{bmatrix} K^T R K + Q & (A + B K)^T \\ A + B K & 0_{n \times n} \end{bmatrix}, \quad \forall \tau \in [t_0, t_f]. \]

Suppose further that there exists a perturbation matrix \( E \) such that \( Q_f = Q + E \succeq Q_f \) with \( E \succ 0_{n \times n} \). For the condition (4.40) choose the terminal conditions to be \( H_1(t_f) = Q_f, \quad \bar{H}_1(t_f) = Q_1 \). If instead the condition (4.41) holds between the gains, choose \( H_1(t_f) = Q_f, \quad \bar{H}_1(t_f) = Q_1 \). Under these conditions, \( \kappa_1(\alpha) - \bar{\kappa}_1(\alpha) \neq 0 \) on \( \alpha \in [t_0, t_f] \), where the cost means are defined by \( \kappa_1(\alpha) = x_0^T H_1(\alpha)x_0 + D_1(\alpha) \) and \( \bar{k}_1(\alpha) = x_0^T \bar{H}_1(\alpha)x_0 + \bar{D}_1(\alpha) \) where \( H_1(\alpha), \bar{H}_1(\alpha) \) follow the equations of motion

\[ \frac{dH_1(\alpha)}{d\alpha} = - (A(\alpha) + B(\alpha)K(\alpha))^T H_1(\alpha) - H_1(\alpha)(A(\alpha) + B(\alpha)K(\alpha)) \]

\[ - K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha) \]

The main result can now be established using the foregoing lemmas.
\[
d\frac{dD_1(\alpha)}{d\alpha} = -\text{Tr}(H_1(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, t_f]
H_1(t_f), \quad D_1(t_f) = 0.
\]

and the functions \(\tilde{H}_1(\alpha), \tilde{D}_1(\alpha)\) are determined by

\[
\frac{d\tilde{H}_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)K(\alpha))^T\tilde{H}_1(\alpha) - \tilde{H}_1(\alpha)(A(\alpha) + B(\alpha)\tilde{K}(\alpha))
- \tilde{K}^T(\alpha)R(\alpha)\tilde{K}(\alpha) - Q(\alpha)
\frac{d\tilde{D}_1(\alpha)}{d\alpha} = -\text{Tr}(\tilde{H}_1(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, t_f]
\tilde{H}_1(t_f), \quad \tilde{D}_1(t_f) = 0.
\]

**Proof** Assume \(\kappa_1(t) - \tilde{\kappa}_1(t) = 0\) for some \(\alpha \in [t_0, t_f]\) and both \(H_1(t_f) = Q_f^* > Q_f = \tilde{H}_1(t_f) \succ 0_{n \times n}\) and assume the following relation holds on \([t, t_f]\),

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0_{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0_{n \times n}
\end{bmatrix}
\]

so that \(\Delta H_1(\tau) \succ 0_{n \times n}\) on the entire interval. Using the inequality (4.42) it follows that,

\[
0 < \lim_{\alpha \to \tau} \text{Tr}(\Delta H_1(t)) - \int_{t_f}^{t} \lambda_1(S(\tau))\text{Tr}(\Delta H_1(\tau))d\tau 
\leq \text{Tr}((x_0x_0^T)\Delta H_1(t)) - \int_{t_f}^{t} \text{Tr}(S(\tau)\Delta H_1(\tau))d\tau 
= \kappa_1(t) - \tilde{\kappa}_1(t) = 0.
\]

This is an apparent contradiction. Assume again \(\kappa_1(t) - \tilde{\kappa}_1(t) = 0\) but the following relation holds instead on \([t, t_f]\),

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0_{n \times n}
\end{bmatrix} \succeq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0_{n \times n}
\end{bmatrix}
\]

168
and $H_1(t_f) = Q_f^* > Q_f = H_1(t_f) > 0^{n \times n}$ and so that $\Delta H_1(\alpha) \prec 0$, then via (4.43) the following can be written

$$0 = \kappa_1(t) - \tilde{\kappa}_1(t) = \text{Tr}((x_0x_0^T)\Delta H_1(t)) - \int_t^{t_f} \text{Tr}(S(\tau)\Delta H_1(\tau))d\tau$$

$$\leq \lambda_1(x_0x_0^T)\text{Tr}(\Delta H_1(t)) - \int_t^{t_f} \lambda_1(S(\tau))\text{Tr}(\Delta H_1(\tau))d\tau$$

Again a contradiction is obtained so it can be concluded that $\kappa_1(t) - \tilde{\kappa}_1(t) \neq 0, \forall t \in [t_0, t_f]$, when either (4.40) or (4.41) is satisfied over the entire interval.

A second zero-crossing condition can be established that does not require a perturbation to the terminal condition of the generating equation for the target cost mean. It stems from consequences from uniqueness and existence properties of the cost cumulant-generating equations, and makes use of the $kCC$ existence criteria of Liberty and Mou [63]. Before stating the result and posing its proof, a few supporting definitions and lemmas are presented that wholly are a result of the referenced work.

However, before proceeding, it’s important to note that the results in [63] make use of a different, though equivalent, definition of the cost cumulants that carries an impact to the cumulant-generating equations of the LQG framework. Given this difference, it is important to establish the equivalency of the two forms of cost cumulants, and show formally how changes are manifest in the cumulant-generating equations. A lemma is presented here before the properties of cost cumulants are given.

**Lemma 4.6.5 (Equivalency of Cost Cumulants Forms)**

When the process having dynamics (4.1) is subject to a linear state-feedback control
The cumulants of (4.3) have the representation,

\[ \kappa_j(\alpha, x) = x^T H_j(\alpha) x + D_j(\alpha) \]

where the functions \( D_i(\alpha) \) and \( H_i(\alpha) \) satisfy the family of differential equations,

\[
\frac{dH_1(\alpha)}{d\alpha} = -F^T(\alpha) H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha) \\
\frac{dH_i(\alpha)}{d\alpha} = -F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha) \\
- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \quad i \geq 2
\]

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}(H_j(\alpha) G(\alpha) W G^T(\alpha)), \quad \alpha \in [t_0, t_f], \quad j \geq 1
\]

\( H_1(t_f) = Q_f, \quad H_i(t_f) = 0^{n \times n}, \quad D_j(t_f) = 0, \quad i \geq 2, \quad j \geq 1. \)

which equivalently can be written as

\[ \kappa_j(\alpha, x) = x^T (j! H_j(\alpha)) x + (j! D_j(\alpha)) \]

where the functions \( D_i(\alpha) \) and \( H_i(\alpha) \) satisfy the family of differential equations,

\[
\frac{dH_1(\alpha)}{d\alpha} = -F^T(\alpha) H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha) \\
\frac{dH_i(\alpha)}{d\alpha} = -F^T(\alpha) H_i(\alpha) - H_i(\alpha) F(\alpha) \\
- 2 \sum_{j=1}^{i-1} H_j(\alpha) G(\alpha) W G^T(\alpha) H_{i-j}(\alpha), \quad i \geq 2
\]

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}(H_j(\alpha) G(\alpha) W G^T(\alpha)), \quad \alpha \in [t_0, t_f], \quad j \geq 1
\]

\( H_1(t_f) = Q_f, \quad H_i(t_f) = 0^{n \times n}, \quad D_j(t_f) = 0, \quad i \geq 2, \quad j \geq 1. \)

**Proof** It is proposed that the following relation hold

\[ H_i(\alpha) = \frac{H_i(\alpha)}{i!}, \quad D_i(\alpha) = \frac{D_i(\alpha)}{i!} \]

which can be established by showing that

\[
\frac{dH_i(\alpha)}{d\alpha} = \left(\frac{1}{i!}\right) \frac{dH_i(\alpha)}{d\alpha}, \quad \frac{dD_i(\alpha)}{d\alpha} = \left(\frac{1}{i!}\right) \frac{dD_i(\alpha)}{d\alpha}
\]
It is evident that

\[
\frac{dH_1(\alpha)}{d\alpha} = \frac{dH_1(\alpha)}{d\alpha}, \quad H_1(t_f) = H_1(t_f) = Q_f \text{ so that } H_1(\alpha) = H_1(\alpha)
\]

from the statement of the lemma. Consider that

\[
\frac{dH_i(\alpha)}{d\alpha} = -F^T(\alpha)H_i(\alpha) - H_i(\alpha)F(\alpha)
\]

\[
- \sum_{j=1}^{i-1} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha)
\]

\[
= -F^T(\alpha) \left( \frac{H_i(\alpha)}{i!} \right) - \left( \frac{H_i(\alpha)}{i!} \right) F(\alpha)
\]

\[
- \frac{2}{i!} \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha) \left( \frac{H_{i-j}(\alpha)}{(i-j)!} \right)
\]

\[
= \left( \frac{1}{i!} \right) \cdot \left( -F^T(\alpha)H_i(\alpha) - H_i(\alpha)F(\alpha) \right)
\]

\[
+ \left( \frac{1}{i!} \right) \cdot \left( -2 \sum_{j=1}^{i-1} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha) \right)
\]

\[
= \left( \frac{1}{i!} \right) \cdot \frac{dH_i(\alpha)}{d\alpha}
\]

The terminal conditions are

\[
H_i(t_f) = \frac{H_i(t_f)}{i!} = 0^{n \times n} = H_i(t_f).
\]

Also, it readily is established that

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}(H_j(\alpha)G(\alpha)WG^T(\alpha))
\]

\[
= \left( \frac{1}{i!} \right) \cdot \left( -\text{Tr}(H_j(\alpha)G(\alpha)WG^T(\alpha)) \right)
\]

\[
= \left( \frac{1}{i!} \right) \cdot \frac{dD(\alpha)}{d\alpha}
\]
and analogously

\[ D_i(t_f) = \frac{D_i(t_f)}{i!} = 0 = D_i(t_f). \]

\[ \square \]

**Definition 4.6.6 (Total Cumulant)**

Let the total cumulant be defined as

\[ \kappa = \sum_{i=1}^{\infty} \mu_i \kappa_i(t_0) \quad (4.44) \]

where \( \kappa_i(t_0) \) is the \( i \)th initial Cost cumulant and \( \{ \mu_i \}_{i=1}^{\infty} \) is some sequence of reals.

**Definition 4.6.7 (Valid \( \rho \)-growth Sequences)**

Let \( \rho > 0 \) and \( \{ \mu_i \}_{i=1}^{\infty} \) be some sequence of reals. Then a sequence of positive reals satisfying a \( \rho \)-growth condition is defined when the sequence satisfies the following condition.

\[ 0 \leq \mu_i \leq \rho \mu_j \mu_{i-j}, \quad i \geq 2, \quad 1 \leq j \leq i - 1 \]

**Remark(s)** The above condition ensures that given a sequence \( \{ \mu_i \}_{i=1}^{\infty} \), that the total cumulant is well-defined (that is, the series \((4.44)\) converges) for some control \( K \in \mathcal{K}_\mu \neq \emptyset \), where \( \mathcal{K}_\mu \) denotes the admissible set. That is, \( \mathcal{K}_\mu \) is the set of all controls \( K \) such that the total cumulant exists.

**Lemma 4.6.8 (Characterization of \( \mathcal{K}_\mu \))**

For any \( L > 0 \) there is a \( \rho > 0 \) such that if \( \{ \mu_i \}_{i=1}^{\infty} \) satisfies a \( \rho \)-growth condition, then the total cumulant \( \kappa \) converges for all continuous functions \( K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n}) \) such that \( \| K \|_\infty < L \). Formally, this is

\[ \mathcal{K}_\mu = \{ K \mid \| K \|_\infty < L, \; L > 0 \}. \]

**Proof** See Liberty and Mou [63].
Definition 4.6.9 (Total H Matrix)
Let \( H_i(\tau) \) denote the solution to the \( i \)th Riccati-like cumulant-generating equation. Then the total \( H \) matrix associated with the sequence \( \{\mu_i\}_{i=1}^{\infty} \) satisfying a \( \rho \)-growth condition is defined as below.

\[
H(\tau) = \sum_{i=1}^{\infty} \mu_i H_i(\tau)
\]

Definition 4.6.10 (Trace Measures)
Let the total \( H \) matrix be defined as above and consider the quantity

\[
z(\tau) = \text{Tr}(H(\tau)) = \text{Tr}(\sum_{i=1}^{\infty} \mu_i H_i(\tau)).
\]

Lemma 4.6.11 (kCC Existence)
Let \( z \) be the trace measure as defined above. Then \( z \) satisfies the differential inequality

\[
- \frac{dz}{d\alpha} \leq a_1(\alpha)z^2 + b_1(\alpha)z + c_1(\alpha)
\]

\[
z(t_f) = \text{Tr}(Q_f), \; \alpha \in [t_0, t_f]
\]

where

\[
a_1(\alpha) \geq 2\rho \text{Tr}(W), \quad b_1(\alpha) \geq 2\lambda_n((A + BK)^*), \quad c_1(\alpha) \geq \text{Tr}(Q + K^T R K).
\]

Suppose now that \( z \) satisfies the differential equation.

\[
- \frac{dz}{d\alpha} = a_1(\alpha)z^2 + b_1(\alpha)z + c_1(\alpha)
\]

\[
z(t_f) = \text{Tr}(Q_f), \; \alpha \in [t_0, t_f]
\]

Then the cumulant-generating equations \( \{H_i(\tau)\}_{i=1}^{\infty} \) and \( \{D_i(\tau)\}_{i=1}^{\infty} \) have a solution on \([t_0, t_f]\) for the provided \( K \parallel \|K\|_{\infty} < L \) where \( L > 0 \).

Proof See Liberty and Mou [63].

Remark(s) The above existence result applies to the cumulant-generating equations \( \{H_i(\alpha)\}_{i=1}^{\infty} \) and \( \{D_i(\alpha)\}_{i=1}^{\infty} \) for a general control \( K \),

\[
\frac{dH_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)K(\alpha))^T H_1(\alpha) - H_1(\alpha)(A(\alpha) + B(\alpha)K(\alpha))
\]
Let

\[
\frac{dH_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)K(\alpha))^T H_1(\alpha) - H_1(\alpha)(A(\alpha) + B(\alpha)K(\alpha)) - K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha)
\]

\[
\frac{dD_1(\alpha)}{d\alpha} = -\text{Tr}(H_1(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
H_1(t_f) = Q_f, \quad D_1(t_f) = 0.
\]

and the functions \( \tilde{H}_1(\alpha), \tilde{D}_1(\alpha) \) are determined by

\[
\frac{d\tilde{H}_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)\tilde{K}(\alpha))^T \tilde{H}_1(\alpha) - \tilde{H}_1(\alpha)(A(\alpha) + B(\alpha)\tilde{K}(\alpha)) - \tilde{K}^T(\alpha)R(\alpha)\tilde{K}(\alpha) - Q(\alpha)
\]

\[
\frac{d\tilde{D}_1(\alpha)}{d\alpha} = -\text{Tr}(\tilde{H}_1(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t, t_f]
\]

\[
\tilde{H}_1(t_f) = Q_f \quad \tilde{D}_1(t_f) = 0.
\]

Assume further that \( \text{Nul}(G) = \emptyset \), and that either of the following conditions hold,

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}
\tag{4.46}
\]

or

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix} \succeq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \quad \forall \tau \in [t, t_f]. \tag{4.47}
\]

Under these restrictions, \( \text{Tr}(H_1(\tau) - \tilde{H}_1(\tau)) = \text{Tr}(\Delta H_1(\tau)) = 0 \) on \( \tau \in [t, t_f] \).

**Proof** Let \( \Delta H_1(\tau) \succ 0^{n \times n} \) by (4.46) so that \( \text{Tr}(H_1(\tau) - \tilde{H}_1(\tau)) = \text{Tr}(\Delta H_1(\tau)) \geq 0, \forall \tau \in [t, t_f] \).
and suppose that \( \kappa_1(t) - \tilde{\kappa}_1(t) = 0 \). Under these conditions, the contradiction is reached, because \( \lambda_1(S(\tau)) > 0, \forall \tau \)

\[
0 < \lambda_1(x_0 x_0^T) \text{Tr}(\Delta H_1(t)) - \int_{t_f}^{t} \lambda_1(S(\tau)) \text{Tr}(\Delta H_1(\tau)) d\tau \\
\leq \text{Tr}(x_0 x_0^T) \Delta H_1(t) - \int_{t_f}^{t} \text{Tr}(S(\tau) \Delta H_1(\tau)) d\tau \\
= \kappa_1(t) - \tilde{\kappa}_1(t) = 0.
\]

Assume again \( \kappa_1(t) - \tilde{\kappa}_1(t) = 0 \) but instead \( \Delta H_1(\tau) \preceq 0^{n \times n} \) by (4.47), so that \( \text{Tr}(H_1(\tau) - \tilde{H}_1(\tau)) = \text{Tr}(\Delta H_1(\tau)) \leq 0, \forall \tau \in [t, t_f] \). Once again, a contradiction is reached

\[
\lambda_1(x_0 x_0^T) \text{Tr}(\Delta H_1(t)) - \int_{t_f}^{t} \lambda_1(S(\tau)) \text{Tr}(\Delta H_1(\tau)) d\tau \\
\geq \text{Tr}(x_0 x_0^T) \Delta H_1(t) - \int_{t_f}^{t} \text{Tr}(S(\tau) \Delta H_1(\tau)) d\tau \\
= \kappa_1(t) - \tilde{\kappa}_1(t) = 0.
\]

Hence under either (4.46) or (4.47) with \( \kappa_1(t) - \tilde{\kappa}_1(t) = 0 \), it must be that \( \text{Tr}(H_1(\tau) - \tilde{H}_1(\tau)) = \text{Tr}(\Delta H_1(\tau)) = 0, \forall \tau \in [t, t_f] \).

\[ \Box \]

**Remark(s)** In the above proof, the integrands \( \lambda_1(S(\tau)) \text{Tr}(\Delta H_1(\tau)) \) are given the indication of strict positivity and negativity, and this reflects the assumption \( \lambda_i(S(\tau)) \text{Tr}(\Delta H_1(\tau)) \in C([t_0, t_f]; \mathbb{R}), \forall i \). Under this membership, the integrands can either be zero everywhere or asymptotically approaching zero. The functions exhibiting certain behavior only on sets of zero measure, that is either equality with zero or strict positivity (respectively, strict negativity), implies that the functions are discontinuous in nature so this is not considered. Given the objective at hand and the fact that a zero integrand represents a trivial case, it is regarded that the integrand makes an asymptotic approach to zero, and therefore indications of strict positivity (negativity) are retained.
Theorem 4.6.13 (Zero-Crossings, Condition 2) Consider the process with linear dynamics and additive noise defined by
\[
dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t), \ t \in [t_0, t_f]
\]
x_0 = E\{x(t_0)\}, \ x_0 \in \mathbb{R}^n
\]
where \( A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m}) \), \( G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p}) \), \( w(t) \) is a \( p \)-dimensional stationary Wiener process having a correlation of increments defined by
\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|_{\tau_1 - \tau_2}|, W > 0^{p \times p}.
\]
Consider a cost \( J[x, u; t_0, x_0] \) that is an integral-quadratic form and defined by
\[
J[x, u; t_0, x_0] = \int_{t_0}^{t_f} (x(\tau)^TQ(\tau)x(\tau) + u(\tau)^TR(\tau)u(\tau)) d\tau + x(t_f)^TQ_fx(t_f).
\]
Subject the process to linear state-feedback control, \( u = Kx \). Let \( \text{Nul}(G) = \emptyset \) and assume \( K \) and \( \tilde{K} \) are two control gains satisfying either
\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
b A + B \tilde{K}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
b A + BK
\end{bmatrix}
\]
(4.48) \( ||(A + BK)^*||_2 - ||(A + B \tilde{K})^*||_2 \geq 0 \)
or
\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
b A + B \tilde{K}
\end{bmatrix} \succeq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
b A + BK
\end{bmatrix}, \ \forall \tau \in [t_0, t_f]
\]
(4.49) \( ||(A + B \tilde{K})^*||_2 - ||(A + BK)^*||_2 \geq 0 \).

Assume further that the following scalar Riccati equations have a solution,
\[
-\frac{dz(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)z^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)K(\alpha))^*)z(\alpha) + \text{Tr}(Q(\alpha) + K^T(\alpha)R(\alpha)K(\alpha))
\]
(4.50)
\[
-\frac{d\hat{z}(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)\hat{z}^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)\tilde{K}(\alpha))^*)\hat{z}(\alpha) + \text{Tr}(Q + K^T R \tilde{K}), \ \alpha \in [t_0, t_f]
\]
z(t_f) = \hat{z}(t_f) = \text{Tr}(Q_f).
\]
If these conditions are met, then \( \kappa_1(\tau) - \tilde{\kappa}_1(\tau) \neq 0 \) on \( \tau \in [t_0, t_f] \), where the means are defined by \( \kappa_1(\alpha) = x_0^T H_1(\alpha) x_0 + D_1(\alpha) \) and \( \tilde{\kappa}_1(\alpha) = x_0^T \tilde{H}_1(\alpha) x_0 + \tilde{D}_1(\alpha) \) and \( H_1(\alpha), D_1(\alpha) \) follow the equations of motion

\[
\frac{dH_1(\alpha)}{d\alpha} = - (A(\alpha) + B(\alpha) K(\alpha))^T H_1(\alpha) - H_1(\alpha) (A(\alpha) + B(\alpha) K(\alpha)) \\
- K^T(\alpha) R(\alpha) K(\alpha) - Q(\alpha) \equiv F_1(H_1(\alpha), K(\alpha))
\]

\[
\frac{dD_1(\alpha)}{d\alpha} = - \text{Tr}(H_1(\alpha) G(\alpha) W G^T(\alpha)), \; \alpha \in [t_0, t_f] \\
H_1(t_f) = Q_f, \; D_1(t_f) = 0.
\]

and the functions \( \tilde{H}_1(\alpha), \tilde{D}_1(\alpha) \) are determined by

\[
\frac{d\tilde{H}_1(\alpha)}{d\alpha} = - (A(\alpha) + B(\alpha) \tilde{K}(\alpha))^T \tilde{H}_1(\alpha) - \tilde{H}_1(\alpha) (A(\alpha) + B(\alpha) \tilde{K}(\alpha)) \\
- \tilde{K}^T(\alpha) R(\alpha) \tilde{K}(\alpha) - Q(\alpha)
\]

\[
\frac{d\tilde{D}_1(\alpha)}{d\alpha} = - \text{Tr}(\tilde{H}_1(\alpha) G(\alpha) W G^T(\alpha)), \; \alpha \in [t_0, t_f] \\
\tilde{H}_1(t_f) = Q_f, \; \tilde{D}_1(t_f) = 0.
\]

**Proof** Consider the sequence \( \{\mu_i\}_{i=1}^\infty \) where \( \mu_1 = 1 \) and \( \mu_i = 0 \) for \( i \geq 2 \). It is obvious that this sequence satisfies a \( \rho \)-growth condition \( \forall \rho > 0 \). This sequence is used to characterize the total \( H \) and total \( \tilde{H} \) matrices,

\[
H(\tau) = \sum_{i=1}^\infty \mu_i H_i(\tau) = H_1(\tau), \; \tilde{H}(\tau) = \sum_{i=1}^\infty \mu_i \tilde{H}_i(\tau) = \tilde{H}_1(\tau)
\]

A contradiction is pursued by assuming that \( \kappa_1(t) - \tilde{\kappa}(t) = 0 \) and \( \kappa_1(\tau) - \tilde{\kappa}(\tau) \neq 0 \) for \( \tau \in (t, t_f] \) under the conditions given in the theorem. Appealing to the necessity result above, it is evident under these conditions that \( \text{Tr}(\Delta H_1(\tau)) = 0, \; \forall \tau \in [t, t_f] \), which means \( z(\tau) = \text{Tr}(H_1(\tau)) = \text{Tr}(\tilde{H}_1(\tau)) = \tilde{z}(\tau), \; \forall \tau \in [t, t_f] \). By the assumption
(4.50), \(z(\tau)\) and \(\tilde{z}(\tau)\) satisfy the scalar Riccati differential equations

\[
-\frac{dz(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)z^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)K(\alpha))^*)z(\alpha) + \text{Tr}(Q(\alpha) + K^T(\alpha)R(\alpha)K(\alpha))
\]

\[
-\frac{d\tilde{z}(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)\tilde{z}^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)\tilde{K}(\alpha))^*)\tilde{z}(\alpha) + \text{Tr}(Q(\alpha) + \tilde{K}^T(\alpha)R(\alpha)\tilde{K}(\alpha)), \ \alpha \in [t, t_f]
\]

\[z(t_f) = \tilde{z}(t_f) = \text{Tr}(Q_f).\]

Since \(z(\tau) = \tilde{z}(\tau)\), it is clear that \(z(\tau)\) will satisfy both equations of motion on \([t, t_f]\).

In particular,

\[
-\frac{dz(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)z^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)K(\alpha))^*)z(\alpha) + \text{Tr}(Q(\alpha) + K^T(\alpha)R(\alpha)K(\alpha))
\]

\[
-\frac{d\tilde{z}(\alpha)}{d\alpha} = 2\rho \text{Tr}(W)\tilde{z}^2(\alpha) + 2\lambda_n((A(\alpha) + B(\alpha)\tilde{K}(\alpha))^*)\tilde{z}(\alpha) + \text{Tr}(Q(\alpha) + \tilde{K}^T(\alpha)R(\alpha)\tilde{K}(\alpha)), \ \alpha \in [t, t_f]
\]

\[z(t_f) = \tilde{z}(t_f) = \text{Tr}(Q_f).\]

Obviously if \(\dot{z} = f_1(t, z) = f_2(t, z)\) it must be that \(f_1(t, z) - f_2(t, z) = 0\). Since the coefficients of \(z^2\) are the same for both equations of motion, subtracting gives

\[
\frac{2(\lambda_n((A + BK)^*) - \lambda_n((A + B\tilde{K})^*))}{\Delta b} z + \frac{\text{Tr}(K^T R K - \tilde{K}^T R \tilde{K})}{\Delta c} = 0
\]

\[\Delta b \cdot z + \Delta c = 0.\]

Without loss of generality, this proof address only the case when (4.48) establishes a contradiction. Note that when the conditions (4.49) are true, similar reasoning results in an analogous contradiction.
Recall that (4.48) states,

\[
\begin{bmatrix}
\hat{K}^T R \hat{K} + Q & (A + B \hat{K})^T \\
A + B \hat{K} & 0^{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}
\]

when \( H_1(t_f) = \tilde{H}_1(t_f) = Q_f \). This alternatively can be written as

\[
\Xi = \begin{bmatrix}
K^T R K - \hat{K}^T R \hat{K} & (A + BK)^T - (A + B \hat{K})^T \\
(A + BK) - (A + B \hat{K}) & 0^{n \times n}
\end{bmatrix} \succeq 0^{2n \times 2n}.
\]

Since each partition matrix is of dimension \( n \times n \), then

\[
\text{Tr}(\Xi) = \text{Tr}(K^T R K - \hat{K}^T R \hat{K}) + \text{Tr}(0^{n \times n})
\]

\[
= \text{Tr}(K^T R K - \hat{K}^T R \hat{K})
\]

\[
= \Delta c \geq 0, \text{ since } \Xi \succeq 0^{2n \times 2n}
\]

Observe that the assumption in the theorem yields,

\[
|| (A + BK)^s ||_2 - || (A + B \hat{K})^s ||_2 = \lambda_n((A + BK)^s) - \lambda_n((A + B \hat{K})^s) = \frac{\Delta b}{2} \geq 0.
\]

Together, these conditions imply

\[
\Delta b \cdot z + \Delta c \geq 0.
\]

Since \( z \geq 0 \), \( \Delta b \cdot z + \Delta c = 0 \) can only have a solution if \( \Delta b = 0 \) and \( \Delta c = 0 \), meaning

\[
\lambda_n((A + BK)^s) = \lambda_n((A + B \hat{K})^s)
\]

\[
\text{Tr}(K^T R K) = \text{Tr}(\hat{K}^T R \hat{K}), \text{ } R > 0^{m \times m}, \text{ } B \text{ arbitrary}.
\]
But for the above equations to hold simultaneously and generally requires that
\( K = \tilde{K} \) on \([t, t_f]\). So for \( t_f > t^* > t \), the following must be true

\[
H_1(t^*) = -\int_{t_f}^{t^*} F_1(H_1(\tau), K(\tau)) d\tau
= -\int_{t_f}^{t^*} F_1(\tilde{H}_1(\tau), \tilde{K}(\tau)) d\tau
= \tilde{H}_1(t^*).
\]

More formally, this stems from the equations

\[
\frac{dH_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)\tilde{K}(\alpha))\tilde{H}_1(\alpha) - H_1(\alpha)(A(\alpha) + B(\alpha)\tilde{K}(\alpha))
- \tilde{K}^T(\alpha) R(\alpha) \tilde{K}(\alpha) - Q(\alpha)
\]

\[
\frac{d\tilde{H}_1(\alpha)}{d\alpha} = -(A(\alpha) + B(\alpha)\tilde{K}(\alpha))^T \tilde{H}_1(\alpha) - \tilde{H}_1(\alpha)(A(\alpha) + B(\alpha)\tilde{K}(\alpha))
- \tilde{K}^T(\alpha) R(\alpha) \tilde{K}(\alpha) - Q(\alpha), \ \alpha \in [t, t_f]
\]

\[
H_1(t_f) = \tilde{H}_1(t_f) = Q_f
\]

from which

\[
H_1(\tau) = \tilde{H}_1(\tau), \ \tau \in [t, t_f].
\]

The above relation follows due to uniqueness of solutions of the cost cumulant-generating equations. It follows immediately that

\[
\kappa_1(t^*) = \text{Tr}(H_1(t^*)x_0x_0^T) - \int_{t_f}^{t^*} \text{Tr}(H_1(\tau)GWG^T) d\tau
= \text{Tr}(\tilde{H}_1(t^*)x_0x_0^T) - \int_{t_f}^{t^*} \text{Tr}(\tilde{H}_1(\tau)GWG^T) d\tau
= \tilde{\kappa}_1(t^*).
\]
Since by selection \( t^* > t \) and \( \kappa_1(t^*) - \tilde{\kappa}(t^*) = 0 \), a contradiction results, hence under the conditions of the theorem on \( K \) and \( \tilde{K} \) there cannot be a zero-crossing time \( t \) such that \( \kappa_1(t) - \tilde{\kappa}(t) = 0 \) and \( \kappa_1(\tau) - \tilde{\kappa}(\tau) \neq 0, \ \tau \in (t, t_f] \) unless \( K = \tilde{K} \) and thus \( \kappa_1(\tau) = \tilde{\kappa}_1(\tau), \ \forall \tau \in (t, t_f] \).

\[ \square \]

### 4.6.2 Volterra Conditions for Delta-Mean Zero-Crossings

Herein, a delta mean zero-crossing result is provided using Volterra Integral equations of the second kind (\( VIE2 \)). The development begins with some assumptions that are necessary in the proofs. In particular, some non-restrictive requirements on the system characterization \((A, B, G)\) and also the control gains \( K \) and \( \tilde{K} \) are made, which ultimately aid the identification of conditions that ensure \( \kappa_1(\tau) - \tilde{\kappa}_1(\tau) \neq 0 \) everywhere on the time horizon \([t_0, t_f]\).

For general control gains \( K \) and \( \tilde{K} \) and fixed \( R, A, B, \) and \( Q \) it is assumed one of the following conditions hold,

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0^{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \ \forall \tau \in [t, t_f]
\]

or

\[
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0^{n \times n}
\end{bmatrix} \succeq \begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \ \forall \tau \in [t, t_f].
\]

The system dynamics are such that the additive noise has independent effects
on the individual dimensions of the process. Formally, this requires that

$$\text{Nul}(G) = \emptyset.$$  

It assumed in the following that $$S \triangleq GWG^T \in \mathcal{C}([t_0, t_f]; \mathbb{R}^n_+)$$ implies that

$$\lambda_i(S) \in \mathcal{C}([t_0, t_f]; \mathbb{R}^+), \ 1 \leq i \leq n.$$  

Finally, it is assumed that the known initial state for the process $$x_0 \neq 0^n$$. This ensures that $$\lambda_n(x_0x_0^T) > 0$$ for the rank-one matrix, positive semi-definite matrix $$x_0x_0^T$$, where $$\lambda_n(x_0x_0^T)$$ denotes its largest eigenvalue.

Under these conditions the value of $$\Delta z(\tau) = z(\tau) - \tilde{z}(\tau)$$ is either strictly positive or negative on $$[t_0, t_f]$$ when the solution of a Volterra integral equation admits a strictly positive solution. This result will be established in the following. Beforehand, however, some definitions are given that are key in what ensues.

**Definition 4.6.14 (Delta Mean Volterra Kernel)**

Let $$K_{\Delta\kappa_1}(t, \tau)$$ denote the delta mean Volterra kernel, which is non-negative and continuous on $$[t, t_f]$$ where $$t \in [t_0, t_f]$$. Define this kernel as below.

$$K_{\Delta\kappa_1}(t, \tau) = \begin{cases} \frac{\lambda_n(GWG^T)(\tau)}{\lambda_n(x_0x_0^T)}, & \tau \in [t, t_f] \\ 0, & \tau \in [t_0, t] \end{cases}$$

**Definition 4.6.15 (Delta Mean Function)**

Let $$f_{\Delta\kappa_1}(\tau)$$ denote the delta mean function, which is continuous on $$[t_0, t_f]$$. Define the function as below.

$$f_{\Delta\kappa_1}(\tau) = \frac{\kappa_1(\tau) - \tilde{\kappa}_1(\tau)}{\lambda_n(x_0x_0^T)}$$

Volterra Integral equations are a topic of considerable subject matter that receive attention in just about any book on functional analysis. See, for instance,
Due to the preference for conciseness here, only some of the most essential definitions and results associated with $VIE2$ are shown below.

**Definition 4.6.16 (Integral Kernel)**

Consider the function $K(t, \tau) \in C([t, t_f]; \mathbb{R})$ where $t \in [t_0, t_f]$ that is non-negative on its domain. This will be known as the Volterra Integral Kernal.

**Theorem 4.6.17 (Unique Solutions)**

Suppose that $f(\tau) \in C([t_0, t_f]; \mathbb{R})$ and $K(t, \tau) \in C([t, t_f]; \mathbb{R})$ for $t \in [t_0, t_f]$. Then $VIE2$ has a unique solution $\alpha(\tau) \in C([t_0, t_f]; \mathbb{R})$ for every constant $c$.

**Proof** See Kreyszig [13], pg. 322.

Comparison lemmas are prevalent for differential equations in the literature, and in fact are the cornerstone of Liberty’s and Mou’s existence result for $kCC$ controllers. Interestingly enough, there are analogous comparison lemmas for integral equations, some of which pertain to integral equations of the Volterra form. It is a comparison theorem attributed to Beesack that is used in the following. This is presented, and then the main results.

**Theorem 4.6.18 (Comparison Theorem for Solutions of $VIE2$)**

Let $f(\tau)$ and $K(t, \tau)$ be defined as above. Let $\alpha(\tau)$ and $\beta(\tau)$ denote the solutions to the following integral equations

\[
\alpha(t) - c \int_{t_f}^{t} K(t, \tau)\alpha(\tau) d\tau = f(t)
\]

\[
\beta(t) - c \int_{t_f}^{t} K(t, \tau)\beta(\tau) d\tau \leq f(t).
\]

Then $\beta(\tau) \leq \alpha(\tau)$ almost everywhere (a.e.) on $[t_0, t_f]$. Conversely, if $\alpha(\tau)$ and $\beta(\tau)$ denote the solutions to the integral equations

\[
\alpha(t) - c \int_{t_f}^{t} K(t, \tau)\alpha(\tau) d\tau = f(t)
\]

\[
\beta(t) - c \int_{t_f}^{t} K(t, \tau)\beta(\tau) d\tau \geq f(t).
\]

then $\beta(\tau) \geq \alpha(\tau)$ a.e. on $[t_0, t_f]$. 

183
Under the assumptions given for the Zero-Crossings, Volterra Condition, the value of \( \Delta z(\tau) = z(\tau) - \hat{z}(\tau) \) is either strictly positive or negative on \([t_0, t_f]\) when the solution of a Volterra integral equation admits a strictly positive solution. This result will be established in the following. The following lemma is needed in order to establish the sufficient condition.

**Lemma 4.6.19 (Solutions to Integral Inequalities)**

The function \( \Delta z(\tau) = \text{Tr}(\Delta H_1(\tau)) \) is the solution to the following Volterra integral inequality.

\[
\Delta z(t) - \int_{t_f}^{t} K_{\Delta \kappa_1}(t, \tau) \Delta z(\tau) d\tau \geq f_{\Delta \kappa_1}(t)
\]

when

\[
\begin{bmatrix}
\hat{K}^T R \hat{K} + Q & (A + B \hat{K})^T \\
A + B \hat{K} & 0^{n \times n}
\end{bmatrix}
\preceq
\begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \forall \tau \in [t, t_f].
\]

If instead

\[
\begin{bmatrix}
\hat{K}^T R \hat{K} + Q & (A + B \hat{K})^T \\
A + B \hat{K} & 0^{n \times n}
\end{bmatrix}
\succeq
\begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \forall \tau \in [t, t_f].
\]

then the function \( \Delta z(\tau) = \text{Tr}(\Delta H_1(\tau)) \) is the solution to the Volterra integral inequality

\[
\Delta z(t) - \int_{t_f}^{t} K_{\Delta \kappa_1}(t, \tau) \Delta z(\tau) d\tau \leq f_{\Delta \kappa_1}(t).
\]

**Proof** Assume for fixed \( A, B, R, \) and control gains \( K \) and \( \hat{K} \),

\[
\begin{bmatrix}
\hat{K}^T R \hat{K} + Q & (A + B \hat{K})^T \\
A + B \hat{K} & 0^{n \times n}
\end{bmatrix}
\preceq
\begin{bmatrix}
K^T R K + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \forall \tau \in [t, t_f].
\]
so that $\Delta H_1(\tau) \preceq 0^{n \times n}$ on $[t, t_f]$. It follows that

$$
\lambda_n(x_0x_0^T)\text{Tr}(\Delta H_1(t)) - \int_{t_f}^t \lambda_n(G(\tau)WG^T(\tau))\text{Tr}(\Delta H_1(\tau))d\tau \geq \kappa_1(t) - \tilde{\kappa}_1(t)
$$

from which

$$
\lambda_n(x_0x_0^T) \left( \text{Tr}(\Delta H_1(t)) - \int_{t_f}^t K_{\Delta \kappa_1}(t, \tau)\text{Tr}(\Delta H_1(\tau))d\tau \right) \geq \lambda_n(x_0x_0^T) \left( \frac{\kappa_1(t) - \tilde{\kappa}_1(t)}{\lambda_n(x_0x_0^T)} \right) = \lambda_n(x_0x_0^T)f_{\Delta \kappa_1}(t)
$$

so that clearly

$$
\Delta z(t) - \int_{t_f}^t K_{\Delta \kappa_1}(t, \tau)\Delta z(\tau)d\tau \geq f_{\Delta \kappa_1}(t).
$$

Reasoning in the same manner, if

$$
\begin{bmatrix}
\tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
A + B \tilde{K} & 0^{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
K^T RK + Q & (A + BK)^T \\
A + BK & 0^{n \times n}
\end{bmatrix}, \forall \tau \in [t, t_f].
$$

then $\Delta H_1(\tau) \preceq 0^{n \times n}$ on $[t, t_f]$. It is immediate that

$$
\lambda_n(x_0x_0^T)\text{Tr}(\Delta H_1(t)) - \int_{t_f}^t \lambda_n(G(\tau)WG^T(\tau))\text{Tr}(\Delta H_1(\tau))d\tau \leq \kappa_1(t) - \tilde{\kappa}_1(t)
$$

from which

$$
\lambda_n(x_0x_0^T) \left( \text{Tr}(\Delta H_1(t)) - \int_{t_f}^t K_{\Delta \kappa_1}(t, \tau)\text{Tr}(\Delta H_1(\tau))d\tau \right) \leq \lambda_n(x_0x_0^T) \left( \frac{\kappa_1(t) - \tilde{\kappa}_1(t)}{\lambda_n(x_0x_0^T)} \right) = \lambda_n(x_0x_0^T)f_{\Delta \kappa_1}(t_0)
$$

185
so that

\[ \Delta z(t) - \int_{t_0}^{t_f} K_{\Delta \kappa_1}(t, \tau) \Delta z(\tau) d\tau \leq f_{\Delta \kappa_1}(t). \]

\[ \square \]

**Theorem 4.6.20** (Zero-Crossings - Volterra Condition)

Consider a process with linear dynamics and additive noise defined by

\[ dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t), \; t \in [t_0, t_f] \]

\[ x(t_0) = x_0, \; x_0 \in \mathbb{R}^n \]

where \( A \in C([t_0, t_f]; \mathbb{R}^{n \times n}), \) \( B \in C([t_0, t_f]; \mathbb{R}^{n \times m}), \) \( G \in C([t_0, t_f]; \mathbb{R}^{n \times p}), \) \( w(t) \) is a \( p \)-dimensional stationary Wiener process having a correlation of increments defined by

\[ E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}. \]

Consider a cost \( J[x, u; t_0, x_0] \) that is an integral-quadratic form and defined by

\[ J[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left( x(\tau)^T Q(\tau)x(\tau) + u(\tau)^T R(\tau)u(\tau) \right) d\tau + x(t_f)^T Q f x(t_f). \]

Subject the process to linear state-feedback control, \( u = K x, \) and assume also that the process may be alternatively controlled by \( u = \tilde{K} x. \) Let \( \text{Nul}(G) = \emptyset \) and \( K \) and \( \tilde{K} \) be two control gains satisfying

\[ \begin{bmatrix} K^T R \tilde{K} + Q & (A + B \tilde{K})^T \\ A + B \tilde{K} & 0^{n \times n} \end{bmatrix} \preceq \begin{bmatrix} K^T R K + Q & (A + BK)^T \\ A + BK & 0^{n \times n} \end{bmatrix}. \]

so that \( \Delta H_1(\tau) \succeq 0^{n \times n} \) and assume further the following integral equation admits a strictly positive solution \( \beta(t), \)

\[ \beta(t) > 0 \mid \beta(t) - \int_{t_0}^{t_f} K_{\Delta \kappa_1}(t, \tau) \beta(\tau) d\tau = f_{\Delta \kappa_1}(t) \tag{4.53} \]

where \( t \in [t_0, t_f]. \) Then the difference \( \kappa_1(t) - \tilde{\kappa}_1(t) > 0. \)
Proof Consider that previously it was shown that
\[
\begin{bmatrix}
  \tilde{K}^T R \tilde{K} + Q & (A + B \tilde{K})^T \\
  A + B \tilde{K} & 0_{n \times n}
\end{bmatrix} \preceq \begin{bmatrix}
  K^T R K + Q & (A + BK)^T \\
  A + BK & 0_{n \times n}
\end{bmatrix}.
\]
is sufficient for $\Delta z(t)$ to be a solution to the following integral equation,
\[
\Delta z(t) - \int_{t_f}^t K_{\Delta \kappa_1}(t, \tau) \Delta z(\tau) d\tau \geq f_{\Delta \kappa_1}(t).
\]
and this guarantees that $\Delta H_1(\tau) \succeq 0_{n \times n}$. Assume as in the theorem statement that the following integral equation admits a strictly positive solution $\beta(t)$,
\[
\beta(t) > 0 \mid \beta(t) - \int_{t_f}^t K_{\Delta \kappa_1}(t, \tau) \beta(\tau) d\tau = f_{\Delta \kappa_1}(t)
\]
where $t \in [t_0, t_f]$. By the Comparison Theorem, it follows immediately that $0 < \beta(t) \leq \Delta z(t)$, and thus
\[
\lambda_1(x_0 x_0^T) \beta(t) - \int_{t_f}^t \lambda_1(G(\tau)WG(\tau)^T) \beta(\tau) d\tau = \lambda_1(x_0 x_0^T) \beta(t) - \int_{t_f}^t \lambda_1(G(\tau)WG(\tau)^T) \beta(\tau) d\tau > 0
\]
for $G(\tau)WG(\tau)^T > 0_{n \times n}$ with $\text{Nul}(G) = \emptyset$. Hence, it follows that
\[
0 < \lambda_1(x_0 x_0^T) \beta(t) - \int_{t_f}^t \lambda_1(G(\tau)WG(\tau)^T) \beta(\tau) d\tau
\]
\[
\leq \lambda_1(x_0 x_0^T) \Delta z(t) - \int_{t_f}^t \lambda_1(G(\tau)WG(\tau)^T) \Delta z(\tau) d\tau
\]
\[
\leq \text{Tr}(x_0 x_0^T \Delta H_1(t)) - \int_{t_f}^t \text{Tr}(G(\tau)WG(\tau)^T \Delta H_1(\tau)) d\tau
\]
\[
= \kappa_1(t) - \tilde{\kappa}(t).
\]
CHAPTER 5
INFINITE-HORIZON MCCDS OPTIMAL CONTROL

5.1 Introduction

Usually no fixed terminal time is specified in control applications. It would therefore be ideal that the MCCDS control have a good long-run performance. This practical situation involves the system being controlled over a time interval extending to infinity. This chapter investigates the aforementioned case in order to complete the theory of linear state-feedback MCCDS control. For completeness, the results of Pham [26] are presented that consider the cost cumulants when \( t_f \to \infty \). In particular, the resulting forms of cost cumulants per unit time are given here, in addition to the limiting forms of the cost cumulant-generating equations of the LQG framework. This theory is necessary to formulate the MCCDS optimization on the infinite-horizon. An adaptation of the Lagrange multiplier theory to the MCCDS problem is used to derive the infinite-horizon MCCDS optimal control in conjunction with an established cancellation property.

After these results are presented, the MCCDS infinite-horizon optimization is formulated and the solution for the state-feedback MCCDS control is established. In conclusion of the chapter, the infinite-horizon MCCDS optimal control algorithm is provided. The first-generation AMD benchmark is used to validate the algorithm. An abbreviated form of this development with the same application can be found in [43].

188
5.2 Preliminaries

For the theory of averaged cost cumulants on the infinite-horizon, the same restriction will apply to the integral-quadratic costs and linear systems with additive white Gaussian noise that constitute the LQG framework. Once again the system and cost are presented, however this time with the assumption that state-feedback control is employed. Hence the development assumes a general closed-loop $F(t)$ matrix versus the closed-loop matrix $A(t) + B(t)K(t)$. As such, the preliminary discussion will not center on $A(t)$, $B(t)$, or $K(t)$ directly. Rather, general conditions will be studied such that the cost cumulant-generating equations of the previously-studied finite-horizon case are bounded above, and convergent to equilibrium solutions.

The following development appears in Pham [26], where an excellent treatment is given to the properties of cost cumulant-generating equations on the infinite-horizon. A subset of lemmas and a theorem is restated to provide a background of the development of infinite-horizon MCCDS control theory. The proofs have been excluded here for conciseness, though Pham’s dissertation can be consulted for the details. To begin, the definitions of the process and cost are reiterated.

**Definition 5.2.1 (Process)**
Consider the stochastic process defined on a separable probability space $(\Omega, F, P)$ with linear dynamics and additive white Gaussian noise

$$dx(t) = F(t)dt + G(t)dw(t), \quad t \in [t_0, t_f]$$

$$x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n$$

where the initial state $x_0$ is assumed to be the constant mean of the process at the initial time, that is $x_0 = E[x(t_0)]$. Above, the matrix $F(t) \in C([t_0, t_f]; \mathbb{R}^{n \times n})$ and $w(t)$ is a $p$-dimensional stationary Wiener process having a correlation of increments defined by

$$E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, \quad W \succ 0^{p \times p}.$$
**Definition 5.2.2 (Cost)**
The cost $J[x,u;t_0,x_0]$ is integral-quadratic and defined by
\[
J[x,u;t_0,x_0] = \int_{t_0}^{t_f} x^T(\tau) N(\tau) x(\tau) d\tau + x^T(t_f) Q_f x(t_f)
\]  
(5.2)
where $N(\tau) = N(\tau)^T \in C([t_0,t_f];\mathbb{S}^n_+)$ and $Q_f = Q_f^T \in \mathbb{S}^n_+$.

For the well-posedness of the associated stochastic optimal control problem, it is required that $N,Q_f \succeq 0^{n \times n}$. A sufficient condition is now presented for the time-variant deterministic linear dynamic system characterized by $F(t)$ and $x_0$, such that the cost cumulant-generating equations are bounded above.

**Definition 5.2.3 (Exponential Stability)**
Let $F \in C([t_0,t_f];\mathbb{R}^{n \times n})$. Define the deterministic system $S$ with no additive noise as
\[
\frac{dx(t)}{dt} = F(t)x(t), \ x(t_0) = x_0.
\]
The system $S$ is said to be exponentially stable if and only if there exists a positive constant $M \in \mathbb{R}_+$ that is independent of $t_0$ and $t_f$ such that
\[
\int_{t_0}^{t_f} \|\Phi(\tau,t_0)\|^2 d\tau \leq M
\]
where $\Phi(t,t_0)$ is the state-transition matrix for the system $S$ that satisfies
\[
\frac{\Phi(t,t_0)}{dt} = F(t)\Phi(t,t_0), \ \Phi(t_0,t_0) = I^{n \times n}.
\]

With the assumption of exponential stability of the system $S$, the upper bounds of the solutions to the cost cumulant-generating equations can be formally established. The following lemma states the result.

**Lemma 5.2.4 (Upper Bounds, Solutions of Cumulant-Generating Equations)**
Let $H_l(\alpha,t_f)$, $1 \leq l \leq r$ denote the solutions to the matrix-differential equations,

\[
\frac{dH_1(\alpha)}{d\alpha} = -F^T(\alpha)H_1(\alpha) - H_1(\alpha)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_l(\alpha)}{d\alpha} = -F^T(\alpha)H_l(\alpha) - H_l(\alpha)F(\alpha)
\]

\[
- \sum_{j=1}^{l-1} H_l(\alpha)G(\alpha)WG^T(\alpha)H_{l-j}(\alpha), \ 2 \leq l \leq r
\]

satisfying the terminal conditions

\[
H_1(t_f) = Q_f, \ H_l(t_f) = 0^{n \times n}, \ 2 \leq l \leq r.
\]

The matrix norm of the solutions $H_l(\alpha,t_f)$, $1 \leq l \leq r$ are bounded above as

\[
||H_l(\alpha)|| \leq \Gamma_l, \ 1 \leq l \leq r
\]

where the bounds $\Gamma_l$ are given as follows

\[
\Gamma_1 = M_2N_{\text{max}}
\]

\[
\Gamma_l = \sum_{j=1}^{l-1} \Gamma_j \Gamma_{l-j}G_{\text{max}}^2 ||W||M_2, \ 2 \leq l \leq r
\]

and the matrices $M_2$, $G_{\text{max}}$, and $N_{\text{max}}$ are defined as

\[
N_{\text{max}} \triangleq \max_{t \in [t_0,t_f]} N(t), \ G_{\text{max}} \triangleq \max_{t \in [t_0,t_f]} G(t)
\]

\[
\lim_{t_f \to \infty} \int_{t_0}^{t_f} ||\Phi(\tau,\alpha)||^2 d\tau \leq M_2.
\]

**Proof** See [26].

In the interest of clarity when presenting results dealing with the nature of solutions to the cost cumulant-generating equations as $t_f \to \infty$, it is expedient to make the terminal time *explicit* in the definition of the equations of motion for the higher-order cost statistics. This is done in the following.

**Definition 5.2.5** *(Cost Cumulant-Generating Equations, Explicit Terminal Time)*
Let $H_l(\alpha, t_f)$, $1 \leq l \leq r$ denote the solutions to the matrix-differential equations,

\[
\frac{dH_1(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_1(\alpha, t_f) - H_1(\alpha, t_f)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_l(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_l(\alpha, t_f) - H_l(\alpha, t_f)F(\alpha)
\]

\[- \sum_{j=1}^{l-1} H_l(\alpha, t_f)G(\alpha)WG^T(\alpha)H_{l-j}(\alpha, t_f), \ 2 \leq l \leq r
\]

These functions satisfy the terminal conditions

\[
H_1(t_f, t_f) = Q_f, \ H_l(t_f, t_f) = 0^{n \times n}, \ 2 \leq l \leq r.
\]

Interestingly enough, the monotonicity of matrix solutions to the cost cumulant-generating equations is preserved as the terminal time is increased without bound. This is the subject of the next lemma.

**Lemma 5.2.6 (Monotonicity of Solutions as $t_f \to \infty$)**

Let $H_l(\alpha, t_f)$, $1 \leq l \leq r$ denote the solutions to the matrix-differential equations,

\[
\frac{dH_1(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_1(\alpha, t_f) - H_1(\alpha, t_f)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_l(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_l(\alpha, t_f) - H_l(\alpha, t_f)F(\alpha)
\]

\[- \sum_{j=1}^{l-1} H_l(\alpha, t_f)G(\alpha)WG^T(\alpha)H_{l-j}(\alpha, t_f), \ 2 \leq l \leq r
\]

and suppose the system $S$ with $F \in C([t_0, t_f]; \mathbb{R}^{n \times n})$ is exponentially stable, with $G \in C([t_0, t_f]; \mathbb{R}^{n \times p})$ defined, and $N \in C([t_0, t_f]; \mathbb{S}_+^n)$, and $W \in C([t_0, t_f]; \mathbb{S}^{n \times n}_{++})$. Then the solutions $\{H_l(\alpha, t_f)\}_{l=1}^r$ and $\{H_l(\alpha, t_f + \Delta t_f)\}_{l=1}^r$ satisfy the relation

\[
H_l(\alpha, t_f + \Delta t_f) \succeq H_l(\alpha, t_f)
\]

for any $\Delta t_f > 0$ and $t_0 \leq \alpha \leq t_f$. That is, the solutions $H_l(\alpha, t_f)$ are monotone increasing as $t_f \to \infty$.

**Proof** See [26].

In summary, the aforementioned results show that the monotonicity properties of solutions to the cost cumulant-generating equations of the LQG framework
are preserved as the final time $t_f \to \infty$. Furthermore, the results give sufficient conditions under which the solutions to the cost cumulant-generating equations are bounded above. The next theorem is a direct result of applying the monotone convergence theorem when the aforementioned conditions are true [26].

**Theorem 5.2.7 (Equilibrium Solutions, Cost Cumulant-Generating Equations)**

Suppose that the system $S$ with $F \in C([t_0, t_f]; \mathbb{R}^{n \times n})$ is exponentially stable, that $G \in C([t_0, t_f]; \mathbb{R}^{n \times p})$ is defined, along with some $N \in C([t_0, t_f]; \mathbb{S}^n_{++})$ and $W \in C([t_0, t_f]; \mathbb{S}^n_{++})$.

Then the following properties can be established for the bounded solutions $H_l(\alpha, t_f), 1 \leq l \leq r$, of the matrix-differential equations,

\[
\frac{dH_l(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_{l-1}(\alpha, t_f) - H_l(\alpha, t_f)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_l(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_l(\alpha, t_f) - H_{l-1}(\alpha, t_f)F(\alpha)
\]

\[
- \sum_{j=1}^{l-1} H_j(\alpha, t_f)G(\alpha)WG^T(\alpha)H_{l-j}(\alpha, t_f), 2 \leq l \leq r
\]

which satisfy the terminal conditions

\[
H_1(t_f, t_f) = Q_f, \quad H_l(t_f, t_f) = 0^{n \times n}, 2 \leq l \leq r. \quad (5.10)
\]

- The equilibrium solutions $\{\bar{H}_l(\alpha)\}_{l=1}^r$ exist,

\[
\{\bar{H}_l(\alpha)\}_{l=1}^r = \lim_{t_f \to \infty} \{H_l(\alpha, t_f)\}_{l=1}^r
\]

- The equilibrium solutions $\{\bar{H}_l(\alpha)\}_{l=1}^r$ satisfy the following equations,

\[
\frac{d\bar{H}_l(\alpha, t_a)}{d\alpha} = -F^T(\alpha)\bar{H}_l(\alpha, t_a) - \bar{H}_l(\alpha, t_a)F(\alpha) - N(\alpha)
\]

\[
\frac{d\bar{H}_l(\alpha, t_a)}{d\alpha} = -F^T(\alpha)\bar{H}_l(\alpha, t_a) - \bar{H}_{l-1}(\alpha, t_a)F(\alpha)
\]

\[
- \sum_{j=1}^{l-1} \bar{H}_j(\alpha, t_a)G(\alpha)WG^T(\alpha)\bar{H}_{l-j}(\alpha, t_a), 2 \leq l \leq r
\]

and the terminal conditions

\[
\bar{H}_1(t_a, t_a) = Q_f, \quad \bar{H}_l(t_a, t_a) = 0^{n \times n}, 2 \leq l \leq r. \quad (5.12)
\]

on the large finite horizon $[\alpha, t_a]$ as $t_a \to \infty$. 

193
Proof See [26].

5.3 Limiting Forms of Cost Cumulants and Generating Equations

Since the matrix function $N(t) \succeq 0^{n \times n}$, as $t_f \to \infty$ the random cost (5.2) may become infinite, and as a result the notion of cost cumulants is without meaning. Indeed, it can easily be verified that the finite-horizon cost cumulants

$$\kappa_l(\alpha) = x_0^T H_l(\alpha)x_0 + D_l(\alpha)$$

contain the term

$$D_l(\alpha) = \int_{\alpha}^{t_f} \text{Tr}(H_l(\alpha)G(\tau)WG(\tau)^T)d\tau$$

where it can be observed that the integrand is non-negative and increasing in $t_f$, given the monotonic-increasing property of $H_l(\alpha)$. It is immediate that for an unbounded random cost functional $J$, the cost cumulants are undefined.

The MCCDS optimization on the infinite-horizon will be posed, but it cannot involve the cost cumulants as defined on the finite-horizon for the above reason. Hence averaged cost cumulants of the infinite-horizon are used in the problem formulation, which were originally studied by Pham. These values are defined on $[t_0, t_f]$ as $t_f \to \infty$, and give meaning to the notion of cost cumulants on the infinite-horizon for a positive cost function. A starting point for this development is the distinction between zero and non-zero mean parts of the cost cumulants, made now.

**Definition 5.3.1** (Averaged Cost Cumulants, Zero and Nonzero-Mean Parts)
Let \( r \in \mathbb{N} \), and define the nonzero-mean part of the \( l \)th cost cumulant as

\[
\{\kappa_{\text{avg},l}\}_{nz} = \frac{1}{t_f - t_0} l! \cdot 2^{l-1} x_0^T H_l(\alpha, t_f) x_0
\]

and the zero-mean part of the \( l \)th cost cumulant is defined as

\[
\{\kappa_{\text{avg},l}\}_z = \frac{1}{t_f - t_0} l! \cdot 2^{l-1} D_l(\alpha, t_f)
\]

where \( H_l(\alpha, t_f) \) and \( D_l(\alpha, t_f) \) are solutions to the following system of matrix differential equations,

\[
\frac{dH_1(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_1(\alpha, t_f) - H_1(\alpha, t_f)F(\alpha) - N(\alpha)
\]

\[
\frac{dH_l(\alpha, t_f)}{d\alpha} = -F^T(\alpha)H_l(\alpha, t_f) - H_l(\alpha, t_f)F(\alpha)
\]

\[
- \sum_{j=1}^{l-1} H_l(\alpha, t_f)G(\alpha)WG^T(\alpha)H_{l-j}(\alpha, t_f), \quad 2 \leq l \leq r
\]

\[
\frac{dD_m(\alpha, t_f)}{d\alpha} = -\text{Tr}(H_m(\alpha, t_f)G(\alpha)WG^T(\alpha)), \quad 1 \leq m \leq r
\]

which satisfy the terminal conditions

\[
H_1(t_f, t_f) = Q_f, \quad H_l(t_f, t_f) = 0^{n \times n}, \quad 2 \leq l \leq r
\]

\[
D_1(t_f, t_f) = 0, \quad D_2(t_f, t_f) = 1, \quad D_m(t_f, t_f) = 0, \quad 3 \leq m \leq r.
\]

It can be shown that the zero-mean part of the \( l \)th cost cumulant vanishes as \( t_f \to \infty \). However, the nonzero-mean part of the \( l \)th cost cumulant converges to a constant as the terminal time increases without bound. This is the result of the next lemma. Before proceeding, define the following matrices in the forward-time variable \( t \triangleq t_f + t_0 - \alpha \),

\[
\hat{F}(t) = F(t_f + t_0 - t), \quad \hat{G}(t) = G(t_f + t_0 - t),
\]

\[
\hat{N}(t) = N(t_f + t_0 - t), \quad \hat{H}_l(t, t_f) = H_l(t_f + t_0 - t, t_f)
\]
where $\hat{H}_l(t, t_f)$ satisfy the forward-in-time equations

$$
\frac{d\hat{H}_1(t, t_0)}{dt} = \hat{F}^T(t)\hat{H}_1(t, t_0) + \hat{H}_1(t, t_0)\hat{F}(t) + \hat{N}(t)
$$

$$
\frac{d\hat{H}_l(t, t_0)}{dt} = \hat{F}^T(t)\hat{H}_l(t, t_0) + \hat{H}_l(t, t_0)\hat{F}(t) + \sum_{j=1}^{l-1} \hat{H}_l(t, t_0)\hat{G}(t)W\hat{G}^T(t)\hat{H}_{l-j}(t, t_0), \ 2 \leq l \leq r
$$

(5.14)

which satisfy the terminal conditions

$$
\hat{H}_1(t_0, t_0) = Q_f, \ \hat{H}_l(t_0, t_0) = 0^{n \times n}, \ 2 \leq l \leq r.
$$

(5.15)

Lastly, assume that

$$
\lim_{t \to \infty} \{\hat{F}(t), \hat{G}(t), \hat{N}(t)\} = \{F_\infty, G_\infty, N_\infty\}.
$$

Theorem 5.3.2 (Averaged Cost Cumulants, Zero Parts)
Consider the process and cost of the LQG framework,

$$
dx(t) = F(t)dt + G(t)dw(t), \ t \in [t_0, t_f]
$$

$$
x_0 = E\{x(t_0)\}, \ x_0 \in \mathbb{R}^n
$$

$$
J[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_fx(t_f)
$$

where $F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$ with $S$ exponentially stable, $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$, $N(\tau) = N(\tau)^T \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_+)$ and $Q_f = Q_f^T \in \mathbb{S}^p_+$ with

$$
\lim_{t \to \infty} \{\hat{F}(t), \hat{G}(t), \hat{N}(t)\} = \{F_\infty, G_\infty, N_\infty\}.
$$

The Wiener process, $w(t)$, is $p$-dimensional with the correlation of increments,

$$
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W[\tau_1 - \tau_2], \ W \succ 0^{p \times p}.
$$

Let $r \in \mathbb{N}$ be fixed, and let the functions $\hat{H}_l(t, t_0)$ satisfy the forward in time equations

$$
\frac{d\hat{H}_l(t, t_0)}{dt} = \hat{F}^T(t)\hat{H}_l(t, t_0) + \hat{H}_l(t, t_0)\hat{F}(t) + \hat{N}(t)
$$

(5.16)
\[
\frac{d\hat{H}_l(t, t_0)}{dt} = \hat{F}(t)\hat{H}_l(t, t_0) + \hat{H}_l(t, t_0)\hat{F}(t) + \sum_{j=1}^{l-1} \hat{H}_l(t, t_0)\hat{G}(t)W\hat{G}^T(t)\hat{H}_{l-j}(t, t_0), \quad 2 \leq l \leq r
\]

which satisfy the initial conditions

\[
\hat{H}_1(t_0, t_0) = Q_f, \quad \hat{H}_l(t_0, t_0) = 0^{n \times n}, \quad 2 \leq l \leq r.
\] (5.17)

Under these conditions, as \( t_f \to \infty \), the following relations hold,

\[
\{\kappa_{\infty,l}\}_{nz} = \lim_{t \to \infty} \{\kappa_{\text{avg},l}\}_{nz} = 0
\]

\[
\{\kappa_{\infty,l}\}_z = \lim_{t \to \infty} \{\kappa_{\text{avg},l}\}_z = l! \cdot 2^{l-1} \text{Tr}(\hat{H}_l(\infty)G_{\infty}WG_{\infty}), \quad 1 \leq l \leq r
\]

where \( \hat{H}_l(\infty) \) denotes the \( l \)th equilibrium solution at infinity.

**Proof** See [26].

The form of cost cumulants of the infinite horizon has been established. The following theorem restates this result, and establishes that the limiting form of the generating equations for the cost cumulants is a family of algebraic (rather than differential) Lyapunov equations. Under certain control inputs, these algebraic equations give rise to coupled algebraic matrix Riccati equations that must be satisfied by the cost cumulant variables \( \hat{H}_l, 1 \leq l \leq r \).

**Theorem 5.3.3 (Infinite-Horizon Cost Cumulants and Generating Equations)**

Consider the process and cost of the LQG framework,

\[
dx(t) = F(t)dt + G(t)dw(t), \quad t \in [t_0, t_f]
\]

\[
x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n
\]

\[
J[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_fx(t_f)
\]

where \( F \in C([t_0, t_f]; \mathbb{R}^{n \times n}) \) with \( S \) exponentially stable, \( G \in C([t_0, t_f]; \mathbb{R}^{n \times p}) \), \( N(\tau) = N(\tau)^T \in C([t_0, t_f]; \mathbb{S}^n_+) \) and \( Q_f = Q_f^T \in \mathbb{S}^p_+ \) with

\[
\lim_{l \to \infty} \{\hat{F}(t), \hat{G}(t), \hat{N}(t)\} = \{F_\infty, G_\infty, N_\infty\}.
\]

The Wiener process, \( w(t) \), is \( p \)-dimensional with the correlation of increments,

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W(\tau_1 - \tau_2), \quad W > 0^{p \times p}.
\]
Under these conditions, as \( t \to \infty \), the \( l \)th infinite-horizon cost cumulant is,

\[
\kappa_{\infty,l} = \text{Tr}(H_l G \infty W G \infty), \quad 1 \leq l \leq r
\]

where for \( r \in \mathbb{N} \) fixed, the function \( H_l \) satisfies the algebraic Lyapunov equations,

\[
\begin{align*}
\dot{F}^T_{\infty} H_1 + H_1 \dot{F}_{\infty} + N_{\infty} &= 0^{n \times n} \\
\dot{F}^T_{\infty} H_l + H_l \dot{F}_{\infty} + 2 \sum_{j=1}^{l-1} \binom{l}{j} H_j G \infty W G^T \infty H_{l-j} &= 0^{n \times n}, \quad 2 \leq l \leq r.
\end{align*}
\] (5.18)

**Proof** See [26].

5.4 MCCDS Infinite-Horizon Problem

5.4.1 Formulation of the MCCDS Infinite-Horizon Problem

5.4.1.1 Preliminaries

The infinite-horizon MCCDS problem formulation uses Pham’s framework for deriving the infinite-horizon \( k \)CC control solution, as is described in this section and the next. It pertains exclusively to the process with linear dynamics and additive noise,

\[
dx(t) = \left( Ax(t) + Bu(t) \right) dt + Gdw(t),
\]

\[
x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n, \quad t \in [t_0, \infty)
\] (5.19)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p} \) and \( w(t) \) is a \( p \)-dimensional stationary Wiener process having a correlation of increments defined by

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, \quad W > 0^{p \times p}.
\] (5.20)

It is assumed that \( u \in L^2_p(\Omega, \mathcal{C}([t_0, \infty); \mathbb{R}^m)) \) so that \( x \in L^2_p(\Omega, \mathcal{C}([t_0, \infty); \mathbb{R}^n)) \). That is, \( E\{\int_{t_0}^{\infty} u^T(t)u(t)dt\} < \infty \) ensures \( E\{\int_{t_0}^{\infty} x^T(t)x(t)dt\} < \infty \). The cost \( J \) is an integral-
quadratic form defined by

\[ J(u) = \int_{t_0}^{\infty} \left( x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right) dt \]  \hspace{1cm} (5.21)

where \( Q = Q^T \in S_n^+ \) and \( R = R^T \in S_m^+ \). Under this membership, \( Q \succeq 0^{n \times n} \) and \( R \succ 0^{m \times m} \).

5.4.1.2 Cost Cumulants

The form of the cost cumulants on the infinite-horizon is reiterated once more.

**Definition 5.4.1 (Stabilizing Control)**
A control \( u(t) = k(t,x(t)) \) is called stabilizing if the system (5.19) is bounded-input/bounded-state stable. In addition, if \( w(t) = 0^p \) then the origin \( x = 0^n \) is asymptotically stable.

**Definition 5.4.2 (Stabilizing Gain)**
A feedback gain \( K \in \mathbb{R}^{n \times m} \) is stabilizing if the following state-feedback control law is stabilizing, \( u(t) = Kx(t), \ t \in [t_0, \infty) \).

**Theorem 5.4.3 (Infinite-Horizon Cost Cumulants)**
Let \( r \in \mathbb{Z}^+ \), and the matrices \( A, B, G, Q \) and \( R \) be as defined previously. Suppose \( (A,B) \) is stabilizable, so that \( \exists K \) such that the closed-loop matrix \( (A + BK) \) has only eigenvalues with negative real parts and the controlled system (5.19) in the absence of disturbances is exponentially stabilized. Under these conditions, the \( r \) cost cumulants are defined by

\[ \kappa_{\infty,l} = \text{Tr}(H_lGWG^T), \ 1 \leq l \leq r \]

where the matrices \( \{H_l\}_{l=1}^r \) are such that \( H_l \succeq 0^{n \times n} \) and satisfy the equations,

\[ F_1(H, K) \triangleq (A + BK)^T H_1 + H_1(A + BK) + Q_\infty + K^T R_\infty K = 0^{n \times n} \]
\[ F_l(H, K) \triangleq (A + BK)^T H_l + H_l(A + BK) + 2 \sum_{j=1}^{l-1} \binom{l}{j} H_j GWG^T H_{l-j} = 0^{n \times n}. \]  \hspace{1cm} (5.22)

where \( Q_\infty = \lim_{t \to \infty} Q(t) \) and \( R_\infty = \lim_{t \to \infty} R(t) \).
5.4.1.3 Notation

Some notation is introduced to make restatements of the above equations more concise in the development. Begin by defining the state variable $\mathcal{H} \in \mathbb{R}^{r \times n}$ as below

$$\mathcal{H} \triangleq (H_1, \ldots, H_r).$$

Using these state variables, define the function

$$\mathcal{F}(\mathcal{H}, K) \triangleq (\mathcal{F}_1(\mathcal{H}, K), \ldots, \mathcal{F}_r(\mathcal{H}, K)).$$

where $\mathcal{F}_i(\cdot)$ in the above definition is defined as in (5.22).

Also, let the vector of cumulants $\kappa_\infty \in \mathbb{R}^r$ be defined as

$$\kappa_\infty = (\kappa_{\infty, 1}, \ldots, \kappa_{\infty, r}).$$

Where appropriate, the dependence of $\kappa_\infty$ on $\mathcal{H}$ will be indicated by $\kappa_\infty(\mathcal{H})$.

5.4.1.4 Target Cost Statistics

Given matrices for a system characterization $(A, B, G)$, an integral-quadratic cost characterization $(Q, R)$, and the second-order statistics of the noise $(W)$, consider the cost cumulants as a result of the alternative (and unknown) *stabilizing* linear state-feedback control $\hat{u}(t) = \tilde{K}\hat{x}(t)$, where $\tilde{K} \in \mathbb{R}^{m \times n}$. Given the preceding
results, $r$ cost cumulants are given by

$$\tilde{\kappa}_{\infty,l} = \text{Tr}(\tilde{\mathcal{H}}_l G W G^T), \ 1 \leq l \leq r$$

where the positive semi-definite matrices $\{\tilde{\mathcal{H}}_l\}_{l=1}^r$ satisfy the algebraic equations $\mathcal{F}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}}) = 0^{r \times n}$. In the development, the quantities $\{\tilde{\kappa}_{\infty,l}\}_{l=1}^r$ will frequently be referred to as the target cost cumulants.

5.4.1.5 Problem Statement

The optimization problem can be formulated by defining a control space over which the infinite-horizon MCCDS performance index can be minimized. The appropriate definitions precede the problem statement.

**Definition 5.4.4 (Well-Posed Control Law)**

A feedback gain $K$ is well-posed if $\mathcal{F}(\mathcal{H}, K) = 0^{r \times n}$ admits unique solutions $\mathcal{H}_l$, $1 \leq l \leq r$.

**Definition 5.4.5 (Admissible Control Gain)**

A feedback gain $K$ is called admissible if $\mathcal{F}(\mathcal{H}, K) = 0^{r \times n}$ admits unique solutions $\mathcal{H}_l$, $1 \leq l \leq r$ that are positive semi-definite. Denote this set of gains as $\mathcal{K}_\infty$.

**Definition 5.4.6 (Infinite-Horizon Performance Index)**

Consider a function $g : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ denoted as $g(\kappa_\infty, \tilde{\kappa}_\infty)$ which satisfies the following properties:

- The function $g$ is analytic in both of its vector arguments
- The function $g$ is convex in $\kappa_\infty$ and the domain $\text{dom} \ g_{\tilde{\kappa}_\infty}$ (e.g. $g$’s restriction to $\tilde{\kappa}_\infty$) is a convex set
- The function $g(\kappa_\infty, \tilde{\kappa}_\infty)$ is non-negative in $\kappa_\infty$ on some neighborhood of $\tilde{\kappa}_\infty$
- The function $g(\kappa_\infty, \tilde{\kappa}_\infty)$ is strictly non-decreasing in $\kappa_\infty$
The MCCDS infinite-horizon performance index is non-negative, and convex in the cost cumulants. As such, it is well-suited for the objective function in the following optimization problem.

**Definition 5.4.7 (MCCDS Infinite-Horizon Optimization)**
The infinite-horizon MCCDS optimization can be stated as,

\[
\min_{K \in \mathcal{K}_\infty} g(\kappa_\infty(\mathcal{H}), \tilde{\kappa}_\infty(\tilde{\mathcal{H}}))
\]

subject to:

\[
\mathcal{F}(\mathcal{H}, K) = 0^{r \times n}, \quad \mathcal{F}(\tilde{\mathcal{H}}, \tilde{K}) = 0^{r \times n}.
\]

5.4.2 Technical Aspects of the MCCDS Infinite-Horizon Problem

Since the equilibrium solutions of the cost cumulant-generating equations \(\{\bar{\mathcal{H}}_l(\alpha, t_f)\}_{l=1}^r\) are continuously differentiable as \(t_f \to \infty\), then the values \(\{\kappa_l(\alpha)\}_{l=1}^r\) are continuous in \(\alpha\). Hence the following result holds.

**Theorem 5.4.8 (Transfer of Limit)**
Let \(g : \mathbb{R}_+^r \times \mathbb{R}_+^r \to \mathbb{R}_+\) be a continuous map from a Cartesian product of two non-negative valued \(r\)-dimensional vectors into the positive reals. Under the aforementioned regulatory conditions for cost cumulants on the infinite horizon, the limits may be defined as

\[
\lim_{t_f \to \infty} \kappa_l(t_f; t_0, K) = \kappa_{\infty,l}, \quad \lim_{t_f \to \infty} \tilde{\kappa}_l(t_f; t_0, K) = \tilde{\kappa}_{\infty,l},
\]

and thus

\[
\lim_{t_f \to \infty} g(\kappa_l(t_f; t_0, K), \tilde{\kappa}_l(t_f; t_0, K)) = g(\lim_{t_f \to \infty} \kappa_l(t_f; t_0, K), \lim_{t_f \to \infty} \tilde{\kappa}_l(t_f; t_0, K)) = g(\kappa_\infty(\mathcal{H}), \tilde{\kappa}_\infty(\tilde{\mathcal{H}})).
\]

**Proof** Consider some neighborhood \(N_\epsilon(\kappa_\infty, \tilde{\kappa}_\infty)\) of \(g(\kappa_\infty, \tilde{\kappa}_\infty)\) in \(\mathbb{R}_+\). Since \(g(\cdot)\) is continuous, the set \(N_\delta = g^{-1}(N_\epsilon)\) is an open set in \(\mathbb{R}_+^r \times \mathbb{R}_+^r\) that contains \((\kappa_\infty, \tilde{\kappa}_\infty)\).

Since \(\lim_{t_f \to \infty} \kappa_l(t_f; t_0, K) = \kappa_{\infty,l}\) and \(\lim_{t_f \to \infty} \tilde{\kappa}_l(t_f; t_0, K) = \tilde{\kappa}_{\infty,l}\), there exists some \(n_0 \in \mathbb{N}\) such that for \(n \geq n_0\) it is true that \((\kappa(t_n; t_0, K), \tilde{\kappa}(t_n; t_0, K)) \in N_\delta\) where \((t_0 < t_1 < t_2 < \cdots < t_n \ldots)\). Then \(g(\kappa(t_n; t_0, K), \tilde{\kappa}(t_n; t_0, K)) \in N_\epsilon\), and thus

\[
\lim_{n \to \infty} g(\kappa(t_n; t_0, K), \tilde{\kappa}(t_n; t_0, K)) = g(\kappa_\infty, \tilde{\kappa}_\infty).
\]
Since $\kappa_{l,\infty} < \infty, 1 \leq l \leq r$ and $\tilde{\kappa}_{l,\infty} < \infty, 1 \leq l \leq r$ and $(\kappa_{\infty}, \tilde{\kappa}_{\infty}) \in \text{dom} \ g$, it must be that $g(\kappa_{\infty}, \tilde{\kappa}_{\infty}) < \infty$.

\[ \square \]

**Definition 5.4.9 (Regular Point Qualification)**

Consider the set of constraints $\{f_i(x)\}_{i=1}^{k}$ with $k \leq n$ that are continuously differentiable mappings, $f_i : \mathbb{R}^n \to \mathbb{R}$ with rules of action $f_i(x) = 0, 1 \leq i \leq k$. Let $F(x)$ denote the function

$$F(x) = \begin{bmatrix} f_1(x) & \cdots & f_k(x) \end{bmatrix}, x \in \mathbb{R}^n$$

and suppose that $x_0 \in \mathbb{R}^n$ is such that $F(x_0) = 0^{k \times 1}$ such that the function $\text{grad} \ F(x_0)$ has full column rank $k$. Under these conditions, $x_0$ is said to be regular.

**Remark(s)** If $\text{grad} \ F(x)$ in an onto mapping, this is equivalent with regularity. When the set of $k$ gradient column vectors $\{\text{grad} \ g_i(x)\}_{i=1}^{k}$ is directly related to the notion of $k$-fold smoothness of the constraint surface $F(x)$.

**Theorem 5.4.10 (Necessary Condition for Optimality)**

Let $x_0 \in \mathbb{R}^n$ be the local optimum for the real-valued functional $f_0(x)$ subject to the constraints $F(x) = 0^{k \times 1}$. Suppose further that $x_0$ is a regular point for the constraint set. Under these conditions, there exists a unique vector of multipliers $\lambda$,

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{R}^k$$

such that the Lagrange functional

$$f_0(x) + F(x)\lambda$$

is stationary at $x_0$, meaning that

$$\text{grad} \ f_0(x_0) + \text{grad} \ F(x_0)\lambda = 0^{n \times 1}$$

**Proof** See [11].

203
Corollary 5.4.11 (Alternative Necessary Condition for Optimality)
Let \( x_0 \in \mathbb{R}^n \) be the local optimum for the real-valued functional \( f_0(x) \) subject to the constraints \( F(x) = 0^{k \times 1} \). Then there exists the vector
\[
\lambda' = \begin{bmatrix} \lambda_0 \\ \lambda \end{bmatrix} \in \mathbb{R}^{k+1}
\]
such that
\[
\lambda_0 \cdot \text{grad} \ f_0(x_0) + \text{grad} \ F(x_0) \lambda = 0^{n \times 1}.
\]


Matrix extensions of the above optimality conditions are now presented, so that these might be used in the development of the control solution to the MCCDS Infinite-Horizon problem. In the following, suppose that \( k = r \) in all of the aforementioned formal statements.

The corresponding constraints \( F(x) = 0 \) in the MCCDS Infinite-Horizon optimization problem are a set of matrix functions. In order to support this inherent difference, the previous results can be extended using the linear isomorphic vec(·) mapping between vectors \( \mathbb{R}^{mn} \) and matrices \( \mathbb{R}^{m \times n} \). It has been shown to preserve the metric distance between elements in the vector space \( \mathbb{R}^{mn} \) and the matrix space \( \mathbb{R}^{m \times n} \), and hence the spaces are algebraically and topologically equivalent. The mapping vec: \( \mathbb{R}^{m \times n} \to \mathbb{R}^{mn} \) is non-singular, so that the results that hold in one space must also hold in the other space and vice-versa. See [26] for more details.

Theorem 5.4.12 (Regularity Condition, Matrix Case)
Let the optimization point \( (\mathcal{H}, K) \) satisfy the equations of motion \( F(\mathcal{H}, K) = 0^{rn \times n} \) and the point \( (\tilde{\mathcal{H}}, \tilde{K}) \) satisfy the equations \( F(\tilde{\mathcal{H}}, \tilde{K}) = 0^{n \times n} \). Then it is said that \( (\mathcal{H}, K) \) and \( (\tilde{\mathcal{H}}, \tilde{K}) \) are regular if the equations
\[
\text{grad}\left\{ \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) + \sum_{k=1}^{r} \text{Tr} \left( F_k(\tilde{\mathcal{H}}, \tilde{K}) \tilde{\Lambda}_k^T \right) \right\} = 0^{2rn \times n}
\]

204
admit the unique solutions
\[ \Lambda = (\Lambda_1, \ldots, \Lambda_r) = (0^{n \times n}, \ldots, 0^{n \times n}), \quad \tilde{\Lambda} = (\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_r) = (0^{n \times n}, \ldots, 0^{n \times n}). \]

**Proof** The relations below are used in the following.

\[
\frac{\partial}{\partial H_l} \left( \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) + \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, \tilde{K}) \tilde{\Lambda}_k^T \right) \right) = \frac{\partial}{\partial H_l} \left( \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) \right)
\]

and

\[
\frac{\partial}{\partial \tilde{H}_l} \left( \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) + \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, \tilde{K}) \tilde{\Lambda}_k^T \right) \right) = \frac{\partial}{\partial \tilde{H}_l} \left( \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) \right)
\]

Consider the partial derivatives in the gradient,

\[
\frac{\partial}{\partial H_l} \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) = (A + BK)\Lambda_l + \Lambda_l(A + BK)^T \quad (5.23)
\]

\[
+ 2 \sum_{k=l+1}^{r} \binom{k}{l} (\Lambda_k \mathcal{H}_{k-1} S + S \mathcal{H}_{k-1} \Lambda_k) = 0^{n \times n}, \quad 1 \leq l \leq r - 1
\]

\[
\frac{\partial}{\partial \tilde{H}_r} \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) = (A + BK)\Lambda_r + \Lambda_r(A + BK)^T = 0^{n \times n}. \quad (5.24)
\]

The last condition (5.24) is satisfied by the unique solution \( \Lambda_r = 0^{n \times n} \), and this can be used in the \((r-1)\)th regulatory condition to yield

\[
\frac{\partial}{\partial \mathcal{H}_{r-1}} \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}, K) \Lambda_k^T \right) = (A + BK)\Lambda_{r-1} + \Lambda_{r-1}(A + BK)^T + 2r (\Lambda_r \mathcal{H}_1 S + S \mathcal{H}_1 \Lambda_r)
\]

\[
= (A + BK)\Lambda_{r-1} + \Lambda_{r-1}(A + BK)^T = 0^{n \times n}. \quad (5.25)
\]

Analogously reasoning gives that \( \Lambda_{r-1} = 0^{n \times n} \) is the unique solution from (5.25).
Using the facts that $\Lambda_r = 0^{n \times n}$ and $\Lambda_{r-1} = 0^{n \times n}$ in the $(r-2)^{th}$ regulatory condition will yield

$$\frac{\partial}{\partial H_{r-2}} \sum_{k=1}^{r} \text{Tr} \left(F_k(H, K)\Lambda_k^T\right) = (A + BK)\Lambda_{r-2} + \Lambda_{r-2}(A + BK)^T = 0^{n \times n}.$$

from which recursive reasoning yields that $\Lambda_i = 0^{n \times n}$, $\forall i$. Similar computation of derivatives $\frac{\partial}{\partial H_i}(\cdot)$ and recursive reasoning shows $\tilde{\Lambda}_i = 0^{n \times n}$, $\forall i$. Hence any pair $(H, K)$ and $(\tilde{H}, \tilde{K})$ is regular when

$$F(H, K) = (0^{n \times n}, \ldots, 0^{n \times n}), \quad F(\tilde{H}, \tilde{K}) = (0^{n \times n}, \ldots, 0^{n \times n}).$$

**Theorem 5.4.13** (Necessary Conditions for Optimality)
Assume $(H^*, K^*) \in (S^n)^r \times K$ and $(\tilde{H}, \tilde{K})$ are regular points for the respective constraint hyper-surfaces

$$F(H, K) = 0^{r \times n}, \quad F(\tilde{H}, \tilde{K}) = 0^{r \times n}$$

for which the functional $g(\kappa_\infty(H), \kappa_\infty(\tilde{H}))$ is minimized. Then there exists matrix multipliers $\Lambda^* = (\Lambda_1^*, \ldots, \Lambda_r^*)$ and $\tilde{\Lambda}^* = (\tilde{\Lambda}_1^*, \ldots, \tilde{\Lambda}_r^*)$ such that the gradient of the Lagrangian

$$\mathcal{L}(H, K, \Lambda, \tilde{\Lambda}) = g(\kappa_\infty(H), \kappa_\infty(\tilde{H})) + \sum_{i=1}^{r} \text{Tr}(F_i(H, K)\Lambda_i^T) + \sum_{i=1}^{r} \text{Tr}(F_i(\tilde{H}, \tilde{K})\tilde{\Lambda}_i^T)$$

vanishes for the optimal 4-tuple $(H^*, K^*, \Lambda^*, \tilde{\Lambda}^*)$, that is

$$\nabla_{\mathcal{L}_{H,K,\Lambda,\tilde{\Lambda}}}(H^*, K^*, \Lambda^*, \tilde{\Lambda}^*) = 0.$$

**Proof** A straight-forward application of Corollary 5.4.11 with Theorem 5.4.12 leads to this result.
In the following, the denotation $S \triangleq GWG^T$ is made. The following lemma is key to the MCCDS infinite-horizon controller derivation.

**Lemma 5.4.14 (Cancellation Properties)**

Let $(A,B)$ be stabilizable and $(Q,A)$ be detectable. Furthermore, let $\mathcal{H}_l$, $1 \leq l \leq r$ be the solutions of the following system of equations $F_l(\mathcal{H}_l, K) = 0^{n \times n}$, $1 \leq l \leq r$ under $K^\dagger$ where the gain $K^\dagger$ for $c_l > 0$, $1 \leq l \leq r$ is defined by

$$K^\dagger = -R^{-1}B^T \left( \mathcal{H}_1^l + \sum_{i=2}^{r} \frac{c_i}{c_1} \mathcal{H}_i^l \right).$$

Suppose the solution $\Gamma$ exists to the following Lyapunov equation

$$(A + BK^\dagger)^T \Gamma + \Gamma (A + BK^\dagger) = -c_1 S.$$

Then the matrices $\mathcal{H}_l$ satisfy

$$\Gamma \mathcal{H}_l S + S \mathcal{H}_l \Gamma = 0^{n \times n}, 1 \leq l \leq r - 1. \quad (5.26)$$

**Proof** It has been shown in Pham [26] that when $(A,B)$ is stabilizable and $(Q,A)$ is detectable, the control

$$K^\dagger = -R^{-1}B^T \left( \mathcal{H}_1^l + \sum_{i=2}^{r} \frac{c_i}{c_1} \mathcal{H}_i^l \right), \; c_l > 0, \; 1 \leq l \leq r$$

is the optimal solution to the problem,

$$\min_K \left\{ \sum_{l=1}^{r} c_l \cdot \kappa_\infty, l(\mathcal{H}; K) \right\}$$

s.t. $F_l(\mathcal{H}, K) = 0^{n \times n}, 1 \leq l \leq r$

Since $K^\dagger$ is optimal, the traditional necessary conditions for optimality must be satisfied. That is, the gradient of the Lagrangian functional $\mathcal{L}(\mathcal{H}, K, \Lambda)$ defined by

$$\mathcal{L}(\mathcal{H}, K, \Lambda) = \sum_{l=1}^{r} c_l \cdot \kappa_\infty, l(\mathcal{H}) + \sum_{l=1}^{r} \text{Tr} \left( F_l(\mathcal{H}, K) \Lambda_l^T \right)$$

207
must vanish at the optimum pair \((\mathcal{H}^\dagger, K^\dagger, \Lambda^\dagger)\). This means that the following conditions hold

\[
\frac{\partial \mathcal{L}(\mathcal{H}^\dagger, K^\dagger, \Lambda^\dagger)}{\partial \Lambda_l} = 0^{n \times n}, \quad 1 \leq l \leq r \tag{5.27}
\]

\[
\frac{\partial \mathcal{L}(\mathcal{H}^\dagger, K^\dagger, \Lambda^\dagger)}{\partial H_m} = 0^{n \times n}, \quad 1 \leq m \leq r \tag{5.28}
\]

\[
\frac{\partial \mathcal{L}(\mathcal{H}^\dagger, K^\dagger, \Lambda^\dagger)}{\partial K} = 0^{n \times n} \tag{5.29}
\]

The relations (5.28) state the following is true for \(1 \leq l \leq r - 1\),

\[
\frac{\partial}{\partial \mathcal{H}_l} \left( \sum_{k=1}^{r} c_k \cdot \kappa_{\infty,k}(\mathcal{H}^\dagger) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}^\dagger, K^\dagger)\Lambda_k^T \right) \right)
= (A + BK^\dagger)\Lambda_l^l + \Lambda_l^l(A + BK^\dagger)^T + c_1 S \tag{5.30}
\]

\[
+ 2 \sum_{k=l+1}^{r} \binom{k}{l} \left( \Lambda_k^l H_{k-l}^l S + S H_{k-l}^l \Lambda_k^l \right) = 0^{n \times n}.
\]

For \(l = r\), the equation below is satisfied

\[
\frac{\partial}{\partial \mathcal{H}_r} \left( \sum_{k=1}^{r} c_k \cdot \kappa_{\infty,k}(\mathcal{H}^\dagger) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}^\dagger, K^\dagger)\Lambda_k^T \right) \right)
= (A + BK^\dagger)\Lambda_r^r + \Lambda_r^r(A + BK^\dagger)^T + c_r S = 0^{n \times n} \tag{5.31}
\]

The derivatives above for \(1 \leq l \leq r - 1\) can be evaluated by observing that

\[
\sum_{i=1}^{r} \mathcal{F}_i(\mathcal{H}^\dagger, K^\dagger)\Lambda_i^T = (A + BK)^T H_1 \Lambda_1^T + H_1(A + BK)\Lambda_1^T + K^T R K \Lambda_1^T + Q \Lambda_1^T
\]

\[
+ \sum_{k=2}^{r} (A + BK)^T H_k \Lambda_k^T + H_k(A + BK) \Lambda_k^T
\]

\[
+ 2 \sum_{k=2}^{r} \sum_{l=1}^{k-1} \binom{k}{l} H_l S H_{k-l} \Lambda_k^T
\]
\[
= \sum_{k=1}^{r} (A + BK)^T \mathcal{H}_k \Lambda_k^T + \mathcal{H}_k (A + BK) \Lambda_k^T \\
+ 2 \sum_{k=2}^{r} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T + K^T RK \Lambda_1^T + Q \Lambda_1^T
\]

From this expansion it is apparent that

\[
\frac{\partial}{\partial \mathcal{H}_j} \left( \sum_{i=1}^{r} \mathcal{F}_i (\mathcal{H}, K) \Lambda_i^T \right) = \frac{\partial}{\partial \mathcal{H}_j} (A + BK)^T \mathcal{H}_j \Lambda_j^T + \mathcal{H}_j (A + BK) \Lambda_j^T \\
+ 2 \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=2}^{r} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T
\]

The first term is equal to \((A + BK) \Lambda_j + \Lambda_j^T (A + BK)\). The expression for the second term follows readily from the following. Expand the first summation over \(k\) to write this as,

\[
\sum_{k=2}^{r} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T = \sum_{k=2}^{j} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T + \sum_{k=j+1}^{r} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T.
\]

(5.32)

Notice that the first of two terms (5.32) vanishes when the derivative is taken,

\[
\frac{\partial}{\partial \mathcal{H}_j} \sum_{k=2}^{j} \sum_{l=1}^{k-1} \binom{k}{l} \mathcal{H}_l \mathcal{H}_{k-l} \Lambda_k^T = 0^{n \times n}.
\]

The Kronecker delta appears above to indicate the summed terms that will remain
after the derivative is taken, and for this purpose it is sparsely used in the following. This denotation \( \delta_{ij} \) bears the meaning

\[
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j.
\end{cases}
\]

Now the relation (5.34) is a direct consequence of the double summation in (5.32). The index \( l \) is at most \( j - 1 \) and \( k - l \) is greatest when \( l \) is minimum (that is \( l = 1 \)) and \( k \) assumes its maximum value \( k = j \). Thus no terms in the sum contain \( \mathcal{H}_i \) matrices beyond \( \mathcal{H}_{j-1} \). Hence, it readily follows that

\[
\frac{\partial}{\partial \mathcal{H}_j} \sum_{k=2}^{r} \sum_{l=1}^{k-1} \left( \begin{array}{c} k \\ l \end{array} \right) \mathcal{H}_i \mathcal{H}_{k-1} \Lambda^T_k = \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \sum_{l=1}^{k-1} \left( \begin{array}{c} k \\ l \end{array} \right) \mathcal{H}_i \mathcal{H}_{k-1} \Lambda^T_k \delta_{lj}
\]

Straightforward mathematics gives the derivative of the remaining term (5.33) as,

\[
\frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \sum_{l=1}^{k-1} \left( \begin{array}{c} k \\ l \end{array} \right) \mathcal{H}_i \mathcal{H}_{k-1} \Lambda^T_k = \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \sum_{l=1}^{k-1} \left( \begin{array}{c} k \\ l \end{array} \right) \mathcal{H}_i \mathcal{H}_{k-1} \Lambda^T_k \delta_{lj}
\]

\[
+ \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \sum_{l=1}^{k-1} \left( \begin{array}{c} k \\ l \end{array} \right) \mathcal{H}_i \mathcal{H}_{k-1} \Lambda^T_k \delta_{l(k-j)}
\]

\[
= \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \left( \begin{array}{c} k \\ j \end{array} \right) \mathcal{H}_j \mathcal{H}_{k-j} \Lambda^T_k
\]

\[
+ \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \left( \begin{array}{c} k \\ k-j \end{array} \right) \mathcal{H}_{k-j} \mathcal{H}_j \Lambda^T_k
\]

\[
= \frac{\partial}{\partial \mathcal{H}_j} \sum_{k=j+1}^{r} \left( \begin{array}{c} k \\ j \end{array} \right) (\mathcal{H}_j \mathcal{H}_{k-j} + \mathcal{H}_{k-j} \mathcal{H}_j) \Lambda^T_k
\]
\[
\sum_{k=j+1}^{r} \binom{k}{j} \left( (S\mathcal{H}_{k-j}\Lambda_k^T)^T + (\mathcal{H}_{k-j}S)^T(\Lambda_k^T)^T \right)
\]

\[
= \sum_{k=j+1}^{r} \binom{k}{j} (\Lambda_k \mathcal{H}_{k-j}S + S\mathcal{H}_{k-j}\Lambda_k).
\]

The constraint equations stem from the relations (5.27),

\[
\frac{\partial}{\partial \Lambda_1^T} \left( \sum_{k=1}^{r} c_k \cdot \kappa_{\infty,k}(\mathcal{H}_1^\dagger) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}_1^\dagger,K_1^\dagger)\Lambda_k^T \right) \right) 
= (A + BK^\dagger)^T \mathcal{H}_1^\dagger + \mathcal{H}_1^\dagger (A + BK^\dagger) + K^T R K^\dagger + Q = 0^{n \times n}
\]

\[
\frac{\partial}{\partial \Lambda_l^T} \left( \sum_{k=1}^{r} c_k \cdot \kappa_{\infty,k}(\mathcal{H}_l^\dagger) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}_l^\dagger,K_1^\dagger)\Lambda_k^T \right) \right) 
= (A + BK^\dagger)^T \mathcal{H}_l^\dagger + \mathcal{H}_l^\dagger (A + BK^\dagger) + 2 \sum_{j=1}^{l-1} \binom{l}{j} \mathcal{H}_j^\dagger S \mathcal{H}_l^\dagger = 0^{n \times n}
\]

and finally (5.29) yields

\[
\frac{\partial}{\partial K^\dagger} \left( \sum_{k=1}^{r} c_k \cdot \kappa_{\infty,k}(\mathcal{H}_l^\dagger) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}_l^\dagger,K_1^\dagger)\Lambda_k^T \right) \right) 
= 2B^T \sum_{k=1}^{r} \mathcal{H}_k^\dagger \Lambda_k^T + 2 R K^\dagger \Lambda_1^T = 0^{n \times n}.
\]

(5.36)

Consider the multipliers below,

\[
\Lambda_1^T = \frac{c_l}{c_1} \Lambda_1^T, \ 1 \leq l \leq r
\]

which turn (5.36) into

\[
\left( B^T \sum_{k=1}^{r} \frac{c_k}{c_1} \mathcal{H}_k^\dagger + R K^\dagger \right) \Lambda_1^T = 0^{n \times n}
\]

(5.38)
For any $\Lambda_1^\dagger$, this equation can be satisfied when $K^\dagger$ is the optimal solution,

$$K^\dagger = -R^{-1}B^T \left( H_1^\dagger + \sum_{i=2}^{r} \frac{c_i}{c_1} H_i^\dagger \right).$$

Since it is true that $K^\dagger$ produces $H^\dagger$ that satisfies the equality constraints $F_l(H^\dagger, K^\dagger) = 0$, it can readily be verified that the Lagrangian $L(\cdot)$ is stationary for any multiplier $\Lambda$ that leads to $K^\dagger$.

$$L(H^\dagger, K^\dagger, \Lambda) = \sum_{i=1}^{r} c_i \cdot \kappa_{\infty,i}(H^\dagger) + \sum_{i=1}^{r} \text{Tr} \left( \underbrace{F_l(H^\dagger, K^\dagger)}_{0^{n \times n}} \Lambda_i^T \right) = \sum_{i=1}^{r} c_i \cdot \kappa_{\infty,i}(H^\dagger)$$

In other words, $L(\cdot)$ is minimized for $K^\dagger$, so the gradient of $L(\cdot)$ will vanish $\forall \Lambda$ yielding $K^\dagger$ according to (5.36). Because of the constraint (5.31), $\Lambda_1^\dagger$ will satisfy the following equation,

$$(A + BK^\dagger) \Lambda_1^\dagger + \Lambda_1^\dagger (A + BK^\dagger)^T + c_1 S$$

$$= (A + BK^\dagger) \left( \frac{c_r}{c_1} \Lambda_1^\dagger \right) + \left( \frac{c_r}{c_1} \Lambda_1^\dagger \right) (A + BK^\dagger)^T + c_1 S$$

$$= (A + BK^\dagger) \left( \frac{1}{c_1} \Lambda_1^\dagger \right) + \left( \frac{1}{c_1} \Lambda_1^\dagger \right) (A + BK^\dagger)^T + S = 0^{n \times n}.$$  

From the last equation, it is clear that $\Gamma = \Lambda_1^\dagger$ and that

$$(A + BK^\dagger)^T \Lambda_1^\dagger + \Lambda_1^\dagger (A + BK^\dagger) = c_1 S.$$

With the multiplier selections, the relations (5.30) become for $1 \leq l \leq r - 1$

$$(A + BK^\dagger) \left( \frac{c_l}{c_1} \Lambda_1^\dagger \right) + \left( \frac{c_l}{c_1} \Lambda_1^\dagger \right) (A + BK^\dagger)^T + c_l S$$
\[ + 2 \sum_{k=l+1}^{r} \binom{k}{l} \left( \left( \frac{c_k}{c_1} \Lambda_1^\dagger \right) \mathcal{H}_{k-l} S + S \mathcal{H}_{k-l}^\dagger \left( \frac{c_k}{c_1} \Lambda_1^\dagger \right) \right) \]

\[ = \frac{c_l}{c_1} \left( (A + BK^\dagger) \Lambda_1^\dagger + \Lambda_1^\dagger (A + BK^\dagger)^T \right) + \frac{c_l}{c_1} c_1 S \]

\[ + 2 \sum_{k=l+1}^{r} \frac{c_k}{c_1} \binom{k}{l} \left( \Lambda_1^\dagger \mathcal{H}_{k-l} S + S \mathcal{H}_{k-l}^\dagger \Lambda_1^\dagger \right) \]

\[ = \frac{c_l}{c_1} \left( (A + BK^\dagger) \Lambda_1^\dagger + \Lambda_1^\dagger (A + BK^\dagger)^T + c_1 S \right) \]

\[ + 2 \sum_{k=l+1}^{r} \frac{c_k}{c_1} \binom{k}{l} \left( \Lambda_1^\dagger \mathcal{H}_{k-l} S + S \mathcal{H}_{k-l}^\dagger \Lambda_1^\dagger \right) \]

\[ = 2 \sum_{k=l+1}^{r} \frac{c_k}{c_1} \binom{k}{l} \left( \Lambda_1^\dagger \mathcal{H}_{k-l} S + S \mathcal{H}_{k-l}^\dagger \Lambda_1^\dagger \right) + \frac{c_l}{c_1} (-c_1 S + c_1 S) \]

\[ = 2 \sum_{k=l+1}^{r} \frac{c_k}{c_1} \binom{k}{l} \left( \Lambda_1^\dagger \mathcal{H}_{k-l} S + S \mathcal{H}_{k-l}^\dagger \Lambda_1^\dagger \right) = 0^{n \times n}. \]

Observe that \( l = r - 1 \) gives the relation

\[ \binom{r}{r-1} \frac{c_r}{c_1} \left( \Lambda_1^\dagger \mathcal{H}_1^\dagger S + S \mathcal{H}_1^\dagger \Lambda_1^\dagger \right) = 0^{n \times n} \]

so that

\[ \Lambda_1^\dagger \mathcal{H}_1^\dagger S + S \mathcal{H}_1^\dagger \Lambda_1^\dagger = 0^{n \times n}. \]

When \( l = r - 2 \),

\[ \binom{r-1}{r-2} \frac{c_{r-1}}{c_1} \left( \Lambda_1^\dagger \mathcal{H}_1^\dagger S + S \mathcal{H}_1^\dagger \Lambda_1^\dagger \right) + \binom{r}{r-2} \frac{c_r}{c_1} \left( \Lambda_1^\dagger \mathcal{H}_2^\dagger S + S \mathcal{H}_2^\dagger \Lambda_1^\dagger \right) = 0^{n \times n} \]

213
so that
\[ \Lambda_1^T H_2^T S + S H_2^T \Lambda_1^T = 0^{n \times n}. \]

Reasoning recursively, leads to
\[ \Lambda_l^T H_l^T S + S H_l^T \Lambda_l^T = 0^{n \times n}, \quad 1 \leq l \leq r - 1. \]

### 5.4.3 Main Result

The main result is now presented, and its formal proof. After this, a presentation of the MCCDS Infinite-Horizon algorithm will be made, and simulation results are shown for validation purposes.

**Theorem 5.4.15 (Infinite-Horizon MCCDS Control)**

Assume that \((A, B)\) is stabilizable and \((Q, A)\) is detectable. Fix \(r \in \mathbb{N}\) and suppose there exist \(r\) cost cumulants of (5.21). Under these conditions, the optimal solution to the Infinite-Horizon MCCDS optimization concerning the process having dynamics (5.19) and the cost (5.21) is given by \(u^*(t) = K^* x(t)\) where the extremalizing gain \(K^*\) is

\[
K^* = -R^{-1} B^T \sum_{i=2}^{r} \left( H_i^* + \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_{\infty, l}} \right) \hat{H}_i^*
\]

which is well-defined when the following equations admit a solution \(F(H^*, K^*) = 0^{r \times n}\), \(F(H, \tilde{K}) = 0^{r \times n}\).

**Proof** The regularity condition has been verified already in Theorem 5.4.12. Now the necessary condition of optimality can be employed to the Lagrange functional \(L(H, K, \Lambda, \tilde{\Lambda})\) defined by

\[
L(H, K, \Lambda, \tilde{\Lambda}) = g(\kappa_\infty(H), \tilde{\kappa}_\infty(\tilde{H})) + \sum_{k=1}^{r} \text{Tr} \left( F_k(H, K) \Lambda_k^T \right) + \sum_{k=1}^{r} \text{Tr} \left( F_k(\tilde{H}, \tilde{K}) \tilde{\Lambda}_k^T \right)
\]
which is \( \nabla L_{\mathcal{H},K,\Lambda,\tilde{\Lambda}}(\mathcal{H}^*,K^*,\Lambda^*,\tilde{\Lambda}^*) = 0 \).

A key assumption for the fixed control \( \tilde{\Lambda} \) is that it satisfies the constraints \( \mathcal{F}(\tilde{\mathcal{H}},\tilde{K}) = 0^{r \times n} \), so that \( \mathcal{L}(\cdot) \) is minimized for any choice of \( \tilde{\Lambda} \) with the optimal triple \( (\mathcal{H}^*,K^*,\Lambda^*) \). Thus choose \( \tilde{\Lambda} = 0^{r \times n} \), and write \( \mathcal{L}^\dagger(\mathcal{H},K,\Lambda) = \mathcal{L}(\mathcal{H},K,\Lambda,0^{r \times n}) \) and consider that \( \nabla L_{\mathcal{H},K,\Lambda}^\dagger(\mathcal{H}^*,K^*,\Lambda^*) = 0 \) will yield the extremal point of the original functional \( \mathcal{L}(\cdot) \). This condition requires that

\[
\frac{\partial L^\dagger(\mathcal{H}^*,K^*,\Lambda^*)}{\partial \Lambda_l} = 0^{n \times n}, \quad 1 \leq l \leq r \tag{5.37}
\]

\[
\frac{\partial L^\dagger(\mathcal{H}^*,K^*,\Lambda^*)}{\partial \mathcal{H}_m} = 0^{n \times n}, \quad 1 \leq m \leq r \tag{5.38}
\]

\[
\frac{\partial L^\dagger(\mathcal{H}^*,K^*,\Lambda^*)}{\partial K} = 0^{n \times n} \tag{5.39}
\]

The equations (5.37) are just the constraints \( \mathcal{F}(\mathcal{H},K^*) = 0^{r \times n} \). The equations (5.38) give the relations \( 1 \leq m \leq r - 1 \),

\[
\frac{\partial}{\partial \mathcal{H}_m} \left( g(\kappa_\infty(\mathcal{H}^*),\tilde{\kappa}_\infty(\tilde{\mathcal{H}})) + \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}^*,K^*) \Lambda_k^T \right) \right) = (A + BK^*) \Lambda_m^* + \Lambda_m^* (A + BK^*)^T + \frac{\partial g(\kappa_\infty,\tilde{\kappa}_\infty)}{\partial \kappa_\infty,m} \Lambda_m^* + 2 \sum_{k=m+1}^{r-1} \binom{k}{m} \left( \Lambda_k^* \mathcal{H}_{k-m}^* S + S \mathcal{H}_{k-m}^* \Lambda_k^* \right) = 0^{n \times n} \tag{5.40}
\]

and for \( m = r \) the relation,

\[
\frac{\partial}{\partial \mathcal{H}_r} \sum_{k=1}^{r} \text{Tr} \left( \mathcal{F}_k(\mathcal{H}^*,K^*) \Lambda_k^T \right) = (A + BK^*) \Lambda_r^* + \Lambda_r^* (A + BK^*)^T + \frac{\partial g(\kappa_\infty,\tilde{\kappa}_\infty)}{\partial \kappa_\infty,r} S = 0^{n \times n}. \tag{5.41}
\]

215
The derivatives are used above,

\[ \frac{\partial g(\kappa_{\infty}(\mathcal{H}), \bar{\kappa}_{\infty}(\bar{\mathcal{H}}))}{\partial H_l} = \frac{\partial g(\kappa_{\infty}, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,l}} \frac{\partial \kappa_{\infty,l}}{\partial H_l} \]
\[ = \frac{\partial g(\kappa_{\infty}, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,l}} \frac{\partial \text{Tr}(\mathcal{H}_l S)}{\partial H_l} \]
\[ = \frac{\partial g(\kappa_{\infty}, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,l}} S. \]

Now propose the multipliers \( \Lambda_l, 2 \leq l \leq r, \)

\[ \Lambda_l^* = \begin{pmatrix} \frac{\partial g(\kappa_{\infty}^*, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,l}} \\ \frac{\partial g(\kappa_{\infty}^*, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,1}} \end{pmatrix} \Lambda_1^*, \ 2 \leq l \leq r. \tag{5.42} \]

From (5.41), clearly \( \Lambda_1^* \) must satisfy

\[ (A + BK^*)\Lambda_l^* + \Lambda_l^*(A + BK^*)^T = -\frac{\partial g(\kappa_{\infty}^*, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,1}} S. \tag{5.43} \]

Above the standard assumption on the finite-horizon has been invoked, more precisely

\[ \left( \frac{\partial g(\kappa_{\infty}^*, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,1}} \right) \neq 0. \]

Note that for \( 1 \leq m \leq r - 1, \) the expression below

\[ (A + BK^*)\Lambda_m^* + \Lambda_m^*(A + BK^*)^T + \frac{\partial g(\kappa_{\infty}, \bar{\kappa}_{\infty})}{\partial \kappa_{\infty,m}} S \]
by (5.42) becomes

\[ \left( (A + BK^*) \Lambda^*_1 + \Lambda^*_1 (A + BK^*)^T \right) \cdot \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, m} \right) + \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, 1} S \]

\[ = - \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, m} \right) S + \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, m} S = 0^{n \times n}. \]

Using the above relations in (5.40) reduces the equations to

\[ \frac{\partial}{\partial \mathcal{H}_m} \left( g(\kappa_\infty(\mathcal{H}^*), \tilde{\kappa}_\infty(\tilde{\mathcal{H}})) + \sum_{k=1}^{r} \text{Tr} \left( F_k(\mathcal{H}^*, K^*) \Lambda_k^T \right) \right) \]

\[ = 2 \sum_{k=m+1}^{r-1} \binom{k}{m} \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, k} \right) \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, 1} \right) \left( \Lambda_k^* \mathcal{H}_{k-m}^* S + S \mathcal{H}_{k-m}^* \mathcal{H}_{k-m} \right) = 0^{n \times n} \quad (5.44) \]

Introduce the notation

\[ c_{k,m} \equiv 2 \binom{k}{m} \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, k} \right) \left( \frac{\partial g(\kappa_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_\infty, 1} \right). \]

The non-zero ratio of derivatives above reflects the assumption that \( g(\kappa_\infty, \tilde{\kappa}_\infty) \) is convex and strictly non-decreasing in \( \kappa_\infty \). Re-write the equations (5.44) as

\[ \sum_{k=m+1}^{r-1} c_{k,m} \left( \Lambda_k^* \mathcal{H}_{k-m}^* S + S \mathcal{H}_{k-m}^* \Lambda_k^* \right) = 0^{n \times n} \quad (5.45) \]

Using the multipliers (5.42), consider now the optimality condition (5.39),

\[ \frac{\partial \mathcal{L} \big| (\mathcal{H}^*, K^*, \Lambda^*)}{\partial K} = 2B^T \sum_{l=1}^{r} \mathcal{H}_l^* \Lambda_l^* + 2RK^* \Lambda_1^* \]
becomes

\[
\left(2B^T \sum_{l=2}^{r} \left( \mathcal{H}_1^* + \left( \frac{\partial g(\kappa^*_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_{\infty, l}} \right) \mathcal{H}_l^* \right) + 2RK^* \right) \Lambda_1^* = 0^{n \times n}
\]

For any \( \Lambda_1^* \), this equation is satisfied when the gain is

\[
K^* = -R^{-1} B^T \sum_{l=2}^{r} \left( \mathcal{H}_1^* + \left( \frac{\partial g(\kappa^*_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_{\infty, l}} \right) \mathcal{H}_l^* \right).
\]  

(5.46)

Since \( \frac{\partial g(\kappa^*_\infty, \tilde{\kappa}_\infty)}{\partial \kappa_{\infty, l}} > 0 \), \( 1 \leq l \leq r \) are just positive constants and given the form of the control gain above, it is known from the previous lemma that the following relations are true for the solution to equation (5.43),

\[
\Lambda_1^* \mathcal{H}_l^* S + S \mathcal{H}_l^* \Lambda_1^* = 0^{n \times n}, 1 \leq l \leq r - 1.
\]

With the above relation, clearly (5.45) is rendered true, which is the ultimate simplification of (5.38) given the consequences of the selection (5.42). The gain (5.46) is thus extremalizing in that (5.37), (5.38), and (5.39) are all satisfied with the triple \( (\mathcal{H}^*, K^*, \Lambda^*) \). When \( \Lambda_1^* \succeq 0^{n \times n} \) this gain is minimizing, as seen by

\[
\frac{\partial^2 \mathcal{L}}{\partial K_2^2}(\mathcal{H}^*, K^*, \Lambda^*) = 2R \otimes \Lambda_1^* \succeq 0.
\]

\( \square \)
5.5 MCCDS Infinite-Horizon Controller Computation

5.5.1 Algorithm

The MCCDS Infinite-Horizon control can be computed according to the algorithm described in this section. The careful reader will note this algorithm reduces to a $k$CC controller computation with optimal MCCDS parameters.

After presenting the algorithm, the system and the cost of the first-generation AMD benchmark problem is used to compute a infinite-horizon $4$CC control, and to generate target cost statistics. Using this information, the MCCDS control is computed via this algorithm, and a comparison is made between the performance of the two control laws.

(Necessary Pre-Conditions)

Determine $K_1$ such that

\[ A + BK_1 \prec 0^{n \times n} \]

Infinite-Horizon MCCDS Iterate Algorithm:

INPUTS: $K_1$, $TOL_1 > 0$, $TOL_2 > 0$, and $\{ \tilde{\kappa}_{\infty,i}^k \}_{k=1}^\infty$, $1 \leq i \leq r$

Let $p = 1$ and $q = 1$ (initially)...

219
STEP 1: Solve the following equations

\[
\begin{align*}
\mathcal{H}_1^p \left| (A + BK_p)^T \mathcal{H}_1^p + \mathcal{H}_1^p (A + BK_p) + \mathcal{K}^T_p \mathcal{R} \mathcal{K}_p + Q \right. & = 0^{n \times n} \\
\mathcal{H}_i^p \left| (A + BK_p)^T \mathcal{H}_i^p + \mathcal{H}_i^p (A + BK_p) + 2 \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} \mathcal{H}_i^p \mathcal{G} \mathcal{W} \mathcal{G}^T \mathcal{H}_i^p \right. & = 0^{n \times n}, \quad 2 \leq i \leq r.
\end{align*}
\]

STEP 2: Compute the cumulants,

\[
k_{i,1}^p \equiv \text{Tr}(\mathcal{H}_i^p \mathcal{G} \mathcal{W} \mathcal{G}^T), \quad 1 \leq i \leq r.
\]

STEP 3: Compute the following \( r - 1 \) parameters

\[
\begin{align*}
\gamma_{2}^{(p,q)} & = \left( \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{2,\infty}^p}, \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{1,\infty}^p} \right), \\
\gamma_{3}^{(p,q)} & = \left( \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{3,\infty}^p}, \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{1,\infty}^p} \right), \\
& \quad \ldots, \\
\gamma_{r}^{(p,q)} & = \left( \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{r,\infty}^p}, \frac{\partial g(\mathbf{k}_{s}^p, \mathbf{k}_{t}^q)}{\partial \mathbf{k}_{1,\infty}^p} \right).
\end{align*}
\]

STEP 4: Set \( l = p \)

\[
K_{l+1} = -R^{-1} B^T \left( \mathcal{H}_1^l + \sum_{i=2}^{r} \gamma_{i}^{(p,q)} \mathcal{H}_i^l \right)
\]

and the matrix

\[
M_l = \sum_{k=2}^{r} \gamma_{k}^{(p,q)} \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} \mathcal{H}_i^l \mathcal{G} \mathcal{W} \mathcal{G}^T \mathcal{H}_{i-j}^l.
\]
STEP 5: Compute the iterate matrices, and then set $p = p + 1$

\[
\mathcal{H}_1^{l+1} | (A + BK_{l+1})^T \mathcal{H}_1^{p+1} + H_1^{l+1}(A + BK_{l+1}) + K_{l+1}^T RK_{l+1} + Q = 0^{n \times n}
\]

\[
\mathcal{H}_1^{l+1} | (A + BK_{l+1})^T \mathcal{H}_1^{l+1} + H_1^{l+1}(A + BK_{l+1})
\]

\[
+ 2 \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} \mathcal{H}_1^{l+1} GWG^T \mathcal{H}_1^{l+1} = 0^{n \times n}, \quad 2 \leq i \leq r.
\]

STEP 6: Compute the control gain

\[
K_{l+2} = -R^{-1} B^T \left( \mathcal{H}_1^{l+1} + \sum_{i=2}^{r} \gamma_i^{(p,q)} H_1^{l+1} \right)
\]

and the matrix

\[
M_{l+1} = \sum_{k=2}^{r} \gamma_k^{(p,q)} \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} \mathcal{H}_1^{l+1} GWG^T \mathcal{H}_1^{l+1}.
\]

STEP 7: IF $M_l \preceq M_{l+1}$, CONTINUE

ELSE QUIT

STEP 8: IF $||K_{l+2} - K_{l+1}||_F > TOL_1$,

then set $l = l + 1$, $K_{l+1} = K_{l+2}$, and $M_l = M_{l+1}$ REPEAT STEP 5
ELSE set \( q = q + 1 \), \( p = p + 1 \), and \( K_p = K_{i+1} \). REPEAT STEP 1

STEP 9: IF \( \| \kappa_{\infty}^q - \kappa_{\infty}^p \| < TOL_2 \), STOP

5.5.2 Validation

The first-generation benchmark problem will be used to validate the infinite-horizon MCCDS control algorithm. In particular, the system matrices \((A, B, G)\) and output matrices \((C, D)\) of the reduced-order model for control design are used. Furthermore, weighting matrices \((Q, R)\) are employed that appear in the original benchmark study. See [26] for more details on the system and cost. The target \( \mathcal{J} \) control is calculated using iterative techniques. With this computation, the infinite-horizon averaged cost cumulants are realized in solving the family of algebraic Riccati equations to arrive at the \( \mathcal{H} \) matrices. In particular, solve the equations

\[
(A + BK)^T \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_1(A + BK) + \hat{K}^T R \hat{K} + \bar{Q} = 0_{n \times n}
\]

\[
(A + BK)^T \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_2(A + BK) + 4 \hat{\mathcal{H}}_1 GWG^T \hat{\mathcal{H}}_1 = 0_{n \times n}
\]

\[
(A + BK)^T \hat{\mathcal{H}}_3 + \hat{\mathcal{H}}_3(A + BK) + 6 \hat{\mathcal{H}}_1 GWG^T \hat{\mathcal{H}}_2
\]

\[
+ 6 \hat{\mathcal{H}}_2 GWG^T \hat{\mathcal{H}}_1 = 0_{n \times n}
\]

\[
(A + BK)^T \hat{\mathcal{H}}_4 + \hat{\mathcal{H}}_4(A + BK) + 8 \hat{\mathcal{H}}_1 GWG^T \hat{\mathcal{H}}_3
\]

\[
+ 12 \hat{\mathcal{H}}_2 GWG^T \hat{\mathcal{H}}_2 + 8 \hat{\mathcal{H}}_3 GWG^T \hat{\mathcal{H}}_1 = 0_{n \times n}
\]

using Pham’s iterate solutions technique with the control gain

\[
\hat{K} = -B^T R^{-1} \left( \hat{\mathcal{H}}_1 + \mu_2 \hat{\mathcal{H}}_2 + \mu_3 \hat{\mathcal{H}}_3 + \mu_4 \hat{\mathcal{H}}_4 \right)
\]
where

\[
(\mu_1, \mu_2, \mu_3, \mu_4) = (1.0 \times 10^{-5}, 9.0 \times 10^{-12}, 2.0 \times 10^{-20})
\]

gives the vector of targets \( \hat{\kappa}_\infty \in \mathbb{R}^4 \) with components

\[
\hat{\kappa}_{\infty, i} = \text{Tr}(\tilde{H}_i GW G^T), \quad 1 \leq i \leq 4.
\] (5.48)

Consider the function

\[
g(\kappa_\infty, \tilde{\kappa}_\infty) = \kappa_{\infty,3}^2 + \tilde{\kappa}_{\infty,3}^2 - 2\kappa_{\infty,3} \tilde{\kappa}_{\infty,3} + \frac{\kappa_{\infty,4}^2 + \tilde{\kappa}_{\infty,4}^2 - 2\kappa_{\infty,4} \tilde{\kappa}_{\infty,4}}{92\tilde{\kappa}_{\infty,2}^2} \right)
+ \frac{21\tilde{\kappa}_{\infty,3}^4 + 7\kappa_{\infty,3} \tilde{\kappa}_{\infty,3}^3}{92\tilde{\kappa}_{\infty,2}^2} + \frac{(\kappa_{\infty,3} \tilde{\kappa}_{\infty,3} - \kappa_{\infty,3}^2) \cdot \tilde{\kappa}_{\infty,4}}{8\tilde{\kappa}_{\infty,2}^6}
+ \frac{(\tilde{\kappa}_{\infty,3}^2 - \kappa_{\infty,3}^2) \cdot \kappa_{\infty,4}}{16\tilde{\kappa}_{\infty,2}^6} + \frac{1}{2} \left( \frac{\kappa_{\infty,2}}{\tilde{\kappa}_{\infty,2}} - 1 \log \left( \frac{\kappa_{\infty,2}}{\tilde{\kappa}_{\infty,2}} \right) + \left( \frac{\kappa_{\infty,1} - \tilde{\kappa}_{\infty,1}}{\tilde{\kappa}_{\infty,2}} \right)^2 \right).
\]

which is a hybrid variant to a cumulant-representation of the KLD. This is derived using techniques described in [24], and used in the optimization

\[
\min_K \left\{ g(\kappa_\infty, \tilde{\kappa}_\infty) \right\}
\]

\[
dx(t) = (A + BK(t))x(t) + Gdw(t), \quad x_0 = E\{x(t_0)\}, \quad t \in [t_0, \infty)
\]

From the proof, the solution is

\[
K = -R^{-1}B^T \left( \mathcal{H}_1^* + \gamma_{\infty,2} \mathcal{H}_2^* + \gamma_{\infty,3} \mathcal{H}_3^* + \gamma_{\infty,4} \mathcal{H}_4^* \right).
\]
with

$$\gamma_{\infty,i} = \begin{pmatrix} \frac{\partial g(\kappa^\ast_{\infty}, \kappa_{\infty})}{\partial \kappa_{\infty,i}} \\ \frac{\partial g(\kappa^\ast_{\infty}, \kappa_{\infty})}{\partial \kappa_{\infty,1}} \end{pmatrix}, \quad 2 \leq i \leq 4$$

where $\mathcal{H}_i$, $1 \leq i \leq 4$ are determined by the algebraic equations

$$(A + BK^*)^T \mathcal{H}_1^* + \mathcal{H}_1^*(A + BK^*) + K^TRK^* + Q = 0^{n \times n}$$

$$(A + BK^*)^T \mathcal{H}_2^* + \mathcal{H}_2^*(A + BK^*) + 4\mathcal{H}_1^*GWG^T \mathcal{H}_1^* = 0^{n \times n}$$

$$(A + BK^*)^T \mathcal{H}_3^* + \mathcal{H}_3^*(A + BK^*) + 6\mathcal{H}_1^*GWG^T \mathcal{H}_2^* + 6\mathcal{H}_2^*GWG^T \mathcal{H}_1^* = 0^{n \times n}$$

$$(A + BK^*)^T \mathcal{H}_4^* + \mathcal{H}_4^*(A + BK^*) + 8\mathcal{H}_1^*GWG^T \mathcal{H}_3^* + 12\mathcal{H}_2^*GWG^T \mathcal{H}_2^* + 8\mathcal{H}_3^*GWG^T \mathcal{H}_1^* = 0^{n \times n}. \quad (5.49)$$

The components of the vector $\kappa^\ast_{\infty} \in \mathbb{R}^4$ can then be computed by

$$\kappa^\ast_{\infty,i} = \text{Tr}(\mathcal{H}_i^*GWG^T), \quad 1 \leq i \leq 4. \quad (5.50)$$

The cumulants and targets are plotted in Figure 5.1. Numerical simulation yields average cost cumulants that align well with the targets, approximately achieving the target statistical characterization for the MCCDS control design. Recall that the MCCDS control seeks to track cost cumulants resultant from the 4CC control.
Figure 5.1. Infinite-Horizon Cost Density-Shaping
CHAPTER 6
CUMULANT-REPRESENTATIONS OF HYBRID VARIANTS TO PROBABILITY DISTANCE MEASURES

6.1 Introduction

The results of Chapter 4 make it evident that probability distance measures are ideal performance indices for the MCCDS optimization. However, analytical expressions of the Kullback-Leibler Divergence (KLD), the Hellinger Distance (HD), the Bhattacharyya Coefficient (BC), and the Bhattacharyya Distance (BD) in terms of a density’s cumulants and the cumulants for some target density are directly available for only the mean-variance case. Representations of the aforementioned distance functions in terms of the higher-order cumulants for a density and a target density have largely not been addressed in the literature.

The work of Lin et. al. [24] appears to be the first to study cumulant representations of the KLD and their application in image processing. The authors’ approach for obtaining the cumulant expressions of the KLD involves the well-known Edgeworth series approximation to density functions. Edgeworth series provide a means to approximate a density function by its cumulants and another baseline density function. It will be shown in this chapter that the Gaussian approximation to the density of the normalized variates considered in Chapter 4 form the base case for a succession of natural approximations to density functions via Edgeworth series.
The derivations in [24] pertain exclusively to the KLD, and the order of error used to truncate the Edgeworth series for the density approximations is so large that the fourth cumulant and the fourth target cumulant do not enter into the authors’ final expressions in a meaningful way. In this chapter, so-called *hybrid representations* of the KLD, HD, BC, and BD are derived that make adequate use of the statistical information in higher-order cumulants of the cost density and target density. It is then demonstrated through a numerical experiment that the *hybrid KLD* can be used in the developed framework of Chapter 3 for MCCDS optimizations. This work applies the same method used in [24], but with a smaller order of error to truncate the Edgeworth series as done in Comon [52].

6.2 Foundational Results

6.2.1 An Alternative Representation for the Random Cost

The representation of the cost as a sum of independent random variables will be made in this section to provide a basis for expressing the density for a normalized cost variate in an Edgeworth expansion. This is essential because the Edgeworth expansion is only a valid asymptotic expansion for sums of random variables and standardized sums of random variables. It is the provision of a central limit theorem that restricts how the cumulants of such sums behave asymptotically. This behavior of the higher-order cumulants makes the density of the sum asymptotically converge to the Gaussian density in distribution. Consequently this enables the approximation error in truncating the Edgeworth series to be quantified and controlled.

A theorem is given and then a conjecture stated in order to motivate an Edge-
worth expansion of the normalized random variate

\[ Z = \frac{\hat{J} - E\{\hat{J}\}}{\left( E\{\hat{J}^2\} - (E\{\hat{J}\})^2 \right)^{1/2}} \]

where \( \hat{J} \) is to be defined. Consider a set of deterministic functions \( \phi_i : \mathbb{R}^+ \to \mathbb{R}^n \) where the functions \( \phi_j(\tau) \) are orthonormal, with respect to the following inner product,

\[ < \phi_i(t), \phi_j(t) > = \int_{t_0}^{t_f} \phi_i^T(\tau)N(\tau)\phi_j(\tau)d\tau + \phi_i^T(t_f)Q_f\phi_j(t_f) = \delta_{ij}, \forall i,j. \]

Above, \( \delta_{ij} \) denotes the Kronecker delta, which is defined as

\[ \delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases} \]

Now assume the process \( x(\tau) \) has a Karhunen-Loeve (K-L) expansion under the defined basis

\[ x(\tau) = \sum_{j=1}^{\infty} x_j \phi_j(\tau), \ \tau \in [t_0, t_f] \]

where convergence is in the mean-square sense. The following relations have been cited in [26]:

\[ x_i = < x(t), \phi_i(t) >, \ m_i = E[x_i], \ E[(x_i - m_i)(x_j - m_j)] = \lambda_i \delta_{ij} \]

It is easy to verify that the variables \( x_i \) are uncorrelated and Gaussian, i.e. \( x_i \sim \)
Recall that an integral-quadratic cost is written generally as follows,

$$J = \int_{t_0}^{t_f} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)d\tau + x^T(t_f)Q_fx(t_f).$$  \hspace{1cm} (6.1)

When linear state feedback control is used, $u = Kx$ and this becomes

$$J = \int_{t_0}^{t_f} x^T(\tau)\left(Q(\tau) + K^T(\tau)R(\tau)K(\tau)\right)x(\tau)d\tau + x^T(t_f)Q_fx(t_f)$$

The motivational result can now be provided.

**Theorem 6.2.1** (Representation of Cost as Infinite Sum of Independent R.V.)

Under linear state feedback control, the integral-quadratic cost (6.1) on the finite-time horizon can expressed as an infinite sum of independent random variables, where convergence is with probability one (w.p. 1).

$$J = \sum_{i=1}^{\infty} x_i^2, \text{ w.p. 1}$$

**Proof** Proceed directly by considering the cost under linear state feedback control.

$$J = \int_{t_0}^{t_f} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)d\tau + x^T(t_f)Q_fx(t_f)$$

$$= \int_{t_0}^{t_f} x^T(\tau)\left(Q(\tau) + K^T(\tau)R(\tau)K(\tau)\right)x(\tau)d\tau + x^T(t_f)Q_fx(t_f)$$

Next express $x(\tau)$ using the K-L expansion and reduce using the orthonormality relations.

$$= \int_{t_0}^{t_f} \left(\sum_{i=1}^{\infty} x_i\phi_i(\tau)\right)^T N(\tau) \left(\sum_{j=1}^{\infty} x_j\phi_j(\tau)\right) d\tau + \left(\sum_{i=1}^{\infty} x_i\phi_i(t_f)\right)^T Q_f \left(\sum_{j=1}^{\infty} x_j\phi_j(t_f)\right)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i x_j \int_{t_0}^{t_f} \phi_i^T(\tau)N(\tau)\phi_j(\tau)d\tau + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i x_j \phi_i^T(t_f)Q_f\phi_j(t_f)$$

229
It is conjectured that the cost can be approximated by a truncated sum of independent random variables \( x_i^2 \). This is stated in the following.

**Conjecture 6.2.2 (Approximation of the Cost by a Truncated Sum)**
There exists a positive integer \( N \) such that the following approximation is reasonable.

\[
J = \sum_{i=1}^{\infty} x_i^2 \approx N \sum_{i=1}^{N} x_i^2 = \hat{J}.
\]

**Remark(s)** In the proceeding development, the distinction between \( J \) and \( \hat{J} \) is not made. Only \( J \) is discussed hereafter.

In Pham [26], it is shown that the cost density has a generalized \( \chi^2 \) distribution, given its expression as an infinite sum of independent squared Gaussian random variates. It is worthwhile to note that researchers have long used Edgeworth expansions to approximate \( \chi^2 \) densities and distribution functions, which provide accurate approximations to within one or two standard deviations of the mean as discussed in Patnaik [56]. This being, any loss of accuracy in \( \chi^2 \) density approximation via Edgeworth expansions is typically in the tails due to the
skewness of $\chi^2$ densities. Despite this, the Edgeworth expansions have seen a va-
riety of corrections for skewness that have compensated quite well for its use with
non-normal densities [53-54].

6.2.2 Gram-Charlier Series, Edgeworth Series, and a Central Limit Theorem

The Edgeworth and Gram-Charlier series are expansions of a probability den-
sity function for a sum of random variables in terms of its cumulants. Consider
the normalized sum of independent random variables $J \approx \sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} Y_i$ (as con-
sidered above) and the variable

$$Z = \frac{(\sum_{i=1}^{N} Y_i - E[\sum_{i=1}^{N} Y_i])}{\sqrt{Var[\sum_{i=1}^{N} Y_i]}}.$$

Its Gram-Charlier series is given below.

$$p_Z(z) = \phi(z)(1 + \frac{\rho_3}{3!} H_3(z) + \frac{\rho_4}{4!} H_4(z) + \ldots)$$

Here the *standardized cumulants* $\rho_i$ are used, which are defined as

$$\rho_i = \frac{\kappa_i}{\kappa_2^{1/2}}.$$

and the Hermite polynomials $H_i(z)$ are defined by the following recursive formula,

$$H_{k+1}(z) = z \cdot H_k(z) - \frac{d}{dz} H_k(z), \ k = 0, 1, 2, \ldots$$

$$H_0(z) = 1$$

The Hermite polynomials are orthogonal polynomials satisfying the orthogonality
relationship

\[ <\mathcal{H}_j(z), \mathcal{H}_k(z)> = \int_{-\infty}^{\infty} \phi(z)\mathcal{H}_j(z)\mathcal{H}_k(z)dz = k! \cdot \delta_{jk}. \]

The Gram-Charlier representation of a density is only valid for the class of functions \( f(z) \) such that the integral

\[ \int_{-\infty}^{\infty} \exp \left( -\frac{z^2}{4} \right) \cdot f(z)dz < \infty \]

converges. For functions not satisfying this sufficient condition, the series is not guaranteed to converge. This result is discussed in Cramér’s classical work [19]. Since the Gram-Charlier series fails as an asymptotic expansion because the resultant error from truncating the series after a fixed number of terms cannot be estimated, another series representation for the density function must be considered.

The limitations of the Gram-Charlier series have made necessary a more useful expansion of a density function, which leads this discussion to the Edgeworth series. It essentially approximates a density as the product of a polynomial and the Gaussian density function. As mentioned before, the Edgeworth series relies on the validity of central limit theorems for a normalized sum of random variables so that the error when truncating the series is well-behaved. The application of Edgeworth series to densities of summed, independent random variables can be made indiscriminately of how the random variables are distributed (e.g. whether identically-distributed or non-identically-distributed). A thorough treatment of these topics is given in Kolassa [22]. Comon uses a central limit theorem attributed to Cramér for a random variable \( Z \) that is a sum of \( N \) independent
random variables with finite cumulants [52], under which the \( i \)th cumulant of \( Z \) is of order,

\[
\kappa_i = o(N^{\frac{(z-i)}{2}}), \quad i = 3, 4, \ldots
\]  

(6.2)

### 6.2.3 Common Material, MCCDS Performance Index Construction

A common framework for constructing hybrid variants to probability distance measures is now presented. Consider the following normalized cost and target cost variates,

\[
Z = \frac{J - \kappa_1}{\kappa_2^{1/2}}, \quad \tilde{Z} = \frac{J - \tilde{\kappa}_1 + \kappa_1 - \tilde{\kappa}_1}{\kappa_2^{1/2}} = \left( \frac{\kappa_2}{\kappa_2} \right)^{1/2} \cdot \frac{J - \kappa_1}{\kappa_2^{1/2}} Z + \frac{\kappa_1 - \tilde{\kappa}_1}{\kappa_2^{1/2}} b
\]

\[
= aZ + b.
\]

Suppose the above variates have the best Gaussian approximations

\[
p_Z(z) \approx \frac{1}{\sqrt{2\pi}} \cdot \exp\left(\frac{-z^2}{2}\right), \quad p_{\tilde{Z}}(\tilde{z}) \approx \frac{a}{\sqrt{2\pi}} \cdot \exp\left(\frac{-(az + b)^2}{2}\right) = ap_{Z}(az + b)
\]

where evidently

\[
\int_{-\infty}^{\infty} p_Z(z)dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(\frac{-z^2}{2}\right) dz = 1
\]

\[
\int_{-\infty}^{\infty} p_{\tilde{Z}}(\tilde{z})dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(\frac{-(az + b)^2}{2}\right) d(az) = 1.
\]

It is now proposed that the Gaussian approximations above be improved upon
by using the successive density approximations below,

\[ p_Z(z) \approx \phi(z) \]
\[ \approx \phi(z)(1 + \frac{\rho_3}{3!} H_3(z)) \]
\[ \approx \phi(z)(1 + \frac{\rho_3}{3!} H_3(z) + \frac{\rho_4}{4!} H_4(z)) \]
\[ \approx \phi(z)(1 + \frac{\rho_3}{3!} H_3(z) + \frac{\rho_4}{4!} H_4(z) + \frac{10}{6!} \rho_5^2 H_6(z)) \]
\[ \approx \phi(z)(1 + \frac{\rho_3}{3!} H_3(z) + \frac{\rho_4}{4!} H_4(z) + \frac{10}{6!} \rho_5^2 H_6(z)) + \ldots \]

The approximations above constitute the well-known Edgeworth expansions, which are realized by truncating an Edgeworth series based upon a specified order of error in truncation that is deemed acceptable. This truncation error is well-behaved when the assumptions of the central limit theorem for sums of independent random variables is satisfied. The Edgeworth expansion up to \( o(N^{-2}) \) for the densities of the normalized cost variate and the target cost variate are given by the following expressions [52],

\[ p_Z(z) = \phi(z)(1 + v(z)) + o(N^{-2}), \quad p_{\tilde{Z}}(z) = \phi(\tilde{z})(1 + \tilde{v}(\tilde{z})) + o(N^{-2}) \]

where the functions \( v(z) \) and \( \tilde{v}(z) \) are defined in terms of the cumulants of \( Z \) and \( \tilde{Z} \), and also the orthogonal Hermite polynomials \( H_i(z) \) as below,

\[ v(z) \triangleq \frac{\rho_3}{3!} H_3(z) + \frac{\rho_4}{4!} H_4(z) + \frac{\rho_5}{5!} H_5(z) + \left( \frac{10}{6!} \rho_3^2 + \frac{\rho_6}{6!} \right) H_6(z) \]
\[ + \frac{35}{7!} \rho_3 \rho_4 H_7(z) + \left( \frac{56}{8!} \rho_3 \rho_5 + \frac{35}{8!} \rho_4^2 \right) H_8(z) + \frac{280}{9!} \rho_5^3 H_9(z) + \frac{2100}{10!} \rho_3^2 \rho_4 H_{10}(z) \]
\[ + \frac{15 \cdot 400}{12!} \rho_3^4 H_{12}(z) + o(N^2) \]
and

\[ \bar{v}(z) \triangleq \frac{\hat{\rho}_3}{3!} \mathcal{H}_3(z) + \frac{\hat{\rho}_4}{4!} \mathcal{H}_4(z) + \frac{\hat{\rho}_5}{5!} \mathcal{H}_5(z) + \left( \frac{10}{6!} \hat{\rho}_3^2 + \frac{\hat{\rho}_6}{6!} \right) \mathcal{H}_6(z) \]

\[ + \frac{35}{7!} \hat{\rho}_3 \hat{\rho}_4 \mathcal{H}_7(z) + \left( \frac{56}{8!} \hat{\rho}_3 \hat{\rho}_5 + \frac{35}{8!} \hat{\rho}_4^2 \right) \mathcal{H}_8(z) + \frac{280}{9!} \hat{\rho}_3^2 \hat{\rho}_9(z) + \frac{2100}{10!} \hat{\rho}_3 \hat{\rho}_4 \mathcal{H}_{10}(z) \]

\[ + \frac{15 \cdot 400}{12!} \hat{\rho}_3^4 \mathcal{H}_{12}(z) + o(N^2). \]

Introduce constants for the coefficients above,

\[ a_3 = \frac{1}{3!}, \quad a_4 = \frac{1}{4!}, \quad a_5 = \frac{1}{5!}, \quad a_6,1 = \frac{10}{6!}, \quad a_6,2 = \frac{1}{6!}, \quad a_7 = \frac{35}{7!}, \]

\[ a_{8,1} = \frac{56}{8!}, \quad a_{8,2} = \frac{35}{8!}, \quad a_9 = \frac{280}{9!}, \quad a_{10} = \frac{2100}{10!}, \quad a_{11} = 0, \quad a_{12} = \frac{15 \cdot 400}{12!} \quad (6.3) \]

and make the designations

\[ \rho_i = \frac{k_i}{\kappa_2^{1/2}}, \quad \hat{\rho}_i = \frac{\hat{k}_i}{\kappa_2^{1/2}}, \quad \hat{\rho}_i = \frac{\hat{k}_i}{\kappa_2^{1/2}}. \]

The Jacobian of the transformation for \( \tilde{Z} \) is \( a \) and the density of \( \tilde{Z} \) can be expressed in terms of \( Z \) as a

\[ p_{\tilde{Z}}(z) = \phi(az + b)(1 + \bar{v}(az + b)). \quad (6.4) \]

6.2.4 Overview of MCCDS Performance Index Construction

It will be seen that so-called hybrid representations of distance measures comprising the Mean-Variance Cost Density-Shaping objective functions in Chapter 4 can be derived to account for higher-order cumulants. This is discussed in the next section. Also, the KLD may be represented in terms of the higher-order cumulants of a density and target density when the densities are expanded in an Edgeworth series. The technique used for both the hybrid representations and
the multiple-cumulant representations of the KLD is the same as that used by Lin and Saito [24]. The material in this section will provide an overview of this procedure.

Representations of probability distance measures are derived when densities are approximated by truncated Edgeworth series and the respective series expansion for each density is inserted into the distance measure of interest, which is then expanded and simplified. Such forms for up to $r$ cumulants may be considered between any probability density functions where $r$ cumulants of the underlying variate exist.

For linear state-feedback control inputs, it is shown in [27] that $J$ is a finite $\chi^2$ random variable on a probability space $(\Omega, \mathcal{F}, P)$. The finiteness of $J$ stems from the fact that for linear state-feedback controls, the “running cost” and “terminal cost” functions of $J$ always satisfy the suitable polynomial growth conditions necessary for boundedness of the expectation of the cost functional [66]. So under this class of control inputs, a finite number of $r$ cumulants exist for $J$. Clearly then, cumulant representations of probability distance measures between the cost density and a target cost density exist when $r$ is sufficiently large.

When the above existence criteria are satisfied, the following enumerated steps can be performed on distance measures that can be approximated as below.

$$D(p_Z(z), p_{\tilde{Z}}(z)) \approx \int_{-\infty}^{\infty} h(\phi(z)) \cdot \sum_{m_1 + m_2 \leq p} c_{v, \bar{v}, m_2} v^{m_1}(z)\bar{v}^{m_2}(az + b)dz, \quad p \in \mathbb{N}$$

where $h(\cdot)$ is a linear function and $p$ is known. Essentially, the required form for a distance measure is that it involve an integral with the Gaussian density kernel in the integrand, multiplied by some polynomial function of $v$ and $\bar{v}$ of degree at most $p$. Below assume $v(z) = \sum_{j=3}^s a_j f_j(\rho_3, \ldots, \rho_s) \mathcal{H}_j(z)$ and $\bar{v}(z) = \sum_{j=3}^s a_j f_j(\tilde{\rho}_3, \ldots, \tilde{\rho}_s) \mathcal{H}_j(z)$.  

236
STEP 1: Derive the approximation

\[ D(p_Z(z), p_Z(\tilde{z})) \approx \int_{-\infty}^{\infty} h(\phi(z)) \cdot \sum_{m_1 + m_2 \leq p} c_{m_1, \tilde{m}_2} v^{m_1}(z) \tilde{v}^{m_2}(az + b) dz, \quad p \in \mathbb{N} \]

STEP 2: Use the multinomial theorem on the products \( v^{m_1}(z) \tilde{v}^{m_2}(az + b) \) to yield the expansion,

\[
\int_{-\infty}^{\infty} h(\phi(z)) \cdot \sum_{m_1 + m_2 \leq p} c_{m_1, \tilde{m}_2} \left( \sum_{\text{multi-terms}} \right) \]  

STEP 3: Use the orders of magnitude property on the standardized cumulants to eliminate terms,

\[
\prod_{k=k_3}^{k_s} \rho_k \cdot \prod_{l=l_3}^{l_s} \tilde{\rho}_l \to 0
\]

such that

\[
\sum_{i=3}^{s} \frac{(2-i)}{2} k_i + \sum_{i=3}^{s} \frac{(2-i)}{2} l_i < m
\]

STEP 4: If \( a \neq 1 \) and \( b \neq 0 \), expand polynomials \( \mathcal{H}_i(az + b) \) in Hermite basis,

\[
\mathcal{H}_i(az + b) = \sum_{j=1}^{i} T_{ij}(a, b) \mathcal{H}_j(z), \quad \text{where } T_{ij}(a, b) = 0 \text{ for } j > i
\]  

237
The weighting coefficients \( T_{ij}(a, b) \) shown above can be collected to form a triangular matrix for Hermite polynomials of the affine transformation \( az + b \). Consider the matrix shown below where \( T_{ij} \triangleq T_{ij}(a, b) \),

\[
\begin{bmatrix}
H_0(az + b) & T_{11} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
H_1(az + b) & T_{21} & T_{22} & 0 & 0 & 0 & 0 & \ldots & 0 \\
H_2(az + b) & T_{31} & T_{32} & T_{33} & 0 & 0 & 0 & \ldots & 0 \\
H_3(az + b) & T_{41} & T_{42} & T_{43} & T_{44} & 0 & 0 & \ldots & 0 \\
H_4(az + b) & T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & 0 & \ldots & 0 \\
H_5(az + b) & T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} & 0 & \ldots & 0 \\
H_6(az + b) & T_{71} & T_{72} & T_{73} & T_{74} & T_{75} & T_{76} & T_{77} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_{n-1}(az + b) & T_{s1} & T_{s2} & T_{s3} & T_{s4} & T_{s5} & T_{s6} & T_{s7} & \ldots & T_{ss} \\
\end{bmatrix}
= 
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
H_2(z) \\
H_3(z) \\
H_4(z) \\
H_5(z) \\
H_6(z) \\
\vdots \\
H_{n-1}(z) \\
\end{bmatrix}
\]

STEP 5: Complete integral evaluations for products of Hermite polynomials

\[
\int_{-\infty}^{\infty} h(\phi(z)) \mathcal{H}_{r_1}(z) \ldots \mathcal{H}_{r_q}(z) dz
\]

The delineated steps are made clearer through concrete examples of their application. The mechanics of this procedure are demonstrated in derivations of hybrid forms for the BC and KLD, along with a full derivation of a four-cumulant representation of the KLD. Outputs are provided in the following from complicated symbolic computations using MATLAB’s symbolic toolbox. Evaluations of products of Hermite polynomials are done in Mathematica.

**Remark(s)** In the development of this chapter, \( m = 2 \) is chosen. This selection results in the density approximations described in Section 6.2.3. The number \( p \) in general depends on how Taylor series expansions are truncated in the original
integrand of the distance function to be approximated. The control designer is free to choose this value. In the following $p = 4$ is chosen often.

6.3 Hybrid Variants of Probability Distance Measures

A *hybrid variant* of a probability distance function involves adding two functions. For the first function, a cumulant representation of the distance measure for a pre-specified amount of higher-order cumulants is derived under the assumption that lower-order cumulants are equal to the corresponding targets. In particular, the hybrid forms derived in this work assume that the variate’s mean and the variance agree with a target mean and variance, but higher-order cumulants disagree. Essentially this first step amounts to quantifying the distance between two different densities that share the same best Gaussian density approximation. For the second function, consider the opposite case. That is, assume the higher-order cumulants of the variate agree and only the best Gaussian density approximations differ, and derive the cumulant expression of the distance function under the assumptions. The derivation of the second function can sometimes be done analytically, as demonstrated in Chapter 4. Once both functions are derived, consider the function that results from summing them together. The new form is called a hybrid variant of the distance function.

Some formalism will the make these ideas clearer. Consider the normalized variates below,

$$Z = \frac{J - \kappa_1}{\kappa_2^{1/2}}, \quad \tilde{Z} = \frac{J - \tilde{\kappa}_1}{\tilde{\kappa}_2^{1/2}} = \frac{J - \kappa_1 + \kappa_1 - \tilde{\kappa}_1}{\tilde{\kappa}_2^{1/2}}$$

$$= \left( \frac{\kappa_2}{\tilde{\kappa}_2} \right)^{1/2} \cdot \frac{J - \kappa_1}{\tilde{\kappa}_2^{1/2}} + \frac{\kappa_1 - \tilde{\kappa}_1}{\tilde{\kappa}_2^{1/2}} + \frac{\kappa_1 - \tilde{\kappa}_1}{\tilde{\kappa}_2^{1/2}}$$
having the best Gaussian approximations

\[ p_Z(z) \approx \frac{1}{\sqrt{2\pi}} \cdot \exp \left( \frac{-z^2}{2} \right), \quad p_{\tilde{Z}}(\tilde{z}) \approx \frac{a}{\sqrt{2\pi}} \cdot \exp \left( \frac{-(az + b)^2}{2} \right) = ap_z(az + b) \]

where evidently

\[ \int_{-\infty}^{\infty} p_Z(z)dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp \left( \frac{-z^2}{2} \right) dz = 1 \]

\[ \int_{-\infty}^{\infty} p_{\tilde{Z}}(\tilde{z})d\tilde{z} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp \left( \frac{-(az + b)^2}{2} \right) d(az) = 1. \]

Consider the two density approximations for \( p_Z(z) \) and \( p_{\tilde{Z}}(\tilde{z}) \) individually,

\[ p_Z(z) \approx \phi(z)(1) \]

\[ \approx \phi(z)(1 + \frac{\rho_3}{3!} \mathcal{H}_3(z) + \frac{\rho_4}{4!} \mathcal{H}_4(z) + \frac{10}{6!} \rho_5^2 \mathcal{H}_6(z) + \ldots) \]

\[ p_{\tilde{Z}}(z) \approx a\phi(az + b) \]

\[ \approx a\phi(az + b)(1 + \frac{\rho_3}{3!} \mathcal{H}_3(az + b) + \frac{\rho_4}{4!} \mathcal{H}_4(az + b) + \frac{10}{6!} \rho_5^2 \mathcal{H}_6(az + b) + \ldots) \]

where \( \rho_i \) denotes the \( i \)th normalized cumulant. In the above expressions, the Gaussian density approximations might be regarded as an approximation that neglects higher-order cumulant information of \( Z \) and \( \tilde{Z} \), respectively. The Edgeworth series truncated at a level of acceptable error by the order of magnitudes property of cumulants for a sum of independent random variables often provides a better density approximation than the former Gaussian approximations, and this error tolerance characterizes \( v(z) \) and \( \bar{v}(z) \). The dependence of \( p_Z(z) \) on \( v \) and \( p_{\tilde{Z}}(z) \) on
$a, b, \bar{v}$ is made explicit in the following denotation,

$$p_Z(z; v) \triangleq \phi(z)(1 + \frac{\rho_3}{3!} \mathcal{H}_3(z) + \frac{\rho_4}{4!} \mathcal{H}_4(z) + \frac{10}{6!} \rho_3^2 \mathcal{H}_6(z) + \ldots)$$

$$p_Z(z; a, b, \bar{v}) \triangleq a\phi(az + b)(1 + \frac{\rho_3}{3!} \mathcal{H}_3(az + b) + \frac{\rho_4}{4!} \mathcal{H}_4(az + b) + \frac{10}{6!} \rho_3^2 \mathcal{H}_6(az + b) + \ldots)$$

Note that the best Gaussian approximation to the density of $Z$ and that of $\tilde{Z}$ correspond to $v(z) = 0$ and $\bar{v}(z) = 0$, respectively. Let $D_{Hybrid}(p_Z(z), p_{\tilde{Z}}(z))$ denote a hybrid probability distance measure between the density of the normalized cost variate $p_Z(z)$ and also the density of the normalized target cost variate $p_{\tilde{Z}}(z)$, which is defined here as

$$D_{Hybrid}(p_Z(z), p_{\tilde{Z}}(z)) = D(p_Z(z; 0), p_{\tilde{Z}}(z; a, b, 0)) + D(p_Z(z; v), p_{\tilde{Z}}(z; 1, 0, \bar{v}))$$

where $v$ and $\bar{v}$ will be defined for a given level of error $o(N^{-m})$ under the orders of magnitude property of the cumulants.

6.3.1 Hybrid Variant, Kullback-Leibler Divergence

Assume the framework of Section 6.2.3. Begin the derivation by inserting the expression (6.4) into the KLD,

$$KLD(p_Z(z), p_{\tilde{Z}}(z)) = \int_{-\infty}^{\infty} p_Z(z) \log\left(\frac{p_Z(z)}{p_{\tilde{Z}}(z)}\right) dz$$

$$= \int_{-\infty}^{\infty} \phi(z)(1 + v(z)) \log\left(\frac{\phi(z)(1 + v(z))}{a\phi(az + b)(1 + \bar{v}(az + b))}\right) dz$$

$$= \int_{-\infty}^{\infty} \phi(z) \log\left(\frac{\phi(z)}{a\phi(az + b)}\right) dz$$

$$+ \int_{-\infty}^{\infty} \phi(z)v(z) \log\left(\frac{\phi(z)}{a\phi(az + b)}\right) dz$$

241
The hybrid KLD is formed by summing the KLD representation in lower-order cumulants with the representation in higher-order cumulants. In particular, the KLD between densities to Gaussian approximations to normalized cost and target cost variates (i.e. the MKLD-CDS of Chapter 4) is added to the KLD between Edgeworth approximations to the aforementioned densities when \( a = 1 \) and \( b = 0 \). Observe that \( a = 1 \) and \( b = 0 \) in the second function will be essentially enforced by the MCCDS minimization by adding the MKLD-CDS performance index to the expression derived below. This enables consideration of the KLD between Edgeworth density approximations with agreement of mean and variance of the respective targets, and disagreement of higher-order cumulants.

So under the assumptions \( a = 1 \) and \( b = 0 \), the final line above becomes

\[
\int_{-\infty}^{\infty} \phi(z) (1 + v(z)) \log (\phi(z)(1 + v(z))) \, dz - \int_{-\infty}^{\infty} \phi(z) (1 + v(z)) \log ((1 + \bar{v}(az + b))) \, dz.
\] (6.5)
Use the Taylor series expansion of \( \log(1 + x) \) about \( x = 0 \),

\[
\log(x) \approx \log(1) + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)
\]

to express the integrands above in (6.5) as,

\[
(1 + v(z))\log(1 + v(z)) \approx v(z) + \frac{v^2(z)}{2} - \frac{v^3(z)}{6} + \frac{v^4(z)}{12} + o(v^4(z)) \tag{6.6}
\]

\[
(1 + \bar{v}(z))\log(1 + \bar{v}(z)) \approx \bar{v}(z) - \frac{\bar{v}^2(z)}{2} + \frac{\bar{v}^3(z)}{3} - \frac{\bar{v}^4(z)}{4}
+ v(z)\bar{v}(z) - \frac{v(z)\bar{v}^2(z)}{2} + \frac{v(z)\bar{v}^3(z)}{3} - \frac{v(z)\bar{v}^4(z)}{4} + o(\bar{v}^4(z)) \tag{6.7}
\]

The strategy is just as that described earlier. Namely, the polynomial approximations above are used in the integrands of the expanded KLD between the Edgeworth expansions to \( o(N^{-2}) \) of the density for the normalized cost and target variates. Essentially, the powers of the series given above can be expanded using the multinomial theorem. An extremely complicated expression results from this operation.

Simplification of the expression is a two-fold process. First, the orders of magnitude property (6.2) of the cumulants is employed to identify terms that asymptotically vanish. This reduction can be done without even considering the integral of the product of Hermite polynomials with the Gaussian kernel for such terms. Of whatever terms are remaining, the Hermite polynomials can be evaluated by using the integral in the KLD and the Gaussian baseline density of the Edgeworth expansion as the kernel function.

These actions are completed on the terms in the expansions of \( v^n(z) \), \( \bar{v}^n(z) \), and \( v^m(z)\bar{v}(z)^n \) via the multinomial theorem. Many of these terms become zero, and in fact the vanishing terms by in large outnumber the nonzero ones. Thus, only the
straightforward though tedious evaluation of each term in (6.6) and (6.7) yields,

\[ \int_{-\infty}^{\infty} v^2(z) \phi(z) dz = a_6^2 \cdot \hat{\rho}_3^2 \cdot Z_6^2 + a_{6,1}^2 \cdot \hat{\rho}_3^2 \cdot Z_6^2 + a_4^2 \cdot \hat{\rho}_3^2 \cdot Z_4^2 + a_3^2 \cdot \hat{\rho}_3^2 \cdot Z_3^2 \]

\[ \int_{-\infty}^{\infty} v^3(z) \phi(z) dz = a_6^2 \cdot a_4 \cdot \hat{\rho}_3 \cdot \hat{\rho}_4 \cdot Z_3^2 Z_4 + a_{6,1}^2 \cdot \hat{\rho}_3^4 \cdot Z_3^2 Z_6 \]

\[ \int_{-\infty}^{\infty} v^4(z) \phi(z) dz = a_4^4 \cdot \hat{\rho}_3^4 \cdot Z_4^4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^2(z) \phi(z) dz = a_6^2 \cdot \bar{\rho}_3^2 \cdot \bar{\rho}_4 \cdot Z_3^2 Z_4 + a_{6,1}^2 \cdot \bar{\rho}_3^2 \cdot Z_3^2 Z_6 \]

\[ \int_{-\infty}^{\infty} \bar{v}^3(z) \phi(z) dz = a_6^2 \cdot a_4 \cdot \bar{\rho}_3 \cdot \bar{\rho}_4 \cdot Z_3^2 Z_4 + a_{6,1}^2 \cdot \bar{\rho}_3^4 \cdot Z_3^2 Z_6 \]

\[ \int_{-\infty}^{\infty} \bar{v}^4(z) \phi(z) dz = a_4^4 \cdot \bar{\rho}_3^4 \cdot Z_4^4 \]

It is easy to verify the following due to the orthogonality properties of the Hermite polynomials,

\[ \int_{-\infty}^{\infty} v(z) \phi(z) dz = 0, \quad \int_{-\infty}^{\infty} \bar{v}(z) \phi(z) dz = 0. \]

Also, it can be verified

\[ \int_{-\infty}^{\infty} \bar{v}^4(z) v(z) \phi(z) dz \to 0 \]
since no term in the expansion survives the orders of magnitude property (6.2). Use the following results concerning the integral evaluation for products of Hermite polynomials with the Gaussian kernel (see [52]) to evaluate the above expressions directly.

\[ Z_p Z_q \triangleq \int_{-\infty}^{\infty} H_p(z) H_q(z) \phi(z) dz = p! \delta_{pq} \]

\[ Z_3^2 Z_4 \triangleq \int_{-\infty}^{\infty} H_3^2(z) H_4(z) \phi(z) dz = (3!)^3 \]

\[ Z_3^2 Z_6 \triangleq \int_{-\infty}^{\infty} H_3^2(z) H_6(z) \phi(z) dz = 6! \]

\[ Z_4^4 \triangleq \int_{-\infty}^{\infty} H_4(z) \phi(z) dz = 93 \cdot (3!)^2 \]

Using (6.10) in (6.8) after applying the values in (6.3), and then simplifying, yields the following expression.

\[
\begin{align*}
\frac{1}{4} \cdot \frac{\kappa_3 \tilde{\kappa}_3 \kappa_4}{\kappa_2^4} - \frac{35}{162} \cdot \frac{\kappa_3^2 \tilde{\kappa}_3 \kappa_4}{\kappa_2^6} + \frac{1}{4} \cdot \frac{\kappa_5 \tilde{\kappa}_3 \kappa_4}{\kappa_2^8} + \frac{1}{8} \cdot \frac{\kappa_4 \tilde{\kappa}_3 \kappa_4}{\kappa_2^6} + \frac{7}{12} \cdot \frac{\kappa_3 \tilde{\kappa}_3 \kappa_4}{\kappa_2^6} - \frac{1}{8} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} + \frac{1}{12} \cdot \frac{\kappa_3^2 \kappa_4}{\kappa_2^6} + \frac{1}{24} \cdot \frac{\kappa_4 \kappa_4}{\kappa_2^6} - \frac{1}{6} \cdot \frac{\kappa_3 \tilde{\kappa}_3}{\kappa_2^6} + \frac{1}{48} \cdot \frac{\kappa_4^2}{\kappa_2^6} + \frac{707}{1296} \cdot \frac{\kappa_4^2}{\kappa_2^6} + \frac{1}{12} \cdot \frac{\kappa_5 \kappa_4}{\kappa_2^6} + \frac{1}{48} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} + \frac{329}{1296} \cdot \frac{\kappa_4^2}{\kappa_2^6} + \frac{1}{24} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} - \frac{1}{6} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} - \frac{1}{8} \cdot \frac{\kappa_4^2}{\kappa_2^6} - \frac{7}{12} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} - \frac{35}{162} \cdot \frac{\kappa_3 \kappa_4}{\kappa_2^6} + \frac{1}{12} \cdot \frac{\kappa_4 \kappa_4}{\kappa_2^6}
\end{align*}
\]

(6.11)

Simple grouping of terms enables the expression to re-written as

\[
\begin{align*}
&\frac{1}{\kappa_2^2} \cdot \left( \frac{1}{12} \cdot \kappa_3^2 + \frac{1}{12} \cdot \tilde{\kappa}_3^2 - \frac{1}{6} \cdot \kappa_3 \tilde{\kappa}_3 \right) \\
&+ \frac{1}{\kappa_2^2} \cdot \left( \frac{1}{48} \cdot \kappa_4^2 + \frac{1}{48} \cdot \tilde{\kappa}_4^2 - \frac{1}{24} \cdot \kappa_4 \tilde{\kappa}_4 \right) \\
&+ \frac{1}{\kappa_2^2} \cdot \left( \frac{1}{4} \cdot \kappa_3 \tilde{\kappa}_3 \tilde{\kappa}_4 - \frac{1}{4} \cdot \tilde{\kappa}_3 \tilde{\kappa}_4 + \frac{1}{8} \cdot \kappa_3 \tilde{\kappa}_4 - \frac{1}{8} \cdot \kappa_3 \kappa_4 \right) \\
&+ \frac{1}{\kappa_2^2} \cdot \left( \frac{707}{1296} \cdot \tilde{\kappa}_3^4 + \frac{329}{1296} \cdot \kappa_4^2 - \frac{7}{12} \cdot \kappa_3 \tilde{\kappa}_3^3 - \frac{35}{162} \cdot \kappa_3 \kappa_4^2 \right)
\end{align*}
\]

(6.12)

The hybrid KLD is formed by a weighted combination of the MKLD-CDS performance index and the above expression, so to force \( a = 1 \) and \( b = 0 \) in an
MCCDS optimization problem. This yields

\[
KLD_{\text{hybrid}}(p_Z(z), p_{\tilde{Z}}(z)) = \frac{c_1}{\kappa_2} \cdot \left( \frac{1}{12} \cdot \kappa_3^2 + \frac{1}{12} \cdot \tilde{\kappa}_3^2 - \frac{1}{6} \cdot \kappa_3 \tilde{\kappa}_3 \right) \\
+ \frac{c_1}{\kappa_2} \cdot \left( \frac{1}{48} \cdot \kappa_2^2 + \frac{1}{48} \cdot \tilde{\kappa}_2^2 - \frac{1}{24} \cdot \kappa_2 \tilde{\kappa}_2 \right) \\
+ \frac{c_1}{\kappa_2} \cdot \left( \frac{1}{4} \cdot \kappa_3 \tilde{\kappa}_4 - \frac{1}{4} \cdot \tilde{\kappa}_3 \kappa_4 + \frac{1}{8} \cdot \tilde{\kappa}_2 \kappa_4 - \frac{1}{8} \cdot \kappa_2 \tilde{\kappa}_4 \right) \\
+ \frac{c_2}{2} \cdot \left( \frac{1}{1296} \cdot \tilde{\kappa}_3^4 + \frac{329}{1296} \cdot \kappa_3^4 - \frac{7}{12} \cdot \kappa_3 \tilde{\kappa}_3^3 - \frac{35}{162} \cdot \kappa_3^2 \tilde{\kappa}_3^2 \right) \\
+ \frac{c_2}{2} \cdot \left( \kappa_2^2 - 1 - \log \left( \frac{\kappa_2^2}{\tilde{\kappa}_2^2} \right) + \left( \frac{\kappa_1 - \tilde{\kappa}_1}{\tilde{\kappa}_2} \right) \right), \quad c_1, c_2 > 0. 
\]  

6.3.2 Hybrid Variant, Bhattacharyya Coefficient

Just as above, assume the framework of Section 6.2.3. Begin the derivation by inserting the expression (6.4) into the BC,

\[
BC(p_Z(z), p_{\tilde{Z}}(z)) = \int_{-\infty}^{\infty} \sqrt{p_Z(z)p_{\tilde{Z}}(z)} dz \\
= \int_{-\infty}^{\infty} \sqrt{\phi(z)(1 + v(z))}\phi(az + b)(1 + \bar{v}(az + b)) d(\sqrt{a}z) 
\]  

The hybrid BC is formed by summing the BC representation in lower-order cumulants with the representation in higher-order cumulants. In particular, the BC between densities to Gaussian approximations to normalized cost and target cost variates (i.e. the MBC-CDS of Chapter 4) is added to the BC between Edgeworth approximations to the aforementioned densities when \( a = 1 \) and \( b = 0 \). Observe that \( a = 1 \) and \( b = 0 \) in the second function will be essentially enforced by the MC-CDS minimization by adding the MBC-CDS performance index to the expression derived below. This enables consideration of the BC between Edgeworth density approximations with agreement of mean and variance of the respective targets,
and disagreement of higher-order cumulants.

Given the above, write

\[
BC(p_Z(z), p_{\tilde{Z}}(z)) = \int_{-\infty}^{\infty} \phi(z) \sqrt{(1 + v(z))(1 + \tilde{v}(z))} \, dz
\]

\[
= \int_{-\infty}^{\infty} \phi(z) \sqrt{(1 + v(z) + \tilde{v}(z) + v(z)\tilde{v}(z))} \, dz
\]

Consider the Taylor series of \( \sqrt{x+1} \) about \( x = 0 \),

\[
\sqrt{x+1} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + o(x^4).
\]

This allows for the expression of the non-Gaussian kernel portion in the integrand above as,

\[
-5 \cdot \frac{v^4(z)}{128} + \frac{v^3(z)\tilde{v}(z)}{32} + \frac{v^3(z)}{16} + \frac{v^2(z)\tilde{v}^2(z)}{64} - \frac{v^2(z)\tilde{v}(z)}{16} - \frac{v^2(z)}{8} + \frac{v(z)\tilde{v}^3(z)}{32}
\]

\[
- \frac{v(z)\tilde{v}^2(z)}{16} + \frac{v(z)\tilde{v}(z)}{4} + \frac{v}{2} - 5 \cdot \frac{\tilde{v}^4(z)}{128} + \frac{\tilde{v}^3(z)}{16} - \frac{\tilde{v}^2(z)}{8} + \frac{\tilde{v}(z)}{2} + 1 + o(v^{s_1}(z)\tilde{v}^{s_2}(z))
\]

where \( s_1 + s_2 = 4 \) and \( s_1, s_2 \in \mathbb{N} \).

The strategy is just as that described earlier. Namely, the polynomial approximations above are used in the integrands of the expanded BC between the Edgeworth expansions to \( o(N^{-2}) \) of the density for the normalized cost and target variates. Essentially, the powers of the series given above can be expanded using the multinomial theorem. An extremely complicated expression results from this operation.

Simplification of the expression is a two-fold process. First, the orders of magnitude property (6.2) of the cumulants is employed to identify terms that asymptotically vanish. This reduction can be done without even considering the
integral of the product of Hermite polynomials with the Gaussian kernel for such terms. Of whatever terms are remaining, the Hermite polynomials can be evaluated by using the integral in the BC and the Gaussian baseline density of the Edgeworth expansion as the kernel function.

These actions are completed on the terms above, many of which become zero. Since the vanishing terms by in large outnumber those of interest, only the ultimate form of the expression is presented after simplification is performed.

\[ \int_{-\infty}^{\infty} v^2(z)\phi(z)dz = a_0^2 \cdot \rho_3^2 \cdot Z_3^2 + a_{6,1}^2 \cdot \rho_3^4 \cdot Z_6^2 + a_4^2 \cdot \rho_4^2 \cdot Z_4^2 + a_3^2 \cdot \rho_3^4 \cdot Z_3^2 \]

\[ \int_{-\infty}^{\infty} v^3(z)\phi(z)dz = a_3^2 \cdot a_4 \cdot \rho_3^2 \cdot \rho_4 \cdot Z_3^2 Z_4 + a_2^2 \cdot a_{6,1} \cdot \rho_3^4 \cdot Z_6^2 Z_4 \]

\[ \int_{-\infty}^{\infty} v^4(z)\phi(z)dz = a_3^4 \cdot \rho_3^4 \cdot Z_3^4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^2(z)\phi(z)dz = a_0^2 \cdot \rho_3^2 \cdot Z_3^2 + a_{6,1}^2 \cdot \rho_3^4 \cdot Z_6^2 + a_4^2 \cdot \rho_4^2 \cdot Z_4^2 + a_3^2 \cdot \rho_3^4 \cdot Z_3^2 \]

\[ \int_{-\infty}^{\infty} \bar{v}^3(z)\phi(z)dz = a_3^2 \cdot a_4 \cdot \rho_3^2 \cdot \rho_4 \cdot Z_3^2 Z_4 + a_2^2 \cdot a_{6,1} \cdot \rho_3^4 \cdot Z_6^2 Z_4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^4(z)\phi(z)dz = a_3^4 \cdot \rho_3^4 \cdot Z_3^4 \]  \hspace{1cm} (6.14)

\[ \int_{-\infty}^{\infty} \bar{v}^2(z)\bar{v}(z)\phi(z)dz = a_0^2 \cdot \rho_3^2 \cdot \rho_4 \cdot Z_3^4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^3(z)v(z)\phi(z)dz = a_3^4 \cdot \rho_3^4 \cdot Z_3^4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^2(z)v^2(z)\phi(z)dz = a_3^4 \cdot \rho_3^2 \cdot \rho_4^2 \cdot Z_3^4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^2(z)v(z)\phi(z)dz = a_3^4 \cdot a_4 \cdot \rho_3 \cdot \rho_3 \cdot \rho_4 \cdot Z_3^2 Z_4 + a_3^2 \cdot a_{6,1} \cdot \rho_3^2 \cdot \rho_3 \cdot \rho_4 \cdot Z_3^2 Z_6 \]

\[ + a_3^2 \cdot a_{6,1} \cdot \rho_3 \cdot \rho_3 \cdot \rho_3 \cdot Z_3^2 Z_6 + a_3^2 \cdot a_4 \cdot \rho_4 \cdot \rho_3 \cdot \rho_3 \cdot Z_3^2 Z_4 \]

\[ \int_{-\infty}^{\infty} \bar{v}^2(z)\bar{v}(z)\phi(z)dz = a_0^2 \cdot a_4 \cdot \rho_3 \cdot \rho_4 \cdot \rho_3 \cdot Z_3^2 Z_6 + a_3^2 \cdot a_{6,1} \cdot \rho_3^2 \cdot \rho_3 \cdot \rho_4 \cdot Z_3^2 Z_4 \]

\[ + a_3^2 \cdot a_{6,1} \cdot \rho_3 \cdot \rho_3 \cdot Z_3^2 Z_6 + a_3^2 \cdot a_4 \cdot \rho_3 \cdot \rho_4 \cdot Z_3^2 Z_4 \]
\[ \int_{-\infty}^{\infty} v(z) \bar{v}(z) \phi(z) dz = a_4^2 \cdot \hat{\rho}_4 \cdot \hat{\rho}_4 \cdot Z_4^2 + a_3^2 \cdot \hat{\rho}_3 \cdot \hat{\rho}_3 \cdot Z_3^2 + a_0^2 \cdot \hat{\rho}_0 \cdot \hat{\rho}_0 \cdot Z_0^2 + a_{6,1}^2 \cdot \hat{\rho}_6 \cdot \hat{\rho}_6 \cdot Z_6^2 \]

It is easy to verify the following due to the orthogonality properties of the Hermite polynomials,

\[ \int_{-\infty}^{\infty} v(z) \phi(z) dz = 0, \quad \int_{-\infty}^{\infty} \bar{v}(z) \phi(z) dz = 0. \]

Also, it can be verified

\[ \int_{-\infty}^{\infty} \bar{v}^4(z) v(z) \phi(z) dz \to 0 \]

since no term in the expansion survives the orders of magnitude property (6.2).

Use the following results concerning the integral evaluation for products of Hermite polynomials (see [52]) with the Gaussian kernel to evaluate the above expression.

\[ Z_p Z_q \triangleq \int_{-\infty}^{\infty} H_p(z) H_q(z) \phi(z) dz = p! \delta_{pq} \]

\[ Z_3^2 Z_4 \triangleq \int_{-\infty}^{\infty} H_3^2(z) H_4(z) \phi(z) dz = (3!)^3 \] \hfill (6.15)

\[ Z_3^2 Z_6 \triangleq \int_{-\infty}^{\infty} H_3^2(z) H_6(z) \phi(z) dz = 6! \]

\[ Z_3^4 \triangleq \int_{-\infty}^{\infty} H_3^4(z) \phi(z) dz = 93 \cdot (3!)^2 \]

Using (6.15) in (6.14) after applying the values in (6.3), and then simplifying, yields the following expression.

\[
1 - \frac{1}{32} \cdot \frac{\kappa_3 \kappa_3 \kappa_4}{\kappa_2^5} - \frac{1}{32} \cdot \frac{\kappa_3 \kappa_4 \kappa_3}{\kappa_2^5} + \frac{1957}{20736} \cdot \frac{\kappa_3^2 \kappa_3^2}{\kappa_2^9} + \frac{3}{64} \cdot \frac{\kappa_3^2 \kappa_4}{\kappa_2^6} - \frac{1}{64} \cdot \frac{\kappa_4 \kappa_4^2}{\kappa_2^5} \\
- \frac{1}{64} \cdot \frac{\kappa_3^2 \kappa_4}{\kappa_2^6} + \frac{53}{1152} \cdot \frac{\kappa_3^3 \kappa_3}{\kappa_2^8} + \frac{53}{1152} \cdot \frac{\kappa_3 \kappa_3^3 \kappa_3}{\kappa_2^8} + \frac{3}{64} \cdot \frac{\kappa_3 \kappa_4^3}{\kappa_2^6} + \frac{1}{96} \cdot \frac{\kappa_4 \kappa_4 \kappa_4}{\kappa_2^6} - \frac{1}{48} \cdot \frac{\kappa_3 \kappa_4^2}{\kappa_2^6} - \frac{1}{192} \cdot \frac{\kappa_3^2 \kappa_4}{\kappa_2^6}
\]

249
Simple grouping of terms enables the expression to be re-written as

\[
1 \plus \frac{1}{\kappa_2} \cdot \left( \frac{1}{2} \cdot \kappa_4 \tilde{k}_4 - \frac{1}{192} \cdot \tilde{k}_4^2 - \frac{1}{192} \cdot \kappa_4^2 \right) + \frac{1}{\kappa_2} \cdot \left( \frac{1}{24} \cdot \kappa_3 \tilde{k}_3 - \frac{1}{48} \cdot \tilde{k}_4 - \frac{1}{48} \cdot \kappa_4^2 \right)
\]

\[
\frac{1}{\kappa_2^2} \left( \frac{1957}{20736} \cdot \kappa_3 \kappa_4^2 + \frac{53}{1152} \cdot \kappa_3 \tilde{k}_3 + \frac{53}{1152} \cdot \tilde{k}_3^2 - \frac{3865}{1152} \cdot \tilde{k}_3 \right) - \frac{3865}{1152} \cdot \kappa_3 \tilde{k}_3 \frac{1}{\kappa_3 \tilde{k}_3} \left( \frac{1}{32} \cdot \kappa_3 \tilde{k}_3 \frac{1}{\kappa_3 \tilde{k}_3} \right)
\]

This hybrid variant of the BC is a weighted combination of the MBC-CDS performance index and the expression above for \( c_1 > 0 \) and \( c_2 > 0 \)

\[
BC_{\text{hybrid}}(p_Z(z), p_{\tilde{Z}}(z)) = e^{\sqrt{2}} \cdot \frac{(\kappa_2 \tilde{k}_2)^{\frac{1}{4}}}{\sqrt{\kappa_2 + \tilde{k}_2}} \cdot \exp \left( -\frac{(\kappa_1 - \tilde{k}_1)^2}{4(\kappa_2 + \tilde{k}_2)} \right)
\]

\[
+ c_1 + \frac{c_1}{\kappa_2} \left( \frac{1}{96} \cdot \kappa_4 \tilde{k}_4 - \frac{1}{192} \cdot \tilde{k}_4^2 - \frac{1}{192} \cdot \kappa_4^2 \right) + \frac{c_1}{\kappa_2} \cdot \left( \frac{1}{24} \cdot \kappa_3 \tilde{k}_3 - \frac{1}{48} \cdot \tilde{k}_4^2 - \frac{1}{48} \cdot \kappa_4^2 \right)
\]

\[
\frac{c_1}{\kappa_2^2} \left( \frac{1957}{20736} \cdot \kappa_3 \kappa_4^2 + \frac{53}{1152} \cdot \kappa_3 \tilde{k}_3 + \frac{53}{1152} \cdot \tilde{k}_3^2 - \frac{3865}{1152} \cdot \tilde{k}_3 \right) - \frac{3865}{1152} \cdot \kappa_3 \tilde{k}_3 \frac{1}{\kappa_3 \tilde{k}_3} \left( \frac{1}{32} \cdot \kappa_3 \tilde{k}_3 \frac{1}{\kappa_3 \tilde{k}_3} \right).
\]

Note the choice \( c_1 = c_2 = \frac{1}{2} \) causes the hybrid BC to evaluate to 1 when \( \kappa_i = \tilde{k}_i, 1 \leq i \leq 4 \). This characteristic is consistent with that of the MBC-CDS performance index, and further discussions of hybrid BD and HD make this selection of weighting constants.

### 6.3.3 Hybrid Variants, Bhattacharyya and Hellinger Distance

Using the hybrid BC as defined in the last section, hybrid forms of the BD and also the HD can now be presented. This development begins with the BD.
Definition 6.3.1 (Hybrid BD)

The Bhattacharyya-Distance (BD) between probability densities \( p_1(x) \) and \( p_2(x) \) is defined as

\[
BD_{\text{Hybrid}}(p_1(x), p_2(x)) = -\ln \left( BC_{\text{Hybrid}}(p_Z(z), p_{\tilde{Z}}(z)) \right).
\]

The development proceeds with the definition of the hybrid HD.

Definition 6.3.2 (Hybrid HD)

The Hellinger-Distance (HD) between probability densities \( p_1(x) \) and \( p_2(x) \) is defined as

\[
HD_{\text{Hybrid}}(p_1(x), p_2(x)) = \sqrt{1 - BC_{\text{Hybrid}}(p_Z(z), p_{\tilde{Z}}(z))}.
\]

6.4 Application of MCCDS, Four-Cumulant Cost Density-Shaping

The first-generation benchmark problem for seismically-excited buildings serves to validate the MCCDS theory with a hybrid distance function between densities. This benchmark involves a 3-story test structure, that is subjected to 1-dimensional ground motion in order to simulate the effects of seismic disturbances. The test structure frame is constructed of steel, and has a mass of 77 kg and a height of 158 cm. The floor masses for the three floors are distributed evenly, and sum to a total mass of 227 kg.

For control purposes, a representative Active Mass Driver (AMD) has been implemented on the third story of the structure. The AMD is comprised of a
single hydraulic actuator with steel masses attached to the end of a piston rod. For this experiment, the moving mass of the AMD was 5.2 kg, which amounted to 1.7% of the total mass of the structure. The goal of the study was to design a control that optimizes consumption of control resources while best attenuating disturbances according to a variety of pre-defined metrics. For further details on the setup of the experiment, consult [3].

System Model

A high-fidelity, linear time-invariant, state-space model for the structure
just described has been developed. The 28-state system has the form

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t) + E\ddot{x}_g(t)
\]

\[
y(t) = C_y x(t) + D_y u(t) + F_y \ddot{x}_g(t) + v(t)
\]

\[
z(t) = C_z x(t) + D_z u(t) + F_z \ddot{x}_g(t)
\]  

(6.20)

Above, \( x \) is the state vector, \( \ddot{x}_g \) is the scalar ground acceleration, \( u \) is the scalar control input, \( y \) is the vector of responses that can be directly measured, \( v \) is the measurement noise, and \( z \) is the vector of responses that can be regulated. The system matrices \( A, B, E, C_y, D_y, C_z, D_z, F_y, \) and \( F_z \) are all of the appropriate dimensions, and were determined by system identification methods applied to experimental data in order to represent the input-output behavior of the system up to 100 Hz. The system identification results are provided in Spencer [3].

**Performance: rms Responses**

Without considering historical earthquake data, one way of simulating the effect of an earthquake is by assuming the excitation \( \ddot{x}_g \) is a stationary random process with spectral density defined by the Kanai-Tajimi (K-T) spectrum:

\[
S_{\ddot{x}_g\ddot{x}_g}(\omega) = \frac{S_0 (4\zeta^2 \omega^2 + \omega^4)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\zeta^2 \omega_g^2 \omega^2}
\]

(6.21)

where

\[
S_0 = \frac{0.03\zeta_g}{\pi \omega_g (4\zeta^2_g + 1)} \quad \text{g}^2 \cdot \text{sec}
\]

and

\[
20 \text{ rad/sec} \leq \omega_g \leq 120 \text{ rad/sec}, \quad 0.3 \leq \zeta_g \leq 0.75.
\]
The first performance measure $J_1$ deals with maximum rms inter-story drift for admissible ground motions, among all three floors. The drifts (between floors) are defined as $d_1(t) = x_1(t)$, $d_2(t) = x_2(t) - x_1(t)$, and $d_3(t) = x_3(t) - x_2(t)$. The second performance measure deals with maximum rms per-story acceleration, analogously among all floors.

$$J_1 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{d_1}}{\sigma_{x_{30}}}, \frac{\sigma_{d_2}}{\sigma_{x_{30}}}, \frac{\sigma_{d_3}}{\sigma_{x_{30}}} \right\}, \quad J_2 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{\dot{x}_{a1}}}{\sigma_{x_{a30}}}, \frac{\sigma_{\dot{x}_{a2}}}{\sigma_{x_{a30}}}, \frac{\sigma_{\dot{x}_{a3}}}{\sigma_{x_{a30}}} \right\}$$  \hspace{1cm} (6.22)

Performance criteria that measure how much control resources are consumed by a particular control paradigm are given below. In particular, three quantities are evaluated in this assessment of the control: $\sigma_{x_m}$, $\sigma_{\dot{x}_m}$, and $\sigma_{\ddot{x}_{am}}$. The first measure $\sigma_{x_m}$ is the rms actuator displacement, and imposes requirements on the physical size of the actuator. The rms actuator velocity $\sigma_{\dot{x}_m}$ corresponds to the control energy used by the actuator. The final measure, $\sigma_{\ddot{x}_{am}}$, which is the rms absolute acceleration indicates the magnitude of the forces the actuator generates when executing the control action.

$$J_3 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{x_m}}{\sigma_{x_{30}}} \right\}, \quad J_4 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{\dot{x}_m}}{\sigma_{x_{30}}} \right\}, \quad J_5 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{\ddot{x}_{am}}}{\sigma_{x_{a30}}} \right\}$$  \hspace{1cm} (6.23)

The above metrics are normalized with respect to worst-case displacement, velocity, and acceleration (respectively) that occur for the third floor of the uncontrolled structure for the excitation parameters $\omega_g = 37.3$ rad/sec and $\zeta_g = 0.3$. The worst-case normalization constants are, $\sigma_{x_{30}} = 1.31$ cm (worst-case rms stationary displacement), $\sigma_{\dot{x}_{30}} = 47.9$ cm/sec (worst-case rms stationary velocity), and $\sigma_{\ddot{x}_{a30}} = 1.79$ g (worst-case rms stationary acceleration).

The following control constraints have been imposed: $\sigma_u \leq 1$ V, $\sigma_{\ddot{x}_{am}} \leq 2$ g, and
σ_{x_m} \leq 3 \text{ cm.} \\

**Performance: Peak Responses**

Here the input excitation $\ddot{x}_g$ is taken to be historical earthquake data. The 1940 El Centro earthquake and also the 1968 Hachinohe earthquake have been used. Because the system is a scaled model, this data must also be scaled in accordance with the time scale for the model.

In the same vein as the rms performance criteria, analogous measures are defined below. First, for the interstory drift and the peak acceleration, respectively,

$$J_6 = \max_{\text{El Centro, Hachinohe}} \left[ \max_t \left\{ \frac{|d_1(t)|}{x_{3o}}, \frac{|d_2(t)|}{x_{3o}}, \frac{|d_3(t)|}{x_{3o}} \right\} \right]$$

$$J_7 = \max_{\text{El Centro, Hachinohe}} \left[ \max_t \left\{ \frac{|\ddot{x}_{a1}(t)|}{\ddot{x}_{a3o}}, \frac{|\ddot{x}_{a2}(t)|}{\ddot{x}_{a3o}}, \frac{|\ddot{x}_{a3}(t)|}{\ddot{x}_{a3o}} \right\} \right].$$

Next, the performance criteria to gauge a control’s consumption of resources (e.g. control energy, physical space in implementation, et cetera),

$$J_8 = \max_{\text{El Centro, Hachinohe}} \left[ \max_t \left\{ \frac{|x_m(t)|}{x_{3o}} \right\} \right]$$

$$J_9 = \max_{\text{El Centro, Hachinohe}} \left[ \max_t \left\{ \frac{|\dot{x}_m(t)|}{x_{3o}} \right\} \right]$$

$$J_{10} = \max_{\text{El Centro, Hachinohe}} \left[ \max_t \left\{ \frac{|\dddot{x}_{am}(t)|}{\dddot{x}_{a3o}} \right\} \right].$$

Here, $x_{3o}$ denotes the worst-case third-story displacement, $\dot{x}_{3o}$ denotes the worst-case third-story velocity, and $x_{a3o}$ is the worst-case third-story acceleration. For the El Centro earthquake these are $x_{3o} = 3.37$ cm, $\dot{x}_{3o} = 131$ cm/sec, and $\dddot{x}_{a3o} = 5.05$ g. For the Hachinohe earthquake these are $x_{3o} = 1.66$ cm, $\dot{x}_{3o} = 58.3$ cm/sec, and $\dddot{x}_{a3o} = 2.58$ g.

Just as for the K-T Spectrum, control constraints have been imposed on the problem: $\max_t |u(t)| \leq 3$ V, $\max_t |x_m(t)| \leq 9$ cm, and $\max_t |\dddot{x}_{am}(t)| \leq 6$ g.
Simulation Results

It has been shown in [27] that a specified 4CC control yields considerable performance gains when compared against control paradigms formulated by professionals in structural engineering whose methods have benefited from deep application experience, such as the Covariance, Multi-objective, and Sliding-Mode controls. Consider the aforementioned 4CC control characterized by the constants $\mu_2 = 1.0 \times 10^{-5}$, $\mu_3 = 9 \times 10^{-12}$, and $\mu_4 = 2 \times 10^{-20}$:

$$\tilde{K}(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( \tilde{H}_1(\alpha) + \mu_2 \tilde{H}_2(\alpha) + \mu_3 \tilde{H}_3(\alpha) + \mu_4 \tilde{H}_4(\alpha) \right)$$

(6.26)

where $\tilde{H}_i(\alpha)$, $1 \leq i \leq 4$ are the target cumulant variables characterized by the system

$$\frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]$$

$$\tilde{H}(t_f) = H_{f,\mathcal{E}^*}, \quad \tilde{D}(t_f) = D_{f,\epsilon^*}$$

with $\mathcal{E}^* = 0^{10 \times 10}$ and $\epsilon^* = 1.0 \times 10^{-9}$. Associated with these variables are the target cumulants $\tilde{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha)$, which can be computed directly. The MCCDS framework can be used to track the approximate target density specified by the initial cumulants that are at the end of these target cumulant trajectories.

The goal of this study is to formulate a MCCDS control that can track the target cost cumulants realized via the 4CC control mentioned above, while giving a comparable performance in terms of the 10 pre-defined metrics. To apply the MCCDS theory for the defined targets, an appropriate function $g(\kappa, \tilde{\kappa})$ that meets all of the aforementioned criteria must be found. Also, it is helpful if this function gives an inherit measure of distance between the initial cost density and the target

256
initial cost density.

Once again, the framework of Section 6.2.3 is assumed. For a performance index, use (6.13) with $c_1 = \frac{1}{\kappa_2(t_0)}$ and $c_2 = 1$. This is,

\[
g(\kappa(t_0), \tilde{\kappa}(t_0)) = \\
\frac{1}{\kappa_2(t_0)} \cdot \left( \frac{1}{12} \cdot \kappa_3(t_0) + \frac{1}{12} \cdot \tilde{\kappa}_3(t_0) - \frac{1}{6} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) \right) \\
+ \frac{1}{\kappa_2(t_0)} \cdot \left( \frac{1}{48} \cdot \kappa_4(t_0) + \frac{1}{48} \cdot \tilde{\kappa}_4(t_0) - \frac{1}{24} \cdot \kappa_4(t_0)\tilde{\kappa}_4(t_0) \right) \\
+ \frac{1}{\kappa_2(t_0)} \cdot \left( \frac{1}{4} \cdot \kappa_3(t_0)\tilde{\kappa}_4(t_0) - \frac{1}{4} \cdot \tilde{\kappa}_3(t_0)\kappa_4(t_0) + \frac{1}{8} \cdot \kappa_3(t_0)\kappa_4(t_0) - \frac{1}{8} \cdot \kappa_3(t_0)\tilde{\kappa}_4(t_0) \right) \\
+ \frac{1}{\kappa_2(t_0)} \cdot \left( \frac{707}{1296} \cdot \tilde{\kappa}_3(t_0) + \frac{329}{1296} \cdot \kappa_3(t_0) - \frac{7}{12} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) - \frac{35}{162} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) \right) \\
+ \frac{1}{2} \cdot \left( \frac{\kappa_2(t_0)}{\kappa_2(t_0)} - 1 - \log \left( \frac{\kappa_2(t_0)}{\kappa_2(t_0)} \right) + \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{\kappa_2(t_0)} \right). \]

Since the KLD is a non-negative, convex measure between probability density functions, positivity of this function in some neighborhood of $\tilde{x}$ is guaranteed when $g(\tilde{\kappa}, \tilde{\kappa}) = 0$ and $\nabla_{\kappa} g(\kappa, \tilde{\kappa}) \big|_{\kappa = \tilde{\kappa}} = 0^{1 \times 4}$ if this function is convex. These conditions are both satisfied for (6.13), so under the assumption of convexity, the MCCDS optimization involving the function above is well-posed.

Now employ the definition of the MCCDS control with the gain

\[
K(\alpha) = -R^{-1}(\alpha)B^T(\alpha) (H_1(\alpha) + \gamma_2(\alpha)H_2(\alpha) + \gamma_3(\alpha)H_3(\alpha) + \gamma_4(\alpha)H_4(\alpha)) \tag{6.27}
\]

with

\[
\gamma_i(\alpha) = \begin{pmatrix} \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_1(\alpha)} & \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_2(\alpha)} \\ \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_3(\alpha)} & \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_4(\alpha)} \end{pmatrix}, \quad 2 \leq i \leq 4
\]

and where the $H_i(\alpha), \ i = 1 \leq i \leq 4$ are computed via the cumulant-generating
equations under this MCCDS optimal gain, and are therefore influenced by the
target cumulants $\tilde{\kappa}_i(\alpha)$, where $1 \leq i \leq 4$.

The cost cumulants under the MCCDS control are plotted in Figure 6.3. These
are overlaid with the target cost cumulants. The plot shows the quantities as be-
ing approximately equal, with a margin of error of less than 5% on the appropriate
scale for each target cumulant. Cost cumulant-tracking is therefore achieved, but
is this capability realized at the expense of performance losses in terms of the 10
metrics when comparing the MCCDS control to the 4CC control? Observing the
performance measures $J_1-J_2$ and $J_6-J_7$ side-by-side answers this question; it can be
seen directly that the performance measures corresponding to the MCCDS control
nearly identically match those for the 4CC control in Figure 6.2, making it clear
that the MCCDS controller provides a cumulant-tracking capability while preserv-
ing the closed-loop stability properties, performance gains, and design tradeoffs as
achieved with the original 4CC control. The parameters $\gamma_i(\alpha)$, $2 \leq i \leq 4$ are key to
meeting the density-shaping objectives and these are well-behaved and bounded
on $[t_0, t_f]$ as evidenced by Figure 6.4. Notice that the parameters are characterized
by the MCCDS performance index shown above as well as the target cumulants.

Since the derived MCCDS control approximately matches the 4CC control,
a comparison between MCCDS control performance (e.g. $J_1-J_2$, $J_6-J_7$) with
that of the Covariance, Multi-objective, and Sliding-Mode controls would yield
identical bar charts as in [27].
Figure 6.2: First-Generation Benchmark, Finite-Horizon MCCDS Four-Cumulant Control, Peak and rms Performance
Figure 6.3: First-Generation Benchmark, Finite-Horizon MCCDS Four-Cumulant Control, Cost-Cumulant Tracking
Figure 6.4: First-Generation Benchmark, Finite-Horizon MCCDS
Four Cost Cumulant Control, Optimal Parameters
CHAPTER 7
COST DENSITY-SHAPING GAMES

7.1 Introduction

Game theory has seen an increasing number of applications in economics, evolutionary biology, engineering, and warfare. More recently, the common formalisms in game theory have been adapted to cost cumulant control paradigms [57] [29]. Statistical control theory applied to games is very effective as it offers the ability to characterize sample paths of the system under different strategies in probability space. Furthermore, in a non-cooperative setting, cost cumulant games capture the inherent nature of uncertainty in predicting characteristics and patterns, and takes into account some degree of unpredictability, irregularity, and intelligence in the behavior of competing agents. The uncertainty in detecting trends of gameplay is a core component of robust decision-making, and in essence this has placed game extensions of statistical control paradigms at some advantage with competing paradigms for applications where uncertainty is inevitable.

While statistical control affords the capability of characterizing paths of the system under different control inputs in a probabilistic way, to date there has been no extension that enables the designer to formulate a team or player strategy so to deliberately shape the density function of a random cost or payoff with respect to how other competing agents influence the process evolution. The goal
of this chapter is to extend the cost density-shaping paradigm of Chapter 3 for the finite-horizon to non-cooperative games. In particular, saddle-point solutions for zero-sum cost density-shaping games (ZS-CDS) and Nash equilibrium solutions for nonzero-sum cost density-shaping (NZS-CDS) games will be presented.

7.2 Cost Cumulants of the LQG Framework, and Games

The dynamic games problem formulations for the Multiple-Cumulant Cost Density-Shaping (MCCDS) problem in this chapter introduce additional separate control inputs in the process dynamics and the random cost(s). The resulting forms still fit the LQG framework, and a natural line of thought leads one to conjecture that the cumulants of the random cost are still quadratic in the known initial state of the process with an additive term. This fact stems from the form of the moment-generating function for the random cost, and consequently the form for its cumulant-generating function. The form of the cost cumulants, given additional control inputs in the process dynamics and also the cost itself, can be formally established by later exploiting this particular representation of the cumulant-generating function of the cost, or more commonly, its second characteristic function.

First, a framework for this discussion must be erected. Assume in the following, that $F \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $N \in \mathcal{C}([t_0, t_f]; \mathbb{S}_+^{n})$, and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$ are known time-varying matrices, and also $W \in \mathbb{S}_+^{p}$, $Q_f \in \mathbb{S}_+^{n}$ are known constant matrices. Consider the stochastic process $x(t)$ having dynamics governed by the linear stochastic differential equation,

$$dx(t) = F(t)x(t)dt + G(t)dw(t), \quad x_0 = E\{x(t)\}$$
where \( w(t) \) is a stationary, \( p \)-dimensional Wiener process satisfying the correlation of increments property

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ \mathbf{0}^{p \times p}.
\]

Associated with the process state is the random cost functional \( J[x,u; t_0, x_0] \) given by,

\[
J = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_f x(t_f).
\]

The cost above possesses a cumulant-generating function with the form given in the following theorem, the proof of which is presented from [26] following the approach in [12].

**Theorem 7.2.1 (Representation, Cumulant-Generating Function for Cost)**

For fixed \( \theta \) let \( S(\theta, \alpha) \) and \( \rho(\theta, \alpha) \) denote functions that satisfy the partial differential equations

\[
\frac{\partial S(\theta, \alpha)}{\partial \alpha} = -F^T(\alpha)S(\theta, \alpha) - S(\theta, \alpha)F(t) - 2S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) - \theta N(\alpha), \quad S(\theta, t_f) = \theta Q_f \tag{7.1}
\]

\[
\frac{\partial \rho(\theta, \alpha)}{\partial \alpha} = -\rho(\theta, \alpha) \text{Tr}(S(\theta, \alpha)G(\alpha)WG^T(\alpha)), \quad \rho(\theta, t_f) = 1. \tag{7.2}
\]

Then the functions \( \varphi(\alpha, x_\alpha; \theta) \) and \( \psi(\alpha, x_\alpha; \theta) \) have the representations,

\[
\begin{align*}
\varphi(\alpha, x_\alpha; \theta) &= \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) \\
\psi(\alpha, x_\alpha; \theta) &= d(\theta, \alpha) + x_\alpha^T S(\theta, \alpha) x_\alpha \tag{7.3}
\end{align*}
\]

where \( d(\theta, \alpha) = \ln \rho(\theta, \alpha) \) is the function that satisfies

\[
\frac{\partial d(\theta, \alpha)}{\partial \alpha} = -\text{Tr}(S(\theta, \alpha)G(\alpha)WG^T(\alpha)), \quad d(\theta, t_f) = 0.
\]

**Proof** Replace the initial time and state \((t_0, x_0)\) with the general pair \((\alpha, x_\alpha)\) and
consider the “cost-to-go” functional,

\[ J(\alpha, x_\alpha) = \int_{\alpha}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_f x(t_f). \]

Using the notion of cost-to-go, define the function,

\[ v(\alpha, x_\alpha; \theta) = \exp(\theta J(\alpha, x_\alpha)) \]

from which the moment-generating function for \( J(\alpha, x_\alpha) \) is given by

\[ \varphi(\alpha, x_\alpha; \theta) = E\left\{ v(\alpha, x_\alpha; \theta) \mid x(\alpha) = x_\alpha \right\}. \]

Its time derivative is given by,

\[
\frac{d}{d\alpha} \varphi(\alpha, x_\alpha; \theta) = E\left\{ \frac{d}{d\alpha} v(\alpha, x_\alpha; \theta) \right\} \\
= E\left\{ \theta v(\alpha, x_\alpha; \theta) \cdot \frac{d}{d\alpha} J(\alpha, x_\alpha) \right\} \\
= E\left\{ \theta v(\alpha, x_\alpha; \theta) \cdot (-x(\alpha)^T N(\alpha)x(\alpha)) \right\} \\
= -\theta \varphi(\alpha, x_\alpha; \theta) \cdot x_\alpha^T N(\alpha)x_\alpha. \tag{7.4}
\]

The second-to-last line follows from the use of the Leibniz integral rule,

\[
\frac{\partial}{\partial \alpha} \int_{\alpha}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau = \int_{\alpha}^{t_f} \frac{dx^T(\tau)N(\tau)x(\tau)}{d\alpha}d\tau \\
+ x^T(t_f)N(t_f)x(t_f) \frac{d}{d\alpha} t_f \\
- x(\alpha)^T N(\alpha)x(\alpha) \frac{d}{d\alpha} \alpha \\
= -x(\alpha)^T N(\alpha)x(\alpha). \]
The relations (7.3) can be verified with stochastic calculus. Start with,

\[ d\varphi(\alpha, x; \theta) = E \left\{ dv(\alpha, x; \theta) \right\}. \]

To derive the expression for \( dv(\alpha, x; \theta) \), use the Ito lemma for stochastic differentiation,

\[
dv(\alpha, x; \theta) = v_\alpha(\alpha, x; \theta) d\alpha + \nabla^T x_\alpha v(\alpha, x; \theta) \cdot dx_\alpha + \frac{1}{2} \text{Tr} \left( \nabla^2 x_\alpha v(\alpha, x; \theta) dx_\alpha d\alpha^T \right).
\]

Taking the expectation above yields,

\[
d\varphi(\alpha, x; \theta) = \varphi_\alpha(\alpha, x; \theta) d\alpha + \nabla^T x_\alpha \varphi(\alpha, x; \theta) F(\alpha) x_\alpha d\alpha + \frac{1}{2} \text{Tr} \left( \nabla^2 x_\alpha \varphi(\alpha, x; \theta) G(\alpha) W G^T(\alpha) \right) d\alpha. \tag{7.5}
\]

Re-write (7.5) as

\[
\frac{d}{d\alpha} \varphi(\alpha, x; \theta) = \varphi_\alpha(\alpha, x; \theta)
+ \nabla^T x_\alpha \varphi(\alpha, x; \theta) F(\alpha) x_\alpha
+ \frac{1}{2} \text{Tr} \left( \nabla^2 x_\alpha \varphi(\alpha, x; \theta) G(\alpha) W G^T(\alpha) \right) .
\]

Assume \( \varphi(\alpha, x; \theta) = \rho(\theta, \alpha) \cdot \exp(\nabla^T x_\alpha S(\theta, \alpha) x_\alpha) \), and use this form to evaluate the
expression above. Its time derivative is

$$\varphi_\alpha(\alpha, x_\alpha; \theta) = \frac{d}{d\alpha} \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right)$$

$$+ \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) \left( x_\alpha^T \frac{d}{d\alpha} S(\theta, \alpha) x_\alpha \right)$$

$$= \frac{d}{d\alpha} \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right)$$

$$+ \varphi(x_\alpha, \alpha; \theta) \left( x_\alpha^T \frac{d}{d\alpha} S(\theta, \alpha) x_\alpha \right).$$

(7.6)

The gradient is,

$$\nabla_{x_\alpha}^T \varphi(\alpha, x_\alpha; \theta) = \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) \cdot \left[ \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right]^T$$

$$= \varphi(\alpha, x_\alpha; \theta) \left[ \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right]^T.$$

(7.7)

The Hessian matrix is,

$$\nabla_{x_\alpha}^2 \varphi(\alpha, x_\alpha; \theta) = \frac{\partial}{\partial x_\alpha} \left( \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) \cdot \left[ \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right]^T \right)$$

$$= \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right)$$

$$\cdot \left( \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right) \left[ \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right]^T$$

$$+ \rho(\theta, \alpha) \cdot \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right)$$

$$\varphi(\alpha, x_\alpha; \theta) \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right) x_\alpha \right)^T$$

$$+ \varphi(\alpha, x_\alpha; \theta) \left( S^T(\theta, \alpha) + S(\theta, \alpha) \right)$$

so that

$$\text{Tr} \left( \nabla_{x_\alpha}^2 \varphi(\alpha, x_\alpha; \theta) G(\alpha) W G^T(\alpha) \right)$$

$$= \varphi(\alpha, x_\alpha; \theta) \left( \text{Tr} \left( S^T(\theta, \alpha)x_\alpha x_\alpha^T S(\theta, \alpha) G(\alpha) W G^T(\alpha) \right) \right)$$

$$+ \text{Tr} \left( S^T(\theta, \alpha)x_\alpha x_\alpha^T S^T(\theta, \alpha) G(\alpha) W G^T(\alpha) \right)$$

(7.8)
\[ + \text{Tr} \left( S(\theta, \alpha)x_\alpha x_\alpha^T S(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]
\[ + \text{Tr} \left( S(\theta, \alpha)x_\alpha x_\alpha^T S(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]
\[ + \text{Tr} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]
\[ + \text{Tr} \left( S^T(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]

which becomes

\[ = \varphi(\alpha, x_\alpha; \theta)\left( \text{Tr} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S^T(\theta, \alpha)x_\alpha x_\alpha^T \right) \right. \]
\[ + \text{Tr} \left( S^T(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)x_\alpha x_\alpha^T \right) \]
\[ + \text{Tr} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)x_\alpha x_\alpha^T \right) \]
\[ + \text{Tr} \left( S^T(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]
\[ + \text{Tr} \left( S^T(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \]

from which it follows that

\[ \text{Tr} \left( \nabla_{x_\alpha}^2 \varphi(\alpha, x_\alpha; \theta)G(\alpha)WG^T(\alpha) \right) = \varphi(\alpha, x_\alpha; \theta)(4x_\alpha^T S(\theta, \alpha)G(\alpha)WG^T(\alpha)S^T(\theta, \alpha)x_\alpha \]
\[ + 2\text{Tr} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha) \right) \frac{d}{d\alpha} S(\theta, \alpha)x_\alpha \]

(7.11)

Above it is assumed that the function \( S(\theta, \alpha) = S^T(\theta, \alpha) \) is a symmetric solution to a Riccati equation (7.1). Using the following properties of trace

\[ \text{Tr}(A) = \text{Tr}(A^T), \quad \text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB) \]

the terms (7.6), (7.11), and (7.9) can be combined as

\[ \frac{d}{d\alpha} \varphi(\alpha, x_\alpha; \theta) = \frac{d}{d\alpha} \rho(\theta, \alpha) \cdot \exp(x_\alpha^T S(\theta, \alpha)x_\alpha) + \varphi(\alpha, x_\alpha; \theta)x_\alpha^T \frac{d}{d\alpha} S(\theta, \alpha)x_\alpha \]
\[ + \varphi(\alpha, x_\alpha; \theta)(x_\alpha^T F^T(\theta) S(\theta, \alpha) x_\alpha + x_\alpha^T S(\theta, \alpha) F(\alpha) x_\alpha) \]
\[ + \varphi(\alpha, x_\alpha; \theta)(2x_\alpha^T S(\theta, \alpha) G(\alpha) W G^T(\alpha) S^T(\theta, \alpha) x_\alpha \]
\[ + \text{Tr } (S(\theta, \alpha) G(\alpha) W G^T(\alpha)) \]
\[ = \left( \frac{d}{d\alpha} \rho(\theta, \alpha) \right) \cdot \rho(\theta, \alpha) \exp \left( x_\alpha^T S(\theta, \alpha) x_\alpha \right) + \varphi(\alpha, x_\alpha; \theta) x_\alpha^T \frac{d}{d\alpha} S(\theta, \alpha) x_\alpha \]
\[ + \varphi(\alpha, x_\alpha; \theta)(x_\alpha^T F^T(\alpha) S(\theta, \alpha) x_\alpha + x_\alpha^T S(\theta, \alpha) F(\alpha) x_\alpha) \]
\[ + \varphi(\alpha, x_\alpha; \theta)(2x_\alpha^T S(\theta, \alpha) G(\alpha) W G^T(\alpha) S^T(\theta, \alpha) x_\alpha \]
\[ + \text{Tr } (S(\theta, \alpha) G(\alpha) W G^T(\alpha)) \].

Insert the expression (7.4) into the one above, to get
\[- \theta \varphi(\alpha, x_\alpha; \theta) \cdot x_\alpha^T N(\alpha) x_\alpha = \left( \frac{d}{d\alpha} \rho(\theta, \alpha) \right) \cdot \varphi(\alpha, x_\alpha; \theta) x_\alpha^T \frac{d}{d\alpha} S(\theta, \alpha) x_\alpha \]
\[ + \varphi(\alpha, x_\alpha; \theta)(x_\alpha^T F^T(\alpha) S(\theta, \alpha) x_\alpha + x_\alpha^T S(\theta, \alpha) F(\alpha) x_\alpha) \]
\[ + \varphi(\alpha, x_\alpha; \theta)(2x_\alpha^T S(\theta, \alpha) G(\alpha) W G^T(\alpha) S^T(\theta, \alpha) x_\alpha + \text{Tr } (S(\theta, \alpha) G(\alpha) W G^T(\alpha)) \].

If \( S(\theta, \alpha) \) and \( \rho(\theta, \alpha) \) satisfy the following differential equations,
\[ \frac{\partial S(\theta, \alpha)}{\partial \alpha} = -F^T(\alpha) S(\theta, \alpha) - S(\theta, \alpha) F(t) \]
\[ - 2S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) - \theta N(\alpha), \quad S(\theta, t_f) = \theta Q_f \]
\[ \frac{\partial \rho(\theta, \alpha)}{\partial \alpha} = -\rho(\theta, \alpha) \text{Tr}(S(\theta, \alpha) G(\alpha) W G^T(\alpha)), \quad \rho(\theta, t_f) = 1. \]
it is immediate that this choice renders the derivative of (7.3) equal to the derivative of the moment-generating function for the cost-to-go.

\[
\frac{d}{d\alpha} \varphi(\alpha, x_\alpha; \theta) = E \left\{ \theta v(\alpha, x_\alpha; \theta) \cdot \frac{d}{d\alpha} J(\alpha, x_\alpha) \right\} = -\theta \varphi(\alpha, x_\alpha; \theta) \cdot x_\alpha^T N(\alpha)x_\alpha
\]

\[
\frac{d}{d\alpha} (\rho(\theta, \alpha) \cdot \exp (x_\alpha^T S(\theta, \alpha)x_\alpha)) = -\theta \varphi(\alpha, x_\alpha; \theta) \cdot x_\alpha^T N(\alpha)x_\alpha
\]

Hence the relationship \( \varphi(\alpha, x_\alpha; \theta) = \rho(\theta, \alpha) \cdot \exp (x_\alpha^T S(\theta, \alpha)x_\alpha) \) clearly holds. By definition, it follows that the cumulant-generating function for the cost-to-go is given by

\[
\psi(\alpha, x_\alpha; \theta) = \ln \varphi(\alpha, x_\alpha; \theta) = \ln \rho(\theta, \alpha) + x_\alpha^T S(\theta, \alpha)x_\alpha = d(\theta, \alpha) + x_\alpha^T S(\theta, \alpha)x_\alpha
\]

where \( d(\theta, \alpha) \) clearly satisfies

\[
\frac{\partial}{\partial \alpha} d(\theta, \alpha) = \frac{1}{\rho(\theta, \alpha)} \cdot \frac{\partial}{\partial \alpha} \rho(\theta, \alpha) = -\text{Tr}(S(\theta, \alpha)G(\alpha)WG^T(\alpha))
\]

with an terminal condition

\[
d(\theta, t_f) = \ln \rho(\theta, t_f) = \ln(1) = 0.
\]

7.3 Zero-Sum Cost Density-Shaping Games

Zero-Sum Cost Density-Shaping (ZS-CDS) games are the subject of this section. Covered topics include the process and cost definition for the ZS-CDS problem, and the associated cost cumulants. A simplified notation for the optimization
is introduced, along with the notion of target cost statistics. Following this, the ZS-CDS problem formulation is made, where a general class of convex functions of initial cost cumulants and target initial cost cumulants is proposed for dynamic optimization. An inherent aspect of the assumed form for viable performance indices is convexity in the state variables. After these topics are touched upon, the development will proceed to the theoretical aspects of the solution concept based in dynamic programming and an adapted form of the Hamilton-Jacobi-Issacs (HJI) equation to the cost cumulant-generating equations of the LQG framework. Directly following the discussion of these topics the minimax solution is derived.

7.3.1 Process and Cost

Let \((t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n\) be fixed, and let \(w(t) = W(t, \omega)\) be a \(p\)-dimensional stationary Wiener process on \([t_0, t_f]\) where \(w : [t_0, t_f] \times \Omega \to \mathbb{R}^p\) on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the following correlation of increments property is satisfied,

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ 0^{p \times p}.
\]

Let \(U_1 \in L^2_\mathcal{F}_t(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1}))\) and \(U_2 \in L^2_\mathcal{F}_t(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2}))\) be Hilbert spaces of \(\mathbb{R}^{m_1}\)-valued and \(\mathbb{R}^{m_2}\)-valued, square-integrable processes \(u_1 \in U_1\) and \(u_2 \in U_2\) that are adapted to the \(\sigma\)-field generated by \(w(t)\), where by their construction

\[
E\left\{ \int_{t_0}^{t_f} u_1^T(\tau)u_1(\tau) d\tau \right\} < \infty \quad \text{and} \quad E\left\{ \int_{t_0}^{t_f} u_2^T(\tau)u_2(\tau) d\tau \right\} < \infty.
\]

Consider the problem of Player 1 choosing strategies \(u_1 \in U_1\) and Player 2 choosing strategies \(u_2 \in U_2\) so to influence the states \(x(t) = X(t, \omega)\) of the following
linear stochastic differential equation, which belong to $L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n))$ and are adapted to the $\sigma$-field generated by $w(t)$,

\[
\begin{align*}
\frac{dx(t)}{dt} &= \left( A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t) \right)dt + G(t)dw(t), \quad t \in [t_0, t_f] \\
x_0 &= E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n
\end{align*}
\] (7.12)

where

\[
A \in C([t_0, t_f]; \mathbb{R}^{n \times n}), \quad B_1 \in C([t_0, t_f]; \mathbb{R}^{n \times m_1}), \quad B_2 \in C([t_0, t_f]; \mathbb{R}^{n \times m_2}), \quad G \in C([t_0, t_f]; \mathbb{R}^{n \times p}).
\]

Players choose strategies such that the statistical characterization of the integral-quadratic random cost functional below is optimized,

\[
J[x, u; t_0, t_0] = \int_{t_0}^{t_f} \left( x^T(\tau)Q(\tau)x(\tau) + u_1^T(\tau)R_1(\tau)u_1(\tau) - u_2^T(\tau)R_2(\tau)u_2(\tau) \right) d\tau \\
+ x^T(t_f)Q_x x(t_f).
\] (7.13)

In the above cost functional $J$, it is understood that $Q \in C([t_0, t_f]; \mathbb{S}^n_+)$ and $R_1 \in C([t_0, t_f]; \mathbb{S}^{m_1}_+), \quad R_2 \in C([t_0, t_f]; \mathbb{S}^{m_2}_+)$ for well-posedness.

Suppose further that players choose their optimal control actions within the class of memoryless, full-observation strategies, or more precisely

\[
\begin{align*}
\xi_1 : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) &\rightarrow L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^{m_1})) \\
\xi_2 : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) &\rightarrow L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^{m_2}))
\end{align*}
\]

and

\[
u_1(t) = \xi_1(t, x(t)) = K_1(t)x(t), \quad u_2(t) = \xi_2(t, x(t)) = K_2(t)x(t).
\] (7.14)
When the process having dynamics (7.12) is subjected to the controls of each player, where $K_1 \in C([t_0, t_f]; \mathbb{R}^{m_1 \times n})$ and $K_2 \in C([t_0, t_f]; \mathbb{R}^{m_2 \times n})$ are admissible control gains with respective compact, allowable sets of gains $\bar{K}_1 \subset \mathbb{R}^{m_1 \times n}$ and $\bar{K}_2 \subset \mathbb{R}^{m_2 \times n}$, it becomes

\[
\begin{align*}
    dx(t) &= \left( A(t) + B_1(t)K_1(t) + B_2(t)K_2(t) \right)x(t)dt + G(t)dw(t), \; t \in [t_0, t_f] \\
    x_0 &= E\{x(t_0)\}, \; x_0 \in \mathbb{R}^n
\end{align*}
\]  

(7.15)

and the cost (7.13) can be written as

\[
J[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Q_f x(t_f)
\]  

(7.16)

where $N(\tau) = K_1^T(\tau)R_1(\tau)K_1(\tau) - K_2^T(\tau)R_2(\tau)K_2(\tau) + Q(\tau)$.

7.3.2 Cost Cumulants (ZS-CDS)

When the process having dynamics (7.15) and cost (7.16) result from the application of linear, state-feedback control, it is known from the work of [61-62] that the cumulants are quadratic in the known initial state of the process. The form of the cost cumulants is formally presented below, and notice how the control inputs $u_1, u_2$ enter into the equations of motion.

**Theorem 7.3.1 (Cost Cumulants, Dynamic Games)**

For the cost (7.16) and the player inputs (7.14) to the system (7.12), the $r$ initial cost cumulants of $J$ have the following quadratic form

\[
\kappa_i(\alpha) = x_0^T H_i(\alpha)x_0 + D_i(\alpha), \; 1 \leq i \leq r
\]
where the functions \(H_i(\alpha)\) and \(D_i(\alpha)\) follow the equations of motion,

\[
\frac{dH_i(\alpha)}{d\alpha} = - \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_i(\alpha)
- H_i(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)
- K_i^T(\alpha)R_1(\alpha)K_1(\alpha) + K_i^T(\alpha)R_2(\alpha)K_2(\alpha) - Q(\alpha) \triangleq \mathcal{F}_i(H(\alpha), K_1(\alpha), K_2(\alpha))
\]

\[
\frac{dH_i(\alpha)}{d\alpha} = - \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_i(\alpha)
- H_i(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)
- \frac{2}{r} \sum_{j=1}^{r} \binom{i}{j} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha) \triangleq \mathcal{G}_j(H(\alpha)), \, \alpha \in [t_0, \, t_f] \, 1 \leq j \leq r.
\]

These functions satisfy the terminal conditions

\[
H_i(t_f) = Q_f, \quad H_i(t_f) = 0^{n \times n}, \quad i \geq 2
\]
\[
D_i(t_f) = 0, \quad D_2(t_f) = 1, \quad D_j(t_f) = 0, \quad j \geq 3.
\]

**Proof** Begin by using the functions \(S(\theta, \alpha)\) and \(d(\theta, \alpha)\) and define,

\[
H_i(\alpha) \triangleq \frac{\partial^i}{\partial \theta^i}S(0, \alpha), \quad D_i(\alpha) \triangleq \frac{\partial^i}{\partial \theta^i}d(0, \alpha).
\]

Write the MacLaurin series for the cumulant-generating function for \(J(\alpha, x_\alpha)\),

\[
\psi(\alpha, x_\alpha; \theta) = \sum_{j=1}^{\infty} \kappa_j(\alpha, x_\alpha) \frac{\theta^j}{j!} = d(\theta, \alpha) + x_\alpha^T S(\theta, \alpha) x_\alpha.
\]

Now consider the MacLaurin series representations of \(x_\alpha^T S(\theta, \alpha) x_\alpha\) and \(d(\theta, \alpha)\) as below,

\[
d(\theta, t) = \sum_{j=1}^{\infty} \left( \frac{\partial^j}{\partial \theta^j}d(0, \alpha) \right) \cdot \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} D_j(\alpha) \frac{\theta^j}{j!}
\]

\[
x_\alpha^T S(\theta, t) x_\alpha = \sum_{j=1}^{\infty} x_\alpha^T \left( \frac{\partial^j}{\partial \theta^j}S(0, \alpha) \right) x_\alpha \cdot \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} x_\alpha^T H_j(\alpha) x_\alpha \frac{\theta^j}{j!}
\]

274
Inserting these representations into the MacLaurin series for \( \psi(\alpha, x; \theta) \) yields,

\[
\psi(\alpha, x; \theta) = \sum_{j=1}^{\infty} \kappa_j(\alpha, x) \frac{\theta^j}{j!} = \sum_{j=1}^{\infty} (D_j(\alpha) + x^T H_j(\alpha) x) \frac{\theta^j}{j!}
\]

from which a comparison of coefficients yields the result,

\[
\kappa_j(\alpha, x) = x^T H_j(\alpha) x + D_j(\alpha).
\]

Now all that is left to show is that the functions \( H_i(\alpha) \) and \( D_i(\alpha) \) satisfy the given family of differential equations. Recall that \( S(\theta, \alpha) \) satisfies the partial differential equation

\[
\frac{\partial S(\theta, \alpha)}{\partial \alpha} = -F^T(\alpha) S(\theta, \alpha) - S(\theta, \alpha) F(\alpha) - 2S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) - \theta N(\alpha)
\]

\[
S(\theta, t_f) = \theta Q_f.
\]

For \( \theta = 0 \), this becomes

\[
\frac{\partial S(0, \alpha)}{\partial \alpha} = -F^T(\alpha) S(0, \alpha) - S(0, \alpha) F(\alpha) - 2S(0, \alpha) G(\alpha) W G^T(\alpha) S(0, \alpha)
\]

\[
S(0, t_f) = 0^{n \times n}.
\]

The unique solution to this equation is \( S(0, \alpha) = 0^{n \times n}, \forall \alpha \in [t_0, t_f] \). The consequence of this for \( \rho(0, \alpha) \) can be understood by considering the equation

\[
\frac{\partial \rho(0, \alpha)}{\partial \alpha} = -\rho(0, \alpha) \text{Tr}(S(0, \alpha) G(\alpha) W G^T(\alpha)) = 0, \ \rho(0, t_f) = 1
\]

from which it is clear that \( \rho(0, \alpha) = 1, \forall \alpha \in [t_0, t_f] \). Now differentiate the equation
giving the time evolution of $S(\theta, \alpha)$ with respect to $\theta$,

$$\frac{\partial}{\partial \alpha} \left( \frac{\partial S(\theta, \alpha)}{\partial \theta} \right) = -F^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} - \frac{\partial S(\theta, \alpha)}{\partial \theta} F(\alpha) - N(\alpha)$$

$$- 2 \frac{\partial}{\partial \theta} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right), \frac{\partial S(\theta, t_f)}{\partial \theta} = Q_f.$$  \hspace{1cm} (7.19)

Let $\theta = 0$ and it is readily observed that this gives,

$$\frac{dH_1(\alpha)}{d\alpha} = -F^T(\alpha) H_1(\alpha) - H_1(\alpha) F(\alpha) - N(\alpha), \ H_1(t_f) = Q_f \hspace{1cm} (7.20)$$

accounting for the definition of $H_1(\alpha)$ and also noting that

$$\frac{\partial}{\partial \theta} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right)
= \frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta}$$

for which it is apparent that $\theta = 0$ renders this expression equal to an $n \times n$ zero matrix. Now differentiate (7.19) with respect to $\theta$,

$$\frac{\partial}{\partial \alpha} \left( \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} \right) = -F^T(\alpha) \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} - \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} F(\alpha)$$

$$- 2 \frac{\partial^2}{\partial \theta^2} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right), \frac{\partial^2 S(\theta, t_f)}{\partial \theta^2} = 0_{n \times n}.$$  \hspace{1cm} (7.21)

where

$$\frac{\partial^2}{\partial \theta^2} \left( S(\theta, \alpha) G(\alpha) W G^T(\alpha) S(\theta, \alpha) \right) = 2 \frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta}$$

$$+ \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2} G(\alpha) W G^T(\alpha) S(\theta, \alpha) + S(\theta, \alpha) G(\alpha) W G^T(\alpha) \frac{\partial^2 S(\theta, \alpha)}{\partial \theta^2}.$$
so that when $\theta = 0$ in (7.21) this yields,

$$
\frac{dH_2(\alpha)}{d\alpha} = - F^T(\alpha)H_2(\alpha) - H_2(\alpha)F(\alpha)
$$

$$
- 4H_1(\alpha)G(\alpha)WG^T(\alpha)H_1(\alpha), \quad H_2(t_f) = 0^{n \times n}.
$$

By similar reasoning,

$$
\frac{\partial}{\partial \alpha} \left( \frac{\partial S(\theta, \alpha)}{\partial \theta^i} \right) = - F^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta^i} - \frac{\partial S(\theta, \alpha)}{\partial \theta^i} F(\alpha)
$$

$$
- \frac{\partial^j}{\partial \theta^i} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right), \quad \frac{\partial^j S(\theta, t_f)}{\partial \theta^i} = 0^{n \times n}.
$$

and it can be verified that

$$
\frac{\partial^i}{\partial \theta^i} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right) \bigg|_{\theta = 0} = \sum_{j=1}^{i-1} \left( \begin{array}{c} i \\ j \end{array} \right) \frac{\partial^j S(0, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(0, \alpha)}{\partial \theta^{i-j}}
$$

The above relation can be established with a simple inductive proof. Use (7.21) as the base case, and suppose that the following holds for $i - 1$,

$$
\frac{\partial^{i-1}}{\partial \theta^{i-1}} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right)
$$

$$
= \sum_{j=1}^{i-2} \left( \begin{array}{c} i - 1 \\ j \end{array} \right) \frac{\partial^{j} S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-(j+1)} S(\theta, \alpha)}{\partial \theta^{i-(j+1)}}
$$

$$
+ \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha)WG^T(\alpha)S(\theta, \alpha) + S(\theta, \alpha)G(\alpha)WG^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}}.
$$

The above form is entirely motivated by the expression (7.21) where $i = 3$. Con-
Consider the derivative for \( i \) and use the result above,

\[
\frac{\partial^i}{\partial \theta^i} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)) = \frac{\partial}{\partial \theta} \left( \frac{\partial^{i-1}}{\partial \theta^{i-1}} (S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha)) \right)
\]

\[
= \frac{\partial}{\partial \theta} \left( \sum_{j=1}^{i-2} \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}} \right)
\]

\[
+ \frac{\partial}{\partial \theta} \left( \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} G(\alpha)WG^T(\alpha)S(\theta, \alpha) + S(\theta, \alpha)G(\alpha)WG^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} \right).
\]

From the above expression, consider the first term

\[
\frac{\partial}{\partial \theta} \left( \sum_{j=1}^{i-2} \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}} \right)
\]

\[
= \sum_{j=1}^{i-2} \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^{i+1} S(\theta, \alpha)}{\partial (i+1) \theta} G(\alpha)WG^T(\alpha) \frac{\partial^{(i-1)-j} S(\theta, \alpha)}{\partial \theta^{(i-1)-j}}
\]

\[
+ \sum_{j=1}^{i-2} \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j+1} S(\theta, \alpha)}{\partial \theta^{i-j+1}}
\]

which leads to

\[
= \sum_{j=2}^{i-1} \begin{pmatrix} i - 1 \\ j - 1 \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}}
\]

\[
+ \sum_{j=1}^{i-2} \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}}
\]

\[
= \sum_{j=2}^{i-2} \begin{pmatrix} i - 1 \\ j - 1 \end{pmatrix} + \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}}
\]

\[
= \sum_{j=2}^{i-2} \begin{pmatrix} i - 1 \\ j - 1 \end{pmatrix} + \begin{pmatrix} i - 1 \\ j \end{pmatrix} \frac{\partial^j S(\theta, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial \theta^{i-j}}
\]

278
\[
\begin{align*}
&+ \begin{pmatrix} i - 1 \\ 1 \end{pmatrix} \frac{\partial S(\theta, \alpha)}{\partial \theta} W G^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial \theta^{i-1}} \\
&+ \begin{pmatrix} i - 1 \\ i - 2 \end{pmatrix} \frac{\partial^{i-1} S(\theta, \alpha)}{\partial^{i-1} \theta} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta} 
\end{align*}
\]

The relations below are not difficult to establish,

\[
\begin{pmatrix} i - 1 \\ j - 1 \end{pmatrix} + \begin{pmatrix} i - 1 \\ j \end{pmatrix} = \frac{(i - 1)!}{(j - 1)! \cdot (i - j)!} + \frac{(i - 1)!}{j! \cdot (i - j - 1)!} = \frac{j \cdot (i - 1)!}{j! \cdot (i - j)!} + \frac{(i - j) \cdot (i - 1)!}{j! \cdot (i - j)!} = \frac{(j + (i - j)) \cdot (i - 1)!}{j! \cdot (i - j)!} = \frac{i \cdot (i - 1)!}{j! \cdot (i - j)!} = \frac{i!}{j! \cdot (i - j)!} = \binom{i}{j}
\]

and

\[
\begin{pmatrix} i - 1 \\ 1 \end{pmatrix} = \binom{i}{1} - 1, \quad \begin{pmatrix} i - 1 \\ i - 2 \end{pmatrix} = \binom{i}{i - 1} - 1.
\]
Using the results (7.25) and (7.26), in (7.22), leads to

\[
\sum_{j=2}^{i-2} \left( \begin{array}{c} i-1 \\ j-1 \end{array} \right) \frac{\partial^i S(\theta, \alpha)}{\partial^i \theta} G(\alpha) W G^T(\alpha) \frac{\partial^{i-j} S(\theta, \alpha)}{\partial^{i-j} \theta} \\
+ \left( \begin{array}{c} i-1 \\ 1 \end{array} \right) \frac{\partial S(\theta, \alpha)}{\partial \theta} G(\alpha) W G^T(\alpha) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial^{i-1} \theta} \\
+ \left( \begin{array}{c} i-1 \\ i-2 \end{array} \right) \frac{\partial^{i-1} S(\theta, \alpha)}{\partial^{i-1} \theta} G(\alpha) W G^T(\alpha) \frac{\partial S(\theta, \alpha)}{\partial \theta}
\]

(7.25)

which completes the inductive proof. Now, use the fact that \( S(0, \alpha) = 0 \), \( \forall \alpha \) from
which it follows that

\[
\frac{\partial^i}{\partial \theta^i} \left( S(\theta, \alpha)G(\alpha)WG^T(\alpha)S(\theta, \alpha) \right) \bigg|_{\theta=0} = \sum_{j=1}^{i-1} \begin{pmatrix} i \cr j \end{pmatrix} \frac{\partial^j S(0, \alpha)}{\partial \theta^j} G(\alpha)WG^T(\alpha) \frac{\partial^{i-j} S(0, \alpha)}{\partial \theta^{i-j}} + \frac{\partial^i S(0, \alpha)}{\partial \theta^i} G(\alpha)WG^T(\alpha) \frac{\partial^i S(0, \alpha)}{\partial \theta^i} = 0_{n \times n}.
\]

\[
= \sum_{j=1}^{i-1} \begin{pmatrix} i \cr j \end{pmatrix} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha).
\]

It follows immediately that for \(2 \leq i \leq r\)

\[
\frac{dH_i(\alpha)}{d\alpha} = -(A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha))^T H_i(\alpha)
\]

\[
- H_i(\alpha)(A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha)) - 2 \sum_{j=1}^{i-1} \begin{pmatrix} i \cr j \end{pmatrix} H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha).
\]

Consider the equation

\[
\frac{\partial}{\partial \alpha} \left( \frac{\partial^i d(\theta, \alpha)}{\partial \theta^i} \right) = -\text{Tr}(\frac{\partial^i S(\theta, \alpha)}{\partial \theta^i} G(\alpha)WG^T(\alpha)), \quad \frac{\partial^i d(\theta, t_j)}{\partial \theta^i} = 0
\]

which makes it immediate that

\[
\frac{dD_i(\alpha)}{d\alpha} = -\text{Tr}(H_i(\alpha)G(\alpha)WG^T(\alpha)), \quad D_i(\alpha) = 0, \quad i \geq 1.
\]

when \(\theta = 0\). The proof is complete.
7.3.3 Notation (ZS-CDS)

Some notation is introduced to make restatements of the above equations more concise in the development. As before, this notation is inspired by that of Pham used in [26]. Begin by defining the state variables $H(\alpha) \in \mathbb{R}^{r \times n}$ and $D(\alpha) \in \mathbb{R}^r$ as below

$$H(\alpha) \triangleq \begin{bmatrix} H_1(\alpha) \\ \vdots \\ H_r(\alpha) \end{bmatrix}, \quad D(\alpha) \triangleq \begin{bmatrix} D_1(\alpha) \\ \vdots \\ D_r(\alpha) \end{bmatrix}. $$

Using these state variables, define the functions

$$\mathcal{F}(H(\alpha), K_1(\alpha), K_2(\alpha)) \triangleq \begin{bmatrix} \mathcal{F}_1(H(\alpha), K_1(\alpha), K_2(\alpha)) \\ \vdots \\ \mathcal{F}_r(H(\alpha), K_1(\alpha), K_2(\alpha)) \end{bmatrix}, \quad \mathcal{G}(H(\alpha)) \triangleq \begin{bmatrix} \mathcal{G}_1(H(\alpha)) \\ \vdots \\ \mathcal{G}_r(H(\alpha)) \end{bmatrix}. $$

Let $\mathcal{F}_i(\cdot)$ and $\mathcal{G}_i(\cdot)$ in the above definitions be defined as beforehand in (7.17). Also to be introduced is a condensed form for the terminal conditions as below.

$$H_f \triangleq \begin{bmatrix} Q_f \\ \mathbf{0}_n \times n \\ \vdots \\ \mathbf{0}_n \times n \end{bmatrix}, \quad D_f \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. $$

282
Finally, denote the vector of cost cumulants $\kappa(\alpha) \in \mathbb{R}^r$ as

$$
\kappa(\alpha) \triangleq \begin{bmatrix}
\kappa_1(\alpha) \\
\vdots \\
\kappa_r(\alpha)
\end{bmatrix}.
$$

### 7.3.4 Target Cost Statistics (ZS-CDS)

Given matrices for a system characterization $(A, B_1, B_2, G)$, an integral-quadratic cost characterization $(Q, R_1, R_2, Q_f)$, and the second-order statistics of the noise $(W)$, consider the cost cumulants as a result of alternative (and unknown) linear state-feedback controls $\tilde{u}_1(t) = \tilde{K}_1(t)\tilde{x}(t)$ and $\tilde{u}_2(t) = \tilde{K}_2(t)\tilde{x}(t)$, where $\tilde{K}_1 \in C([t_0, t_f]; \mathbb{R}^{m_1 \times n})$ and $\tilde{K}_2 \in C([t_0, t_f]; \mathbb{R}^{m_2 \times n})$. The initial cost cumulants are given by

$$
\bar{\kappa}_i(\alpha) = x_0^T \tilde{H}_i(\alpha)x_0 + \tilde{D}_i(\alpha), \quad 1 \leq i \leq r \tag{7.29}
$$

when $\alpha = t_0$. Let this set of numbers be regarded as target cost cumulants. Here the functions $\tilde{H}_i(\alpha)$ are determined by the same system of backwards-in-time, matrix differential equations as (7.17). The dynamics of $\tilde{D}_i(\alpha)$ will also be as before,

$$
\frac{d\tilde{H}(\alpha)}{d\alpha} = \mathcal{F}(\tilde{H}(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{H}(\alpha)) \tag{7.30}
$$

$$
\tilde{H}(t_f) = H_{f,\epsilon^*}, \quad \tilde{D}(t_f) = D_{f,\epsilon^*}, \quad \alpha \in [t_0, t_f]
$$

where

$$
\tilde{H}_1(t_f) = Q_f + \mathcal{E}^*, \quad \tilde{H}_i(t_f) = 0^{n \times n}, i \geq 2 \tag{7.31}
$$

$$
\tilde{D}_1(t_f) = \epsilon^*, \quad \tilde{D}_2(t_f) = 1, \quad \tilde{D}_j(t_f) = 0, \quad j \geq 3
$$
and the short-hand notation is used,

\[
H_{f,E^*} \triangleq \begin{bmatrix}
Q_f + E^*
& 0^{n \times n}
& \cdots
& 0^{n \times n}
\end{bmatrix},
\quad D_{f,E^*} \triangleq \begin{bmatrix}
\epsilon^*
& 1
& 0
& \vdots
& 0
\end{bmatrix}.
\]

Here \( \epsilon^* > 0 \) is a small perturbation constant, and \( E^* \succ 0^{n \times n} \) is a positive-definite perturbation matrix. As with the cost cumulants, compose a vector of target initial cost cumulants \( \tilde{\kappa}(\alpha) \in \mathbb{R}^r \) defined as below,

\[
\tilde{\kappa}(\alpha) \triangleq \begin{bmatrix}
\tilde{\kappa}_1(\alpha)
& \vdots
& \tilde{\kappa}_r(\alpha)
\end{bmatrix}.
\]

**Remark(s)** Given a fixed pair of “target” control gains \((\tilde{K}_1, \tilde{K}_2)\) for the unknown linear controls underlying the target initial cost cumulants, the selection of \((K_1, K_2)\) is sufficient to uniquely characterize the trajectories of the controlled state variables \((H(t_0), D(t_0))\).

### 7.3.5 Problem Formulation (ZS-CDS)

The problem statement provided for the MCCDS theory of Chapter 3 is now adapted for the ZS-CDS optimization. Before proceeding, two important definitions must be introduced: the target set and admissible feedback gains. Namely, the target set is a space that the end value of the state trajectories must lie in. Put somewhat loosely, the set of admissible control gains are those gains that can steer the state variables into the target set. The following definitions formalize
these ideas.

**Definition 7.3.2 (Target Set - ZS-CDS)**

Let \((t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}\), where \(\mathcal{M}\) denotes the target set which is a closed subset of \([t_0, t_f] \times (\mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times \mathbb{R}^r \times (\mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times \mathbb{R}^r\).

**Remark(s)** Note that the goal of tracking target cumulants suggests that predetermined trajectories of \(\dot{H}(\alpha)\) and \(D(\alpha)\) will inherently have initial values in \(\mathcal{M}\) by their definition.

For given terminal conditions, the sets of admissible feedback gains are denoted as

\[
K_{t_f, H(t_f), D(t_f), \dot{H}(t_f), \dot{D}(t_f)}^i, \quad 1 \leq i \leq 2
\]

and contain matrices \(K_i \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i \times n}), 1 \leq i \leq 2\) such that values

\[
(t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}
\]

are obtained at the end of the trajectories for the state equations (7.17) and (7.30).

This is formally stated in the following definition.

**Definition 7.3.3 (Admissible Feedback Gains - ZS-CDS)**

Denote the allowable sets of control gain values by \(K_1 \subset \mathbb{R}^{m_1 \times n}\) and \(K_2 \subset \mathbb{R}^{m_2 \times n}\), and let these sets be compact. For fixed \(r \in \mathbb{N}\) let \(K_{t_f, H(t_f), D(t_f), \dot{H}(t_f), \dot{D}(t_f)}^1 \triangleq K_1^1(t_f)\) characterize a class of \(\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1 \times n})\) and let \(K_{t_f, H(t_f), D(t_f), \dot{H}(t_f), \dot{D}(t_f)}^2 \triangleq K_2^2(t_f)\) characterize a class of \(\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2 \times n})\) such that for any \(K_1 \in K_1^1(t_f)\) and \(K_2 \in K_2^2(t_f)\) the solutions to

\[
\frac{dH(\alpha)}{d\alpha} = \mathcal{F}(H(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{d\dot{H}(\alpha)}{d\alpha} = \mathcal{F}(\dot{H}(\alpha), \dot{K}_1(\alpha), \dot{K}_2(\alpha))
\]

\[
\frac{dD(\alpha)}{d\alpha} = \mathcal{G}(H(\alpha)), \quad \frac{d\dot{D}(\alpha)}{d\alpha} = \mathcal{G}(\dot{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
H(t_f) = H_f, \quad \dot{H}(t_f) = H_{f; \varepsilon}, \quad D(t_f) = D_f, \quad \dot{D}(t_f) = D_{f; \varepsilon}.
\]
exist and the initial values of the state trajectories satisfy
\[(t_0, H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \in \mathcal{M}.\]

Consider a scalar function \(g : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}\) with vector arguments as a general performance index, which is denoted by \(g(x, \bar{x})\). For fixed \(\bar{x}\), the function becomes \(g_\bar{x} : \mathbb{R}^r \to \mathbb{R}\). Analogously for fixed \(x\), the function becomes \(g_x : \mathbb{R}^r \to \mathbb{R}\). Impose the following restrictions on \(g_\bar{x}(x)\) and \(g_x(\bar{x})\) to ensure that the ensuing optimization problem is well-posed:

- The function \(g_\bar{x}\) is analytic on \(\text{dom } g_\bar{x}\) and \(g_x\) is analytic on \(\text{dom } g_x\)
- The function \(g_\bar{x}\) is convex in \(x\) and its domain \(\text{dom } g_\bar{x}\) is a convex set
- The function \(g_x\) is non-negative in \(x\) on some neighborhood of \(\bar{x}\)

**Definition 7.3.4 (ZS-CDS Performance Index)**

Let the ZS-CDS performance index be defined as the function
\[
\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) = g(\kappa(t_0), \dot{\kappa}(t_0)).
\]

**Remark(s)** It is not difficult to see that the above ZS-CDS performance index satisfies a notion of convexity for every pair of controlled states \((H(t_0), D(t_0))\). In particular, consider that for \(0 \leq \zeta \leq 1\), the relations
\[
H(t_0) = \zeta H_1(t_0) + (1 - \zeta) H_2(t_0)
\]
and
\[
D(t_0) = \zeta D_1(t_0) + (1 - \zeta) D_2(t_0)
\]
together mean that \(\kappa(t_0) = \zeta \kappa_1(t_0) + (1 - \zeta) \kappa_2(t_0)\) since the convexity of the states guarantees that
\[
\kappa_i(t_0) = x_0^T (\zeta H_{i,1}(t_0) + (1 - \zeta) H_{i,2}(t_0)) x_0 + \zeta D_{i,1}(t_0) + (1 - \zeta) D_{i,2}(t_0)
\]
\[
= \zeta x_0^T H_{i,1}(t_0)x_0 + (1 - \zeta)x_0^T H_{i,2}(t_0)x_0 + \zeta D_{i,1}(t_0) + (1 - \zeta) D_{i,2}(t_0)
\]
\[
= \zeta (x_0^T H_{i,1}(t_0)x_0 + D_{i,1}(t_0)) + (1 - \zeta) (x_0^T H_{i,2}(t_0)x_0 + D_{i,2}(t_0))
\]
\[
= \zeta \kappa_i(t_0) + (1 - \zeta) \kappa_{i+2}(t_0), \ 1 \leq i \leq r.
\]
So clearly,
\[
H(t_0) = \zeta H_1(t_0) + (1 - \zeta) H_2(t_0) \quad \text{and} \quad H(t_0) = \zeta D_1(t_0) + (1 - \zeta) D_2(t_0)
\]
and by the convexity of \(g(\kappa(t_0), \dot{\kappa}(t_0))\) it is true that
\[
g(\zeta \kappa_1(t_0) + (1 - \zeta) \kappa_2(t_0), \dot{\kappa}(t_0)) \leq \zeta g(\kappa(t_0), \dot{\kappa}(t_0)) + (1 - \zeta) g(\kappa_2(t_0), \dot{\kappa}(t_0)).
\]

286
Consider the following optimization problem concerning the function \( g(\kappa(t_0), \tilde{\kappa}(t_0)) \) of the initial cost cumulants for a given set of target initial cost cumulants, over the admissible space of controls defined above. Ideally, the function \( g(\kappa(t_0), \tilde{\kappa}(t_0)) \) will be some positive measure between probability density functions recast in terms of initial cost cumulants and target initial cost cumulants for those densities. Deriving a control input (i.e. \( u_1 = K_1x \)) which minimizes this function will drive initial cost cumulants closer to the targets, and in effect will drive the initial cost density closer to the target initial cost density; achieving a nominal statistical characterization for the initial cost \( J \) fits the natural context of the objective for Player 1. On the other hand, a control input that maximizes the performance index (i.e. \( u_2 = K_2x \)) moves the density of \( J \) further away from the nominal statistical characterization of the initial cost given by \( \tilde{\kappa}(t_0) \); this is the competing objective of Player 2.

**Definition 7.3.5 (ZS-CDS Optimization)**

For every \( \tilde{\kappa}(t_0) \), let \( g(\kappa(t_0), \tilde{\kappa}(t_0)) \) be an analytic function, convex in \( \kappa(t_0) \), defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of \( \tilde{\kappa}(t_0) \). Let \( r \in \mathbb{N} \) be a fixed positive integer, where \( \kappa(t_0), \tilde{\kappa}(t_0) \in \mathbb{R}^r \) are the vectors of initial cost cumulants and target initial cost cumulants, respectively. Then the ZS-CDS optimization can be formulated as,

\[
\max_{K_2 \in K_2(t_f)} \min_{K_1 \in K_1(t_f)} \left\{ \phi(H(t_0), D(t_0), H(t_0), D(t_0)) \right\}
\]

subject to:

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha))
\]

\[
\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
\]

\[
H(t_f) = H_f, \quad \tilde{H}(t_f) = H_{f, \tilde{\varepsilon}}, \quad D(t_f) = D_f, \quad \tilde{D}(t_f) = D_{f, \tilde{\varepsilon}}
\]

where the initial values of the state trajectories must satisfy

\[
(t_0, H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0)) \in \mathcal{M}.
\]
When a saddle-point exists for the ZS-CDS optimization above, this is necessary and sufficient for the interchangability of the \( \min(\cdot) \) and \( \max(\cdot) \) operations above without changing the optimization problem. The notion of a saddle point is defined in the following, where the implicit dependence of \((H(t_0), D(t_0))\) on \((K_1, K_2)\) is made. In particular, the notation \( H(t_0; K_1, K_2) \) and \( D(t_0; K_1, K_2) \) is used below, and for the most part will be abandoned hereafter.

**Definition 7.3.6 (Saddle Point for the ZS-CDS Optimization)**

Consider a pair of controls \((K_1^*, K_2^*)\) such that \( K_1^* \in \mathcal{K}^1(t_f) \) and \( K_2^* \in \mathcal{K}^2(t_f) \). Suppose \( \forall K_1 \neq K_1^* \mid K_1 \in \mathcal{K}^1(t_f) \), it is true that

\[
\phi(H(t_0; K_1^*, K_2^*), D(t_0; K_1^*, K_2^*), H(t_0), \bar{D}(t_0)) \leq \phi(H(t_0; K_1, K_2^*), D(t_0; K_1, K_2^*), H(t_0), \bar{D}(t_0)).
\]

Analogously, suppose also \( \forall K_2 \neq K_2^* \mid K_2 \in \mathcal{K}^2(t_f) \), it is true that

\[
\phi(H(t_0; K_1^*, K_2), D(t_0; K_1^*, K_2), H(t_0), \bar{D}(t_0)) \geq \phi(H(t_0; K_1^*, K_2), D(t_0; K_1^*, K_2), H(t_0), \bar{D}(t_0)).
\]

Under the above conditions, the control gains \((K_1^*, K_2^*)\) constitute a saddle-point solution to the ZS-CDS optimization.

It is assumed that a minimax saddle-point solution to the ZS-CDS optimization exists in the following development. Under this assumption, the optimization given before is recast.

**Definition 7.3.7 (ZS-CDS Optimization, Minimax)**

For every \( \tilde{\kappa}(t_0) \), let \( \phi(\kappa(t_0), \tilde{\kappa}(t_0)) \) be an analytic function, convex in \( \kappa(t_0) \), defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of \( \kappa(t_0) \). Let \( r \in \mathbb{N} \) be a fixed positive integer, where \( \kappa(t_0), \tilde{\kappa}(t_0) \in \mathbb{R}^r \) are the vectors of initial cost cumulants and target initial cost cumulants, respectively. Then the ZS-CDS (minimax) optimization can be formulated as,

\[
\max_{K_2 \in \mathcal{K}^2(t_f)} \min_{K_1 \in \mathcal{K}^1(t_f)} \left\{ \phi(H(t_0), D(t_0), H(t_0), \bar{D}(t_0)) \right\}
\]

\[
= \min_{K_1 \in \mathcal{K}^1(t_f)} \max_{K_2 \in \mathcal{K}^2(t_f)} \left\{ \phi(H(t_0), D(t_0), H(t_0), \bar{D}(t_0)) \right\}
\]

subject to:

\[
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha))
\]

288
\[
\begin{align*}
\frac{d\mathbf{D}(\alpha)}{d\alpha} &= \mathcal{G}(\mathbf{H}(\alpha)), \quad \frac{d\tilde{\mathbf{D}}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{\mathbf{H}}(\alpha)), \quad \alpha \in [t_0, t_f] \\
\mathbf{H}(t_f) &= \mathbf{H}_f, \quad \tilde{\mathbf{H}}(t_f) = \tilde{\mathbf{H}}_{f, \varepsilon^*}, \quad \mathbf{D}(t_f) = \mathbf{D}_f, \quad \tilde{\mathbf{D}}(t_f) = \tilde{\mathbf{D}}_{f, \varepsilon^*},
\end{align*}
\]

where the initial values of the state trajectories must satisfy

\[
(t_0, \mathbf{H}(t_0), \mathbf{D}(t_0), \tilde{\mathbf{H}}(t_0), \tilde{\mathbf{D}}(t_0)) \in \mathcal{M}.
\]

7.3.6 A Dynamic Programming Framework (ZS-CDS)

A solution to the ZS-CDS (minimax) optimization is now derived by employing the traditional techniques of dynamic programming for Mayer Form problems adapted for games. A crucial assumption in this development are that strategies of Player 1 and Player 2 are non-anticipative and only known to each player at each instant of time. Furthermore, it is assumed that strategies follow the model of Varaiya and Elliott-Kalton, where players know a priori their counter-strategy or response for every possible strategy of the opposing player. In some sense, this model mitigates any inherent advantage that Player 1 could have over Player 2, and vice versa.

Despite the aforementioned model for playability of the game, there is the possibility that whichever player makes the first move has the upper hand in the game. When this is the case, the upper and lower value functions for the game do not agree. It is the case when agreement of upper and lower value functions is realized that a saddle-point minimax solution can be obtained. The work of [31] showed that for Mayer problems with Lipschitz continuous terminal cost, the upper and lower value functions are the unique viscosity solutions of two Hamilton-Jacobi-Bellman (HJB) equations. Under the minimax assumption that the Hamiltonians agree, the two value functions become equal. The consequential uniqueness of the value function stems from uniqueness results for viscosity solutions of the
HJB equations. When the terminal cost of a Mayer problem is discontinuous, the uniqueness results for viscosity solutions of partial differential equations cannot be used. Nevertheless, when the terminal cost is at least semicontinuous, uniqueness of the solution to the Hamilton-Jacobi-Issacs (HJI) equation is guaranteed as well, as shown by [64]. The HJI equation is a generalization of the HJB equations for zero-sum dynamic games.

The HJI equation adapted to the cost cumulant-generating equations is presented in this section, and going forward, the discussion addresses an initial cost problem akin to the initial cost problem studied in Chapter 3. As before, the fundamental difference between this problem and the traditional terminal cost Mayer problem is a direct result of the backwards-in-time equations giving the dynamics of the game.

The following variables will be used in the solution derivation. Define the block matrices \( Y(\epsilon) \), \( \tilde{Y}(\epsilon) \) ∈ \( \mathbb{R}^{rn \times n} \) and the vectors \( Z(\epsilon) \), \( \tilde{Z}(\epsilon) \) ∈ \( \mathbb{R}^r \) as below,

\[
Y(\epsilon) = \begin{bmatrix} Y_1(\epsilon) \\ Y_2(\epsilon) \\ \vdots \\ Y_r(\epsilon) \end{bmatrix}, \quad \tilde{Y}(\epsilon) = \begin{bmatrix} \tilde{Y}_1(\epsilon) \\ \tilde{Y}_2(\epsilon) \\ \vdots \\ \tilde{Y}_r(\epsilon) \end{bmatrix}, \quad Z(\epsilon) = \begin{bmatrix} Z_1(\epsilon) \\ Z_2(\epsilon) \\ \vdots \\ Z_r(\epsilon) \end{bmatrix}, \quad \tilde{Z}(\epsilon) = \begin{bmatrix} \tilde{Z}_1(\epsilon) \\ \tilde{Z}_2(\epsilon) \\ \vdots \\ \tilde{Z}_r(\epsilon) \end{bmatrix}.
\]

\( Y_i(\epsilon) = H_i(\epsilon), \quad Z_i(\epsilon) = D_i(\epsilon), \quad \tilde{Y}_i(\epsilon) = \tilde{H}_i(\epsilon), \quad \tilde{Z}_i(\epsilon) = \tilde{D}_i(\epsilon), \quad 1 \leq i \leq r \)

References to states \( Y(\tau), \tilde{Y}(\tau), Z(\tau), \) and \( \tilde{Z}(\tau) \) will be made, and the above definitions of block matrices and vectors apply \( \forall \tau \in [t_0, \epsilon) \). Before continuing, note that in this development it is assumed that the terminal time \( t_f \) and the
associated boundary conditions have been displaced to corresponding ones at some
time $\epsilon$, and the nature of the control that drives the state from $t_f$ to $\epsilon$ must be
regarded. In cases where a control is assumed part of the saddle-point solution,
a * designation will appear with the control denotation, for example $K_1^*$. The
state variables under this control will receive a similar distinction. For instance,
$(\epsilon, \mathbf{y}^*(\epsilon), \mathbf{z}^*(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$ refer to
"displaced" terminal conditions arrived at from the saddle-point solution $(K_1^*, K_2^*)$ on $(\epsilon, t_f)$, and $(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$ will refer to terminal conditions arrived at from any strategy pair $(K_1, K_2)$ on $(\epsilon, t_f)$.

It is appropriate now to define the playable set $\mathcal{Q}$ and the value function
$V(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$, which are core constructs to this
dynamic programming formulation.

**Definition 7.3.8 (Value Function - ZS-CDS)**

Let $(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon)) \in [t_0, t_f] \times (S^n)^r \times \mathbb{R}^r \times (S^n)^r \times \mathbb{R}^r$ and let
$V(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$ be a scalar function

$$V : [t_0, t_f] \times (S^n)^r \times \mathbb{R}^r \times (S^n)^r \times \mathbb{R}^r \to \mathbb{R}$$

such that

$$V(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$$

$$= \left\{ \max_{K_2 \in \mathcal{K}_2(\epsilon)} \left\{ \min_{K_1 \in \mathcal{K}_1(\epsilon)} \phi(\mathbf{H}(t_0), \mathbf{D}(t_0), \hat{\mathbf{H}}(t_0), \hat{\mathbf{D}}(t_0)) \right\}, \mathcal{K}_1(\epsilon) \times \mathcal{K}_2(\epsilon) \neq \emptyset \right\} \cup \left\{ \min_{K_1 \in \mathcal{K}_1(\epsilon)} \left\{ \max_{K_2 \in \mathcal{K}_2(\epsilon)} \phi(\mathbf{H}(t_0), \mathbf{D}(t_0), \hat{\mathbf{H}}(t_0), \hat{\mathbf{D}}(t_0)) \right\}, \mathcal{K}_1(\epsilon) \times \mathcal{K}_2(\epsilon) = \emptyset \right\}$$

where $\mathcal{K}(\epsilon) \triangleq \mathcal{K}_1(\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon))$.

**Definition 7.3.9 (Playable Set - ZS-CDS)**

Define the playable set as the set of terminal values from which there exists a
control that can take the system to the target set. More formally, this is

$$\mathcal{Q} = \{ (\epsilon, \mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \hat{\mathbf{y}}(\epsilon), \hat{\mathbf{z}}(\epsilon)) \mid \mathcal{K}_1(\epsilon) \times \mathcal{K}_2(\epsilon) \neq \emptyset \}.$$
Under certain conditions, the value function defined above will be continuously differentiable on sets within $Q$. The following theorem defines these conditions, and it can be proven via the approach used in Chapter 3.

**Theorem 7.3.10 (Differentiability of Value Function)**

Suppose that controls $(K_1(\alpha), H(\alpha), D(\alpha), \tilde{H}(\alpha), \tilde{D}(\alpha), K_2(\alpha), H(\alpha), D(\alpha), \tilde{H}(\alpha), \tilde{D}(\alpha))$ are given, and let the initial time $t_0(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))$ and

$$H(t_0(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon)), D(t_0(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon)))$$

denote initial states of the trajectories

$$\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K_1(\alpha), K_2(\alpha)), \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha))$$

$$\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \alpha \in [t_0, \epsilon]$$

$$H(\epsilon) = \mathcal{Y}(\epsilon), \tilde{H}(\epsilon) = \check{Y}(\epsilon), D(\epsilon) = Z(\epsilon), \tilde{D}(\epsilon) = \check{Z}(\epsilon).$$

The value function $V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))$ is differentiable in $(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon))$ when the terminal states and initial time are also differentiable in those variables.

**Proof** Identical to the approach used in Chapter 3.

**Theorem 7.3.11 (HJI Equation, ZS-CDS Optimization)**

Let $(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))$ be an interior point of the playable set $Q$ where the value function $V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))$ is differentiable. If a control pair $(K_1^*, K_2^*) \in K_1^*(\epsilon) \times K_1^*(\epsilon)$ is a saddle-point solution, then the scalar function $V(\cdot)$ satisfies the HJI equation below

$$-\frac{\partial V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))}{\partial \epsilon} = \max_{K_2 \in K_2^*} \min_{K_1 \in K_1^*} \left\{ \frac{\partial V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))}{\partial Z(\epsilon)} G(\check{Y}(\epsilon)) + \frac{\partial V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))}{\partial \mathcal{Y}(\epsilon)} G(\mathcal{Y}(\epsilon)) + \frac{\partial V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))}{\partial \mathcal{Z}(\epsilon)} G(\check{Z}(\epsilon)) + \frac{\partial V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon))}{\partial vec(\mathcal{Y}(\epsilon))} vec(F(\check{Y}(\epsilon), K_1^*(\epsilon), K_2^*(\epsilon)) \right\}.$$

with the boundary condition,

$$V(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon)) = \phi(H(t_0), D(t_0), \tilde{H}(t_0), \tilde{D}(t_0))$$

$$(\epsilon, \mathcal{Y}(\epsilon), Z(\epsilon), \check{Y}(\epsilon), \check{Z}(\epsilon)) \in \mathcal{M}. \quad (7.32)$$

292
\[ W(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) = \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \]

(7.33)

then \((K_1^*, K_2^*)\) is a saddle-point solution and it must be that

\[ W(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) = \mathcal{V}(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)). \]

**Proof** See [26].

A candidate value function of the following form is proposed.

**Definition 7.3.13 (Candidate Value Function)**

Consider a solution of the form

\[ W(\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) = \eta(\epsilon) - \phi(Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) \]

(7.34)

where the function \(\eta(\tau) \in C^1([t_0, t_f], \mathbb{R})\) is to be determined.

For any point \((\epsilon, Y(\epsilon), Z(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon))\) in the playable set \(\mathcal{Q}\) where the candidate value function is differentiable, it may differentiated directly. The following will
establish the form of the total time derivative of $W(\varepsilon, \mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))$, which essentially is constrained by the HJI equation.

**Lemma 7.3.14 (Derivative of Candidate Value Function)**

Let $r \in \mathbb{N}$ be fixed and let $(\varepsilon, \mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon)) \in \mathcal{Q}$ be an interior point of the set at which the candidate value function

$$ W(\varepsilon, \mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon)) = \eta(\varepsilon) - \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon)) $$

is differentiable. Then its total time derivative takes the form

$$ \frac{dW(\varepsilon, \mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{d\varepsilon} = - \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\varepsilon)} g(\kappa(\varepsilon), \dot{\kappa}(\varepsilon)) \mathbf{x}_0^T \mathbf{F}_i(\mathbf{y}(\varepsilon), K_1(\varepsilon), K_2(\varepsilon)) \mathbf{x}_0 $$

$$ - \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\varepsilon)} g(\kappa(\varepsilon), \dot{\kappa}(\varepsilon)) \mathbf{x}_0^T \mathbf{F}_i(\dot{\mathbf{y}}(\varepsilon), \dot{K}_1(\varepsilon), \dot{K}_2(\varepsilon)) \mathbf{x}_0 $$

$$ - \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\varepsilon)} g(\kappa(\varepsilon), \dot{\kappa}(\varepsilon)) \mathbf{g}_i(\mathbf{y}(\varepsilon)) $$

$$ - \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\varepsilon)} g(\kappa(\varepsilon), \dot{\kappa}(\varepsilon)) \mathbf{g}_i(\dot{\mathbf{y}}(\varepsilon)) + \frac{d\eta(\varepsilon)}{d\varepsilon}. $$

**Proof** Proceed by directly differentiating (7.34) to obtain the following formal expression,

$$ \frac{dW(\varepsilon, \mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{d\varepsilon} = - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \mathbf{y}_i(\varepsilon)} \mathbf{y}_i(\varepsilon) \mathbf{y}_i(\varepsilon)^T $$

$$ - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \dot{\mathbf{y}}_i(\varepsilon)} \dot{\mathbf{y}}_i(\varepsilon) \dot{\mathbf{y}}_i(\varepsilon)^T $$

$$ - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \mathbf{z}_i(\varepsilon)} \mathbf{z}_i(\varepsilon) \mathbf{z}_i(\varepsilon)^T $$

$$ - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \dot{\mathbf{z}}_i(\varepsilon)} \dot{\mathbf{z}}_i(\varepsilon) \dot{\mathbf{z}}_i(\varepsilon)^T - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \mathbf{z}_i(\varepsilon)} \mathbf{g}_i(\mathbf{y}(\varepsilon)) $$

$$ - \sum_{i=1}^{r} \frac{\partial \phi(\mathbf{y}(\varepsilon), \mathbf{z}(\varepsilon), \dot{\mathbf{y}}(\varepsilon), \dot{\mathbf{z}}(\varepsilon))}{\partial \dot{\mathbf{z}}_i(\varepsilon)} \mathbf{g}_i(\dot{\mathbf{y}}(\varepsilon)) + \frac{d\eta(\varepsilon)}{d\varepsilon} $$

$$ \forall K_1 \in \mathcal{K}^1(\varepsilon), \ K_2 \in \mathcal{K}^2(\varepsilon) $$

Note the cumulant and target cost cumulant forms when the cost is integral quadratic and the control input is of the linear state feedback type. In partic-
\[ \kappa_i(\epsilon) = x_0^T \lambda_i(\epsilon)x_0 + \lambda_i(\epsilon), \quad \tilde{\kappa}_i(\epsilon) = x_0^T \tilde{\lambda}_i(\epsilon)x_0 + \tilde{\lambda}_i(\epsilon) \]

Note also the properties below of the vec(·) and Tr(·) operators for \( x \in \mathbb{R}^r \) and \( A, B \in \mathbb{R}^{n \times n} \),

\[ x^T Ax = \text{Tr}(Axx^T) = \text{Tr}(xx^TA), \quad \text{vec}(B)^T \text{vec}(A) = \text{Tr}(AB) = \text{Tr}(BA). \]

Recall that the trace operator \( \text{Tr}(\cdot) \) is defined for square matrices \( U \in \mathbb{R}^{n \times n} \) with components \( [u_{ij}]_{i,j=1}^n \) as

\[ \text{Tr}(U) = \sum_{i=1}^n \sum_{j=1}^n u_{ij}\delta_{ij} = \sum_{i=1}^n u_{ii} \]

The vec(·) operator is defined for the matrix

\[
U = \begin{bmatrix}
    u_{11} & u_{12} & \cdots & u_{1n} \\
    u_{21} & u_{22} & \cdots & u_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{n1} & u_{n2} & \cdots & u_{nn}
\end{bmatrix}
\]

as the \( n^2 \times 1 \) column vector formed by stacking the columns of \( U \),

\[ \text{vec}(U) = \begin{bmatrix}
    u_{11} & u_{21} & \cdots & u_{n1} & u_{1n} & u_{2n} & \cdots & u_{nn}
\end{bmatrix}^T \]

Using the cumulant forms, the chain rule of differentiation, and the properties
above, write

\[
\frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\bar{y}}(\epsilon), \mathbf{\bar{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \frac{\partial \kappa_i(\epsilon)}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} \text{vec}(\mathbf{f}_i(\mathbf{y}(\epsilon), K_1(\epsilon), K_2(\epsilon)))
\]

(7.37)

\[
\frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\bar{y}}(\epsilon), \mathbf{\bar{z}}(\epsilon))}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \frac{\partial \kappa_i(\epsilon)}{\partial \text{vec}(\mathbf{y}_i(\epsilon))} \text{vec}(\mathbf{f}_i(\mathbf{y}(\epsilon), K_1(\epsilon), K_2(\epsilon)))
\]

(7.38)

\[
\frac{\partial \phi(\mathbf{y}(\epsilon), \mathbf{z}(\epsilon), \mathbf{\bar{y}}(\epsilon), \mathbf{\bar{z}}(\epsilon))}{\partial \mathbf{z}_i(\epsilon)} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \frac{\partial \kappa_i(\epsilon)}{\partial \mathbf{z}_i(\epsilon)} = \frac{\partial g(\kappa(\epsilon), \bar{\kappa}(\epsilon))}{\partial \bar{\kappa}_i(\epsilon)} \frac{\partial \bar{\kappa}_i(\epsilon)}{\partial \mathbf{z}_i(\epsilon)}
\]

(7.39)

Inserting the relations (7.37), (7.38), and (7.39) into the expression (7.36) gives the desired form (7.35), and the proof is complete.

7.3.7 Problem Solution (ZS-CDS)

The discussion will now focus upon the solution to the ZS-CDS optimization, using the HJI verification lemma.
Theorem 7.3.15 (Saddle-Point Solution to ZS-CDS Optimization)
Consider the LQG stochastic optimal control problem involving the process having dynamics (7.15) and the cost (7.16). Then the linear state-feedback, finite-horizon, optimal control solutions to the ZS-CDS optimization are characterized by the optimal gains

\[ K_1^*(\alpha) = -R_1^{-1}(\alpha)B_1^T(\alpha) \left( H_1^*(\alpha) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa^*(\alpha), \hat{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \right) H_i^*(\alpha) \right) \]

\[ K_2^*(\alpha) = R_2^{-1}(\alpha)B_2^T(\alpha) \left( H_1^*(\alpha) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa^*(\alpha), \hat{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \right) H_i^*(\alpha) \right) \]

where the optimal cost cumulants and target cost cumulants are defined by

\[ \kappa_i^*(\alpha) = x_0^T H_i^*(\alpha)x_0 + D_i^*(\alpha) \quad \text{and} \quad \hat{\kappa}_i(\alpha) = x_0^T \bar{H}_i(\alpha)x_0 + \bar{D}_i(\alpha), \quad 1 \leq i \leq r. \]

The optimal state variables \( \mathbf{H}^*(\alpha), \mathbf{D}^*(\alpha) \) and \( \mathbf{H}(\alpha), \mathbf{D}(\alpha) \) follow the equations of motion

\[ \frac{d\mathbf{H}^*(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}^*(\alpha), K_1^*(\alpha), K_2^*(\alpha)), \quad \frac{d\mathbf{H}(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}(\alpha), \bar{K}_1(\alpha), \bar{K}_2(\alpha)) \]

\[ \frac{d\mathbf{D}^*(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}^*(\alpha)), \quad \frac{d\mathbf{D}(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}(\alpha)), \quad \alpha \in [t_0, t_f] \]

\[ \mathbf{H}^*(t_f) = \mathbf{H}_f, \quad \mathbf{H}(t_f) = \mathbf{H}_{f,\mathbf{x}}, \quad \mathbf{D}^*(t_f) = \mathbf{D}_f, \quad \mathbf{D}(t_f) = \mathbf{D}_{f,\mathbf{x}}. \]

exist and also the initial values of the state trajectories satisfy

\[ (t_0, \mathbf{H}(t_0), \mathbf{D}(t_0), \bar{\mathbf{H}}(t_0), \bar{\mathbf{D}}(t_0)) \in \mathcal{M}. \]

**Proof** The objective is to identify control gains \( K_1^*, K_2^* \) and a function \( \eta(\epsilon) \) such that

\[ \frac{d\eta(\epsilon)}{d\epsilon} = \max_{\kappa_1 \in \mathcal{K}_1, \kappa_2 \in \mathcal{K}_2} \min_{\kappa_1 \in \mathcal{K}_1, \kappa_2 \in \mathcal{K}_2} \left\{ \sum_{i=1}^{r} \frac{\partial g(\kappa(\epsilon), \hat{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} x_0^T \mathcal{F}_i(\mathbf{Y}(\epsilon), K_1(\epsilon), K_2(\epsilon))x_0 \right. \]

\[ + \sum_{i=1}^{r} \frac{\partial g(\kappa(\epsilon), \hat{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} x_0^T \mathcal{F}_i(\tilde{\mathbf{Y}}(\epsilon), \tilde{\kappa}_1(\epsilon), \tilde{\kappa}_2(\epsilon))x_0 \]

\[ + \sum_{i=1}^{r} \frac{\partial g(\kappa(\epsilon), \hat{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \mathcal{G}_i(\mathbf{Y}(\epsilon)) \]

297
\[\sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\tilde{Y}(\epsilon))\].

Such selections of \(K_1^*\) and \(K_2^*\) along with \(\eta(\epsilon)\) will ensure that the candidate value function \(W(\epsilon, Y^*(\epsilon), Z^*(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))\) satisfies the HJI equation. The proof begins with the optimizations,

\[
\max_{K_2 \in \tilde{K}} \min_{K_1 \in \tilde{K}} \left\{ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T F_i(Y(\epsilon), K_1(\epsilon), K_2(\epsilon)) x_0 \right. \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) x_0^T \mathcal{F}_i(\tilde{Y}(\epsilon), \tilde{K}_1(\epsilon), K_2(\epsilon)) x_0 \right. \\
+ \left. \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(Y(\epsilon)) \right. \\
+ \left. \sum_{i=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)) \mathcal{G}_i(\tilde{Y}(\epsilon)) \right\}.
\]

In the following differentiation, denote the partial derivatives of \(\phi(Y(\epsilon), Z(\epsilon), \tilde{Y}(\epsilon), \tilde{Z}(\epsilon))\) as shown below to simplify the notation,

\[c_i(\epsilon) = \frac{\partial}{\partial \kappa_i(\epsilon)} g(\kappa(\epsilon), \tilde{\kappa}(\epsilon)), \quad 1 \leq i \leq r.\]

Now differentiate the expression in braces with respect to \(K_1\) and \(K_2\) in the final expression of (7.41) and set the resultant equal to a zero matrix with the appropriate dimension. This is the necessary condition for the expression to take an extremal value on the interior of its domain.

\[ -2B_1^T(\epsilon) \sum_{i=1}^{r} c_i(\epsilon) Y_i(\epsilon) (x_0 x_0^T) - 2c_1(\epsilon) R_1(\epsilon) K_1(\epsilon) (x_0 x_0^T) = 0^{m_1 \times n} \]

\[ -2B_2^T(\epsilon) \sum_{i=1}^{r} c_i(\epsilon) Y_i(\epsilon) (x_0 x_0^T) + 2c_1(\epsilon) R_2(\epsilon) K_2(\epsilon) (x_0 x_0^T) = 0^{m_2 \times n} \]

Assume that \(c_1(\epsilon) \neq 0, \quad \forall \epsilon \in [t_0, t_f]\). Since \(x_0 x_0^T\) is a fixed rank-one matrix, it must
be that

\[
K_1^*(\epsilon) = -R_1^{-1}(\epsilon) B_1^T(\epsilon) \left( \mathcal{Y}_1(\epsilon) + \sum_{i=2}^{r} \frac{c_i(\epsilon)}{c_1(\epsilon)} \mathcal{Y}_i(\epsilon) \right)
\]

\[
= -R_1^{-1}(\epsilon) B_1^T(\epsilon) \left( \mathcal{Y}_1(\epsilon) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \right) \mathcal{Y}_i(\epsilon) \right)
\]

\[
K_2^*(\epsilon) = R_2^{-1}(\epsilon) B_2^T(\epsilon) \left( \mathcal{Y}_1(\epsilon) + \sum_{i=2}^{r} \frac{c_i(\epsilon)}{c_1(\epsilon)} \mathcal{Y}_i(\epsilon) \right)
\]

\[
= R_2^{-1}(\epsilon) B_2^T(\epsilon) \left( \mathcal{Y}_1(\epsilon) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i(\epsilon)} \right) \mathcal{Y}_i(\epsilon) \right)
\]

Let \( \mathcal{Y}^*(\tau) \) and \( \mathcal{Z}^*(\tau) \) denote the solutions of the equations of motion under this control selection,

\[
\frac{dY^*_i(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau) K_1^*(\tau) + B_2(\tau) K_2^*(\tau) \right) \mathcal{Y}_i^*(\tau)
\]

\[
- \mathcal{Y}_i^*(\tau) \left( A(\tau) + B_1(\tau) K_1^*(\tau) + B_2(\tau) K_2^*(\tau) \right)
\]

\[
- K_1^T(\tau) R_1(\tau) K_1^*(\tau) + K_2^T(\tau) R_2(\tau) K_2^*(\tau) - Q(\tau)
\]

\[
\frac{dY^*_i(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau) K_1^*(\tau) + B_2(\tau) K_2^*(\tau) \right) \mathcal{Y}_i^*(\tau)
\]

\[
- \mathcal{Y}_i^*(\tau) \left( A(\tau) + B_1(\tau) K_1^*(\tau) + B_2(\tau) K_2^*(\tau) \right)
\]

\[
- K_1^T(\tau) R_1(\tau) K_1^*(\tau) + K_2^T(\tau) R_2(\tau) K_2^*(\tau) - Q(\tau)
\]

\[
\frac{dZ^*_i(\tau)}{d\tau} = - \text{Tr} \left( \mathcal{Y}_i^*(\tau) G(\tau) W G^T(\tau) \right)
\]

\[
\mathcal{Y}_i^*(t_f) = H_i(t_f), \quad Z_i^*(t_f) = D_i(t_f), \quad \tau \in [\epsilon, t_f], \quad 1 \leq i \leq r.
\]

It is assumed that the solutions to the above equations exist. Return again to the
minimized expression (7.41) inserting the determined controller,

\[
\frac{d\eta(e)}{d\epsilon} = \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i^*(e)} g(\kappa^*(e), \bar{\kappa}(e)) x_0^T F_i(Y^*(e), K_1^*(e), K_2^*(e)) x_0 \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i^*(e)} g(\kappa^*(e), \bar{\kappa}(e)) x_0^T F_i(\dot{Y}(e), \dot{K}_1(e), \dot{K}_2(e)) x_0 \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i^*(e)} g(\kappa^*(e), \bar{\kappa}(e)) \bar{G}_i(Y^*(e)) \\
+ \sum_{i=1}^{r} \frac{\partial}{\partial \kappa_i^*(e)} g(\kappa^*(e), \bar{\kappa}(e)) \bar{G}_i(\dot{Y}(e)) \\
= \sum_{i=1}^{r} \frac{\partial \phi(Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e))}{\partial \text{vec}(Y^*_i(e))} \text{vec}(F_i(Y^*(e), K_1^*(e), K_2^*(e))) \\
+ \sum_{i=1}^{r} \frac{\partial \phi(Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e))}{\partial \text{vec}(Y^*_{i}(e))} \text{vec}(F_i(\dot{Y}(e), \dot{K}_1(e), \dot{K}_2(e))) \\
+ \sum_{i=1}^{r} \frac{\partial \phi(Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e))}{\partial \dot{Y}^*_i(e)} \bar{G}_i(Y^*(e)) \\
+ \sum_{i=1}^{r} \frac{\partial \phi(Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e))}{\partial \dot{Z}^*_i(e)} \bar{G}_i(\dot{Y}(e)) \\
= \frac{d\phi(Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e))}{d\epsilon}
\]

Note that \((e, Y(e), Z(e), \dot{Y}(e), \dot{Z}(e)) \in Q\), for some \((K_1, K_2) \in K^3(e) \times K^2(e)\). Recall by assumption, \(K_1^*\) and \(K_2^*\) are admissible on \([t_0, \epsilon]\). Thus \((e, Y^*(e), Z^*(e), \dot{Y}(e), \dot{Z}(e)) \in Q\) since \(K_1^* \in K^1(e)\) and \(K_2^* \in K^2(e)\). Consider now that any displaced terminal conditions at \(\tau < \epsilon\) along the respective state trajectories resultant from \(K_i^*\) will also be in \(Q\) due to the fact that the restrictions of the control \(K_i^*\) to \([t_0, \tau]\) will be in \(K^i(\tau)\), where of course \(1 \leq i \leq 2\). Clearly then, it can be similarly argued that \((\tau, Y^*(\tau), Z^*(\tau), \dot{Y}(\tau), \dot{Z}(\tau)) \in Q\), \(\forall \tau \in [t_0, \epsilon]\). It is desired to determine \(\eta(e)\) so to enforce the above relationship for all possible displaced terminal times \(\tau < \epsilon\) and the associated terminal conditions, gotten from anywhere along the optimal
trajectory of the state equations.

\[
\frac{d\phi(Y^*(\tau), Z^*(\tau), \dot{Y}(\tau), \dot{Z}(\tau))}{d\tau} = \frac{d\eta(\tau)}{d\tau}, \quad \tau \in [t_0, \epsilon]
\]

This differential equation can be integrated over a reduced time-horizon since the state equations \(Y_i^*(\tau), Z_i^*(\tau), \dot{Y}_i(\tau), \dot{Z}_i(\tau)\) for \(1 \leq i \leq r\) are continuously differentiable. By the Fundamental Theorem,

\[
\phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)) - \phi(Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) = -\int_{t_0}^{\epsilon} \frac{d}{d\tau} \left( \phi(Y^*(\tau), Z^*(\tau), \dot{Y}(\tau), \dot{Z}(\tau)) \right) \eta(\epsilon) - \eta(t_0) = -\int_{t_0}^{\epsilon} \frac{d}{d\tau}(\eta(\tau))
\]

from which it is immediate that

\[
\eta(\epsilon) = \eta(t_0) - \phi(Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)).
\]

Since \((t_0, Y^*(t_0), Z^*(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) \in M\), the target set, the above equation can be re-written as

\[
\eta(\epsilon) = \eta(t_0) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) + \phi(Y^*(\epsilon), Z^*(\epsilon), \dot{Y}(\epsilon), \dot{Z}(\epsilon)).
\]

The function \(\eta(\epsilon)\) is determined up to its initial condition, which remains to be constrained for this problem. To achieve this end, the Mayer-form boundary condition (7.32) of the HJI verification lemma is used, which is

\[
\mathcal{W}(t_0, Y(t_0), Z(t_0), \dot{Y}(t_0), \dot{Z}(t_0)) = \eta(t_0) - \phi(Y(t_0), Z(t_0), \dot{Y}(t_0), \dot{Z}(t_0))
\]

301
= \eta(t_0) - \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0))
= \phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)).

This condition requires \( \eta(t_0) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) \) and thus the function is determined completely as

\[ \eta(\epsilon) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) + \phi(\mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)). \]

Our candidate value function becomes

\[ W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) = 2\phi(H(t_0), D(t_0), \dot{H}(t_0), \dot{D}(t_0)) - \phi(H^*(t_0), D^*(t_0), \dot{H}(t_0), \dot{D}(t_0)) + \phi(\mathcal{Y}^*(\epsilon), \mathcal{Z}^*(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) - \phi(\mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) \]

The selection of \( \eta(\epsilon) \) as above makes \( W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) \) satisfy the requirements of the HJI verification lemma by construction. This confirms that

\[ W(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) = V(\epsilon, \mathcal{Y}(\epsilon), \mathcal{Z}(\epsilon), \dot{\mathcal{Y}}(\epsilon), \dot{\mathcal{Z}}(\epsilon)) \]

and the determined ZS-CDS controls are optimal,

\[ K_1^*(\epsilon) = -R_i^{-1}(\epsilon)B_i^T(\epsilon) \left( \mathcal{Y}_i^*(\epsilon) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa^*(\epsilon), \tilde{\kappa}(\epsilon))}{\partial \kappa_i^*(\epsilon)} \right) \mathcal{Y}_i^*(\epsilon) \right) \]
\[ K_2^*(e) = R_2^{-1}(e)B_2^T(e) \left( \mathcal{Y}_1^*(e) + \sum_{i=2}^{r} \left( \frac{\partial g(\kappa^*(e), \tilde{\kappa}(e))}{\partial \kappa_1^*(e)} \right) \mathcal{Y}_i^*(e) \right). \]

7.4 Non-Zero-Sum Cost Density-Shaping Games

The ZS-CDS optimization is of the traditional min-max form, where two players are concerned with competing objectives concerning the same cost functional. A different sort of game is one where players form strategies from optimizing the statistical characterization of distinct cost functionals. The following 2-player, Nonzero-Sum, Cost Density-Shaping (NZS-CDS) game will be discussed in this section.

7.4.1 Process and Cost (NZS-CDS)

Let \((t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n\) be fixed, and let \(w(t) = W(t, \omega)\) be a \(p\)-dimensional stationary Wiener process on \([t_0, t_f]\) where \(w : [t_0, t_f] \times \Omega \to \mathbb{R}^p\) on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the following correlation of increments property is satisfied,

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, \quad W > \mathbf{0}^{p \times p}.
\]

Let \(\mathcal{U}_1 \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1}))\) and \(\mathcal{U}_2 \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2}))\) be Hilbert spaces of \(\mathbb{R}^{m_1}\)-valued and \(\mathbb{R}^{m_2}\)-valued, square-integrable processes \(u_1 \in \mathcal{U}_1\) and \(u_2 \in \mathcal{U}_2\) that are adapted to the \(\sigma\)-field generated by \(w(t)\), where by their construction

\[
E\left\{ \int_{t_0}^{t_f} u_1^T(\tau)u_1(\tau) d\tau \right\} < \infty \quad \text{and} \quad E\left\{ \int_{t_0}^{t_f} u_2^T(\tau)u_2(\tau) d\tau \right\} < \infty.
\]
Consider the problem of Player 1 choosing strategies \( u_1 \in U_1 \) and Player 2 choosing strategies \( u_2 \in U_2 \) so to influence the states \( x(t) = X(t, \omega) \) of the following linear stochastic differential equation, which belong to \( L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \) and are adapted to the \( \sigma \)-field generated by \( w(t) \),

\[
\begin{align*}
\frac{dx(t)}{dt} &= \left(A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t)\right)dt + G(t)dw(t), \quad t \in [t_0, t_f] \\
x_0 &= E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n
\end{align*}
\] (7.42)

where

\[
A \in C([t_0, t_f]; \mathbb{R}^{n \times n}), \quad B_1 \in C([t_0, t_f]; \mathbb{R}^{n \times m_1}), \quad B_2 \in C([t_0, t_f]; \mathbb{R}^{n \times m_2}), \quad G \in C([t_0, t_f]; \mathbb{R}^{n \times p}).
\]

In particular, Player 1 chooses \( u_1 \) to optimize the statistical characterization of the integral-quadratic cost functional given below,

\[
J_1[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left(x^T(\tau)Q_1(\tau)x(\tau) + u_1^T(\tau)R_{11}(\tau)u_1(\tau) + u_2^T(\tau)R_{12}(\tau)u_2(\tau)\right)d\tau + x^T(t_f)Q_{1f}x(t_f).
\] (7.43)

Analogously, Player 2 chooses \( u_2 \) so to optimize the statistical characterization of the integral-quadratic cost functional given below,

\[
J_2[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left(x^T(\tau)Q_2(\tau)x(\tau) + u_1^T(\tau)R_{21}(\tau)u_1(\tau) + u_2^T(\tau)R_{22}(\tau)u_2(\tau)\right)d\tau + x^T(t_f)Q_{2f}x(t_f).
\] (7.44)

It is understood that \( Q_1, Q_2 \in C([t_0, t_f]; S^n_+), R_{11}, R_{21} \in C([t_0, t_f]; S^{m_1}_+), \) and \( R_{12}, R_{22} \in C([t_0, t_f]; S^{m_2}_+) \) for well-posedness of the problem. Suppose further that players choose their optimal control actions within the class of memoryless, full-observation
strategies, or more precisely

\[ \xi_1 : [t_0, t_f] \times L^2_{\mathbb{P}}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \rightarrow L^2_{\mathbb{P}}(\Omega; C([t_0, t_f]; \mathbb{R}^{m_1})) \]

\[ \xi_2 : [t_0, t_f] \times L^2_{\mathbb{P}}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \rightarrow L^2_{\mathbb{P}}(\Omega; C([t_0, t_f]; \mathbb{R}^{m_2})) \]

and

\[ u_1(t) = \xi_1(t, x(t)) = K_1(t)x(t), \quad u_2(t) = \xi_2(t, x(t)) = K_2(t)x(t). \quad (7.45) \]

When the process having dynamics (7.42) is subjected to the controls of each player, where \( K_1 \in C([t_0, t_f]; \mathbb{R}^{m_1 \times n}) \) and \( K_2 \in C([t_0, t_f]; \mathbb{R}^{m_2 \times n}) \) are admissible control gains with respective compact, allowable sets of gains \( \bar{K}_1 \subset \mathbb{R}^{m_1 \times n} \) and \( \bar{K}_2 \subset \mathbb{R}^{m_2 \times n} \), it becomes

\[ dx(t) = \left( A(t) + B_1(t)K_1(t) + B_2(t)K_2(t) \right)x(t)dt + G(t)dw(t), \quad t \in [t_0, t_f] \]

\[ x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n \]

and the costs (7.43) and (7.44) can be written as

\[ J_1[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N_1(\tau)x(\tau)d\tau + x^T(t_f)Q_1f(t_f) \quad (7.47) \]

\[ J_2[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N_2(\tau)x(\tau)d\tau + x^T(t_f)Q_2f(t_f) \]

where

\[ N_1(\tau) = K_1^T(\tau)R_{11}(\tau)K_1(\tau) + K_2^T(\tau)R_{12}(\tau)K_2(\tau) + Q_1(\tau) \]

\[ N_2(\tau) = K_1^T(\tau)R_{21}(\tau)K_1(\tau) + K_2^T(\tau)R_{22}(\tau)K_2(\tau) + Q_2(\tau). \]
7.4.2 Cost Cumulants, NZS-CDS Optimization

The form of the cost cumulants for both cost functionals (7.47) can readily be found from the same approach given earlier in this chapter. The following theorem captures the result.

**Theorem 7.4.1 (Cost Cumulants, NZS-CDS Dynamic Game)**

For the costs (7.47) and linear state-feedback control inputs $u_1$ and $u_2$ to the system (7.42), the $r$ initial cost cumulants of $J_1$ have the following quadratic form

$$\kappa_i^1(\alpha) = x_0^T H_i^1(\alpha) x_0 + D_i^1(\alpha), \quad 1 \leq i \leq r$$

where the functions $H_i^1(\alpha)$ and $D_i^1(\alpha)$ follow the equations of motion,

$$\frac{dH_i^1(\alpha)}{d\alpha} = -\left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_i^1(\alpha)$$

$$- H_i^1(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)$$

$$- K_i^T(\alpha)R_{11}(\alpha)K_1(\alpha) - K_i^T(\alpha)R_{12}(\alpha)K_2(\alpha) - Q_1(\alpha) \triangleq F_i(H_i^1(\alpha), K_1(\alpha), K_2(\alpha))$$

$$\frac{dH_j^1(\alpha)}{d\alpha} = -\left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_j^1(\alpha)$$

$$- H_j^1(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)$$

$$- 2 \sum_{j=1}^{i-1} \binom{i}{j} H_j^1(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}^1(\alpha) \triangleq F_i(H_i^1(\alpha), K_1(\alpha), K_2(\alpha)), \quad 2 \leq i \leq r$$

$$\frac{dD_i^1(\alpha)}{d\alpha} = -\text{Tr} \left( H_j^1(\alpha)G(\alpha)WG^T(\alpha) \right) \triangleq G_j(H_i^1(\alpha)), \quad \alpha \in [t_0, t_f] \quad 1 \leq j \leq r.$$ 

These functions satisfy the terminal conditions

$$H_i^1(t_f) = Q_i f, \quad H_i^1(t_f) = 0^{n \times n}, \quad i \geq 2$$

$$D_i^1(t_f) = 0, \quad D_i^1(t_f) = 1, \quad D_i^1(t_f) = 0, \quad j \geq 3.$$ 

(7.48)

In a completely analogous way, the $r$ initial cost cumulants of $J_2$ have the following quadratic form

$$\kappa_i^2(\alpha) = x_0^T H_i^2(\alpha) x_0 + D_i^2(\alpha), \quad 1 \leq i \leq r$$
where the functions $H_i^2(\alpha)$ and $D_i^2(\alpha)$ follow the equations of motion,

$$\frac{dH_i^2(\alpha)}{d\alpha} = - \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_i^2(\alpha) - H_i^2(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right) - K_i^T(\alpha)R_{21}(\alpha)K_1(\alpha) - K_i^T(\alpha)R_{22}(\alpha)K_2(\alpha) - Q_2(\alpha) = \mathcal{F}_i(H^2(\alpha), K(\alpha), K_2(\alpha))$$

and

$$\frac{dH_i^2(\alpha)}{d\alpha} = - \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right)^T H_i^2(\alpha) - H_i^2(\alpha) \left( A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha) \right) - 2 \sum_{j=1}^{i-1} H_j^2(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}^2(\alpha) = \mathcal{F}_i(H^2(\alpha), K_1(\alpha), K_2(\alpha)), \ 2 \leq i \leq r$$

$$\frac{dD_i^2(\alpha)}{d\alpha} = - \text{Tr} \left( H_j^2(\alpha)G(\alpha)WG^T(\alpha) \right) = \mathcal{G}_j(H^2(\alpha)), \ \alpha \in [t_0, t_f] \ 1 \leq j \leq r.$$

and satisfy similar terminal conditions

$$H_i^2(t_f) = Q_{2f}, \ H_i^2(t_f) = 0^{n \times n}, \ i \geq 2$$

$$D_i^2(t_f) = 0, \ D_i^2(t_f) = 1, \ D_i^2(t_f) = 0, \ j \geq 3.$$  

**Proof** Analogous to the approach used for the ZS-CDS case. Make the following definitions,

$$N_1(\alpha) = Q_1(\alpha) + K_1^T(\alpha)R_{11}(\alpha)K_1(\alpha) + K_2^T(\alpha)R_{12}(\alpha)K_2(\alpha)$$

$$F(\alpha) = A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha), \ Q_f = Q_{1f}$$

and the equations (7.48) follow from (7.20), (7.27), and (7.28). In a similar fashion, define

$$N_2(\alpha) = Q_2(\alpha) + K_2^T(\alpha)R_{21}(\alpha)K_1(\alpha) + K_2^T(\alpha)R_{22}(\alpha)K_2(\alpha)$$

$$F(\alpha) = A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha), \ Q_f = Q_{2f}$$

and likewise the equations (7.50) follow from (7.20), (7.27), and (7.28).
7.4.3 Notation (NZS-CDS)

The presented notation is reintroduced for the purposes of clarity. As before, begin by defining the state variables $H_i(\alpha) \in \mathbb{R}^{r \times n}$ and $D_i(\alpha) \in \mathbb{R}^r$ as below

$$
H_i(\alpha) \triangleq \begin{bmatrix}
H_i^1(\alpha) \\
\vdots \\
H_i^i(\alpha)
\end{bmatrix},
D_i(\alpha) \triangleq \begin{bmatrix}
D_i^1(\alpha) \\
\vdots \\
D_i^i(\alpha)
\end{bmatrix}, \quad 1 \leq i \leq 2.
$$

Using these state variables, define the functions

$$
\mathcal{F}(H^i(\alpha), K_1(\alpha), K_2(\alpha)) \triangleq \begin{bmatrix}
\mathcal{F}_1(H^i(\alpha), K_1(\alpha), K_2(\alpha)) \\
\vdots \\
\mathcal{F}_r(H^i(\alpha), K_1(\alpha), K_2(\alpha))
\end{bmatrix},
\mathcal{G}(H^i(\alpha)) \triangleq \begin{bmatrix}
\mathcal{G}_1(H^i(\alpha)) \\
\vdots \\
\mathcal{G}_r(H^i(\alpha))
\end{bmatrix}, \quad 1 \leq i \leq 2.
$$

Let $\{\mathcal{F}_j(\cdot)\}_{j=1}^r$ and $\{\mathcal{G}_j(\cdot)\}_{j=1}^r$ in the above definitions be defined as beforehand in (7.48) and (7.50). Also a condensed form for the terminal conditions is introduced as below.

$$
H_f^1 \triangleq \begin{bmatrix}
Q_{1f} \\
0^{n \times n} \\
\vdots \\
0^{n \times n}
\end{bmatrix}, \quad H_f^2 \triangleq \begin{bmatrix}
Q_{2f} \\
0^{n \times n} \\
\vdots \\
0^{n \times n}
\end{bmatrix}, \quad D_f \triangleq \begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix}.
$$
Finally, denote the cost cumulant vectors $\kappa^1(\alpha), \kappa^2(\alpha) \in \mathbb{R}^r$ as

\[
\kappa^1(\alpha) \triangleq \begin{bmatrix}
\kappa^1_1(\alpha) \\
\vdots \\
\kappa^1_r(\alpha)
\end{bmatrix}, \quad \kappa^2(\alpha) \triangleq \begin{bmatrix}
\kappa^2_1(\alpha) \\
\vdots \\
\kappa^2_r(\alpha)
\end{bmatrix}.
\]

Using this notation, the equations (7.48) and (7.50) and their associated terminal condition systems can be written concisely as

\[
\frac{dH^1(\alpha)}{d\alpha} = F(H^1(\alpha), K^1(\alpha), K^2(\alpha)), \quad \frac{dD^1(\alpha)}{d\alpha} = G(H^1(\alpha))
\]

\[
H^1(t_f) = H^1_f, \quad D^1(t_f) = D_f
\]

\[
\frac{dH^2(\alpha)}{d\alpha} = F(H^2(\alpha), K^1(\alpha), K^2(\alpha)), \quad \frac{dD^2(\alpha)}{d\alpha} = G(H^2(\alpha))
\]

\[
H^2(t_f) = H^2_f, \quad D^2(t_f) = D_f, \quad \alpha \in [t_0, t_f].
\]

**Remark(s)**

- The equations (7.48) and (7.50) uniquely characterize the solutions $H^1(\alpha)$, $H^2(\alpha)$ and $D^1(\alpha)$, $D^2(\alpha)$ when control gains $(K_1, K_2)$ are specified. Hence the ensuing Nash game is subject to the typical interpretation in terms of $(K_1, K_2)$ rather than $(u_1, u_2)$.

- The cost cumulant-generating equations for $\kappa^1(\alpha), \kappa^2(\alpha)$ admit unique solutions when the equations are solvable on the interval $[t_0, t_f]$. This suggests that the game having dynamics (7.48) and (7.50) and gains $(K_1, K_2)$ corresponding to strategies $(\xi_1, \xi_2)$ is globally playable on $[t_0, t_f]$ and admits a unique Nash equilibrium solution.

7.4.4 Target Cost Statistics (NZS-CDS)

Given matrices for a system characterization $(A, B_1, B_2, G)$, an integral-quadratic cost characterization $(Q, R_1, R_2, Q_f)$, and the second-order statistics of the noise $(W)$, consider the cost cumulants of $J_1$ as a result of alternative (and un-
known) linear state-feedback controls \( \tilde{u}_1(t) = \tilde{K}_1(t)\tilde{x}(t) \) and \( \tilde{u}_2(t) = \tilde{K}_2(t)\tilde{x}(t) \), where \( \tilde{K}_1 \in C([t_0,t_f];\mathbb{R}^{m_1 \times n}) \) and \( \tilde{K}_2 \in C([t_0,t_f];\mathbb{R}^{m_2 \times n}) \). The initial cost cumulants of \( J_1 \) when \( \alpha = t_0 \) are given by

\[
\tilde{\kappa}_1^i(\alpha) = x_0^T \tilde{H}_1^i(\alpha)x_0 + \tilde{D}_1^i(\alpha), \quad 1 \leq i \leq r
\] (7.53)

Analogously, for \( J_2 \) we have the initial cost cumulants

\[
\tilde{\kappa}_2^i(\alpha) = x_0^T \tilde{H}_2^i(\alpha)x_0 + \tilde{D}_2^i(\alpha), \quad 1 \leq i \leq r
\] (7.54)

when \( \alpha = t_0 \). Let this set of \( 2r \) numbers be regarded as the target cost cumulants.

Here the functions \( \tilde{H}_1^i(\alpha), \tilde{H}_2^i(\alpha) \) and \( \tilde{D}_1^i(\alpha), \tilde{D}_2^i(\alpha) \) are determined by the same system of backwards-in-time, matrix differential equations as (7.48) and (7.50).

More precisely, this is

\[
\begin{align*}
\frac{d\tilde{H}_1^i}{d\alpha} &= F(\tilde{H}_1^i, \tilde{K}_1, \tilde{K}_2), \quad \frac{d\tilde{D}_1^i}{d\alpha} = G(\tilde{H}_1^i) \\
\tilde{H}_1^i(t_f) &= \tilde{H}_1^i, \quad \tilde{D}_1^i(t_f) = \tilde{D}_1^i
\end{align*}
\] (7.55)

\[
\begin{align*}
\frac{d\tilde{H}_2^i}{d\alpha} &= F(\tilde{H}_2^i, \tilde{K}_1, \tilde{K}_2), \quad \frac{d\tilde{D}_2^i}{d\alpha} = G(\tilde{H}_2^i) \\
\tilde{H}_2^i(t_f) &= \tilde{H}_2^i, \quad \tilde{D}_2^i(t_f) = \tilde{D}_2^i \quad \alpha \in [t_0,t_f].
\end{align*}
\]

where

\[
\begin{align*}
\tilde{H}_1^i(t_f) &= Q_{1f} + E_{1}, \quad \tilde{H}_1^i(t_f) = 0^{n \times n}, i \geq 2 \\
\tilde{D}_1^i(t_f) &= e_1, \quad \tilde{D}_1^i(t_f) = 1, \tilde{D}_1^i(t_f) = 0, \quad j \geq 3
\end{align*}
\] (7.56)

\[
\begin{align*}
\tilde{H}_2^i(t_f) &= Q_{2f} + E_{2}, \quad \tilde{H}_2^i(t_f) = 0^{n \times n}, i \geq 2 \\
\tilde{D}_2^i(t_f) &= e_2, \quad \tilde{D}_2^i(t_f) = 1, \tilde{D}_2^i(t_f) = 0, \quad j \geq 3
\end{align*}
\]

310
and the short-hand notation is used,

\[
H_{f,\mathcal{E}_i}^\dagger \triangleq \begin{bmatrix}
Q_{i_f} + \mathcal{E}_i^\dagger \\
0_{n \times n} \\
\vdots \\
0_{n \times n}
\end{bmatrix}, \quad D_{f,\mathcal{E}_i}^\dagger \triangleq \begin{bmatrix}
\epsilon_i^* \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad 1 \leq i \leq 2.
\]

Here \(\epsilon_1^*, \epsilon_2^* > 0\) are small perturbation constants, and \(\mathcal{E}_1^*, \mathcal{E}_2^* > 0_{n \times n}\) are positive-definite perturbation matrices. As with the cost cumulants, compose vectors of target initial cost cumulants \(\tilde{\kappa}_1^i(\alpha), \tilde{\kappa}_2^i(\alpha) \in \mathbb{R}^r\) defined as below,

\[
\tilde{\kappa}_i^i(\alpha) \triangleq \begin{bmatrix}
\tilde{\kappa}_i^1(\alpha) \\
\vdots \\
\tilde{\kappa}_i^r(\alpha)
\end{bmatrix}, \quad 1 \leq i \leq 2.
\]

### 7.4.5 Problem Formulation (NZS-CDS)

The problem statement provided for the MCCDS theory of Chapter 3 is now adapted for the NZS-CDS optimization. Before proceeding, two important definitions must be introduced - the target set and admissible feedback gains. Namely, the target set is a space that the end value of the state trajectories must lie in. Put somewhat loosely, the set of admissible control gains are those gains that can steer the state variables into the target set. The following definitions formalize these ideas.

**Definition 7.4.2 (Target Sets - NZS-CDS)**

Let \((t_0, H^i(t_0), D^i(t_0), \bar{H}^i(t_0), \bar{D}^i(t_0)) \in \mathcal{M}_i\), where \(\mathcal{M}_i\) denotes the target set for Player
i which is a closed subset of 

$$[t_0, t_f] \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times R^r \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}) \times R^r.$$

**Remark(s)**

- Note that the goal of tracking target cumulants suggests that predetermined trajectories of $\tilde{H}_i^i(\alpha)$, $\tilde{D}_i^i(\alpha)$, $1 \leq i \leq 2$ will inherently have initial values in $M_i$ by their definition.

- The combined target space $M_1 \cup M_2$ is closed, since the union of a finite number of closed sets is also closed.

For given terminal conditions, the sets of admissible feedback gains are denoted as

$$K_{t_f, \tilde{H}_i^i(t_f), \tilde{D}_i^i(t_f), \tilde{H}_i^i(t_0), \tilde{D}_i^i(t_0)}, \ 1 \leq i \leq 2$$

and contain matrices $K_i \in C([t_0, t_f]; \mathbb{R}^{m_i \times n})$ such that

$$(t_0, H_i^i(t_0), D_i^i(t_0), \tilde{H}_i^i(t_0), \tilde{D}_i^i(t_0)) \in M_i, \ 1 \leq i \leq 2$$

is obtained at the end of the trajectories for the state equations (7.48), (7.50), and (7.55). This is formally stated in the following definition.

**Definition 7.4.3 (Admissible Feedback Gains - NZS-CDS)**

Denote the allowable set of control gain values by $K_1 \subset \mathbb{R}^{m_1 \times n}$ and $K_2 \subset \mathbb{R}^{m_2 \times n}$, and let these sets be compact. For fixed $r \in \mathbb{N}$ let $K_{t_f}^{1, H_i^i(t_f), \tilde{H}_i^i(t_f), D_i^i(t_f), \tilde{D}_i^i(t_f)} \triangleq K_{t_f}(t_f)$ characterize a class of $C([t_0, t_f]; \mathbb{R}^{m_1 \times n})$ and let $K_{t_f}^{2, H_i^i(t_f), \tilde{H}_i^i(t_f), D_i^i(t_f), \tilde{D}_i^i(t_f)} \triangleq K_{t_f}(t_f)$ characterize a class of $C([t_0, t_f]; \mathbb{R}^{m_2 \times n})$ such that for any $K_1 \in K_{t_f}(t_f)$ and $K_2 \in K_{t_f}(t_f)$ the solutions to

$$\frac{dH_i^1(\alpha)}{d\alpha} = F(H_i^1(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{dD_i^1(\alpha)}{d\alpha} = G(H_i^1(\alpha))$$

$$H_i^1(t_f) = H_i^1, \quad D_i^1(t_f) = D_i$$

$$\frac{dH_i^2(\alpha)}{d\alpha} = F(H_i^2(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{dD_i^2(\alpha)}{d\alpha} = G(H_i^2(\alpha))$$

312
\[ H^2(t_f) = H_f^2, \quad D^2(t_f) = D_f \]

and

\[
\frac{d\tilde{H}^1(\alpha)}{d\alpha} = F(\tilde{H}^1(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha)), \quad \frac{d\tilde{D}^1(\alpha)}{d\alpha} = G(\tilde{H}^1(\alpha))
\]

\[
\tilde{H}^1(t_f) = \tilde{H}^1_{f;e};^1, \quad \tilde{D}^1(t_f) = \tilde{D}^1_{f;e};^1
\]

\[
\frac{d\tilde{H}^2(\alpha)}{d\alpha} = F(\tilde{H}^2(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha)), \quad \frac{d\tilde{D}^2(\alpha)}{d\alpha} = G(\tilde{H}^2(\alpha))
\]

\[
\tilde{H}^2(t_f) = \tilde{H}^2_{f;e};^2, \quad \tilde{D}^2(t_f) = \tilde{D}^2_{f;e};^2
\]

exist on \( \alpha \in [t_0, t_f] \) and the initial values of the state trajectories satisfy

\[
(t_0, H^1(t_0), D^1(t_0), H^1(t_0), D^1(t_0)) \in M_i, \quad 1 \leq i \leq 2.
\]

Consider scalar functions \( g^i : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R} \) with vector arguments as a general performance index, which is denoted by \( g^i(x, \bar{x}) \). For fixed \( \bar{x} \), the function becomes \( g^i_x : \mathbb{R}^r \to \mathbb{R} \). Analogously for fixed \( x \), the function becomes \( g^i_x : \mathbb{R}^r \to \mathbb{R} \). Impose the following restrictions on \( g^i_x(x) \) and \( g^i_x(\bar{x}) \) to ensure that the ensuing optimization problem is well-posed:

- The function \( g^i_x \) is analytic on \( \text{dom} \ g^i_x \) and \( g^i_x \) is analytic on \( \text{dom} \ g^i_x \)
- The function \( g^i_x \) is convex in \( x \) and its domain \( \text{dom} \ g^i_x \) is a convex set
- The function \( g^i_x \) is non-negative in \( x \) on some neighborhood of \( \bar{x} \)

**Definition 7.4.4 (NZS-CDS Performance Indices)**
Let the NZS-CDS performance indices be defined as

\[
\phi^1(H^1(t_0), D^1(t_0), H^1(t_0), D^1(t_0)) = g^1(\kappa^1(t_0), \bar{\kappa}(t_0))
\]

\[
\phi^2(H^2(t_0), D^2(t_0), H^2(t_0), D^2(t_0)) = g^2(\kappa^2(t_0), \bar{\kappa}(t_0)).
\]

**Remark(s)** It is not difficult to see that the above NZS-CDS performance indices satisfy a notion of convexity for every pair of controlled states \((H^i(t_0), D^i(t_0))\). In particular, consider that for \( 0 \leq \zeta \leq 1 \), the relations \( H^i(t_0) = \zeta H^i_1(t_0) + (1 - \zeta)H^i_2(t_0) \)

313
and $D^i(t_0) = \zeta \mathbf{H}_i^1(t_0) + (1 - \zeta) \mathbf{D}_i^2(t_0)$ together mean that $\kappa^i(t_0) = \zeta \kappa_1^i(t_0) + (1 - \zeta) \kappa_2^i(t_0)$ since the convexity of the states guarantees that
\[
\kappa_1^i(t_0) = x_0^T (1 - \zeta) H_{j,1}^i(t_0) + (1 - \zeta) H_{j,2}^i(t_0)) x_0 + \zeta D_{j,1}^i(t_0) + (1 - \zeta) D_{j,2}^i(t_0) = \zeta x_0^T H_{j,1}^i(t_0) x_0 + (1 - \zeta) (x_0^T H_{j,2}^i(t_0)) x_0 + \zeta D_{j,1}^i(t_0) + (1 - \zeta) D_{j,2}^i(t_0) = \zeta \kappa_1^i(t_0) + (1 - \zeta) \kappa_2^i(t_0), \quad 1 \leq i \leq 2, \quad 1 \leq j \leq r.
\]
and by the form of $g^i(\kappa^i(t_0), \kappa^i(t_0))$ it is true that
\[
g^i(\kappa_1^i(t_0) + (1 - \zeta) \kappa_2^i(t_0), \kappa^i(t_0)) \leq \zeta g^i(\kappa_1^i(t_0), \kappa^i(t_0)) + (1 - \zeta) g^i(\kappa_2^i(t_0), \kappa^i(t_0)).
\]

The concept of a Nash equilibrium is now defined as the solution sought for the NZS-CDS optimization.

**Definition 7.4.5 (Nash Equilibrium for the NZS-CDS Optimization)**

Consider a pair of controls $(K_1^*, K_2^*)$ such that $K_1^* \in \mathcal{K}^1(t_f)$ and $K_2^* \in \mathcal{K}^2(t_f)$. The control gains $(K_1^*, K_2^*)$ constitute a Nash equilibrium solution to the NZS-CDS optimization, if the following conditions are satisfied.

Suppose $\forall K_1 \neq K_1^* \mid K_1 \in \mathcal{K}^1(t_f)$, it is true that
\[
\phi^1(\mathbf{H}^1(t_0; K_1^*, K_2^*), \mathbf{D}^1(t_0; K_1, K_2^*), \mathbf{H}^1(t_0), \mathbf{D}^1(t_0)) 
\leq \phi^1(\mathbf{H}^1(t_0; K_1, K_2^*), \mathbf{D}^1(t_0; K_1, K_2^*), \mathbf{H}^1(t_0), \mathbf{D}^1(t_0)).
\]

Suppose $\forall K_2 \neq K_2^* \mid K_2 \in \mathcal{K}^2(t_f)$, it is true that
\[
\phi^2(\mathbf{H}^2(t_0, K_1^*, K_2^*), \mathbf{D}^2(t_0, K_1^*, K_2^*), \mathbf{H}^2(t_0), \mathbf{D}^2(t_0)) 
\leq \phi^2(\mathbf{H}^2(t_0, K_1^*, K_2), \mathbf{D}^2(t_0, K_1^*, K_2), \mathbf{H}^2(t_0), \mathbf{D}^2(t_0)).
\]

**Remark(s)** The above conditions can be understood conceptually as the pair $(K_1^*, K_2^*)$ is such that when Player 1 chooses strategy $K_1^*$, then Player 2 cannot reduce the cost $\phi^2(\cdot)$ below its value under the strategy $K_2^*$. By similar means, when Player 2 chooses strategy $K_2^*$, Player 1 cannot reduce the cost $\phi^1(\cdot)$ below its value under the strategy $K_1^*$. So when $(K_1^*, K_2^*)$ is the pair of strategies played, neither player has incentive (e.g. a more minimal cost) to play some strategy $K_i \neq K_i^*$.

The NZS-CDS optimization can now be introduced given the preceding development.
Definition 7.4.6 (NZS-CDS Optimization)

For every $\kappa^i(t_0)$, let $g^i(\kappa^i(t_0), \hat{\kappa}^i(t_0))$ be an analytic function, convex in $\kappa^i(t_0)$, defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of $\hat{\kappa}^i(t_0)$. Let $r \in \mathbb{N}$ be a fixed positive integer, where $\kappa^i(t_0)$, $\hat{\kappa}^i(t_0) \in \mathbb{R}^r$ are the vectors of initial cost cumulants and target initial cost cumulants, respectively, for Player $i$. Then the 2-Player NZS-CDS optimization can be formulated as,

$$
\min_{K_1 \in K^1(t_f)} \phi^1(H^1(t_0), D^1(t_0), \hat{H}^1(t_0), \hat{D}^1(t_0))
$$

$$
\min_{K_2 \in K^2(t_f)} \phi^2(H^2(t_0), D^2(t_0), \hat{H}^2(t_0), \hat{D}^2(t_0))
$$

subject to:

$$
\frac{dH^1(\alpha)}{d\alpha} = F(H^1(\alpha), K_1(\alpha), K_2(\alpha)) \quad \frac{dD^1(\alpha)}{d\alpha} = G(H^1(\alpha))
$$

$$
H^1(t_f) = H^1_f, \quad D^1(t_f) = D_f
$$

$$
\frac{d\hat{H}^1(\alpha)}{d\alpha} = \tilde{F}(\hat{H}^1(\alpha), \hat{K}_1(\alpha), \hat{K}_2(\alpha)) \quad \frac{d\hat{D}^1(\alpha)}{d\alpha} = \tilde{G}(\hat{H}^1(\alpha))
$$

$$
\hat{H}^1(t_f) = \hat{H}^1_f; \hat{D}^1(t_f) = \hat{D}_f
$$

$$
\frac{dH^2(\alpha)}{d\alpha} = F(H^2(\alpha), K_1(\alpha), K_2(\alpha)) \quad \frac{dD^2(\alpha)}{d\alpha} = G(H^2(\alpha))
$$

$$
H^2(t_f) = H^2_f; \quad D^2(t_f) = D_f
$$

$$
\frac{d\hat{H}^2(\alpha)}{d\alpha} = \tilde{F}(\hat{H}^2(\alpha), \hat{K}_1(\alpha), \hat{K}_2(\alpha)) \quad \frac{d\hat{D}^2(\alpha)}{d\alpha} = \tilde{G}(\hat{H}^2(\alpha))
$$

$$
\hat{H}^2(t_f) = \hat{H}^2_f; \quad \hat{D}^2(t_f) = \hat{D}_f
$$

where the initial values of the state trajectories must satisfy

$$(t_0, H^1(t_0), D^1(t_0), \hat{H}^1(t_0), \hat{D}^1(t_0)) \in \mathcal{M}_i, \quad 1 \leq i \leq 2.$$

7.4.6 A Dynamic Programming Framework (NZS-CDS)

A HJB equation for each player is adapted to that player's cost cumulant-generating equations. Such equations will be presented in this section. Going forward, the discussion will address an initial cost problem akin to the initial cost problem studied in Chapter 3. As before, this fundamental difference corresponds to the backwards-in-time equations giving the dynamics of the game.

The following variables will be used in the solution derivation. Notice that
since two pairs of differential equations (e.g. one per player) are specified in (7.57), distinct dynamic programming variables are maintained for the associated set of terminal conditions. Define the block matrices $\mathbf{Y}(\epsilon), \mathbf{\dot{Y}}(\epsilon) \in \mathbb{R}^{n \times n}$ and the vectors $\mathbf{Z}(\epsilon), \mathbf{\dot{Z}}(\epsilon) \in \mathbb{R}^r$ as below,

$$
\begin{align*}
\mathbf{Y}(\epsilon) &= \begin{bmatrix} \mathbf{Y}_1^i(\epsilon) \\ \mathbf{Y}_2^i(\epsilon) \\ \vdots \\ \mathbf{Y}_j^i(\epsilon) \end{bmatrix}, \quad \mathbf{\dot{Y}}(\epsilon) &= \begin{bmatrix} \mathbf{\dot{Y}}_1^i(\epsilon) \\ \mathbf{\dot{Y}}_2^i(\epsilon) \\ \vdots \\ \mathbf{\dot{Y}}_j^i(\epsilon) \end{bmatrix}, \\
\mathbf{Z}(\epsilon) &= \begin{bmatrix} \mathbf{Z}_1^i(\epsilon) \\ \mathbf{Z}_2^i(\epsilon) \\ \vdots \\ \mathbf{Z}_j^i(\epsilon) \end{bmatrix}, \quad \mathbf{\dot{Z}}(\epsilon) &= \begin{bmatrix} \mathbf{\dot{Z}}_1^i(\epsilon) \\ \mathbf{\dot{Z}}_2^i(\epsilon) \\ \vdots \\ \mathbf{\dot{Z}}_j^i(\epsilon) \end{bmatrix}.
\end{align*}
$$

$$
\begin{align*}
\mathbf{Y}_i^j(\epsilon) &= \mathbf{H}_i(\epsilon), \quad \mathbf{Z}_i^j(\epsilon) = \mathbf{D}_i(\epsilon), \quad \mathbf{\dot{Y}}_i^j(\epsilon) = \mathbf{\dot{H}}_i(\epsilon), \quad \mathbf{\dot{Z}}_i^j(\epsilon) = \mathbf{\dot{D}}_i(\epsilon), \quad 1 \leq i \leq r, \quad 1 \leq j \leq 2.
\end{align*}
$$

References to states $\mathbf{Y}(\tau), \mathbf{\dot{Y}}(\tau), \mathbf{Z}(\tau), \mathbf{\dot{Z}}(\tau)$ will be made, and the above definitions of block matrices and vectors apply $\forall \tau \in [t_0, \epsilon)$. Before continuing, note that in this development it is assumed that the terminal time $t_f$ and the associated boundary conditions have been displaced to corresponding ones at some time $\epsilon$, and the nature of the control that drives the state from $t_f$ to $\epsilon$ must be regarded. In cases where a control is assumed part of the Nash equilibrium, a $\ast$ designation will appear with the control denotation, for example $K_i^\ast$. The state variables under this control will receive a similar distinction. For instance, $(\epsilon, \mathbf{Y}_i(\epsilon), \mathbf{Z}_i(\epsilon), \mathbf{\dot{Y}}_i(\epsilon), \mathbf{\dot{Z}}_i(\epsilon))$ will refer to “displaced” terminal conditions arrived at from the Nash solution $(K_1^*, K_2^*)$ on $(\epsilon, t_f)$, and $(\epsilon, \mathbf{Y}_i(\epsilon), \mathbf{Z}_i(\epsilon), \mathbf{\dot{Y}}_i(\epsilon), \mathbf{\dot{Z}}_i(\epsilon))$ will refer to terminal conditions arrived at from any general pair of strategies $(K_1, K_2)$ on $(\epsilon, t_f)$.
It is time to define the playable sets \( Q_i \) and the value functions,

\[
V^i(\epsilon, Y^i(\epsilon), Z^i(\epsilon), \hat{Y}^i(\epsilon), \hat{Z}^i(\epsilon)), \ 1 \leq i \leq 2
\]

which are core constructs to this dynamic programming formulation.

**Definition 7.4.7 (Value Function - NZS-CDS)**

Let \((\epsilon, Y^i(\epsilon), Z^i(\epsilon), \hat{Y}^i(\epsilon), \hat{Z}^i(\epsilon)) \in [t_0, t_f] \times (S^n)^r \times \mathbb{R}^r \times (S^n)^r \times \mathbb{R}^r\) and let

\[
V^i(\epsilon, Y^i(\epsilon), Z^i(\epsilon), \hat{Y}^i(\epsilon), \hat{Z}^i(\epsilon)) \text{ for } 1 \leq i \leq 2 \text{ be scalar functions}
\]

such that

\[
V^1(\epsilon, Y^1(\epsilon), Z^1(\epsilon), \hat{Y}^1(\epsilon), \hat{Z}^1(\epsilon)) = \left\{ \begin{array}{ll}
\min_{K_1 \in K^1(\epsilon)} \phi^1(H^1(t_0), D^1(t_0), \hat{H}^1(t_0), \hat{D}^1(t_0)) & , K^1(\epsilon) \neq \emptyset \\
\infty, & K^1(\epsilon) = \emptyset
\end{array} \right.
\]

\[
V^2(\epsilon, Y^2(\epsilon), Z^2(\epsilon), \hat{Y}^2(\epsilon), \hat{Z}^2(\epsilon)) = \left\{ \begin{array}{ll}
\min_{K_2 \in K^2(\epsilon)} \phi^2(H^2(t_0), D^2(t_0), \hat{H}^2(t_0), \hat{D}^2(t_0)) & , K^2(\epsilon) \neq \emptyset \\
\infty, & K^2(\epsilon) = \emptyset
\end{array} \right.
\]

where \( K^i(\epsilon) \triangleq K^i(\epsilon, Y^i(\epsilon), Z^i(\epsilon), \hat{Y}^i(\epsilon), \hat{Z}^i(\epsilon)) \) for \( 1 \leq i \leq 2 \).

**Definition 7.4.8 (Playable Set - NZS-CDS)**

Define the playable sets as the sets of terminal values from which there exists a control that can take the system to the target set. More formally, this is

\[
Q_i = \{(\epsilon, Y^i(\epsilon), Z^i(\epsilon), \hat{Y}^i(\epsilon), \hat{Z}^i(\epsilon)) \mid K^i(\epsilon) \neq \emptyset \}, \ 1 \leq i \leq 2.
\]

Under certain conditions, the value function defined above will be continuously differentiable within \( Q_i \), \( 1 \leq i \leq 2 \). The following theorem defines these conditions, and it can be proven via the approach used in Chapter 3.

**Theorem 7.4.9 (Differentiability of Value Function - NZS-CDS)**

Suppose that controls \((K_1(\alpha, H^1(\alpha), D^1(\alpha), \hat{H}^1(\alpha), \hat{D}^1(\alpha)), K_2(\alpha, H^2(\alpha), D^2(\alpha), \hat{H}^2(\alpha), \hat{D}^2(\alpha)))\)
constitute a Nash equilibrium solution. For Player $i$, let the initial time $t_0(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))$

and

$$H^i(t_0(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))), \; D^i(t_0(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon)))$$

denote initial states of the trajectories

$$\frac{dH^i(\alpha)}{d\alpha} = F(H^i(\alpha), K_1(\alpha), K_2(\alpha)), \; \frac{d\bar{H}^i(\alpha)}{d\alpha} = F(\bar{H}^i(\alpha), \bar{K}_1(\alpha), \bar{K}_2(\alpha))$$

$$\frac{dD^i(\alpha)}{d\alpha} = G(H^i(\alpha)), \; \frac{d\bar{D}^i(\alpha)}{d\alpha} = G(\bar{H}^i(\alpha)), \; \alpha \in [t_0, \epsilon]$$

$$H^i(\epsilon) = \mathbf{y}^i(\epsilon), \; \bar{H}^i(\epsilon) = \tilde{\mathbf{y}}^i(\epsilon), \; D^i(\epsilon) = \mathbf{z}^i(\epsilon), \; \bar{D}^i(\epsilon) = \tilde{\mathbf{z}}^i(\epsilon).$$

Analogously, for Player 2 the initial time is $t_0(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon))$ and the initial states are

$$H^2(t_0(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon))), \; D^2(t_0(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon)))$$

for the trajectories

$$\frac{dH^2(\alpha)}{d\alpha} = F(H^2(\alpha), K_1(\alpha), K_2(\alpha)), \; \frac{d\bar{H}^2(\alpha)}{d\alpha} = F(\bar{H}^2(\alpha), \bar{K}_1(\alpha), \bar{K}_2(\alpha))$$

$$\frac{dD^2(\alpha)}{d\alpha} = G(H^2(\alpha)), \; \frac{d\bar{D}^2(\alpha)}{d\alpha} = G(\bar{H}^2(\alpha)), \; \alpha \in [t_0, \epsilon]$$

$$H^2(\epsilon) = \mathbf{y}^2(\epsilon), \; \bar{H}^2(\epsilon) = \tilde{\mathbf{y}}^2(\epsilon), \; D^2(\epsilon) = \mathbf{z}^2(\epsilon), \; \bar{D}^2(\epsilon) = \tilde{\mathbf{z}}^2(\epsilon).$$

The value function $V^i(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))$ is differentiable in $(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon))$ when initial states and initial times are also differentiable in those variables for $1 \leq i \leq 2$.

For points in $Q_1$ and $Q_2$ satisfying the above conditions, a sufficient condition can be verified to confirm whether or not a given control is optimal. This sufficient condition requires the following HJB equation.

**Theorem 7.4.10 (HJB Equation - NZS-CDS Problem)**

Suppose $(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{z}^1(\epsilon), \tilde{\mathbf{y}}^1(\epsilon), \tilde{\mathbf{z}}^1(\epsilon))$ and $(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon))$ are points in the Playable sets $Q_1$ and $Q_2$ where the value functions

$$V^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{z}^1(\epsilon), \tilde{\mathbf{y}}^1(\epsilon), \tilde{\mathbf{z}}^1(\epsilon)), \; V^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon))$$

318
are differentiable. Then, if \((K_1^*, K_2^*)\) is a Nash equilibrium, the value function \(V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e))\) for \(K_1(e) = K_1^*(e)\) satisfies

\[
- \partial V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e)) \quad \frac{\partial}{\partial e} \\
= \min_{K_1 \in K_1} \left\{ \frac{\partial V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e))}{\partial Z^1(e)} \mathcal{G}(\dot{Y}^1(e)) \right. \\
+ \frac{\partial V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e))}{\partial \vec{Y}^1(e)} \mathcal{G}(\dot{Z}^1(e)) \right. \\
+ \frac{\partial V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e))}{\partial \vec{Z}^1(e)} \mathcal{G}(\dot{Z}^1(e)) \\
\left. \left( \partial V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e)) \right) \mathcal{G}(\dot{Z}^1(e)) \right) \\
V^1(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e)) = \phi^1(H^1(t_0), D^1(t_0), H^1(t_0), D^1(t_0)) \\
(e, Y^1(e), Z^1(e), \dot{Y}^1(e), \dot{Z}^1(e) \in \mathcal{M}_1.
\]

Likewise, for a Nash equilibrium \((K_1^*, K_2^*)\), the value function \(V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e))\) satisfies for \(K_2(e) = K_2^*(e)\)

\[
- \partial V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e)) \quad \frac{\partial}{\partial e} \\
= \min_{K_2 \in K_2} \left\{ \frac{\partial V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e))}{\partial Z^2(e)} \mathcal{G}(\dot{Y}^2(e)) \right. \\
+ \frac{\partial V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e))}{\partial \vec{Y}^2(e)} \mathcal{G}(\dot{Z}^2(e)) \\
+ \frac{\partial V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e))}{\partial \vec{Z}^2(e)} \mathcal{G}(\dot{Z}^2(e)) \\
\left. \left( \partial V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e)) \right) \mathcal{G}(\dot{Z}^2(e)) \right) \\
V^2(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e)) = \phi^2(H^2(t_0), D^2(t_0), H^2(t_0), D^2(t_0)) \\
(e, Y^2(e), Z^2(e), \dot{Y}^2(e), \dot{Z}^2(e) \in \mathcal{M}_2.
\]

**Proof** Identical to the approach used in Chapter 3.

With the HJB equation for the NZS-CDS 2-Player Nash game at hand, the following verification lemma can be posed.
**Theorem 7.4.11 (HJB Verification, NZS-CDS Problem)**

Suppose \((e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))\) and \(((e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))\) are points in the

Playable sets \(Q_1\) and \(Q_2\) where the non-increasing, scalar functions

\[
W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e)), \quad W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))
\]

are differentiable. Suppose that for a nominal pair \((K_1^*, K_2^*)\), the function

\[
W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e)) \text{ satisfies }
\]

\[
- \frac{\partial W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))}{\partial e} = \min_{K_1 \in K_1^*} \left\{ \frac{\partial W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))}{\partial \dot{Z}^1(e)} g(\dot{y}^1(e)) + \frac{\partial W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))}{\partial \dot{y}^1(e)} g(y^1(e)) + \frac{\partial W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))}{\partial \text{vec}(\dot{y}^1(e))} \text{vec}(F(\dot{y}^1(e), K_1(e), K_2(e))) + \frac{\partial W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e))}{\partial \text{vec}(\dot{y}^1(e))} \text{vec}(F(y^1(e), K_1(e), K_2(e))) \right\}, \quad K_1(e) = K_1^*(e)
\]

with the boundary condition,

\[
W^1(e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e)) = \phi^3(H^1(t_0), D^1(t_0), \dot{H}^1(t_0), \dot{D}^1(t_0))
\]

\((e, y^1(e), Z^1(e), \dot{y}^1(e), \dot{Z}^1(e)) \in M_1. \quad (7.58)\)

Likewise, suppose that for \((K_1^*, K_2^*)\), the function \(W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))\) sat-

ises

\[
- \frac{\partial W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))}{\partial e} = \min_{K_2 \in K_2^*} \left\{ \frac{\partial W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))}{\partial \dot{Z}^2(e)} g(\dot{y}^2(e)) + \frac{\partial W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))}{\partial \dot{y}^2(e)} g(y^2(e)) + \frac{\partial W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))}{\partial \text{vec}(\dot{y}^2(e))} \text{vec}(F(\dot{y}^2(e), K_1(e), K_2(e))) + \frac{\partial W^2(e, y^2(e), Z^2(e), \dot{y}^2(e), \dot{Z}^2(e))}{\partial \text{vec}(\dot{y}^2(e))} \text{vec}(F(y^2(e), K_1^*(e), K_2^*(e))) \right\}, \quad K_2(e) = K_2^*(e)
\]

320
with the boundary condition,

\[ W^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) = \phi^2(\mathbf{H}^2(t_0), \mathbf{D}^2(t_0), \mathbf{H}^2(t_0), \mathbf{D}^2(t_0)) \]

(7.59)

\[ (\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) \in M_2. \]

Under these conditions, it must be true that

\[ W^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{Z}^1(\epsilon), \mathbf{Y}^1(\epsilon), \mathbf{Z}^1(\epsilon)) = V^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{Z}^1(\epsilon), \mathbf{Y}^1(\epsilon), \mathbf{Z}^1(\epsilon)) \]

\[ W^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) = V^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) \]

and that \((K^*_1, K^*_2)\) is a Nash equilibrium.

**Definition 7.4.12 (Candidate Value Function)**

Consider candidate value functions of the form

\[ W^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{Z}^1(\epsilon), \mathbf{Y}^1(\epsilon), \mathbf{Z}^1(\epsilon)) = \nu^1(\epsilon) - \phi^1(\mathbf{y}^1(\epsilon), \mathbf{Z}^1(\epsilon), \mathbf{Y}^1(\epsilon), \mathbf{Z}^1(\epsilon)) \]

(7.60)

\[ W^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) = \nu^2(\epsilon) - \phi^2(\mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon)) \]

(7.61)

where the functions \(\nu^i(\tau) \in C^1([t_0, t_f], \mathbb{R})\), \(1 \leq i \leq 2\), are to be determined.

For any point \((\epsilon, \mathbf{y}^1(\epsilon), \mathbf{Z}^1(\epsilon), \mathbf{Y}^1(\epsilon), \mathbf{Z}^1(\epsilon))\) in \(Q_1\) or \((\epsilon, \mathbf{y}^2(\epsilon), \mathbf{Z}^2(\epsilon), \mathbf{Y}^2(\epsilon), \mathbf{Z}^2(\epsilon))\) in \(Q_2\) where the candidate value functions are differentiable, they may differentiated directly. The following will establish the form of the total time derivative of \(W^i(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{Z}^i(\epsilon), \mathbf{Y}^i(\epsilon), \mathbf{Z}^i(\epsilon))\), which essentially is constrained by the HJB equations given in the preceding.

**Lemma 7.4.13 (Derivative of Candidate Value Function)**

Let \(r \in \mathbb{N}\) be fixed and let \((\epsilon, \mathbf{y}^i(\epsilon), \mathbf{Z}^i(\epsilon), \mathbf{Y}^i(\epsilon), \mathbf{Z}^i(\epsilon)) \in Q_i\) be an interior point of the set at which the candidate value function

\[ W^i(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{Z}^i(\epsilon), \mathbf{Y}^i(\epsilon), \mathbf{Z}^i(\epsilon)) = \nu^i(\epsilon) - \phi^i(\mathbf{y}^i(\epsilon), \mathbf{Z}^i(\epsilon), \mathbf{Y}^i(\epsilon), \mathbf{Z}^i(\epsilon)) \]
is differentiable. Then its total time derivative takes the form

$$
\frac{dV^i(e, Y^i(e), Z^i(e), \tilde{Y}^i(e))}{de} = -\sum_{j=1}^{r} \frac{\partial}{\partial \kappa_{ij}^i(e)} g^i(\kappa^i(e), \tilde{\kappa}^i(e)) x_0^T F_j(Y^i(e), K_1(e), K_2(e)) x_0
$$

$$
- \sum_{j=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_{ij}^i(e)} g^i(\kappa^i(e), \tilde{\kappa}^i(e)) x_0^T F_j(\tilde{Y}^i(e), \tilde{K}_1(e), \tilde{K}_2(e)) x_0
$$

$$
- \sum_{j=1}^{r} \frac{\partial}{\partial \kappa_{ij}^i(e)} g^i(\kappa^i(e), \tilde{\kappa}^i(e)) G_j(Y^i(e))
$$

$$
- \sum_{j=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_{ij}^i(e)} g^i(\kappa^i(e), \tilde{\kappa}^i(e)) G_j(\tilde{Y}^i(e)) + \frac{dv^i(e)}{de}, \tag{7.62}
$$

where $1 \leq i \leq 2$.

**Proof** Use the same approach as was used for ZS-CDS.

### 7.4.7 Problem Solution (NZS-CDS)

The discussion will now focus upon the solution to the NZS-CDS optimization, using the HJB verification lemma.

**Theorem 7.4.14 (Nash Equilibrium Solution, NZS-CDS Optimization)**

Consider the LQG stochastic optimal control problem involving the process having dynamics (7.46) and the costs (7.47). Then the linear state-feedback, finite-horizon, optimal control solutions to the NZS-CDS optimization are characterized by the optimal gains

$$
K^1_1(\alpha) = -R_{11}^{-1}(\alpha)B^T_1(\alpha) \left( H^{1*}_1(\alpha) + \sum_{i=2}^{r} \left( \frac{\partial g^1(\kappa^{1*}(\alpha), \tilde{\kappa}^1(\alpha))}{\partial \kappa^1(\alpha)} \right) H^{1*}_i(\alpha) \right)
$$

$$
K^2_2(\alpha) = -R_{22}^{-1}(\alpha)B^T_2(\alpha) \left( H^{2*}_1(\alpha) + \sum_{i=2}^{r} \left( \frac{\partial g^2(\kappa^{2*}(\alpha), \tilde{\kappa}^2(\alpha))}{\partial \kappa^2(\alpha)} \right) H^{2*}_i(\alpha) \right)
$$

where the optimal cost cumulants for Players 1 and 2 are defined by

$$
\kappa^{1*}_i(\alpha) = x_0^T H^{1*}_i(\alpha) x_0 + D^{1*}_i(\alpha), \quad \kappa^{2*}_i(\alpha) = x_0^T H^{2*}_i(\alpha) x_0 + D^{2*}_i(\alpha)
$$
such that
\[ \tilde{\kappa}_i^1(\alpha) = x_0^T \tilde{H}_i^1(\alpha) x_0 + \tilde{D}_i^1(\alpha), \quad \tilde{\kappa}_i^2(\alpha) = x_0^T \tilde{H}_i^2(\alpha) x_0 + \tilde{D}_i^2(\alpha) \]

where \( 1 \leq i \leq r \). The optimal state variables \( H^1(\alpha), H^2(\alpha) \) and \( D^1(\alpha), D^2(\alpha) \) follow the equations of motion

\[
\begin{align*}
\frac{dH^1(\alpha)}{d\alpha} & = F(H^1(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{dD^1(\alpha)}{d\alpha} = G(H^1(\alpha)) \\
H^1(t_f) & = H_f^1, \quad D^1(t_f) = D_f \\
\frac{dH^2(\alpha)}{d\alpha} & = F(H^2(\alpha), K_1(\alpha), K_2(\alpha)), \quad \frac{dD^2(\alpha)}{d\alpha} = G(H^2(\alpha)) \\
H^2(t_f) & = H_f^2, \quad D^2(t_f) = D_f
\end{align*}
\]

and

\[
\begin{align*}
\frac{d\tilde{H}^1(\alpha)}{d\alpha} & = F(\tilde{H}^1(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha)), \quad \frac{d\tilde{D}^1(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)) \\
\tilde{H}^1(t_f) & = \tilde{H}^1_{f;e_1}, \quad \tilde{D}^1(t_f) = \tilde{D}_{f;e_1} \\
\frac{d\tilde{H}^2(\alpha)}{d\alpha} & = F(\tilde{H}^2(\alpha), \tilde{K}_1(\alpha), \tilde{K}_2(\alpha)), \quad \frac{d\tilde{D}^2(\alpha)}{d\alpha} = G(\tilde{H}^2(\alpha)) \\
\tilde{H}^2(t_f) & = \tilde{H}^2_{f;e_2}, \quad \tilde{D}^2(t_f) = \tilde{D}_{f;e_2}.
\end{align*}
\]

**Proof** The objective is to identify control gains \( K_1^*, K_2^* \) and functions \( v^1(\epsilon), v^2(\epsilon) \) such that

\[
\frac{dv^j(\epsilon)}{d\epsilon} = \min_{K_i \in K_i} \left\{ \sum_{j=1}^r \frac{\partial}{\partial K^j_i(\epsilon)} g^i(K^j_i(\epsilon), \tilde{K}^j_i(\epsilon)) x_0^T F_j(Y^j(\epsilon), K_1(\epsilon), K_2(\epsilon)) x_0 \\
+ \sum_{j=1}^r \frac{\partial}{\partial \tilde{K}^j_i(\epsilon)} g^i(K^j_i(\epsilon), \tilde{K}^j_i(\epsilon)) x_0^T \tilde{F}_j(Y^j(\epsilon), \tilde{K}_1(\epsilon), \tilde{K}_2(\epsilon)) x_0 \\
+ \sum_{j=1}^r \frac{\partial}{\partial K^j_i(\epsilon)} g^i(K^j_i(\epsilon), \tilde{K}^j_i(\epsilon)) \tilde{G}_j(Y^j(\epsilon)) \\
+ \sum_{j=1}^r \frac{\partial}{\partial \tilde{K}^j_i(\epsilon)} g^i(K^j_i(\epsilon), \tilde{K}^j_i(\epsilon)) \tilde{G}_j(Y^j(\epsilon)) \right\}
\]

Such selections ensure that the candidate value function

\[ W^i(\epsilon, Y^i(\epsilon), Z^i(\epsilon), \tilde{Y}^i(\epsilon), \tilde{Z}^i(\epsilon)) \]

satisfies the HJB equation for Player \( i \). The proof
begins with the optimization,

\[
\begin{align*}
\min_{\kappa_i \in \mathcal{K}} \left\{ \sum_{j=1}^{r} \frac{\partial}{\partial \kappa_i^j(\epsilon)} g^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)) x_0^T F_j(Y^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)), K_1(\epsilon), K_2(\epsilon)) x_0 \\
+ \sum_{j=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i^j(\epsilon)} g^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)) x_0^T F_j(Y^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)), K_1(\epsilon), K_2(\epsilon)) x_0 \\
+ \sum_{j=1}^{r} \frac{\partial}{\partial \tilde{\kappa}_i^j(\epsilon)} g^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)) \tilde{G}_j(Y^i(\epsilon)) \\
+ \sum_{j=1}^{r} \frac{\partial}{\partial \kappa_i^j(\epsilon)} g^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)) \tilde{G}_j(Y^i(\epsilon)) \right\} 
\end{align*}
\]

(7.63)

In the following differentiations, denote the partial derivatives in the above expression as shown below to simplify the notation,

\[ c_{ij}(\epsilon) = \frac{\partial}{\partial \kappa_i^j(\epsilon)} g^i(\kappa_i^j(\epsilon), \tilde{\kappa}_i^j(\epsilon)), \quad 1 \leq i \leq 2, \quad 1 \leq j \leq r. \]

Now differentiate the expression in braces (7.63) with respect to \( K_1 \) and set the result equal to a zero matrix with the appropriate dimension. Then differentiate the expression in braces (7.63) with respect to to \( K_2 \), and set the result to an appropriately-sized zero matrix. The resulting equations are shown below.

\[
-2B_1^T(\epsilon) \sum_{j=1}^{r} c_{1j}(\epsilon) \tilde{J}_1^j(\epsilon)(x_0x_0^T) - 2c_{11}(\epsilon) R_{11}(\epsilon) K_1(\epsilon)(x_0x_0^T) = 0^{m_1 \times n}
\]

\[
-2B_2^T(\epsilon) \sum_{j=1}^{r} c_{2j}(\epsilon) \tilde{J}_2^j(\epsilon)(x_0x_0^T) - 2c_{11}(\epsilon) R_{22}(\epsilon) K_2(\epsilon)(x_0x_0^T) = 0^{m_2 \times n}
\]

These are necessary conditions for each respective expression to take an extremal value on the interior of its domain. Assume that \( c_1(\epsilon) \neq 0, \quad \forall \epsilon \in [\epsilon_0, \tau_f] \). Since \( x_0x_0^T \)
is a fixed rank-one matrix, it must be that

\[ K_i^*(c) = -R_{11}^{-1}(c)B_1^T(c) \left( \mathcal{V}_1^1(c) + \sum_{j=2}^r \frac{e_j^*(c)}{c_j^1(c)} \mathcal{V}_j^1(c) \right) \]

\[ = -R_{11}^{-1}(c)B_1^T(c) \left( \mathcal{V}_1^1(c) + \sum_{j=2}^r \left( \frac{\partial g^1(\kappa_1^1(c), \kappa_1^1(c))}{\partial \kappa_1^1(c)} \right) \mathcal{V}_j^1(c) \right) \]

\[ K_2^*(c) = -R_{22}^{-1}(c)B_2^T(c) \left( \mathcal{V}_1^2(c) + \sum_{j=2}^r \frac{e_j^*(c)}{c_j^2(c)} \mathcal{V}_j^2(c) \right) \]

\[ = -R_{22}^{-1}(c)B_2^T(c) \left( \mathcal{V}_1^2(c) + \sum_{j=2}^r \left( \frac{\partial g^1(\kappa_2^1(c), \kappa_2^1(c))}{\partial \kappa_2^1(c)} \right) \mathcal{V}_j^2(c) \right) \]

Let \( \mathcal{Y}^1*(\tau) \) and \( \mathcal{Z}^1*(\tau) \) denote the solutions of the equations of motion under this control selection,

\[
\frac{d\mathcal{V}_j^1*(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)^T \mathcal{V}_j^1*(\tau) \\
- \mathcal{V}_j^1*(\tau) \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right) \\
- K_1^{T}(\tau)R_{11}(\tau)K_1^*(\tau) - K_2^{T}(\tau)R_{12}(\tau)K_2^*(\tau) - Q_1(\tau) \\
(7.64)
\]

\[
\frac{d\mathcal{V}_j^1*(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)^T \mathcal{V}_j^1*(\tau) \\
- \mathcal{V}_j^1*(\tau) \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right) \\
- 2 \sum_{j=1}^{i-1} \frac{i}{j} \mathcal{V}_j^1*(\tau) G(\tau) W G^T(\tau) \mathcal{V}_j^1*(\tau) \\
\frac{d\mathcal{Z}_i^1*(\tau)}{d\tau} = - \text{Tr} \left( \mathcal{V}_i^1*(\tau) G(\tau) W G^T(\tau) \right) \]

\[ \mathcal{V}_i^1*(t_f) = H_i(t_f), \quad \mathcal{Z}_i^1*(t_f) = D_i(t_f), \quad \tau \in [\epsilon, t_f], \quad 1 \leq i \leq r. \]

Analogously, let \( \mathcal{Y}^2*(\tau) \) and \( \mathcal{Z}^2*(\tau) \) denote the solutions of the equations of motion...
under this control selection,

\[
\frac{dY^i_2(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)^T Y^i_2(\tau)
\]

\[
- Y^i_2(\tau) \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)
\]

\[
- K_1^T(\tau)R_{21}(\tau)K_1^*(\tau) - K_2^*(\tau)R_{22}(\tau)K_2^*(\tau) - Q_2(\tau)
\]

\[
\frac{dY^i_2(\tau)}{d\tau} = - \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)^T Y^i_2(\tau)
\]

(7.65)

\[
- Y^i_2(\tau) \left( A(\tau) + B_1(\tau)K_1^*(\tau) + B_2(\tau)K_2^*(\tau) \right)
\]

\[
- 2 \sum_{j=1}^{i-1} \begin{pmatrix} i \\ j \end{pmatrix} Y^j_2(\tau) G(\tau) W G^T(\tau) Y^{i-j}_2(\tau)
\]

\[
\frac{dZ^i_2(\tau)}{d\tau} = - \text{Tr}\left( Y^i_2(\tau) G(\tau) W G^T(\tau) \right)
\]

\[
Y^i_2(t_f) = H(t_f), \quad Z^i_2(t_f) = D_i(t_f), \quad \tau \in [\epsilon, t_f], \quad 1 \leq i \leq r.
\]

It is assumed that the solutions to the above equations exist. Return again to the minimized expression (7.63) inserting the determined controllers,

\[
\frac{dv^i(\epsilon)}{d\epsilon} = \sum_{j=1}^{r} \frac{\partial}{\partial \kappa^i_j(\epsilon)} g^i(\kappa^i(\epsilon), \bar{\kappa}^i(\epsilon)) x_0^T \mathcal{F}_j(\bar{\mathbf{Y}}^i(\epsilon), K_1^i(\epsilon), K_2^i(\epsilon)) x_0
\]

\[
+ \sum_{j=1}^{r} \frac{\partial}{\partial \bar{\kappa}^i_j(\epsilon)} g^i(\kappa^i(\epsilon), \bar{\kappa}^i(\epsilon)) x_0^T \mathcal{F}_j(\bar{\mathbf{Y}}^i(\epsilon), \bar{K}_1(\epsilon), \bar{K}_2(\epsilon)) x_0
\]

\[
+ \sum_{j=1}^{r} \frac{\partial}{\partial \kappa^i_j(\epsilon)} g^i(\kappa^i(\epsilon), \bar{\kappa}^i(\epsilon)) \mathcal{G}_j(\mathbf{Y}^i(\epsilon))
\]

\[
+ \sum_{j=1}^{r} \frac{\partial}{\partial \bar{\kappa}^i_j(\epsilon)} g^i(\kappa^i(\epsilon), \bar{\kappa}^i(\epsilon)) \mathcal{G}_j(\mathbf{Y}^i(\epsilon))
\]

\[
+ \sum_{j=1}^{r} \frac{\partial \phi^i(\mathbf{Y}^i(\epsilon), \bar{\mathbf{Z}}^i(\epsilon), \bar{\mathbf{Y}}^i(\epsilon), \bar{\mathbf{Z}}^i(\epsilon))}{\partial \text{vec}(\mathbf{Y}^i(\epsilon))} \text{vec}(\mathcal{F}_j(\mathbf{Y}^i(\epsilon), K_1^i(\epsilon), K_2^i(\epsilon)))
\]

\[
+ \sum_{j=1}^{r} \frac{\partial \phi^i(\mathbf{Y}^i(\epsilon), \bar{\mathbf{Z}}^i(\epsilon), \bar{\mathbf{Y}}^i(\epsilon), \bar{\mathbf{Z}}^i(\epsilon))}{\partial \text{vec}(\bar{\mathbf{Y}}^i(\epsilon))} \text{vec}(\mathcal{F}_j(\mathbf{Y}^i(\epsilon), \bar{K}_1(\epsilon), \bar{K}_2(\epsilon)))
\]

326
\[
\begin{align*}
+ \sum_{j=1}^{r} \frac{\partial \phi^{i}(Y^{i*}(\epsilon), Z^{i*}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon))}{\partial Z^{i*}_{j}(\epsilon)} g_j(Y^{i*}(\epsilon)) \\
+ \sum_{j=1}^{r} \frac{\partial \phi^{i}(Y^{i*}(\epsilon), Z^{i*}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon))}{\partial Z_{j}(\epsilon)} g_j(Y^{i}(\epsilon)) \\
= \frac{d\phi^{i}(Y^{i*}(\epsilon), Z^{i*}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon))}{d\epsilon}, \quad 1 \leq i \leq 2
\end{align*}
\]

Note that \((\epsilon, Y^{i}(\epsilon), Z^{i}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon)) \in Q_i\), for some \((K_1, K_2) \in \mathcal{K}_1(\epsilon) \times \mathcal{K}_2(\epsilon)\). Recall by assumption, \(K_1^*\) and \(K_2^*\) are admissible on \([t_0, \epsilon]\). Thus \((\epsilon, Y^{i*}(\epsilon), Z^{i*}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon)) \in Q_i\) since \(K_1^* \in \mathcal{K}_1(\epsilon)\) and \(K_2^* \in \mathcal{K}_2(\epsilon)\). Consider now that any displaced terminal conditions at \(\tau < \epsilon\) along the respective state trajectories resultant from \(K_i^*\) will also be in \(Q_i\), due to the fact that the restrictions of the control \(K_i^*\) to \([t_0, \tau]\) will be in \(\mathcal{K}_i(\tau)\), where of course \(1 \leq i \leq 2\). Clearly then, it can be similarly argued that \((\tau, Y^{i*}(\tau), Z^{i*}(\tau), \hat{Y}^{i}(\tau), \hat{Z}^{i}(\tau)) \in Q_i, \forall \tau \in [t_0, \epsilon]\). It is desired to determine \(v^{i}(\epsilon)\) so to enforce the above relationship for all possible displaced terminal times \(\tau < \epsilon\) and the associated terminal conditions, gotten from anywhere along the optimal trajectory of the state equations.

\[
\frac{d\phi^{i}(Y^{i*}(\tau), Z^{i*}(\tau), \hat{Y}^{i}(\tau), \hat{Z}^{i}(\tau))}{d\tau} = \frac{dv^{i}(\tau)}{d\tau}, \quad \tau \in [t_0, \epsilon], 1 \leq i \leq 2
\]

These differential equations can be integrated over a reduced time-horizon since the state equations \(Y^{i*}_{j}(\tau), Z^{i*}_{j}(\tau), \hat{Y}^{j}(\tau), \hat{Z}^{j}(\tau)\) for \(1 \leq j \leq r\) are continuously differentiable. By the Fundamental Theorem,

\[
\phi^{i}(Y^{i*}(\epsilon), Z^{i*}(\epsilon), \hat{Y}^{i}(\epsilon), \hat{Z}^{i}(\epsilon)) - \phi^{i}(Y^{i*}(t_0), Z^{i*}(t_0), \hat{Y}^{i}(t_0), \hat{Z}^{i}(t_0)) = - \int_{\epsilon}^{t_0} \frac{d}{d\tau} \left( \phi^{i}(Y^{i*}(\tau), Z^{i*}(\tau), \hat{Y}^{i}(\tau), \hat{Z}^{i}(\tau)) \right) \\
v^{i}(\epsilon) - v^{i}(t_0) = - \int_{\epsilon}^{t_0} \frac{d}{d\tau} (v^{i}(\tau))
\]

327
from which it is immediate that

\[ v^i(\epsilon) = v^i(t_0) - \phi^i(\mathbf{y}^{i*}(t_0), \mathbf{z}^{i*}(t_0), \mathbf{\dot{y}}^{i}(t_0), \mathbf{\dot{z}}^{i}(t_0)) + \phi^i(\mathbf{y}^{i*}(\epsilon), \mathbf{z}^{i*}(\epsilon), \mathbf{\dot{y}}^{i}(\epsilon), \mathbf{\dot{z}}^{i}(\epsilon)). \]

Since \((t_0, \mathbf{y}^{i*}(t_0), \mathbf{z}^{i*}(t_0), \mathbf{\dot{y}}^{i}(t_0), \mathbf{\dot{z}}^{i}(t_0)) \in \mathcal{M}_i\), the target set, the above equation can be re-written as

\[ v^i(\epsilon) = v^i(t_0) - \phi^i(\mathbf{H}^{i*}(t_0), \mathbf{D}^{i*}(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0)) + \phi^i(\mathbf{y}^{i*}(\epsilon), \mathbf{z}^{i*}(\epsilon), \mathbf{\dot{y}}^{i}(\epsilon), \mathbf{\dot{z}}^{i}(\epsilon)). \]

The function \(\eta(\epsilon)\) is determined up to its initial condition, which remains to be constrained for this problem. To achieve this end, the Mayer-form boundary conditions (7.58) and (7.59) of the HJB verification lemma are used. Individually, this gives

\[ W^i(t_0, \mathbf{y}^i(t_0), \mathbf{z}^i(t_0), \mathbf{\dot{y}}^i(t_0), \mathbf{\dot{z}}^i(t_0)) = v^i(t_0) - \phi^i(\mathbf{y}^i(t_0), \mathbf{z}^i(t_0), \mathbf{\dot{y}}^i(t_0), \mathbf{\dot{z}}^i(t_0)) \]
\[ = v^i(t_0) - \phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0)) \]
\[ = \phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0)). \]

This condition requires \(v^i(t_0) = 2\phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0))\) and thus the function is determined completely as

\[ v^i(\epsilon) = 2\phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{\dot{H}}(t_0), \mathbf{\dot{D}}(t_0)) - \phi^i(\mathbf{H}^{i*}(t_0), \mathbf{D}^{i*}(t_0), \mathbf{\dot{H}}^i(t_0), \mathbf{\dot{D}}^i(t_0)) \]
\[ + \phi^i(\mathbf{y}^{i*}(\epsilon), \mathbf{z}^{i*}(\epsilon), \mathbf{\dot{y}}^i(\epsilon), \mathbf{\dot{z}}^i(\epsilon)). \]
The candidate value functions become

$$W^1(\epsilon, Y^1(\epsilon), Z^1(\epsilon), \tilde{Y}^1(\epsilon), \tilde{Z}^1(\epsilon)) = 2\phi^1(H^1(t_0), D^1(t_0), \tilde{H}^1(t_0), \tilde{D}^1(t_0)) - \phi^1(H^{1*}(t_0), D^{1*}(t_0), \tilde{H}^1(t_0), \tilde{D}^1(t_0)) + \phi^1(Y^{1*}(\epsilon), Z^{1*}(\epsilon), \tilde{Y}^1(\epsilon), \tilde{Z}^1(\epsilon)) - \phi^1(Y^1(\epsilon), Z^1(\epsilon), \tilde{Y}^1(\epsilon), \tilde{Z}^1(\epsilon))$$

and

$$W^2(\epsilon, Y^2(\epsilon), Z^2(\epsilon), \tilde{Y}^2(\epsilon), \tilde{Z}^2(\epsilon)) = 2\phi^2(H^2(t_0), D^2(t_0), \tilde{H}^2(t_0), \tilde{D}^2(t_0)) - \phi^2(H^{2*}(t_0), D^{2*}(t_0), \tilde{H}^2(t_0), \tilde{D}^2(t_0)) + \phi^2(Y^{2*}(\epsilon), Z^{2*}(\epsilon), \tilde{Y}^2(\epsilon), \tilde{Z}^2(\epsilon)) - \phi^2(Y^2(\epsilon), Z^2(\epsilon), \tilde{Y}^2(\epsilon), \tilde{Z}^2(\epsilon)).$$

It is not difficult to see that when

$$(\epsilon, Y^1(\epsilon), Z^1(\epsilon), \tilde{Y}^1(\epsilon), \tilde{Z}(\epsilon)) \in M_1$$

the following relations hold

$$Y^1(\epsilon) = H^1(t_0), \ Y^{1*}(\epsilon) = H^{1*}(t_0), \ \tilde{Y}^1(\epsilon) = \tilde{H}^1(t_0)$$
$$Z^1(\epsilon) = D^1(t_0), \ Z^{1*}(\epsilon) = D^{1*}(t_0), \ \tilde{Z}^1(\epsilon) = \tilde{D}^1(t_0)$$

from which it is apparent that the boundary condition is satisfied,

$$W^1(t_0, H^1(t_0), D^1(t_0), \tilde{H}^1(t_0), \tilde{D}^1(t_0)) = 2\phi^1(H^1(t_0), D^1(t_0), \tilde{H}^1(t_0), \tilde{D}^1(t_0))$$
\[-\phi^1(\mathbf{H}^1(t_0), \mathbf{D}^1(t_0), \tilde{\mathbf{H}}^1(t_0), \tilde{\mathbf{D}}^1(t_0))
+ \phi^1(\mathbf{H}^1(t_0), \mathbf{D}^1(t_0), \tilde{\mathbf{H}}^1(t_0), \tilde{\mathbf{D}}^1(t_0))
- \phi^1(\mathbf{H}^1(t_0), \mathbf{D}^1(t_0), \tilde{\mathbf{H}}^1(t_0), \tilde{\mathbf{D}}^1(t_0))
= \phi^1(\mathbf{H}^1(t_0), \mathbf{D}^1(t_0), \tilde{\mathbf{H}}^1(t_0), \tilde{\mathbf{D}}^1(t_0)).\]

Since both candidate value functions have the same form, this result holds for \(W^2(t_0, \mathbf{H}^2(t_0), \mathbf{D}^2(t_0), \tilde{\mathbf{H}}^2(t_0), \tilde{\mathbf{D}}^2(t_0))\) as well. Now the determination of \(v^i(\epsilon)\) as above makes \(W^i(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))\) satisfy the requirements of the HJB verification lemma by construction, and it is also so that the value functions are non-increasing.

To verify this, consider that

\[
\frac{d}{d\epsilon}W^i(\epsilon, \mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon)) = \frac{dv^i(\epsilon)}{d\epsilon} - \frac{d\phi^i(\mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))}{d\epsilon}
= \min_{K_i \in K_i} \left\{ \frac{d\phi^i(\mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))}{d\epsilon} \right\}
- \frac{d\phi^i(\mathbf{y}^i(\epsilon), \mathbf{z}^i(\epsilon), \tilde{\mathbf{y}}^i(\epsilon), \tilde{\mathbf{z}}^i(\epsilon))}{d\epsilon} \leq 0.
\]

Thus all required conditions of the HJB verification lemma have been met. This confirms that

\[
W^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{z}^1(\epsilon), \tilde{\mathbf{y}}^1(\epsilon), \tilde{\mathbf{z}}^1(\epsilon)) = V^1(\epsilon, \mathbf{y}^1(\epsilon), \mathbf{z}^1(\epsilon), \tilde{\mathbf{y}}^1(\epsilon), \tilde{\mathbf{z}}^1(\epsilon))
\]

\[
W^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon)) = V^2(\epsilon, \mathbf{y}^2(\epsilon), \mathbf{z}^2(\epsilon), \tilde{\mathbf{y}}^2(\epsilon), \tilde{\mathbf{z}}^2(\epsilon))
\]

and the determined NZS-CDS controls are optimal,

\[
K^*_1(\epsilon) = -R^1_1(\epsilon)B^T_1(\epsilon) \left( \gamma^1(\epsilon) + \sum_{j=2}^r \left( \frac{\partial g^1(\kappa^1_j(\epsilon), \kappa^1_j(\epsilon))}{\partial \kappa^1_j(\epsilon)} \right) \gamma^1_j(\epsilon) \right)
\]

330
\[
K_1^2(\epsilon) = -R_{22}^{-1}(\epsilon)B_2^2(\epsilon) \left( \mathcal{Y}_{11}^2(\epsilon) + \sum_{j=2}^r \left( \frac{\partial g^2(\kappa^{2+}(\epsilon), \kappa^{2^*}(\epsilon))}{\partial \kappa^{2^*}(\epsilon)} \right) \mathcal{Y}_{2j}^2(\epsilon) \right).
\]

7.4.8 Cost Density-Shaping, N-Player Game

The \textit{N-Player} Nash game is the generalization of the NZS-CDS optimization just considered, and its solution is presented here in a way that parallels the development in [44]. Much of the development follows the solution for the 2-\textit{Player} Nash game, so considerable details are omitted for the sake of brevity. This is merely a sketch of the problem formulation and a statement of the main result. First the process and costs are discussed.

Let \((t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n\) be fixed, and let \(w(t) = W(t, \omega)\) be a \(p\)-dimensional stationary Wiener process on \([t_0, t_f]\) where \(w : [t_0, t_f] \times \Omega \to \mathbb{R}^p\) on the complete probability space \((\Omega, \mathcal{F}, \mathcal{P})\) and the following correlation of increments property is satisfied,

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|, W \succ \mathbf{0}^{p \times p}.
\]

Let \(U_i \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i}))\), \(1 \leq i \leq N\) be Hilbert spaces of \(\mathbb{R}^{m_i}\)-valued, square-integrable processes \(u_i \in U_i\), where by their construction

\[
E\left\{ \int_{t_0}^{t_f} u_i^T(\tau) u_i(\tau) d\tau \right\} < \infty.
\]

Further, let the processes in \(U_i\) be adapted to the \(\sigma\)-field generated by \(w(t)\), \(\mathcal{F}_t\). Consider the problem of \textit{Player} \(i\) choosing strategies \(u_i \in U_i\) so to influence the
states \( x(t) = X(t, \omega) \) of the following linear stochastic differential equation, which belongs to \( L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \) and is adapted to the \( \sigma \)-field generated by \( w(t), \)

\[
\begin{align*}
  dx(t) &= \left( A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) \right) dt + G(t)dw(t), \quad t \in [t_0, t_f] \\
  x_0 &= E\{x(t_0)\}, \; x_0 \in \mathbb{R}^n \\
\end{align*}
\]

where

\[
A \in C([t_0, t_f]; \mathbb{R}^{n \times n}), \; B_i \in C([t_0, t_f]; \mathbb{R}^{n \times m_i}) \quad (1 = 1, 2, \ldots, N), \; G \in C([t_0, t_f]; \mathbb{R}^{n \times p}).
\]

In particular, Player \( i \) chooses \( u_i \) to optimize the statistical characterization of the integral-quadratic cost functional given below,

\[
J_i[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left( x^T(\tau)Q_i(\tau)x(\tau) + \sum_{j=1}^{N} u_j(\tau)^T R_{ij}(\tau)u_j(\tau) \right) d\tau + x^T(t_f)Q_{if}x(t_f). \tag{7.67}
\]

It is understood that \( Q_i \in C([t_0, t_f]; \mathbb{S}^n_+) \), \( R_{ij} \in C([t_0, t_f]; \mathbb{S}^{m_i}_+) \), and \( Q_{if} \in \mathbb{S}^n_+ \) for well-posedness of the problem. Suppose further that players choose their optimal control actions within the class of memoryless, full-observation strategies, or more precisely

\[
\xi_i : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^n)) \to L^2_{\mathcal{F}_t}(\Omega; C([t_0, t_f]; \mathbb{R}^{m_i})).
\]

and

\[
u_i(t) = \xi_i(t, x(t)) = K_i(t)x(t). \tag{7.68}
\]
When the process having dynamics (7.66) is subjected to the controls of each player, where $K_i \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m_i \times n})$ are the admissible control gains with respective compact, allowable sets of gains $\bar{K}_i \subset \mathbb{R}^{m_i \times n}$, it becomes

$$dx(t) = \left( A(t) + \sum_{i=1}^{N} B_i(t)K_i(t) \right) x(t)dt + G(t)dw(t), \quad t \in [t_0,t_f]$$

(7.69)

and the costs (7.67) can be written as

$$J_i[x,u;t_0,x_0] = \int_{t_0}^{t_f} x^T(\tau)N_i(\tau)x(\tau)d\tau + x^T(t_f)Q_{if}x(t_f)$$

(7.70)

where

$$N_i(\tau) = \sum_{j=1}^{N} K_j(\tau)^T R_{ij}(\tau)K_j(\tau) + Q_i(\tau).$$

In the following, the denotation $K_{N-i}$ will be made on occasion and this refers to the set of control gains excluding the $i$th or more precisely,

$$K_{N-i} = \times_{j \neq i} K_j = \underbrace{K_1 \times K_2 \times \cdots \times K_{i-1} \times K_{i+1} \times \cdots \times K_N}_{N-1 \text{ times}}.$$

With this apparatus in place, the Cartesian product of all $N$ control gains can be abbreviated by $K_i \times K_{N-i}$. It is to be understood that $K_{N-i}$ refers to the situation when all players except the $i$th play their Nash strategy. Define variables $H^i(t_0), D^i(t_0), \tilde{H}^i(t_0), \tilde{D}^i(t_0))$ in the same sense as for NZS-CDS, but accounting for $N$ players.
The $r$ cost cumulants for Player $i$ take the following form,

$$\kappa_i^k(\alpha) = x_0^T H_i^k(\alpha) x_0 + D_i^k(\alpha), \ 1 \leq k \leq r$$

where the $H_i^k(\alpha)$ and $D_i^k(\alpha)$ functions satisfy the system of differential equations,

$$\frac{dH_i^1(\alpha)}{d\alpha} = - \left( A(\alpha) + \sum_{j=1}^N B_j(\alpha) K_j(\alpha) \right)^T H_i^1(\alpha)$$

$$- H_i^1(\alpha) \left( A(\alpha) + \sum_{j=1}^N B_j(\alpha) K_j(\alpha) \right)$$

$$- N_i(\alpha) \triangleq F_1(H_i^1(\alpha), K_i(\alpha), K_{N-i}(\alpha))$$

$$\frac{dH_i^k(\alpha)}{d\alpha} = - \left( A(\alpha) + \sum_{j=1}^N B_j(\alpha) K_j(\alpha) \right)^T H_i^k(\alpha)$$

$$- H_i^k(\alpha) \left( A(\alpha) + \sum_{j=1}^N B_j(\alpha) K_j(\alpha) \right)$$

$$- 2 \sum_{j=1}^{k-1} \binom{k}{j} H_j^1(\alpha) G(\alpha) W G^T(\alpha) H_{k-j}(\alpha),$$

$$\triangleq F_k(H_i^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)), \ 2 \leq k \leq r$$

$$\frac{dD_i^1(\alpha)}{d\alpha} = - \text{Tr}(H_i^1(\alpha) G(\alpha) W G^T(\alpha))$$

$$\triangleq G_k(H_i^i(\alpha)), \ \alpha \in [t_0, t_f], \ 1 \leq k \leq r.$$
\[ H^i(\alpha) \triangleq \begin{bmatrix} H^i_1(\alpha) \\ \vdots \\ H^i_r(\alpha) \end{bmatrix}, \quad D^i(\alpha) \triangleq \begin{bmatrix} D^i_1(\alpha) \\ \vdots \\ D^i_r(\alpha) \end{bmatrix}, \quad 1 \leq i \leq N. \]

Using these state variables, define the functions

\[ \mathcal{F}(H^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)) \triangleq \begin{bmatrix} \mathcal{F}_1(H^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)) \\ \vdots \\ \mathcal{F}_r(H^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)) \end{bmatrix}, \quad \mathcal{G}(H^i(\alpha)) \triangleq \begin{bmatrix} \mathcal{G}_1(H^i(\alpha)) \\ \vdots \\ \mathcal{G}_r(H^i(\alpha)) \end{bmatrix}, \quad 1 \leq i \leq N. \]

Let \( \{\mathcal{F}_j(\cdot)\}_{j=1}^r \) and \( \{\mathcal{G}_j(\cdot)\}_{j=1}^r \) in the above definitions be defined as beforehand in (7.71). Also a condensed form for the terminal conditions is introduced as below.

\[ H^i_f \triangleq \begin{bmatrix} Q_{i_f} \\ 0^{n \times n} \\ \vdots \\ 0^{n \times n} \end{bmatrix}, \quad D_f \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \]

Finally, denote the cost cumulant vectors \( \kappa^i(\alpha) \) as

\[ \kappa^i(\alpha) \triangleq \begin{bmatrix} \kappa^i_1(\alpha) \\ \vdots \\ \kappa^i_r(\alpha) \end{bmatrix}, \quad 1 \leq i \leq N. \]
Using this notation, the equations (7.71) and their associated terminal condition systems can be written concisely as

\[
\frac{dH^i(\alpha)}{d\alpha} = F(H^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)), \quad \frac{dD^i(\alpha)}{d\alpha} = G(H^i(\alpha))
\]

\[
H^i(t_f) = H_f^i, \quad D^i(t_f) = D_f, \quad \alpha \in [t_0, t_f].
\]

Also, the target statistics for Player \( i \) can be written as follows,

\[
\frac{d\tilde{H}^i(\alpha)}{d\alpha} = F(\tilde{H}^i(\alpha), \tilde{K}_i(\alpha), \tilde{K}_{N-i}(\alpha)), \quad \frac{d\tilde{D}^i(\alpha)}{d\alpha} = G(\tilde{H}^i(\alpha))
\]

\[
\tilde{H}^i(t_f) = \tilde{H}_f^i, \quad \tilde{D}^i(t_f) = \tilde{D}_f, \quad \alpha \in [t_0, t_f]
\]

(7.73)

where

\[
\tilde{H}_1(t_f) = Q_{if} + \varepsilon_i^*, \quad \tilde{H}_j(t_f) = 0^{n \times n}, \quad j \geq 2
\]

\[
\tilde{D}_1(t_f) = \varepsilon_i^*, \quad \tilde{D}_2(t_f) = 1, \quad \tilde{D}_j(t_f) = 0, \quad j \geq 3
\]

(7.74)

and the short-hand notation is used,

\[
H_{f,\varepsilon_i}^i \triangleq \begin{bmatrix}
Q_{if} + \varepsilon_i^*\\
0^{n \times n}\\
\vdots\\
0^{n \times n}
\end{bmatrix}, \quad D_{f,\varepsilon_i}^i \triangleq \begin{bmatrix}
\varepsilon_i^*\\
1\\
0\\
\vdots\\
0
\end{bmatrix}, \quad 1 \leq i \leq N.
\]

Here \( \varepsilon_i^* > 0 \) are small perturbation constants, and \( \varepsilon_i^* > 0^{n \times n} \) are positive-definite perturbation matrices. As with the cost cumulants, compose vectors of
target initial cost cumulants $\tilde{\kappa}^i(\alpha) \in \mathbb{R}^r$ defined as below,

$$\tilde{\kappa}^i(\alpha) \triangleq \begin{bmatrix} \tilde{\kappa}_1^i(\alpha) \\ \vdots \\ \tilde{\kappa}_r^i(\alpha) \end{bmatrix}, \quad 1 \leq i \leq N.$$

The target set and admissible space of control gains characterizing linear controls for which the evolution equations are solvable have been presented previously for the ZS-CDS and NZS-CDS optimizations, and natural analogues are made here.

**Definition 7.4.15 (Target Sets, N-Player CDS Game)**

Let $(t_0, H^i(t_0), D^i(t_0), \tilde{H}^i(t_0), \tilde{D}^i(t_0)) \in \mathcal{M}_i$, where $\mathcal{M}_i$ denotes the target set for the Player $i$ which is a closed subset of

$$[t_0, t_f] \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \ldots \mathbb{R}^{n \times n}) \times \mathbb{R}^r \times (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \ldots \mathbb{R}^{n \times n}) \times \mathbb{R}^r.$$

The combined target space is closed, $\bigcup_{i=1}^N \mathcal{M}_i$.

For given terminal conditions, the sets of admissible feedback gains are denoted as

$$K_{t_f, H^i(t_f), D^i(t_f), \tilde{H}^i(t_f), \tilde{D}^i(t_f)}, \quad 1 \leq i \leq N$$

and contain matrices $K_i \in C([t_0, t_f]; \mathbb{R}^{m_i \times n})$ such that

$$(t_0, H^i(t_0), D^i(t_0), \tilde{H}^i(t_0), \tilde{D}^i(t_0)) \in \mathcal{M}_i, \quad 1 \leq i \leq N$$

is obtained at the end of the trajectories for the state equations (7.71) and (7.73). This is formally stated in the following definition.

**Definition 7.4.16 (Admissible Feedback Gains, N-Player CDS Game)**
Denote allowable sets of control gain values by $K_i \subset \mathbb{R}^{m_i \times n}$ and let these sets be compact. For fixed $r \in \mathbb{N}$ let $\mathcal{K}_i^{t_j,\mathbf{H}^{(t_j)},\mathbf{D}^{(t_j)},\mathbf{H}^{(t_f)},\mathbf{D}^{(t_f)}} \triangleq \mathcal{K}^i(t_f)$ characterize a class of $\mathcal{C}(t_0, t_f; \mathbb{R}^{m_i \times n})$ such that for $K_i \in \mathcal{K}^i(t_f)$, $1 \leq i \leq N$ the solutions to

$$
\frac{d\mathbf{H}^i(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}^i(\alpha), K_i(\alpha), K_{-i}(\alpha)), \quad \frac{d\mathbf{D}^i(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}^i(\alpha))
$$

exist on $\alpha \in [t_0, t_f]$ and the initial values of the state trajectories satisfy

$$(t_0, \mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{H}^i(t_0), \mathbf{D}^i(t_0)) \in \mathcal{M}_i, \quad 1 \leq i \leq N.$$

Consider scalar functions $g^i : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ with vector arguments as general performance indices, which are denoted by $g^i(\mathbf{x}, \mathbf{\hat{x}})$. For fixed $\mathbf{\hat{x}}$, the function becomes $g^i_{\mathbf{\hat{x}}} : \mathbb{R}^r \to \mathbb{R}$. Analogously for fixed $\mathbf{x}$, the function becomes $g^i_{\mathbf{x}} : \mathbb{R}^r \to \mathbb{R}$. Impose the following restrictions on $g^i_{\mathbf{\hat{x}}}(\mathbf{x})$ and $g^i_{\mathbf{x}}(\mathbf{\hat{x}})$ to ensure that the ensuing optimization problem is well-posed:

- The function $g^i_{\mathbf{\hat{x}}}$ is analytic on $\text{dom} \ g^i_{\mathbf{\hat{x}}}$ and $g^i_{\mathbf{x}}$ is analytic on $\text{dom} \ g^i_{\mathbf{x}}$
- The function $g^i_{\mathbf{\hat{x}}}$ is convex in $\mathbf{x}$ and its domain $\text{dom} \ g^i_{\mathbf{\hat{x}}}$ is a convex set
- The function $g^i_{\mathbf{x}}$ is non-negative in $\mathbf{x}$ on some neighborhood of $\mathbf{\hat{x}}$

**Definition 7.4.17 (Performance Indices, N-Player CDS Game)**

Let the N-Player CDS performance indices be defined as

$$
\phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \mathbf{H}^i(t_0), \mathbf{D}^i(t_0)) = g^i(\mathbf{\kappa}^i(t_0), \mathbf{\bar{\kappa}}^i(t_0)), \quad 1 \leq i \leq N.
$$

The concept of an N-Player Nash equilibrium solution is defined in the following.

**Definition 7.4.18 (Nash Equilibrium, N-Player CDS Game)**

Consider a set of player strategies $\{K^*_j\}_{j=1}^N$ such that $K^*_j \in \mathcal{K}^j(t_f), 1 \leq j \leq N$. The strategies $\{K^*_j\}_{j=1}^N$ constitute a Nash equilibrium solution to the N-Player CDS game, if the following conditions are satisfied.

Suppose $\forall K_i \neq K^*_i \mid K_i \in \mathcal{K}^i(t_f)$ and for $1 \leq i \leq N$, it is true that

$$
\phi^i(\mathbf{H}^i(t_0; K^*_i, K^*_{-i}), \mathbf{D}^i(t_0; K^*_i, K^*_{-i}), \mathbf{H}^i(t_0), \mathbf{D}^i(t_0))
$$
\[ \leq \phi^i(H^i(t_0; K_i, K^*_{N-1}), D^i(t_0; K_i, K^*_{N-1}), \tilde{H}^i(t_0), \tilde{D}^i(t_0)). \]

The N-Player CDS game can now be introduced given the framework that has been presented.

**Definition 7.4.19 (N-Player CDS Game)**
For every \( \tilde{\kappa}^i(t_0) \), let \( g^i(\kappa^i(t_0), \tilde{\kappa}^i(t_0)) \) be an analytic function, convex in \( \kappa^i(t_0) \), defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of \( \tilde{\kappa}^i(t_0) \). Let \( r \in \mathbb{N} \) be a fixed positive integer, where \( \kappa^i(t_0), \tilde{\kappa}^i(t_0) \in \mathbb{R}^r \) are the vectors of initial cost cumulants and target initial cost cumulants, respectively, for Player \( i \). Then the N-Player CDS game can be formulated as,

\[
\begin{align*}
\min_{\kappa^i \in \mathbb{K}^i(t_f)} & \quad \phi^i(H^i(t_0), D^i(t_0), \tilde{H}^i(t_0), \tilde{D}^i(t_0)), \quad 1 \leq i \leq N \\
\text{subject to:} & \\
\frac{dH^i(\alpha)}{d\alpha} = & \mathcal{F}(H^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)), \quad \frac{dD^i(\alpha)}{d\alpha} = \mathcal{G}(H^i(\alpha)) \\
H^i(t_f) = & H^i_f, \quad D^i(t_f) = D_f \\
\frac{d\tilde{H}^i(\alpha)}{d\alpha} = & \mathcal{F}(\tilde{H}^i(\alpha), \tilde{K}_i(\alpha), \tilde{K}_{N-i}(\alpha)), \quad \frac{d\tilde{D}^i(\alpha)}{d\alpha} = \mathcal{G}(\tilde{H}^i(\alpha)), \quad 1 \leq i \leq N \\
\tilde{H}^i(t_f) = & \tilde{H}^i_{f; \xi}, \quad \tilde{D}^i(t_f) = \tilde{D}_{f; \xi}; \quad (7.75)
\end{align*}
\]

exist on \( \alpha \in [t_0, t_f] \) and the initial values of the state trajectories satisfy

\[ (t_0, H^i(t_0), D^i(t_0), \tilde{H}^i(t_0), \tilde{D}^i(t_0)) \in \mathcal{M}_i, \quad 1 \leq i \leq N. \]

The discussion will now focus upon the Nash solution to the N-Player CDS game.

**Theorem 7.4.20 (Nash Equilibrium Solution, N-Player CDS Game)**
Consider the LQG stochastic optimal control problem involving the process having dynamics (7.69) and the costs (7.70). Then Player \( i \)’s Nash equilibrium solution to the N-Player game is characterized by the optimal gain

\[
K^*_i(\alpha) = -R^{-1}_i(\alpha)B_i^T(\alpha) \left( H^*_i(\alpha) + \sum_{j=2}^r \frac{\partial g^i(\kappa^*_i(\alpha), \tilde{\kappa}^i(\alpha))}{\partial \kappa^i_j(\alpha)} \frac{\partial \kappa^i_j(\alpha)}{\partial \kappa^i_j(\alpha)} H^*_j(\alpha) \right)
\]
where the optimal cost cumulants for Player $i$ are defined by
\[
\kappa^*_j(\alpha) = x_0^T H^*_j(\alpha)x_0 + D^*_j(\alpha)
\]
and the target cost cumulants for Player $i$ are,
\[
\tilde{k}^*_j(\alpha) = x_0^T \tilde{H}^*_j(\alpha)x_0 + \tilde{D}^*_j(\alpha)
\]
where $1 \leq i \leq r$. The optimal state variables $H^*_i(\alpha)$ and $D^*_i(\alpha)$ for $1 \leq i \leq N$ follow the equations of motion
\[
\frac{dH^*_i(\alpha)}{d\alpha} = F(H^*_i(\alpha), K_i(\alpha), K_{N-i}(\alpha)), \quad \frac{dD^*_i(\alpha)}{d\alpha} = G(H^*_i(\alpha)), \quad \alpha \in [t_0, t_f]
\]
\[
H^*_i(t_f) = H^*_f, \quad D^*_i(t_f) = D^*_f
\]
and the target cost statistics for all $N$ players evolve according to
\[
\frac{d\tilde{H}^*_i(\alpha)}{d\alpha} = F(\tilde{H}^*_i(\alpha), \tilde{K}_i(\alpha), \tilde{K}_{N-i}(\alpha)), \quad \frac{d\tilde{D}^*_i(\alpha)}{d\alpha} = G(\tilde{H}^*_i(\alpha)), \quad \alpha \in [t_0, t_f]
\]
\[
\tilde{H}^*_i(t_f) = \tilde{H}^*_f, \quad \tilde{D}^*_i(t_f) = \tilde{D}^*_f
\]

\textbf{Proof} The proof follows from a straightforward extension of approach used for NZS-CDS in Theorem 7.4.14.

7.5 Simulation Results

In [68] and [35], a 4-story structure is considered. A cost density-shaping game will be posed using this system, which has two disturbances - the ground excitation, and also uncertainties in the system. These system uncertainties pertain to the stiffness and damping matrices, and also those dealing with the control input itself. How the aforementioned system uncertainties influence the system state will be shown below. The system matrices in the model are given in the following.

First consider the parameters $k = 350 \times 10^6$ N/m, $m = 1.05 \times 10^6$ kg, and $c = 1.575 \times 10^6$ Ns/m. Using these values, the stiffness, damping, and mass matrices
(denoted as $K$, $C$, $M$ respectively) can be defined as

$$ K = \begin{bmatrix} 4k & -2k & 0 & 0 \\ -2k & 3k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix}, \quad C = \begin{bmatrix} 2c & -c & 0 & 0 \\ -c & 2c & -c & 0 \\ 0 & -c & 2c & -c \\ 0 & 0 & -c & c \end{bmatrix}, \quad M = \begin{bmatrix} 2m & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}. $$

In terms of these matrices, the system matrices are

$$ A = \begin{bmatrix} 0^{4\times4} & I^{4\times4} \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B = \begin{bmatrix} 0^{4\times4} \\ -M^{-1}B_{ch} \end{bmatrix}, \quad G = \begin{bmatrix} 0^{4\times1} \\ F_w \end{bmatrix} $$

with $B_{ch}$ and $F_w$ given as

$$ B_{ch} = I^{4\times4} + \begin{bmatrix} 0^{3\times1} & -I^{3\times3} \\ 0 & 0^{1\times3} \end{bmatrix}, \quad F_w = \frac{1}{m} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. $$

To represent how uncertainties impact the dynamics of the system, the following matrix is introduced as in [58].

$$ \begin{bmatrix} 0^{4\times4} & 0^{4\times4} & 0^{4\times4} \\ -M & -M & -M^{-1}B_{ch} \end{bmatrix} $$

The dynamics of the 4-story structure with the ground acceleration $\ddot{x}_g(t)$ due to the earthquake, accounting for system uncertainties, are given by the model,

$$ dx(t) = \left( Ax(t) + Bu_1(t) + Du_2(t) \right) dt + G\ddot{x}_g(t), x_0 = E\{x(t_0)\} $$
\[ z_1(t) = H_1 x(t) + G_1 u_1(t) \]
\[ z_2(t) = H_2 x(t) + G_2 u_1(t), \quad t \in [t_0, t_f] \]

Above, the regulated outputs \( z_1(t) \) and \( z_2(t) \) are characterized by \( H_1, H_2 \) and \( G_1, G_2 \) given below as

\[
H_1 = 10^6 \begin{bmatrix} I_{8 \times 8} \ 0_{4 \times 8} \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0_{8 \times 4} \ I_{4 \times 4} \end{bmatrix}
\]
\[
H_2 = \begin{bmatrix} 0.1 K & 0_{4 \times 4} \\ 0_{4 \times 4} & -0.1 C \\ 0_{4 \times 4} & 0_{4 \times 4} \end{bmatrix}, \quad G_2 = \frac{1}{8 \times 10^5} \begin{bmatrix} 0_{8 \times 4} \ I_{4 \times 4} \end{bmatrix}
\]

Now, form a NZS-CDS game as follows. Let Player 1 be the control designer, who is interested in choosing \( u_1(t) = K_1(t)x(t) \) to optimize the statistical characterization (the probability density) of his/her random payoff,

\[ J_1 = \int_{t_0}^{t_f} z_1^T(\tau) z_1(\tau) d\tau \]

On the other hand, let Player 2 be the second disturbance (e.g. the system uncertainties), who is interested in choosing \( u_2(t) = K_2(t)x(t) \) to optimize the statistical characterization (the probability density) of its random payoff,

\[ J_2 = \int_{t_0}^{t_f} \delta^2 u_2^T(\tau) u_2(\tau) - z_2^T(\tau) z_2(\tau) d\tau, \quad \delta = 20 \]

With this choice of \( J_2 \), note that its cumulants are no longer guaranteed to be positive. Since the MCCDS theory doesn’t directly minimize cost (or payoff) cumulants, but rather a distance function between finite-cumulant approximations.
to the cost density and the target density. Since the distance function as the MC- CDS performance index is bounded below, the negativeness of \( J_2 \)'s cumulants (e.g. its mean) presents no compromise to well-posedness of the cost density-shaping optimization. This formulation for costs is taken from the generalization of \( H_2/H_\infty \) control with multi-objective, multi-cumulant control attributed to Diersing [58].

The elements of the NZS-CDS game that remain to be considered are the performance index of each player, and also each player’s target distribution for \( J_1 \) and \( J_2 \). To facilitate the discussion, once again normalized cost and target cost variates are introduced for \( j = 1, 2 \),

\[
Z_i = \frac{J_i - \kappa_1^i(t_0)}{\kappa_2^i(t_0)^{1/2}}
\]

\[
\tilde{Z}_i = \frac{J_i - \tilde{\kappa}_1^i(t_0)}{\tilde{\kappa}_2^i(t_0)^{1/2}} = \left( \frac{\kappa_2^i(t_0)}{\tilde{\kappa}_2^i(t_0)} \right)^{1/2} \cdot J_1 \cdot \frac{\kappa_1^i(t_0) - \tilde{\kappa}_1^i(t_0)}{\tilde{\kappa}_2^i(t_0)^{1/2} \kappa_i} \right)
\]

\[= a_i Z_i + b_i.\]

with their best Gaussian density approximations,

\[
p_{Z_1}(z) \approx \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right), \quad p_{\tilde{Z}_1}(\tilde{z}) \approx \frac{a_i}{\sqrt{2\pi}} \exp \left( -\frac{(a_i \tilde{z} + b_i)^2}{2} \right) = a_i p_{Z_1}(a_i \tilde{z} + b_i).
\]

Consider the performance index for Player 1 first,

\[
KLD(p_{Z_1}(z), p_{\tilde{Z}_1}(\tilde{z})) = \int_{-\infty}^{\infty} p_{Z_1}(z) \log \left( \frac{p_{Z_1}(z)}{p_{\tilde{Z}_1}(\tilde{z})} \right) dz
\]

\[= \frac{1}{2} \left( \frac{\kappa_2^i(t_0)}{\tilde{\kappa}_2^i(t_0)} + 1 - \log \left( \frac{\kappa_2^i(t_0)}{\tilde{\kappa}_2^i(t_0)} \right) + \frac{(\kappa_1^i(t_0) - \tilde{\kappa}_1^i(t_0))^2}{\tilde{\kappa}_2^i(t_0)^{1/2} \kappa_i} \right)
\]

\[= g_1\left( \begin{bmatrix} \kappa_1^i(t_0) \\ \kappa_2^i(t_0) \end{bmatrix}, \begin{bmatrix} \tilde{\kappa}_1^i(t_0) \\ \tilde{\kappa}_2^i(t_0) \end{bmatrix} \right).\]
Next, consider the performance index for Player 2,

\[
HD^2(p_{Z_2}(z), p_{\tilde{Z}_2}(z)) = 1 - \int_{-\infty}^{\infty} \sqrt{p_{Z_2}(z)p_{\tilde{Z}_2}(z)} \, dz
\]

\[
= 1 - \sqrt{2} \cdot \frac{(\kappa_2(t_0)\tilde{\kappa}_2(t_0))^4}{\kappa_2^4(t_0) + \tilde{\kappa}_2^4(t_0)} \cdot \exp \left( -\frac{(\kappa_1^2(t_0) - \tilde{\kappa}_1^2(t_0))^2}{4(\kappa_2^2(t_0) + \tilde{\kappa}_2^2(t_0))} \right)
\]

\[
= g_2\left( \begin{bmatrix} \kappa_1^2(t_0) \\ \kappa_2^2(t_0) \end{bmatrix}, \begin{bmatrix} \tilde{\kappa}_1^2(t_0) \\ \tilde{\kappa}_2^2(t_0) \end{bmatrix} \right).
\]

In the above expressions, the cumulants of each player’s cost are,

\[
\kappa_j^i(t_0) = x_0^T H_1^i(\alpha)x_0 + D_1^j(\alpha), \quad 1 \leq i, j \leq 2
\]

where for Player 1,

\[
F(\alpha) = \left( A + BK_1(\alpha) + DK_2(\alpha) \right)
\]

\[
\frac{dH_1^i(\alpha)}{d\alpha} = -F^T(\alpha)H_1^i(\alpha) - H_1^i(\alpha)F(\alpha) - Q_1
\]

\[
- K_1^T(\alpha)R_{11}K_1(\alpha) - K_2^T(\alpha)R_{12}K_2(\alpha)
\]

\[
\frac{dH_2^i(\alpha)}{d\alpha} = -F^T(\alpha)H_2^i(\alpha) - H_2^i(\alpha)F(\alpha) - 4H_1^i(\alpha)GWG^T H_1^i(\alpha)(\alpha)
\]

\[
\frac{dD_1^j(\alpha)}{d\alpha} = -\text{Tr}(H_1^j(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, \ t_f], \ 1 \leq j \leq 2
\]

These functions satisfy the terminal conditions

\[
H_1^1(t_f) = Q_f, \quad H_2^1(t_f) = 0^{n \times n}, \quad D_1^1(t_f) = 0, \quad D_2^1(t_f) = 1.
\]

For Player 2, the equations for the cost cumulant-generating equations are

\[
F(\alpha) = \left( A + BK_1(\alpha) + DK_2(\alpha) \right)
\]

344
\[
\begin{align*}
\frac{dH_{21}^2(\alpha)}{d\alpha} &= -F^T(\alpha)H_{12}^2(\alpha) - H_{12}^2(\alpha)F(\alpha) - Q_2 \\
&- K_{12}^T(\alpha)R_{21}K_{11}(\alpha) - K_{12}^T(\alpha)R_{22}K_{22}(\alpha) \\
\frac{dH_{22}^2(\alpha)}{d\alpha} &= -F^T(\alpha)H_{22}^2(\alpha) - H_{22}^2(\alpha)F(\alpha) - 4H_{12}^2(\alpha)GWG^T H_{11}^2(\alpha)(\alpha) \\
\frac{dD_{2j}^2(\alpha)}{d\alpha} &= -\text{Tr}(H_{2j}^2(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, \ t_f], \ 1 \leq j \leq 2
\end{align*}
\]

These functions satisfy the terminal conditions

\[
H_{12}^2(t_f) = Q_f, \quad H_{22}^2(t_f) = 0, \quad D_{11}^2(t_f) = 0, \quad D_{22}^2(t_f) = 1.
\]

Above, the optimal NZS-CDS controls are taken,

\[
K_{1i}(\alpha) = -R_{11}^{-1}B^T \left( H_{1i}^1(\alpha) + \frac{\partial g_1(\kappa_1(\alpha), \tilde{\kappa}_1(\alpha))}{\partial \kappa_1^1(\alpha)} \right) \frac{H_{1i}^2(\alpha)}{\partial \kappa_1^1(\alpha)}
\]

\[
K_{2j}(\alpha) = -R_{22}^{-1}D^T \left( H_{2j}^2(\alpha) + \frac{\partial g_2(\kappa_2(\alpha), \tilde{\kappa}_2(\alpha))}{\partial \kappa_2^1(\alpha)} \right) \frac{H_{2j}^2(\alpha)}{\partial \kappa_2^1(\alpha)}.
\]

The optimal gains involve the target cumulants for each player, which have been computed according to

\[
\tilde{\kappa}_i^j(t_0) = x_0^T \tilde{H}_i^j(\alpha)x_0 + \tilde{D}_j^i(\alpha), \quad 1 \leq i, j \leq 2
\]

where the functions \(\tilde{H}_i^j(\alpha)\) and \(\tilde{D}_j^i(\alpha)\) are determined by

\[
\tilde{F}(\alpha) = \left( A + B\tilde{K}_1(\alpha) + \tilde{D}\tilde{K}_2(\alpha) \right)
\]
\[
\frac{d\tilde{H}_1^1(\alpha)}{d\alpha} = -\tilde{F}^T(\alpha)\tilde{\tilde{H}}_1^1(\alpha) - \tilde{\tilde{H}}_1^1(\alpha)\tilde{F}(\alpha) - Q_1 \\
- \tilde{K}_1^T(\alpha)R_{11}\tilde{K}_1(\alpha) - \tilde{K}_2^T(\alpha)R_{12}\tilde{K}_2(\alpha)
\]

\[
\frac{d\tilde{H}_1^2(\alpha)}{d\alpha} = -\tilde{F}^T(\alpha)\tilde{\tilde{H}}_2^1(\alpha) - \tilde{\tilde{H}}_2^1(\alpha)\tilde{F}(\alpha) - 4\tilde{H}_1^2(\alpha)GWG^T\tilde{\tilde{H}}_1^1(\alpha)(\alpha)
\]

\[
\frac{d\tilde{D}_1^j(\alpha)}{d\alpha} = -\text{Tr}(\tilde{\tilde{H}}_1^j(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, t_f], \quad 1 \leq j \leq 2
\]

with terminal conditions

\[
\tilde{H}_1^1(t_f) = Q_f + \mathcal{E}^*, \quad \tilde{H}_1^2(t_f) = 0^{n \times n}, \quad \tilde{D}_1(t_f) = \epsilon^*, \quad \tilde{D}_2(t_f) = 1
\]

and also \( \tilde{H}_2^j(\alpha) \) and \( \tilde{D}_2^j(\alpha) \) are given by

\[
\tilde{F}(\alpha) = \left( A + B\tilde{K}_1(\alpha) + \tilde{D}\tilde{K}_2(\alpha) \right)
\]

\[
\frac{d\tilde{H}_2^2(\alpha)}{d\alpha} = -\tilde{F}^T(\alpha)\tilde{\tilde{H}}_2^2(\alpha) - \tilde{\tilde{H}}_2^2(\alpha)\tilde{F}(\alpha) - Q_2 \\
- \tilde{K}_1^T(\alpha)R_{21}\tilde{K}_1(\alpha) - \tilde{K}_2^T(\alpha)R_{22}\tilde{K}_2(\alpha)
\]

\[
\frac{d\tilde{H}_2^2(\alpha)}{d\alpha} = -\tilde{F}^T(\alpha)\tilde{\tilde{H}}_2^2(\alpha) - \tilde{\tilde{H}}_2^2(\alpha)\tilde{F}(\alpha) - 4\tilde{H}_1^2(\alpha)GWG^T\tilde{\tilde{H}}_1^1(\alpha)(\alpha)
\]

\[
\frac{d\tilde{D}_2^2(\alpha)}{d\alpha} = -\text{Tr}(\tilde{\tilde{H}}_2^j(\alpha)G(\alpha)WG^T(\alpha)), \quad \alpha \in [t_0, t_f], \quad 1 \leq j \leq 2
\]

with terminal conditions

\[
\tilde{H}_2^1(t_f) = Q_f + \mathcal{E}^*, \quad \tilde{H}_2^2(t_f) = 0^{n \times n}, \quad \tilde{D}_1(t_f) = \epsilon^*, \quad \tilde{D}_2(t_f) = 1
\]

where \( \mathcal{E}^* = 0 \) and \( \epsilon^* = 1.0 \times 10^{-9} \). For Player 1 and Player 2, choose the control
gains to drive the target cost cumulants as below, with $\mu_1^2 = 1.0 \times 10^{-5}$, $\mu_2^2 = 0$. 

$$\tilde{K}_1(\alpha) = -R_{11}^{-1}B^T\left(\bar{H}_1^1(\alpha) + \mu_1^2\bar{H}_2^1(\alpha)\right), \quad \tilde{K}_2(\alpha) = -R_{22}^{-1}D^T\left(\bar{H}_1^2(\alpha) + \mu_2^2\bar{H}_2^2(\alpha)\right).$$

By numerical simulation, it can be verified that the NZS-CDS controls approximately realize the target mean-variance approximations to the target densities for the random costs $J_1$ and $J_2$. The target cumulants are shown with the broken-dotted red line, while the computed cumulants under the NZS-CDS controls above are represented by the blue line in Figure 7.1.
Figure 7.1: Cumulant Trajectories, NZS-CDS Game
8.1 Introduction

To this point, the presented Cost Density-Shaping (CDS) theory has mainly been supported by numerical experiments that validate the method. More precisely, known $k$CC controls have been used to compute cumulants of the “cost-to-go” using the cumulant-generating equations of the Linear Quadratic Gaussian (LQG) framework. These cumulants are fed back into one of the many Multiple-Cumulant Cost Density-Shaping (MCCDS) controls studied in this work, and the resulting cumulants of the cost-to-go are compared side-by-side with the cost cumulants. It has been observed that the performance of MCCDS control approximately matches that of the controls generating the target statistics. Furthermore, the robust stability of the control underlying the targets is not compromised with the MCCDS control. Together, these results suggest that the performance and robust stability of a controller underlying the cost cumulants are embedded in the statistical characterization achieved for the cost.

The observations noted above are positive ones from a theoretical standpoint. By in large, it seems likely that the traditional cumulant optimizations such as LQG, Risk-Sensitive (RS) control, and “$k$ Cost Cumulant” ($k$CC) control have all led to high-performance, stable controllers where the achieved control system
behavior stems more from an optimal “shape” of the cost density versus optimal cost cumulants (e.g. the minimal mean cost of LQG, et cetera). To reinforce this point, the goal of this chapter is to propose a methodology by which alternative statistical characterizations of the cost may be identified under which there exists considerable performance gains on nominal high-performance controllers used for target cost cumulant generation. The procedure developed in this chapter provides a way for the control designer to formulate linear control laws that are optimal with respect to a class of target cost density functions. Formulating control laws with the aforementioned optimality among a family of distributions might be regarded as a new kind of robustness for decision under uncertainty.

This chapter is organized as follows. An overview of the Statistical Target Selection (STS) is provided, followed by some theoretical developments that explain why the computations are feasible in practice. After this, some material is dedicated to the numerical experiments that illustrate how STS is used in practice, and what performance gains are possible on nominal $kCC$ controls. This material establishes the STS theory as applied in [45-46], and [48].

8.2 Overview

An overview of STS will now be provided. This discussion begins with the presentation of the framework for the problem. As before, consider the process $x(t)$ determined by the linear stochastic differential equation

$$dx(t) = \left( A(t)x(t) + B(t)u(t) \right) dt + G(t)dw(t), \quad t \in [t_0, t_f]$$

(8.1)

$$x_0 = E\{x(t_0)\}, \quad x_0 \in \mathbb{R}^n$$
where \( A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m}) \), \( G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p}) \) and \( w(t) \) is a \( p \)-dimensional Wiener process having a correlation of increments defined by

\[
E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W_{|\tau_1 - \tau_2|}, W \succ 0^{p \times p}.
\]

Control inputs are assumed to be \( \mathbb{R}^{m} \)-valued, square-integrable, non-anticipating processes \( u \in U \subset L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m})) \) adapted to the filtration \( \mathcal{F}_t \) generated by \( w(t) \). Under this assumption, \( x \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{n})) \). The control input \( u \) is chosen with respect to the cost \( J[x, u; t_0, x_0] \), which is integral-quadratic and defined by

\[
J[x, u; t_0, x_0] = \int_{t_0}^{t_f} \left( x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau) \right) d\tau + x^T(t_f)Q_fx(t_f)
\]  

(8.2)

where \( Q \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n}) \), \( R \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times m}) \), and \( Q_f \in \mathbb{R}^{n \times n} \). For the well-posedness of the associated stochastic optimal control problem, it is imposed further that \( Q, Q_f \succeq 0^{n \times n} \), \( R \succ 0^{m \times m} \).

8.2.1 Cost Cumulants

The process is subjected to linear state-feedback control \( u = Kx \), \( K \in \mathcal{C}([t_0, t_f]; \mathbb{K}) \), where \( \mathbb{K} \subset \mathbb{R}^{m \times n} \) with \( \mathbb{K} \) compact. Under this control input, the equations governing the dynamics of the process become

\[
dx(t) = \left( A(t) + B(t)K(t) \right)x(t)dt + G(t)dw(t), \ t \in [t_0, t_f]
\]

\[
x_0 = E\{x(t_0)\}, \ x_0 \in \mathbb{R}^{n}
\]
and the cost can be written as

\[ J[x, u; t_0, x_0] = \int_{t_0}^{t_f} x^T(\tau)N(\tau)x(\tau)d\tau + x^T(t_f)Qx(t_f) \]

where \( N(\tau) = K^T(\tau)R(\tau)K(\tau) + Q(\tau) \).

The work of Liberty [61-62] established that under the foregoing assumptions of the cost being IQF and the control being linear state-feedback, the initial cost cumulants have the following quadratic form

\[ \kappa_i(\alpha) = x_0^T H_i(\alpha)x_0 + D_i(\alpha), \quad 1 \leq i \leq r \]

when \( \alpha = t_0 \). The functions \( H_i(\alpha) \) satisfy the following system of backwards-in-time, matrix differential equations, and the dynamics of the functions \( D_i(\alpha) \) depend upon the \( H_i(\alpha) \) functions,

\[
\frac{dH_1(\alpha)}{d\alpha} = -\left(A(\alpha) + B(\alpha)K(\alpha)\right)^T H_1(\alpha) - H_1(\alpha)\left(A(\alpha) + B(\alpha)K(\alpha)\right) \\
- K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha) \triangleq F_1(\mathbf{H}(\alpha), K(\alpha))
\]

\[
\frac{dH_i(\alpha)}{d\alpha} = -\left(A(\alpha) + B(\alpha)K(\alpha)\right)^T H_i(\alpha) - H_i(\alpha)\left(A(\alpha) + B(\alpha)K(\alpha)\right) \\
- 2 \sum_{j=1}^{i-1} \left( \begin{array}{c} i \\ j \end{array} \right) H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha) \triangleq F_i(\mathbf{H}(\alpha), K(\alpha)), \quad 2 \leq i \leq r
\]

\[
\frac{dD_j(\alpha)}{d\alpha} = -\text{Tr}\left(H_j(\alpha)G(\alpha)WG^T(\alpha)\right) \triangleq G_j(\mathbf{H}(\alpha)), \quad \alpha \in [t_0, t_f], 1 \leq j \leq r.
\]
These functions satisfy the terminal conditions

\[
H_1(t_f) = Q_f, \quad H_i(t_f) = 0^{n \times n}, \quad i \geq 2 \tag{8.4}
\]

\[
D_1(t_f) = 0, \quad D_2(t_f) = 1, \quad D_j(t_f) = 0, \quad j \geq 3.
\]

For linear state-feedback control inputs, it is shown in [27] that \( J \) is a finite \( \chi^2 \) random variable on a probability space \((\Omega, \mathcal{F}, P)\). The finiteness of \( J \) stems from the fact that for linear state-feedback controls, the “running cost” and “terminal cost” functions of (8.2) always satisfy the suitable polynomial growth conditions necessary for boundedness of the expectation of the cost functional [66]. So under this class of control inputs, a finite number of \( r \) cumulants exist for \( J \).

The initial cumulants of (8.2) are given explicitly by the following well-known recursive relationship,

\[ \kappa_1(t_0) \triangleq E\{J\}, \quad \kappa_r(t_0) \triangleq E\{J^r\} - \sum_{i=1}^{r-1} \binom{r-1}{i-1} \kappa_i(t_0) E\{J^{r-i}\}, \quad r \geq 2. \]

### 8.2.2 Notation

Some notation is introduced to make restatements of the above equations more concise in the development. Begin by defining the state variables \( \mathbf{H}(\alpha) \in \mathbb{R}^{rn \times n} \) and \( \mathbf{D}(\alpha) \in \mathbb{R}^r \) as below

\[
\mathbf{H}(\alpha) \triangleq \begin{bmatrix} H_1(\alpha) \\ \vdots \\ H_r(\alpha) \end{bmatrix}, \quad \mathbf{D}(\alpha) \triangleq \begin{bmatrix} D_1(\alpha) \\ \vdots \\ D_r(\alpha) \end{bmatrix}.
\]
Using these state variables, define the functions

\[
\mathcal{F}(H(\alpha), K(\alpha)) \triangleq \begin{bmatrix}
F_1(H(\alpha), K(\alpha)) \\
\vdots \\
F_r(H(\alpha), K(\alpha))
\end{bmatrix},
\mathcal{G}(H(\alpha)) \triangleq \begin{bmatrix}
G_1(H(\alpha)) \\
\vdots \\
G_r(H(\alpha))
\end{bmatrix}.
\]

Let \( \mathcal{F}_i(\cdot) \) and \( \mathcal{G}_i(\cdot) \) in the above definitions be defined as beforehand in (8.3). Also to be introduced is a condensed form for the terminal conditions as below.

\[
H_f \triangleq \begin{bmatrix}
Q_f \\
0^{n \times n} \\
\vdots \\
0^{n \times n}
\end{bmatrix},
D_f \triangleq \begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix}.
\]

Finally, denote the vector of cost cumulants \( \kappa(\alpha) \in \mathbb{R}^r \) as

\[
\kappa(\alpha) \triangleq \begin{bmatrix}
\kappa_1(\alpha) \\
\vdots \\
\kappa_r(\alpha)
\end{bmatrix}.
\]

### 8.2.3 Target Cost Cumulants

The target cumulants for the STS technique are a unique feature from previous definitions of target cost cumulants. Up until now, targets have been determined by cost cumulant-generating equations under the influence of some linear control, assumed unknown. There has been no connection made between the target cost statistics in time, and the statistics of any other variate besides the cost. Here, the expectation of an introduced target random variable is used as a target, which
allows the control designer to “mix” target cost cumulants resulting from nominal controls.

Consider $r$ random variables on a common space $(\Omega, \mathcal{F}, P_\theta)$. Let the probability space be the same $\forall \alpha \in [t_0, t_f]$ and denote the variables as

$$\tilde{\kappa}_i^\xi(\omega, \alpha), \; 1 \leq i \leq r$$

where

$$\tilde{\kappa}_i^\xi : \Omega \times [t_0, t_f] \rightarrow \mathbb{R}^+.$$ 

Let the variables $\tilde{\kappa}_i^\xi(\omega, \alpha)$ assume $N$ values

$$(\tilde{\kappa}_i(\omega_1, \alpha), \ldots, \tilde{\kappa}_i(\omega_N, \alpha)), \; \tilde{\kappa}_i(\omega_j, \alpha) \in \mathbb{R}^r$$

Assume the values $\tilde{\kappa}_i(\omega, \alpha)$ occur with frequencies

$$(\tilde{\kappa}_i(\omega_1, \alpha), \ldots, \tilde{\kappa}_i(\omega_N, \alpha)) \rightarrow (\theta_1, \ldots, \theta_N), \forall i$$

As defined, $\tilde{\kappa}_i^\xi(\omega, \alpha)$ follows a discrete distribution, or in other words, a $N$-point probability mass function. Definition will now be provided for the sample space elements $\tilde{\kappa}_i(\omega_j, \alpha), \; 1 \leq j \leq N$ for each random variable $1 \leq i \leq r$.

Given matrices for a system characterization $(A, B, G)$, an integral-quadratic cost characterization $(Q, R, Q_f)$, and the second-order statistics of the noise $(W)$, the generating-equations for the quantity $\tilde{\kappa}_i(\omega_j, \alpha)$ can be defined, the solutions of which are uniquely determined by a control gain $\tilde{K}_{\theta_j}(\alpha)$ (potentially unknown) that characterizes a linear control input $\tilde{u}_j(t) = \tilde{K}_{\theta_j}(t)\tilde{x}(t)$ to the process. For $1 \leq j \leq N$
and \( K_\theta (\alpha) \in C([t_0, t_f]; \mathbb{R}^{m \times n}) \), the initial cost cumulants take the familiar form,

\[
\tilde{\kappa}_i^\xi (\omega_j, \alpha) = x_0^T \tilde{H}_j^i (\alpha) x_0 + \tilde{D}_j^i (\alpha), \quad 1 \leq i \leq r
\]  

(8.5)

when \( \alpha = t_0 \). Let this set of numbers be regarded as 
\textit{target cost cumulants}. Here the functions \( \tilde{H}_j^i (\alpha) \) are determined by the same system of backwards-in-time, matrix differential equations as (8.3). The dynamics of \( \tilde{D}_j^i (\alpha) \) will also be as before,

\[
\frac{d\tilde{H}_j^i (\alpha)}{d\alpha} = F(\tilde{H}_j^i (\alpha), \tilde{K}_\theta (\alpha)), \quad \frac{d\tilde{D}_j^i (\alpha)}{d\alpha} = G(\tilde{H}_j^i (\alpha))
\]  

(8.6)

where

\[
\tilde{H}_j^i (t_f) = H_{j, \varepsilon_j^*}, \quad \tilde{D}_j^i (t_f) = D_{j, \varepsilon_j^*}, \quad \alpha \in [t_0, t_f]
\]

(8.7)

and the short-hand notation is used,

\[
H_{j, \varepsilon_j^*} \triangleq \begin{bmatrix} Q_f + E_j^* \\ 0^{n \times n} \\ \vdots \\ 0^{n \times n} \end{bmatrix}, \quad D_{j, \varepsilon_j^*} \triangleq \begin{bmatrix} \varepsilon_j^* \\ 1 \\ \vdots \\ 0 \end{bmatrix}.
\]

Here \( \varepsilon_j^* > 0 \) is a small perturbation constant, and \( E_j^* \succ 0^{n \times n} \) is a positive-definite perturbation matrix. As with the cost cumulants, compose a vector of
target initial cost cumulants $\tilde{\kappa}(\omega_j, \alpha) \in \mathbb{R}^r$ defined as below,

$$
\tilde{\kappa}(\omega_j, \alpha) \triangleq \begin{bmatrix}
\tilde{\kappa}_1(\omega_j, \alpha) \\
\vdots \\
\tilde{\kappa}_r(\omega_j, \alpha)
\end{bmatrix}.
$$

Remark(s) The control $\tilde{K}_{\theta_j}(\alpha)$ is a deterministic quantity used to generate cost cumulants $\tilde{\kappa}(\omega_j, \alpha)$ that occur with probability $\theta_j$ via $\tilde{\kappa}(\omega, \alpha)$.

Assume that the solutions to the equations (8.6) exist for $1 \leq j \leq N$ so that the cumulants $\tilde{\kappa}(\omega_j, \alpha)$ exist $\forall \alpha \in [t_0, t_f]$. It should be noted that the vector $\tilde{\kappa}(\omega_j, \alpha)$ is known, since it is determined exactly by the solutions of (8.6). Given how the random vector $\tilde{\kappa}(\omega, \alpha)$ has been defined above, $\tilde{\kappa}(\omega_j, \alpha)$ is a possible value of $\tilde{\kappa}(\omega, \alpha)$ that occurs with probability $\theta_j$. Targets are defined below.

Consider now the random variable $\tilde{\kappa}_i(\omega, \alpha)$, $1 \leq i \leq r$ with a time-invariant probability mass function $P_\theta$ where the sample space contains $\{\tilde{\kappa}_i(\omega, \alpha)\}_{j=1}^N$, and the $ith$ target cumulant $\tilde{\kappa}_i(\alpha)$ as the expectation $\tilde{\kappa}_i(\alpha) = E_{P_\theta}[\tilde{\kappa}_i(\omega, \alpha)]$. The main idea is depicted in Figure 8.1.

Consider the special case where the $ith$ random target $\tilde{\kappa}_i(\omega, \alpha)$ is made up of a sample space $\{\tilde{\kappa}_{i,k,CC}(\alpha)\}_{j=1}^{N_{j}}$, where cumulants from $k_j$CC controls have replaced for the former, more general, cumulants that compose the sample space. Suppose that for each $ith$ cost cumulant corresponding to the $k_j$CC control that there exists parameters $\{\mu_i\}_{j=1}^{k_j}$ satisfying

$$
\mu_i \leq \rho \mu_{j-1} \mu_{i-j}, \ 2 \leq i \leq k_j, \ 1 \leq j \leq i-1, \ \rho > 0.
$$

From [63], the above conditions ensure that the admissible control space of the
\[ \sum_{j=1}^{N} \theta_j = 1; \tilde{\kappa}_i(\alpha) = E_{\hat{\theta}}[\kappa_{i}^{\xi}(\omega, \alpha)] = \sum_{j=1}^{N} \theta_j \tilde{\kappa}_i^{\xi}(\omega_j, \alpha), \theta \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{bmatrix} \]

Figure 8.1. Statistical Target Selection, General Case

associated \( k_j \)CC optimization with weights \( \{\mu_i\}_{i=1}^{k_j} \) is non-empty. Without any loss of generality, it is assumed that \( \{\mu_i\}_{i=1}^{k_j} \) will also be chosen such that the sufficient conditions for existence of solutions to the cumulant-generating equations in [63] can be verified. This is consistent with the previous assumption for the general case. For the remainder of this chapter, the above special case will specifically consider \( N \) distinct \( k \)CC controls generating the sample space, \( \{\tilde{\kappa}_{i,CC}(\alpha)\}_{i=1}^{N} \), or equivalently \( k_i = i \). See Figure 8.2 for an illustration.

8.3 Theory of Statistical Target Selection

This section provides a brief overview of the theory that makes STS computations feasible for a sample space of cost cumulant trajectories, when each sample point (e.g. possible cost cumulant) can be realized by a given MCCDS control
\[
\sum_{j=1}^{N} \theta_j = 1; \ \bar{k}_i(\alpha) = \mathbb{E}_{\theta} [\bar{k}_i^\xi(\omega, \alpha)] = \sum_{j=1}^{N} \theta_j \bar{k}_{i,j,CC}(\alpha), \ \theta \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{bmatrix}
\]

Figure 8.2: Statistical Target Selection, Special \( k_{CC} \) Case
paradigm. It is highlighted that existence of sample space elements and their realization are very important assumptions to be made, and also a valid probability mass function $P_{\theta}$ must be specified. It is by taking a convex combination of the sample space elements corresponding to the expectation of $P_{\theta}$, the MCCDS performance index $g(\kappa(t_0), \tilde{\kappa}(t_0))$ is bounded for the expected targets as long as the function $g(\cdot)$ is convex in $\tilde{\kappa}(t_0)$.

Parametric targets formed by taking convex combinations of cost statistics resulting from nominal control laws yield a bounded performance index, as long as the performance index is bounded for each set of cost statistics per nominal control law, in addition to the convexity assumption above holding. This result is stated in the following theorem.

**Theorem 8.3.1 (STS Solution, Existence)**

Let $g : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^+$ be a scalar function $g(\kappa(\alpha), \tilde{\kappa}(\alpha))$ that is

1) convex in $\kappa(\alpha)$ and $\tilde{\kappa}(\alpha)$
2) analytic in $\kappa(\alpha)$ and $\tilde{\kappa}(\alpha)$
3) non-negative, $g(\kappa(\alpha), \tilde{\kappa}(\alpha)) \geq 0$

Assume for the $N$ time-varying deterministic target vectors in $\mathbb{R}^r$, $\{\tilde{\kappa}^\xi(\omega, \alpha)\}_{j=1}^N$ and the random quantity $\tilde{\kappa}^\xi(\omega, \alpha) \in \mathbb{R}^r$, the following holds true:

1) $g(\kappa(\alpha), \tilde{\kappa}^\xi(\omega, \alpha)) < \infty$
2) $\{\kappa^\xi(\omega, \alpha)\}_{i=1}^r$ constitute a set of independent random variables

Above, the cumulants $\kappa(\alpha), \tilde{\kappa}^\xi(\omega, \alpha)$ are composed of functions $H(\alpha), D(\alpha), \tilde{H}(\alpha), \tilde{D}(\alpha)$ that satisfy for $K \in \mathcal{K}(t_f)$ and $1 \leq j \leq N$,

$$
\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K(\alpha)), \quad \frac{d\tilde{H}(\alpha)}{d\alpha} = F(\tilde{H}(\alpha), \tilde{K}_j(\alpha))$$
$$
\frac{dD(\alpha)}{d\alpha} = G(H(\alpha)), \quad \frac{d\tilde{D}(\alpha)}{d\alpha} = G(\tilde{H}(\alpha)), \quad \alpha \in [t_0, t_f]
$$

$H(t_f) = H_f, \quad \tilde{H}(t_f) = \tilde{H}_f, \tilde{D}(t_f) = D_f, \quad \tilde{D}(t_f) = D_j$.
Under these conditions, it must be that
\[ g(\kappa(\alpha), EP_{\theta}[\tilde{\kappa}(\omega, \alpha)]) < \infty. \]

**Proof** Begin by using Jenson’s equality to write,

\[ g(\kappa(\alpha), EP_{\theta}[\tilde{\kappa}(\omega, \alpha)]) \leq EP_{\theta}[g(\kappa(\alpha), \tilde{\kappa}(\omega, \alpha))]. \]

For any function \( f \) of a discrete random variable \( X \), the following is true,

\[ EP[f(X)] = \sum_{i} f(x_i) \cdot P_i\{X = x_i\} \]

Use this rule to write,

\[
\begin{align*}
EP_{\theta}[g(\kappa(\alpha), \tilde{\kappa}(\omega, \alpha))] &= \sum_{i=1}^{N} g(\kappa(\alpha), \tilde{\kappa}(\omega_i, \alpha)) \cdot P(\bigcup_{j=1}^{r} \{ \tilde{\kappa}_j(\alpha) = \tilde{\kappa}_j(\omega_i, \alpha) \}) \\
&= \sum_{i=1}^{N} g(\kappa(\alpha), \tilde{\kappa}(\omega_i, \alpha)) \cdot \prod_{j=1}^{r} P(\{ \tilde{\kappa}_j(\alpha) = \tilde{\kappa}_j(\omega_i, \alpha) \}) \\
&= \sum_{i=1}^{N} g(\kappa(\alpha), \tilde{\kappa}(\omega_i, \alpha)) \cdot \left( \prod_{j=1}^{r} \theta_j \right) \\
&\leq \sum_{i=1}^{N} g(\kappa(\alpha), \tilde{\kappa}(\omega_i, \alpha)) \cdot (1) \leq \infty.
\end{align*}
\]

The first to third lines follow from the fact that \( \{\kappa_i^\xi(\omega, \alpha)\}_{i=1}^{r} \) constitutes a set of independent random variables. It follows immediately that,

\[ g(\kappa(\alpha), EP_{\theta}[\tilde{\kappa}(\omega, \alpha)]) \leq EP_{\theta}[g(\kappa(\alpha), \tilde{\kappa}(\omega, \alpha))] < \infty \]

and the proof is complete.

\[ \square \]
compute nominal targets

choose new $(\theta_1, \ldots, \theta_N)$ for targets $\kappa_1, \ldots, \kappa_r$

reduce time-step $\Delta \alpha$

compute CDS control for target

did numerical integration converge?

no

is set of possible $\theta$ exhausted?

no

iterate $\theta$ ...

yes

iterate $K_\theta$ ...

use control $u = K_\theta x$ into evaluation model

more disturbance data $dw(t)$?

no

yes

iterate $dw(t)$ ...

more controllers $K_\theta$?

no

done controller comp. & eval.

yes

Figure 8.3. Flow Chart for STS Controller Computation and Evaluation
8.4 Numerical Experiments

8.4.1 Statistical Target Selection and MHD\textsuperscript{2}-CDS

This computation follows the procedure shown in the flow chart of Figure 8.3. The first-generation benchmark serves to validate the STS methodology using two cost cumulants, and the associated system matrices, weighting matrices (from the baseline LQG control), and the structure’s evaluation model are used for the analysis in this section.

Target statistics for the cost are chosen as the expectation of a nominal 2-point probability mass function on a random target with a sample space that contains the LQG and 2CC cost statistics. This essentially amounts to choosing the target mean as a convex combination of cost means resultant from LQG and 2CC controllers. Analogously we take an identical convex combination of the cost variances resultant from 2CC and LQG controls as the cost’s target variance.

Consider the following control to generate the nominal statistics for the STS design.

\[
\tilde{K}^x(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( \tilde{H}_1(\alpha) + \mu_2^e \tilde{H}_2(\alpha) \right)
\]

Let \(\mu_2^{\text{LQG}} = 0\) and \(\mu_2^{\text{2CC}} = 1.0 \times 10^{-5}\) characterize the LQG and 2CC control gain, respectively. Using both controls, integrate the equations below using \(\epsilon_\alpha = 1.0 \times 10^{-9}\) and \(\mathcal{E}_x = 0^{10 \times 10}\),

\[
\begin{align*}
\frac{d\tilde{H}^x(\alpha)}{d\alpha} &= \mathcal{F}(\tilde{H}^x(\alpha), \tilde{K}^x(\alpha)), \quad \frac{d\tilde{D}^x(\alpha)}{d\alpha} = \mathcal{G}(\tilde{H}^x(\alpha)) \\
\tilde{H}^x(t_f) &= H_{f,\mathcal{E}_x}, \quad \tilde{D}^x(t_f) = D_{f,\epsilon_\alpha}, \quad \alpha \in [t_0, t_f]
\end{align*}
\]

(8.8)
where

\[ \begin{align*}
\tilde{H}_x(t_f) &= Q_f, \quad \bar{H}_x(t_f) = 0^{n \times n}, \quad \bar{D}_x(t_f) = \epsilon_x^*;
\end{align*} \tag{8.9} \]

This produces target cumulants \( \{\tilde{\kappa}_{iQ}^L(\alpha)\}_{i=1}^2 \) and \( \{\tilde{\kappa}_{iCC}^2(\alpha)\}_{i=1}^2 \) that will be used to produce a family of parametric targets for \( 0 \leq \theta \leq 1 \) as below,

\[ \begin{align*}
\tilde{\kappa}_1(\alpha) &= \theta \cdot \tilde{\kappa}_{iQ}^L(\alpha) + (1 - \theta) \cdot \tilde{\kappa}_{iCC}^1(\alpha) \\
\tilde{\kappa}_2(\alpha) &= \theta \cdot \tilde{\kappa}_{iQ}^L(\alpha) + (1 - \theta) \cdot \tilde{\kappa}_{iCC}^2(\alpha).
\end{align*} \tag{8.10} \]

Consider normalized cost and target cost variates as below,

\[ \begin{align*}
Z &= \frac{J - \kappa_1(t_0)}{\kappa_2(t_0)^{1/2}} \\
\tilde{Z} &= \frac{J - \tilde{\kappa}_1(t_0)}{\tilde{\kappa}_2(t_0)^{1/2}} = \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} \right)^{1/2} \frac{J - \kappa_1(t_0)}{\kappa_2(t_0)^{1/2}} + \frac{\kappa_1(t_0) - \tilde{\kappa}_1(t_0)}{\tilde{\kappa}_2(t_0)^{1/2}} \\
&= aZ + b,
\end{align*} \]

with the best Gaussian density approximations,

\[ \begin{align*}
p_Z(z) &\approx \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-z^2}{2} \right), \quad p_{\tilde{Z}}(\tilde{z}) \approx \frac{a}{\sqrt{2\pi}} \exp \left( \frac{-(az + b)^2}{2} \right) = ap_Z(az + b).
\end{align*} \]

The STS design seeks to minimize the function of cost cumulants and target cost cumulants

\[ \begin{align*}
HD^2(p_Z(z), p_{\tilde{Z}}(z)) &= 1 - \int_{-\infty}^{\infty} p_Z(z)p_{\tilde{Z}}(z)dz \\
&= 1 - \sqrt{Z} \cdot \left( \frac{(\kappa_2(t_0)\tilde{\kappa}_2(t_0))^{1/4}}{\sqrt{\kappa_2(t_0) + \tilde{\kappa}_2(t_0)}} \right) \cdot \exp \left( - \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))}{4(\kappa_2(t_0) + \tilde{\kappa}_2(t_0))} \right)
\end{align*} \]

\[ \text{364} \]
which from Chapter 4, it is known the optimal linear MHD$^2$-CDS control solution is

$$K_\theta(\alpha) = -R^{-1}(\alpha)B^T(\alpha)
\left(H_1(\alpha) + \gamma_{HD^2}(\alpha)H_2(\alpha)\right)$$

where the optimal parameter is

$$\gamma_{HD^2}(\alpha) = \frac{1}{2} \cdot \left(\frac{\kappa_2^2(\alpha) - \tilde{\kappa}_2^2(\alpha) - \kappa_2(\alpha)(\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha))^2}{\kappa_2(\alpha)(\kappa_2(\alpha) + \tilde{\kappa}_2(\alpha))(\kappa_1(\alpha) - \tilde{\kappa}_1(\alpha))}\right).$$

This control is computed by integrating the equations of motion for $r = 2$,

$$\frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K_\theta(\alpha)), \quad \frac{dD(\alpha)}{d\alpha} = G(H(\alpha))$$

$$H(t_f) = H_{f, \gamma}, \quad D(t_f) = D_{f, \gamma}, \quad \alpha \in [t_0, t_f].$$

It should be noted that the parametric targets can actually be obtained for some but not all choices of $\theta$ through use of the MHD$^2$-CDS paradigm. Nevertheless all calculations for control gains with the chosen targets converge using Runge-Kutta numerical integration with a fixed step of $\Delta\alpha = 50 \mu\text{sec}$.

Now, the fullness of the family of control gains above depends upon the increments $\Delta \theta$ chosen in the parameter $\theta \in [0, 1]$. Each control gain $K_\theta$ in the family will characterize a linear control input to the high-fidelity evaluation model for the test structure under different excitations - the KT spectrum, Hachinohe (HA) earthquake, and El Centro (EC) earthquake. The increment in $\theta$ is chosen as $\Delta \theta = 0.005$ to limit the computational overhead. This results in a family of over
200 controllers, and subsequently over 200 different target cost densities.

When controllers $K_{\theta}$ are used in the evaluation model, the control limits must be checked before control performance and control effort are considered. This is essential so that it can be determined whether or not the controller is valid and that there is no actuator saturation. In Figure 8.4, it can be seen that the entire family of MHD$^2$-CDS controllers $K_{\theta}$ for parametric targets (8.15) are within the control limits of the AMD Benchmark for peak and rms responses:

$$\max_t |u(t)| \leq 3\, V, \max_t |x_{m}(t)| \leq 9\, cm, \max_t |a_{m}(t)| \leq 6\, g$$

$$\sigma_u \leq 1\, V, \sigma_{x_m} \leq 3\, cm, \sigma_{a_m} \leq 2\, g$$

In Figure 8.5, normalized $J_1$ and $J_2$ and also $J_6$ and $J_7$ is shown versus the parameter $\theta$. Normalization is done with respect to the corresponding $J_i$ value achieved under 2CC control, which generally is the higher-performing control of both LQG and 2CC controls. The net effect of the normalization is that for control performance, values below 1.0 for $\theta$ values indicate greater reduction in the structure’s response during seismic excitation than 2CC control. Conversely, for control implementation costs and measures of control effort, values above 1.0 for $\theta$ values indicate greater actuator action than 2CC control.

Again, the quantities to measure control performance are defined as,

$$J_1 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{x_1}}{\sigma_{x_{3o}}}, \frac{\sigma_{x_2}}{\sigma_{x_{3o}}}, \frac{\sigma_{x_3}}{\sigma_{x_{3o}}} \right\}, J_2 = \max_{\omega_g, \zeta_g} \left\{ \frac{\sigma_{a_1}}{\sigma_{a_{3o}}}, \frac{\sigma_{a_2}}{\sigma_{a_{3o}}}, \frac{\sigma_{a_3}}{\sigma_{a_{3o}}} \right\}$$

$$J_6 = \max_{H_{A, EC}} \left\{ \frac{|d_1(t)|}{x_{3o}}, \frac{|d_2(t)|}{x_{3o}}, \frac{|d_3(t)|}{x_{3o}} \right\}, J_7 = \max_{H_{A, EC}} \left\{ \frac{|a_1(t)|}{a_{3o}}, \frac{|a_2(t)|}{a_{3o}}, \frac{|a_3(t)|}{a_{3o}} \right\}$$

where $d_i(t) = x_i(t) - x_{i-1}(t), \ 1 \leq i \leq 3$, is the $i$th inter-story drift with $x_0(t) = 0$, and $a_i(t) = \ddot{x}_i(t)$ is the acceleration for the $i$th story of the structure. These
are used to measure peak response. Likewise, $\sigma_{x_i}$ and $\sigma_{a_i}$, $1 \leq i \leq 3$ are the rms response for inter-story drift and per-story acceleration, respectively. The normalization in these metrics is done with respect to worst-case values of the appropriate quantity measured for the uncontrolled structure. The quantities above are used to measure control performance achieved under a particular control law, whereas the quantities $J_3$, $J_4$, and $J_5$ and also quantities $J_8$, $J_9$, and $J_{10}$ (not shown) pertain to control effort and controller implementation costs to the designer.

Relative to 2CC (equivalently, $\theta = 0.0$) and LQG (equivalently, $\theta = 1.0$), we
can see that the control gain $K_{\theta}$ with $\theta = 0.215$ achieves the greatest reduction of peak structural responses among the controllers that aim to achieve the remaining parametric targets. So for this family of MHD$^2$-CDS controllers, the MHD$^2$-CDS ($\theta = 0.215$) control is optimal.

The third-story drift and third-story acceleration are shown in Figures 8.7 and 8.8, and also Figure 8.9 (a) in order to examine how structural dynamics differ under the infinite-horizon 2CC and MHD$^2$-CDS ($\theta = 0.215$) controllers. It can be seen that response is mostly lower under the MHD$^2$-CDS ($\theta = 0.215$) control than the 2CC control throughout the time horizon. The reductions in response are particularly evident for the highest-peaked vibrations. Specifically, there is a 2.94% reduction in $J_6$ for Hachinohe (9.73% for El Centro) and a 3.04% reduction in $J_7$ (respectively, 11.7%). For rms response, MHD$^2$-CDS ($\theta = 0.215$) control provides a 11.83% reduction in $J_1$, and a 10.82% reduction in $J_2$. This experiment illustrates that given the statistical characterizations of the costs resultant from two stabilizing linear state-feedback compensations, it is possible to derive controls for alternative statistical characterizations of the random cost functional and investigate their performance.

Peak performance has clearly been improved in terms of $J_1$, $J_2$ and also $J_6$, $J_7$ with the MHD$^2$-CDS ($\theta = 0.215$) control, but it should be verified that the stability margins have not moved much versus the LQG or 2CC controls. In Figure 8.9 (c), the Nyquist plot of the determinant of the return difference matrix shows that the stability margins are essentially unchanged using this new control, which indicates that there is no compromise in stability with the MHD$^2$-CDS ($\theta = 0.215$) control.
Figure 8.5: STS Control Performance, MHD²-CDS
Figure 8.6: STS Control Effort, MHD²-CDS
Figure 8.7: Third-Story Acceleration, MHD$^2$-CDS STS Design (El Centro)
Figure 8.8: Third-Story Acceleration, MHD$^2$–CDS STS Design (Hachinohe)
Figure 8.9: Analysis Artifacts, MHD$^2$-CDS STS Design
8.4.2 Statistical Target Selection using Four Cost Cumulants

This computation follows the procedure shown in the flow chart of Figure 8.3. The first-generation benchmark serves to validate the STS methodology using four cost cumulants. This is a more detailed development of the discussion in [48]. This problem involves a 3-story test structure, that is subjected to 1-dimensional ground motion in order to simulate the effects of seismic disturbances. In particular, the structure is on a shaking table that is excited according to historical data from the El Centro and Hachinohe earthquakes. Also, excitation from a random process is used, where it has the spectral density of the Kanai-Tajimi spectrum. The test structure frame is constructed of steel, and has a mass of 77 kg and a height of 158 cm. The floor masses for the three floors are distributed evenly, and sum to a total mass of 227 kg. On each floor of the structure, accelerometers are mounted to record accelerations.

For control purposes, a representative Active Mass Driver (AMD) has been deployed on the third story of the structure, which is comprised of a single hydraulic actuator with steel masses attached to the end of a piston rod. For this experiment, the moving mass of the AMD was 5.2 kg, which amounted to 1.7% of the total mass of the structure. Separate sensors are used to record the displacement and acceleration of the AMD device. For further details on the experiment’s setup, the system model, and controller evaluations, consult [27] and [3]. The goal of the initial study was to design a control that optimizes consumption of control resources while best attenuating disturbances according to 10 performance measures \( \{J_i\}_{i=1}^{10} \). These measures pertain to the structural dynamics of the test building when protected by the AMD under a given control actuation, and the quality and costs associated with the control effort. In particular, \( J_1 \), \( J_6 \) pertain
to maximum, normalized inter-story drifts and $J_2$, $J_7$ pertain to maximum, normalized per-story accelerations of the structure. Respectively, the performance measures $J_3$, $J_4$, $J_5$ as well as $J_8$, $J_9$, $J_{10}$ deal with the physical size of the actuator, the control energy expended by a given control law, and the magnitude of control forces generated by the actuator. The first five performance measures $\{J_i\}_{i=1}^5$ measure rms response of the structure when subjected to excitations from the Kanai-Tajimi (KT) spectrum. On the other hand, $\{J_i\}_{i=6}^{10}$ assess peak response when the structure is excited with historical data from the El Centro and Hachinohe earthquakes. In general, lower values of $J_1$ and $J_2$ are a good tradeoff for increased values of $J_3$, $J_4$, and $J_5$, within acceptable limits on the control effort. Likewise, decreased $J_6$ and $J_7$ are considered a good tradeoff for increased values of $J_8$, $J_9$, and $J_{10}$. A full description of these performance metrics and the restrictions on control implementation, control energy, and control force can be found in [27].

A family of stabilizing MCCDS controls computed via the STS methodology is now evaluated based upon the amount of reduction in $J_1$, $J_6$ and $J_2$, $J_7$ in comparison to the levels of these metrics achieved with the 4CC control of [27]. All controllers in the following are computed using the parameters of the linear time-invariant, state-space model for the structure and the cost weighting matrices. The target cost statistics are generated from controls of the form,

$$
\hat{K}(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\left(\hat{H}_1(\alpha) + \mu_2^x\hat{H}_2(\alpha) + \mu_3^x\hat{H}_3(\alpha) + \mu_4^x\hat{H}_4(\alpha)\right). 
$$

(8.11)

In order to characterize LQG, 2CC, 3CC, and 4CC control gains, define the following parameters based on those Pham used in [27]:

$$
\mu_{2}^{LQG} = 0
$$
\[ \mu_{2}^{CC} = \mu_{2}^{3CC} = \mu_{2}^{2CC} = 1.0 \times 10^{-5} \]

\[ \mu_{3}^{3CC} = \mu_{3}^{LQG} = 0 \]

\[ \mu_{4}^{3CC} = 9.0 \times 10^{-12} \]

\[ \mu_{4}^{3CC} = \mu_{4}^{2CC} = \mu_{4}^{LQG} = 0 \]

\[ \mu_{4}^{4CC} = 2.0 \times 10^{-20} \]

Using these four controls, integrate the equations below using \( \epsilon^{*} = 1.0 \times 10^{-9} \),

\[ \frac{d\tilde{H}^{x}(\alpha)}{d\alpha} = \mathcal{F}(\tilde{H}^{x}(\alpha), \tilde{K}^{x}(\alpha)), \quad \frac{d\tilde{D}^{x}(\alpha)}{d\alpha} = \mathcal{G}(\tilde{H}^{x}(\alpha)) \] (8.12)

where

\[ \tilde{H}^{x}(t_{f}) = H_{f_{i}y_{j}}, \quad \tilde{D}^{x}(t_{f}) = D_{f_{i}y_{j}}, \quad \alpha \in [t_{0}, t_{f}] \]

\[ \tilde{H}_{1}^{x}(t_{f}) = Q_{f}, \quad \tilde{H}_{2}^{x}(t_{f}) = 0^{n \times n}, \quad \tilde{H}_{3}^{x}(t_{f}) = 0^{n \times n}, \quad \tilde{H}_{4}^{x}(t_{f}) = 0^{n \times n} \] (8.13)

\[ \tilde{D}_{1}^{x}(t_{f}) = \epsilon^{*}, \quad \tilde{D}_{2}^{x}(t_{f}) = 1, \quad \tilde{D}_{3}^{x}(t_{f}) = 0, \quad \tilde{D}_{4}^{x}(t_{f}) = 0 \]

to produce cumulants \( \{\tilde{\kappa}_{i}^{LQG}(\alpha)\}_{i=1}^{4}, \{\tilde{\kappa}_{i}^{2CC}(\alpha)\}_{i=1}^{4}, \{\tilde{\kappa}_{i}^{3CC}(\alpha)\}_{i=1}^{4}, \) and \( \{\tilde{\kappa}_{i}^{4CC}(\alpha)\}_{i=1}^{4} \).

Consider \( r = 4 \) random quantities \( \tilde{\kappa}_{i}^{x}(\omega, \alpha) \) that assume values \( \tilde{\kappa}_{i}^{LQG}(\alpha), \tilde{\kappa}_{i}^{2CC}(\alpha), \tilde{\kappa}_{i}^{3CC}(\alpha), \) and \( \tilde{\kappa}_{i}^{4CC}(\alpha) \) at specified frequencies. Use as the \( ith \) target the expectation of \( \tilde{\kappa}_{i}^{x}(\omega, \alpha) \) given below, where \( 1 \leq i \leq 4 \).

\[ \tilde{\kappa}_{i}(\alpha) = E_{P_{\theta}}[\tilde{\kappa}_{i}^{x}(\omega, \alpha)] = \theta_{1} \cdot \tilde{\kappa}_{i}^{LQG}(\alpha) + \theta_{2} \cdot \tilde{\kappa}_{i}^{2CC}(\alpha) + \theta_{3} \cdot \tilde{\kappa}_{i}^{3CC}(\alpha) + \theta_{4} \cdot \tilde{\kappa}_{i}^{4CC}(\alpha), \quad \sum_{j=1}^{4} \theta_{j} = 1 \] (8.14)
This leads to parametric targets as below,

\[
\tilde{\kappa}_1(\alpha) = \theta_1 \cdot \tilde{\kappa}_{1LQG}(\alpha) + \theta_2 \cdot \tilde{\kappa}_{1CC}(\alpha) + \theta_3 \cdot \tilde{\kappa}_{1CC}(\alpha) + \theta_4 \cdot \tilde{\kappa}_{1CC}(\alpha) \\
\tilde{\kappa}_2(\alpha) = \theta_1 \cdot \tilde{\kappa}_{2LQG}(\alpha) + \theta_2 \cdot \tilde{\kappa}_{2CC}(\alpha) + \theta_3 \cdot \tilde{\kappa}_{2CC}(\alpha) + \theta_4 \cdot \tilde{\kappa}_{2CC}(\alpha) \\
\tilde{\kappa}_3(\alpha) = \theta_1 \cdot \tilde{\kappa}_{3LQG}(\alpha) + \theta_2 \cdot \tilde{\kappa}_{3CC}(\alpha) + \theta_3 \cdot \tilde{\kappa}_{3CC}(\alpha) + \theta_4 \cdot \tilde{\kappa}_{3CC}(\alpha) \\
\tilde{\kappa}_4(\alpha) = \theta_1 \cdot \tilde{\kappa}_{4LQG}(\alpha) + \theta_2 \cdot \tilde{\kappa}_{4CC}(\alpha) + \theta_3 \cdot \tilde{\kappa}_{4CC}(\alpha) + \theta_4 \cdot \tilde{\kappa}_{4CC}(\alpha)
\]

(8.15)

The approach above follows the Statistical Target Selection (STS) methodology as originally introduced in [45], which is the subject of this chapter. Here target cost statistics are chosen as the expectation of a family of 4-point probability mass functions. See Figure 8.10 for an graphical illustration.

\[
\tilde{\kappa}_i(\alpha) = E_{\mathcal{P}_\theta}[\tilde{\kappa}_i^\xi(\omega, \alpha)] = \sum_{j=1}^{4} \theta_j \kappa_{i,jCC}(\alpha), \quad \theta \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}
\]

Figure 8.10. Statistical Target Selection, Four kCC Targets

For the MCCDS performance index \( \phi(\cdot) = g(\cdot) \), consider a “hybrid” version of
the KLD, where the KLD is the non-negative measure between probability density
functions \( p_Z \) and \( \tilde{p}_Z \) given by

\[
\text{KLD}(p_Z(z), \tilde{p}_Z(\tilde{z})) = \int_{-\infty}^{\infty} p_Z(z) \log \left( \frac{p_Z(z)}{\tilde{p}_Z(\tilde{z})} \right) dz.
\] (8.16)

This function \( g(\kappa(t_0), \tilde{\kappa}(t_0)) \) has been derived using the techniques described in
Lin and Saito [24], and is shown below,

\[
g(\kappa(t_0), \tilde{\kappa}(t_0)) = \frac{1}{\tilde{\kappa}_2(t_0)} \cdot \left( \frac{1}{12} \cdot \kappa_3(t_0) + \frac{1}{12} \cdot \tilde{\kappa}_3(t_0) - \frac{1}{6} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) \right) \\
+ \frac{1}{\tilde{\kappa}_2(t_0)} \cdot \left( \frac{1}{48} \cdot \kappa_4(t_0) + \frac{1}{48} \cdot \tilde{\kappa}_4(t_0) - \frac{1}{24} \cdot \kappa_4(t_0)\tilde{\kappa}_4(t_0) \right) \\
+ \frac{1}{\tilde{\kappa}_2(t_0)} \cdot \left( \frac{1}{4} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0)\tilde{\kappa}_4(t_0) - \frac{1}{4} \cdot \tilde{\kappa}_3(t_0)\tilde{\kappa}_4(t_0) \right) \\
+ \frac{1}{\tilde{\kappa}_2(t_0)} \cdot \left( \frac{1}{8} \cdot \tilde{\kappa}_4(t_0)\kappa_4(t_0) - \frac{1}{8} \cdot \kappa_4(t_0)\kappa_4(t_0) \right) \\
+ \frac{1}{\tilde{\kappa}_2(t_0)} \cdot \left( \frac{707}{1296} \cdot \kappa_3(t_0) + \frac{329}{1296} \cdot \kappa_3(t_0) - \frac{7}{12} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) - \frac{35}{162} \cdot \kappa_3(t_0)\tilde{\kappa}_3(t_0) \right) \\
+ \frac{1}{2} \cdot \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} - 1 - \log \left( \frac{\kappa_2(t_0)}{\tilde{\kappa}_2(t_0)} \right) + \frac{(\kappa_1(t_0) - \tilde{\kappa}_1(t_0))^2}{\tilde{\kappa}_2(t_0)} \right). \] (8.17)

Use of this MCCDS performance index assumes the framework of Section 6.2.3
in Chapter 6. For each \( \theta_i = (\theta_1, \theta_2, \theta_3, \theta_4) \), consider the optimal linear MCCDS con-
trol solution to the following optimization,

\[
\min_{\kappa \theta} \left\{ g(\kappa(t_0), \tilde{\kappa}(t_0)) \right\} \\
\quad dx(t) = (A(t) + B(t)K(t))x(t) + G(t)dw(t), \ x_0 = E\{x(t_0)\}, \ t \in [t_0, t_f]
\]
which from Chapter 3 has been determined to be

\[ K_\theta(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \left( H_1(\alpha) + \sum_{i=2}^{r} \gamma_i(\alpha)H_i(\alpha) \right) \]

where the optimal parameters are

\[ \gamma_i(\alpha) = \begin{pmatrix} \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_i(\alpha)} \\ \frac{\partial g(\kappa(\alpha), \tilde{\kappa}(\alpha))}{\partial \kappa_1(\alpha)} \end{pmatrix}, \quad 1 \leq i \leq r, \quad \alpha \in [t_0, t_f]. \]

This control is computed by integrating the equations of motion for \( r = 4 \),

\[ \frac{dH(\alpha)}{d\alpha} = F(H(\alpha), K_\theta(\alpha)), \quad \frac{dD(\alpha)}{d\alpha} = G(H(\alpha)) \]

\[ H(t_f) = H_{f,x^*_j}, \quad D(t_f) = D_{f,x^*_j}, \quad \alpha \in [t_0, t_f]. \]

The objective function (8.17) can be used to compute the controller gain above for each target (8.14) to make a comparison between over 250 stabilizing MCCDS controllers, considering parameter sets \( \{\theta_j\}_{j=1}^{4} \) where \( \theta_j = j \cdot \Delta \theta \) with an increment \( \Delta \theta = 0.1 \). Relative to the 4 CC control with parameters \( (\mu_1, \mu_2, \mu_3, \mu_4) = (1.0, 1.0 \times 10^{-5}, 9.0 \times 10^{-12}, 2.0 \times 10^{-20}) \), it can be verified in Figure 8.11 that the MCCDS \( (\theta_1 = 0.3, \theta_2 = 0.1, \theta_3 = 0.0, \theta_4 = 0.6) \) control has better peak response and rms response, and stays within the control limits:

\[ \max_t |u(t)| \leq 3 \text{ V}, \quad \max_t |x_m(t)| \leq 9 \text{ cm}, \quad \max_t |a_m(t)| \leq 6 \text{ g} \]

\[ \sigma_u \leq 1 \text{ V}, \quad \sigma_{x_m} \leq 3 \text{ cm}, \quad \sigma_{a_m} \leq 2 \text{ g} \]

For this MCCDS control, there is a 2.11% (respectively, 6.89%) reduction in peak inter-story drift and a 2.24% (8.39%) reduction in peak acceleration or the Hachi-
nohe (respectively, El Centro) earthquake. As for the K-T spectrum case, \( J_1 \) and \( J_2 \) are lowered by 7.60% and 6.64% respectively.

Table 8.2 shows data for 4 additional MCCDS \((\theta_1, \theta_2, \theta_3, \theta_4)\) controls that reduce \( J_1, J_6 \) and \( J_2, J_7 \) more than \( 4\text{CC} \) does while meeting the established control constraints. Indicated is the percentage change to \( J_i, \ 1 \leq i \leq 10 \) achieved with \( 4\text{CC} \). This family of controls is rich in tradeoffs between rms/peak performance and controller implementation costs. Additionally, the best-performing MCCDS control has similar robust stability properties as the nominal \( 4\text{CC} \) control, as evidenced by Figure 8.15 (c). Our experiment illustrates that given the statistical characterizations of the costs resulting from stabilizing linear state-feedback compensations - here LQG, 2CC, 3CC, 4CC controls - it is possible to derive controls for alternative statistical profiles of the random cost \( J \) and investigate the resulting closed-loop system behavior.

### TABLE 8.1
CONTROL LIMITS, TOP-FIVE MCCDS \((\theta_1, \theta_2, \theta_3, \theta_4)\) CONTROLS

<table>
<thead>
<tr>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(\theta_3)</th>
<th>(\theta_4)</th>
<th>(\max_t u)</th>
<th>(\max_t x_m)</th>
<th>(\max_t a_m)</th>
<th>(\sigma_u)</th>
<th>(\sigma_{x_m})</th>
<th>(\sigma_{a_m})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>0.6</td>
<td>0.3</td>
<td>1.2656</td>
<td>4.5626</td>
<td>5.9733</td>
<td>0.2901</td>
<td>1.1340</td>
<td>1.5215</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>0.3</td>
<td>0.6</td>
<td>1.2656</td>
<td>4.5634</td>
<td>5.9377</td>
<td>0.2902</td>
<td>1.1341</td>
<td>1.5216</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.7</td>
<td>1.2422</td>
<td>4.4756</td>
<td>5.8238</td>
<td>0.2846</td>
<td>1.1136</td>
<td>1.4924</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
<td>1.2319</td>
<td>4.4444</td>
<td>5.8418</td>
<td>0.2827</td>
<td>1.1095</td>
<td>1.4947</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>0.4</td>
<td>0.1</td>
<td>1.2305</td>
<td>4.4335</td>
<td>5.8194</td>
<td>0.2821</td>
<td>1.1062</td>
<td>1.4872</td>
</tr>
</tbody>
</table>
TABLE 8.2: CONTROL PERF./EFFORT, TOP-FIVE MCCDS $(\theta_1, \theta_2, \theta_3, \theta_4)$ CONTROLS

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\Delta J_1$</th>
<th>$\Delta J_2$</th>
<th>$\Delta J_3$</th>
<th>$\Delta J_4$</th>
<th>$\Delta J_5$</th>
<th>$\Delta J_6$</th>
<th>$\Delta J_7$</th>
<th>$\Delta J_8$</th>
<th>$\Delta J_9$</th>
<th>$\Delta J_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>0.6</td>
<td>0.3</td>
<td>-7.60</td>
<td>-6.64</td>
<td>8.85</td>
<td>8.55</td>
<td>7.45</td>
<td>-2.11</td>
<td>-2.24</td>
<td>17.51</td>
<td>14.65</td>
<td>7.62</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>0.3</td>
<td>0.6</td>
<td>-7.61</td>
<td>-6.64</td>
<td>8.87</td>
<td>8.56</td>
<td>7.45</td>
<td>-2.10</td>
<td>-2.15</td>
<td>17.58</td>
<td>14.70</td>
<td>8.82</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.7</td>
<td>-6.08</td>
<td>-5.09</td>
<td>6.90</td>
<td>6.73</td>
<td>5.39</td>
<td>-1.65</td>
<td>-1.56</td>
<td>13.68</td>
<td>11.92</td>
<td>5.64</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
<td>-5.69</td>
<td>-5.04</td>
<td>6.51</td>
<td>6.27</td>
<td>5.55</td>
<td>-1.54</td>
<td>-1.54</td>
<td>12.70</td>
<td>10.70</td>
<td>6.41</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>0.4</td>
<td>0.1</td>
<td>-5.46</td>
<td>-4.70</td>
<td>6.19</td>
<td>6.00</td>
<td>5.02</td>
<td>-1.46</td>
<td>-1.41</td>
<td>12.16</td>
<td>10.49</td>
<td>6.05</td>
</tr>
</tbody>
</table>
Figure 8.11: Control Limits, Hybrid KLD and STS
Figure 8.12: Control Performance, Hybrid KLD
Figure 8.13: Control Effort, Hybrid KLD
Figure 8.14: Third-Story Acceleration, Hybrid KLD STS Design
Figure 8.15: Analysis Artifacts, Hybrid KLD STS Design
CHAPTER 9

CONCLUSIONS

9.1 Summary

This work has been motivated largely by the lacking capability in current cost cumulant control paradigms to shape the probability density function of the random cost functional in a deliberate way. The performance gains of cost cumulant controllers [26], [57] on other competing control methods have also motivated this work. Such is empirical evidence that the cost density differs appreciably under one control law versus the other, and that there are observable effects in the closed-loop system behavior directly associated to cost density “shape”. Several times in this work, validation exercises have been performed which show that performance and robust stability properties of $kCC$ cumulant controls are embedded in the achieved cumulants for the random cost functional, and thus a finite-cumulant approximation to the cost density. This further corroborates the claim that the cost’s statistical character is intimately related to the closed-loop behavior of the system. Therefore good reason exists to develop control laws which shape the cost density function.

The Multiple-Cumulant Cost Density-Shaping (MCCDS) control is derived by adapting classical dynamic programming techniques to the cost cumulant-generating equations of the LQG framework. The derived control relies on the
cost being integral-quadratic, and the process having dynamics subject to additive white Gaussian noise under linear state-feedback control. The MCCDS control law is the linear optimal control that minimizes a general scalar convex function of a fixed number of cost cumulants and target cumulants. The MCCDS control paradigm enables the designer to transform the shape of a target density for the random cost functional into a linear control law.

For Gaussian best density approximations for normalized cost and target cost variates, the MCCDS control paradigm specializes to Mean-Variance Cost Density-Shaping (MVCDS) control theory. Several demonstrated MVCDS controls show the power of the subsuming MCCDS control paradigm, in that probability distance functions can be expressed as a function of density and target density cumulants. In particular, this was demonstrated for the Kullback Leibler-Divergence (KLD), the Bhattacharyya Distance (BD), and the squared Hellinger Distance (HD$^2$). Convexity and non-negativity properties of these functions have been formally established. Further work has detailed a process for constructing cumulant representations of probability distance functions in higher-order statistics beyond the mean and variance. Hybrid variants for KLD, BD, and HD have been proposed. The preliminary findings in this work are promising that a rich class of functions exist that are suitable for performance indices in MCCDS optimization problems.

The finite-horizon MCCDS control theory is extended to the infinite-horizon, in the interest of tackling cost density-shaping problems where long-run performance is necessary. A cancellation property is derived that allows for the infinite-horizon MCCDS controller to be derived via Lagrange multiplier theory applied to a constrained optimization, which involves a general, convex form of cost and target cumulants as well as a set of algebraic matrix equations. Simulation results
are shown that indicate cost density-shaping objectives for a nominal target are approximately achieved on the infinite-horizon.

Both minimax and Nash cost density-shaping games are explored using the MCCDS theory as a foundation for the investigation. Via this control design, a non-cooperative agents can influence the process evolution and effectively the shape of the agent’s cost density while accounting for the shape(s) of opponent cost densities. Both minimax and Nash equilibrium solutions are derived. A non-trivial structural controls problem has been used to apply the theory and validate it. In retrospect of this development, cost density-shaping games offer the control designer the ability to formulate a control law that achieves an equilibrium statistical characterization for the cost(s) amid competing forces.

A Statistical Target Selection (STS) methodology is proposed for MCCDS controller evaluations. This procedure addresses the challenge of finding a target density for the cost that corresponds to a high-performance, stabilizing control input to the controlled system. The STS procedure enables control design that is robust and optimally-performing according to a specified class of target densities for the random cost functional. STS is applied to a non-trivial structural controls application gave rise to a MCCDS controller with higher performance than the nominal $kCC$ controls used for the STS design. This control respects the established control constraints for the problem, and retains similar robust stability characteristics to LQG and the highest-performing nominal $kCC$ control(s) that generated the family of targets.
9.2 Future Work

Perhaps the biggest challenges associated with obtaining the widespread acceptance and use of this theory in the research community are finding more application areas for MCCDS and also establishing additional procedures for selecting target statistics of the cost functional, and hence target densities. So far, this research has applied MCCDS to vibration suppression problems involving lightly-damped structures. Structural control has served as an excellent domain for the empirical evaluation of cost cumulant controls [26], [58], [40], and naturally has continued to be for this work. Yet this is not the sole type of problem where MCCDS can be used. New applications for this theory might include satellite structure and attitude control, which have been fruitful application areas for cost cumulant controls in previous work [8], [23]. Many design problems involving the mitigation of vibrations or hysteresis are good candidates for the MCCDS paradigm, where it is ideal to formulate a control input that can achieve an acceptable distribution of energy in the system, as represented by an appropriate random cost. Mathematical finance can potentially be another field where MCCDS can successfully be applied. Relevant challenges would be the linearization of nonlinear stochastic pricing/asset models, choosing the weighting matrices in the random cost to optimally capture the properties of a positive performance functional, and understanding what effects hedging would have on the additive disturbance in the asset/pricing model. This last point is essential to whether or not the cost cumulant-generating equations of the LQG framework would be valid for suitable approximations to the system and cost for the problem in question. Finally, it should be emphasized that extending the MCCDS theory to the partial observation, output-feedback case would open up many application areas where
measurement noise is nearly always present, such as missile interception problems [49].

The STS design methodology involves computing a family of MCCDS controllers for parametric target statistics, and then evaluating the ensemble of control laws in the evaluation model for the controlled system. While this is a straightforward technique, there may be other ways of considering a group of target densities for the cost functional. For instance, it may be that the necessary conditions of the problem of cumulants (see Chapter 2) can be used to classify viable target cumulants based upon some nominal statistical characterization for the cost, say cost cumulants achieved under LQG control. On the other hand, target statistics might be adjusted dynamically with the process evolution to achieve target cost density at a future point in time. And these are just two possibilities of perhaps many. Target selection for cost density-shaping seems to be a problem deserving its own dedicated research, and this will likely be a focal point for investigations in the continued development of MCCDS theory.


