THE EFFECT OF FLOW GEOMETRY ON SHEAR-INDUCED PARTICLE SEGREGATION AND RESUSPENSION

Abstract

by

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In this thesis, we investigate through theory and experiment the influence of the flow geometry on various shear-induced migration phenomena such as particle demixing, viscous resuspension and meniscus accumulation for a suspension of rigid, non-colloidal particles.

The major focus of this thesis is the elucidation of a particle flux mechanism that has been ignored in the literature in the theoretical calculation of the concentration distribution in shear-induced migration phenomena. This mechanism is the convective flux due to the secondary currents arising from the non-Newtonian rheology of suspensions. Historically, suspensions have been modeled as Newtonian fluids with concentration dependent viscosities when calculating velocity distributions due to the tremendous simplification of the governing equations. The results presented in this thesis, however, demonstrate that it is critical to consider the complete rheology of a concentrated suspension when modeling flows in complex geometries. While the magnitude of the secondary currents is small, in many cases they are the dominant mechanism governing the resulting particle concentration distribution. In chapters 2 through 4, we investigate the impact of these secondary currents on the concentration profiles developed in suspension
flow through conduits of arbitrary geometry, and in resuspension flow through a tube.

In chapter 5, we examine the radial segregation of particles in the squeeze flow of concentrated suspensions. This flow is identical to that produced in loading suspensions on to a parallel plate viscometer and thus the concentration inhomogeneities generated during the loading phenomenon may play a role in the well known scatter of torque measurements in this system. We develop a criterion in terms of the experimental parameters in a parallel plate experiment for the onset of radial inhomogeneities.

In the final investigation reported in this thesis, we develop a theoretical model for describing the droplet distribution in the Poiseuille flow of an emulsion through a tube. We show that the mathematical problem that results from this model is amenable to self-similar analysis via the trial function approach. The self-similar solution so obtained is used to evaluate oscillatory flows as a possible technique for separation of the dispersed phase from suspending fluid.
DEDICATION

To my parents Geetha Lakshmi and Subramaniam Ramachandran for their love, support and sacrifices that made this thesis possible.
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ACKNOWLEDGMENTS

I would first like to express my gratitude to my advisor Prof. David Leighton for his continuous support and encouragement. I had the opportunity to imbibe the many qualities that make him an excellent researcher: strong theoretical knowledge, acute physical insight, innovative experimental skills, keen analytical skills and lucid technical writing. I was able to work with him on a variety of problems during my stay here, and I leave Notre Dame having absorbed a truly unique body of knowledge.

I would also like to thank Prof. Hseuh-Chia Chang for the many interactions we have had during the course of my Ph.D. His boundless enthusiasm, outstanding mathematical skills and hunger for learning are truly inspirational. I wish to acknowledge the other members of my committee Prof. Mark McCready, Prof. Davide Hill and Prof. Andre Palmer for their valuable time and patience.

I have also been fortunate to have some great friends during my stay at Notre Dame. In particular, I would like to thank my roommates Debashis Dutta, Manish Kelkar and Jagadish Venkataraman who had to endure my eccentricities. Other close friends include Siddharth Maheshwari, Shramik Sengupta, Gaurav Arya, Shyni Varghese and Amaresh Malipatil. I would also like to thank the following graduate students who entered the department with me: Peter Erri, Jason Gordon, Ryan Gwaltney, Philip Wingert and Cyril Lesage. They helped make the transition in a new country very smooth. I must acknowledge my lab mates Mike
Lundin, Philip Schonewill and Eric Smith for some interesting conversations and also for lending an extra pair of hands in some of the experiments. I am also very grateful to Dian Arifin for her love and encouragement. Her company during the early years of my Ph.D. helped me overcome some difficult periods. I would also like to acknowledge Mrs. Vasanthy Kumar, who’s music lessons provided a soothing and often much needed break from work. Last, but definitely not the least, I would like to thank my sister Anitha Karthik for her love and support.

Finally, I would like to acknowledge the support from the Army (grant number DAAB07-03-3-K414) for the work on drop/bubble migration discussed in Chapter 6.
A suspension may be defined as the dispersion of rigid or deformable particles in a liquid medium. The field of suspensions has always been one of active research, simply because of the ubiquity of suspensions not only in day to day life but also in industry [51]. Many food items consumed by us are suspensions: mustard, mayo, ice cream, batters, puddings, sauces, etc. For example, mustard is a paste of ground mustard seeds in a mixture of oil, vinegar and salt water, while mayo is a suspension of deformable oil droplets in a mixture of vinegar and egg yolk. Personal care products such as toothpastes, moisturizers and face washes are used by us everyday. Toothpastes contain gentle abrasives like silica spheres which lift off yellowing stains to give whiter teeth. Nowadays, moisturisers and face washes contain small beads that break and release encapsulated vitamins on application. The blood we carry in our bodies is a suspension of red blood cells, white blood cells and platelets. Inks and paints also belong to the class of suspensions. For example, latex paints consist of sub-micron polymer beads suspended in water, which coalesce and form a polymer film on the applied surface as the water is evaporated. Suspensions are also widely found in industrial processes such as molding of polymer composites, powder injection molding and slurry transport. Suspensions are particularly widespread in the petroleum industry [82]. Drilling muds, well stimulation and fracturing suspensions, well cementing slurries, disper-
sions of asphaltenes in crude oils and emulsified oil obtained in tertiary recovery processes constitute a few examples.

The study of the rheology of suspensions is important in order to understand and modulate the flow properties for a desired application. For example, in moisturizers and face washes with vitamin-encapsulated beads, the stability of the suspension against settling is an important factor in design. The stabilization of colloidal suspensions such as fabric enhancers, liquid soaps and liquid detergents is still a challenge faced by the soap industry, as it affects the shelf life of these products. Artificial red blood cell substitutes and artificial arterial valves need to be designed taking into consideration their contribution to the total pressure drop to prevent excessive pumping loads on the heart. In the injection molding of polymers containing glass beads, the particle distribution during the mold filling procedure should be homogeneous in order to obtain uniform strength in the finished object. For the transport of slurries, the flow conditions should be such that the particles being carried remain in suspension and do not sediment within the conduit.

In order to characterize the rheology of suspensions, we need to know the total stress in the suspension as a function of the applied flow. For a suspension undergoing shear, the total stress in a suspension consists of contributions from both fluid and particulate phases [7]. As shown by Brady [8], the particulate contribution to the stress arises from Brownian, hydrodynamic and other interparticle interactions (e.g. electrostatic interactions). The balance between hydrodynamic and Brownian particle diffusivities is given by the Peclet number $\text{Pe}_\gamma$.

$$\text{Pe}_\gamma = \frac{\dot{\gamma} \mu_0 a^3}{kT} \tag{1.1}$$
This dissertation deals with the rheology of a very simple class of suspensions: non-colloidal particles in a Newtonian fluid at low Reynolds numbers. In such suspensions, inertial effects are absent, and Brownian motion of the particles is negligible compared to hydrodynamic motion, i.e. $\text{Pe}_\gamma \gg 1$. The latter is typically satisfied when the particle size is greater than 10 µm. A number of industrially important suspensions are non-colloidal suspensions: drilling muds, solid rocket fuel simulants, proppants used for hydraulic fracturing in oil wells, and feeds used in powder injection molding.

In spite of being a small subset of the entire class of suspensions, a suspension of non-colloidal particles in a Newtonian fluid under shear shows very rich and complex behavior. A fascinating phenomenon associated with the shear flow of suspensions is the migration of particles across streamlines, formally known as shear-induced migration due to the seminal work of Leighton and Acrivos [55]. (To appreciate the history of the discovery of shear-induced migration, look at the 1994 Bingham award lecture of Acrivos [2]). When a concentrated non-Brownian suspension is subjected to an inhomogeneous shear field, particle migration occurs from regions of high shear stress to low, high concentration to low and high streamline curvature to low. This shear-induced diffusion may be explained by simple kinematic arguments [55]. Consider two spheres approaching each other in an unbounded simple shear flow along streamlines that are close to each other as shown in figure 1.1. If the interaction is reversible, the spheres should return to their original incident streamlines at the end of the interaction. However, the interaction is rendered irreversible due to a variety of perceived reasons (e.g. surface roughness [23, 55, 89], residual Brownian motion [10], multiparticle interactions [70]), the spheres are displaced from their original streamlines by a distance of
Figure 1.1. Two spheres interacting in a simple shear flow. Due to the irreversible nature of the interaction, the spheres which initially at a separation $2\Delta d_1$ are displaced further from each other to a separation $2\Delta d_2$. The increase in separation between the sphere centers $\Delta d_2 - \Delta d_1$ is of the order of the sphere radius $a$.

the order of the radius $a$ of the spheres. Since the relative velocity between the spheres is of the order $\dot{\gamma}a$, where $\dot{\gamma}$ is the shear rate corresponding to the simple shear flow, the shear-induced particle diffusivity is proportional to $\dot{\gamma}a^2$. The rate of such irreversible interactions is proportional to the local shear rate and the local concentration $\phi$. Thus, if there is a gradient in the shear rate across a given test particle, then the particle experiences more irreversible interactions from the side of high shear rate compared to low, and is therefore displaced towards the region of lower shear rate. Similarly, if there is a gradient in concentration across the particle, the particle experiences a net displacement towards the region of lower concentration.

The shear-induced diffusion of particles leads to the several important physical phenomena such as demixing of particles as a suspension is flowed through a conduit [38, 55, 73], blunting of the velocity profile in tube and channel flow [38, 45, 49, 60], viscous resuspension [3, 5, 53], meniscus accumulation [18, 46] and improved mass transport [14, 102]. Some fundamental and practical aspects
of a select few of these phenomena have been investigated in this thesis, and they are outlined below.

In chapter 2, we study the fully developed concentration and velocity distributions for the pressure-driven flow of a concentrated suspension of particles through conduits of arbitrary cross-section. In this chapter, two intuitions developed in the shear-induced migration literature over the years are challenged. First, the fully developed concentration distribution is expected to be independent of the particle size when wall effects are negligible, since the scaling for the diffusive terms $\dot{\gamma}_0 a^2$ is eliminated from the governing equations when the divergence of the diffusive flux vector in the cross-section is set to zero. Here, $\dot{\gamma}_0$ is the characteristic shear rate for the flow geometry. Second, a local concentration maximum is always expected to occur for such flows in regions of low shear stress. For example, consider the cross-section depicted in figure 1.2. This geometry consists of two parallel, large-aspect ratio channels P and Q of dimensions $a_1 \times b_1$ and $a_2 \times b_2$ respectively, such that the thin edges satisfy $a_1 < a_2$ and fluid communication is permitted between their edges. For a given applied pressure drop, the shear stresses in channels P and Q scale as $a_1$ and $a_2$ respectively. Therefore, migration of particles should occur not only to the centerlines of both channels which are the local regions of low shear stress, but also from the global high-shear stress regions of channel Q to the global low-shear stress regions of channel P. If the dimensions $b_1$ and $b_2$ are chosen properly, more particles will occupy the channel P, leading to a concentration maximum in this region.

These intuitions are, however, based on an isotropic description of particle migration. It is well known in the literature that concentrated suspensions are anything but isotropic. Experimental measurements [6, 32, 88, 97] have shown that a
Figure 1.2. Schematic of a geometry that does not show a concentration distribution expected from intuition. The geometry consists of two rectangular channels \( P \) and \( Q \) of dimensions \( a_1 \times b_1 \) and \( a_2 \times b_2 \) with fluid communication allowed between their edges. Since the average shear stress is lower in the thinner channel \( P \), it is expected that this will be area of higher concentration.
concentrated suspension of rigid particles in a shear flow shows a strong, negative second normal stress difference and a weak, negative first normal stress difference. Using the suspension balance model of Nott and Brady [70], the shear-induced particle diffusive flux can be related to these normal stresses. The anisotropic particulate normal stresses influence the fully developed concentration distribution in two ways. First, any curvature in the geometry of the flow leads to additional particle migration due to the anisotropy of the shear-induced diffusive flux. For example, in the parallel plate geometry, the shear stress is large near the outer edge of the plates due to the high rotation velocities in that region, but is zero at the center of the plates. The isotropic diffusive flux model [73] thus predicts a radially inward migration of particles. However, experiments show that there is negligible radial migration in this geometry [11, 18, 21, 50, 61]. The explanation for this discrepancy is that the curvature of the flow field in the parallel plate geometry results in an additional outward radial flux driven by the anisotropy of the normal stresses that negates the inward radial flux due to shear stress gradients. This aspect of suspension modeling has received a fair amount of attention in the literature.

The second way in which particulate stress anisotropy affects the concentration distribution is via the incorporation of this anisotropy in the calculation of the rheology within the cross-section. Any curvature in the flow field within the cross-section will result in second normal stress difference induced secondary currents. This was first demonstrated experimentally by Geisekus [34] in 1965 for a viscoelastic polymer, which also exhibits a negative second normal stress difference. Surprisingly, the significance of this effect has not been recognized in the literature. Till now, the anisotropy of particle migration has been incorporated
for predicting the fully developed concentration distributions only in geometries where the secondary currents vanish due to symmetry. The flux due to the secondary currents scales as $B^2/a^2$, where $B$ is the characteristic length scale in the cross-section. In order to suppress wall effects and enable modeling of the suspension as a continuum, the ratio $B/a$ is kept high, usually 10 or more. Therefore, even though the magnitude of these secondary currents is small, their impact on the concentration distribution can be strong. In chapter 2, we explore the effects of the normal stress difference induced secondary currents on the fully developed concentration profile for suspension flow through different conduits via simulations, and show that the intuitions elucidated above are not always true. For example, with the inclusion of secondary currents, the concentration distribution in the geometry depicted in figure 1.2 shows a global concentration minimum in channel P instead of the expected maximum.

Shear-induced diffusion also explains the resuspension of concentrated slurries under creeping flow conditions. Slurry resuspension is typically associated with turbulence and high Reynolds numbers, but it was first demonstrated by Gadala-Maria [32] that a settled bed of coal particles in a viscous fluid could be resuspended even at low Reynolds numbers by simply applying shear. As explained by Leighton and Acrivos [53], viscous resuspension occurs because of a balance between the gravitational settling flux of particles and shear-induced diffusive flux. Since the settling velocity of particles and the shear-induced diffusive velocity both scale as $a^2$, the effect of particle size should scale out of the concentration balance equation. This is indeed true for the plane Couette and plane Poiseuille resuspension geometries. However, for more complicated geometries like tube Poiseuille flow, the breaking of the symmetry of the flow provided by gravity leads to sec-
ondary currents within the cross-section. The effect of these secondary currents on the concentration distribution equation scales as $R^2/a^2$ where $R$ is the radius of the tube and therefore, as discussed above, their effect on the concentration distribution cannot be ignored. Isotropic particle migration models predict that the only source for these secondary currents is the gravitational head difference that arises within the cross-section due to shear-induced migration. In chapter 3, we show using simulations that incorporation of the particle stress anisotropy in the calculation of the rheology results in a dominant contribution to secondary currents from second normal stress differences. These normal stress difference induced secondary currents are opposite in direction to the currents expected from gravity alone, and render the suspension-suspending fluid interface concave downward instead of concave upward. In addition to these qualitative changes, the inclusion of the non-Newtonian rheology leads to resuspension in the tube geometry that is stronger by about a factor of 4 at 40% average concentration. The predictions of our anisotropic particle migration model agree very well with the MRI measurements of Altobelli et al. [5].

Another consequence of shear-induced migration that is not very well recognized in the literature is the phenomenon of meniscus accumulation. When a suspension is drawn into an empty tube beyond a critical length called the induction length, a packed layer of particles is observed to grow continuously at the advancing meniscus [18, 46]. For example, when the finger tip is punctured with a needle, the first few drops of blood contain more cells and appear brighter than the subsequent drops, and this is due to meniscus accumulation of red blood cells [96]. Due to shear-induced migration of particles from the slow moving wall regions to the fast moving regions near the tube center, the particle average ve-
locity exceeds the fluid average velocity. Since the meniscus moves at the average velocity of the suspension, this results in a net flux of particles towards the meniscus, causing accumulation at the meniscus. The appearance of the packed layer of particles at the meniscus results in a sharp increase in the pressure drop required to pump the suspension through the tube. Meniscus accumulation is thus an important issue in capillary wetting based microdevices. For example, whole blood fails to completely penetrate an empty capillary if the capillary diameter is smaller than 50 \( \mu \text{m} \)\(^1\). In chapter 4, we investigate the effect of gravity on the meniscus accumulation phenomenon. In the presence of gravity normal to the flow, particles are drawn by gravity to the slow moving regions near the wall of the tube, leading to reduction in the flux of particles towards the meniscus. If gravity is sufficiently strong compared to shear-induced diffusion, there is a negative flux of particles towards the meniscus which results in the growth of a particle free suspending fluid layer at the meniscus. Because the rate of accumulation or depletion is steady after some initial transient, the length of the meniscus layer grows linearly with the total length of suspension through the tube. The accumulation/depletion phenomenon can thus be characterized by measuring the meniscus growth rate and induction length, the slope and intercept of this linear relationship. These experimental measurements are compared with the theoretical predictions from particle migration models. It is shown that while the results of the isotropic and anisotropic particle flux models are both in qualitative agreement with experiments, the isotropic model is found to strongly underpredict the meniscus growth rate, but the anisotropic model with the normal stress differences included captures the experiments to within 10%.

While chapter 4 looks at the meniscus accumulation phenomenon in tube,
chapter 5 examines this phenomenon in the more complicated case of squeeze flow. Squeeze flow is a geometry frequently employed for rheometry [30]. In fact, the coarse technique of pressing a fluid between the index finger and the thumb in order to gain a measure of its viscosity is also a form of squeeze flow rheometry. Squeeze flow also occurs when a fluid sample is loaded on to a parallel plate rheometer. In chapter 5, we study by experiment the radial segregation of particles due to the squeeze flow occurring during the loading of a concentrated suspension in the parallel plate geometry. Typically, torque measurements are made in the parallel plate system assuming that the initial distribution is radially uniform, which is not necessarily true. Inhomogeneous concentrations upon loading have been characterized in the Couette geometry [55], but the literature appears to be lacking in analogous effects in the parallel plate geometry. We show that in this case the inhomogeneity may be described in terms of a single parameter $\Theta$ coupling the plate radius, the gap separation between the plates and the particle size in the suspension. Depending on different values of $\Theta$, the behavior of the torque can be any of a short term increase, a short term decrease, a long term increase or a long term decrease. All these different behaviors have been reported in the literature for concentrated suspensions, and the experimental parameters in these measurements appear to fall in the correct range of $\Theta$ for the respective behaviors.

In chapter 6, we study the time dependent evolution of the droplet distribution in the pressure-driven flow of a dilute emulsion through a tube. The steady state concentration distribution of droplets has been modeled as a balance between the droplet drift away from the walls of the tube and the shear-induced diffusion of droplets resisting this drift [100]. At sufficiently high flow rates, this balance
results in a droplet-free layer of suspending fluid near the walls of the tube. In this chapter, we demonstrate that the unsteady concentration distribution of droplets resulting from a time-dependent average velocity obeys a self-similar solution, provided the thickness of the droplet-depleted region near the walls is always non-zero. The self-similar solution is used to evaluate oscillatory pressure-driven flows as a means of separation of the dispersed phase from the suspending fluid, and it is shown that a significant net flux of the dispersed phase may result over a cycle with an appropriate choice of the oscillatory flow field and operating parameters.

The final chapter in this thesis contains suggestions for future work.
2.1 Introduction

Over the past three decades, the migration of particles across streamlines in concentrated suspensions has received a great deal of attention. The flow geometries in which particle migration has been studied experimentally are channel flow \([49, 55, 60, 80]\), Couette flow \([55, 85, 95]\), wide-gap Couette flow \([1, 93]\), tube flow \([38]\), parallel-plate flow \([11, 18, 21, 50, 61]\), cone and plate flow \([20, 31]\), eccentric cylinder flow \([72]\) and resuspension \([3, 5, 19, 53, 69]\). These studies have shown that particles under shear generally migrate from regions of high concentration to low, from regions of high shear stress to low, and from regions of high streamline curvature to low.

In order to model the migration of particles across streamlines, two major approaches have been employed in the literature: the trajectory model (also known as the diffusive flux model), and the suspension balance model. The trajectory approach is based on a simple view of the kinematics of particle interactions. In this model, when a given test particle in a simple shear flow interacts with another particle approaching from above it, the test particle is displaced downwards. On the other hand, when the second particle approaches the given test
particle from below, the test particle gets displaced upwards. The microscopic reversibility normally expected for Stokes flow is presumably broken by irreversible non-hydrodynamic interactions via surface roughness [23, 89]. If there is a gradient in shear or gradient in concentration of the particles across the streamlines, there is a gradient in the rate of interactions experienced by the particle, which results in migration of the particle across the streamline. In addition, viscosity gradients result in displacement of particles from regions of higher viscosity to regions of lower viscosity. Using these simple intuitive arguments, Leighton and Acrivos [55] proposed the following particle flux constitutive equation for channel flow

$$N_y = -K_{\sigma} \frac{\phi^2}{\tau} \frac{\partial \gamma a^2}{\partial y} - K_{\parallel} \frac{\phi^2}{\mu} \frac{\partial \phi}{\partial y} \frac{\partial \gamma a^2}{\partial \phi}$$

(2.1)

The shear-induced gradient diffusion coefficient thus scales as $\gamma a^2$, where $\gamma$ is the local shear rate, and $a$ is the particle radius. $\tau$ is the magnitude of the local shear stress. $\mu$ is the viscosity of the suspension as a function of the volume fraction $\phi$. The parameters $K_{\sigma}$ and $K_{\parallel}$ were experimentally estimated to be 0.7 and 0.6 respectively at 45% volume fraction. This model was extended to more general two dimensional shear flows by Phillips et al.[73].

$$N = -K_c a^2 [\dot{\gamma} \phi \nabla \dot{\gamma} + \gamma \phi \nabla \phi] - K_\eta \frac{\phi^2}{\mu} \frac{\partial \mu}{\partial \phi} \gamma a^2 \nabla \phi$$

(2.2)

The constants $K_c$ and $K_\eta$ were empirically determined to be 0.43 and 0.65 respectively. The model was further modified by Tetlow et al. [93] by determining $K_c$ and $K_\eta$ as linear functions of $\phi$, but retained its essential form. Since the model devised by Leighton and Acrivos was applicable only to simple planar unidirectional
flows, the extension by Phillips et al. to more complicated flows implicitly assumes that particle migration is isotropic. Since this model always predicts migration down a shear gradient, it is unable to explain the apparent lack of migration in parallel-plate flow [11, 18, 61] in which the shear rate increases radially outwards, and the outward migration of particles in cone and plate flow [20, 31], in which the stress is constant. To account for the effects of streamline curvature, Krishnan et al. [50] proposed an additional particle flux term in the diffusive flux model which causes particle migration due to gradients in streamline curvature. Their modified constitutive equation for the radial flux balance in the parallel-plate geometry is

\[ N_r = K_{r\perp} \frac{\partial}{\partial r} \gamma a^2 - K_{\sigma r\perp} \phi^2 a^2 \frac{\partial^2 \gamma}{\partial r^2} - K_{\perp} \frac{\phi^2}{\mu} \frac{\partial \phi}{\partial \phi} \frac{\partial \gamma}{\partial r} a^2 \] (2.3)

The symbol \( \perp \) denotes migration perpendicular to the plane of shear. The constant \( K_{r\perp} \) was taken to be 0.7, thus balancing the shear rate gradient term exactly. In recent work, Merhi et al. [61] suggested a slightly greater value for \( K_{r\perp} \) leading to a small outward radial migration. A similar curvature term can be added to the radial flux balance in the case of wide-gap Couette flow. However, in this case the migration is in the plane of shear from streamlines of higher curvature to streamlines of lower curvature, and therefore should be characterized by a different coefficient which also may be a function of concentration. Thus, it can be seen that although the diffusive flux model is simple and intuitive, it entails several coefficients which must be determined empirically.

The suspension balance model, on the other hand, provides a more unified approach for describing particle migration [31, 66, 70]. In this model, particle migration is ascribed to gradients in the particle stress tensor \( \Sigma_{ij}^p \). As described in section 2.2 of this chapter, with the inclusion of the anisotropic terms in the par-
ticle stress tensor, the suspension balance model successfully explains the migration behaviors in all perturbations of simple shear flows, including the curvature-induced migration in the parallel-plate and cone and plate geometries. What has not been previously appreciated is the direct effect of the non-Newtonian particle stress on the convection within the cross-section of a conduit.

Consider the simple example of unidirectional flow of a suspension through a conduit of arbitrary cross-section. The total stress $\Sigma_{ij}$ in a suspension is the sum of the stress contributions from the suspending liquid and the particles. At high concentrations, the particle stress and consequently the total stress is anisotropic in nature, rendering concentrated suspensions non-Newtonian. If the particle stress tensor is taken to be isotropic (thus assuming that the suspension is Newtonian), it can be shown that the cross-sectional components of velocity are identically zero. However, if the anisotropy of the particle stress tensor is factored in, then this trivial solution no longer satisfies the flow equations, the components of the velocity vector in the plane of the cross-section of the conduit are necessarily non-zero, and this results in a helicoidal flow of the suspension at steady state.

The existence of such flows is already quite well established in polymer literature. Green and Rivlin [36] were the first to conclude that it is in general not possible to have steady and rectilinear flow of a viscoelastic polymer through a channel whose cross-section was not circular. They used a fourth-order fluid model to show that the second normal stress differences could produce a secondary flow in the steady flow of the fluid in a non-circular conduit. This was first verified experimentally by Geisekus [34] in 1965. He injected a stream of dyed polymer into a 5% solution of polyacrylamide flowing in an elliptical channel and observed that the dye exhibited a swirling motion as the polymer flowed through the channel.
at a Reynolds number of order $10^{-4}$. Secondary currents in polymer flows were also studied experimentally by Semjonow [83] and Dodson et al. [27] by similar visual techniques. More recently, Dooley’s research group has demonstrated the strong impact of secondary flows in the coextrusion process by studying the deformation of the interface between two contrastingly pigmented layers of LDPE during flow through a square channel [24, 25]. Brady and Carpen [9] also showed that in two-layer Couette flow and the falling film geometry, the flow field for a suspension of non-colloidal particles (modeled with constant concentration) was unstable to spanwise perturbations, the instability being driven by a second normal stress difference jump between the suspension and Newtonian phases. The secondary currents in Brady and Carpen’s work appear as a perturbation from an unstable steady state that does not have any circulation. However, in this chapter, we demonstrate that the steady state distribution itself is accompanied by secondary currents for suspension flow through conduits. The similarity between the secondary motions induced in viscoelastic polymer flow and suspension flow in channels is that their magnitudes are inherently weak, typically two to three orders of magnitude lower than the axial velocity. The subtle difference, however, is that in suspension flow, these weak secondary currents are linked to the concentration distribution equation by a convective term that scales as the Peclet number, which for typical experimental conditions is of order $10^2$ or higher. In this chapter, we demonstrate the significant and sometimes counter-intuitive effects that these circulation velocities can have on the steady-state particle distribution. Note that the Peclet number we refer to in this chapter is not the ratio of hydrodynamic to Brownian diffusivity in Eq. 1.1, but rather the ratio of the convective velocity scaling to the shear-induced hydrodynamic diffusivity as
defined later in Eq. 2.59.

This chapter is laid out as follows. In section 2.2, the suspension balance model is formulated using the constitutive equations of Zarraga et al. [97], and the capability of this anisotropic model for describing particle migration behaviors in curvilinear flows is described. In section 2.3, the effect of the geometry of the conduit cross-section on the velocity distribution is analyzed assuming a constant concentration throughout the cross-section of the conduit. This would be the initial velocity distribution expected in the case of a homogeneous suspension of very small particles for which the long time to approach the non-uniform steady concentration distribution has not yet elapsed. In section 2.4, the effects of particle stress anisotropy on the concentration and velocity distributions respectively are integrated; i.e. the full solutions of the governing equations for fully developed flow in elliptical, rectangular (with and without side walls), polygonal and wedge shaped cross-sections are analyzed. Finally, we conclude by summarizing our results.

2.2 Concentration profiles in the absence of secondary flows

The suspension balance approach for modeling the particle and velocity distributions in concentrated suspension flows [31, 43, 66, 70] performs mass and momentum balances on both particulate and suspension phases. The coupling between the two momentum balances is achieved via the particle stress tensor \( \Sigma_{ij}^p \), and any non-zero gradients of the particle stress tensor result in particle migration and in a net normal stress being exerted on the fluid phase.

In a suspension, as shown by Nott and Brady [70], the flux of particles is given
by the average motion of all the particles

\[ N_i = u_i \phi + \langle M_{ij} F_j \rangle \phi \]  

(2.4)

where \( \langle \cdot \rangle \) denotes an ensemble average over all the particles contained in a volume \( V \). \( u_i \) is the volume averaged suspension velocity and \( M_{ij} \) is the mobility function that determines the velocity of the particles in response to a force \( F_i \). Invoking the key approximation that the ensemble average of the mobility times the force on the particles is equal to the product of the separate ensemble averages of the mobility and the force [70], i.e.

\[ \langle M_{ij} F_j \rangle \approx \langle M_{ij} \rangle \langle F_j \rangle \]  

(2.5)

and approximating the mobility \( \langle M_{ij} \rangle \) by an isotropic function that describes the hindered motion of a particle surrounded by other particles (namely the hindered Stokes sedimentation velocity),

\[ \langle M_{ij} \rangle \approx \frac{1}{6 \pi \mu_0 d} f(\phi) \delta_{ij} \]  

(2.6)

the particle flux is obtained as

\[ N_i = u_i \phi + \frac{1}{6 \pi \mu_0 d} f(\phi) \langle F_i \rangle \phi \]  

(2.7)

Note that the approximations in equations 2.5 and 2.6 are rigorously valid for a dilute suspension of particles which interact solely through a long range repulsive hard-sphere potential. They also appear, however, to be remarkably robust even for highly concentrated anisotropic suspensions [98]. Applying a momentum bal-
ance over the particulate phase at steady state under creeping flow conditions, we get

\[
\frac{\partial \Sigma_{ij}}{\partial x_j} = \langle F_i \rangle \frac{\phi}{\phi/4/3 \pi a^3} \tag{2.8}
\]

where \(\phi/(4/3)\pi a^3\) is the number of particles per unit volume. In this chapter, we assume that the particles are neutrally buoyant so that there are no body forces applied on the particulate phase. Combining equations 2.7 and 2.8, the particulate flux is given by

\[
N_i = \underbrace{u_i \phi}_{\text{Convective flux } N_{Ci}} + \underbrace{2 \frac{a^2}{9 \mu_0} f(\phi) \frac{\partial \Sigma_{ij}}{\partial x_j}}_{\text{Shear-induced diffusive flux } N_{Di}} \tag{2.9}
\]

Thus, as was shown by Nott and Brady [70], the shear-induced diffusive particle flux is directly proportional to gradients in the particle stress tensor \(\Sigma_{ij}\).

The model is completed by assuming an appropriate constitutive equation for the particle stress. Several constitutive equations have been developed to describe the particle stress in terms of particle concentration and the shear geometry. Nott and Brady developed a constitutive model in which they defined a suspension temperature and pressure in terms of velocity fluctuations akin to granular flow, but they assumed the suspension temperature and pressure to be isotropic. Mills and Snabre [64] derived an isotropic model relating the particle stress to the mean stress in the suspension by considering the lubrication interaction between colliding spheres. Unfortunately, these constitutive equations have the same deficiency as any isotropic model: failure to explain the observed particle migration behaviors in geometries with curvature (e.g. parallel-plate and cone and plate flow).
The simple doublet model of Nir and Acrivos [68] shows that if two spherical particles in contact are oriented in the compressional quadrant, then the first normal stress difference \( N_1 = \Sigma_{mm} - \Sigma_{nn} \) is small and negative, while the second normal stress difference \( N_2 = \Sigma_{nn} - \Sigma_{pp} \) is large and negative, where \( \Sigma_{ij} \) is the total stress in the suspension and \( m, n \) and \( p \) represent the flow, velocity gradient and vorticity directions of a simple shear flow respectively. It has been shown both by simulations [10] and by experiment [71, 78] that the pair distribution function of a concentrated suspension shows more particles in the compression quadrant than the extensional quadrant, indicating that the normal stresses are compressive in nature. Unlike polymers where the normal stress scales as the square of the average stress, the normal stress in suspensions is directly proportional to the stress. Morris and Boulay [66] used these physical and scaling arguments to propose a particle stress tensor that included a constant anisotropy tensor, and determined the components of this tensor by fitting existing steady state and transient experimental data. Fang et al. [31] similarly incorporated particle stress anisotropy into the suspension temperature formulation of Nott and Brady [70] and provided a more general framework for defining a flow-aligned particle stress tensor.

Rather than using the existing theoretical or experimental developments of the constitutive equations for the particle stress tensor, we find it convenient in this thesis to use the measurements of Zarraga et al. [97] directly for describing the particle stresses, yielding a model with zero adjustable parameters. Zarraga et al. [97] characterized the dimensionless normal stresses \((N_1 - N_2)/\tau, N_1/\tau \) and \((N_2 + 1/2N_1)/\tau \) in the parallel-plate, cone and plate and rotating rod geometries respectively. These normal stress difference results were connected with the particle stress \(-\Sigma_{pp}/\tau \) extracted from the data collected by Acrivos et al. [3] for
resuspension in a Couette device. Their results for the particle stress in the \( \mathbf{m}, \mathbf{n} \) and \( \mathbf{p} \) directions are as follows

\[
\begin{align*}
\Sigma_{mm}^p &= -1.15\alpha(\phi)\tau = -(1 - b)\alpha(\phi)\tau \\
\Sigma_{nn}^p &= -\alpha(\phi)\tau \\
\Sigma_{pp}^p &= -0.46\alpha(\phi)\tau = -(1 + d)\alpha(\phi)\tau
\end{align*}
\]

(2.10)

where \( \alpha(\phi) \) is the reduced second normal stress given by

\[
\alpha(\phi) = 2.17\phi^3 \exp(2.34\phi)
\]

(2.11)

with \( b = -0.15 \) and \( d = -0.54 \) being the first and second normal stress difference coefficients normalized by the total particle stress in the gradient direction, i.e. \( b = N_1/\alpha\tau \) and \( d = N_2/\alpha\tau \). In practice, of course, the structure (and hence anisotropy) of a suspension depends on the concentration \( \phi \). Zarraga et al., however, found their data to be reasonably described by taking \( b \) and \( d \) to be constant. As expected from the doublet model, the second normal stress difference coefficient is much larger than the first. The local shear stress \( \tau \) is defined as

\[
\tau = \mu_0 \mu_r \dot{\gamma}
\]

(2.12)

where \( \mu_0 \) is the viscosity of the suspending fluid and \( \dot{\gamma} \) is the local shear rate. For a general shear flow, \( \dot{\gamma} \) is given by

\[
\dot{\gamma} = \sqrt{2\epsilon_{ij}\epsilon_{ji}}
\]

(2.13)
\( e_{ij} \) being the rate of strain tensor defined as

\[
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

(2.14)

\( \mu_r \) is the suspension relative viscosity given by the correlation [97]

\[
\mu_r = \frac{\exp(-2.34\phi)}{\left(1 - \frac{\phi}{\phi_m}\right)^3}
\]

(2.15)

where \( \phi_m = 0.62 \) is the maximum packing fraction. The approach of Morris and Boulay [66] is equivalent to that determined from rheological measurements described here except for a different constitutive relationship for \( \alpha \) and where \( b = -0.25 \) and \( d = -0.375 \). Similarly, in the model of Fang et al., \( b \) is set to 0, while \( d \) is taken to be \(-0.5\).

The particle stress tensor is thus written as

\[
\Sigma_{ij}^p = \Sigma_{mm}^p m_i m_j + \Sigma_{nn}^p n_i n_j + \Sigma_{pp}^p p_i p_j + 2[\mu_r - 1]e_{ij}
\]

\[
= -\alpha \tau [(1 - b)m_i m_j + n_i n_j + (1 + d)p_i p_j] + 2[\mu_r - 1]e_{ij}
\]

\[
= -\alpha \tau H_{ij} + 2[\mu_r - 1]e_{ij}
\]

(2.16)

\( H_{ij} \) is a symmetric tensor that represents the flow geometry weighted with the normal stress difference coefficients.

\[
H_{ij} = (1 - b)m_i m_j + n_i n_j + (1 + d)p_i p_j
\]

(2.17)

Now let us consider steady-state concentration distributions in viscometric geometries. At steady state, it is common to assume that both convective and diffusive particle fluxes are identically zero. Under these conditions, the steady
state concentration profile can be obtained from the simple equation

\[
\frac{\partial \Sigma_{ij}^p}{\partial x_j} = 0 \quad (2.18)
\]

As we shall see in sections 2.3 and 2.4, this assumption is true for isotropic suspension models and for anisotropic suspensions only for the commonly used viscometric geometries (parallel-plate flow, Couette flow, cone and plate flow, etc.) that are discussed in this section. This assumption will be shown in section 2.4 to break down for more complicated geometries.

First, let us examine migration in viscometric flows under the assumption that the particle stress tensor is isotropic. A key feature of these geometries is that the steady-state concentration profiles are invariant in the flow direction. In this case, the geometry tensor \( H_{ij} \) reduces to

\[
H^{(f)}_{ij} = \frac{1}{3} [(1 - b) + 1 + (1 + d)] \delta_{ij} = 0.87 \delta_{ij} \quad (2.19)
\]

The steady state solution to the particle flux balance equations for fully developed unidirectional flow through any geometry with arbitrary (but uniform) cross-section is very simple with this isotropic approximation:

\[
\alpha \tau = \text{constant} \quad (2.20)
\]

i.e. the particle stress is predicted to be identical at all points in the cross-section. Since \( \alpha \) is a monotonically increasing function of \( \phi \), the concentration in any region in the conduit is inversely related to the shear stress in that region.

The simple approach described above is strictly a result of ignoring the anisotropy of the particle stress tensor. Unfortunately, this solution holds true only for the
case of plane Poiseuille flow in a channel without side-walls where the streamlines are not curved. The outward migration of particles observed in the cone and plate and parallel-plate geometries cannot be explained by the above solution. Even for circular tubes, the particle migration towards the center of the tube is stronger than that calculated from the isotropic model. To see how anisotropy influences migration, consider the case of migration in a circular tube [31].

\[
N_r = \frac{2a^2}{9\mu_0} f(\phi) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \Sigma^p_{rr} \right) - \frac{\Sigma^p_{\theta\theta}}{r} \right] = 0 \quad (2.21)
\]

For this geometry, the \( r \) direction represents the velocity gradient direction and the \( \theta \) direction represents the vorticity direction. Therefore, from Eq. 2.10, we have

\[
\Sigma_{rr} = -\alpha(\phi)\tau \quad \text{and} \quad \Sigma_{\theta\theta} = -(1 + d)\alpha\tau \quad (2.22)
\]

Substitution of these particle stress components into Eq. 2.21 yields

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r\alpha\tau \right) - (1 + d)\frac{\alpha\tau}{r} = 0 \quad (2.23)
\]

which can be simplified to

\[
\frac{\partial \alpha\tau}{\partial r} - d\frac{\alpha\tau}{r} = 0 \quad (2.24)
\]

The above equation can be solved to yield

\[
\alpha\tau = \text{constant} \times r^d \quad (2.25)
\]
Since \( d \) is negative, this results in stronger radial concentration gradients than for the isotropic model (which predicts a constant particle pressure).

As shown by Morris and Boulay [66] and Fang et al. [31], the inclusion of the particle stress anisotropy into the constitutive equation also enables the model to explain the observed concentration distributions in the parallel-plate and cone and plate geometries. For example, in the parallel-plate geometry, a flux balance in the \( r \) direction at steady state gives

\[
N_r = \frac{2}{9} \frac{a^2 f(\phi)}{\mu_0} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \Sigma_{rr}^p \right) - \frac{\Sigma_{\theta\theta}^p}{r} \right] = 0 \tag{2.26}
\]

In this geometry, the \( \theta \) direction represents the flow direction, the \( z \) direction is the velocity gradient direction, while the \( r \) direction is the vorticity direction. Using the constitutive equations of Zarraga et al. [97], we have from equations (2.10) and (2.16) that

\[
\Sigma_{rr}^p = -(1 + d) \alpha(\phi) \tau, \quad \Sigma_{\theta\theta}^p = -(1 - b) \alpha(\phi) \tau \tag{2.27}
\]

Also, the shear rate increases linearly with the radial position \( r \). This yields the radial concentration gradient as

\[
\frac{\partial \phi}{\partial r} = -\frac{1 + b + 2d}{1 + d} \frac{\alpha \mu_r}{(\alpha \mu_r)'} \frac{1}{r} \tag{2.28}
\]

where the prime denotes the derivative with respect to concentration. Integrating the above equation with \( r \) and using the measured dependences of \( \alpha \) and \( \mu_r \) yields
the concentration distribution up to an arbitrary constant.

\[ \phi = \frac{c\phi_m r^{-\frac{1+b+2d}{3(1+d)}}}{\phi_m + cr^{-\frac{1+b+2d}{3(1+d)}}} \]  

(2.29)

This constant \( c \) is set by choosing the average concentration in the device. The exponent \(-\frac{(1+b+2d)}{3(1+d)}\) is equal to 0.17 using the values for the normal stress difference coefficients from Eq. 2.10. One can thus see that the model predicts a very weak outward migration of particles, which is consistent with the apparent lack of migration observed in parallel-plate experiments [18]. It is still not clear whether migration actually occurs in the parallel-plate geometry. Recently, Merhi et al. [61] reported the experimental measurement of an outward migration of particles in their parallel-plate experiments. This could not be confirmed by Bricker and Butler [11], who did not observe any change in the apparent viscosity of the suspension during their parallel-plate measurements. The model does, however, correctly predict outward migration of particles in the cone and plate geometry reported by Chow et al. [20] and Fang et al. [31].

A compact and sensitive way of summarizing the steady-state concentration distributions in viscometric flows is to determine the dimensionless concentration gradient \( r\partial\phi/\partial r \) described in table 2.1 using the constitutive equations of Zarraga et al. [97] for these geometries. Note that the dimensionless concentration gradient can be predicted with no adjustable parameters using the suspension balance formulation discussed above. The results from various experimental studies in different geometries are presented in figure 2.1. We see that the agreement is excellent within the experimental scatter of these studies.

In the viscometric geometries considered above, both components of the particle flux vector were zero at steady state, leading to a simple prediction for the
TABLE 2.1
THE DIMENSIONLESS CONCENTRATION GRADIENT IN VARIOUS VISCOMETRIC GEOMETRIES

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Dimensionless concentration gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel flow</td>
<td>( y \frac{\partial \phi}{\partial y} = -\frac{\alpha}{\alpha'} )</td>
</tr>
<tr>
<td>Tube Poiseuille flow</td>
<td>( r \frac{\partial \phi}{\partial r} = -(1 - d) \frac{\alpha}{\alpha'} )</td>
</tr>
<tr>
<td>Wide Gap Couette flow</td>
<td>( r \frac{\partial \phi}{\partial r} = (2 - b) \frac{\alpha}{\alpha'} )</td>
</tr>
<tr>
<td>Cone and Plate flow</td>
<td>( r \frac{\partial \phi}{\partial r} = \frac{b + 2d}{1 + d} \frac{\alpha \mu_r}{\alpha'} )</td>
</tr>
<tr>
<td>Parallel plate geometry</td>
<td>( r \frac{\partial \phi}{\partial r} = \frac{1 + b + 2d}{1 + d} \frac{\alpha \mu_r}{\alpha'} )</td>
</tr>
</tbody>
</table>

steady concentration distribution even including the effects of anisotropy. We now consider the more complex situation of unidirectional flow through a conduit of arbitrary cross-section where this is no longer the case.

2.3 The origin of secondary flows in particulate suspensions

Consider the flow of a suspension of rigid, non-colloidal particles through a conduit of arbitrary cross-section (Figure 2.2). Let the \( z \) axis represent the axial direction and let \( x \) and \( y \) represent the cross-sectional co-ordinates. Since our focus in this work is on the fully-developed distributions of particle concentration \( \phi \) and velocities \([u, v, w]\) within the cross-section \((x - y \text{ plane})\), we invoke the quasi steady-state approximation, i.e. all variations in the axial direction are ignored.

The variables in this problem are rendered dimensionless as follows:

\[
[x, y, z] = [x^*, y^*, z^*]B, \quad [u, v, w] = [u^*, v^*, w^*]GB^2/\mu_0, \quad P = P^*GB \quad (2.30)
\]
Figure 2.1. A comparison between the experimental data and theoretical prediction of the dimensionless concentration gradient in various geometries from table 2.1. Wide gap Couette data of Phillips et al. [73]: triangle pointing upwards - 45%, triangle pointing downwards - 50%, star - 55%, solid line - theory. Tube data obtained by Hampton et al. [38]: circle - 20%, square - 30%, diamond - 45%, dotted line - theory. Plane Poiseuille flow of Leighton and Acrivos [55]: plus - 45%, dash-dotted line - theoretical result. Truncated cone and plate data of Chow et al. [20]: asterisk - 50% concentration, dashed line - prediction from theory.
Here, the pressure $P$ in the conduit is rendered dimensionless by the scaling $GB$, $G$ being the pressure gradient applied in the flow direction. The velocity components $u$, $v$ and $w$ are non-dimensionalized by the scaling $U_c = GB^2/\mu_0$, where $B$ is the characteristic length scale in the cross-section and $\mu_0$ is the viscosity of the suspending fluid. For convenience, the asterisks accompanying the non-dimensionalized variables will be dropped henceforth. The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.31)$$

A momentum balance over the suspension yields

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = 0 \quad (2.32)$$
Here, the total suspension stress $\Sigma_{ij}$ is given by

$$\Sigma_{ij} = -P\delta_{ij} + 2\varepsilon_{ij} + \Sigma^p_{ij}$$  \hspace{1cm} (2.33)

The particle stress tensor $\Sigma^p_{ij}$ is defined using Eq. 2.16 as

$$\Sigma^p_{ij} = -\alpha\tau H_{ij} + 2(\mu_r - 1)e_{ij}$$  \hspace{1cm} (2.34)

Combining equations 2.32, 2.33 and 2.34, we obtain

$$\frac{\partial}{\partial x_j}[2\mu_re_{ij}] = \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}[\alpha\tau H_{ij}]$$  \hspace{1cm} (2.35)

The tensor $H_{ij}$ as defined in Eq. 2.17 represents the geometry of the flow field. If the flow were purely unidirectional, the flow $m_i$, velocity gradient $n_i$ and the vorticity $p_i$ directions would be given by

$$m_i = \delta_{i3}$$  
$$n_i = \frac{\partial w/\partial x_i}{\sqrt{\partial w/\partial x_k \partial w/\partial x_k}}$$  
$$p_i = \frac{\Omega_i}{\sqrt{\Omega_k\Omega_k}}$$  \hspace{1cm} (2.36)

where $\Omega_i$ is the vorticity vector given by

$$\Omega_i = \epsilon_{ij3}\frac{\partial w}{\partial x_j}$$  \hspace{1cm} (2.37)

Using Eq. 2.17, the geometry tensor can be written for purely unidirectional flows.
as

\[ H_{ij} = (1 - b)\delta_{i3}\delta_{j3} + n_in_j + (1 + d)p_ip_j \] (2.38)

The secondary currents induced by the non-Newtonian rheology are very small in magnitude compared to the axial velocity, and thus to leading order, we may ignore their effect on the flow geometry tensor. Therefore, the expression for \( H_{ij} \) in Eq. 2.38 may also be used for the quasi-unidirectional flows expected here.

Since the velocity and vorticity vectors defined above are always orthogonal, we can write

\[ \delta_{i3}\delta_{j3} + n_in_j + p_ip_j = \delta_{ij} \] (2.39)

The geometry tensor thus reduces to

\[ H_{ij} = \delta_{ij} - b\delta_{i3}\delta_{j3} + \frac{\Omega_i\Omega_j}{\Omega_k\Omega_k} \] (2.40)

The stress \( \tau \) is defined as

\[ \tau = \mu_r\dot{\gamma} \] (2.41)

while \( \dot{\gamma} \) is defined to leading order as

\[ \dot{\gamma} = \sqrt{w_x^2 + w_y^2} \] (2.42)

Because of the assumption of quasi-unidirectional flow, the total pressure \( P \) in
the system can be decomposed into an axial component \( \tilde{P} \) that varies linearly in axial position (constant axial pressure gradient) and a cross-sectional component \( \overline{P} \).

\[
P(x, y, z) = \overline{P}(z) + \tilde{P}(x, y) = -z + \tilde{P}(x, y) \tag{2.43}
\]

With this simplification, the momentum equations in the \( x \), \( y \) and \( z \) directions can be written separately as

\[
\frac{\partial}{\partial x} \left[ 2\mu_r \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu_r \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] = \frac{\partial \tilde{P}}{\partial x} + \frac{\partial}{\partial x} [\alpha \tau H_{11}] + \frac{\partial}{\partial y} [\alpha \tau H_{12}] \tag{2.44a}
\]

\[
\frac{\partial}{\partial x} \left[ \mu_r \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ 2\mu_r \frac{\partial v}{\partial y} \right] = \frac{\partial \tilde{P}}{\partial y} + \frac{\partial}{\partial x} [\alpha \tau H_{21}] + \frac{\partial}{\partial y} [\alpha \tau H_{22}] \tag{2.44b}
\]

\[
\frac{\partial}{\partial x} \left[ \mu_r \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu_r \frac{\partial w}{\partial y} \right] = -1 \tag{2.44c}
\]

Consider the case when the concentration distribution in the cross-section of the conduit is uniform. This situation is physically realizable in the limit of very large conduit to particle size ratios, i.e. \( B/a >> 1 \). In this limit, the characteristic shear-induced migration velocity of the particle is very small, and it takes extremely long channels for the concentration distribution to evolve from the
uniform distribution at the inlet to the non-uniform fully developed concentration
distribution. Therefore, the concentration profile can be assumed to be constant
at the inlet concentration $\phi_f$ and axially invariant in this asymptotic limit. With
this assumption, the momentum equations are simplified considerably.

\[
\nabla^2 u = \frac{1}{\mu_r} \frac{\partial \hat{P}}{\partial x} + \alpha \left[ \frac{\partial (\dot{\gamma} H_{11})}{\partial x} + \frac{\partial (\dot{\gamma} H_{12})}{\partial y} \right] \quad (2.45a)
\]

\[
\nabla^2 v = \frac{1}{\mu_r} \frac{\partial \hat{P}}{\partial y} + \alpha \left[ \frac{\partial (\dot{\gamma} H_{21})}{\partial x} + \frac{\partial (\dot{\gamma} H_{22})}{\partial y} \right] \quad (2.45b)
\]

\[
\nabla^2 w = -\frac{1}{\mu_r} \quad (2.45c)
\]

Consider the governing equations for $u$ and $v$ in Eq. 2.45a and 2.45b respectively.
The driving terms for these velocity components in the conduit are gradients in
the particle stress. Note that the isotropic part of the particle stress tensor can
always be balanced by an appropriate pressure gradient by defining an augmented
pressure $\hat{P}$.

\[
\hat{P} = \hat{P} + \alpha \tau \quad (2.46)
\]

The momentum equations in the $x$ and $y$ directions can be recast in terms of the
augmented pressure $\hat{P}$.

\[
\nabla^2 u - \frac{1}{\mu_r} \frac{\partial \hat{P}}{\partial x} = \kappa \sigma_e \left[ \frac{\partial}{\partial x} \left( \frac{\gamma_1}{|\Omega|^2} \right) + \frac{\partial}{\partial y} \left( \frac{\gamma_1 \Omega_2}{|\Omega|^2} \right) \right] \quad (2.47a)
\]
\[
\n\nabla^2 v - \frac{1}{\mu_r} \frac{\partial P}{\partial y} = d\alpha \left[ \frac{\partial}{\partial x} \left( \gamma \frac{\Omega_2 \Omega_1}{|\Omega|^2} \right) + \frac{\partial}{\partial y} \left( \gamma \frac{\Omega_2^2}{|\Omega|^2} \right) \right] \tag{2.47b}
\]

The definition of \( H_{ij} \) in 2.40 has been applied in the above equations. If we differentiate Eq. 2.47a with respect to \( y \), Eq. 2.47b with respect to \( x \) and subtract the two, we get

\[
\nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -\nabla^2 \omega = \frac{d\alpha}{\mu_r} \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \left( \gamma \frac{\Omega_2 \Omega_1}{|\Omega|^2} \right) + \frac{\partial}{\partial y} \left( \gamma \frac{\Omega_2^2}{|\Omega|^2} \right) \right] - \frac{d\alpha}{\mu_r} \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \left( \gamma \frac{\Omega_2 \Omega_1}{|\Omega|^2} \right) + \frac{\partial}{\partial y} \left( \gamma \frac{\Omega_2^2}{|\Omega|^2} \right) \right] = \xi \tag{2.48}
\]

Here, \( \omega \) is the \( z \) component of the vorticity vector that involves only the transverse components of velocity. \( \xi \) therefore represents the source term for the vorticity equation. Because the boundary conditions for \( u \) and \( v \) are homogeneous, Eq. 2.48 will be satisfied by the trivial solution \( u = v = 0 \) only if \( \xi \) is identically zero. This can occur in two ways. First, note that \( \xi \) is proportional to the reduced second normal stress coefficient \( d \). Thus, if the suspension were Newtonian \((d = 0)\), the secondary currents would vanish as expected. Alternatively, even for non-zero values of \( d \), secondary currents vanish if the director derivative terms in \( \xi \) are zero by symmetry. This actually occurs for the geometries listed in table 2.1. For all other geometries (e.g. rectangular ducts, ellipsoidal tubes, etc.), non-zero second normal stress differences must give rise to secondary currents.

To appreciate the role of the director derivative term, consider the curvature term \(-d\alpha \tau/r\) in the particle flux balance equation for a circular tube at steady state (Eq. 2.24) which is the analog of the terms on the RHS in equations 2.47a and 2.47b. This term is directly proportional to the second normal stress differ-
ence coefficient \( d \), the local particle pressure \( \alpha \tau \) and inversely proportional to the local radius of curvature. This term, which is responsible for the sharper particle migration towards the tube center in anisotropic suspensions, also represents the forcing function for the secondary currents and establishes a force that drives a flow along the local radial vector normal to the axial streamsurfaces. For the tube geometry, this driving force is identical from all directions, i.e. the driving force is invariant in the \( \theta \) direction, as shown schematically in figure 2.3 and therefore there is zero net flow in the cross-section. However, when there is a breaking of symmetry in the cross-section, as would occur with the introduction of an eccentricity to the circle, this term produces a net force that drives circulation currents. Since the driving force is directly proportional to the local curvature, the anisotropic particle stress terms drive a stronger flow from the side regions of the ellipse than from the top and bottom of the ellipse. This results in a non-zero secondary current that flows in from the side regions along the major axis of the ellipse and flows back along the minor axis and the top and bottom walls.

To arrive at a scaling for the magnitude of the circulation velocity, let us examine the solution of the momentum equations in Eq. 2.45 for different geometries, beginning with the elliptical geometry shown in figure 2.4. The quantity \( \psi/(-d)\alpha \), where the streamfunction \( \psi \) is obtained by solving the equation (for this and subsequent geometries)

\[
\nabla^2 \psi = -\frac{1}{\langle w \rangle} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]

(2.49)

is shown in figure 2.6 for different aspect ratios. \( \psi \) is set to 0 at all boundaries in the cross-section. For the uniform concentration distribution assumed here, the
Figure 2.3. Explanation of the origin of the secondary flow in an ellipse. The relative sizes of the arrows (not to any scale) convey the driving force for circulation in the conduits represented by the curvature term in Eq. 2.24. For the circular cross-section, the driving force is identical in all directions and therefore there is zero net circulation in the geometry. For the elliptical cross-section, the driving force from the shallow side regions is the strongest and therefore produces a secondary current that flows towards the center of the geometry and back to the sides along the top (and bottom) walls.

Figure 2.4. Conduit with elliptical cross-section of aspect ratio $W$. 
Figure 2.5. The variation of the function $k$ as defined in Eq. 2.50 with the inverse of the aspect ratio of the conduit $1/W$. The circles and the crosses are the results for rectangles and ellipses of different aspect ratios respectively.
magnitude of the average circulation velocity $u_{\text{CIRC}}$ scales as

$$
 u_{\text{CIRC}} = \left\langle \sqrt{u^2 + v^2} \right\rangle = (-d)k\alpha U
$$

(2.50)

where $U$ is the average axial velocity $\langle w \rangle$ of the suspension through the tube. $k$ is a function of the geometry of the channel only.

In figure 2.5, we have shown the variation of $k$ with the inverse of the aspect ratio $1/W$ for the elliptical geometry. As the aspect ratio is increased, $k$ increases and reaches a maximum value of order $10^{-2}$ when the aspect ratio is roughly 2. This increase is due to the symmetry breaking provided by the introduction of the second length scale in the cross-section. If the aspect ratio is increased further, $k$ decreases and assumes an asymptotic value of zero as the aspect ratio tends to infinity. The flow is driven by the suspension anisotropy and the gradient in the directors describing the flow geometry. Since the difference in these directors from the center to the side regions of the conduit is fixed, increasing the horizontal length scale decreases the magnitude of the gradient and hence the magnitude of the secondary current. Examination of the streamfunction depicted in figure 2.6 shows that the shape of the velocity distribution is roughly independent of the aspect ratio; rather, the magnitude simply vanishes as $1/W$ or $1 - 1/W$ approach zero. The numerical value of $(-d)k\alpha$ is also of interest as it represents the ratio of the circulation velocity to that of the mean flow. The distance $L$ the suspension is convected in the axial direction during the time required for the transverse circulation in the cell scales roughly as

$$
 \frac{L}{B} \sim \frac{1}{(-d)k\alpha}
$$

(2.51)
Figure 2.6. Normalized streamfunction $\psi(x,y)/(-d)\alpha(\phi_f)$ (contour profile) and secondary current profiles $[u(x,y), v(x,y)]$ (quiver profile) assuming the concentration distribution to be constant at $\phi_f$ for an ellipse with (a) $W = 1.5$ (b) $W = 3.0$ (c) $W = 5.0$
Figure 2.7. Conduit with rectangular cross-section of aspect ratio $W$.

where $B$ is the characteristic depth of the channel. Since $\alpha$ is $O(1)$ for reasonably concentrated suspensions (e.g. $\alpha(\phi = 0.45) \sim 0.6$) and $k \sim 10^{-2}$, $L$ is approximately $O(10^2)$ times the channel depth. Thus, as in the case of polymeric systems, the secondary flow is apparent only for very long conduits.

For the rectangular channel shown in figure 2.7, the variation of $k$ with the aspect ratio is superficially similar to that of elliptical channels. As shown in figure 2.5, $k$ has a maximum at an aspect ratio of around 2. Examination of the streamfunction profiles depicted in figure 2.8, however, shows that there are two circulation cells in each quadrant rather than the single circulation cell in elliptical geometries. In general, the streamline curvature is largest near the corners, weakest near the top and bottom walls at the center, and intermediate near the side walls. For a square channel ($W = 1$), the two circulation flows from the side walls and the top and bottom walls are identical, and a secondary flow pattern with 8 equal cells is set up in the cross-section. For large aspect ratios, the flow in the side regions reaches an asymptote with a weak circulation cell adjacent to the side wall and a stronger cell promoting convection towards the center. Because this
convection dies off exponentially away from the side regions, the overall average magnitude of the convection vanishes as $1/W \to 0$.

Consider now the velocity distribution in a cross-section with the shape of a regular polygon for a uniform concentration distribution. If $n$ is the number of sides of the regular polygon, then the cross-section can be represented by a right-triangular unit cell of hypotenuse 1 and angle $\theta$ as shown in figure 2.9 such that

$$\theta = \left(\frac{n - 2}{n}\right) \frac{\pi}{2}$$

(2.52)

The regular polygon can be constructed from $2n$ such unit cells. For example, $n = 3$, $n = 4$ and $n = 10$ correspond to the triangle, the square and the decagon. $n = \infty$ is the limiting case of a circle. The boundary conditions on the hypotenuse (which is also the radius of the polygon) and the height of the triangle are the symmetry conditions while the boundary condition on the triangle base is the no-slip condition. The variation of $k$ in this geometry as a function of the normalized polygon index $3/n$ is shown in figure 2.10. It can be seen that $k$ decreases monotonically as $n$ is increased, and asymptotes to 0 as $n \to \infty$, the case of a circle. The normalized streamfunction $\psi/(−d)\alpha$ is shown in figure 2.11 for the triangle ($n = 3$), pentagon ($n = 5$) and decagon ($n = 10$). The secondary currents flow from the corners of the polygon towards the center and back towards the midpoint of the polygon side. The flow is driven by the difference in the streamline curvatures at the polygon corners and the center of each polygon side. As $n$ is increased, this curvature difference and consequently the magnitude of the secondary currents decreases. In the case of the circle ($n = \infty$), there is no azimuthal variation in the curvature and therefore the secondary currents are zero.
Figure 2.8. Normalized streamfunction $\psi(x, y)/(-d)\alpha(\phi_f)$ (contour profile) and secondary current profiles $[u(x, y), v(x, y)]$ (quiver profile) assuming the concentration distribution to be constant at $\phi_f$ for a rectangular cross-section of aspect ratio $W$ (a) $W=1$ (b) $W=3$ (c) $W=5$. The separating streamlines are indicated by bold black lines.
Figure 2.9. The unit cell for the polygonal cross-section. The unit cell is a right triangle with a characteristic angle $\theta$ defined in Eq. 2.52. The boundary conditions for this geometry are zero flux and no-slip across the solid walls for particle flux and velocity field respectively, and the symmetry condition for the symmetry axes.

Figure 2.10. The function $k$ as defined in Eq. 2.50 for regular polygons as a function of the normalized polygon index $3/n$, where $n$ is the number of sides of the polygon.
Figure 2.11. Normalized streamfunction $\psi(x, y)/(−d)\alpha(\phi_f)$ (contour profile) and secondary current profiles $[u(x, y), v(x, y)]$ (quiver profile) assuming the concentration distribution to be constant at $\phi_f$ for different polygons.
It must be noted here that the secondary flow profiles observed experimentally in polymer literature for elliptical [34], square [24, 25, 27] and rectangular [24] ducts appear to be qualitatively the same as the ones discussed here for suspension flow, although the constitutive equations governing the secondary flow in polymers are completely different. The number of cells and the direction of flow in figures 2.6 and 2.8 for the elliptical and rectangular geometries observed for suspensions are identical to what has been observed experimentally and theoretically for viscoelastic polymers, and the magnitudes are again proportional to the second normal stress difference.

Consider a comparison of the scalings of the shear-induced migration velocity and the circulation velocity. From the definition of the flux due to shear-induced migration in Eq. 2.9, the shear-induced migration velocity scales as

\[ u_{\text{SIM}} = U \frac{a^2 f_\alpha \mu_r}{B^2 \phi_f} \]  

(2.53)

The circulation velocity relative to the shear-induced migration velocity of the particle is then

\[ \frac{u_{\text{CIRC}}}{u_{\text{SIM}}} = \frac{(-d)k\alpha U}{U \frac{a^2 f_\alpha \mu_r}{\phi_f}} = \frac{(-d)k}{a^2 f_\mu_r} \]  

(2.54)

In figure 2.12, we have shown the concentration dependence of the RHS of Eq. 2.54 as a function of concentration. At low concentrations, the function asymptotes to zero because the second normal stress difference driving the secondary flow goes to zero in the dilute limit faster than the shear-induced migration velocity of the particles. For very large concentrations, the shear-induced migration velocity again dominates the circulation velocity due to its stronger dependence on con-
centration. The function shows a maximum value of $O(1)$ at around 40% average concentration. Therefore, for the circulation velocity to have the same magnitude as the shear-induced migration velocity, the quantity $(-d)kB^2/a^2$ should be $O(1)$, i.e. the Peclet number $B^2/a^2$ should be $O(100)$, since $(-d)k$ is of order $10^{-2}$. For example, the circulation velocity relative to the shear-induced migration velocity has a value of 3 at 40% concentration for a $B/a$ ratio of just 20. This indicates that the circulation currents can be comparable or even stronger than the shear-induced migration velocities at high Pe numbers. For vanishingly small particles, shear-induced migration is negligible, but the effect of the secondary flow still persists. The flow average concentration in most of the simulations in this chapter is taken as 0.4 in order to capture the maximum effect of the secondary currents.

In this section, the velocity distributions for fully developed flow of a suspension through channels of different cross-sections were examined and a scaling for the magnitude of the secondary currents was also established. In the next section, we consider the impact of the secondary currents on the particle distribution at different Peclet numbers and examine the complete solutions of the momentum and particle flux balance equations for different geometries.

2.4 Concentration and velocity distributions for different geometries

2.4.1 Governing equations and solution procedure

In section 2.3, we eliminated the governing equation for particle distribution from the analysis by assuming that $\phi(x,y) = \phi_f$ in the limit of negligible shear-induced migration, which is appropriate when the axial length of the conduit $L$ is much smaller than the shear-induced migration length scale $B^3/a^2$. Here, we examine the opposite limit of the particle distribution for finite Peclet numbers
Figure 2.12. The function $\phi/f\mu_r$ which is directly related to the ratio of the scalings of the secondary current and the shear-induced migration velocity as defined in Eq. 2.54.
when \( L >> \frac{B^3}{a^2} \). To do this, in addition to solving the continuity (Eq. 2.31) and momentum (Eq. 2.44) equations, we need to obtain the solution to the steady-state particle flux balance equation.

For general flows, the following flux balance equation must be solved to determine the concentration distribution

\[
\frac{\partial \phi}{\partial t} + \frac{\partial N_i}{\partial x_i} = 0 \quad (2.55)
\]

Time is rendered dimensionless with the diffusive scaling \( \mu_0 B/Ga^2 \). The particle flux vector \( N_i \) comprises the convective and diffusive contributions shown earlier in Eq. 2.9,

\[
N_i = Pe u_i \phi + N_{D_i} \quad (2.56)
\]

where \( N_D \) is the shear-induced diffusive flux vector.

\[
N_{D_i} = \frac{2}{9} f \frac{\partial \Sigma_{ij}^p}{\partial x_j} = -\frac{2}{9} f \frac{\partial \alpha \tau H_{ij}}{\partial x_j} + \frac{2}{9} f \frac{\partial}{\partial x_j} [2(\mu_r - 1) e_{ij}] \quad (2.57)
\]

\( f \), the hindered settling factor, is given by the Richardson-Zaki correlation.

\[
f(\phi) = (1 - \phi)^{5.1} \quad (2.58)
\]

Here we use the exponent 5.1 employed by Chapman and Leighton [19]. The Peclet number \( Pe \) is given by

\[
Pe = \frac{B^2/[(U/B)a^2]}{B/U} = \frac{B^2}{a^2} \quad (2.59)
\]

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Pe may be interpreted as the ratio of the diffusive time scale to the convective
time scale, both defined in the above equation with respect to the characteristic
length scale \(B\) of the conduit.

In this chapter, we are restricting our analysis to axially invariant flows in
the steady asymptotic limit \(L \gg B^3/a^2\). In this limit, the particle conservation
equation reduces to

\[
Pe \frac{\partial \phi}{\partial x} + Pe \frac{\partial \phi}{\partial y} + \frac{\partial N_{D_1}}{\partial x} + \frac{\partial N_{D_2}}{\partial y} = 0
\]  

(2.60)

Since we are considering particle flux within the cross-section of the conduit, the
deviatoric stress term in Eq. 2.57 involves gradients of the transverse velocity com-
ponents only. We have already seen in section 2.3 that although the secondary
velocity components have the same scaling as the axial velocity, they are numer-
ically three orders of magnitude smaller. Therefore the effect of the secondary
currents on the particle stress is negligible, and to a high degree of approximation
we may take the flux \(N_D\) as

\[
N_{Di} = \frac{2}{9} f \frac{\partial \Sigma_{ij}}{\partial x_j} \approx -\frac{2}{9} f \frac{\partial \alpha \tau H_{ij}}{\partial x_j}
\]  

(2.61)

where \(\tau\) and \(H_{ij}\) may be calculated solely from the axial flow field \(w(x, y)\). On
the other hand, the convective term involving the transverse velocity must be
preserved because it is multiplied by the scaling Pe which is usually kept high in
experiments \((10^2\) or greater) in order to enable the modeling of the suspension as
a continuum. Thus, the particle flux can be reduced to

\[
N_i = Pe u_i \phi - \frac{2}{9} f \frac{\partial \alpha \tau H_{ij}}{\partial x_j}
\]  

(2.62)
The boundary condition for the concentration balance equation is the zero-flux condition normal to the walls. The velocity field \([u, v, w]\) satisfies the no-slip condition on the walls, which is justified for moderate concentrations and large \(B/a\) ratios. For high concentrations or smaller \(B/a\), these boundary conditions could be modified by the introduction of wall slip [42]. Such a modification, however, is beyond the scope of this thesis.

The solution of the non-linear, steady-state problem in Eq. 2.60 can be determined as the asymptotic limit of the entrance flow problem, as the asymptotic limit of the startup flow problem or by a direct, iterative method. For the problem of entrance flows, the steady, spatially-varying concentration distribution is governed by the following simpler form of Eq. 2.55

\[
Pew \frac{\partial \phi}{\partial z} = -Peu \frac{\partial \phi}{\partial x} - Pev \frac{\partial \phi}{\partial y} - \frac{\partial N_{D_1}}{\partial x} - \frac{\partial N_{D_2}}{\partial y} \tag{2.63}
\]

Here the axial length scale \(L\) is assumed to be much greater than the characteristic cross-sectional dimension \(B\), as would occur for large values of \(B/a\), so that diffusive terms containing derivatives in \(z\) produce negligible flux. This equation is normally solved by marching forward in the axial (\(z\)) direction. Note that for entrance flows, both convective terms on the RHS should be preserved even in the absence of secondary currents driven by normal stress differences. This is because in entrance flow the migration leads to an axial variation of the axial velocity, and hence from continuity both \(u\) and \(v\) are necessarily non-zero even for an isotropic suspension.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \tag{2.64}
\]
The axial velocity variation is caused by the well-known blunting of the velocity profile as particles migrate to the low shear stress streamlines. Inclusion of the source term is necessary to close the mass balance of particles. This source also produces a transverse velocity which, in general, resists shear-induced migration. However, while these continuity-driven secondary currents are of the same order as the particle migration velocities which produce the blunting, they are numerically much smaller. Therefore, neglect of this term [63, 86] produces negligible change in the entrance length. Besides, since the axial variation of \( w \) vanishes for fully-developed flow, ignoring the source term does not affect the asymptotic concentration distribution. The continuity driven transverse velocity may be avoided entirely if instead we consider the axially-invariant, time-dependent startup flow problem:

\[
\frac{\partial \phi}{\partial t} = -Pe u \frac{\partial \phi}{\partial x} - Pe v \frac{\partial \phi}{\partial y} \frac{\partial N_{D_1}}{\partial x} - \frac{\partial N_{D_2}}{\partial y} \tag{2.65}
\]

The steady distribution can be computed by integrating Eq. 2.63 forward in space or Eq. 2.65 forward in time until the profiles become fully developed. Unfortunately, this is actually quite difficult for complex geometries due to singularities in the constitutive equations at zero shear stress regions in the flow, such as at the center of a tube. While various fixes for this problem have been used [66, 70, 99], here we side-step these problems by directly computing the fully developed concentration and velocity distributions through the following iterative algorithm:

1. Begin with a uniform concentration profile \( \phi = \phi_f \) and transverse velocity field \( u = v = 0 \).
2. Compute the axial velocity field $w$ from Eq. 2.45c.

3. With the axial velocity field from step 2, calculate $\dot{\gamma}$ and the geometry tensor $H_{ij}$ from their definitions in equations 2.42 and 2.40 respectively.

4. Solve for the particle pressure $\alpha\tau$ in the cross-section using equation 2.60 with the flow average concentration constrained at $\phi_f$.

$$\phi_f = \frac{\int \int u\phi dS}{\int \int u dS}$$ \hspace{1cm} (2.66)

5. Update the concentration field using the equation

$$\phi = \frac{\phi_m (\alpha \tau)^{1/3}}{\phi_m (2.17 \dot{\gamma})^{1/3} + (\alpha \tau)^{1/3}}$$ \hspace{1cm} (2.67)

which is obtained by inverting the definition of $\alpha\tau$ as shown below in Eq. 2.68.

6. Calculate the transverse velocity field $[u, v]$ from 2.44 with the knowledge of the particle pressure field $\alpha\tau$ from step 4

7. Repeat steps 2 through 6 beginning with the concentration and transverse velocity fields obtained from steps 5 and 6 respectively, and iterate until convergence is achieved.

The key to the robustness of this iterative algorithm is the use of the natural dependent variable of the system $\alpha\tau$ to map the concentration $\phi$.

$$\alpha\tau = \alpha\mu_r \dot{\gamma} = 2.17 \dot{\gamma} \frac{\phi^3}{\left(1 - \frac{\phi}{\phi_m}\right)^3}$$ \hspace{1cm} (2.68)
A simple analytical transformation thus connects $\alpha \tau$, $\dot{\gamma}$ and $\phi$. Therefore, once the particle pressure field is obtained from the solution of Eq. 2.60, the concentration field can be determined seamlessly using Eq. 2.67 with knowledge of $\dot{\gamma}$. Effectively, the iterative solution for $\phi$ provided by Eq. 2.67 simply requires that the concentration distribution be consistent with the calculated shear rate fields. The functional relationship, however, avoids the numerical difficulties typically encountered as $\dot{\gamma} \rightarrow 0$. Additionally, implementation of this algorithm is roughly an order of magnitude faster than the solution of the transient problem to the same tolerance; however it loses information on the rate of approach to steady state.

In the case of the isotropic model, the particle pressure is a constant as presented earlier in Eq. 2.20, rendering step 4 in the solution technique redundant. The concentration distribution obeys the simple relationship

$$\phi = \frac{c\phi_m}{\phi_m (2.17\dot{\gamma})^{1/3} + c} \quad (2.69)$$

where $c$ is a constant adjusted to constrain the flow average concentration to be $\phi_f$. Because of the weak dependence of concentration on the shear rate ($\dot{\gamma}^{1/3}$ dependence), convergence is achieved rapidly.

Implementation of this iterative solution procedure based on the calculation of the particle pressure is more complicated in cases where, for the anisotropic model, $\alpha \tau$ has singularities due to curvature. At the centerline of tube flow, for example, $\alpha \tau \sim r^d$ and thus, since $d$ is negative for the observed anisotropy, $\alpha \tau$ diverges. This solution for $\alpha \tau$ is non-physical, and this issue may be resolved by calculating the non-local finite particle interaction effects which are not accounted for in the governing equations. At high concentrations, particles interact over a length scale
that may be significantly greater than the particle radius \( a \). Therefore, the stress experienced by a particle in the flow is not the stress at the local particle position, but rather the stress averaged over a length scale of the order of the particle size around the particle location. Thus, at the centerline of a flow where the local shear stress goes to zero, the stress experienced by the particles will be non-zero and this effect can be captured if a non-local description of the particle stress is used. The most physically reasonable method of introducing non-locality in the particle stress is to average the stress over a volume whose radius is of the order of a few particle radii as discussed by Mills and Snabre [64]. However, this technique is computationally difficult to implement. Therefore, other ad-hoc fixes have been employed in the literature. For example, the local shear rate \( \dot{\gamma} \) can be augmented by a small constant \( \dot{\gamma}_0 \) as suggested by Morris and Boulay [66] and Miller and Morris [63] so that the shear rate is non-zero everywhere in the cross-section. This technique restricts the maximum concentration that can be achieved at the flow center. An alternative fix to this problem is to recognize that the curvature-induced singularity is directly associated with concentrations at maximum packing. Therefore, the numerical difficulty of \( \alpha \tau \) becoming singular at the flow center due to the anisotropy may be overcome by allowing the reduced second normal stress difference coefficient \( d \) to vanish at a high concentration \( \phi_0 \) short of maximum packing rather than treating it strictly as a constant (e.g. \( d = -0.54 \)). In our simulations, we have used the smooth approximation to a step function

\[
    d = \frac{-0.54}{2} \left[ 1 - \tanh \left( k_0 \left( \frac{\phi}{\phi_0} - 1 \right) \right) \right] 
\]  
(2.70)

where the constants \( k_0 \) and \( \phi_0 \) were taken as 100 and 0.58 respectively. Because
Eq. 2.70 modifies $d$ only at very high concentrations where the viscosity is extremely large, any choice of $\phi_0$ above 0.58 was not found to have a significant effect on calculated profiles. The value of $k_0$ and $\phi_0$ were varied from 0.58 to 0.615 and 100 to 600 respectively, but this produced a less than a 2% change in the magnitude of the secondary currents, while the concentration distribution was virtually unaffected, although the convergence was expectedly slower.

As is typical in convection-diffusion problems, an under-relaxation approach [76] had to be employed to achieve convergence whenever $u$ and $v$ were non-zero and the Peclet number was sufficiently high.

\[
(u, v, w, \phi)^{(n+1)} = \epsilon(u, v, w, \phi)^{\text{predicted}} + (1 - \epsilon)(u, v, w, \phi)^{(n)}
\]  

(2.71)

where $(u, v, w, \phi)^{\text{predicted}}$ are the fields calculated from the governing equations by using the solutions $(u, v, w, \phi)^{(n)}$ from the $n$th iteration. The under-relaxation coefficient $\epsilon$ depended on the Peclet number in the problem; larger Peclet numbers required smaller under-relaxation coefficients. For large values of the Peclet number (usually greater than 1000), the under-relaxation coefficient had to be set to as low as 0.0025 to force convergence. Such small values of $\epsilon$ resulted in a large number of iterations required to reach the solution with the desired tolerance. The loop was terminated when the error $\Delta \phi$ defined below was under $2 \times 10^{-4}$.

\[
\Delta \phi = \sqrt{\frac{\iint (\phi^{\text{predicted}} - \phi^{(n)})^2} {\iint dS}}
\]  

(2.72)

All the computations in this chapter were performed using the linear solvers in COMSOL 3.2. Some computational requirements of this technique are discussed in chapter 3.
In the subsections below, we will discuss the steady state concentration and velocity profiles for different geometries by comparing three cases: (a) the isotropic model, (b) the anisotropic model with $\text{Pe} = 0$ and (c) the anisotropic model again, but with a finite $\text{Pe}$ appropriately chosen to elucidate the effect of the secondary currents. Although the $\text{Pe} = 0$ case is non-physical (this would require a particle size infinitely greater than the conduit size), we still display the results for this asymptotic case, as it represents the limit when secondary currents are turned off, but the anisotropy-induced particle diffusion is not. It should be noted here that the area average concentration is not conserved for the three cases. This is because the simulations have been performed not for a fixed area average concentration, but for a fixed flow average or bulk concentration $\phi_f$. 

Figure 2.13. Conduit with trapezoidal cross-section of aspect ratio $W$. 

$1$ $2W$ $y$
Figure 2.14. Concentration (contour plot) and secondary current (quiver plot) profiles for the wedge geometry computed for the (a) isotropic model and the anisotropic model with (b) Pe= 0 and (c) Pe=250. The wedge has an aspect ratio of 4 over which the change in depth is 80%.
Figure 2.15. Axial (a) concentration ($\phi(x, 0)$) and (b) velocity ($w(x, 0)/\langle w \rangle$) profiles along the wedge axis for the isotropic model (dashed line) and for the anisotropic model with $Pe=0$ (dash-dotted line) and $Pe=250$ (solid line). The wedge has an aspect ratio of 4 over which the change in depth is 80%.
Figure 2.16. The concentration maximum along the midplane near the shallow end of the wedge as a function of the Peclet number. The wedge has an aspect ratio of 4 over which the change in depth is 80%. As can be seen from the graph, the concentration is much higher than the bulk concentration (30%, shown by the horizontal line) for Pe = 0, but decreases gradually as the Peclet number is increased and eventually drops below the bulk concentration for a Peclet number of approximately 235.
2.4.2 Wedge shaped channels

A simple geometry that demonstrates the strong effect of circulation within
the cross-section on the concentration profile is the wedge geometry (figure 2.13).
In this geometry, the depth varies linearly along the $x$ direction. For large aspect
ratios, the depth averaged velocity at any position scales as the square of the
thickness of the wedge at that position, while the stress scales directly as the wedge
thickness. Therefore, intuitively, one should expect migration of particles from the
thick, high shear stress regions of the wedge to the thin low shear stress regions of
the wedge, leading to a high concentration of particles in the thin regions. This
is exactly what is predicted by the isotropic model as can be seen from figure
2.14(a) and 2.15(a) for a wedge with an aspect ratio of 4 and a change of depth
of 80% over this width at a flow average concentration of 30%. Since the thin
regions also move with slower average velocities, the area average concentration
is 0.29, which is greater than the value of 0.28 calculated for a plain rectangular
channel with the same mean depth. If we compute the distributions with the
anisotropic model but with the Peclet number set to zero to include just the
migration effects of the anisotropy of the particle stress, then the concentration in
the shallow regions increases further [see figures 2.14 (b) and 2.15 (a)] yielding an
area average concentration of 0.31, which is greater than the bulk concentration
($\phi_f = 0.3$). If the complete effect of the anisotropy of the particle stress (to include
both secondary currents and particle diffusion) is considered by setting the Peclet
number to 250 [figure 2.14 (c)], it is observed that the particles actually migrate
out of the shallow regions into the deeper regions of the channel, a trend completely
opposite to what one would expect from intuition. Due to the symmetry breaking
provided by the wedge shape of the cross-section, transverse currents induced by
the anisotropy of the particle stress tensor pump particles from the thin regions out into the thicker regions of the wedge, yielding an area average concentration of 0.27. As can be seen from figure 2.15(a), the concentration near the shallow region (before the sharp drop near the right edge) for the isotropic model is around 40%, that predicted by the anisotropic model for Pe = 0 is 55%, while for Pe = 250, this concentration is just 28%. One anticipates that for larger Peclet numbers, this draining of particles out of the side pockets of the wedge will be stronger. In figure 2.16, we have shown the variation of the concentration maximum near the shallow region as a function of the Peclet number. This figure clearly shows the changing nature of the shallow region from being a pocket of high concentration to being a pocket of low concentration with increase in Peclet number. As a result of this transition, the velocity profiles get progressively more blunted with increase in the Peclet number as can be seen in figure 2.15(b).

2.4.3 Polygonal channels

Let us now consider the quasi-steady state flow of suspensions through conduits whose cross-sections are regular polygons. In figures 2.17 and 2.18, we have shown the concentration and axial velocity profiles along the polygon radius and the height of the polygon unit cell (see figure 2.9) for \( \phi_f = 0.4 \) and for Pe = 0 and Pe = 7500 for an equilateral triangle \((n = 3)\). Also shown in figure 2.19 are contour plots of the concentration along with quiver plots of the secondary current profiles for the isotropic model and the anisotropic model with Pe = 0 and Pe = 7500. Because of the close proximity of the two surfaces, the average stress near a polygon corner is smaller than the average stress based on the radius of the polygon. Therefore, the corner regions are predicted to be pockets of high concentration according
Figure 2.17. Concentration $[\phi(x,0)]$ and axial velocity $[w(x,0)/\langle w \rangle]$ profiles for the equilateral triangle geometry ($\theta = \pi/6$ as defined in Eq. 2.52) along its radius computed using the isotropic model (dashed line) and the anisotropic model with $Pe=0$ (dash-dotted line) and $Pe=7500$ (solid line). The co-ordinate $s$ is measured linearly along the radius of the polygon, and has values of 0 and 1 at the polygon center and corner respectively.
Figure 2.18. Concentration and axial velocity profiles for the equilateral triangle geometry ($\theta = \pi/6$ as defined in Eq. 2.52) along its height computed using the isotropic model (dashed line) and the anisotropic model with $Pe= 0$ (dash-dotted line) and $Pe=7500$ (solid line). The co-ordinate $y$ is measured linearly along the height of the triangular unit cell, and has values of 0 and 0.5 at the triangle center and base center respectively.
Figure 2.19. Concentration $\phi$ (contour profile) and secondary current profiles $[u, v]$ (quiver profile) for the equilateral triangle geometry ($\theta = \pi/6$ as defined in Eq. 2.52) at $\phi_I = 0.4$ computed using the (a) isotropic model and with the anisotropic model with (b) $Pe= 0$ and (c) $Pe=7500$. 
Figure 2.20. Concentration and axial velocity profiles for different regular polygons for $\phi_f = 0.4$ and $Pe = 1000$ along the radius of the polygon. $n = 3$ (Triangle) - solid line, $n = 4$ (Square) - dashed line, $n = 6$ (Hexagon) - dash-dotted line, $n = 10$ (Decagon) - parallel line. The co-ordinate $s$ is measured linearly along the radius of the polygon, and has values of 0 and 1 at the polygon center and corner respectively.
to the isotropic model as can be seen in figures 2.17(a) and 2.19(a). Also, for the isotropic model, the transverse velocity components are identically zero since normal stress differences have been ignored. If the particle stress anisotropy is switched on but the effect of the secondary currents is suppressed by setting Pe = 0, there is stronger particle migration towards the center of the polygon as is observed in the case of a circular tube, which can be seen by examining figure 2.17(a). For the anisotropic model, irrespective of the Peclet number, the second normal stress difference drives a secondary current that flows from the corner of the polygon to the center and then back to the center of the polygon base as was seen in section 2.3. Thus, as the Peclet number is increased from zero, the secondary flow pumps particles out of the corner into the center of the polygon [see figure 2.19(c)]. In figure 2.17, we can see that while the minimum concentration along the radius of the triangle is approximately 40% for the isotropic model, it drops to 32% close to the corner for the anisotropic model with Pe = 7500, showing that there is strong convection of the particles out of the corner. On the other hand, the secondary current increases the concentration along the height of the triangle as can be seen in figure 2.18.

It is interesting to note that much higher Peclet numbers are required to see significant modification of the concentration distribution in the triangle than in the wedge geometry. This is in part due to the way the Peclet number is defined in the two geometries. For a more direct comparison, Pe for the wedge geometry could be defined in terms of the width rather than the half-depth. With this definition, the effect of Pe is comparable in both geometries.

In figure 2.20, we have shown the variation of the concentration along the polygon radius for different regular polygons. With an increase in the polygon
index \( n \), the characteristic angle \( \theta \) in the corner increases and the region of reduced stress vanishes. As \( n \) increases, therefore, both the difference in these length scales and the local concentration maximum in the corners decreases in magnitude. Ultimately, for a circle, the corners disappear and there remains only the concentration maximum in the center. The magnitude of the circulation velocity likewise vanishes.

2.4.4 Elliptical channels

Consider the elliptical channel of aspect ratio \( W \) shown in figure 2.4. In figure 2.21, we have shown the concentration contours and the secondary velocity profiles for the flow of a 40% suspension (\( \phi_f = 0.4 \)) in an elliptical channel of aspect ratio 2 for the isotropic model and the anisotropic model with \( \text{Pe} = 0 \) and \( \text{Pe} = 1600 \). Shear-induced migration results in the diffusion of particles from the top and bottom of the tube towards the midplane (major axis) of the ellipse. The second normal stress induces a convection from the high curvature regions near the sides towards the center. At high \( \text{Pe} \) [e.g. figure 2.21(c)], the combination of these effects results in a significant depletion of particles from the side regions. This effect is more clearly seen in figure 2.22, where we have plotted the concentration distribution along the major axis of the ellipse. For \( \text{Pe} = 1600 \), the particles are convected from the side walls towards the center of the ellipse more strongly relative to the isotropic and the \( \text{Pe} = 0 \) concentration profiles, causing a “shoulder” in the profile. The concentration predicted by the isotropic model near the side wall is around 37%, while that predicted by the anisotropic model at \( \text{Pe} = 1600 \) is just 27%. The depletion near the side wall with respect to the 40% flow average concentration is thus nearly 4 times that predicted by the isotropic model. The
Figure 2.21. Concentration $\phi$ (contour profile) and secondary current profiles $[u,v]$ (quiver profile) for the elliptical geometry with an aspect ratio of 2 computed using the (a) isotropic model and the anisotropic model with (b) $\text{Pe} = 0$ and (c) $\text{Pe} = 1600$. 

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Figure 2.22. (a) Concentration $\phi(x, 0)$ and (b) axial velocity profiles $w(x, 0)/\langle w \rangle$ along the centerline for the elliptical geometry with an aspect ratio of 2: dash line - isotropic model, dash-dot line - Pe = 0 and solid line - Pe = 1600.
axial velocity $w/\langle w \rangle$ along the major axis is shown for the three cases in figure 2.22(b). While very similar, the velocity profile for the anisotropic model with Pe = 1600 shows the maximum blunting among the three due to the stronger migration of the particles towards the center.

For an aspect ratio of 2 and a flow average concentration of 40\%, the average circulation velocity $\langle \sqrt{u^2 + v^2} \rangle / \langle w \rangle$ is shown as a function of Peclet number in figure 2.23. The magnitude of the circulation velocity drops sharply as the Peclet number is increased from zero to small values, and then decreases gradually to a non-zero asymptote for larger values of Pe, but the impact of the secondary flow on the concentration profile increases progressively. Since the depth of the channel
Figure 2.24. Variation of the average magnitude $\langle \sqrt{u^2 + v^2} \rangle / \langle w \rangle$ of the secondary currents with the flow average concentration $\phi_f$ for an elliptical channel of aspect ratio 2 for $Pe = 400$. 
Figure 2.25. (a) Concentration $[\phi(x, 0)]$ and (b) axial velocity $[w(x, 0)/\langle w \rangle]$ profiles along the horizontal axis for an elliptical channel of aspect ratio 2 for different flow average concentrations. Dashed-dotted line - Pe = 0 and solid line - Pe = 400.
is small in the side regions, the curvature of the streamlines is greater near the side walls than near the center of the ellipse. This leads to large circulation cells that flow in from the side walls to the center along the horizontal axis and weak circulation cells near the top and bottom of the minor axis, resulting in 8 total cells in the cross-section as shown in figure 2.21, which is twice the number of cells observed in the case of a constant concentration field in section 2.3. As the Peclet number is increased, the depletion of particles from the side regions leads to a relative decrease in the driving force for the cell along the major axis since this driving force is directly related to \( \alpha \), a monotonically increasing function of concentration. This results in a decrease in the size of this cell and an increase in the size of the cell along the minor axis with increase in Peclet number. The interaction of the two convection cells leads to a stretching out of the high concentration region along the dividing streamline [see figure 2.21(c)].

Next, we examine the variation of the concentration and velocity distributions with the average concentration \( \phi_f \). We see in figure 2.24 that the magnitude of the secondary currents increases monotonically with concentration, which again is because the secondary currents scale as the reduced normal stress \( \alpha \). In figure 2.25, we have shown the concentration and axial velocity profiles along the major axis of an ellipse of aspect ratio 2 for \( \phi_f = 0.1, 0.3 \) and \( 0.5 \) at \( \text{Pe} = 0 \) and \( \text{Pe} = 400 \). The deviation between the two curves at \( \text{Pe} = 0 \) and \( \text{Pe} = 400 \) shows the additional segregation produced by the secondary currents. We can see that the difference between the curves at the two Peclet numbers is small for the low 10% and the high 50% average concentrations and is large for the intermediate 30% concentration. This can be explained on the basis of the scaling analysis presented in Eq. 2.54. For low concentrations, the normal stresses are negligible and therefore there is
little difference in the concentration profile with the inclusion of the secondary currents. At high concentration, shear-induced migration increases more rapidly with concentration than do the secondary currents, thus the effect of convection shows a maximum at intermediate concentrations.

Finally, we study the effect of the aspect ratio of the ellipse on the concentration and velocity distributions. The concentration and axial velocity profiles along the major axis of the ellipse are shown for three different aspect ratios: 2, 3 and 5 in figure 2.26 for $Pe = 400$. We see in this case that for the anisotropic model with $W = 5$ [figure 2.26(c)], the concentration curve shows two maxima. This can also be seen in the contour plot of the concentration profile in figure 2.27 for this aspect ratio. In fact, the concentration variation for $W = 5$ appears to be an abrupt change from the monotonic variations observed for the smaller aspect ratios of 2 and 3 in figures 2.26(a) and 2.26(b) respectively. The appearance of the two concentration maxima along the major axis of the ellipse is not a numerical artifact, but the result of an instability due to second normal stress difference induced secondary currents. In the limit of large aspect ratios where the effect of the side regions is weak, weak cells are formed away from the side regions whose wavelengths are comparable to the depth of the ellipse. To understand this phenomenon, we turn to the limiting geometry of plane-Poiseuille flow.

2.4.5 Plane Poiseuille flow

Consider the plane Poiseuille flow of a suspension through a rectangular channel of aspect ratio $W$ without sidewalls as depicted in figure 2.28. The solution of the particle distribution equation for this simple geometry is quite straightforward: $\alpha \tau = \text{constant}$, as noted in section 2.2. Since $\tau$ is a function of $y$ only, the
Figure 2.26. (a) Concentration \([\phi(x, 0)]\) and (b) axial velocity \([w(x, 0)/\langle w \rangle]\) profiles along the horizontal axis for elliptical channels of different aspect ratios with \(\phi_f = 0.4\) and \(Pe = 400\). Dashed line - \(W = 2\), dashed-dotted line - \(W = 3\) and solid line - \(W = 5\).
Figure 2.27. Concentration $\phi$ (contour profile) and secondary current profiles $[u, v]$ (quiver profile) for the elliptical geometry with an aspect ratio of 5 for $\text{Pe} = 400$ and $\phi_f = 0.4$. 
Figure 2.28. Schematic for the plane Poiseuille flow of a suspension. The cross-section is described by the co-ordinates \(x\) and \(y\), \(y\) being 0 at the midplane, and \(z\) represents the flow direction. The two plates are separated by a distance \(2B\).

The concentration profile does not vary in the vorticity direction. Also, this solution is satisfied by the governing equations for all Peclet numbers, because the secondary current is identically zero for this solution. We shall call this solution \(\Phi(y)\). At low Peclet numbers, it was possible to retrieve this trivial solution via the iterative solution technique proposed earlier in this section. However, for Peclet numbers greater than 400, a solution with the required tolerance was not accessible with this technique. The similarity between the iterative solution with underrelaxation employed here and the marching solution to the transient problem suggests that this trivial solution \(\Phi\) could be unstable at large Peclet numbers.

To investigate the stability of \(\Phi\), we integrated the axially invariant time dependent problem defined in Eq. 2.65 with time to determine the asymptotic behavior for an area average concentration of 37.5% and an aspect ratio of 5 at different Peclet numbers. In this startup flow problem, we circumvented the issue of concentrations exceeding maximum packing by solving for the dependent variable \(\Psi\).
which is defined as

$$\Psi = \log \left[ \frac{\alpha \tau}{2.17 \dot{\gamma}} \right]$$  \hspace{1cm} (2.73)

The concentration distribution can be obtained by inverting the above map:

$$\phi = \frac{\phi_m \exp(\Psi/3)}{\phi_m + \exp(\Psi/3)}$$  \hspace{1cm} (2.74)

To avoid complications due to the vanishing shear-rates at flow centers, the shear rate $\dot{\gamma}$ was augmented by a constant $\dot{\gamma}_0$ as suggested by Miller and Morris [63].

$$\dot{\gamma} = \sqrt{w_x^2 + w_y^2 + \dot{\gamma}_0^2}$$  \hspace{1cm} (2.75)

$\dot{\gamma}$ was taken as $10^{-4}$ in all the simulations. The practical implication of this numerical fix is that concentrations never quite reach maximum packing, but rather are limited to approximately 58%. Apart from this, the computed profiles were insensitive to the choice of $\dot{\gamma}_0$. Because this method of non-localization of the particle stress was employed, the reduced second normal stress difference coefficient $d$ was kept unmodified at the constant value of $-0.54$. Symmetry boundary conditions were applied at the two side boundaries of the computational domain.

The variation of the flow average concentration with time is shown in figure 2.29 for a range of Peclet numbers. At $\text{Pe} = 0$, the flow average concentration increases with time because of particle migration towards the centerline, and reaches an asymptote. It can be seen that the curves for $\text{Pe} = 25$ and 100 lie on top of the curve at $\text{Pe} = 0$, which represents the case when secondary currents are turned off. For $\text{Pe} = 400$, however, the flow average concentration follows the $\text{Pe} = 0$ curve.
Figure 2.29. The variation of the flow average concentration with time for the startup flow with a initially uniform concentration distribution ($\phi_a = 0.375$) for different Peclet numbers. For $Pe = 0, 25$ and 100, the flow average concentration curves lie on top of each other and produce the steady state case of $\alpha \tau = \text{constant}$ (solid line). For $Pe = 400, 1000$ and 2500 (dash, dot and dash-dot lines respectively), the flow average concentration curves stay along the $Pe = 0$ initially, but fall off eventually, displaying an oscillatory behavior.
Figure 2.30. The concentration ($\phi$) profiles at $tGa^2/\mu_0B = 50$ for Peclet numbers of 0, 25, 100, 400 and 1000. The area average concentration is 0.375 for all the simulations.
for a short while, and then falls off the curve to exhibit an oscillatory variation. The reduction in the flow average concentration is due to mixing induced by the secondary currents within the cross-section. A similar behavior is observed at Pe = 1000 and Pe = 2500. The concentration distributions at $tGa^2/(\mu_0B) = 50$ for different Peclet numbers are shown in figure 2.30. The concentration profiles for Pe = 0, 25 and 100 are invariant in the vorticity ($x$) direction, which is consistent with the solution $\Phi$. However, for larger Peclet numbers, the solution is unstable with periodic structures in the $x$ direction. These periodic structures translate in the $x$ direction, and thus there is no steady state for these Peclet numbers (see figure 2.31).

In retrospect, the instability of $\Phi$ at high Peclet numbers should not be surprising, since it has been shown theoretically by Brady and Carpen [9] that the Couette flow of a combination of a suspension (modeled with constant concentration) and a Newtonian fluid superposed over each other is unstable to spanwise perturbations due to a jump in the second normal stress difference between the two fluids. It is possible to show using similar modeling that the same is true for plane Poiseuille flow of a suspension and a Newtonian fluid. Performing the complete stability analysis for plane Poiseuille flow of a suspension is, however, beyond the scope of this thesis. We make a note of this instability here, however, as it is necessary to understand the steady results for conduits of large aspect ratios.

2.4.6 Rectangular channels

Consider the rectangular channel of aspect ratio $W$ shown in figure 2.7. In figure 2.32, we have shown the concentration and secondary current profiles along
Figure 2.31. The concentration profiles in plane Poiseuille flow at a Peclet number of 400 for times $tGa^2/\mu_0 B$ equal to (a) 38 (b) 44 (c) 50. The figures show waves traveling from the center to both left and right with time.
Figure 2.32. (a) Concentration and (b) axial velocity profiles along the horizontal axis $\phi(x, 0)$ for the rectangular channel with aspect ratio $W = 2$ and $\phi_f = 0.4$ for the isotropic model (dashed line) and the anisotropic model with $Pe = 400$ (dash-dotted line) and $Pe = 1000$ (solid line).
Figure 2.33. Concentration $\phi$ and secondary current profiles $[u, v]$ (quiver profile) for a rectangular channel with an aspect ratio of 2 at 40% bulk concentration for (a) $Pe = 0$ (b) $Pe = 400$ (c) $Pe = 1000$
the horizontal symmetry axis of a rectangular channel with an aspect ratio of $W = 2$ at a flow average concentration of 40%. As seen in the case of the ellipse, the concentration profiles are shifted away from the side walls towards the center of the channel for the anisotropic model relative to the isotropic model. For large Peclet numbers (e.g. $Pe = 1000$ as shown in the figure), one observes the emergence of a second maximum in the concentration. This can be seen more clearly in figure 2.33, where we have shown the contours of the concentration and velocity profiles for $Pe = 0$, $= 400$ and $= 1000$ for $\phi_f = 0.4$ and aspect ratio of $W = 2$. The secondary maximum is driven by the same curvature effects that produce the instability at high $Pe$ for unbounded plane Poiseuille flow. For the rectangular channel at small aspect ratios, however, the presence of the side walls fixes the location of the maximum in the channel and convergence is observed even at $Pe = 1000$. Note that the concentration between the two concentration maxima varies from around 58% to maximum packing, which is not large compared to the the variation between the concentrations at the wall and near the center of the channel. Therefore, it is unlikely that these concentration maxima will be accessible by experiment.

Next, we investigate the effect of aspect ratio on the concentration and secondary current profiles in figure 2.34 with $\phi_f = 0.4$ and $Pe = 100$. Note that this Peclet number is well below the critical Peclet number for the onset of the instability in unbounded plane Poiseuille flow, and thus convergent steady solutions could be obtained at all aspect ratios. For low aspect ratios, there are three circulation cells in each quadrant of the rectangle. For the uniform concentration asymptote considered in section 2.3, there are only two cells per quadrant. The third cell at the top of the channel, similar to the one observed in the case of the
Figure 2.34. Concentration $\phi$ (contour profile) and secondary current profiles $[u, v]$ (quiver profile) for the rectangular channel with aspect ratios (a) $W = 2$ (b) $W = 3$ and (c) $W = 5$ for $Pe = 100$ and $\phi_f = 0.4$. 
ellipse, appears because of the competition between the driving forces for circulation near the channel top & bottom and the channel sides. As the aspect ratio increases, this third cell decreases in size and eventually vanishes due to negligible curvature of the axial streamsurfaces in this region. For large aspect ratios [e.g. figure 2.34(c)] at this small Peclet number, a secondary maximum is observed near the side walls from both the iterative and time-dependent solutions. Thus for low Peclet numbers, the concentration profile obtained is steady, with a secondary maximum obtained at larger aspect ratios.

For high Peclet numbers and large aspect ratios, a steady solution was not obtained with either the iterative or the time integration techniques. Analogous to the case of plane Poiseuille flow, spatially-periodic concentration profiles translate within the cross-section, resulting in an unsteady variation of the average concentration for the startup flow problem.

2.5 Conclusions

In this chapter, the significance of the effect of second normal stress difference induced secondary currents on the steady state concentration profile for the pressure-driven flow of a suspension through a conduit of arbitrary cross-section is demonstrated through simulations. Traditionally, suspensions have been modeled as isotropic Newtonian fluids with effective viscosities that vary with concentration when calculating the velocity distribution through a conduit. The results presented in this chapter, however, demonstrate that it is critical to consider the complete non-Newtonian rheology of a concentrated suspension when modeling flows in complex geometries. To the best of our knowledge, this is an aspect of continuum suspension modeling that has not been previously recognized with the
notable exception of the stability analyses of two layer Couette and falling film
flows by Brady and Carpen [9].

For unidirectional flow, the anisotropic normal stress differences exhibited by
suspensions cannot be balanced by an isotropic fluid pressure, and therefore these
particle stress gradients will induce non-zero secondary currents. While the mag-
nitude of the secondary currents due to non-Newtonian effects is small, in many
cases they are the dominant mechanism governing the resulting particle concentra-
tion distribution and lead to counterintuitive results. For example, it was shown
for the wedge geometry that the concentration profile obtained from the isotropic
shear-induced migration argument turns out to be the exact opposite of what is
obtained in the presence of the secondary flows in the cross-section. Contrary
to the intuition that particles should migrate into shallow regions and corners in
a geometry on account of their relatively lower average shear stresses, the com-
plete solution of the governing equations shows that secondary currents at high
Pe actually flush particles out of these regions. An important implication of this
observation is that secondary currents can circumvent the plugging of suspen-
sions in the notches and corners of a conduit. Also, the secondary currents may
provide a mechanism for enhancement in the mass transfer of solutes within the
cross-section, and this is discussed using scaling arguments in the appendix to this
chapter.

The secondary currents reach their largest magnitudes when the aspect ratio
of the conduit is approximately 2. With increase in Peclet number, the magnitude
of the secondary currents decreases, but the effect of the secondary currents on
the concentration profile increases progressively. The secondary flow effects are
most likely to be observed when the average concentration ranges from 30% to
50%, when the magnitude of the currents is high relative to the shear-induced migration velocity for reasonably large Peclet numbers.

The concentration distributions for plane Poiseuille flow of a suspension obtained by the time integration of the startup flow problem are invariant in the vorticity direction only for Peclet numbers below a critical value. Beyond this critical Peclet number, this solution is unstable and the secondary flow induced by second normal stress differences leads to upwelling and downwelling regions, producing concentration maxima and minima. Also, these upwelling and downwelling regions are found to translate within the cross-section, and thus these solutions are unsteady.

The origin of the secondary currents in the flow of suspensions through conduits at steady state is attributed to the breaking of symmetry caused by an azimuthal gradient in the curvature of the flow streamlines such as occurs in any non-axisymmetric flow. This form of symmetry breaking is weak and leads to small magnitudes of the circulation currents. Therefore, the effects of these currents is realized only at high Peclet numbers. Because conduits are typically 10 to 50 particle diameters across, however, the Peclet number in experiments usually ranges from 100 to 2500, leading secondary currents to dominate the migration process.

As is well known, in the absence of both secondary currents and wall-slip effects, the steady concentration distribution of a suspension flowing through a straight conduit of arbitrary cross-section is independent of the particle size. With the inclusion of non-Newtonian secondary currents, however, we have shown that there is a pronounced effect of the size ratio $B/a$ for all non-axisymmetric flows.
2.6 Acknowledgements

We thank Prof. Ronald Larson for the helpful discussions and for providing references that discuss the secondary flow profiles observed in non-circular geometries for polymer flows.

2.7 Appendix

It has been long known that the diffusivity of solutes in suspensions of both rigid and deformable particles is enhanced by shear-induced self diffusion[14, 102]. The augmentation $A$ of the solutal diffusivity, which is defined as $\frac{D_{\text{eff}}}{D_0} - 1$, scales as $\frac{\gamma a^2}{D_0} = \frac{U a^2}{R D_0}$, for instance, in tube flow. Here $D_{\text{eff}}$ is the observed diffusivity, $D_0$ is the molecular diffusivity and $R$ is the radius of the tube. In the presence of circulation within the conduit, there will be further enhancement in mass transfer due to convection, which depends on the thickness of the boundary layer next to the wall and in turn, on the Peclet number defined on the basis of the characteristic velocity scaling $U_{cs}$ of the circulation currents.

$$\text{Pe} = \frac{U_{cs} R}{\tilde{D}(\phi_f) U a^2 / R}$$ (2.76)

Larger the Peclet number, thinner is the boundary layer and greater is the mass transfer at the wall. We compare here two sources of convection within the cross-section: Dean vortices and second normal stress difference induced secondary currents.

For curved tubes with sufficient diameter, centrifugal forces induce secondary flows called Dean vortices within the cross-section of the tube which are responsible for enhanced heat and mass transfer in coiled tubes. The dimensionless
number that characterizes the flow through a coiled tube is the Dean number:

\( K = 2(R/R_c)^{1/2}\text{Re} \),

where \( R_c \) and \( \text{Re} = RU \rho/\mu_0 \mu_r(\phi_f) \) are the radius of the coil and the Reynolds number respectively. For a loosely coiled pipe, the cross-sectional secondary currents scales as \( \text{Re}(R/R_c)^{1/2}U \). If the radius of the curved tube is reduced, the enhancement in mass transfer due to Dean vortices diminishes. For small tube radii, the enhancement in mass transfer is solely due to the shear-induced diffusion \([14, 102]\). There exists, however, an intermediate length scale for the radius \( R \) of the curved tube where the secondary currents due to the second normal stress differences are significant compared to the Dean currents. The scaling for the secondary currents discussed in this chapter is \((-d)k\alpha(\phi_f)U\) (Eq. 2.50) and these will be significant compared to the magnitude of the Dean vortices provided

\[
(-d)k\alpha U \gg \frac{RU \rho}{\mu_0 \mu_r} \left( \frac{R}{R_c} \right)^{1/2} U \tag{2.77}
\]

or

\[
R \ll \left[ \frac{R_c \mu_0^2 (-d)^2 k^2 (\alpha \mu_r)^2}{\rho^2 U^2} \right]^{1/3} \tag{2.78}
\]

On the other hand, the effect of the secondary currents on the concentration and velocity profiles is realized only when \( R >> a \). This gives us the domain of validity of the impact of the second normal stress difference induced circulation.

\[
a \ll R \ll \left[ \frac{R_c \mu_0^2 (-d)^2 k^2 (\alpha \mu_r)^2}{\rho^2 U^2} \right]^{1/3} \tag{2.79}
\]

Let us examine the above condition for the case of blood flowing through a coiled artery. Arteries do not have circular cross-sections; they are roughly oval in cross-section \([44]\) and therefore can provide sufficient symmetry breaking to give rise to secondary currents. Although the normal stress difference measurements for
blood are difficult to carry out and constitutive equations for blood in terms particulate stress are not known (to the best of the authors’ knowledge), the second normal stress difference coefficient for an emulsion of surfactantless, monodisperse, Newtonian drops under isothermal Stokes flow conditions has been determined by numerical simulations [101] to be strongly negative and a fairly insensitive function the capillary number. We conjecture that second normal stress difference coefficient for blood should have a behavior intermediate to that of rigid particle suspensions and emulsions, so that the second normal stress difference coefficient is still significant and negative. Also for blood, \( \mu/\rho \sim 0.04 \text{ cm}^2/\text{sec} \), and \( \phi_f = 0.45 \) (the volume fraction of RBCs). If the coil radius \( R_c \) is assumed to be 1 cm and \( U \sim 0.1 \text{ cm/s} \), then for the normal stress difference induced secondary currents to be the dominant mode of mass transfer enhancement, we must have

\[
4\mu m \ll R \ll 170\mu m \tag{2.80}
\]

This range of radii coincides well with that of arterioles in the blood vessel system. Thus, second normal stress difference induced secondary currents are most likely to be dominant in arterioles.
3.1 Introduction

When a shear stress is applied to an initially settled bed of particles, the height of the settled bed increases, thereby entraining the unpacked particles into the fast-moving suspending fluid above it. This is the phenomenon of resuspension and though it is commonly associated with turbulence and high Reynolds numbers, it has also been observed at vanishingly small Reynolds numbers. In the last two decades, this viscous resuspension has received a fair amount of attention in the literature. Experimental and theoretical investigations of this phenomenon have been carried out in different geometries such as pressure-driven plane channel flow [67, 79, 80], cylindrical Couette flow [3], tube flow [5, 69, 94, 99] and parallel plate flow [18, 19, 32, 50, 53, 57].

Viscous resuspension is caused by the phenomenon of shear-induced migration, whereby irreversible particle interactions lead to migration across the flow streamlines. Shear-induced migration [55] is characterized by the diffusive scaling $\dot{\gamma}a^2$, where $\dot{\gamma}$ is the local shear rate, and $a$ is the particle radius. In fully developed resuspension flow through a conduit, it is the flux due to the shear-induced migration of particles that balances the gravitational flux of the particles within
the plane of shear and can be characterized by the Shields parameter $\psi$ which is a dimensionless number defined as

$$
\psi = \frac{\text{Shear-induced migration velocity}}{\text{Stokes sedimentation velocity}} = \frac{\dot{\gamma} a^2 / B}{a^2 \Delta \rho g / \mu_0} = \frac{U_B a^2 / B}{a^2 \Delta \rho g / \mu_0} = \frac{U \mu_0}{B^2 \Delta \rho g} \quad (3.1)
$$

where $B$ is the characteristic length scale of the conduit, $U$ is the average velocity of the suspension through the conduit, $\dot{\gamma}$ is the shear rate with the characteristic scaling $U/B$, $\mu_0$ is the viscosity of the suspending fluid, $\Delta \rho$ is the difference between the densities of the particles and the suspending fluid and $g$ is the acceleration due to gravity.

As discussed in chapter 2, two major approaches have been adopted in the literature in order to model the phenomenon of shear-induced migration: the diffusive-flux model (or the trajectory model) and the suspension balance model. The diffusive flux model [55, 73] involves migration terms due to gradients in shear stress and gradients in concentration. Based on this model, Leighton and Acrivos [53] also proposed the following flux balance for resuspension in simple shear flow:

$$
\frac{2}{9} f \frac{a^2 g \Delta \rho}{\mu_0} \phi = -\dot{\gamma} a^2 \hat{D}_|| \frac{\partial \phi}{\partial z} \quad (3.2)
$$

where $\phi(z)$ is the vertical concentration profile, $f$ is the hindered settling factor and $\hat{D}_||$ is the dimensionless shear-induced diffusivity in the velocity gradient direction. They also experimentally measured the expansion $\Delta h$ of a settled layer of particles in an annular parallel plate device as a function of the applied shear rate, and found that for large Shields parameters (large shear rates), the bed expansion was approximately proportional to $\psi$, thus proving that the viscous resuspension process is consistent with the phenomenon of shear-induced migration.
The shear-induced diffusion model of Leighton and Acrivos [55] was extended by Phillips et al. [73] to more general two dimensional shear flows. Following this formulation, Zhang and Acrivos [99] described the cross-sectional diffusive particle flux vector $\mathbf{N}_D$ in resuspension flow through a tube with the equation:

$$N_{D_i} = -\hat{D}_c(\phi)\gamma_j \frac{\partial \phi}{\partial x_j} - \hat{D}_s(\phi)\frac{\partial \gamma_j}{\partial x_j} - \frac{2 a^2}{9 \mu_0} f \Delta \rho g \phi \delta_{ij}$$  \hspace{1cm} (3.3)

Here $\hat{D}_c$ and $\hat{D}_s$ are the scalar shear-induced diffusion coefficients that describe particle migration due to gradients in concentration and shear rate respectively. Zhang and Acrivos [99] also assumed Newtonian rheology for the suspension, and consequently the secondary currents in the cross-section of the tube were driven purely by body forces in their model. They qualitatively compared the results of their model to the MRI measurements of Altobelli et al. [5]. A key feature of pressure-driven resuspension flow in a conduit is that in the limit of large Shields parameters (weak gravitation), the high shear stress regions near the walls lead to a concentration profile such that particles are highly concentrated near the center, a result that was noted in the false color images of Altobelli et al. [5]. For negatively buoyant particles, such a stratification should lead to a buoyancy-driven downward convection in the center and a corresponding upwelling region near the side walls, as was indeed predicted by Zhang and Acrivos [99]. This convection pattern leads to a suspension-clear fluid interface that is concave upward. Unfortunately, however, the experimentally observed interface is concave downward, the exact reverse of what was predicted by the simulations. Zhang and Acrivos [99] noted that such a profile could only occur if the convective flow was turned off or was upward rather than downward in the center, however they were unable to explain why this would occur.
Since the original model devised by Leighton and Acrivos [55] was for planar
unidirectional flows only, the extension due to Phillips et al. [73] used by Zhang
and Acrivos [99] makes the implicit assumption that shear-induced migration is
isotropic. However, suspensions are known to be non-Newtonian, exhibiting, in
particular, a strong second normal stress difference. As discussed in detail in
chapter 2, the non-Newtonian stress tensor for suspensions can be integrated into
the description of the particle flux by the suspension balance model. However,
the complete anisotropic suspension balance model has not been applied to study
resuspension in a tube; the resuspension phenomenon has been investigated theo-
retically in the literature only using isotropic suspension balance models. Morris
and Brady [67] used the isotropic pressure/temperature formulation to obtain the
steady state distributions of velocity and concentration for the entire range of
Shields parameters in plane Poiseuille flow, and found good agreement with the
results of their Stokesian dynamics simulations. They also considered the ignited
state, i.e. when the resuspension is almost complete ($\psi >> 1$) and when migra-
tion due to stress and concentration gradients are both important. Since they
were considering deviations of the concentration and velocity distributions from
the neutrally buoyant case, they characterized their flows by the inverse of the
Shields parameter, called the buoyancy number $N_b$, which they defined as

$$ N_b = \frac{2B^2 \Delta \rho g}{9U\mu_0} \quad (3.4) $$

where $B$ is the half depth of the channel. For small $N_b$, as in the case of tube
flow, the steady state profiles of Morris and Brady [67] predict a dense suspension
flowing over a light suspension. Such adverse density arrangements are, however,
susceptible to a Rayleigh-Taylor-like instability in channel flow. Carpen and Brady
investigated the stability of the steady state profiles for pressure-driven flow through two parallel plates to spanwise perturbations, again using the suspension balance approach with the particle pressure/temperature formulation. Their simulations indicate that the small $N_b$ profiles are unstable to buoyancy-driven convection, although with a very small growth rate. Norman et al. [69] modeled the resuspension problem in a tube using the suspension balance approach. They used the isotropic version of the empirical constitutive equations of Zarraga et al. [97] for the particle stress tensor. Norman et al. [69] also measured the concentration profiles using electrical impedance tomography (EIT) measurements and obtained qualitative agreement of their measurements with their simulations.

In this chapter, we investigate the fully developed concentration and velocity profiles in a tube as functions of the buoyancy number $N_b$ using the complete anisotropic constitutive equations of Zarraga et al. [97]. In particular, we focus on the variation of these profiles with the tube radius to particle radius ratio $R/a$. Note that the particle radius $a$ scales out of the buoyancy number in Eq. 3.1 because the shear-induced migration velocity and gravitational settling velocity both scale as the square of the particle radius. However, as was shown in chapter 2, the particle radius also appears in the concentration balance equation in the form of the scaling $R^2/a^2$, which is the Peclet number $Pe$ of the cross-sectional currents. For plane Poiseuille flow and plane Couette flow, the secondary currents are identically zero and therefore the profiles are indeed independent of the particle radius in the absence of instabilities. For other geometries, especially when the aspect ratio is order 1, the secondary currents are non-zero in general. The magnitude of the secondary currents is small compared to the magnitude of the axial velocity. However, typical experiments are performed with the ratio of the
characteristic conduit dimension and the particle radius selected as 20 or more in order to enable modeling of the suspension as a continuum. Therefore, as was also shown in chapter 2, the impact of the secondary currents on the concentration profile can be very strong.

In the next section, the particle stress model based on the constitutive equations of Zarraga et al. [97] combined with the continuity and momentum equations is laid out for the tube resuspension problem under creeping flow conditions. These governing equations are used to determine the concentration and velocity profiles as a function of the buoyancy number \( N_b \) for different \( a/R \) ratios for a fixed flow average or bulk concentration \( \phi_f \). In section 3.3, we contrast the results of the anisotropic and isotropic models and also compare these with the experimental results of Altobelli et al. [5]. We conclude in section 3.4 by summarizing our findings.

3.2 Theory

3.2.1 Governing equations

Consider the resuspension of a slurry of non-colloidal, negatively-buoyant particles in the tube geometry shown schematically in figure 3.1. The axial direction is represented by \( z \) while the co-ordinates in the cross-section are \( x \) and \( y \) with gravity pointing in the negative \( y \) direction. The components of the velocity vector in the three directions \( x, y \) and \( z \) are \( u, v \) and \( w \) respectively. The governing equations are rendered dimensionless as follows:

\[
[x, y, z] = [x^*, y^*, z^*]R; [u, v, w] = [u^*, v^*, w^*]GR^2/\mu_0; P = P^*GR \tag{3.5}
\]
Figure 3.1. Schematic of a conduit with circular cross-section. The flow is in the $z$ direction, while $x$ and $y$ are the cross-sectional co-ordinates. Gravity $g$ points downward along the negative $y$ direction. $u,v$ and $w$ are the components of the velocity field along the $x$, $y$ and $z$ axes respectively.
The co-ordinates are rendered dimensionless with the tube radius $R$ while the velocities are non-dimensionalized using the scaling $U_c = GR^2/\mu_0$ arising from the pressure gradient $G$ imposed in the flow direction. The pressure $P$ and the stresses are scaled with the pressure drop $GR$. For convenience, the asterisks accompanying the non-dimensionalized variables will be dropped henceforth. For fully developed axially invariant flow, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$ (3.6)

A momentum balance over the suspension yields

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = \left[ \rho_l + \Delta \rho \phi \right] g \frac{G}{R} \delta_{i2}$$ (3.7)

The total stress $\Sigma_{ij}$ used above has already been defined in Eq. 2.33 as consisting of contributions from the fluid and the particles. As was done in chapter 2, the particle stress tensor $\Sigma_{ij}^p$ can be described using the constitutive equations of Zarraga et al [97] (Eq. 2.16). Using these definitions, we get

$$\frac{\partial}{\partial x_j} \left[ 2\mu_re_{ij} \right] = \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \alpha \tau H_{ij} \right) + \left[ \rho_l + \Delta \rho \phi \right] g \frac{G}{R} \delta_{i2}$$ (3.8)

Here $H_{ij}$ is the flow geometry tensor and as defined in Eq. 2.17 is anisotropic in general and weighted with the appropriate coefficients in the flow, velocity gradient and vorticity directions. However, in this chapter, we also examine the effect of assuming an isotropic particle stress tensor on the fully developed profile, and these computations are made according to the isotropic version of $H_{ij}$ in Eq. 2.19 as done previously by Norman et al. [69].

Because of the assumption of quasi-unidirectional flow, the total pressure $P$ in
the system can be decomposed into an axial component $\bar{P}$ that varies linearly in
axial position (constant axial pressure gradient) and a cross-sectional component
$\tilde{P}$.

\begin{equation}
P(x, y, z) = \bar{P}(z) + \tilde{P}(x, y) = -z + \tilde{P}(x, y)
\end{equation}

Furthermore, the cross-sectional component $\tilde{P}$ can be simplified by eliminating
the static head contribution from the liquid.

\begin{equation}
\tilde{P}(x, y) = \hat{P} + \frac{\rho g}{G} y
\end{equation}

With these simplifications, the momentum equations in the $x$, $y$ and $z$ directions
can be written separately as

\begin{align}
\frac{\partial}{\partial x} \left[ 2\mu_r \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu_r \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] = \frac{\partial \tilde{P}}{\partial x} + \frac{\partial}{\partial x} (\alpha \tau H_{11}) + \frac{\partial}{\partial y} (\alpha \tau H_{12}) \\
\frac{\partial}{\partial y} \left[ \mu_r \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left[ 2\mu_r \frac{\partial v}{\partial y} \right] = \frac{\partial \tilde{P}}{\partial y} + \frac{\partial}{\partial x} (\alpha \tau H_{21}) + \frac{\partial}{\partial y} (\alpha \tau H_{22}) + \tilde{N}_b \phi \\
\frac{\partial}{\partial x} \left[ \mu_r \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu_r \frac{\partial w}{\partial y} \right] = -1
\end{align}

The buoyancy number that appears in the above equation $\tilde{N}_b = \Delta \rho g / G$ is based
on the pressure gradient and not on the average velocity as defined in equation
3.1 for a rectangular channel. The average velocity $U$, however, (as also noted by Norman et al. [69]) has a one to one correspondence with the pressure gradient $G$ for a given $\phi_f$, $R/a$ and $\tilde{N}_b$, so that the buoyancy number in equation 3.4 can be backed out as

$$N_b = \frac{2\Delta \rho g R^2}{9U \mu_0} = \frac{2GR^2/\mu_0}{9U} \tilde{N}_b$$  \hspace{1cm} (3.12)$$

The final equation required to complete the definition of this quasi steady state problem is the particle flux balance equation

$$\frac{\partial N_i}{\partial x_i} = 0$$  \hspace{1cm} (3.13)$$

where the particle flux $N_i$ with $i = 1$ to 2 is composed of convective, shear-induced diffusive and gravity contributions.

$$N_i = Pe u_i \phi + 2f(\phi) \frac{\partial \Sigma_{ij}^p}{\partial x_j} - \frac{2}{9} \tilde{N}_b f(\phi) \phi \delta_{i2}$$  \hspace{1cm} (3.14)$$

$f$ is the hindered settling factor defined earlier in Eq. 2.58.

The Peclet number $Pe$ which is the ratio of the diffusive time scale to the convective time scale, both defined with respect to the radius $R$ of the tube, is given by

$$Pe = \frac{R^2/[(U_c/R)a^2]}{R/U_c} = \frac{R^2}{a^2}$$  \hspace{1cm} (3.15)$$

Since we are considering particle flux within the cross-section of a conduit, the deviatoric component of the particle stress term (see Eq. 2.57) involves gradients
of the transverse velocity components only. We anticipate that the secondary velocity components are much smaller in magnitude compared to the axial velocity. Therefore the effect on the particle flux due to the additional deviatoric stress corresponding to the secondary currents is negligible. Thus, to a high degree of approximation, the shear-induced diffusive flux is given by

\[
\frac{2}{9} f(\phi) \frac{\partial \Sigma_{ij}^p}{\partial x_j} \approx -\frac{2}{9} f(\phi) \frac{\partial \alpha \tau H_{ij}}{\partial x_j}
\]  

(3.16)

where \( \tau \) and \( H_{ij} \) may be calculated from the axial flow field \( w(x,y) \). In other words, the transverse diffusive flux is driven solely by the dominant axial flow. Thus, the particle flux equation can be reduced to

\[
N_i = Pe u_i \phi - \frac{2}{9} f(\phi) \frac{\partial \alpha \tau H_{ij}}{\partial x_j} - \frac{2}{9} \tilde{N}_b f(\phi) \phi \delta_{i2}
\]  

(3.17)

The boundary conditions for the problem are the zero particle flux restriction and no-slip on the walls of the tube, and the symmetry condition on the vertical center-line.

In performing the simulations, the slip of the suspension at the walls has been neglected. Wall slip comes into play when the particle concentration near the walls is high, typically greater than 50%, and when the particle size to tube radius ratio is relatively large. Following the work of Jana et al. [42], at high concentrations, wall slip can be modeled by replacing the no-slip condition at the walls by

\[
w_{r=1} = \frac{1}{8} \mu_r(\phi) \frac{a}{R} \dot{\gamma}
\]  

(3.18)

The effect of wall slip will be significant only for high concentrations and large \( a/R \) ratios. Because particles tend to migrate away from high shear stress regions.
near the walls, the concentration at the walls is high only for very large values of $N_b$, where a concentrated sediment layer forms at the bottom of the tube.

### 3.2.2 Solution technique

In order to determine the fully developed concentration and velocity profiles, we use the iterative solution scheme developed in Chapter 2, but modify it slightly as explained below to overcome the issue of negative particle pressures associated with the clear fluid region at the top of the tube for large buoyancy numbers.

1. Guess the concentration profile $\phi$ and transverse velocity field $[u, v]$.

2. Compute the axial velocity field $w$ from Eq. 3.11c.

3. With the axial velocity field from step 2, calculate $\dot{\gamma}$ and the geometry tensor $H_{ij}$ from their definitions in equations 2.42 and 2.40 respectively.

4. Solve for the particle pressure $\alpha \tau$ in the cross-section using equations 3.13 and 3.17 with the flow average concentration constrained at $\phi_f$.

$$\phi_f = \frac{\int \int u\phi dS}{\int \int udS}$$  \hspace{1cm} \text{(3.19)}

5. Update the concentration field using the equation

$$\phi = \frac{\phi_m [\max(0, \alpha \tau)]^{1/3}}{\phi_m (2.17\dot{\gamma})^{1/3} + [\max(0, \alpha \tau)]^{1/3}}$$  \hspace{1cm} \text{(3.20)}

which is obtained by inverting the definition of $\alpha \tau$ as shown below in Eq. 2.68.
6. Calculate the transverse velocity field \([u, v]\) from equations 3.11a and 3.11b with the knowledge of the particle pressure field \(\alpha \tau\) from step 4.

7. Repeat steps 2 to 6 beginning with the concentration and transverse velocity fields obtained from steps 5 and 6 respectively, and iterate until convergence is achieved.

For the simulations presented in this chapter, a variety of initializations were employed. The symmetric solution in the absence of gravity \((N_b = 0)\) was computed using the iterative scheme described above starting from a uniform concentration distribution \(\phi = \phi_f\). Using this solution as a new starting point, the solution for any \(N_b\) or Pe could be computed by bootstrapping forward using the same iterative scheme. The same final solution for arbitrary \(N_b\) and Pe was obtained using a uniform concentration as an initial guess, however the iterative procedure took much longer to converge.

The solution procedure was started with a uniform mesh for small buoyancy numbers, or a mesh from a lower \(N_b\) or lower Pe. This base mesh and therefore the number of finite elements usually dictated the achievable tolerance. As the concentration profile evolved, the positions of the concentration maximum and/or the suspension-suspending fluid interface within the cross-section were noted, and the mesh was reworked with mesh refinements in these areas made to the base uniform mesh.

As is typical in convection-diffusion problems, an under-relaxation approach had to be employed to achieve convergence as discussed in the previous chapter. For the high Peclet number simulations, especially with the appearance of a suspension-fluid interface, the under-relaxation coefficient had to be set to as low as \(5 \times 10^{-4}\) to force convergence, consequently leading to a large number of
iterations required to reach the desired tolerance. The iterative algorithm was implemented numerically using the linear solvers in COMSOL 3.2. The loop was terminated when the error $\Delta \phi$ defined in Eq. 2.72 was under $5 \times 10^{-4}$.

The number of iterations required for convergence depended on the buoyancy number and the Peclet number. For sufficiently large buoyancy numbers, the suspension-suspending fluid interface appears, and gradients near the interface are sharp. Convergence in this case was achieved by improving the mesh near the interface and reducing the under-relaxation coefficient, and this again led to a large number of iterations. For large Peclet numbers, the under-relaxation coefficient was kept small, and hence a large number of iterations was required to achieve the tolerance. The number of iterations also depended on the model. The anisotropic model generally converged more slowly than the isotropic model due to the singularity near the flow centerline produced by the anisotropic model as explained above.

Consider some typical computational requirements of this numerical technique for two cases with $\phi_f = 0.4$ and $\text{Pe} = 400$, one each at a low and a high buoyancy number. The particle stress model employed for the two cases was the anisotropic model, and the initial concentration profile was uniform at $\phi_f = 0.4$. For $N_b = 2.6$ and a mesh with 3500 elements, the convergence was essentially linear for values of $\epsilon$ below 0.1. The number of iterations required for convergence with $\epsilon = 0.1$ was approximately 120, which took approximately 2 hours on a 2.2 GHz Dual-Core AMD Opteron (Model 175, 32 bit) with 2 GB RAM. The algorithm failed to converge for coarser meshes and higher relaxation parameters. For $N_b = 35.6$, the final converged solution shows a suspension-clear fluid interface, and therefore the memory and relaxation requirements were more stringent. When the algo-
Algorithm was started with a relaxation parameter of 0.1 and mesh of 3500 elements, the error $\Delta \phi$ decreased with the number of iterations, but subsequently settled into an oscillatory behavior due to a traveling wave of high concentration propagating along the interface with the secondary currents. These oscillations were suppressed by improving the discretization locally near the interface, and consequently the number of elements increased to approximately 16000. In addition, relaxation coefficient was reduced to 0.001. With these parameters, convergence was achieved after approximately 8000 iterations, and this took nearly 6 days with the same computational power. Thus, the solution of the governing equations is relatively straightforward for low buoyancy numbers, but much more computationally expensive for large buoyancy numbers.

The algorithm presented above was found to be less successful at high Peclet numbers, typically greater than 10000, especially when the buoyancy number was large enough for the emergence of the suspension-fluid interface. This is due to the large memory requirements set by the fine discretization required for convergence at high Pe.

3.3 Numerical results

In this section, we compare the tube resuspension results of the isotropic and anisotropic suspension balance models. To this end, consider the variation of the concentration and velocity profiles for a flow average concentration of 40% and $R/a = 20$ predicted by the isotropic model and the anisotropic model for different values of the buoyancy number $N_b$ as shown in figures 3.2 and 3.3. For every sub-plot in the figure, the left semicircle shows the velocity profile, while the right semicircle shows the concentration profile.
Figure 3.2. The fully developed concentration and velocity profiles in the tube cross-section as a function of the buoyancy number $N_b$ for $\phi_f = 0.4$ and $Pe = 400$. In every subplot, the left semicircle shows the axial velocity profile $w/U$ with the secondary velocity field $[u, v]/\sqrt{u^2 + v^2}$ (shown with the black arrows) superimposed on it, while the right semicircle shows the concentration profile $\phi$. On the left column [subfigures (a), (c), (e)] are results from the anisotropic suspension balance model with the constitutive equations of Zarraga et al. [97]. On the right column [subfigures (b), (d), (f)], we have shown the profiles resulting from the isotropic suspension balance model with the isotropic version of Zarraga’s model [Eq. 3.3].
Figure 3.3. The fully developed concentration and velocity profiles in the tube cross-section as a function of the buoyancy number $N_b$ for $\phi_f = 0.4$ and $Pe = 400$ (continued from the previous page). In every subplot, the left semicircle shows the axial velocity profile $w/U$ with the secondary velocity field $[u, v]/\sqrt{u^2 + v^2}$ (shown with the black arrows) superimposed on it, while the right semicircle shows the concentration profile $\phi$. On the left column [subfigures (a), (c) and (e)] are results from the anisotropic suspension balance model with the constitutive equations of Zarraga et al. [97]. On the right column [subfigures (b), (d), (f)], we have shown the profiles resulting from the isotropic suspension balance model with the isotropic version of Zarraga’s model [Eq. 3.3].

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Let us examine the effect of the buoyancy number on the area average concentration. The deviation between the flow average and area average concentrations gives a useful quantitative measure of the non-uniformity of the concentration profile. For $N_b = 0$, i.e. in the absence of gravity effects, the concentration and velocity profiles are independent of the azimuthal co-ordinate $\theta$. Shear-induced migration causes the concentration to be higher near the center of the cross-section compared to the regions near the wall. Since the center of the cross-section is also the low shear, high velocity region, the area average concentration is less than the flow average concentration (figure 3.4). As $N_b$ is increased gradually from zero to introduce gravity [figures 3.2 (c) through (f) and figures 3.3 (a) through (f)], the concentration profile shifts downwards, which is expected as gravity points in the negative $y$ direction. Similarly, the velocity maximum (and hence the zero shear stress streamline) moves up due to the greater viscosity resulting from the higher concentration of suspension near the tube bottom. Because particles concentrate in regions of low shear stress, this leads to the paradoxical situation that the concentration maximum shifts upwards with finite $N_b$ rather than downwards as one might expect from intuition [5, 13, 67, 99].

With the higher concentration regions shifting down to the bottom of the tube and the high velocity regions moving up towards the top of the tube, there are more particles in the slow moving regions of the cross-section at the tube bottom, leading to a reduction in the excess of the flow average concentration over the area average concentration as $N_b$ increases. For large effects of gravity [figures 3.3 (c) through (f)], the suspending fluid simply channels over the settled bed of particles, and the area average concentration becomes greater than the flow average concentration. Thus, at a critical intermediate value of the buoyancy
number $N_b$, the difference between the flow average concentration and the area average concentration is identically zero. This trend is summarized in figure 3.4, which shows how the area average concentration $\phi_a$ varies with $N_b$ for the isotropic and anisotropic models for the same set of the fixed parameters: $\phi_f = 0.4$ and $R/a = 20$. We see that the trend discussed above applies qualitatively to both the isotropic and anisotropic models. However, this similarity in the trends is only superficial; there are in fact some strong qualitative and quantitative differences between the two sets of predictions.

First, note that for lower values of $N_b$, the migration of the particles towards the center is much stronger for the anisotropic model than the isotropic model. The curvature of the flow geometry results in an additional flux towards the center relative to the isotropic model that is proportional to the second normal stress difference coefficient [31]. Since the isotropic model ignores any curvature-induced flux, the migration of particles towards the tube center is weaker than that predicted by the anisotropic model, resulting in a higher area average concentration for the same bulk concentration.

Second, we see that for similar values of $N_b$, the extent of stratification due to gravity is much greater for the isotropic model than for the anisotropic model. This can be seen quantitatively in figure 3.4, where we have shown the area average concentration as a function of the buoyancy number for the isotropic and anisotropic models at $Pe = 0$ and $Pe = 400$. Note that $Pe = 0$ represents the non-physical case where the particles are actually much larger than the tube. From Eq. 3.17, however, we see that $Pe = 0$ simply corresponds to the concentration profile in the absence of secondary currents. The isotropic model in general predicts a sharper increase of the area average concentration with respect to $N_b$, i.e. in
Figure 3.4. The variation of the area average concentration with the buoyancy number $N_b$ at a bulk concentration of 40% (gray horizontal line) for both isotropic and anisotropic models at $Pe = 0$ and $Pe = 400$. Hollow symbols - Anisotropic model, Filled symbols - Isotropic model. Circles - $Pe = 0$; Triangles - $Pe = 400$. 

\[ \phi \]
the absence of convection, resuspension is stronger for the anisotropic model than the isotropic model for the curvature reasons discussed above. Of more interest, however, is that an increase in Peclet number from 0 shifts the curves from the two models in different directions: for the isotropic model, an increase in Peclet number results in an increase in the area average concentration for all buoyancy numbers, while the reverse is true for the anisotropic model.

Third, the suspension-suspending fluid interface, when it exists, is concave upward for the isotropic model, but concave downward for the anisotropic model. This is because the secondary currents predicted by the two models are opposite in direction: the isotropic model predicts a secondary flow that is downwards near the center and upwards near the wall, while the anisotropic model yields a surprising result: a current that flows upwards near the center and downwards near the walls. Note that the direction of the currents mentioned here for the two models applies to the region occupied by the suspension only. For low, non-zero buoyancy numbers, the models predict only one cell in each half of the cross-section. However, if the buoyancy number is sufficiently large, then the circulation in the area of the suspension (which shows the sense of direction discussed above) drives a counter-rotating cell in the area of the clear fluid, leading to two cells in each semi-circle of the cross-section as shown in figure 3.5. The sense of rotation of the cells is, however, predicted to be opposite by the two models. The discrepancies in the predictions of the two models occur because the secondary currents arising from the anisotropic suspension rheology overwhelm those due to buoyancy effects (see the appendix at the end of this chapter for a semi-analytical demonstration of the reversal of secondary flow by the second normal stress differences for resuspension in rectangular slot flow).
Figure 3.5. The concentration and secondary current profiles (white quiver plots) at Pe = 1500 and $\phi_a = 0.2$ for the anisotropic (left semicircle) and isotropic (right semicircle) models for buoyancy numbers ($N_b$) of 1.4 and 1.7 respectively. It can be seen that both models predict two circulation cells in each half of the cross-section; the direction of rotation is, however, opposite. For the anisotropic model, the secondary currents flow upwards near the center and downwards near the side walls in the area occupied by the suspension, and this drives a circulation cell in the reverse direction near the top of the tube.

The isotropic model predicts the exact opposite.
Figure 3.6. The magnitude of the secondary currents $\left\langle \sqrt{u^2 + v^2} \right\rangle / U$ as a function of the buoyancy number for the isotropic and anisotropic models. Filled triangles - Isotropic model, $Pe = 400$; Hollow triangles - Anisotropic model, $Pe = 400$. 
To see this, let us first analyze the isotropic model. In this case, the driving force for the secondary currents is solely the body force $\bar{N}_b \phi$ on the RHS of the $y$ momentum equation 3.11b. Because the particle stress tensor is isotropic, the contribution of the particle stress to the total stress in the momentum equation can simply be absorbed into the fluid pressure $\hat{P}$ (see section 2.3), and thus the particulate normal stress has no effect on the secondary current profile. Therefore, according to the isotropic model, the circulation velocity profile arises in the tube geometry because of the horizontal concentration and buoyancy gradients in the cross-section. Due to the phenomenon of shear-induced migration, the particle concentration is higher near the center of the tube than the tube walls. Consequently, the secondary currents flow downwards from the center of the tube and towards the tube bottom, and then back upwards near the walls of the tube as can be seen in figure 3.2(d). This also leads to a suspension-suspending fluid interface which is concave upward in shape (see figures 3.2(f), 3.3(b) and 3.3(d)). In figure 3.6, we have shown the variation of the magnitude of the secondary currents with the buoyancy number $N_b$. For relatively small gravity effects, the circulation currents are weak because the secondary currents scale directly with $N_b$. For relatively large effects of gravity ($N_b \gg 1$), most of the particles are settled and the horizontal concentration gradients due to shear-induced migration are not strong. Therefore the circulation currents vanish in this limit as well. The magnitude of the in-plane velocities thus shows a maximum with respect to the buoyancy number as shown in figure 3.6. This effect can also be seen in the relationship of $\phi_a$ with $N_b$ in figure 3.4, where the plots for the isotropic model for $Pe = 0$ and $Pe = 400$ virtually lie on top of each other for low and high $N_b$.

To understand the effect of the velocity components in the cross-section on the
concentration profile, consider the concentration and secondary current profiles for $N_b = 8.0$, $\phi_f = 0.4$ and $R/a = 20$ for the isotropic model in figure 3.2(f). The circulation currents bring the high concentration regions near the center down to the bottom of the tube, and carry the particle lean regions near the side walls of the tube up towards the top. Thus, there is a greater flux of particles due to the secondary currents towards the tube bottom from the center than the flux exiting from the tube bottom near the tube walls, leading to a net downward flux of particles. Therefore the circulation currents aid gravity in bringing about greater stratification of particle concentration in the cross-section. Consequently, the area average concentration for a given value of $N_b$ is expected to increase with an increase in the Peclet number. Due to the small magnitude of the secondary currents, however, this effect is relatively minor as may be seen in figure 3.4.

Now consider the secondary current profile predicted by the anisotropic model. The driving force for the secondary currents is no longer just the body force $\tilde{N}_b \phi$ on the RHS of the $y$ momentum equation 3.11b, but also includes the gradients in the particle stress tensor $\partial \Sigma^p_{ij}/\partial x_j$ [see Eqs. 3.11a and 3.11b].

$$\frac{\partial}{\partial x_j} \left[ \mu_r \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{\partial \tilde{P}}{\partial x_j} = \tilde{N}_b \phi \delta_{i2} + \frac{\partial \alpha \tau H_{ij}}{\partial x_j}$$

(3.21)

To identify these two sources more clearly, we take the curl of above momentum equation and examine its component in the $z$ direction, e.g.,

$$\varepsilon_{3ij} \frac{\partial^2}{\partial x_i \partial x_k} \left[ \mu_r \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right] = -\xi$$

(3.22)

If the viscosity were not a function of concentration, the LHS of Eq. 3.22 would simply be the Laplacian of the scalar vorticity ($z$ component of the vorticity...
Figure 3.7. The fully developed concentration profile $\phi$ and the scalars $\xi_1$ and $\xi_2$ for $\tilde{N}_b = 1.3$, $\phi_f = 0.3$ and $Pe = 100$. In subfigure (a), we see that the secondary currents $\left(\frac{u,v}{\sqrt{u^2 + v^2}}\right)$ shown with black arrows) flow upwards near the center and downwards near the walls, while the concentration profile $\phi$ exhibits a concave downward interface. In subfigures (b) and (c) we have compared the two scalars $\xi_1$ and $\xi_2$ representing the two driving forces for the secondary currents due to normal stress and buoyancy gradients respectively. One can see that over the major portion of the cross-section, $\xi_1$ is negative and larger in magnitude than $\xi_2$. Note that in the calculation of $\xi_1$ and $\xi_2$, the variables have not been normalized by the average velocity $U$, as is done for switching between the two definitions of the buoyancy number $\tilde{N}_b$ and $\tilde{N}_b$. 

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vector) of the transverse flow field. The RHS \(-\xi\) may thus be interpreted as the scalar driving force or source term for the vorticity of the secondary velocity field. If \(\xi\) is zero, then the secondary currents are identically zero because of the homogeneous velocity boundary conditions. If \(\xi\) is positive in the cross-section, then the secondary currents flow in the counter-clockwise direction, similar to the isotropic model results in the right half of the tube cross-section. If \(\xi\) is negative, then the secondary currents flow in the clockwise direction, akin to the prediction of the anisotropic model in the right semicircle. From Eq. 3.21, it can be shown that the scalar \(\xi\) consists of two contributions \(\xi_1\) and \(\xi_2\) due to normal stress gradients and buoyancy gradients respectively.

\[
\xi = \left[\frac{\partial^2}{\partial x\partial y} [\alpha \tau (H_{11} - H_{22})] + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right) (\alpha \tau H_{12})\right] - \tilde{N}_b \frac{\partial \phi}{\partial x} \quad (3.23)
\]

One obvious case when \(\xi\) is zero is when the buoyancy number \(\tilde{N}_b\) and the second normal stress difference coefficient \(d\) are both zero. Since both \(H_{11} - H_{22}\) and \(H_{12}\) are proportional to \(d\) (see Eq. 2.40), setting \(d = 0\) nullifies the contribution of the normal stresses in Eq. 3.23. However, with \(\tilde{N}_b = 0\), \(\xi_1\) can still be zero for non-zero \(d\) provided \(\alpha \tau\) and \(H_{ij}\) are axisymmetric. In fact, it is relatively straightforward to show that \(\xi_1\) is identically zero for any arbitrary function \(\alpha \tau\) that is azimuthally independent, as would occur in the start-up transient evolution of an axisymmetric concentration profile in a tube or an annulus. Thus normal stress differences do not lead to transverse convection in either case, as expected from symmetry.

The introduction of gravity (non-zero \(\tilde{N}_b\)), however, breaks this symmetry and \(\xi\) is non-zero in the cross-section. For the isotropic model (e.g. for \(d = 0\)), \(\xi\) is
simply $\xi_2 = -\tilde{N}_b \partial \phi / \partial x$. For small values of $\tilde{N}_b$, shear-induced migration results in a concentration profile such that $\partial \phi / \partial x$ is always less than zero in the right semicircle of the cross-section, and therefore $\xi_2$ is always greater than zero for the isotropic model, thus resulting in a counter-clockwise current as explained above.

Unfortunately, the case when both $d$ and $\tilde{N}_b$ are non-zero is more complicated to analyze. It can only be commented that at reasonably high concentrations, when normal stresses are strong, the contribution of the normal stress term in Eq. 3.23 to $\xi$ is negative and greater than the contribution of the buoyancy term, resulting in a negative $\xi$. This establishes a clockwise circulation pattern in the right half of the cross-section. In figure 3.7, we have shown the variation of the two components of $\xi$ over the cross-section for a bulk concentration of 30% and $\text{Pe} = 100$ at $\tilde{N}_b = 1.3$ for the anisotropic model. It can be seen that over a major portion of the cross-section where the concentrations are non-zero, $\xi_1$ is negative and larger in magnitude than $\xi_2$ which is positive throughout, and therefore this results in a flow that is upward near the center and downward near the sides. The only region where $\xi_2$ dominates is near the interface, where the gradients in concentration are large but gradients in $\alpha \tau$ are not, both of which occur due to the small magnitudes of normal stresses at low concentrations. However, the thickness of this region is small and it has a negligible effect on the currents.

The reversal of the direction of the secondary currents with the inclusion of the non-Newtonian rheology may be crudely understood by examining the orientation of the geometry tensor within the cross-section [see figure 3.8]. Near the center of the tube, the velocity gradient direction is along the $y$ axis. Close to the side walls, it is vorticity that is directed vertically due to the curvature of the geometry. Thus, resuspension near the tube center occurs in the velocity gradient direction, while
Figure 3.8. Contour plot of the axial velocity $w/\langle w \rangle$ for $N_b = 2.5$, $Pe = 400$ and $\phi_f = 0.4$. Also shown are the local velocity gradient and vorticity directions near the center and near the sidewalls. The velocity gradient direction is denoted by the V-shaped arrowhead, while the vorticity direction is denoted by the diamond shaped arrow head. It can be seen that the resuspension occurs in the velocity gradient direction near the center and in the vorticity direction close to the side wall for low buoyancy numbers.
resuspension close to the tube walls occurs in the vorticity direction. In rigid particle suspensions, resuspension is stronger in the velocity gradient direction than in the vorticity direction by roughly a factor of \(1/(1 + d) = 2.17\) as can be seen from the definition of the particle stress tensor in Eqs. 2.34 and 2.38. Thus, the vertical concentration gradient due to gravity is stronger near the side walls than in the center. When again coupled with the non-zero normal stress difference, this concentration gradient differential leads to a strong gradient in \(\Sigma_{xx}\) along the bottom of the tube and a weak gradient in \(\Sigma_{xx}\) along the top. It is these normal stress gradients that drive the circulation.

Since the particle concentration is higher near the tube center than near the side walls, for clockwise currents in the right semicircle the flux of particles out of the bottom of the tube towards the center is greater than the flux of particles into the tube bottom coming from the side walls. This results in a net upward flux of particles that aids the resuspension process. As the effect of gravity is increased, gradients in the particle stress become stronger, and this results in stronger secondary currents as can be seen in figure 3.6, where we have shown the variation of the magnitude of the secondary currents with \(N_b\). In contrast, the buoyancy-induced secondary currents predicted for the isotropic model vanish at large \(N_b\).

The effect of the secondary currents for the two models can be summarized in figure 3.9, where we have shown the variation of the critical buoyancy number \(N_{b_c}\) with the Peclet number \(Pe\) for a bulk concentration of 40\%. For the isotropic model, the critical buoyancy number slightly decreases with an increase in Peclet number, while the anisotropic model predicts a sharp increase in \(N_{b_c}\) with \(Pe\); two completely opposite trends. Also note that \(N_{b_c}\) obtained from the anisotropic
model is always greater than that from the isotropic model. For a Peclet number of 400 at a bulk concentration of 40%, the critical buoyancy number predicted by the anisotropic model is around 24, while that predicted by the isotropic model is just 6, a factor of 4 lower.

To conclude this section, we shall compare the results of our anisotropic model for resuspension flow through a tube to the MRI measurements of the concentration and velocity profiles by Altobelli et al. [5]. They reported these distributions for a suspension of heavy divinyl-benzene styrene copolymer particles in lubricating oil for different area average concentrations (since they had a closed loop system) and buoyancy numbers. The data extracted from their images at 39% and 23% area average concentrations is shown in the left columns of figures 3.10 through 10. The tube radius $R$ (1.27 cm) in their experiments was 33 times the
particle radius $a$ (381 µm) yielding a Peclet number of around 1100. Because of the complete information on average concentration, average velocity and suspension properties provided by the investigators, it is possible to do an exact comparison with theory with no adjustable parameters.

On the right columns of these figures are shown the results of the numerical simulations of the anisotropic (left semicircle) and isotropic (right semicircle) models at the corresponding experimental value of $N_b$. For the 39% data, Altobelli et al. [5] reported a small region above the centerline of the tube where the concentration was approximately equal to the maximum packing concentration measured for a static, settled suspension. This result of heavy material flowing over light is captured by both the isotropic and anisotropic simulations. This result was also reproduced by Zhang and Acrivos [99] with their isotropic simulations. However, the anisotropic model predicts the position and size of this region of maximum packing much better than the isotropic model. In fact, for $N_b = 8.4$, the isotropic simulation of the concentration profile [right semicircle in figure 3.10(f)] does not show a region of maximum packing. In contrast, the anisotropic simulation for this case [left semicircle in figure 3.10(f)] quantitatively reproduces the experimental concentration profile. The stretching of the contours near the top of the cross-section in a concave-downward fashion that can be observed in figure 3.10(c) and 3.10(e) is most likely due to secondary currents flowing up from the center and down from the sides, as predicted by the anisotropic simulations, especially in figure 3.10(f).

The 23% concentration data from experiment and simulations are shown in figure 3.11. In sub-figures 3.11(c) and 3.11(e) that correspond to the experimental measurements, one can see that the interface shape is concave downward. This
Figure 3.10. Comparison of the concentration data $\phi(x, y)$ from the NMR images [sub-figures (a), (c) and (e)] of Altobelli et al. [5] with the simulation results using the anisotropic and isotropic models [left and right semicircles respectively in sub-figures (b), (d) and (f)] for an area average concentration of 39% and $\text{Pe} = 1100$. $N_b = 0.8$ for sub-figures (a) and (b), $N_b = 2.0$ for sub-figures (c) and (d) and $N_b = 8.4$ for sub-figures (e) and (f). The black concentration contour lines are provided to elucidate the shape of these contours.
Figure 3.11. Comparison of the concentration data $\phi(x,y)$ from the NMR images [sub-figures (a), (c) and (e)] of Altobelli et al. [5] with the simulation results using the anisotropic and isotropic models [left and right semicircles respectively in sub-figures (b), (d) and (f)] for an area average concentration of 23% and Pe = 1100. $N_b = 0.6$ for sub-figures (a) and (b), $N_b = 2.0$ for sub-figures (c) and (d) and $N_b = 3.9$ for sub-figures (e) and (f). The black concentration contour lines are provided to elucidate the shape of these contours.
shape is captured by the anisotropic simulations, but not by the isotropic ones. The simulations of Zhang and Acrivos [99] with the isotropic formulation also incorrectly predict a concave upward or flat suspension-clear fluid interface. The anisotropic simulations show a region of high concentration above the tube center for $N_b = 2.0$, but not for $N_b = 4.0$. Both predictions are consistent with the experimental concentration profile. The axial velocity profiles $w/U$ from experiment and theory are compared in figure 3.11. As can be seen, the anisotropic model again quantitatively matches the experimental velocity distribution.

The only conditions under which there was significant disagreement between the anisotropic model and experiment was at an area average concentration of 0.23 and $N_b = 0.6$. These results are depicted in figures 3.11(a) and 3.11(b). The experimental concentration profile [figure 3.11(a)] does not show any contours representing the packed region near the location of the zero shear stress streamline [see figure 3.11(a)]. The anisotropic model, however, shows a significant region of maximum packing above the center, thus predicting stronger resuspension than the experimental results. For this value of $N_b$, the anisotropic model also predicts a concave upward interface. For the anisotropic model, under these simulation conditions, there are four eddies within the entire cross-section with the eddies near the top of the tube flowing up from the sides and down along the center. Since the interface coincides with the downwelling region of these currents, it is concave upward in shape.

These disagreements could be attributed to two reasons. First, the normal stress measurements of Zarraga et al. [97] were made only for concentrations greater than 30%. Thus, the rheological model used in our calculations is simply extrapolated to the low 23% average concentration used in this experiment.
Figure 3.12. Comparison of the axial velocity data $w(x, y)/U$ from the NMR images [sub-figures (a), (c) and (e)] of Altobelli et al. [5] with the simulation results using the anisotropic and isotropic models [left and right semicircles respectively in sub-figures (b), (d) and (f)] for an area average concentration of 23% and $Pe = 1100$. $N_b = 0.6$ for sub-figures (a) and (b), $N_b = 2.0$ for sub-figures (c) and (d) and $N_b = 3.9$ for sub-figures (e) and (f).
At larger $N_b$, this is less of an issue due to the increased concentration of the suspension near the bottom of the tube where the secondary circulation is most significant.

Second, it is also possible that the experimental concentration profile at the lowest buoyancy number had not reached steady state. The length preceding the test section in the setup of Altobelli et al. was 117 cm. At zero buoyancy number, the length required to reach steady state is given by $k(\phi)R^3/a^2$, where $k$ is a constant inversely related to the dimensionless shear-induced diffusivity. The constant $k$ was measured by Chapman [18] to be 0.25 and 0.063 at concentrations of 30% and 40% respectively, giving zero buoyancy number equilibration lengths of 350 cm and 90 cm at these two concentrations. At 23% concentration, the induction length should be higher than 400 cm due to the smaller shear-induced diffusivities at lower concentrations. Even though this length decreases with the introduction of gravity, it is not clear whether the entrance length of 117 cm was sufficient to achieve the fully developed concentration profile at the lowest buoyancy number used in their experiment. A profile that has not reached steady state would feature a reduced concentration (compared to the expected maximum packing) near the zero shear stress (and hence zero shear rate) streamlines, precisely as was observed in the experiment.

3.4 Conclusions

The results presented in this chapter corroborate the observation in chapter 2 that it is vital to incorporate the anisotropy of suspensions not only in the particle flux balance equation but also in calculating the velocity profiles. Once the symmetry of the concentration and velocity profiles is broken by introducing gravity,
the particle stress gradient vector is in general non-zero within the cross-section, and this necessarily leads to a non-zero secondary current contribution from particle stress gradients. The effect of secondary flow on the concentration balance equation scales as the Peclet number $R^2/a^2$, which is kept large in experiments so that the continuum approximation can be applied to model the suspension. It was shown that even though the magnitude of these secondary currents is small, the flux of particles due to the secondary currents becomes significant at large Peclet numbers and therefore secondary currents can affect the concentration profile considerably. In the case of tube resuspension flow, these currents are opposite in direction to those expected from simple buoyancy arguments. Also, they produce a suspension-suspending liquid interface that is concave downward in shape and not concave upward as predicted by the isotropic model. The reversed nature of the secondary currents contributes an upward flux that results in stronger resuspension for a given buoyancy number. It was shown that for a 40% suspension at $\text{Pe} = 400$ (which corresponds to $R/a = 20$), the critical buoyancy number predicted by the isotropic model is around 6, while that predicted by the anisotropic model is around 24, thus changing the velocity at which particles resuspend by a factor of 4. Finally, the suspension balance model of Nott and Brady [70] with the constitutive equations of Zarraga et al. [97] is able to predict the experimental measurements of Altobelli et al. [5] reasonably quantitatively.

3.5 Acknowledgements

We are grateful to Dr. Steve Altobelli, Dr. Rick Givler and Dr. Eiichi Fukushima for providing us with the data from the resuspension measurements reported in their paper in 1991.
3.6 Appendix

Consider the fully developed flow of a suspension through a large aspect ratio rectangular channel such as shown schematically in figure 3.13. The suspension flows in the $z$ direction, while $x$ and $y$ are the cross-sectional co-ordinates. Gravity points in the negative $x$ direction. The variables have been rendered dimensionless as follows:

$$
x = x^* W; \quad y = y^* B; \quad z = z^* L; \quad w = w^* U_c;
$$

$$
u = u^* U_x; \quad v = v^* \epsilon U_x; \quad \tau = \tau^* \mu_0 U_c / B
$$ (3.24)

Here $B$ is the half-depth of the channel, $W$ is the width of the channel and $L$ is the length scale in the flow direction. $\epsilon$ is the aspect ratio $B/W$. $U_c = GB^2/\mu_0$ is the axial velocity scaling derived from the constant pressure gradient $G$ imposed in the flow direction. The scaling for $u$ is taken to be $U_x$, while the scaling for $v$ is derived from the continuity equation. $\mu_0$, as usual, is the viscosity of the suspending fluid.

For fully developed flow in the $z$ direction, the particle flux balance becomes

$$
\epsilon \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0
$$ (3.25)

The particle flux vector $\mathbf{N}$ has been non-dimensionalized by $a^2 U_c / B^2$. The flux $\mathbf{N}$ consists of convective and diffusive contributions

$$
N_x = \frac{U_x}{U_c} Pe \epsilon \phi - \epsilon \mathbf{f} \frac{\partial}{\partial x} [\alpha \tau H_{11}] - \frac{2}{9} \mathbf{f} \frac{\partial}{\partial y} [\alpha \tau H_{12}] - \frac{2}{9} \epsilon \bar{N}_b \mathbf{f} \phi
$$

$$
N_y = \epsilon \frac{U_x}{U_c} Pe \epsilon \phi - \epsilon \mathbf{f} \frac{\partial}{\partial x} [\alpha \tau H_{12}] - \frac{2}{9} \mathbf{f} \frac{\partial}{\partial y} [\alpha \tau H_{22}]
$$ (3.26)
Here we have ignored the deviatoric stress arising from the circulation velocity gradient tensor, since the magnitude of these terms is expected to be negligible. $\tilde{N}_b$ is the buoyancy number defined as

$$\tilde{N}_b = \frac{\Delta \rho g W}{G B} \quad (3.27)$$

This definition of the buoyancy number is derived by balancing the upward flux due to the particle stress gradient in the vorticity direction and the downward settling flux due to gravity. The components of the flow geometry tensor $H$ are

$$H_{11} = 1 + d \frac{(\partial w/\partial y)^2}{\epsilon^2(\partial w/\partial x)^2 + (\partial w/\partial y)^2}$$

$$H_{12} = -d \frac{\epsilon(\partial w/\partial x)(\partial w/\partial y)}{\epsilon^2(\partial w/\partial x)^2 + (\partial w/\partial y)^2}$$

$$H_{22} = 1 + d \frac{\epsilon^2(\partial w/\partial x)^2}{\epsilon^2(\partial w/\partial x)^2 + (\partial w/\partial y)^2} \quad (3.28)$$

Substituting equations 3.26 and 3.28 in Eq. 3.25, we get

$$\epsilon \frac{U}{U_c} \text{Pe} \frac{\partial (\upsilon \phi)}{\partial x} + \epsilon \frac{U}{U_c} \text{Pe} \frac{\partial (\upsilon \phi)}{\partial y} = \epsilon^2 \frac{2}{9} \frac{\partial}{\partial x} \left( \alpha \tau H_{11} \right)$$

$$+ \frac{2}{9} \frac{\partial}{\partial x} \left[ f \frac{\partial}{\partial y} (\alpha \tau H_{12}) \right] + \frac{2}{9} \frac{\partial}{\partial y} \left[ f \frac{\partial}{\partial x} (\alpha \tau H_{22}) \right] + \frac{2}{9} \tilde{N}_b \frac{\partial}{\partial x} (f \phi) \quad (3.29)$$

The $x$, $y$ and $z$ momentum equations are

$$\epsilon^2 \frac{U}{U_c} \frac{\partial}{\partial x} \left[ 2 \mu_r \frac{\partial u}{\partial x} \right] + \frac{U}{U_c} \frac{\partial}{\partial y} \left[ \mu_r \left( \frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x} \right) \right]$$

$$= \epsilon \frac{\partial \tilde{P}}{\partial x} + \epsilon \frac{\partial}{\partial x} [\alpha \tau H_{11}] + \frac{\partial}{\partial y} [\alpha \tau H_{12}] + \epsilon \tilde{N}_b \phi \quad (3.30a)$$
Figure 3.13. Resuspension in a rectangular channel along its width. The flow is through the plane of the figure. The depth of the channel is $2B$, while its width is $W$. The $x$ and $y$ co-ordinates are along the width and depth respectively. Gravity points in the negative $x$ direction.
The scaling for the cross-sectional pressure has been chosen as \( \tau_0 = \mu_0 U_c / B = GB \), since, in the lubrication limit, the suspension cannot support a pressure gradient in the shallow direction.

Let us perturb the governing equations in the aspect ratio \( \epsilon \).

\[
\chi = \chi^{(0)} + \epsilon \chi^{(1)} + \epsilon^2 \chi^{(2)} + \cdots
\]  

(3.31)

where \( \chi \) is the vector \((\phi, \alpha \tau, u, v, w, p)\). We will need only the zeroth order expansion to prove our point, and this is considered below.

### 3.6.1 Order \( \epsilon^0 \) expansion

First, let us examine the particle flux equation at this order. The concentration distribution can be determined at this order from Eq. 3.26.

\[
\frac{\partial}{\partial y} \left[ f(\phi^{(0)}) \frac{\partial}{\partial y} \left[ (\alpha \tau)^{(0)} H_{22}^{(0)} \right] \right] = 0
\]  

(3.32)

The components of the geometry tensor \( \mathbf{H} \) at order \( \epsilon^0 \) from Eq. 3.28 are

\[
H_{11}^{(0)} = 1 + d \\
H_{12}^{(0)} = 0
\]
\[ H_{22}^{(0)} = 1 \] (3.33)

Recognizing that the no-flux condition applies at \( y = 0 \) and \( y = 1 \), equations 3.32 and 3.33 suggest that the particle pressure \((\alpha \tau)^{(0)}\) is constant across the thin gap.

\[(\alpha \tau)^{(0)} = (\alpha \tau)^{(0)} \langle \phi \rangle = \alpha_w \] (3.34)

Here \( \langle \phi \rangle \) is the average concentration across the gap which is a function of \( x \) only for fully developed flow.

The momentum equations reduce to the following simple forms:

\[
\frac{U_x}{U_c} \frac{\partial}{\partial y} \left[ \mu_r \left( \frac{\partial (w(0))}{\partial y} \right) \right] = \epsilon \frac{\partial \tilde{P}}{\partial x} + \epsilon \frac{\partial}{\partial x} \left[ (\alpha \tau)^{(0)} H_{11}^{(0)} \right] + \epsilon \frac{\partial}{\partial y} \left[ (\alpha \tau)^{(0)} H_{12}^{(0)} \right] + \epsilon \tilde{N}_b \phi^{(0)} \] (3.35a)

\[ 0 = \frac{\partial \tilde{P}^{(0)}}{\partial y} + \frac{\partial}{\partial y} \left[ (\alpha \tau)^{(0)} H_{22}^{(0)} \right] \] (3.35b)

\[
\frac{\partial}{\partial y} \left[ \mu_r \frac{\partial w^{(0)}}{\partial y} \right] = -1 \] (3.35c)

The \( x \) momentum equation in Eq. 3.35a suggests that the scaling for \( U_x \) is \( \epsilon U_c \).

Note that the particle stress gradient containing the off-diagonal component \( H_{12} \) is retained at order \( \epsilon \), because \( H_{12}^{(0)} \) is zero as indicated by Eq. 3.33. The \( x \) momentum equation thus simplifies to

\[
\frac{\partial}{\partial y} \left[ \mu_r (\phi^{(0)}) \frac{\partial (w(0))}{\partial y} \right] = \frac{\partial \tilde{P}}{\partial x} + (1 + d) \frac{\partial}{\partial x} \left[ (\alpha \tau)^{(0)} H_{11}^{(0)} \right]
\]

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\[ + \frac{\partial}{\partial y} \left[ (\alpha \tau)^{(0)} H_{12}^{(1)} \right] + \tilde{N}_b \phi^{(0)} \] (3.36)

The z momentum equation on the other hand can be integrated to yield the stress \( \tau^{(0)} \).

\[ \tau^{(0)} = -\mu_r \frac{\partial w^{(0)}}{\partial y} = y \] (3.37)

The shear stress at this order can be used to determine the concentration distribution from Eq. 3.32.

\[ \phi^{(0)} = \alpha^{-1} \left( \frac{\alpha w}{y} \right) \] (3.38)

Note that the above equation suggests a region of maximum packing near the center of the gap of thickness \( 2y_c \), where \( y_c \) is defined as

\[ y_c = \frac{\alpha w}{\alpha(\phi_m)} \] (3.39)

The concentration distribution can thus be obtained at this order as a function of the average concentration at every vertical position \( x \), i.e. \( \phi^{(0)} = \phi^{(0)}(\langle \phi \rangle(x), y) \). The axial velocity \( w^{(0)} \) can be determined from Eq. 3.37 as

\[ w^{(0)} = F(\langle \phi \rangle, y) = \int_y^1 \frac{\zeta}{\mu_r(\phi^{(0)})} d\zeta \] (3.40)

Now consider the x momentum equation in Eq. 3.36. This equation can be integrated once with respect to \( y \) to yield

\[ \frac{\partial u^{(0)}}{\partial y} = \frac{\partial \hat{P}}{\partial x} \frac{y}{\mu_r(\phi^{(0)})} + (\alpha \tau)^{(0)} H_{12}^{(1)} \frac{H_{12}^{(1)}}{\mu_r(\phi^{(0)})} + \tilde{N}_b \frac{q_1}{\mu_r(\phi^{(0)})} \] (3.41)

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In the above integration, the symmetry boundary condition at $y = 0$ has been applied to both $u^{(0)}$ and $H_{12}^{(0)}$. Also, since the particle pressure $(\alpha \tau)^{(0)}$ is constant across the gap (i.e. independent of $y$), the off-diagonal term of the particle pressure gradient has been absorbed into the pressure gradient to give a modified pressure $\hat{\mathbf{P}}$ defined as

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}^{(0)} + (1 + d)(\alpha \tau)^{(0)}$$  \hspace{1cm} (3.42)

The function $q_1$ is defined as

$$q_1(\langle \phi \rangle, y) = \int_0^y \phi^{(0)}(\langle \phi \rangle, \zeta) d\zeta$$  \hspace{1cm} (3.43)

Consider the off-diagonal term $H_{12}^{(1)}$ of the geometry tensor

$$H_{12}^{(1)} = (-d)\frac{\partial w^{(0)}}{\partial x} \frac{\partial w^{(0)}}{\partial y}$$

Using the definition of $w$ from Eq. 3.40, we get

$$\frac{\partial w^{(0)}}{\partial x} = \frac{\partial F}{\partial \langle \phi \rangle} \frac{\partial \langle \phi \rangle}{\partial x}$$

$$\frac{\partial w^{(0)}}{\partial y} = -\frac{y}{\mu_r(\phi^{(0)})}$$  \hspace{1cm} (3.45)

Substituting these expressions in Eq. 3.44, we get

$$H_{12}^{(1)} = -(-d)\frac{\partial F}{\partial \langle \phi \rangle} \frac{y}{\mu_r(\phi^{(0)})} \frac{\partial \langle \phi \rangle}{\partial x}$$

$$\epsilon^2 \left( \frac{\partial F}{\partial \langle \phi \rangle} \frac{\partial \langle \phi \rangle}{\partial x} \right)^2 + \left( \frac{y}{\mu_r(\phi^{(0)})} \right)^2 \frac{\partial \langle \phi \rangle}{\partial x}$$  \hspace{1cm} (3.46)

Since the concentration is equal to maximum packing for $y \leq y_c$, where $y_c$ is
defined in Eq. 3.39, $H_{12}$ may be written as following step function:

$$H_{12} = q_2(\langle \phi \rangle, y) \frac{\partial \langle \phi \rangle}{\partial x} = \begin{cases} 0 & y \leq y_c \\ -(-d) \frac{\partial F}{\partial \langle \phi \rangle} \frac{\partial \langle \phi \rangle}{\partial x} & y > y_c \end{cases}$$

(3.47)

Eq. 3.43 can be integrated to yield

$$u^{(0)} = -\frac{\partial \hat{P}}{\partial x} F - (-d)q_3 \frac{\partial \langle \phi \rangle}{\partial x} - \tilde{N}_b q_4$$

(3.48)

where $F$ is defined in Eq. 3.40, and $q_3$ and $q_4$ are defined as

$$q_3(\langle \phi \rangle, y) = \int_y^1 q_2(\langle \phi \rangle, \zeta) \mu_r[\langle \phi^{(0)} \rangle(\langle \phi \rangle, \zeta)] d\zeta$$

$$q_4(\langle \phi \rangle, y) = \int_y^1 \alpha(\phi^{(0)}) \left(-\frac{\partial F}{\partial \langle \phi \rangle}\right) d\zeta$$

(3.49)

Applying the zero net flow condition $\int_0^1 u^{(0)} dy = 0$ for the transverse flow across the thin gap, the $x$ component of velocity becomes

$$u^{(0)} = -(-d)q_3 \frac{\partial \langle \phi \rangle}{\partial x} - \tilde{N}_b q_6$$

(3.50)

where the functions $q_5$ and $q_6$ are

$$q_5(\langle \phi \rangle, y) = q_3 - \frac{\langle q_3 \rangle}{\langle F \rangle} F$$

$$q_6(\langle \phi \rangle, y) = q_4 - \frac{\langle q_4 \rangle}{\langle F \rangle} F$$

(3.51)

To examine the direction of flow of the secondary current $u^{(0)}$ derived above, we need the concentration gradient $\partial \langle \phi \rangle / \partial x$. Although this can be derived more rigorously, here we avoid the additional complicated algebra and use the approx-
Figure 3.14. The functions $-q_6(\langle \phi \rangle, y)$ [subfigure (a)] and $q_7(\langle \phi \rangle, y)$ [subfigure (b)] for different average concentrations: 2% (solid line), 17% (dashed line), 40% (dotted line) and 50% (dashed-dotted line).
Figure 3.15. The function \( q_7(\langle \phi \rangle, y) - q_6(\langle \phi \rangle, y) \) [subfigure (c)] for different average concentrations: 2\% (solid line), 17\% (dashed line), 40\% (dotted line) and 50\% (dashed-dotted line).
imation that in the limit of large $\epsilon$, the vertical concentration balance will be governed by a balance between the upward shear-induced diffusive flux in the vorticity direction and the downward buoyancy driven flux.

$$\frac{\partial}{\partial x} \left[ (\alpha \tau)^{(0)} H_{11}^{(0)} \right] \approx -\hat{N}_b \langle \phi \rangle \quad (3.52)$$

Using equations 3.33 and 3.34, the above equation yields the vertical concentration gradient as

$$\frac{\partial \langle \phi \rangle}{\partial x} \approx -\hat{N}_b \frac{\langle \phi \rangle}{(1 + d)\alpha_w'} \quad (3.53)$$

Substituting the above vertical concentration gradient in Eq. 3.50, we have

$$u^{(0)} = \hat{N}_b \left[ \frac{(-d) \langle \phi \rangle}{(1 + d)\alpha_w'} q_5 - q_6 \right]$$

$$= \hat{N}_b (q_7 - q_6) \quad (3.54)$$

The function $q_7$ is given by

$$q_5(\langle \phi \rangle, y) = \frac{(-d) \langle \phi \rangle}{(1 + d)\alpha_w'} q_5(\langle \phi \rangle, y) \quad (3.55)$$

As expected, the secondary current $u^{(0)}$ is proportional to the buoyancy number. It consists of two contributing parts: a horizontal buoyancy gradient contribution $q_6(\langle \phi \rangle, y)$, and a second normal stress difference contribution $q_7(\langle \phi \rangle, y)$. The functions $q_6$ and $q_7$ are shown in figures 3.14(a) and (b) for different average concentrations (2%, 17%, 40% and 50%). The second normal stress differences induce secondary currents that flow upwards near the center of the channel and downwards near the sides, while the buoyancy driven currents show the opposite
direction: downwards near the center and upwards near the sides. The magnitude of the normal stress difference induced secondary currents is small compared to the buoyancy gradient induced currents, at low concentrations (typically 10% or less), but quickly overtakes the latter as the concentration is raised. Thus, the flow is upwards near the sides of the channel at low concentrations, but downwards near the sides at higher concentrations as can be seen in figure 3.15.

The balance used to determine the vertical concentration gradient in Eq. 3.53 neglects the contribution of the flux arising from the secondary currents. Effectively, we are taking the vertical concentration gradient expected in the absence of secondary currents and second normal stress differences and examining the secondary current pattern resulting from this concentration gradient. The actual concentration gradient established will be weaker than that predicted by Eq. 3.53 when the currents produce an upward flux (when normal stress difference induced secondary currents dominate), or stronger when the currents produce a downward flux (when buoyancy driven currents dominate).
CHAPTER 4

THE EFFECT OF GRAVITY ON THE MENISCUS ACCUMULATION
PHENOMENON IN A TUBE

4.1 Introduction

When a concentrated suspension of neutrally-buoyant rigid particles is drawn into an empty tube, a packed layer of particles starts growing at the advancing meniscus. The meniscus accumulation effect for suspension flow in a tube was first investigated in detail by Karnis and Mason [46]. They attributed the accumulation phenomenon to entrance effects and to the finite ratio of the particle radius $a$ with the tube radius $R$. When a particle is swept out from the center of the fountain-flow region at the meniscus towards the walls, the finite size of the particle implies that the particle suffers an inward displacement relative to its original incident streamline as it exits the fountain-flow region. This exclusion of particles from the slow moving wall regions of the cross-section should result in a net flux of particles into the meniscus region, causing accumulation of particles at the meniscus. They supported this explanation with the observation that as the particle size becomes small relative to the tube size, i.e., as wall effects become weak, the meniscus accumulation phenomenon vanishes. In a related investigation, Seshadri and Sutera [84] found that when a suspension from a well-mixed reservoir was pumped through a tube, the concentration of particles
exiting from the tube was equal to the original concentration of particles in the reservoir. However, when the flow was stopped and the suspension within the tube was analyzed, the concentration of particles within the tube was found to be less than the concentration of particles in the reservoir. This difference in the flow average concentration and the volume average concentration of particles was explained as due to focusing of particles towards the center of the tube during the flow. Such focusing leads to the well-known blunting of the velocity profile within the tube \[38, 45\]. They again attributed this focusing to wall exclusion effects, since the difference between the flow and volume average concentrations vanished in their experiments as the particle size was reduced relative to the tube diameter, thus corroborating the works of Karnis et al. [45] and Karnis and Mason [46].

More recent experiments by Chapman [18], however, revealed that the steady accumulation of particles is governed \textit{not} by the complex flow occurring at the meniscus and wall exclusion but rather by the steady state concentration and velocity profiles developed far upstream of the meniscus. Chapman demonstrated that the steady accumulation rate is an increasing function of \(a/R\), however it has a \textit{non-zero} asymptote at \(a/R = 0\) (see figure 4.1). Thus, even in the limit of vanishing wall effects, one should still observe the accumulation phenomenon at the meniscus. In this asymptotic limit, far away from the meniscus, shear-induced migration [55] causes particles to migrate down shear-stress gradients from the low velocity streamlines near the walls to the high velocity streamlines near the center of the tube [see figure 4.2(a)]. This leads to a particle average velocity greater than the fluid average velocity and consequently there is a net flux of particles towards the meniscus relative to the fluid. Since the meniscus moves at the average velocity of the suspension, this net flux of particles results in the continuous accumulation.
Figure 4.1. The accumulation rate of particles at the meniscus as a function of $a/R$ [18]. Diamonds - $\phi_f = 30\%$, Squares - $\phi_f = 40\%$, Triangles - $\phi_f = 50\%$. The hollow and the solid symbols correspond to data measured with 49µm and 110µm glass spheres respectively.
Figure 4.2. Models of the cross-section averaged concentration profile for the cases of (a) accumulation and (b) depletion at the meniscus.

of particles at the meniscus.

The key aspect of the meniscus accumulation phenomenon which Karnis and Mason [46] did not recognize is the induction length $L_I$ required to observe this effect. Using the scaling $\dot{\gamma}a^2$ for the shear-induced diffusivity [55], where $\dot{\gamma}$ is the characteristic shear rate $U/R$, the induction length $L_I$ reduces to the familiar result [18, 70]

$$L_I = h(\phi_f) \frac{R^3}{a^2} (4.1)$$

where $\phi_f$ is the concentration at the inlet of the tube. The proportionality factor $h(\phi_f)$ is inversely related to the magnitude of dimensionless shear-induced diffusivity in the plane of shear which is a monotonically increasing function of concentration. This scaling was verified by Chapman (1990), and the scaling fac-
tor $h(\phi_f)$ was determined as 0.25, 0.063 and 0.009 for inlet concentrations of 30%, 40% and 50% respectively. Eq. 4.1 suggests that the induction length is large for small particles ($a^{-2}$ dependence), large tubes ($R^3$ dependence) and lower concentrations. Karnis and Mason[46] thus failed to observe meniscus accumulation for their small particles, since their tubes were not long enough for the inception of this phenomenon. Likewise, the deviation between flow and area average concentrations as observed by Seshadri and Sutera [84] vanished for small particles since the radial distribution had not reached steady state.

For neutrally buoyant suspensions, the steady state flow average concentration always exceeds the volume average concentration in a tube and meniscus accumulation eventually occurs. If the suspension is not neutrally buoyant, and there is a component of gravity transverse to the flow, this is not necessarily true. Gravity causes particles to be displaced away from the tube center into regions of lower velocities. Thus, for sufficiently large effects of gravity, the particle average velocity is less than the fluid average velocity in the cross-section, which results in a net flux of particles away from the meniscus. This leads to the continuous growth of a particle-free fluid layer at the meniscus.

The dimensionless parameter that quantifies the effect of gravity perpendicular to the flow is the buoyancy number $N_b$ defined in Eq. 3.4 which measures the gravitational settling velocity relative to the shear-induced resuspending velocity. Accumulation occurs at the meniscus for small values of $N_b$ (relatively weak effects of gravity) while depletion occurs for large values of $N_b$ (relatively strong effects of gravity), implying that there exists a critical buoyancy number $N_{bc}$ which represents a cross-over between the accumulation and depletion regimes.

Since the behavior at the meniscus can be predicted by comparing the particle
average velocity \( \langle w\phi \rangle / \langle \phi \rangle \) and the fluid average velocity \( \langle w(1 - \phi) \rangle / \langle(1 - \phi) \rangle \) corresponding to the fully developed axial velocity \( w \) and concentration \( \phi \) profiles over the cross-section far upstream of the meniscus, the critical buoyancy number represents the case when the two average velocities are equal, i.e.

\[
\frac{\langle w\phi \rangle}{\langle \phi \rangle} = \frac{\langle w(1 - \phi) \rangle}{\langle(1 - \phi) \rangle}
\] (4.2)

which can be reduced to

\[
\langle w\phi \rangle = \langle w \rangle \langle \phi \rangle
\] (4.3)

Here \( < \cdot > \) denotes spatial averaging over the cross-section of the tube. The difference \( \langle w\phi \rangle - \langle w \rangle \langle \phi \rangle \) represents the net flux of particles through a cross-section towards the meniscus in the frame of reference of the meniscus, and therefore is positive for accumulation at the meniscus and negative for depletion. The same interpretation applies to the difference \( \langle w\phi \rangle / \langle w \rangle - \langle \phi \rangle \), where the flow average concentration \( \phi_f = \langle w\phi \rangle / \langle w \rangle \) is compared with the area average concentration \( \phi_a = \langle \phi \rangle \). Note that in the absence of gradients in either the concentration profile or the velocity profile, the flow and area average concentrations are identical and neither accumulation nor depletion should be observed at the meniscus.

Looking at the meniscus accumulation and depletion for suspension flow through a tube in the presence of gravity appears to be quite an uninteresting project. The latent reason for this research, however, is to indirectly examine how resuspension is affected by the cross-sectional components of the velocity vector, which, as seen in chapter 3, are, in general, non-zero. The effect of these secondary currents on the concentration balance equation scales as the Peclet number \( R^2/a^2 \).
In order to model the suspension with continuum equations, the Peclet number is kept high, usually 100 or more. Therefore, the secondary currents can have considerable impact on the concentration profile. Typically, the rheology of the suspension is modeled with a simple Newtonian viscosity \[69, 99\], which is a result of the assumption that the contribution of the particle stress to the overall stress is isotropic and hence Newtonian. With this assumption, the only source term for the secondary flow within the cross-section is the body force term due to buoyancy gradients. This source term produces secondary currents that flow downwards near the center of the tube and upwards near the side walls, thus producing a net downward flux of particles that inhibits resuspension. Tirumukudulu [94] measured the resuspension height for slurry flow through a tube using Laser Doppler velocimetry for different buoyancy numbers and average concentrations. He found that the diffusive flux model (see also [99]) for particle flux combined with a Newtonian rheology for the momentum equations resulted in an underprediction of the experimentally observed resuspension height and also overpredicted the stratification of concentration produced by gravity within the cross-section and therefore the value of \(\phi_a\) for a given value of \(N_b\) and \(\phi_f\). As shown in chapter 3, this discrepancy arises due to the incorrect assumption of Newtonian rheology for suspensions. It is well known in the literature that suspensions are non-Newtonian in nature [33, 87, 88, 97]. The normal stress differences observed in suspensions imply that there is an additional source term in the momentum equations, which, for concentrated suspensions, gives rise to secondary currents that flow upwards near the center and downwards near the sides. This reversal of the secondary current profile enhances resuspension and therefore strongly affects the area average concentration and the meniscus growth rate as compared to the Newtonian result.
In section 4.2, we discuss simple models of the cross-section averaged axial concentration profile for accumulation and depletion that connect the area and flow average concentrations to the meniscus growth rate. This meniscus growth rate is the slope of the linear relationship between the length of the accumulated/depleted layer to the total length of the suspension drawn into the tube, and is measured as a function of the buoyancy number at two different flow average concentrations: $\phi_f = 0.3$ and $0.4$. In section 4.3, the growth rate data is used to determine the variation of $\phi_a$ with $N_b$ using the models in section 4.2. This variation is compared with the behavior of $\phi_a$ with $N_b$ predicted by the isotropic and anisotropic suspension balance models. In addition to this, the variation of the induction length $L_I$ as a function of the buoyancy number and the average axial concentration profiles measured using a simple settling technique are also discussed. Finally, we conclude by summarizing our findings.

4.2 Theory

In chapter 3, we obtained the area average concentration as a function of the buoyancy number for a fixed flow average concentration and Peclet number via simulations for isotropic and anisotropic suspension balance models. Here, we show that area average concentration can also be obtained experimentally by observing the rate of growth of the meniscus layer for a fixed flow average concentration and buoyancy number. The meniscus growth rate $\lambda$ is defined as the rate of increase of the meniscus layer length $L_{\text{men}}$ (packed or empty depending on accumulation or depletion) relative to the rate of increase of the total suspension length $L$ drawn into the tube in the limit of drawn lengths much greater than the
induction length $L_I$.  

$$\lambda = \lim_{L \gg L_I} \frac{dL_{\text{men}}}{dL}$$  \hspace{1cm} (4.4)  

The sign of $\lambda$ is taken to be positive for accumulation and negative for depletion. The growth rate $\lambda$ can be related to the flow average and area average concentrations by performing the mass balance for a simple, representative model of the area average concentration profile along the tube length, and this is described below.

Consider the cross-section averaged concentration profile along the length of the tube for $L \gg L_I$ shown schematically in figure 4.3(a) for the case of accumulation of particles at the interface. The concentration profile falls from the inlet concentration $\phi_f$ to a reduced area averaged concentration $\phi_a$ over a length $L_{I_1}$.  

Figure 4.3. A schematic diagram of the area averaged axial concentration profiles for (a) accumulation and (b) depletion for drawn lengths much greater than the induction length $L_I$.  

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This area averaged concentration $\phi_a$ extends for a length $L_a$ along the tube before spiking to the meniscus packing fraction $\phi_{\text{men}}$ over a length $L_{I_2}$ and ending in the meniscus layer of length $L_{\text{men}}$. The induction length $L_I$ is nothing but the sum of $L_{I_1}$ and $L_{I_2}$. As the suspension is drawn further into the tube, it is assumed that the only regions that increase in length are the bulk length $L_a$ and the meniscus length $L_{\text{men}}$. Using this pseudo steady-state model, a total material balance and particle balance yield

\[
\frac{dL}{dt} = \frac{dL_a}{dt} + \frac{dL_{\text{men}}}{dt} \quad (4.5)
\]

\[
\phi_f \frac{dL}{dt} = \phi_a \frac{dL_a}{dt} + \phi_{\text{men}} \frac{dL_{\text{men}}}{dt} \quad (4.6)
\]

Here $\phi_{\text{men}}$ is the particle concentration in the meniscus layer. Eliminating $dL_a/dt$ from the above equations, we have

\[
\lambda = \frac{dL_{\text{men}}/dt}{dL/dt} = \frac{\phi_f - \phi_a}{\phi_{\text{men}} - \phi_a} \quad (4.7)
\]

Similarly, for the case of particle depletion at the meniscus [see figure 4.3(b)],

\[
\frac{dL}{dt} = \frac{dL_a}{dt} + \frac{dL_{\text{men}}}{dt} \quad (4.8)
\]

\[
\phi_f \frac{dL}{dt} = \phi_a \frac{dL_a}{dt} \quad (4.9)
\]

\[
\lambda = \frac{-dL_{\text{men}}/dt}{dL/dt} = \frac{-\phi_a - \phi_f}{\phi_a} \quad (4.10)
\]

since in this case the meniscus concentration $\phi_{\text{men}}$ is zero. It is thus possible to use the meniscus growth rate to back-calculate the area average concentration knowing the flow average concentration $\phi_f$ and meniscus concentration $\phi_{\text{men}}$ using
the following relationships:

\[
\phi_a = \frac{\phi_f - \lambda \phi_{\text{men}}}{1 - \lambda} \quad \text{for accumulation}
\]

\[
= \frac{\phi_f}{1 + \lambda} \quad \text{for depletion}
\]

Note that in the accumulation regime, the meniscus concentration \( \phi_{\text{men}} \) is required to calculate the area average concentration from the meniscus growth rate. It is difficult to predict the meniscus concentration theoretically, because it is unclear as to precisely what mechanism determines this concentration. In the packed meniscus layer, the concentration distribution is such that the particle and fluid average velocities are equal, i.e. meniscus accumulation ceases. The suspension balance model with Zarraga’s constitutive equations described earlier in this section suggests that this could happen only at maximum packing, where the entire cross-section is at \( \phi_m \), and at zero concentration, where the particles are absent. While the meniscus concentration \( \phi_{\text{men}} \) is close to the maximum packing fraction \( \phi_m \), in practice it is found to be slightly less than this value. In this work, \( \phi_{\text{men}} \) was determined experimentally by measuring the height of the settled meniscus layer using a technique described in the experimental section. Examination of the data in this work and also that of Chapman [18] reveals that the meniscus concentration varies between 0.54 and 0.56 with an average value of 0.55, which is the value employed in calculating the area average concentration from the meniscus growth rate data. The deviation of the meniscus concentration from the maximum packing fraction could be due to the following reasons:

1. The model discussed in this section is not valid at such high average concentrations, since it does not account for the slip between the particulate and fluid phases. Close to maximum packing, the particulate phase may also be
regarded as a packed bed of particles (e.g. a porous solid), thus any pressure gradient will give rise to an additional fluid velocity due to the seepage of fluid through the porous network of particles. Thus the fluid average velocity is augmented by a pressure-driven permeate fluid flux, and therefore the particle and fluid average velocities will become equal at a concentration less than the maximum packing fraction.

2. The flow leading up to the meniscus is a complex, fountain flow that will not satisfy the governing equations laid out in this section. In the fountain flow region, the particle concentration is expected to be high near the center due to shear-induced migration and also near the walls due to concentrated suspension being returned along the walls by the fountain flow effect. Such a particle distribution could satisfy the condition of particle and fluid average velocities being equal.

The resolution of this meniscus packing phenomenon is beyond the scope of this thesis.

In the next section, the experimental setup and procedure used to determine the area average concentration are described. The experimental results are then compared with the theoretical predictions of $\phi_a$ from the isotropic and anisotropic models using Eq. 4.11.

4.3 Experiment

4.3.1 Materials

A variety of combinations of particles, tubes and fluids were used in this study to characterize the meniscus accumulation/depletion phenomenon. The
glass tubes used in the experiments were of two different diameters: 2.00 mm and 6.35 mm. The mean diameters of the tubes were measured by injecting known volumes of water through the empty tubes and measuring the interface displacement. The particles used in the experiments were Class IV glass beads obtained from Mo-Sci Corporation in two sizes: 93.6 ± 9.2 µm and 334.3 ± 14.8 µm. The particle size distributions were measured optically, and were found to be Gaussian. The 94 µm spheres were used with the 2.00 mm tube, while the 334 µm spheres were used with the 6.35 mm tube giving $R/a$ ratios of 21 and 19 respectively for the two combinations. The density of glass spheres was measured via water displacement. The measured densities corresponded with the reported value of 2.48 g/cm$^3$ for both batches of glass spheres. The fluids used were glycerin-water mixtures (88% and 98% by wt.) and the silicone fluid Dow Corning 710. The properties of the suspending fluids used are summarized in table I. The viscosity measure-
Figure 4.4. Experimental setup (a) The angle of the tube $\theta$ with the horizontal is varied to change $N_b$. (b) The tube is kept horizontal and the average velocity of the suspension is varied to change $N_b$.

Experiments were performed using a Carri-Med Controlled Stress rheometer with a 4 cm parallel plate geometry. The viscosities of the suspending fluids were measured both before and after the meniscus experiments and found to be consistent to within 2%, confirming that water absorption or evaporation was minimal. The suspensions were prepared by carefully adding the glass spheres to the suspending fluid while the latter was being stirred slowly by a spatula. Air bubbles entrained during the mixing procedure were reduced by degassing the stirred mixture and allowing it settle for several hours.
4.3.2 Setup and procedure

In this study, the buoyancy number $N_b$ was varied in three different ways. In one setup (figure 4.4 a), the angle of the tube was varied with the horizontal, thus changing the component of gravity normal to the flow direction. The flow rate was kept sufficiently high so that the sedimentation velocity of the suspension along the flow direction was much lower than the average velocity of the suspension through the tube. The buoyancy number of $N_b = 0$ was achieved by drawing the suspension up a vertical tube. In the alternative setup shown in figure 4.4(b), the tube was kept horizontal and the flow rate of the suspension through the tube was varied to change the buoyancy number. The viscosity of the suspending fluid provided the third option for adjusting the buoyancy number. In all cases, the suspension was drawn from a reservoir which was kept unstirred. When an initially uniform suspension with concentration $\phi_0$ settles under the influence of gravity, the concentration profile typically shows three layers (figure 4.5): a clear fluid layer on the top, a layer at the original particle concentration $\phi_0$ and finally a packed layer at the bottom at concentration $\phi_m$. The position of the inlet of the tube and the initial volume of homogeneous suspension was appropriately selected so that the suspension was drawn from the intermediate layer of concentration $\phi_0$ for the entire duration of the experiment. The temperature of the suspension was measured during the experiment, and the viscosity of the fluid was measured at this temperature.

The suspension was drawn into the tube using a syringe pump at a fixed flow rate. The actual flow rate was determined by measuring the total suspension length as a function of time. This flow rate was constant as expected in the depletion region. In the accumulation region, the large resistance of the packed
Figure 4.5. Stratification of particle concentration in the reservoir.

The meniscus layer resulted in a slight drop in the velocity due to the compressibility of the air in the syringe, and the variation of the average velocity as the meniscus accumulation proceeded at a constant rate was less than 3%. In all cases, the measured average flow rate was used in calculating $N_b$.

After the formation of a packed or depleted meniscus layer, the length of the meniscus layer ($L_{men}$) and the total length of the suspension in the tube ($L$) were measured at regular intervals of time. In the case of depletion, the abrupt demarcation of the depleted layer from the flowing suspension was very clear. In the case of accumulation of particles at the meniscus, the observation of the boundary of the packed layer was more difficult. A small fraction of the glass spheres used in these experiments is black in color. The length of the packed layer at the meniscus was recorded at the position where a black sphere approaching the meniscus would turn around due to the fountain flow and start receding away from the meniscus. Whenever this was not possible to observe, then the experiment for a given buoyancy number was repeated for different total lengths of suspension drawn into the tube and the length of the meniscus layer was noted after complete settling of the suspension in the tube. The final meniscus length after settling was also observed for those experiments where the edge of the fountain flow region
Figure 4.6. The meniscus length $L_{m}$ measured as a function of the total length of the suspension $L$ drawn into the tube at a bulk concentration of 30% for (a) accumulation: $N_b = 7.7$ and (b) depletion: $N_b = 24.2$. The slope of the graph represents the growth rate of the meniscus layer, while the $x$-intercept represents the induction length $L_I$. The measurements of the packed meniscus length in subfigure (a) were made by observing the fountain flow of marked particles behind the packed layer. The last reading represented by the triangle was taken right at the end of the experimental run and was verified by allowing the suspension to settle and measuring the length of the packed layer.
could be determined, and was found to be consistent (figure 4.6). In either case, the objective was to determine the meniscus layer length \( L_{\text{men}} \) as a function of the total length \( L \) of the suspension drawn into the tube for a given buoyancy number. As discussed in the theory, a graph of \( L_{\text{men}} \) against \( L \) is a simple measure of the particle distribution far upstream of the meniscus. The asymptote of this graph for large \( L \) is a straight line whose slope gives the growth rate \( \lambda \). Two such data sets, one each for accumulation and depletion, are shown in figure 4.6. As can be seen in this figure, the length of the meniscus layer increases from zero at a finite length of suspension drawn into the tube, and rapidly reaches an asymptotic linear behavior with the length \( L \) of the suspension drawn into the tube. The slope and the intercept of this asymptote are taken as the growth rate \( \lambda \) and the induction length \( L_I \) respectively. In this way, \( \lambda \) and \( L_I \) can be empirically obtained as functions of the buoyancy number. The data for \( L_{\text{men}} \) vs. \( L \) was collected for different values of buoyancy numbers for inlet concentration \( \phi_f \) of 0.3 and 0.4. Whenever the induction length was small, especially at 40% concentration, the larger 334 \( \mu \)m particles were used in combination with the 6.5 mm diameter tube.

The area averaged axial concentration distribution was studied by stopping the flow and allowing the particles to settle over a time much greater than the hindered settling time across the diameter of the tube. The height of the settled bed \( H \) was determined using a horizontal traveling microscope. The measurement of the settled height of the suspension is complicated by the refraction caused by the curvature of the tube wall. Chapman [18] showed that the settled height \( H \) could be determined by turning the tube about its axis so that the interface between the fluid and the settled bed appears horizontal and then measuring the
Figure 4.7. Measurement of the settled height in a tube.
apparent height of this interface in this position from a reference level as shown in figure 4.7. He demonstrated that the difference $\Delta y$ between the height of the interface $y_i$ in this turned position and the height of the center of the tube $y_c$ is directly related to the settled height $H$ as

$$H = R + \frac{n_a}{n_f} \Delta y = R + \frac{n_a}{n_f} (y_i - y_c)$$  \hspace{1cm} (4.12)$$

where $n_a$ and $n_f$ are the refractive indices of air and the suspending fluid respectively. Surprisingly, this relationship is independent of the refractive index $n_g$ of the glass tube and the wall thickness $\Delta R$. For successful implementation of this technique for large average concentrations, however, the tube wall thickness must exceed a minimum thickness $\Delta R_{min}$ in order to avoid total internal reflection at the inner wall of the tube:

$$\Delta R > \Delta R_{min} = R \left( \frac{n_f}{n_a} - 1 \right)$$  \hspace{1cm} (4.13)$$

The local area average concentration profile may be determined from the settled height $H$ using the relationship

$$\phi = \frac{\phi_{m_{exp}}}{\pi} \left[ \frac{\pi}{2} + \left( \frac{H}{R} - 1 \right) \sqrt{1 - \left( \frac{H}{R} - 1 \right)^2} + \sin^{-1} \left( \frac{H}{R} - 1 \right) \right]$$  \hspace{1cm} (4.14)$$

Here $\phi_{m_{exp}}$ is the settled packing fraction measured experimentally by closing the mass balance in the entire tube knowing that the average concentration is $\phi_f$. $\phi_{m_{exp}}$ was found to vary between 0.56 to 0.58 for $R/a = 20$.

The measurement of the area average concentration along the length of the tube using the settling technique discussed above relies on the establishment of a
perfectly flat layer of particles. An error of one particle radius for $R/a = 20$ can result in an error in the measured concentration of around ±0.01. In addition, while the settled interface appeared to be flat, any non-uniformity would lead to a further underestimation of the maximum packing fraction. It is for this reason that the meniscus length provides a much more accurate measure of the small deviation between $\phi_a$ and $\phi_f$ than direct measurement.

4.3.3 Results and discussion

The area-averaged axial concentration profiles for $\phi_f = 0.3$ and $R/a = 20$ are shown in figure 4.8 for a range of buoyancy numbers. As discussed in the previous section, there is significant local scatter in the measurements, but the data does reveal some general trends. When $N_b = 0$, the area averaged concentration decreases from the inlet concentration to a plateau, and then sharply rises to the meniscus concentration. The rapid variation of the concentration profile near the meniscus layer is due to the strong dependence of the shear-induced diffusivity on concentration. The plateau concentration is below the inlet concentration, although, as expected, the difference is very small. As gravity is introduced, the essential features of the zero gravity case are retained, but the difference between the inlet concentration and the plateau concentration decreases further [figure 4.8(b)]. Also, the length of the meniscus layer diminishes as the critical buoyancy number is approached [compare subfigures (b) and (c)]. Close to the critical buoyancy number [figure 4.8 (c) and (d)], the concentration profile is almost flat at the inlet concentration, except at the meniscus, where the concentration rises to maximum packing in the case of accumulation, or drops to zero in the case of depletion. For $N_b >> N_b_c$ [figure 4.8(e)], the concentration increases from the inlet
Figure 4.8. The variation of the cross-section averaged concentration with the axial position as a function of the buoyancy number for a bulk concentration of 30%. The light horizontal line in each plot represents the flow average concentration for the experimental set. Measurements in subfigures (a), (b), (d) and (e) were performed using a suspension of 94 μm glass spheres in DC710, while the measurement in subfigure (c) was performed using a suspension of 94 μm glass spheres in 98% glycerin. The buoyancy numbers ($N_b$) for these measurements are (a) 0 (b) 5.6 (c) 10.4 (d) 23.9 (e) 46.9. The total length of the suspension in each of these measurements are (a) 59.3 cm (b) 44.1 cm (c) 79.4 cm (d) 46.0 cm (e) 16.5 cm.
to the plateau concentration, and falls off to zero as the meniscus is approached, which is a typical feature of the depletion phenomenon. Also, the length of the depleted meniscus layer increases as we move further into the depletion regime away from the critical buoyancy number [compare subfigures (d) and (e)]. These measurements are consistent with our models of the meniscus accumulation and depletion axial profiles in figure 4.3 and therefore our expressions for the area average concentration in Eq. 4.11.

Shown in figure 4.9(a) and 4.9(b) are plots of the meniscus growth rate $\lambda$ against the buoyancy number $N_b$ for $\phi_f = 0.4$ and $\phi_f = 0.3$ respectively. The growth rate is taken to be positive in the accumulation regime and negative in the depletion regime. As expected, the growth rate decreases in the accumulation regime as the buoyancy number is raised, drops to zero at the critical buoyancy number while crossing over to the depletion regime and again continues to decrease monotonically.

Let us now look at the variation of area average concentration as a function of the buoyancy number. In principle, the area average concentration can be determined as the plateau concentration in the area average axial concentration profiles shown in figure 4.8. The meniscus growth rate, however, provides a much more accurate measure of the quantity via Eq. 4.11. The area average concentration calculated from the meniscus growth rate data is shown as function of the buoyancy number in figure 4.10. The dotted and solid lines in each graph are the behaviors predicted from the isotropic and anisotropic models respectively. At a bulk concentration of 40%, the isotropic model overpredicts the experimental measurements of the area average concentration by as much as $\Delta \phi = 0.035$. The isotropic model thus gives a value of 6 for $N_{bc}$ which is more than a factor of 4.
Figure 4.9. The meniscus growth rate \( \lambda \) as a function of the buoyancy number \( N_b \) from experiment for \( R/a = 20, \text{Pe} = 400 \) and a bulk concentration of (a) 40\% and (b) 30\%. Circles - Setup II with 88\% glycerin by wt. with 94 \( \mu \)m particles, Squares - Setup I with 98\% glycerin by wt. with 334 \( \mu \)m particles, Upright triangles - Setup II with DC710 with 94 \( \mu \)m particles, Triangles pointing downward - Setup II with 98\% with 94 \( \mu \)m particles, Diamond - Setup I with 98\% glycerin by wt. with 94 \( \mu \)m particles, Asterisks - Setup I with DC710 with 94 \( \mu \)m particles.
Figure 4.10. The area average concentration $\phi_a$ as a function of the buoyancy number $N_b$ from both experiment and theory for $R/a = 20$ and bulk concentrations of 30% and 40%. The lines are the results from the simulations: solid line - Anisotropic model with $Pe = 400$, dotted line - Isotropic model with $Pe = 400$. The symbols are the results from meniscus growth experiments for $Pe = 400$: Circles - Setup II with 88% glycerin by wt. with 94 $\mu$m particles, Squares - Setup I with 98% glycerin by wt. with 334 $\mu$m particles, Upright triangles - Setup II with DC710 with 94 $\mu$m particles, Triangles pointing downward - Setup II with 98% with 94 $\mu$m particles, Diamonds - Setup I with 98% glycerin by wt. with 94 $\mu$m particles, Asterisks - Setup I with DC710 with 94 $\mu$m particles.
lower than the value of approximately 26 observed in the experiments. On the other hand, \( \phi_a \) predicted by the anisotropic model shows excellent agreement with the experimental data in the depletion regime and an error of \( \Delta \phi < 0.01 \) in the accumulation regime. The value \( N_{b_c} = 23.5 \) obtained from the anisotropic model lies within 10% error of the experimental measurement.

For the 30% data [figure 4.10(b)], the anisotropic model predicts \( N_{b_c} \) to be 6.5, which is a factor of 2.5 smaller than the experimental value of 15, although much closer than the value of \( N_{b_c} = 1.5 \) predicted by the isotropic model. Examination of figure 4.10 shows that the anisotropic model overpredicts \( \phi_a \) by approximately \( \Delta \phi = 0.015 \) at all values of \( N_b \) for this feed concentration. Even though this is much better than isotropic model which overpredicts \( \phi_a \) by as much as \( \Delta \phi = 0.06 \), the discrepancy is still somewhat puzzling, particularly in light of the close agreement at \( \phi_f = 0.4 \). Possible explanations include errors in the constitutive equations (normal stresses are very difficult to measure at low concentrations and exhibit great experimental scatter) or the failure to include non-continuum effects in the model. The determination of the causes of this residual disagreement is left to future work.

The results of the normalized induction length \( L_I/(R^3/a^2) \) as a function of buoyancy number \( N_b \) are shown in figure 4.11 for \( \phi_f = 0.4 \) and 0.3, the trends being similar in both cases. The induction lengths for \( \phi_f = 0.3 \) are more reliable as they are numerically much larger at the lower concentration and thus more easily measured. As the buoyancy number increases from zero in the accumulation regime, the induction length appears to decrease weakly and exhibit increased scatter near the critical buoyancy number. In the depletion regime, just beyond \( N_{b_c} \), the induction length rises to a large value discontinuously after which it
Figure 4.11. The normalized induction length $L_1/(R^3/a^2)$ as a function of the buoyancy number $N_b$ from the meniscus growth experiments for a bulk concentration of (a) 40% and (b) 30%. The symbols are explained in the caption of figure 4.10.
gradually drops with further increase in $N_b$.

The key to understanding the discontinuous behavior of the induction length is to recognize that the length measured in the experiment is not the length required to achieve fully developed flow, but rather the apparent length required to achieve a steady accumulation or depletion of particles at the meniscus. When the suspension is drawn into the tube from an initially mixed condition, particles migrate quickly to the zero shear stress, high velocity streamlines in the cross-section due to shear-induced diffusion. Thus the initial transient as the suspension enters the tube is always such that there is a net transport of particles towards the meniscus, thus increasing the concentration at the meniscus. As the effect of gravity sets in, the particle distribution reaches its steady configuration at a rate determined by $N_b$. If the buoyancy number is slightly less than the critical value, i.e. in the accumulation regime, the steady state distribution is such that the particles are added to an already concentrated meniscus, thus continuously increasing the meniscus length, albeit very slowly. Because the rate of accumulation resulting from the transient initial concentration distribution is greater than the small asymptotic rate, the apparent induction length is reduced and actually may be zero or negative close to $N_{bc}$.

Now consider the situation when $N_b$ is very close to $N_{bc}$, but in the depletion regime. In this case, the particles accumulated at the meniscus due to the initial transient have to be removed completely before a steady production of clear fluid can begin. Since the depletion rate for values of $N_b$ slightly greater than $N_{bc}$ is very small, the complete evacuation of particles from the meniscus leads to large values of $L_I$. Such transient accumulations for $N_b > N_{bc}$ were actually observed in our experiments in the depletion regime.
4.4 Conclusions

In this chapter, we have analyzed the effect of gravity on the meniscus accumulation phenomenon. For small buoyancy numbers, the focusing of particles near the fast-moving, zero shear stress streamlines leads a net flux of particles towards the interface, causing accumulation at the interface. As the strength of gravity relative to shear-induced migration increases, a balance which is quantified by the buoyancy number, more particles are drawn towards the slow-moving wall regions. The growth rate of the packed layer of particles at the meniscus decreases, therefore, with increase in the buoyancy number, eventually leading to a negative growth rate and a pure suspending fluid layer growing continuously at the meniscus above the critical buoyancy number.

The variation of the area average concentration as a function of the buoyancy number was calculated using the suspension balance model with the constitutive equations of Zarraga et al. [97] in both isotropic and anisotropic forms. These results were compared with the experimental behavior of $\phi_a$ with $N_b$ which was measured by observing the rate of the continuous growth of the packed (accumulated) or pure fluid (depleted) layer at the meniscus. At an inlet concentration of 40%, $\phi_a$ calculated from the anisotropic model agrees extremely well with the experimental data, and the critical buoyancy number is obtained is within 10% of the experimental value. On the other hand, the isotropic model grossly underpredicts $N_{bc}$ by a factor of 4. At 30% bulk concentration, however, the anisotropic model does not perform as well, overpredicting $N_{bc}$ by a factor of 2.5. However, the isotropic model deviates from the experimental value of $N_{bc}$ by an order of magnitude, and therefore, the anisotropic model still represents a marked improvement over the isotropic model.
The experimental results in this chapter corroborate indirectly the simulation results of chapter 3. The secondary currents with the inclusion of particle stress anisotropy flow upwards near the tube center and downwards near the side walls, which is completely opposite in direction to the buoyancy gradient driven secondary currents predicted by an isotropic model. The reversed nature of the secondary currents combined with the center-heavy concentration profile produced by shear-induced migration leads to a net upward flux of particles. The isotropic model, on the other hand, predicts a net downward flux due to the gravitationally-induced secondary currents. Although the magnitude of the secondary currents is small, the flux of particles due to the secondary currents scales as the Peclet number $R^2/a^2$ which is kept large in experiments, and therefore the effect of these currents on the concentration distribution is strong. The anisotropic model thus predicts much stronger resuspension (lower $\phi_a$) for a given buoyancy number compared to the isotropic model, which was verified by experiment in this chapter.

In measuring the accumulation/depletion rate at the meniscus, we also measured the induction length required before this phenomenon reached steady state. The induction length was found to be discontinuous at the critical buoyancy number, assuming a small value on the accumulation side, but jumping to a large value on the depletion side. The large induction lengths close to the critical buoyancy number in the depletion regime result from a transient initial accumulation of particles at the meniscus as the suspension is first drawn into the tube, which is difficult to deplete to zero with the small steady negative growth rates for $N_b$ close to $N_{bc}$.

While the meniscus accumulation/depletion phenomenon provides a simple and accurate method for examining the concentration distribution in a tube, it
is also of more direct interest. For example, Zhou and Chang [100] showed that this phenomenon leads to capillary failure in efforts to use capillary forces to draw blood into capillaries less than 50 \( \mu \text{m} \) in diameter for blood diagnostic kits. Also, it may be possible to develop separation schemes based on inducing flow above and below the critical buoyancy number.
CHAPTER 5

PARTICLE MIGRATION IN CONCENTRATED SUSPENSIONS UNDERGOING SQUEEZE FLOW

5.1 Introduction

The parallel plate geometry (figure 5.1) is a popular geometry used to study the rheology of concentrated suspensions [11, 18, 21, 50, 61, 97]. Of particular interest here is the possibility of a non-uniform particle concentration distribution in this geometry which has been a subject of debate in the literature for quite some time. As discussed in chapter 2, the diffusive flux model [55, 73] suggests that particles should migrate from the high shear stress regions near the outer edge of the plate to the low shear stress regions near the center of the plate, giving rise to a maximum in the radial concentration profile at the center. However, when Chapman [18] investigated the torque measured in the parallel plate geometry for a 45% suspension, he found very little change in the torque signal for large strains of order $10^4$. This final strain value was chosen to ensure that particle migration would occur over half of the plate radius. This was a surprising result at that time, since an inward particle migration from an initially homogeneous particle distribution should lead a significant decrease in the torque, thus implying that there is little or no migration of particles in the parallel plate geometry. This suggestion was supported by Chow et al. [21], who measured the radial concentration
Figure 5.1. The classic parallel plate geometry used in suspension experiments. The flow is in $\theta$ direction, while concentration and velocity gradients are established in the $r$ direction.
profile for a 50% suspension via NMR imaging and found no significant concentration inhomogeneities across the radius of the device. However, their torque measurements showed a completely different behavior; it increased sharply over small strains ($\sim 3$) to a maximum value, then decreased strongly to around 80% of the peak value for strains up to 100, and then decreased gradually to around 75% of the maximum signal for a strain of around $10^5$.

In more recent experiments, Merhi et al. [61] reported a weak outward migration of particles for a 40% suspension by measuring light transmission between the parallel plates as a function of the radial position. They supported this observation by measuring the torque signal as a function of time for a suspension of smaller particles and found the torque to increase by almost 40% after a total strain of $2 \times 10^6$. However, the experiments of Bricker and Butler [11] show that the long term behavior of the torque may be due to the change in the composition of the suspending fluid. Since the observation of radial particle migration requires very long experimental run times (16 hours or more), it is extremely difficult to maintain a constant experimental environment over such extended periods. Bricker and Butler noted that suspensions composed of water-based suspending fluids showed a long term increase or decrease in the torque, possibly due to the loss or absorption of moisture from the environment. They repeated their experiments with a suspension made using polyalkylene glycol (not water-based), and found virtually no change in the torque.

In this chapter, we investigate the loading procedure of a suspension on to a parallel plate rheometer as a possible explanation for the observed scatter in the torque measurements. A typical measurement using the parallel plate geometry begins by placing a known volume of the sample on the bottom plate, and then
bringing the two plates together so that the sample is squeezed to fill the gap between the plates. Thus, the loading procedure is essentially a squeeze flow process. Squeeze flow is, in its own right, a rheometric technique (see the comprehensive review of Engmann [30]). In this technique (see figure 5.2), the test material is placed between two parallel plates converging under a constant force or a constant velocity. With knowledge of the normal force $F$ on the plates and the separation $h$ between the plates as functions of time, it is possible to determine rheological properties such as viscosity and yield stress for simple fluids.

Relatively few studies of the squeeze flow of concentrated suspensions are available in the literature. Most of these studies have been performed on particle pastes, where the concentration of the particles is usually greater than 55% by volume fraction. In the squeeze flow of a paste, the relative motion between the binder or the suspending phase and the particles results in interesting segregation effects. Experimental studies [22, 26, 75] have revealed that at low squeeze velocities of the plates, the liquid phase filters through the solid phase, resulting in a high concentration of the particles near the center of the plates and a lower concentration at the outer edge. In this chapter, we discuss the squeeze flow of suspensions at more moderate concentrations of particles. Unlike the research work discussed above, in our experiments, the concentration of particles near the center decreases as the suspension is squeezed, and an enrichment of particles is observed near the interface. The cause of this segregation is the well known meniscus accumulation phenomenon that was first studied in detail by Karnis and Mason [46] for suspension flow through a tube. More interestingly, as the packed layer of particles is formed at the interface, the suspension-air interface breaks into unstable fingers. This curious manifestation of the classical viscous fingering instability was first
Figure 5.2. Schematic of squeeze flow, which is implemented in two configurations: (a) Constant volume mode, where the volume of the test material remains constant throughout the experiment, and (b) Constant area mode, where the area of plates occupied by the material is constant during the experiment, and excess material is squeezed out from the sides. Here $F(t)$ is the normal force applied on the top plate, while $h(t)$ is the separation between the plates as a function of time.
reported by Tang et al. [92] for radial source flow between two parallel plates with a fixed separation.

The objective of this chapter is to characterize the squeeze flow of a moderately concentrated suspension of rigid, non-colloidal particles and to determine the critical radius at which non-uniformities occur. This chapter is organized as follows. In section 5.2, we discuss the theory behind the steady accumulation of particles behind the meniscus, and derive a relationship for the characteristic radius $R_m$ associated with the inception of the meniscus accumulation phenomenon in the squeeze flow geometry. In section 5.3, we describe the simple experimental setup used to study the squeeze flow of suspensions in the constant mass mode (see figure 5.2). We then discuss results of the squeeze flow of 30% and 40% (by volume) suspensions and compare it with pure fluid results. We also verify the induction radius scaling derived in section 5.2. In section 5.4, we discuss the implications of these results on the measured torque in the parallel plate geometry.

5.2 Theory

Consider a pure fluid of viscosity $\mu$ being squeezed in between two parallel plates under the weight of the top plate of mass $M$. Employing the lubrication approximation and assuming negligible surface tension effects, it can be shown that the radius $R$ of the fluid-air interface propagates with time $t$ as

$$R^i = \frac{8MgV_0^2}{3\pi^3\mu}(t - t_0)$$

(5.1)

The quantity $t_0 = -3\pi^3R_0^8\mu/(8MgV_0^2)$ represents the time necessary for the plates to reach the initial value of $R_0$ from a large separation. $V_0$ is the initial volume of the suspension charged between the parallel plates. Thus, a graph of $R^i$ for a
pure fluid being squeezed out between two parallel plates is linear with time.

Now consider the squeeze flow of a concentrated suspension. If the particle distribution in the suspension remains homogeneous during the squeezing procedure, the suspension-air interface should still obey the linear relationship of $R^8$ with $t$ in Eq. 5.1, but with a viscosity $\mu$ given by

$$\mu = \mu_0 \mu_r(\phi)$$  \hspace{1cm} (5.2)$$
where $\mu_r$ is the suspension relative viscosity given in Eq. 2.15. The concentration, however, does not remain uniform in the suspension phase during the squeezing process. Analogous to the tube flow of a suspension discussed in chapter 4, the concentration at the advancing meniscus increases continuously as the squeezing proceeds, eventually reaching a meniscus packing concentration at a critical radius $R_m$. Since viscosity is a highly non-linear function of concentration, the resistance of the suspension to the applied constant force should increase sharply once the meniscus layer is formed, and the graph of $R^8$ vs. $t$ should deviate from the linear behavior. The scaling for $R_m$ at which this deviation occurs is derived below.

For squeeze flow, the approach to steady state may be regarded as the result of a diffusion process across the spacing between the plates. Consider the cross-stream shear-induced diffusion of particles in a reference frame moving with the local average velocity of the suspension. The progress to the steady state distribution is governed by the variable $\chi$ defined as

$$\chi = \int_0^t \frac{D}{(h/2)^2} dt$$  \hspace{1cm} (5.3)$$
where $h$ is the local depth of the channel and $D$ is the shear-induced diffusion
coefficient in the plane of shear. For a suspension, \( D = \hat{D}_|| \dot{\gamma} a^2 \) where \( \dot{\gamma} \) is the local shear rate. The dimensionless diffusivity in the plane of shear was measured by Leighton and Acrivos [55] and found to be well correlated by

\[
\hat{D}_|| = \frac{\phi^2}{3} \left[ 1 + \frac{1}{2} \exp 8.8\phi \right]
\]  

(5.4)

\( \hat{D}_|| \) is a strong function of concentration and reaches an \( O(1) \) value of 0.95 for \( \phi = 0.4 \). If we replace \( \dot{\gamma} \) with its average value \( \langle \dot{\gamma} \rangle \) across the channel, we obtain

\[
\chi = \int_0^t \hat{D}_|| \frac{a^2}{(h/2)^2} \langle \dot{\gamma} \rangle \, dt
\]  

(5.5)

Since \( \langle \dot{\gamma} \rangle \) is given by \( 3 \langle u \rangle / h \) for a Newtonian fluid in the lubrication limit, where \( \langle u \rangle \) is the average velocity, \( \chi \) becomes

\[
\chi = \int_0^t 12 \hat{D}_|| \frac{a^2 \langle u \rangle}{h^2} \frac{\langle \dot{u} \rangle}{h} \, dt
\]  

(5.6)

For the analogous development in tube flow, we have \( \langle \dot{\gamma} \rangle = \frac{8 \langle u \rangle}{3R_t} \), where \( R_t \) is the tube radius. \( \chi \) can therefore be written for the tube geometry as

\[
\chi = \int_0^L \frac{\hat{D}_|| a^2}{R_t^3} \, dL = \frac{\hat{D}_|| La^2}{R_t^3}
\]  

(5.7)

where \( \langle u \rangle \) has been substituted by \( dL/dt \), \( L \) being the length of the suspension in the tube. This leads to the usual scaling for the meniscus length \( L_m \sim \chi_m R_t^3/a^2 \).

From the data of Chapman [18] shown in figure 5.3, \( \chi_m \) is a weakly decreasing function of concentration varying from 0.2 at 30% to 0.15 at 40% flow average concentration. Meniscus accumulation may thus be expected to begin in the tube geometry for a mean value of \( \chi_m \) of 0.175.
Figure 5.3. The quantity $\chi$ calculated from the experimental results of [18] using Eq. 5.7 as a function of $a/R$ at two different average concentrations 30% and 40%. The triangles and the diamond are from data measured at 40% inlet concentration, while the remaining points are from the 30% data. For the hollow symbols, $2a = 49\mu m$, while for the dark symbols, $2a = 115\mu m$. 
For the case of squeeze flow between two parallel plates, the calculation of the progress $\chi$ of the squeezing is more complicated since the depth $h$ is no longer a constant. Applying a mass balance over the depth at a radial position $r$, we obtain

$$\langle u \rangle = \frac{r(-dh/dt)}{2h} \quad (5.8)$$

where $\langle u \rangle$ is the depth-averaged suspension velocity. Substituting the above equation in Eq. 5.6, we get

$$\chi = \int_0^h 12 \hat{D}_r \frac{a^2}{h^2} \frac{1}{2h} \left( \frac{dh}{dt} \right) dt \quad (5.9)$$

This equation applies to both the interior and the outer edge $r = R$. By conservation of mass, the radius $r$ at any time $t$ may be related to the depth $h$ as

$$r = r_0 \left( \frac{h}{h_0} \right)^{-1/2} \quad (5.10)$$

where $r_0$ and $h_0$ are the initial local radius and plate separation respectively. Eq. 5.9 may therefore integrated to provide $\chi$ as

$$\chi = \frac{12 a^2 \hat{D}_r}{7 h^3} \left[ 1 - \left( \frac{h}{h_0} \right)^{7/2} \right] = \frac{12 a^2 \hat{D}_r}{7 h^3} \left[ 1 - \left( \frac{r_0}{r} \right)^7 \right] \quad (5.11)$$

At the suspension-air interface ($r = R$), $\chi$ becomes

$$\chi = \frac{12 a^2 \hat{D}_r R}{7 h^3} \left[ 1 - \left( \frac{R_0}{R} \right)^7 \right] \quad (5.12)$$
If $V_0$ is the volume of the suspension initially loaded in between the plates, then

$$h = \frac{V_0}{\pi R^2} \quad (5.13)$$

thus reducing $\chi$ to

$$\chi = \frac{12\pi^3 a^2 \hat{D}_{\parallel} R^7}{7 V_0^3 \left[ 1 - \left( \frac{R_0}{R} \right)^7 \right]} \quad (5.14)$$

It is interesting to note that since the initial radius $R_0$ is always less than $R$ by some amount and also because of the large exponent of the ratio $(R_0/R)^7$ in the expression for $\chi$, the correction to $\chi$ for the initial radius is nearly always negligible. For example, if $R/R_0 > 2$, taking $R_0 = 0$ in Eq. 5.14 modifies $\chi$ by less than 1%. In other words, the approach to the steady state distribution is an abrupt function of $R$ and is dominated by the later stages of squeezing process. At any stage, a 10% increase in $R$ appropriately doubles $\chi$.

From Eq. 5.14, we may define a characteristic radius $R_c$ associated with the migration process as

$$R_c = \left( \frac{7 V_0^3}{12\pi^3 a^2 \hat{D}_{\parallel}} \right)^{1/7} \quad (5.15)$$

In terms of the dimensionless radius $R^* = R/R_c$, $\chi$ is simply

$$\chi = R^{*7} - R_0^{*7} \quad (5.16)$$

For $R^*$ sufficiently larger than $R_0^*$, the dimensionless strain $\chi$ may be approximated
as

\[ \chi \approx R^7 \]  \hspace{1cm} (5.17)

By analogy with the results of Chapman’s experiments with a tube, we anticipate that for \( \chi \ll 0.175 \) the suspension is essentially homogeneous within the gap, while for \( \chi \sim 0.175 \), a packed meniscus layer should be formed.

We may also use \( R_c \) to construct a characteristic time. From Eq. 5.1, it is apparent that \( R^8 \) varies linearly with time for a Newtonian fluid. Using \( R_c \) as the characteristic radius, the characteristic time scale \( t_c \) may be written from this equation as

\[ t_c = \frac{3\pi^3 \mu_0 \mu_r R^8_c}{8MgV_0^2} \]  \hspace{1cm} (5.18)

If we define a dimensionless time \( t^* \) as

\[ t^* = \frac{t - t_0}{t_c} \]  \hspace{1cm} (5.19)

we obtain the linear relationship \( R^8/t^* = 1 \) for a fluid with constant viscosity. Any deviation from this relationship is a measure of concentration inhomogeneities.

The squeeze flow of suspensions is further complicated by the fact that the packed meniscus layer does not grow continuously as observed in a tube. The key difference between the tube geometry and geometries such as the rectangular channel, radial source flow and squeeze flow is the existence of multiple length scales in the cross-section. The cross-section of a tube is characterized just by its radius \( R \), whereas the cross-section of a rectangular channel, for example, is characterized by a depth \( 2b \) and a width \( W \) while that of radial source flow is
Figure 5.4. An image of the concentration distribution near the meniscus after the inception of the instability in a rectangular slot of depth $2B = 2$ mm and width $W = 5.08$ cm with the depth of the channel directed into the plane of the paper. The image shown is taken 20 cm downstream of the entrance. The suspension used for the experiment was a 40% suspension of 150 µm PMMA particles in Triton X-100. As can be seen, the wavelength of the instability is roughly half of the width of the channel. The arrows depict the flow of the suspension relative to the average motion of the meniscus. Particles are convected from packed meniscus region (which appears as a dark gray band) to the sidewalls and to the center of the channel.
Figure 5.5. An image of the instability observed in the squeeze flow of a 40% suspension of 100 µm glass spheres in glycerin. The image was taken approximately 8 min after the squeezing was started. The plate radius is 23 cm, the mass of the top plate is $M = 987.3$ gm, and the volume of the suspension in between the plates is $V_0 = 8.5$ ml. The black marker at the centers of the two plates helps in aligning the plates.
characterized by the separation $2b$ between the plates and the circumference $2\pi r$ at any radial position $r$ of the plate. As explained by Tang et al. [92] who studied the flow of a suspension through the radial Hele-Shaw geometry at a constant flow rate, the accumulation of particles at the interface produces a sharp increase in concentration at the meniscus. On account of the highly non-linear relationship between concentration and viscosity, the concentration gradient at the meniscus represents a steep viscosity gradient. This unfavorable viscosity contrast at the miscible interface which separates the packed, highly viscous meniscus layer and the particle-lean, less viscous suspension following it, is stable for the tube geometry but susceptible to a miscible viscous fingering instability for geometries with more than one length scale in their cross-section. The unstable miscible interface in turn drives the instability synchronously on the outer, immiscible, suspension-air interface. In figures 5.4 and 5.5, we have shown images of the instability for suspension flow through a large aspect ratio rectangular channel and the squeeze flow geometry respectively. In this chapter, we determine the radius $R_i$ at which the instability at the suspension-air interface first becomes observable. Although $R_i$ is not identical to $R_m$, the radius at which the packed meniscus layer forms, it will be shown that $R_i$ also satisfies the scaling in Eq. 5.17 derived for the critical radius $R_m$.

We end this section by observing that the induction length derived above assumes that shear-induced migration is the dominant mode of cross-stream particle migration in the geometry. This may not be true, especially for larger $a/b$ ratios, where wall exclusion effects are important. In our experiments we take care that at the critical radius, the ratio $h/2a$ is large enough (15 or more) so that the contribution of the wall exclusion phenomenon to the induction radius is not the
dominant contribution. In the next section, the squeezing flows of pure fluid and suspension are investigated experimentally in the constant volume configuration. The mobility of the material is captured by monitoring the average radial position of the material-air interface with time. The experiments also capture critical radius $R_i$ for the observation of the instability on the suspension-air interface by studying the distortion of the interface from its stable circular shape.

5.3 Experiment

5.3.1 Materials and procedure

Two liquids were used in the pure fluid experiments: glycerin-water-dye mixture and a triton-zinc chloride-water-dye mixture. The glycerin-water-dye solution was prepared by mixing 96% glycerin, 2% water and 2% dye, while the triton-zinc chloride-water solution was prepared by mixing 75% of Triton X-100, 14% zinc chloride, 9% water with a few drops of hydrochloric acid and 2%. (all percentages are based on weight). The dye employed was a red dye manufactured by Durkee and was added for the purpose of imaging. The viscosities of the fluids were measured at the operating temperature of 23°C with a CarriMed controlled stress rheometer. For the experiments with suspensions, mixtures of glass spheres in the glycerin-water-dye solution were employed for the 40% measurements, while a 69-29-2 % mixture by weight of karo syrup, glycerin and dye respectively was used at 30% concentration. The properties of various fluids are summarized in table 5.1. The glass spheres were obtained from Mo-Sci Corporation in the Class V size range. Three different sizes of glass spheres were used: $47.2 \pm 2.2 \, \mu m$, $93.6 \pm 9.2 \, \mu m$ and $188.9 \pm 16.9 \, \mu m$. The uncertainty provided here is the population standard deviation. In all experiments, care was taken that the settling time of the
TABLE 5.1
SUSPENDING FLUIDS USED IN THE SQUEEZE FLOW MEASUREMENTS AND THEIR PROPERTIES

<table>
<thead>
<tr>
<th>Suspending fluid</th>
<th>Density at 23°C (gm/cm³)</th>
<th>Viscosity at 23°C (poise)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dyed Glycerin-water mixture</td>
<td>1.24</td>
<td>8.0</td>
</tr>
<tr>
<td>Dyed Karo syrup-glycerin mixture</td>
<td>1.34</td>
<td>32.5</td>
</tr>
<tr>
<td>Dyed Triton-ZnCl₂-Water mixture</td>
<td>1.18</td>
<td>26.2</td>
</tr>
</tbody>
</table>

Suspension over the gap width between the plates was not comparable to the experimental run time. The suspensions were prepared by carefully adding the glass spheres to the suspending fluid while slowing stirring with a spatula. In spite of the careful addition of the glass spheres, some air bubbles were entrained during the mixing procedure especially for the smaller 50 µm spheres. The fraction of air bubbles was reduced by degassing the stirred mixture and by allowing it settle for times of the order of a few days. The glass plates used in the experiment were polished soda-lime plates and their properties are summarized in Table 5.2.

In a typical experiment (see Figure 5.6), the bottom plate was placed on a light table. Three heavy circular blocks on the side of the plates served as guides to prevent the drift of the settling plate during the experiment. A known volume of pure fluid/suspension (with red dye) was injected in a circular holder at the center of the bottom plate. A drop of the suspension was also placed at the center of the top plate. The circular holder was removed and the top plate was quickly placed on the suspension on bottom plate. The squeezing of the suspension was imaged top down using a SONY CCD camera and digitized in MATLAB. The
Figure 5.6. Schematic of the experimental setup used to obtain squeeze flow data. (a) Top view (b) Front view
TABLE 5.2
DETAILS OF THE GLASS PLATES USED IN THE SQUEEZE FLOW EXPERIMENTS

<table>
<thead>
<tr>
<th>Plate number</th>
<th>Plate diameter (in)</th>
<th>Plate thickness (in)</th>
<th>Mass M (gm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>1/8</td>
<td>139.3</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>1/4</td>
<td>586.1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>3/8</td>
<td>987.3</td>
</tr>
</tbody>
</table>

Image processing toolbox in MATLAB was used to determine the boundary of the pure fluid or suspension area with air.

Using the pixels constituting the boundary of the fluid, an interpolating function $R_b(s)$ was constructed to describe the boundary, where $R$ is the distance of the boundary point from the center of mass $(x_c, y_c)$ of the area occupied by the suspension while $s$ is the curvilinear co-ordinate defined along the boundary. The effective radius $R$ at a given instant of time is then given by

$$R = \sqrt{\frac{A}{\pi}}$$

(5.20)

where $A$ is the area occupied by the suspension. In order to determine the onset of the instability, the deviation of the interface from a perfect circle with radius equal to the effective radius needs to be tracked. As the instability is initiated and the fingers grow in amplitude, we expect this deviation to increase in magnitude. For our purposes, this deviation is most easily described via the non-dimensionalized
quantity \( \Lambda \) given by

\[
\Lambda = \frac{1}{S} \int_0^S \left( 1 - \frac{R_b(s)}{R} \right)^2 ds
\]

(5.21)

where \( S \) is the perimeter of the boundary of the suspension-air interface.

5.3.2 Results

Let us first examine the results of the squeezing flow of a pure fluid. In figure 5.7, we have shown the variation of \( R^8 \) and \( R^8/(t - t_0) \) with time \( t \) for the different pure fluids. As expected, these curves obey the straight line relationship in Eq. 5.1 obtained from the lubrication approximation. The slight deviation in the first few seconds of each experiment resulted from the relaxation of any non-uniformities of the initial shape occupied by the fluid to a perfect circle. For longer times, the experimental data follows the lubrication result quite well [figure 5.7 (b)]. We also performed the squeezing experiment with a mixture of Triton X-100, water and zinc chloride with different initial volumes and the same set of plates. It can be seen in figure 5.8 that these measurements satisfy the no-adjustable parameter slope calculated from the viscosity reported in table 5.1.

Now consider the squeezing flow of a suspension. In figure 5.9(a), we have shown the variation of \( R^8 \) with dimensionless time \( t^* \) for a 40% suspension for different combinations of particle sizes and initial volumes. It can be seen that the curves of \( R^8 \) vs. \( t^* \) are linear for a short period of time, but then fall off from the straight line relationship. The slope of the linear regime is close to the slope of 1 expected from theory. Note that the scalings \( R_c \) and \( t_c \) used to obtain the dimensionless radius \( R^* \) and time \( t^* \) respectively were calculated from experimental parameters (except for \( \mu_r \), which was calculated from using from Eq.
Figure 5.7. Verification of the lubrication result in Eq. 5.1 for a pure fluid with (a) $R^8$ vs. $t$, and (b) $R^8/[8MgV_0^2(t-t_0)/(3\pi^3\mu)]$ vs. $t$.

Circles - 98% glycerin, $V_0 = 2$ ml, $M = 139.3$ gm. Triangles - 98% glycerin, $V_0 = 7.8$ ml, $M = 586.1$ gm. Squares - Triton-Water-ZnCl$_2$ mixture, $V_0 = 12.5$ ml, $M = 586.1$ gm. Asterisks - Triton-Water-ZnCl$_2$ mixture, $V_0 = 16.5$ ml, $M = 586.1$ gm. The horizontal line represents a linear relationship. The viscosities used in the curves in subfigure (b) are the measured viscosities reported in table 5.1.
Slope = \( 8MgV_0^2/(3\pi^2\mu) \)

Figure 5.8. Verification of the dependence of the slope of \( R^8 \) v/s \( t \) on \( V_0 \) for a pure fluid. Circles - experimental values of the slope obtained with the triton-water-ZnCl\(_2\) mixture, \( M = 586.1 \) gm. The straight line represents \( 8MgV_0^2/(3\pi^2\mu) \) from the lubrication result in Eq. 5.1 calculated using the viscosity of the liquid reported in table 5.1.
Figure 5.9. Test of the 40% suspension squeeze flow data against the lubrication result in Eq. 5.1. The solid line is the theoretical linear result of slope 1. The data follows the linear relationship between $R^*$ and $t^*$ for a short period, but then falls off this relationship indicating a decrease in mobility.
2.15 using \( \phi = 0.4 \). The quantity \( t_0 \) is obtained as the \( x \) intercept of the linear regime of each plot of \( R^8 \) vs. \( t \). The deviation from the straight line relationship can also be seen from a plot of \( R^8 - t^* \) with \( \chi = R^7 \) in figure 5.9(b). The data can be collapsed better if the slopes of the linear regions of the \( R^8 \) vs. \( t^* \) are normalized to 1. The slopes deviate slightly from 1 as can be seen in figure 5.9(b) due to variations in the viscosity from one experiment to another, and also due to errors in the measured volumes. The maximum error in the slope in these experiments was less than 5%. The data is replotted in figure 5.10 with the normalized linear regimes. One can see from this figure that the deviation from the Newtonian behavior begins at a value of \( \chi \) which varies from 0.07 to 0.15 for all the data sets. This transition, which occurs due to the inception of the meniscus accumulation, will be taken to occur at an average value of \( \chi_m = 0.1 \) (solid line in figure 5.10). This critical value of \( \chi \) is not very different from corresponding value \( \chi_m = 0.175 \) for the inception of meniscus accumulation in a tube, which is indicated by the dashed vertical line in figure 5.10(b). From the definition of \( \chi \) in Eq. 5.17, we can write the critical radius \( R_m \) for meniscus accumulation in squeeze flow as from Eq. 5.23 as

\[
R_m^7 = \frac{7V_0^3}{12\pi^3D\|a^2} \chi_m = 1.88 \times 10^{-3} \frac{V_0^3}{D\|a^2}
\] (5.22)
Figure 5.10. Test of the 40% suspension squeeze flow data against the lubrication result in Eq. 5.1. The data sets have been normalized to have the same initial linear regime but adjusting the initial volumes $V_0$ to the values shown in the inset. In subfigure (b), the solid, dashed and dashed-dotted vertical lines represent $\chi_m$ for squeeze flow, $\chi_m$ for tube flow and $\chi_i$ for squeeze flow respectively.
Figure 5.11. The variation of the deviation from the circular shape \( \Lambda \) (defined in Eq. 5.21) with \( R \) for a 30% suspension of 188 \( \mu \)m glass spheres in Karo syrup-glycerin-dye mixture with \( V_0 = 16.1 \) ml. The solid line is a sixth-order polynomial used to fit the data. The critical radius \( R_i \), which is the minimum of this curve, is around 7.95 cm.

In figure 5.9(b), we have also shown the value of \( \chi \) corresponding to the manifestation of viscous fingers on the suspension-air interface with a vertical dashed-dotted line. This transition \( \chi_i \) can be obtained from a plot of the deviation \( \Lambda \) (Eq. 5.21) of the shape of the interface from a perfect circle with \( R \). As shown in the example in figure 5.11, it can be seen the variation of \( \Lambda \) with \( R \) shows a minimum. At the beginning of the squeezing experiment, the shape of the suspension/air interface is not circular. This causes relatively large deviations for values of \( R \) close to the loading radius. As the squeezing of the suspension proceeds, the stability of the suspension/air interface in the absence of the enrichment at the interface leads to a reduction in this deviation. As the viscous fingering instability sets in,
Figure 5.12. Verification of the scaling for the critical radius $R_i$ in Eq. 5.10. Hollow symbols - 30% data, Filled symbols - 40% data. Triangles - 50 µm particles, Squares - 200 µm particles and Circles - 100 µm particles.

the deviation $\Lambda$ begins to grow due to the perturbation of the interface by the instability. This leads to the minimum observed in the variation of $\Lambda$ with $R$ at $R = R_i$.

We have shown the variation of $R_i^7$ with $V_0^3/a^2$ for a variety of initial volumes and particle radii in figure 5.12 at 30% and 40% bulk concentrations. As can be seen in the figure, the behavior is linear, suggesting that induction radius for viscous fingering also satisfies the scaling derived for meniscus accumulation in Eq. 5.10. The slopes of the two lines are $2.72 \times 10^{-2}$ and $7.29 \times 10^{-3}$ at 30% and 40% respectively, giving a ratio of 3.7 between the slopes at the two concentrations. Eq. 5.10 suggests that the ratio of these slopes should be proportional to the ratio of the shear-induced diffusivities in the plane of shear at these two concentrations.
The shear-induced diffusion coefficient in the plane of shear can be obtained from the correlation of Leighton and Acrivos [55] in Eq. 5.4. $\hat{D}_||$ assumes values of 0.240 and 0.954 at $\phi = 0.3$ and $\phi = 0.4$ respectively which gives a ratio of 3.97 at the two concentrations. This ratio is quite close to the experimentally obtained ratio of the slopes at these two concentrations. The critical radius $R_i$ for observing fingering can therefore be written as

$$R_i^7 = 0.00675 \frac{V_0^3}{\hat{D}_||a^2}$$  \hspace{1cm} (5.23)

The coefficient in the above equation has been obtained as a mean of the values at the two concentrations. Eq. 5.23 corresponds to a value of $\chi$ of

$$\chi_i = \frac{12\pi^3\hat{D}_||a^2R_i^7}{7V_0^3} = 0.36$$  \hspace{1cm} (5.24)

Since $\chi_i$ is greater than $\chi_m$ calculated for both the tube and squeezing flow geometries, it suggests that viscous fingering does not occur instantaneously as meniscus accumulation begins, i.e. a certain thickness of the meniscus layer is allowed to build before fingering can be observed on the suspension-air interface. But note that the interface radius associated with a given value of $\chi$ scales as $\chi^{1/7}$. The critical radii for meniscus accumulation and viscous fingering are in the ratio $(0.36/0.1)^{1/7} \approx 1.2$. In a typical squeeze flow experiment, the radius $R_i$ at which fingers were observed was about 9 cm, which means that the radius $R_m$ for meniscus accumulation should be approximately 7.5 cm. Since typical growth rates of the meniscus are of the order 0.1, the meniscus grows to a thickness of $0.1 \times (9 - 7.5) \approx 0.15$ cm. Thus the thickness of the meniscus layer required to observe the instability on the immiscible interface for the parameters in used in
our experiments is small enough so that it would have been difficult to capture by experiment. Note that it was not possible for us to record the shape of the miscible interface because of the presence of the dye. The difference in the mobilities of the packed meniscus layer and the lean suspension that follows it decreases as the average concentration increases, leading to lower growth rates of the instability at higher average concentrations. The system should thus be able to support thicker meniscus layers at high average concentrations and the detection of the instability from the images with the available image resolution is delayed.

5.4 Discussion

5.4.1 Development of criterion for demixing

The scaling for the critical radius for meniscus accumulation obtained in the preceding section can be used to design a parallel plate viscometer that ensures homogeneous loading of a suspension. In a typical parallel plate experiment, a known volume of the material to be analyzed is loaded on to the bottom plate, and the two plates are brought together so that the material fills the gap in between the plates. Let $R_p$ and $h_p$ be the radius and plate separation of the parallel plate viscometer. If the initial volume $V_0$ loaded on to the bottom plate is squeezed exactly into the gap between the plates, we can write

$$V_0 = \pi R_p^2 h_p$$

(5.25)

Substituting this in Eq. 5.22, we get

$$R_m^7 \approx 0.0875 \frac{(R_p^2 h_p)^3}{D_{||}a^2}$$

(5.26)
### Table 5.3

THE FIGURE OF MERIT $\Theta$ FOR VARIOUS EXPERIMENTAL INVESTIGATIONS IN THE PARALLEL PLATE GEOMETRY FROM THE LITERATURE.

<table>
<thead>
<tr>
<th>Work</th>
<th>$\overline{\phi}$</th>
<th>$2a$ (µm)</th>
<th>$R_p$ (cm)</th>
<th>$h_p$ (mm)</th>
<th>$\Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapman (1990)</td>
<td>45%</td>
<td>87</td>
<td>5.0</td>
<td>2.0</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>45%</td>
<td>500</td>
<td>9.53</td>
<td>10</td>
<td>0.19</td>
</tr>
<tr>
<td>Chow et al. (1994)</td>
<td>50%</td>
<td>196</td>
<td>1.75</td>
<td>1.85</td>
<td>1.58</td>
</tr>
<tr>
<td>Merhi et al. (2005)</td>
<td>40%</td>
<td>225</td>
<td>5.0</td>
<td>1.0</td>
<td>10.35</td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>10</td>
<td>3.0$^\dagger$</td>
<td>0.5</td>
<td>$\leq 0.10$</td>
</tr>
<tr>
<td>Bricker and Butler (2006)</td>
<td>40%</td>
<td>100</td>
<td>2.5</td>
<td>1.0</td>
<td>1.02</td>
</tr>
<tr>
<td>Zarraga et al. (2000)</td>
<td>30%</td>
<td>43</td>
<td>2.5</td>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>35%</td>
<td>43</td>
<td>2.5</td>
<td>1.0</td>
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<td>43</td>
<td>2.5</td>
<td>0.25</td>
<td>23.30</td>
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</table>
From the above equation, we see the radial concentration gradients will be absent if \( R_p << R_c \). On the other hand, there will be strong concentration inhomogeneities in the loaded suspension if \( R_p >> R_c \). We can thus arrive at a figure of merit \( \Theta \) to determine whether a given combination of a suspension and parallel plate geometry will show radial segregation or not.

\[
\Theta = 17.14 \hat{D}_0 \frac{R_p a^2}{h_p^3}
\]  

(5.27)

For small values of \( \Theta \), the loaded suspension is essentially uniform, while for large values of \( \Theta \), the loaded suspension is demixed.

In table 5.3, we have summarized various experimental studies of suspension rheology performed with the parallel plate viscometer and presented the value of \( \Theta \) for each of these investigations. In most of these studies, \( \Theta \) is much smaller than 1, which means that the formation of the meniscus layer should have been avoided. There are, however, several notable exceptions: the work of Merhi et al. [61] with the 200 \( \mu \)m particles and \( h_p = 1 \) mm, the works of Chow et al. [21] and Bricker and Butler [11], and the experiments of Zarraga et al. [97] at 50\% and 55\% and at 45\% for small gap widths (less than 1 mm). The actual values of \( \Theta \) for the work of Zarraga et al. should be smaller than the values indicated in table 5.3, because they mention that the initial volume taken was larger than the volume required to fill the gap between the plates. This means that they could have avoided retaining the packed meniscus region between the plates, but still could have had a radial concentration variation between the plates. Zarraga et al. also performed their parallel plate experiments for a variety of \( h/2a \) ratios at 45\% in

\(^1\)(Referring to the table on the previous page) For this set, we have assumed the largest parallel plate radius available with the Carrimed controlled stress rheometer, which was the rheometer used by Merhi et al. [61]
order to characterize the effect of this aspect ratio. This is briefly discussed in the 
next subsection. It should be noted here however, that in all of their experiments, 
the asymptotic value of viscosity was obtained by extrapolating the torque to the 
limit $2a/h_p \to 0$.

5.4.2 Effect of different loading conditions on the behavior of the torque

In this subsection, we analyze the effect of the concentration inhomogeneities 
induced by the loading procedure on the torque. The torque can show several 
different behaviors depending on the experimental parameters, and we study each 
case below.

The simplest possible case exists when the loaded suspension is essentially 
homogeneous both over the depth and the radius of the geometry. This happens 
for very small values of $\Theta$, where the extent of particle migration is weak during 
the squeezing process. In this case, any variation in the torque will occur only 
due to the evolution of the concentration profile to the steady-state distribution 
predicted for parallel plate flow via the models described in chapter 2 rather than 
due to the loading procedure. The torque measured by the parallel plate rheometer 
is given by

$$\Gamma = \frac{2\pi \mu_0 \Omega R_p^4}{h_p} \int_0^1 \mu_r(\phi) r^* d r^*$$  (5.28)

Thus, if the particle migration is indeed radially outward as predicted by the 
constitutive equations of Zarraga et al. [97] (see chapter 2), then the torque will 
increase with time, and this increase will be significant due to the longer lever 
arm and larger shear rate in the outer region. Following the work of Chapman 
[18], the characteristic elapsed strain at the outer edge required for observing this
particle migration is

$$\dot{\gamma}t = \frac{1}{\sigma_1^2 D_\perp a^2} R^2$$  \hspace{1cm} (5.29)

Here $\hat{D}_\perp$ is the diffusivity normal to the plane of shear, which may be given the correlation of Leighton and Acrivos [55]:

$$\hat{D}_\perp = K_\perp \phi^2 \frac{d\hat{\mu}_r}{d\phi}$$  \hspace{1cm} (5.30)

$\sigma_1$ is the leading eigenvalue, which was calculated by Chapman to be 2.568. The constant $K_\perp$ was measured to be 0.7, while the viscosity $\hat{\mu}_r$ is given by

$$\hat{\mu}_r = \left[1 + \frac{1.5\phi}{1 - \frac{\phi}{0.58}}\right]^2$$  \hspace{1cm} (5.31)

At 40% concentration, the characteristic strain $\dot{\gamma}ta^2/R^2$ required to observe the outward migration of particles can be calculated from Eq. 5.29 to 0.13. In addition, for the radial concentration distribution determined in chapter 2 (Eq. 2.29), the increase in torque for a 40% suspension is 7.5%. This seems to contradict the measurements of Merhi et al. [61] who observed an increase of 18% in the torque over a scaled strain $\gamma_0 a^2/R^2$ of approximately 24 for their 10$\mu$m particle suspensions. It is not clear as to what leads to such strong deviations from the predictions of theory.

Another simple case occurs when the $\chi$ is much greater than the characteristic strain $\chi_m$ required for observing the meniscus layer, but the loaded volume is much greater than the volume of the geometry itself. In this case, any meniscus layer formed is squeezed out of the geometry, and the excess suspension can be
Figure 5.13. The ratio $\Gamma_f/\Gamma_i$ in Eq. 5.36 using a value of $\Delta\phi = 0.025$ for an initial concentration profile that is a step down function in the radial direction (Eq. 5.33). Solid line - $\bar{\phi} = 0.5$, dashed line - $\bar{\phi} = 0.4$, dotted line - $\bar{\phi} = 0.3$. 
scraped off the edge of the plate [97]. The concentration profile in this case shows inhomogeneities both along the depth and in the radial direction. Due to particle migration towards the center of the plates away from the walls during the outward Poiseuille flow generated by the squeezing process, the particle concentration is higher in the center of the gap between the plates than near the plates themselves. Therefore, when the suspension is sheared from this initial condition, there is a short term increase in the measured torque corresponding to the relaxation of the concentration profile over the depth. This is analogous to the viscosity increase observed by Leighton and Acrivos [55] in a Couette device. The characteristic strain for observing this short term viscosity increase is given by

$$\gamma = \frac{1}{\pi^2 D_{||} a^2} \frac{h_p}{p}$$  \hspace{1cm} (5.32)

Since $R_p/h_p \gg 1$ in a typical parallel plate system, the strain required to see this viscosity increase is much lower than the strain required to observe any radial migration of particles. Locally, the strain $\chi$ varies from $\chi_R$ at the outer edge to zero in the center. Thus, particle segregation over the depth is absent near the center of the geometry, but strong near the outer edge. Consequently, the particle concentration is close to the loading concentration $\bar{\phi}$ near the center, but equal to the area average concentration $\bar{\phi} - \Delta\phi$ near the outer edge corresponding to the concentration profile developed over the depth. Since the strain $\chi$ varies as the seventh power of $R$, the transition of $\chi$ from zero to $\chi_m$ represents a sharp transition in $R$ for $\chi_R \gg \chi_m$, so that the radial concentration profile may be represented as a step function for initial volumes much larger than the volume of
the geometry.

\[
\phi = \phi - \Delta \phi H(r - r_s R_p)
\] (5.33)

where \( H \) is the Heaviside step function and \( r_s R_p \) is the radius at which the concentration steps down. The average concentration of particles in the geometry is therefore

\[
\langle \phi \rangle = \bar{\phi} r_s^4 + (\bar{\phi} - \Delta \phi) \left(1 - r_s^4\right)
\] (5.34)

and the measured torque upon loading is

\[
\Gamma_i = \frac{2\pi \mu_0 \Omega^4 R_p^4}{h} \left[ \mu_r \left(\bar{\phi}\right) \frac{r_s^4}{4} + \mu_r \left(\bar{\phi} - \Delta \phi\right) \frac{1 - r_s^4}{4} \right]
\] (5.35)

If the final concentration distribution is homogeneous, then the ratio of the final torque to the initial torque will be

\[
\frac{\Gamma_f}{\Gamma_i} = \frac{\mu_r \left(\langle \phi \rangle\right)}{\mu_r \left(\bar{\phi}\right) r_s^4 + \mu_r \left(\bar{\phi} - \Delta \phi\right) \left(1 - r_s^4\right)}
\] (5.36)

This ratio is shown in figure 5.13. As can be seen, the largest increase in the torque for \( \bar{\phi} \) varying from 0.3 to 0.5 occurs when \( r_s \) is between 0.6 and 0.7. This maximum increase is weak (\( \sim 3.5\%\)) for 30% suspensions, but not negligible (\( \sim 10\%\)) for 50% suspensions. The characteristic strain for observing this long term viscosity increase is again given by Eq. 5.29.

Consider the variation of the measured torque upon loading as a function of the particle size to gap-width ratio after the short term viscosity increase and the homogenization across the gap, as was done by Zarraga et al. [97] (Appendix C of...
their paper). They measured the viscosity of a 45% suspension as a function of the gap spacing and found that the viscosity decreases as the gap spacing is reduced. For example, at a shear rate of 10 s$^{-1}$, their measured relative viscosity using a parallel plate viscometer for a 45% suspension with $h_p/2a = 47$ was 16.5, which is close to the relative viscosity of 16.9 predicted by the correlation of Zarraga et al. (Eq. 2.15). For $h_p/2a = 6$, however, the measured relative viscosity was 9.4, a factor of 1.8 lower. In order to explain this decrease in viscosity with $a/h_p$, Zarraga et al. incorporated the wall slip correlation developed by Jana et al. [42] in their calculations, which resulted in the following equation for the apparent relative viscosity

$$\mu_{r_{\text{app}}} = \frac{\mu_r}{1 + \frac{n-1}{8} \frac{2a}{h_p}}$$  \hspace{1cm} (5.37)

where $\mu_r$ is the suspension relative viscosity in Eq. 2.15. Their experimental data and the prediction of Eq. 5.37 with $\mu_r$ evaluated at a concentration $\phi = 0.45$ are shown in figure 5.14. Although the model predicts the apparent viscosity reasonably for $h/2a$ ratios of 47, 34, 29, and 23, it overpredicted the apparent viscosity for the smaller $h/2a$ ratios. Particularly, for $h/2a = 6$, the measured relative viscosity is 1.3 times that obtained from the model. The discrepancy between the prediction and the experiment may be attributed, at least for the largest $a/R$ employed, to the loading procedure. Zarraga et al. reported that while loading the suspension on to the parallel plate, they used more volume than the final volume of the device $\pi R_p^2 h_p$, and scraped the excess suspension off the edge of the gap. This means that they possibly squeezed out any portion of the packed meniscus layer. The expulsion of the highly concentrated meniscus layer implies a reduction in the average concentration within the device. For the smallest gap
\( h_p/2a = 6 \) and therefore the largest \( \Theta \) they used, one can strongly expect that the concentration within the plates was not the initial loading concentration (0.45), but more likely the area average concentration \( (\sim 0.425) \) corresponding to the fully developed plane Poiseuille flow at the loading concentration. Thus, properly, Eq. 5.37 should evaluated at the area average concentration for large \( 2a/h_p \), and this is shown with the dashed line in figure 5.14. One can see that the prediction is excellent with this concentration for \( h/2a = 6 \), but is off for intermediate aspect ratios. For these aspect ratios, the radial concentration distribution should be modeled as a step function between the loading and the area average concentration as shown in Eq. 5.33 with \( r_s \) being a function of the aspect ratio \( 2a/h \) and the initial volume \( V_0 \). Since we do not have the initial volume data employed by Zarraga et al. in their characterization, we cannot make this prediction here.

Now consider the case when the parameter \( \chi \) is of order \( \chi_m \) and the initial volume is such that the meniscus layer is trapped between the plates near the outer edge after loading. Due to the long lever arm, high shear rate and the large viscosity contribution from the meniscus layer in the outer regions of the geometry, the measured apparent viscosity upon loading is high. As shearing proceeds from this initially inhomogeneous concentration distribution, particles will migrate out of the high concentration regions near the outer edge of the plates into the interior. However, due to the large shear-induced diffusion coefficients for concentrations corresponding to the meniscus packing fraction coupled with the fact that the gradient in the concentration distribution leading up to the meniscus packing region is steep (see figures 4.3 and 4.8), the rate of migration of particles out of the outer region of the plates is rapid, thus leading to sharp reduction in the torque over time. This is exactly what was observed by Bricker and Butler [11]
Figure 5.14. The variation of the apparent relative viscosity with the aspect ratio $2a/h_p$ at 45% concentration. Squares - experimental data of Zarraga et al., solid line - Eq. 5.37 evaluated at the loading concentration $\bar{\phi} = 0.45$, dashed line - Eq. 5.37 evaluated at the area average concentration $\bar{\phi} = 0.425$ corresponding to the loading concentration.
and Chow et al. [21]. Thus, even though the characteristic strain to observe the radial migration is the same as before (Eq. 5.29) for radial particle migration, the magnitude of the strain is numerically much smaller.

Finally, if the loading is carried out such that meniscus layer is expelled but portions of the high concentration regions left behind by the instability are retained within the cross-section, the torque will be high at first, but decrease with time over a strain corresponding the scaling in Eq. 5.29. As the shearing is commenced, these trapped channels of high concentration formed by the instability are first homogenized rapidly in the azimuthal direction. Therefore, the magnitude of the initial torque will be much lower than that in the case when the entire meniscus layer is trapped, and the strain over which the apparent viscosity decreases will be comparatively much larger.

The simplest way of circumventing this initial radial segregation of particles is to load the suspension under shear. The application of shear prevents the evolution of the concentration over the depth of the geometry and eliminates any net migration towards the advancing meniscus. To achieve this successfully, the shear rate produced by the applied rotation must be much greater than that generated by the squeeze flow, i.e.

\[
\frac{\Omega R}{h} \gg \frac{(-dh/dt)R}{h^2} \quad \text{or} \quad \Omega \gg \frac{1}{h} \left(\frac{-dh}{dt}\right)
\]

where \(\Omega\) is the rate of rotation of the plate in rad/sec. Thus, provided the rate of convergence of the plate is maintained to be small, concentration gradients in the thin direction will be weak, the net flux of particles towards the meniscus will be negligible and meniscus accumulation will be avoided.

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5.5 Conclusions

In this chapter, we have shown that when a concentrated suspension is squeezed between two parallel plates, a radial concentration gradient is developed in the device due to shear-induced migration in the shallow direction. After an initial radial transient $R_m$, a highly packed layer of particles is formed at the meniscus followed by a particle lean suspension phase, and this is susceptible to viscous miscible fingering due to the strong, monotonically increasing dependence of viscosity on concentration and thus renders the system unstable. The critical radius $R_m$ at which meniscus accumulation begins and the radius $R_i$ at which the instability forms in this geometry were both shown to obey the scaling as $R^7 \sim \frac{V^3_0}{a^2}$ based on a shear-induced migration argument.

The squeeze flow of pure fluids and suspensions were examined in this work. For a pure Newtonian fluid, the variation of $R^8$ with time was found to be linear with the slope being inversely related to the viscosity, where $R$ is the radius of the suspension-air interface. For suspensions, however, $R^8$ was found to be linear with time only for a short period, beyond which the mobility of the suspension decreased continuously due to meniscus accumulation effects. In order to monitor the progress of squeezing towards meniscus accumulation, a variable $\chi$ was formulated which is directly related to $R^7$. The deviation from the linear regime was observed at $\chi = 0.1$, while viscous fingering was observed to begin at $\chi = 0.36$.

Since the loading of a suspension on to a parallel plate device is essentially a squeeze flow, the segregation observed in this chapter may manifest itself in the parallel plate geometry. The measured dependence of the critical radius for meniscus accumulation on the various system parameters was used to devise a figure of (de)merit $\Theta$. For values of $\Theta$ much smaller than 1, an initially well
mixed suspension will remain homogeneous during the loading procedure on to a parallel plate device. If $\Theta$ is of order 1 or much greater than 1, significant radial inhomogeneity can be expected in the device. It was also shown that depending on the dimensions of the parallel plate geometry, the initially loaded suspension volume and the particle size, the apparent viscosity may show any of the following behaviors: short term decrease, short term increase, long term decrease and long term decrease. This may explain some of the puzzling and contrasting torque measurements in the literature. The radial concentration inhomogeneities may be avoided by loading the suspension slowly under shear.
CHAPTER 6

A SELF SIMILAR SOLUTION FOR THE DROP DISTRIBUTION IN THE
POISEUILLE FLOW OF A DILUTE EMULSION

6.1 Introduction

The hydrodynamic migration of particles across streamlines, be it rigid particles, drops or cells, has been a subject of intense investigation over the past few decades as discussed in the introduction to this thesis. In this chapter, we look at the specific case of hydrodynamic migration of drops across streamlines for tube Poiseuille flow at low Reynolds numbers, and model the resulting concentration distribution for a dilute emulsion of droplets.

The first study of droplet migration for flow through tubes was conducted by Goldsmith and Mason [35]. They observed that individual droplets of water suspended in viscous silicone oil migrated away from the wall of the tube towards the center. They explained this drift using a simple model that accounted for the deformation, rotation and variation of the shear rate across the drop. Chaffey et al. [15] derived the migration velocity of a Newtonian drop away from a single plane wall in a Newtonian fluid undergoing Couette flow using the method of reflections. However, since the Couette flow apparatus of Karnis and Mason [47] had two enclosing boundaries, the expression due to Chaffey et al. was found to overpredict the drift velocity. The result of Chaffey et al. was also used by Karnis

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and Mason [47] to compare their experimental drop-migration measurements in a tube and again, quite understandably, their theoretical predictions could come only within an order of magnitude of their data.

In a detailed theoretical analysis, Chan and Leal [16] examined the migration of a droplet in unidirectional shearing flow with and without a shearing gradient. Employing the reciprocal theorem with the method of reflections (to account for wall effects), they used a perturbation expansion to derive the first effects of droplet deformation and non-Newtonian nature of the droplet and the suspending fluid on the droplet drift velocity normal to the flow. They performed these calculations for a variety of flows including the Couette geometry with both bounding walls, and 3-D Poiseuille flow. They also compared their theoretical predictions in the plane Couette geometry to experimental measurements in a cylindrical Couette device in a follow-up paper [17]. They found excellent agreement with theory for their narrow gap Couette experiments with Newtonian fluids. Significant deviations, however, were found for their wide gap Couette data, which is expected since their theory was developed for the plane Couette geometry. A useful observation made by Chan and Leal [16] is that for a Newtonian droplet in a Newtonian fluid undergoing shear, the contribution of the shearing gradient of the imposed flow to the droplet migration velocity has a scaling that is order $R/a$ higher than that of the contribution of the interaction of the droplet with the wall. Here $a$ is the droplet radius, and $R$ is the length scale over which the gradient in shear rate is imposed. For plane Couette flow, the shear rate is constant and droplet drift is solely due to interaction with the wall. For tube Poiseuille flow, however, the shear rate is highest near the wall and zero near the center. Thus, if the droplet is sufficiently far away from the wall, the droplet migration velocity is dominated
by the contribution of the curvature of the applied flow.

More recently, Smart and Leighton [90] used the image systems of a stresslet about rigid and free surfaces to derive expressions for the drift velocity away from a rigid wall and a free surface respectively. The drift velocities were connected to the total stress in the emulsion derived by Schowalter et al. [81] using the development of Batchelor [7] for the Couette geometry. The experimentally measured drift velocities agreed reasonably well with these theoretical predictions.

Based on the theoretical and experimental results for the motion of a single droplet in Couette flow, droplets are expected to migrate to the center of the gap in the flow of a dilute emulsion as well. As this migration proceeds, the local droplet concentration near the center of the gap increases, and pairwise droplet-droplet interactions become important. A pair of rigid spheres in a simple shear flow under creeping flow conditions experiences zero net displacement over an interaction event because of Stokes flow reversibility. For a pair of drops, however, the governing equations are non-linear, and reversibility does not apply in general. It was shown by Loewenberg and Hinch [59] using boundary element calculations that the interaction of two drops in simple shear flow leads to a net crossflow displacement of the drops. The displacement of the drops is of the order of the particle radius $a$, and since the relative velocity of the drops scales as $\dot{\gamma}a$, the self-diffusion [29, 54] corresponding to this cross-streamline displacement scales as $\dot{\gamma}a^2$. The self-diffusion coefficient of droplets was found to be large in the velocity gradient direction, but weak in the vorticity direction. It was also found to be sensitive to the drop viscosity; highly viscous drops show negligible diffusivities. These numerical observations were supported by the experiments of Guido and Simeone [37]. By following the trajectories of drops in a parallel plate device using
video-enhanced contrast optical microscopy, they were able to show that drops are indeed displaced from their original incident streamlines after an interaction. They were able to confirm that this displacement was irreversible by following repeated collision events between two drops and noting that the difference in the velocity gradient co-ordinates of the drop centers $\Delta y$ continued to increase till the drop interaction was negligible. They also observed larger values of $\Delta y$ along the velocity gradient direction than the vorticity direction, thus corroborating the anisotropy of the self-diffusion coefficient predicted by Loewenberg and Hinch.

Under steady state conditions, the distribution of droplets in a dilute emulsion undergoing flow in a Couette cell or a tube will be determined by a balance between the convective flux due to deformation induced drift and the shear-induced diffusive flux. These combined effects were first investigated by King and Leighton [48]. They modeled the evolution of the concentration distribution of a dilute emulsion in a Couette device using the convective radial droplet flux due to the expression of Chan and Leal [16] and a diffusive radial flux driven by gradients in concentration (not self-diffusion) that scales as $\lambda \phi \dot{\gamma} a^2$, where $\phi$ is the droplet volume fraction. They were able to show that by retaining only the linear term for the droplet drift (which implies that their theory was applicable only for short distances from the Couette center), the concentration profile obeys a self-similar evolution such that the droplet distribution is always a parabola. The theory was valid, however, only if the edge of the concentration distribution was always within the tube walls. They compared this parabolic self-similar solution with their experimental, time-dependent concentration profiles and found reasonably good agreement. The value of the dimensionless gradient diffusivity $\lambda$ estimated from their experiments was of the order $10^{-2}$ and found to be relatively indepen-
dent of the capillary number for Ca < 1. A related study by Hudson [40] was performed with clean, surfactant free fluids with larger viscosities and for a wider range of shear rates in a parallel plate device. They measured λ to be 0.2, which is an order of magnitude higher than the measurements of King and Leighton.

In more recent work, Hollingsworth and Johns [39] examined the concentration distribution of a dilute emulsion in a Couette device via MRI. They found that the extent of migration of the dispersed phase towards the Couette center increased with increasing shear rate, but eventually reached a saturation. They also found the droplet migration to be stronger for higher viscosities of the continuous phase and lower emulsion concentrations, which is consistent with the model of King and Leighton.

In this chapter, we develop a theoretical model for Poiseuille flow of an emulsion in a tube (section 6.2), closely following the work of King and Leighton [48]. We demonstrate that a self-similar solution is possible in this case as well for arbitrary time dependent amplitude of the applied Poiseuille flow. This solution helps us to write the concentration evolution of a dilute emulsion for an oscillatory flow field. In particular, we determine the effect of an asymmetric oscillatory flow field on the total flux of the dispersed phase over one period. By an asymmetric flow field, we mean that the average shear rate in the forward pumping phase is different from the average shear rate in the reverse pumping phase, and this leads to different degrees of focusing of the suspended phase in the two pumping phases. We examine the potential of such a flow field as a technique for the separation of the dispersed phase from the suspending fluid.
6.2 Theory

Consider the flow of a dilute emulsion of neutrally buoyant droplets through a tube of radius $R$ at low Reynolds numbers. Both the droplet and the suspending fluids are Newtonian. The droplets are monodisperse with radius $a$ and have a viscosity of $\kappa \mu_0$, where $\mu_0$ is the viscosity of the suspending fluid. The coordinates in the radial and flow directions are $r$ and $z$ respectively. We assume that the flow is quasi-unidirectional, so that the concentration and flow distributions are independent of the $z$ co-ordinate. Also, the distributions are assumed to be axisymmetric and hence independent of the azimuthal co-ordinate $\theta$.

Since the emulsion is dilute, we ignore the effect of the concentration of the droplets on the emulsion viscosity. This, coupled with the quasi-unidirectional flow assumption, results in a simple solution for the momentum equations in the creeping flow limit with the axial velocity $u$ given by the simple parabolic profile

$$u_z = 2U_0 U(t/t_c) \left( 1 - \frac{r^2}{R^2} \right)$$

(6.1)

Here, we allow the average velocity $U_0 U$ to be a function of time. $t_c$ is a time scale that will be defined later (Eq. 6.18). As mentioned in the introduction, we will explore the effect of applying an oscillatory axial flow on the net axial droplet flux over a period of oscillation. Equation 6.1 assumes, of course, that the time scale of variation of $U$ is much greater than the characteristic time scale $R^2 \rho / \mu_0$ for diffusion of momentum across the tube radius.

The droplet conservation equation is given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rN_r) = 0$$

(6.2)
Here $\phi(r, t)$ is the concentration (volume fraction) distribution of the droplets. $N_r$ is the radial flux of the droplets. Note that the flux term $\partial N_z / \partial z$ in the flow direction is absent, and this, again, is because of the quasi-unidirectional flow approximation. The radial flux of droplets $N_r$ comprises two parts: a convective flux $N_{rC}$ and a diffusive flux $N_{rD}$. We discuss each of these fluxes below separately.

The convective flux $N_{rC}$ arises from the drift of the droplet due to its deformability. Following the work of Chan and Leal [16], for small droplet deformations, the droplet drift velocity in the pressure-driven flow of an emulsion through a tube of radius $R$ is given, to leading order in $a/R$, by

$$u_d(r) = -g(\kappa) \frac{a^2}{R^2} \frac{Ca}{U_0} \frac{r}{R}$$

where

$$g(\kappa) = \frac{8}{(1 + \kappa)^2(2 + 3\kappa)} \left[ \frac{3}{14} \frac{16 + 19\kappa}{2 + 3\kappa} (1 - \kappa - 2\kappa^2) \right]$$

$$+ \frac{(10 + 11\kappa)}{140} (8 - \kappa + 3\kappa^2)$$

(6.4)

$Ca$ is the capillary number given by

$$Ca = \frac{U_0 a \mu_0}{R \sigma}$$

(6.5)

where $\sigma$ is the surface tension at the interface. The function $g(\kappa)$ is positive for small and large $\kappa$, and therefore the drift of the droplet is indeed away from the walls. For intermediate viscosities of the droplet (i.e. $\kappa \sim O(1)$), however, the droplets drift towards the wall under the applied shear flow. In this chapter, we restrict ourselves to the situations where $\kappa << 1$ or $\kappa >> 1$, in which case the
droplet deformation produces a flux towards the tube center. We are interested in this limit, because, as we shall see in the next section, this negative flux leads to a region near the walls devoid of droplets, and this droplet deficient region forms the basis of the development of a self-similar solution. The convective radial flux is therefore

\[ N_r = -g(\kappa)U_0 \frac{a^2}{R^2} \text{Ca} U_0^2 \frac{r}{R} \phi \]  

(6.6)

While the convective flux due to deformation induced drift in a dilute emulsion is well understood from both theory and experiment, the diffusive flux due to drop interactions is less clear. In general, the diffusive flux may be modeled as arising from irreversible displacements which occur as droplets interact in a shear flow. In its simplest form, this gives rise to a shear-induced diffusivity \( D \) proportional to both the local concentration and shear rate, e.g.:

\[ N_r = -D \frac{\partial \phi}{\partial r} = -\lambda \phi \dot{\gamma} a^2 \frac{\partial \phi}{\partial r} \]  

(6.7)

It is this constitutive equation which was used to describe radial migration in a narrow gap Couette device by King and Leighton [48], and then subsequently by Zhou and Chang [100] to model radial migration in the tube Poiseuille flow of red blood cell suspensions. Note that for the plane Couette geometry, \( \dot{\gamma} \) is constant across the gap, whereas for tube flow, \( \dot{\gamma} \) is linearly related to the radial co-ordinate \( r \). For concentrated suspensions of rigid particles, it is well known that irreversible interactions give rise to a migration from regions of high shear rate to low in addition to the diffusive flux described by Eq. 6.7. Thus, even in the absence of deformation-induced drift, this yields a higher concentration at
the centerline of Poiseuille flow. The shear-induced particle migration of rigid particles is best described by the suspension balance model [70], which attributes shear-induced particle flux to gradients in the particle stress. For example, the radial particle flux in the tube flow of a rigid particle suspension may be written as [31, 77]

\[ N_{rd}^{\text{rigid}} = -\frac{2}{9} \frac{a^2}{\mu_0} f(\phi) \left( \frac{\partial \alpha^{\text{rigid}} \tau}{\partial r} + \left( -d \right) \frac{\alpha^{\text{rigid}} \tau}{r} \right) \]  

(6.8)

The first diffusive term in the parentheses in the above equation arises from the combined effects of gradients in concentration and shear rate, while the second term arises due to the curvature of streamsurfaces in tube flow. In Eq. 6.8, \( f \) is the hindered settling factor which for dilute suspensions is approximately 1. \( \tau \) is the magnitude of the local shear stress. For flow through a tube, \( \tau \) is directly proportional to the local radial position.

\[ \tau = \frac{4 U_0 |U| \mu_0}{R} \frac{r}{R} \]  

(6.9)

\( \alpha^{\text{rigid}} = 2.17 \phi^3 \exp(2.34\phi) \) is the reduced normal stress and \( d^{\text{rigid}} \) is the reduced second normal stress difference coefficient. For rigid particle suspensions, \( d^{\text{rigid}} \) was measured by Zarraga et al. [97] for volume fractions ranging from 0.3 to 0.55 and found to be well represented by a constant value of −0.54. For deformable particles, the constitutive equations will be different. Nevertheless, for low capillary numbers, the radial flux may still be governed to leading order by Eq. 6.8, but with different expressions for \( \alpha \) and \( d \). For rigid particles, in the dilute limit, two particle interactions in an unbounded shear flow are symmetric, and therefore the shear-induced diffusivity should scale as \( \phi^2 \), which leads to the \( \phi^3 \) scaling of
Since pair interactions of droplets are not symmetric in the dilute limit, we expect that for a dilute emulsion, $\alpha$ should scale as $\phi^2$.

$$\alpha = \lambda_0 \phi^2$$

where $\lambda_0$ is expected to be a function of the capillary number $Ca$ and the viscosity ratio $\kappa$ [59]. We are not aware of any experimental measurements of the second normal stress difference of dilute emulsions. The second normal stress difference was calculated for emulsions by Loewenberg and Hinch [58] and again by Zinchenko and Davis [101]. However, they did not present their data in the dilute limit in the form $N_2/\phi^2\tau$ for $\kappa < 1$ or $\kappa >> 1$. For simplicity, we assume here that $d$ is constant for dilute emulsions as well. Thus, the radial flux for the tube Poiseuille flow of a dilute emulsion may be written as

$$N_{rd} = -\frac{2}{9} a^2 \mu_0 \lambda_0 \left( \frac{\partial (\phi^2 \tau)}{\partial r} + (-d) \frac{\phi^2 \tau}{r} \right)$$

$$= -\frac{4}{9} a^2 \lambda_0 \phi \tau \frac{\partial \phi}{\partial r} - \frac{2}{9} a^2 \mu_0 \lambda_0 (1 - d) \frac{\phi^2 \tau}{r}$$

The shear-induced diffusivity $D$ is thus

$$D = \frac{4}{9} \frac{\tau}{\mu_0} a^2 \phi = \frac{4}{9} \lambda_0 \dot{\gamma} a^2 \phi$$

The above expression for $D$ can be compared $D$ from the flux expression 6.7 used by King and Leighton [48] to yield an estimate for $\lambda_0$.

$$\lambda_0 = \frac{9}{4} \lambda$$

King and Leighton measured $\lambda$ from the Couette experiments of their surfactant
laden drops to be about 0.02, while Hudson \[40\] measured $\lambda$ to be 0.2 for his clean droplet emulsions. The values of $\lambda_0$ corresponding to these measurements are 0.045 and 0.45 respectively. Note that due to the linear radial dependence of the shear stress in tube Poiseuille flow, the flux model of Zhou and Chang \[100\] may be recovered by simply choosing $d = 1$. For this value, the anisotropy induced migration is radially outward, and exactly balances the inward migration due to gradients in shear stress. The resulting flux is thus solely due to the diffusive term, as proposed in their model. Examination of the concentration distribution as a function of $d$ thus offers a simple way of contrasting between these flux models.

The total radial flux may thus be written using equations 6.6 and 6.11 as

$$N_r = -g(\kappa) U_0 \frac{a^2}{R^2} C a U^2 \frac{r}{R} \phi - \frac{2}{9} \frac{a^2}{\mu_0} \lambda_0 \left( \frac{\partial (\phi^2 \tau)}{\partial r} + (-d) \frac{\phi^2 \tau}{r} \right) \tag{6.14}$$

Combining equations 6.2 and 6.14, the droplet concentration balance is written as

$$\frac{\partial \phi}{\partial t} = g(\kappa) U_0 \frac{a^2}{R^3} C a U^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \phi \right) + \frac{2}{9} \frac{a^2}{\mu_0} \lambda_0 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial (\phi^2 \tau)}{\partial r} + (-d) \phi^2 \tau \right) \tag{6.15}$$

The boundary conditions for this problem are the no-flux condition at the tube wall and the symmetry condition at the center. Also, the average concentration of particles across the cross-section remains a constant at all times, i.e.

$$\int_0^1 2 \pi r \phi dr = \frac{4 \pi R^2 \phi_0}{\pi R^2} = \phi_0 \tag{6.16}$$

Let us now render the above governing equation dimensionless.

$$t = t^* t_c; \quad r = r^* R; \quad \phi = \phi^* \phi_0 \tag{6.17}$$

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The time scale $t_c$ used for non-dimensionalization is the convective time scale arising from the droplet drift

$$t_c = \left( g(\kappa)U_0 \frac{a^2}{R^3 \text{Ca}} \right)^{-1} \quad (6.18)$$

With the above scalings, the droplet distribution equation 6.16 reduces to

$$\frac{\partial \phi^*}{\partial t^*} = U^2 \frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^{*2} \phi^* \right) + \Lambda |U| \frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^* \frac{\partial (\phi^{*2} r^*)}{\partial r^*} + (-d) \phi^{*2} r^* \right) \quad (6.19)$$

where $\Lambda$ may be regarded as a dimensionless diffusivity and is given by

$$\Lambda = \frac{8}{9} \frac{\lambda_0 \phi_0}{g(\kappa) \text{Ca}} \quad (6.20)$$

The total droplet conservation condition in Eq. 6.16 reduces to

$$2 \int_0^1 r^* \phi^* dr^* = 1 \quad (6.21)$$

Consider the simple case when the average velocity is a constant, i.e. $U^* = 1$. In this case, it is possible to determine the steady state distribution by setting the radial flux to zero. The solution of the resulting first order differential equation depends on the value of $\Lambda$. When $\Lambda > \Lambda_c$, where $\Lambda_c$ is given by

$$\Lambda_c = \frac{16}{(3 - d)^2(7 - d)} \quad (6.22)$$

the following solution is obtained:

$$\phi^* = -\frac{1}{(3 - d)\Lambda} r^* + \frac{3 + d}{2} \left[ \frac{1}{3(3 - d)\Lambda} + \frac{1}{2} \right] r^*^{-(1-d)/2} \quad (6.23)$$
For $\Lambda < \Lambda_c$, however, we get a clear fluid region of thickness $1 - r_e$ near the tube walls, and the solution for the droplet distribution is

$$
\phi^* = \frac{r_e^*}{\Lambda(3-d)} \left[ -\frac{r^*}{r_e^*} + \left( \frac{r^*}{r_e^*} \right)^{-1/(1-d)/2} \right] \quad \text{for} \quad r^* \leq r_e^*
$$

$$
= 0 \quad \text{for} \quad r^* > r_e^*
$$

(6.24)

where $r_e^*$ is the radial location of the edge of the concentration profile and is given by

$$
r_e^* = \left[ \frac{3\Lambda(3 + d)}{2} \right]^{1/3}
$$

(6.25)

Observe that in both cases the droplet concentration is singular at the center $r^* = 0$. This is a commonly encountered drawback of diffusive flux models in the literature [63, 66, 70, 73, 77] and is a consequence of the non-inclusion of finite size effects. We note, however, that the singularity is integrable and the amount of material in the region near the center is still finite, as is required by the condition in Eq. 6.21. The steady-state concentration distribution of droplets in Eq. 6.24 is similar to that obtained by modeling the shear-induced diffusion solely by the diffusion term [100]. In that case, the solution may be obtained by setting $d = 1$ arbitrarily as discussed earlier, and consequently the singular term in the equation is replaced by a constant. The concentration distribution in this case becomes triangular in shape [100].

We see from the solutions in equations 6.23 and 6.24 that the length scale of the concentration profile depends on the value of the dimensionless diffusivity $\Lambda$. For $\Lambda > \Lambda_c$, this length scale is nothing but the radius of the tube. For $\Lambda < \Lambda_c$, however, the length scale changes to the profile edge $r_e$, which itself depends on
the diffusivity $\Lambda$. In other words, the length scale vanishes for sufficiently strong focusing conditions. The absence of a physical length scale usually implies the existence of a self-similar solution to the problem, and we explore this in the following section.

6.3 Development of the self-similar solution

Following the work of King and Leighton [48], we adopt the trial function approach of determining self-similar solutions.

$$F(\eta) = \frac{\phi^*}{h(t^*)}; \quad \eta = \frac{r^*}{s(t^*)}$$  \hspace{2cm} (6.26)

To obtain a relationship between the concentration scaling $h(t)$ and radial co-ordinate scaling $s(t)$, we apply the integral condition in Eq. 6.21.

$$2hs^2 \int_0^{\eta_c} F \eta d\eta = 1$$  \hspace{2cm} (6.27)

The scaling function $h$ can therefore be written as

$$h = s^{-2}$$  \hspace{2cm} (6.28)

and the droplet conservation equation reduces to

$$2 \int_0^{\eta_c} F \eta d\eta = 1$$  \hspace{2cm} (6.29)

The governing equation for droplet distribution reduces to

$$-(\eta^2 F)' \left( \frac{1}{s^3} \frac{ds}{dt^*} + \frac{U^2}{s^2} \right) = \frac{\Lambda |U|}{s^5} \left[ (\eta(\eta F^2)')' + (-d) (\eta F^2)' \right]$$  \hspace{2cm} (6.30)
The prime symbol denotes differentiation with respect to $\eta$. The above equation suggests that a self-similar solution is possible if we set

$$s^2 \frac{ds}{dt^*} + s^3 U^2 = \Lambda |U| \quad (6.31)$$

With the length scale $s(t^*)$ obeying this special relationship, Eq. 6.30 reduces to

$$\left(\eta^2 F + \eta(\eta F^2)' + (-d)\eta F^2\right)' = 0 \quad (6.32)$$

Solution of the above equation with the no-flux condition at $\eta = 0, \eta_e$ and the integral condition in 6.29 is analogous to that in Eq. 6.24.

$$F = \frac{\eta_e}{(3-d)} \left[ -\frac{\eta}{\eta_e} + \left( \frac{\eta}{\eta_e} \right)^{-(1-d)/2} \right] \quad \eta \leq \eta_e$$

$$= 0 \quad \eta > \eta_e \quad (6.33)$$

where $\eta_e$ is given by

$$\eta_e = \left[ \frac{3(3+d)}{2} \right]^{1/3} \quad (6.34)$$

The scaling $s$ for the radial length which obeys the simple ODE in Eq. 6.31 can also be solved for analytically.

$$s = \exp \left[ -\int_0^{t^*} U(\hat{t})^2 d\hat{t} \right] \left\{ s_0^3 + 3\Lambda \int_0^{t^*} |U(t')| \exp \left[ \int_0^{t'} 3U(\tilde{t})^2 d\tilde{t} \right] d\tilde{t}' \right\}^{1/3} \quad (6.35)$$

Here $s_0$ is the initial value of $s$.

Before we proceed, it is important to recognize a couple of issues with the self-
similar solution laid out above. First, this solution is subject to the restriction that the edge of the concentration profile is always within the tube, i.e.

\[ s\eta_e < 1 \]  

(6.36)

Note also that a choice of different models for the shear-induced diffusive flux between the one presented here and that due to Zhou and Chang \((d = 1)\) only produces a change in the shape of the self-similar concentration profile \(F(\eta)\), and not in the variation of the scaling length \(s(t^*)\).

Let us calculate the flux of the dispersed phase across the cross-section of the tube at any instant of time. It is known that a sphere suspended in a quadratic flow moves slower than the local velocity of the fluid at the center of the sphere, which may be easily demonstrated by applying Faxen’s law. The velocity \(u_p\) of a drop in a quadratic flow may be written according to Chan and Leal [16], for example, as

\[ u_p = 2U_0U(t/t_c) \left( g_2(\kappa) - \frac{r^2}{R^2} \right) \]  

(6.37)

where \(g_2(\kappa)\) is given by

\[ g_2(\kappa) = 1 - \frac{2\kappa}{2 + 3\kappa} \frac{a^2}{R^2} \]  

(6.38)

The depth-averaged flux of droplets is then given by

\[
\langle u\phi \rangle = \frac{1}{\pi R^2} \int_0^R 2\pi r u\phi dr = 4U_0U(t^*)\phi_0 \int_0^1 r^* \left( g_2(\kappa) - r^{*2} \right) \phi^* dr^*
\]

\[
= 4U_0U(t^*)\phi_0 \int_0^{\eta_c} \eta \left( g_2(\kappa) - \eta^2 s^2 \right) Fd\eta
\]
\[
\begin{align*}
&= 4U_0 U(t^*) \phi_0 \left( g_2(\kappa) \int_0^{\eta_c} \eta F d\eta - s^2 \int_0^{\eta_c} \eta^3 F d\eta \right) \\
\end{align*}
\] (6.39)

Applying the integral condition in Eq. 6.29, we obtain

\[
U_p^*(t^*) = \frac{\langle u \phi \rangle}{U_0 \phi_0} = 2U(t^*) \left( g_2(\kappa) - \frac{2s^2 \eta_e^5}{5(7 + d)} \right) \\
\] (6.40)

The quantity \( U_p^* \) on the LHS is a dimensionless average velocity of the droplet phase at any instant of time. This average velocity may be positive or negative depending on the sign of \( U(t^*) \). The usefulness of the self-similar solution can be demonstrated by considering some test cases for the flow field \( U(t^*) \).

6.3.1 The applied flow is unidirectional and steady \( [U(t^*) = 1] \).

In this case, it is possible to obtain a simple analytical solution for the scaling length \( s(t^*) \).

\[
s = \left\{ \left( s_0^3 - \Lambda \right) \exp \left( -3t^* \right) + \Lambda \right\}^{1/3} \\
\] (6.41)

Surprisingly, this result is identical to that obtained by King and Leighton [48] for simple shear flow of a dilute emulsion. Note that the asymptotic value for \( s \) is \( s_\infty = \Lambda^{1/3} \). Thus, if \( s_0 \) is a steady state solution as well, it may be written as \( s_0 = \Lambda_0^{1/3} \).

In figures 6.1 and 6.2, we have shown the shape \( F \) of the concentration profile, the average velocity of the dispersed phase \( U_p^* \) in Eq. 6.40 and the thickness \( \eta_e s \) of the concentration profile for Zhou and Chang’s model \( (d = 1) \), the isotropic model \( (d = 0) \) and Zarraga et al.’s model \( (d = -0.54) \). It can be seen that with the reduction in the dimensionless diffusivity \( \Lambda \), the droplets focus more sharply
Figure 6.1. (a) The variation of the dimensionless thickness $\eta_s s$ of the concentration profile with time and (b) the shape $F(\eta)$ of the concentration profile. Solid line: $d = -0.54$, dashed line: $d = 0$, dotted line: $d = 1$ (model of Zhou and Chang [100]). $s_0$ was chosen so that $\eta_s s_0 = 1$. Also, $\Lambda$ was taken to be $0.1/\eta_s^3$. These choices collapse the $\eta_s s$ relationships for the different models to one single curve.
Figure 6.2. The variation of the average axial droplet velocity $U_p^*$ with time. Solid line: $d = -0.54$, dashed line: $d = 0$, dotted line: $d = 1$ (model of Zhou and Chang [100]). $s_0$ was chosen so that $\eta_s s_0 = 1$. Also, $\Lambda$ was taken to be $0.1/\eta_s^3$. 
towards the tube center, leading to a thicker clear fluid region near the wall. With
greater focusing of the droplets, the flux also increases monotonically with time
as may be seen from figure 6.1.

Thus the measurement of the steady state width of the distribution yields the
reduced droplet interaction pressure $\lambda_0$ for the dilute emulsion from Eq. 6.20. The
thickness of the concentration profile $s\eta_e$ can be monitored as a function of time
from experiment. The deviation of $s^3$ from its asymptotic value is expected to be
an exponentially decreasing function of time with an exponent of $-3t_c^{-1}$.

Let us examine the length $L$ of tube required to observe the focusing of drops.
$L$ scales approximately as

$$L \sim U_0 \times 3t_c = \frac{3R^3}{a^2 Cag(\kappa)}$$

or

$$\frac{L}{R} = \frac{3}{Cag(\kappa)} \frac{R^2}{a^2}$$

(6.42)

For a tube to particle radius of $R/a = 20$, a value of $\kappa = 0$ and a capillary number
of 0.1 (which is a relatively large number for the small drop deformations assumed
here), the aspect ratio of the tube required for observing the focusing is large
($\sim 1500$), which is more easily realized for capillaries. Recall, however that the
model was derived for a quasi-unidirectional, time dependent startup flow rather
than an entrance flow. In practice, this may be achieved if length of the tube is
much greater than the length required for the focusing of the drops. Therefore,
the actual aspect ratio required to avoid entrance effects will be much greater than
that predicted from the above equation, rendering such an experiment difficult to
perform. We explore oscillatory flows as a solution to this problem in the next
subsection.
6.3.2 \( U(t^*) \) is an oscillatory flow field with zero net flow

The applied flow field in this case is periodic with a dimensionless period of \( T^* \) and a zero net flow over this period, i.e.

\[
U(t^*) = U(t^* + T^*)
\]  

(6.43)

and

\[
\int_0^{T^*} U(t^*)dt^* = 0
\]  

(6.44)

The integral of the average velocity of the droplets over one time period can now be calculated from the result of the self-similar solution in Eq. 6.40.

\[
\frac{U_p^*}{U_0\phi_0} = -\frac{4\eta_c^5}{5(7 + d)T^*} \int_0^{T^*} s^2 U dt^*
\]  

(6.45)

Let us first analyze the simple case of a square wave. In this case, \( U(t^*) \) is defined as

\[
U = 1 \quad 0 < t^* < T^*/2
\]

\[
= -1 \quad T^*/2 < t^* < T^*
\]  

(6.46)

This velocity distribution would correspond to a triangular wave in the average displacement. One can see that as far as the model is concerned, this case is the same as the steady, unidirectional flow discussed previously with an identical solution for \( s(t^*) \) (Eq. 6.41), since the equations only feature the magnitude of \( U(t^*) \) and not its sign. Practically, this will be true provided that the time period
of oscillation is greater than time required for the drop microstructure to flip to its image [56] which scales as $\dot{\gamma}^{-1}$, or if the amplitude of the oscillation is much greater than the drop size [12]. Now if the amplitude of the oscillation is also much smaller than the length of the tube, then we effectively have an infinitely long tube to work with, and this circumvents the large aspect ratio problem elucidated before for a steady unidirectional flow. The model can thus be verified using an oscillatory flow relatively easily as compared to unidirectional steady flow.

A similar analysis may be performed for a sinusoidal average velocity. The average velocity for this flow may be defined as

$$U = \sin \left( \frac{2\pi t^*}{T^*} \right)$$

(6.47)
The asymptotic thickness of the concentration profile for long times averaged over one time period in this case is \( s_\infty = (4\Lambda/\pi)^{1/3} \). In fact, for any oscillatory field \( U(t^*) \), the average asymptotic thickness is given by

\[
s_\infty = (\bar{\Lambda})^{1/3} = \left( \bar{\Lambda} \frac{\int_0^{T^*} U dt^*}{\int_0^{T^*} U^2 dt^*} \right)^{1/3}
\]

(6.48)

Here \( \bar{\Lambda} \) can be considered as an average dimensionless diffusivity.

The advantage of the symmetric square wave and sinusoidal wave is that focusing of the dispersed phase is achieved with zero amplitude of oscillation. As can be seen in figure 6.3, the magnitude of the variation in the thickness \( s \) of the concentration profile about its local mean decreases as the time period of oscillation decreases. For the lowest frequency shown \( (T^* = 0.1) \), \( s \) shows an exponentially decaying behavior as for the steady unidirectional case and decays to the asymptotic value given in Eq. 6.48 which correspond to the average dimensionless diffusivity \( \bar{\Lambda} \).

6.3.3 Asymmetric oscillatory fields

Let us now examine the possibility of separation of droplets from the suspending fluid by using an oscillatory field. We know that the degree of focusing and therefore the flux depends on the magnitude of the average velocity \( U(t^*) \). Therefore, if we impose different average velocities during the forward and reverse pumping strokes, the different degrees of focusing in the two phases should yield a net flux of droplets over a cycle. For example, consider the following asymmetric oscillatory flow with constant but different velocities in the forward and reverse
Figure 6.4. Variation of the scaling length $s(t^*)$ with the normalized time $t^*/T^*$ for different values of the time period $T^*$ with $\zeta = 0.25$, $\Lambda = 0.05$ and $d = -0.54$. Dotted line : $T^* = 0.1$, dashed line : $T^* = 1$, Solid line : $T^* = 10$. 
Figure 6.5. Variation of the scaling length $\eta_{es}(t^*)$ with the normalized time period $t^*/T^*$ for different values of $\zeta$ with $T^* = 10$, $\Lambda = 0.05$ and $d = -0.54$. Solid line : $\zeta = 0.05$, dashed line : $\zeta = 0.25$, dotted line : $\zeta = 0.45$, dashed-dotted line : $\zeta = 0.5$. 
Figure 6.6. Variation of the scaling length $\eta_s(t^*)$ with the normalized time $t^*/T^*$ for different values of the dimensionless diffusivity $\Lambda$ with $T^* = 1$, $\zeta = 0.25$ and $d = -0.54$. Solid line : $\Lambda = 0.1$, dashed line : $\Lambda = 0.05$, dotted line : $\Lambda = 0.01$. 
phases such that the total flow over a cycle is zero.

\[
U = 1 \quad 0 < t^* < \zeta T^*
\]

\[
= -\frac{\zeta}{1 - \zeta} \quad \zeta T^* < t^* < T^*
\] (6.49)

Here \( \zeta \) is a constant that represents the asymmetry of the oscillatory flow and varies between 0 and 0.5. The case of symmetric square wave in Eq. 6.46 can be recovered by setting \( \zeta = 0.5 \). It is possible to write an analytical expression for the scaling length \( s(t^*) \), but the value of \( s_0 \) has to be determined by solving a transcendental equation using the condition \( s(t^* = 0) = s(t^* = T^*) \). In figure 6.4, we have shown \( \eta s(t^*) \) as a function of time \( t^* \) for different values of \( T^* \) at fixed values of \( \zeta \) and \( \lambda \). It can be seen that the variation in the edge of the concentration profile focusing \( \Delta s = (s_0 - s_{\text{min}}) \) is directly related to the time period since greater time is allowed for focusing. Here \( s_{\text{min}} \) is the value of \( s \) at \( t^* = \zeta T^* \). The variation of \( \eta s(t^*) \) with \( \zeta \) is shown in figure 6.5. For small values of \( \zeta \), the degree of focusing in the forward pumping phase is small, and therefore \( \Delta s \) is small. For values of \( \zeta \) close to the symmetric limit of 0.5, \( \Delta s \) is small again. \( \Delta s \) thus shows a maximum with \( \zeta \).

In the limit \( T^* >> 1 \), the drops can be assumed to be completely focused in each phase, and an analytical solution can be written for the time averaged particle velocity over a cycle from Eq. 6.45.

\[
\bar{U}_p = -\frac{4\eta e^5}{5(7 + d)} \frac{1}{T^*} \left( s_f^2 \zeta T^* - s_b^2 \frac{\zeta}{1 - \zeta} (1 - \zeta) T^* \right) = \frac{4\eta e^5}{5(7 + d)} (s_b^2 - s_f^2) \zeta \] (6.50)

Here \( s_f \) and \( s_b \) are the dimensionless thicknesses of the concentration profile in the forward and backward focusing phases. From Eq. 6.41, these thicknesses can
be obtained as

\[ s_f = \Lambda^{1/3}, \quad s_b = (1 - \zeta)\Lambda/\zeta^{1/3} \]  

(6.51)

Substituting these in Eq. 6.50, we get

\[ U_p^* = \frac{4\eta_s^5}{5(7 + d)} \Lambda^{2/3}G(\zeta) \]  

(6.52)

where the function \( G \) is given by the expression

\[ G(\zeta) = \zeta^{1/3}(1 - \zeta)^{2/3} - \zeta \]  

(6.53)

In figure 6.7, we have shown the variation of the time averaged dispersed phase velocity with \( \zeta \) for different time periods of oscillation bounded by the theoretical curve in Eq. 6.52. \( \overline{U_p^*} \) was calculated for a value of \( \lambda \) such that \( \eta_s s_0 = 1 \) for the simulations and \( \eta_e s_b = 1 \) for the theoretical curve, and represents the maximum flux allowable (for e.g., see figure 6.8) before the self-similar solution becomes invalid. As anticipated from the variation \( s \) with \( \zeta \), the net flux of the dispersed phase shows a maximum with \( \zeta \). The velocity of dispersed phase increases with an increase in \( T^* \), since a larger time period allows greater time for focusing and results in a greater \( \Delta s^* \).

As can be seen in figure 6.52 and also as discussed in subsection 6.3.1, the problem with the flow field in Eq. 6.49, of course, is that it takes extremely long times and therefore long channels for the droplets to focus during the forward and reverse phases. Just as oscillatory flow can be used to reduce the equilibration length for steady flows, so we may use a second, higher frequency symmetric oscillation to reduce the equilibration lengths for asymmetric oscillations. Effectively,
Figure 6.7. Variation of the average velocity of the dispersed phase over one period of the flow cycle with $\zeta$ for different values of $T^*$ with $d = -0.54$. Solid line: theoretical curve in Eq. 6.46, dotted line: $T^* = 1$, dashed line: $T^* = 10$, dashed-dotted line: $T^* = 100$. For the simulations, the dispersed phase velocity was evaluated at a value of $\Lambda$ such that $\eta_e s_0 = 1$. For the theoretical curve, $\Lambda$ was chosen by setting $\eta_e s_b = 1$. These limitations on $\Lambda$ are set by the requirement of validity of the self-similar solution.
Figure 6.8. (a) The thickness of the concentration profile as a function of \( \Lambda \). The self-similar solution developed in this chapter is applicable only up to \( \eta_\infty s_0 = 1 \), so that the edge of the concentration profile is always confined within the tube. (b) The variation of the average dispersed phase velocity \( \bar{U}_p^* \) with \( \Lambda \). For both curves, \( T^* = 10 \) and \( \zeta = 0.25 \).
Figure 6.9. The applied oscillatory field $U(t^*)$ [subfigure (a)] in Eq. 6.54 for $T^* = 30$, $t_1^* = 10$, $n = 20$ and $\zeta = 0.3$, and the corresponding profile of $s$ [subfigure (b)]. The arrows marked A, B, C and D correspond to the focusing, forward, relaxing and backward phases respectively.
Figure 6.10. The averaged droplet phase velocity for the complicated oscillatory field $U(t^*)$ in Eq. 6.5 with $t^*_1 = 10$, $n = 20$ and $\zeta = 0.3$. The arrows marked A, B, C and D correspond to the focusing, forward, relaxing and backward phases respectively.
we use the high frequency (and thus small amplitude tidal oscillations for a given
average shear rate) to achieve the focusing, and then low frequency asymmetric
oscillations to induce separation. One such flow field is given by

\[ U = \sin \left( \frac{2\pi n t^*}{t_1^*} \right) \]

\[ = \frac{\pi}{4} + \frac{\zeta}{1 - \zeta} \sin \left[ \frac{2\pi n}{t_1^*} (t^* - t_1^* - \zeta T^*) \right] \]

\[ = -\frac{\pi}{4} \frac{\zeta}{1 - \zeta} \]

\[ 0 < t^* < t_1^* \zeta T^* \]

\[ t_1 < t^* < t_1^* + \zeta (T^* - 2t_1^*) \]

\[ t_1^* + \zeta (T^* - 2t_1^*) < t^* < 2t_1^* + \zeta (T^* - 2t_1^*) \]

\[ 2t_1^* + \zeta (T^* - 2t_1^*) < t^* < T^* \]

(6.54)

It is relatively straightforward to compute the concentration profile thickness \( s \) and
the dispersed phase velocity \( U_p^* \) numerically by applying the results from the
self-similar solution. For example, the average velocity field, \( s \) and \( U_p^* \) are shown
in figure 6.10 for \( n = 20, T^* = 30, t_1^* = 10, \) and \( \zeta = 0.3 \).

A plot of the time-averaged mean droplet velocity with \( \zeta \) is shown in figure 6.11
for \( n = 20, T^* = 30, t_1^* = 10 \). Again, the fluxes were evaluated for a value of \( \Lambda \) so
that \( s_0 \eta e = 1 \). Note that the focusing and relaxing oscillatory phases account for
two-thirds of the total cycle, and the net flux is obtained only over a dimensionless
period of 10. Therefore, the flux shown in the figure for this cycle is averaged over
the effective period of 10. Also shown in the figure are the time-averaged mean
droplet velocities for the asymmetric flow defined in Eq. 6.49 for \( T^* = 10 \), which
has the same amplitude as the cycle being discussed, and for \( T^* = 30, \) which
has the same total time period. It can be seen that except for large \( \zeta \) where we
approach the symmetric limit, the flow cycle in Eq. 6.54 produces an improved
flux compared to the flux in Eq. 6.49 for \( T^* = 10 \). This, of course, is because of
Figure 6.11. The time averaged droplet velocity $\overline{U_p^*}$ as a function of $\zeta$. The solid line with circles is the result for the oscillatory field in Eq. 6.54 with $T^* = 30, t^*_1 = 10, n = 20$. In order to calculate this average velocity, the time period used for normalization was 10 as explained in the text. The dashed and dotted line are the results for the asymmetric oscillatory field in Eq. 6.49 with $T^* = 30$ and $T^* = 10$ respectively.
the dedicated focusing and relaxing phases of the flow. The curve falls below the
curve corresponding to the asymmetric oscillatory field in Eq. 6.49 with $T^* = 30$,
because the unidirectional steady flow is more efficient in focusing droplets than
the sinusoidal wave.

6.4 Conclusions

In this chapter, we have developed a model for the evolution of the concen-
tration distribution during the startup flow for an emulsion in a tube. The con-
centration profile is determined by a balance between the convective flux arising
from the droplet drift towards the tube center on account of its deformation, and
a shear-induced diffusive flux arising on account of droplet-droplet interactions.
The droplet drift was modeled using the small deformation theory of Chan and
Leal [16], while the shear-induced diffusion was modeled with the suspension bal-
ance representation of Nott and Brady [70]. It was shown that if the droplet drift
is strong enough for a clear suspending fluid layer to appear near the wall, then
the length scale in the radial direction vanishes. This extra degree of freedom
was used to construct a self-similar solution using the trial function method [48]
for arbitrary time dependent amplitude of the applied pressure driven flow. An
analogous procedure may be applied to derive the self-similar solution for plane
Poiseuille flow of a dilute emulsion.

The self-similar solution developed in this chapter may be applied to experi-
mentally determine the shear-induced diffusion coefficient of droplets in a dilute
emulsion undergoing oscillatory Poiseuille flow. The slow migration velocity of
droplets implies that a steady unidirectional flow cannot be used for such an
experiment, because it would require a very long tube. We were able to show
through the self-similar solution that the behavior in unidirectional flow can be reproduced by employing a high frequency, oscillatory flow with zero net flow. Since the amplitude of such an oscillatory flow is small, it requires much shorter tubes to achieve the same focusing.

It was also shown that an asymmetric oscillatory flow with different average shear rates during the pumping and withdrawal phase could be used to separate droplets from the suspending phase. The technique relies on the different extents of focusing obtained in the forward and reverse pumping phases of the flow to produce a net flux of droplets over one period of the flow. This method could also be extended to separate red blood cells from plasma, since red blood cells are known to be extremely deformable.
CHAPTER 7

FUTURE DIRECTIONS

As is the never-ending reality in research, the quest for answers often leads to many more questions. In this concluding chapter, we discuss some issues and possible future investigations required to improve our understanding of the research presented in this thesis.

7.1 Measurement of the concentration and velocity distributions in different geometries

In chapter 2, we showed the simulation results for the fully developed concentration and velocity distributions for suspension flow through conduits of arbitrary geometry. In particular, we looked at some counterintuitive results such as particles being drained out of notches and corners by secondary currents. The effect is strong in geometries where the magnitude of the secondary currents is large, such as the equilateral triangular conduit. From isotropic particle migration models, we expect a local maximum near the corners of a triangular conduit, since the shear stress in these areas is small due to their smaller length scales. But the inclusion of the normal stress differences showed that secondary currents actually flush particles out of the corners. The change in concentration near the corners, which can be as much as a volume fraction of 0.2 for sufficiently high Peclet numbers [observe the large depleted zone near the triangle corner in figure 2.19 (c)],
should be very easy to detect using standard imaging techniques such as MRI. Alternatively, one could use the experimental technique of Debbaut et al. [24]. A polymer may be used as a Newtonian suspending fluid as long as the shear rates are much smaller than the inverse of the characteristic relaxation time. After the suspension is flowed completely through a sufficiently long, sacrificial triangular conduit, the polymer can be cured. This sacrificial conduit can then be sliced at different axial lengths, and the concentration distribution can be studied in each slice. Irrespective of the method, the presence of a local minimum near the corners should corroborate the theoretical predictions of the concentration distribution discussed for non-axisymmetric geometries.

7.2 Stability analysis of plane Poiseuille flow of suspensions

Another surprising result we obtained in chapter 2 was that the plane Poiseuille flow of concentrated suspensions is unstable. The stability analysis of startup plane Poiseuille flow is an interesting theoretical problem in its own right. It should be noted that the wavelengths we observed for this instability (see figure 2.30) are of the order of the depth of the channel. That very short waves show a negative growth rate is not surprising, since these are stabilized by transverse shear-induced diffusion. The stability of long waves, however, is not very obvious. This stability can be attributed to the fact that for long waves, shear-induced diffusion renders the particle pressure uniform across the gap. In fact, using an analysis similar to that presented in the appendix of chapter 3, it should be possible to show that the depth averaged concentration $\phi$ obeys the equation

$$\frac{\partial \bar{\phi}}{\partial t} = \frac{\partial}{\partial x} \left[ \tilde{D}(\bar{\phi}) \frac{\partial \bar{\phi}}{\partial x} \right]$$

(7.1)
Figure 7.1. The model problem for the stability analysis of plane Poiseuille flow of a suspension. A layer of the non-Newtonian suspension C is sandwiched symmetrically between two layers of a Newtonian fluid A. The fluids A and C are immiscible, with the interfaces between them being described by \( y = \pm h(x) \). The flow is pressure-driven and into the plane of the paper. The depth of the channel is \( 2B \).

in the long wave limit with a positive and therefore stabilizing effective diffusion coefficient \( \tilde{D} \). It is thus the non-uniformity in the particle pressure and therefore the normal stress across the gap of the channel that leads to the instability.

Considerable insight may be gained into this complicated stability problem by analyzing a model problem analogous to that of Brady and Carpen [9]. The model consists of a sandwich of a fluid layer C between two layers of fluid A as shown in figure 7.1. Fluid A is Newtonian, while fluid C is a suspension with the same suspending fluid, but at uniform concentration \( \phi_0 \). Since there is a jump in the second normal stress difference across the fluid interfaces, the system is susceptible to an instability in the spanwise direction as discussed by Brady and Carpen. In fact, due to the simplified nature of the problem, it may also be possible to obtain an analytical expression for the dispersion relationship (growth rate vs. wavenumber). Using the insight gathered from this analysis, one can attack the more complicated stability analysis of plane Poiseuille flow of a suspension.

The preliminary results discussing the stability of plane Poiseuille flow pre-
sented in chapter 2 were for the startup flow problem. The more practically relevant problem, however, is the entrance flow problem. An additional mode of stabilization exists in this flow problem, which is Taylor dispersion. It is instructive to examine here if Taylor dispersion will wash out the instability arising from the second normal stress difference. As laid out in chapter 2, the time $t_{CIRC}$ required for the secondary currents to circulate within the cross-section obeys the following scaling

$$t_{CIRC} \sim \frac{B}{(-d)k\alpha U}$$

(7.2)

The Taylor dispersion coefficient $D_{eff}$ for suspensions may be written as

$$D_{eff} \sim \frac{2}{105} \frac{Pe^2 D\dot{\gamma} a^2}{1} \left( \frac{B^2}{Da^2} \right)^2 \frac{D}{B} \frac{U a^2}{\dot{\gamma} a} = \frac{1}{210} \frac{B^3}{Da^2} U$$

(7.3)

Here $\dot{D}$ is the dimensionless shear-induced diffusivity in the plane of shear. In time $t_{CIRC}$, the dispersive length scale $L_D$ in the flow direction is given by

$$L_D \sim \sqrt{t_{CIRC}D_{eff}} = \frac{2}{\sqrt{105(-d)k\alpha \dot{D}}} \frac{B^2}{a}$$

(7.4)

On the other hand, the wavelength $\lambda_{CIRC}$ of the instability scales as

$$\lambda_{CIRC} \sim Ut_{CIRC} = \frac{B}{(-d)k\alpha}$$

(7.5)

The ratio $L_D/\lambda_{CIRC}$ is therefore

$$\frac{L_D}{\lambda_{CIRC}} \sim \frac{B}{a} \sqrt{2 \frac{(-d)k\alpha}{105\dot{D}}} = \frac{B}{a} \frac{G(\phi)}{a}$$

(7.6)
If the dispersive length scale $L_D$ is much smaller than the wavelength $\lambda_{\text{CIRC}}$, then the instability will persist in the flow direction. On the other hand, if $L_D >> \lambda_{\text{CIRC}}$, Taylor dispersion will wipe out the instability. The function $G(\phi)$ is shown in figure 7.2 using values of $k = 0.01$ and $d = -0.54$. In calculating $G$, we have employed the suspension balance model and the constitutive equations of Zarraga et al. to write $\hat{D}$ as (see Eq. 2.53)

$$\hat{D} = \frac{2}{9} f \mu_r \frac{d\alpha}{d\phi} \quad (7.7)$$

It can be seen that $G$ is a weak function of $\phi$, assuming of maximum value of around 0.01. Thus, even for $B/a = 50$, which is a relatively large aspect ratio, the ratio $L_D/\lambda_{\text{CIRC}}$ is still only an $O(1)$ quantity. Taylor dispersion should not, therefore, suppress the instability.

### 7.3 Resuspension in other geometries

In light of the results in the tube resuspension geometry, it is instructive to examine the effects of secondary currents in other resuspension geometries such as the parallel plate, Couette and rectangular slot geometries. Take, for instance, resuspension in the wide gap Couette geometry as shown in figure 7.3. Here gravity points vertically downward along the height of the Couette device. In this geometry, the shear stress decreases as we move from the inner cylinder to the outer, and therefore, at steady state, the concentration should increase with radial position. If the buoyancy number is non-zero, a head difference will be established across the gap giving rise to a buoyancy driven secondary current flowing downward near the outer cylinder and upwards near the inner cylinder. This will result in a net downward flux of particles, which will inhibit resuspension. This argument is
made, of course, without considering normal stress differences. In this geometry, both first and second normal stress differences will play a role in affecting the concentration distribution by directly producing a diffusive particle flux and also indirectly through secondary currents. The first normal stress difference appears because of the curvature of the geometry, which leads a radially increasing particle pressure $\alpha \tau$ instead of a constant as expected from the isotropic model. The second normal stress difference comes into play because of the modification of the flow geometry tensor with the introduction of gravity. It will be interesting to determine how each normal stress difference affects the concentration profile. It should be straightforward to determine these effects analytically by performing a regular perturbation expansion in the parameter $\Delta R/R$ about the plane Couette flow base state for arbitrary buoyancy number in the limit $\Delta R/H << 1$. Here $\Delta R$
Figure 7.3. The wide-gap Couette geometry. The outer cylinder rotates with an angular velocity of $\Omega$. The radii of the inner and outer cylinders are $\kappa R$ and $R$ respectively and the spacing between these cylinders is $\Delta R$. 

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is the spacing between the cylinders, $R$ is the radius of the outer cylinder, while $H$ is the height of the Couette device. These results can be confirmed with full scale simulations using a suitable PDE solver (e.g. COMSOL 3.2). Ultimately, these results could be used to produce a plot of the resuspension height as a function of buoyancy number and subsequently compared with the experimental curve of Acrivos et al. [3].

Now consider the torsional flow of a suspension in the parallel plate geometry with gravity pointing vertically downward along the thin gap. In this geometry, we expect that for non-zero buoyancy numbers, the secondary currents will be non-zero due to the increasing degree of resuspension as one moves radially outward. These secondary currents will thus produce a radial flux which may not be negligible owing to the large $h^2/a^2$ scaling of this convective flux, $h$ being the separation between the plates. In the only attempt at determining the concentration and velocity distributions in parallel plate resuspension using the anisotropic suspension balance model, Lenoble et al. [57] took the secondary currents in this geometry to be identically zero by assuming a very specific form for the normal stress difference coefficients: $b + d = 0$. Both experiments and Stokesian dynamics simulations show that the first and second normal stress differences for concentrated, rigid particle, non-colloidal suspensions are negative, and that the sum of the two normal stress difference coefficients is not negligible ($b + d \approx -0.7$). We believe that it is not possible for the system of equations to be closed with the observed suspension anisotropy unless the secondary velocity field is non-zero. This could easily be demonstrated by perturbing the convection-diffusion and momentum equations for small buoyancy numbers.

Resuspension can also be studied in the rectangular slot geometry shown in
figure 3.13. We have already proved in chapter 3 that the second normal stress difference driven secondary currents are much stronger than buoyancy-induced currents. It should be possible to extend this analysis to determine the resuspension height as a function of the buoyancy number. As before, we expect that the resuspension height predicted by the anisotropic suspension balance model will be much higher than that predicted from the isotropic model. The resuspension height-buoyancy number relationship can also be determined experimentally for this geometry for fixed Peclet number and flow average concentration, and then compared with the predictions of the two models.

As a final comment, note that the concentration segregation in the thin direction is much stronger in the rectangular slot than the Couette geometry, for example, and the parallel plate resuspension geometry for low buoyancy numbers. The magnitude of the secondary currents and resulting convective flux strongly correlates with the degree of segregation in the thin direction. The effect of the secondary currents on the concentration distribution should therefore be realized in rectangular slot geometry even for moderate Peclet numbers. Much higher Peclet numbers may be required in wide gap Couette resuspension to see the same degree of effect.

7.4 Improvements to the particle migration model

In this thesis, we have used the combination of the suspension balance model with the anisotropic constitutive equations of Zarraga et al. [97] to predict the concentration and velocity distribution in different geometries. The simulations of the fully developed concentration and velocity profiles for tube resuspension using our model matched the experimental measurements of Altobelli et al. [5]
both qualitatively and quantitatively, especially at high concentrations. For the 23% data, the simulations agreed less favorably with experiment, although the predicted profiles were much better than those from the isotropic model. A similar trend was observed with the meniscus accumulation/depletion measurements. This implies that the constitutive equations get it more or less right when applied to the tube resuspension geometry, but only require some fine tuning at lower concentrations.

Let us now consider the broader scenario of modeling migration in general 3-D flows. In both chapters 2 and 3, we have used the physical approach of first defining the flow, velocity gradient and vorticity directions, and then constructing the anisotropic particle stress tensor based on these vectors. In the simple flows discussed in this thesis, the directors are either constant or spatially varying vectors, but in all cases, they form an orthogonal set at every point in the cross-section. It is this orthogonality that enables us to write the particulate normal stresses separately along each director. It is not obvious, however, whether the same approach can be applied to more complicated and fully 3-D ([u, v, w]) flows, and if the directors in such flows are orthogonal. In fact, it is not even clear how one can define the velocity gradient and vorticity directors for fully 3-D flows. It is, therefore, not a great surprise that suspensions are assumed to be isotropic in the literature.

Consulting the literature suggests this issue has been addressed only by Fang et al. [31], whose procedure is a generalization of the work of Morris and Boulay [66]. In their approach, the vorticity direction is chosen as that eigenvector of the rate of strain tensor which has the least eigenvalue (in magnitude). The other two eigenvectors of the rate of strain tensor give the flow and velocity gradient
directions. In their model, the first normal stress difference is assumed to be zero and the choice of one or the other as the flow or velocity gradient direction is immaterial. The first normal stress difference is, however, not zero as measured by Zarraga et al., and the choice of the flow and velocity gradient directors is now an issue. Also, according to them, this procedure may be applied to any flow, even extensional flows. It is not clear if this is correct. The fundamental underlying assumption in all particle migration models developed to date is that locally, the flow should be a perturbation of simple shear flow. In fact, the assumptions used to derive the suspension balance model are exact for a dilute suspension of hard spheres undergoing simple shear flow at low Reynolds number [98]. It is not immediately obvious that the empirical expressions of normal stresses for simple shear flow can be simply carried over to more general linear flows like extensional flows. It is not even clear if the microstructure responsible for the normal stress differences [71, 74, 78] in simple shear flow is retained in more general flows. These difficulties are compounded by the fact that there are very few experimental investigations of particle migration in extensional and other linear flows [4, 41, 62, 65]. Furthermore, the general linear flow is almost always associated with a stagnation point, and there is very little understanding of shear-induced migration in the vicinity of stagnation points.

These issues suggest that we need a more general framework to model particle migration. The construction of such a framework would require reliable experimental data of particle migration in different linear flows. An experimental setup that is capable of a variety of 2-D linear flows ranging from a purely rotational flow to a purely extensional flow is the four roll mill (e.g. Leal (2004) [52]). It should be relatively straightforward to study the concentration and velocity distributions
in this setup using particle image velocimetry or other imaging techniques. These results, combined with the existing knowledge in simple shear flows can be used to modify or update the existing model. Alternatively, a new model can be developed which describes particle migration in arbitrary flows.

7.5 Study of the instability associated with the meniscus accumulation phenomenon

In chapter 5, we experimentally characterized the inception of meniscus accumulation in the squeeze flow of suspensions. However, we did not characterize the ensuing miscible fingering instability. Here, we lay out a plan for the preliminary investigation of this problem.

This complex problem is simplified to some extent if we consider the stability of suspension flow in the Hele-Shaw geometry first. In the limit of small channel depth to wavelength ratios, the problem may be viewed from a depth-averaged perspective as the flow of four fluid layers in series: a depleted suspension layer at concentration $\phi_0$, a layer of thickness $l_1$ scaling with the induction length $B^{3/2}/a^2$ based on the channel depth in which the concentration varies from $\phi_0$ to the meniscus packing fraction $\phi_{\text{men}}$, a layer of thickness $l$ corresponding to the packed meniscus layer at concentration $\phi_{\text{men}}$, and finally air, which is essentially inviscid. This is shown schematically in figure 7.4. Even this problem, however, is quite challenging. We need to write a depth-averaged particle transport equation for the suspension phase with Taylor dispersion in the flow direction and shear-induced diffusion normal to the flow. To the best of our knowledge, such an effective equation has not been derived in the literature for a suspension of non-colloidal particles, and is a significant challenge in its own right.
Figure 7.4. The four-layer stability problem for a suspension flowing through an empty Hele-Shaw Cell. The concentration varies from $\phi_0$ in the bulk to the meniscus packing concentration $\phi_{\text{men}}$ over the induction length $l_v$. The thickness of the meniscus layer is $l$. 
We can therefore begin the analysis on a simple model fluid instead of the suspension. The depth-averaged mobility of the model fluid depends on the concentration $c$ of the passive tracer carried by the fluid. The effective transport equation for a passive tracer flowing through a channel is known widely in the literature (e.g. Dutta et al. [28]). In order to mimic the suspension, the depth-averaged mobility of the suspension can be assumed to be a suitable monotonically decreasing function of concentration. For example, one may assume the mobility to be a simple exponential function of concentration as was done by Tan and Homsy [91]. Also, the depth-averaged convective flux can be written as $u g(c)$ instead of $uc$, where $u$ is the depth-averaged velocity field and $g$ is a function of concentration $c$ of the tracer such that $g(c) \geq c$ with $g(c) = c$ in the packed layer. This ensures that the particle average velocity is always greater than or equal to the fluid average velocity. Since the meniscus moves at the average velocity and $g(c) = c$ in the meniscus layer, there will be a continuous enrichment of the tracer at the interface supplied by the model fluid upstream of the interface. This particle accretion is also expected to affect the stability of the system. By considering several subproblems shown schematically in figure 7.5, we can gain useful insight into the stability problem. We can then proceed to study the full stability problem for a concentrated suspension in the plane Poiseuille flow geometry, and then finally characterize the stability problem in squeezing flow.

7.6 Summary

We believe that we have pointed out in this thesis an important mechanism of particle flux in the shear flow of non-colloidal suspensions, namely the convective flux due to secondary currents. Since this mechanism may also be the dominant
Figure 7.5. (a) The two layer problem, which is similar to the one solved by Tan and Homsy [91], but with the additional feature of accumulation at the miscible interface due to the particle average velocity being greater than the fluid average velocity. The concentration undergoes a step change at the miscible interface, and thus represents the sharpest possible change in mobility. (b) The three layer problem consisting of a miscible interface between the tracer lean fluid and concentrated tracer fluid, and an immiscible interface between the fluid and air. The thickness $l$ of the intermediate meniscus layer (layer with $c = 1$) is finite. This problem examines the impact of the finiteness of the meniscus layer as opposed to case (a) where the thickness is much larger than the wavelength of the imposed disturbance. (c) The four layer problem, which is an extension of the three layer problem with a diffuse miscible interface. This problem examines the effect of the induction length $l_v$ on the stability characteristics.
mechanism of particle transport, it is necessary to study the effects of these secondary currents in greater detail by both experiment and theory for suspensions and slurries as described in this chapter.

We also recognize that the current particle migration models need to be improved to describe migration in general three dimensional flows. Unfortunately, the suspension literature is strongly lacking in experimental and numerical studies even for general linear flows. There is tremendous scope for fundamental research in this area.

Finally, the viscous miscible fingering associated with the pressure-driven flow of concentrated suspensions through empty, large aspect ratio conduits is a beautiful hydrodynamic stability problem waiting to be analyzed by theory and experiment.
REFERENCES


