ALEXANDROV GEOMETRY OF LEAF SPACES AND APPLICATIONS

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by
Adam J. Moreno

Karsten Grove, Director

Graduate Program in Mathematics
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Abstract

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We develop a number of tools to analyze the geometry and topology of leaf spaces - quotients of singular Riemannian foliations with closed leaves. This expands upon and gives a purely geometric footing to similar tools used to study orbit spaces of isometric group actions. When applied to a given leaf space, these tools not only help describe the geometry/topology of the quotient, but can also reveal information about the leaves of the singular Riemannian foliation and the manifolds which admit such foliations. The majority of the work is done for leaf spaces with positive curvature (in the comparison sense), as the original motivation was to systematically study singular Riemannian foliations as was done for positively curved manifolds with symmetry.
For my family
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1.1. Positive Curvature

It has been well established that the symmetries of a Riemannian manifold encode topological information about that manifold. For example, positively curved homogeneous manifolds (i.e. those on which a group $G$ acts with a single orbit) have been classified through the combined work of Aloff, Berard-Bergery, Berger, and Wallach \[3, 6, 5, 32\] (and later Wilking and Ziller in \[35\]), giving us in particular:

"Other than in dimensions 6, 7, 12, 13, and 24, all compact, simply connected, positively curved homogeneous manifolds are diffeomorphic to a compact rank one symmetric space (CROSS)."

In fact, all known (compact, simply connected) positively curved manifolds (homogeneous or not) are either CROSSes or exist in the dimensions excluded above. Notably, two infinite families of nonhomogeneous examples (called biquotients) exist in dimensions 7 and 13, due to Eschenburg \[9\] and Bazaikan \[4\], respectively. Are there others? If so, finding examples would be great. If not, understanding why not seems a worthwhile pursuit.

Every known positively curved manifold is in some sense or another, “highly symmetric.” Exploring this link was the idea behind the Symmetry Program initiated by Karsten Grove in 1991. The basic start-up kit is this: (1) Pick a notion of high symmetry, then (2) say as much as you can topologically about positively curved manifolds admitting the chosen type of symmetry. Perhaps the most appealing feature
is that the user is free to choose the notion of symmetry, at which point “classification” means the best you can do (e.g. homeo/diffeomorphism, tangential homotopy equivalence, etc.). This systematic approach has yielded several interesting results, for which the reader is encouraged to see [16] and [34]. Finding new examples of positively curved manifolds is only one goal of this program; perhaps one could find obstructions to positive curvature (beyond those given by the theorems of Bonnet-Myers & Synge). Additionally, one hopes to find topological invariants which distinguish (simply connected) manifolds admitting positive curvature metrics from those admitting nonnegative curvature metrics. Currently, no such invariant is known. 

One natural notion of “large” symmetry is given by the cohomogeneity of an action, defined to be the dimension of the orbit space: \( \text{cohom}(M, G) := \dim(M/G) \). The largest symmetry in this sense belongs to the cohomogeneity-0 (i.e. homogeneous) manifolds. Among cohomogeneity-1 manifolds (those whose orbit space is 1-dimensional), two infinite families of “candidates” emerged in dimension 7 (\( P_k \) and \( Q_k, k = 1, 2, \ldots \), where \( P_1 = S^7 \) and \( Q_1 = W_{1,1}^7 \), the “first” Aloff-Wallach example). In [8] and [21], invariant metrics of positive curvature were constructed on \( P_2 \), which by Goette [13] was shown to be an exotic \( T_1S^4 \). Clearly, there is much to be gained by understanding the behavior of special types of symmetries.

1.2. Isometric Group Actions

A very general question one asks is, given a Lie group \( G \) acting by isometries on a Riemannian manifold \((M, g)\), what information about \( M \) can be extracted from properties of \( G \) and how it acts on \( M \) (i.e. the interplay of \( M, G \) and \( g \))? 

As mentioned above, manifolds with low cohomogeneity are a natural starting point for studying manifolds via symmetry. The presence of fixed points certainly restricts this measure, since the fixed point set embeds isometrically into the orbit space \( M/G \). This led Grove and Searle [19] to analyze and classify so-called fixed-
point homogeneous manifolds - those with fixed points and largest possible orbits (so as to have smallest possible orbit space). They achieved the following

**Theorem** (Grove, Searle ’97). *A simply connected, positively curved, fixed-point homogeneous manifold is diffeomorphic to a CROSS.*

So no new examples of positive curvature, but we see that this type of symmetry belongs only to the CROSSes.

The defining property of these fixed-point homogeneous actions is a local one, invoking the infinitesimal behavior of the action. Namely, a fixed-point homogeneous action is one for which the *isotropy representation* acts transitively on the normal spheres to (a component of) the set of fixed points. This property can be generalized to a setting where no group is present. In fact, many of the techniques that Grove and Searle used to study such actions were purely geometric in nature, depending not on the group $G$ itself, but on how $M$ is decomposed into the $G$-orbits. This *orbit decomposition* is an inherent property of the Riemannian manifold $M$ regardless of whether we have a group $G$ which induces it (or if such a $G$ even exists!).

1.3. Singular Riemannian Foliations

Roughly speaking, a *singular Riemannian foliation* $(M, \mathcal{F})$ is a partition of a manifold into submanifolds which are equidistant from each other, much like the orbits of an isometric group action. In fact, such orbit decompositions are a special type of singular Riemannian foliation (SRF) known as *homogeneous* (referring to the fact that the leaves in this context are all homogeneous manifolds, but need not be in general).

The converse is not true. A most notable example of such an *inhomogeneous* SRF is the foliation of $\mathbb{S}^{15}$ by 7-spheres given by the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$. Other classical examples come from special isoparametric hypersurfaces in spheres, constructed by Ozeki and Takeuchi in [28] and later expanded upon by Ferus, Karcher,
and M"užner using representations of Clifford algebras [10]. This technique was recently generalized to arbitrary codimension by Radeschi in [31]. In section 5.3 below, we exhibit inhomogeneous SRFs on spherical space forms. Thus, SRFs are a strictly more general “notion of symmetry” than orbit decompositions by group actions.

Although SRFs are more general, their geometry closely resembles that of the orbits of a group action, and many existing tools can be adapted to the more general setting (sections 2 and 4). In particular, the (normal) infinitesimal behavior at a point $p$ in a leaf $L_p$ is very similar to what one sees when studying the (normal) isotropy representation. That is to say, in the same way a group action induces an ‘infinitesimal group action’ on the normal spaces to the orbits, an SRF induces an infinitesimal SRF (the infinitesimal foliation) on the normal spaces to leaves. This is used heavily below, as the horizontal geometry between the leaves is essentially the geometry of the leaf space $M/F$, and much information can be gathered by studying the structure of this quotient space.

The analogue to transitive group actions (i.e. homogeneous manifolds) in the context of singular Riemannian foliations reveals an interesting difference between the two settings. Evidently, having a transitive group action (one with a single orbit) provides enough information to achieve a classification in positive curvature. However, any manifold can be endowed with a single leaf foliation, so such foliations (when not induced by a group action) are trivial and provide no topological information. Nonetheless, one can still ask a similar question to the one Grove and Searle explored above. Namely, “how large can the leaves of a singular Riemannian foliation be in the presence of point leaves, and what does this say about the manifold?” We refer to such foliations as point leaf maximal (see section 5). We achieve the following:

**Main Theorem.** Let $M$ be a closed, connected, positively curved manifold admitting a point leaf maximal singular Riemannian foliation. If $M$ is simply connected, then $M$ is foliated homeomorphic to a sphere or complex projective space or has the cohomology
ring of a CROSS. If $M$ is not simply connected, then $M$ is either a topological space form or is a $\mathbb{Z}_2$-quotient of an odd dimensional cohomology complex projective space.

As mentioned before, in the non-simply connected case, we find examples not previously covered by the analogous results for isometric group actions. That is, we find examples of point leaf maximal SRFs that are not induced by fixed point homogeneous isometric group actions.
CHAPTER 2

BACKGROUND

2.1. Group Actions

Though the main results of this work deal with Singular Riemannian foliations, much of the geometric intuition can be gathered from the realm of isometric group actions. We will cover some basics here mainly to compare and contrast with the content of later sections. The concepts here should also serve as motivation for defining analogous concepts for foliations. For an excellent and much more thorough treatment, see [1].

2.1.1. Basics and Examples

Starting from the ground, we have

**Definition.** Let $G$ be a Lie group and $M$ a smooth manifold. A smooth map $\mu : G \times M \to M$ is called a (left) $G$-action on $M$ if

(i) $\mu(e, p) = p$, for all $p \in M$, where $e \in G$ is the identity element.

(ii) $\mu(g_1, \mu(g_2, p)) = \mu(g_1 g_2, p)$ for all $g_1, g_2 \in G$ and $p \in M$.

For brevity, we will write $\mu(g, p)$ as simply $g \cdot p$. We also use the term (left) $G$-manifold to refer to a manifold with a smooth (left) $G$-action. The following is a notion of equivalence among $G$-manifolds that we will see in later sections.

**Definition.** Let $M, N$ be two (left) $G$-manifolds with actions $\mu_M$ and $\mu_N$. A map
$f : M \to N$ is said to be $G$-equivariant if for all $x \in M$ and $g \in G$

$$f(\mu_M(g, x)) = \mu_N(g, f(x))$$

Given a (left) $G$-action one can fix an element of either $G$ or $M$ and consider the auxiliary maps:

$$\mu^g : M \to M$$
$$p \mapsto \mu(g, p)$$

$$\mu_p : G \to M$$
$$g \mapsto \mu(g, p).$$

The map $\mu^g$ defines a diffeomorphism of $M$. If $M$ is given a Riemannian metric $\tilde{g}$, one may consider groups $G$ that act isometrically on $(M, \tilde{g})$ (i.e. $\mu^g$ is an isometry of $(M, \tilde{g})$ for all $g \in G$). Such a metric $\tilde{g}$ is said to be $G$-invariant. A smooth $G$-action on $M$ will not in general act by isometries of $(M, \tilde{g})$, but for compact Lie groups we do have the following:

**Theorem** (Invariant Metrics). If $G$ is a compact Lie group acting smoothly on a smooth manifold $M$, then there exists a $G$-invariant Riemannian metric on $M$.

Given a left $G$-action on a manifold $M$ and a point $p \in M$, the orbit of $p$ is the set in $M$,

$$G \cdot p = \{g \cdot p : g \in G\}.$$  

The isotropy group at $p$ is the subgroup of $G$,

$$G_p = \{g \in G : g \cdot p = p\}$$

If $G \cdot p = p$ (or equivalently, $G_p = G$), then $p$ is called a fixed point of the action.

**Proposition.** For all $g \in G$ and $p \in M$, $G_{gp} = gG_pg^{-1}$. That is, isotropy groups along an orbit are conjugate.
The (closed, normal) subgroup of $G$,

$$\bigcap_{p \in M} G_p$$

is called the *ineffective kernel* of the action and consists of the group elements which fix all points of $M$.

**Definition.** A $G$-action is said to be

- *effective* if the ineffective kernel is $\{e\}$
- *free* if $G_p = \{e\}$ for all $p \in M$.
- *almost free* if $G_p$ is finite for all $p \in M$.
- *trivial* if $G_p = G$ for all $p \in M$.
- *transitive* if for all $p, q \in M$, there exists $g \in G$ such that $g \cdot p = q$. (i.e. there is only one orbit)
- *proper* if the map from $G \times M \to M \times M$ given by

$$(g, p) \mapsto (g \cdot p, p)$$

is proper (preimage of compact is compact).

**Remark.** Note that any action induces an effective action by modding out by the ineffective kernel. Also, given a proper action, the preimage of $(p, p)$ is compact (i.e. isotropy groups are compact).

The following fact is a combination of statements from Corollary 3.38 and Proposition 3.41 in [1]:

**Proposition.** For a proper action $\mu : G \times M \to M$, the orbit $G \cdot p$ is an embedded submanifold diffeomorphic to $G/G_p$.

It is easy to show that the orbits of $G$ partition the manifold $M$. That is, $M = \cup_{p \in M} (G \cdot p)$ and if $G \cdot p \cap G \cdot q \neq \emptyset$, then $G \cdot p = G \cdot q$. Thus it makes sense to consider the *orbit space*

$$M/G := \{G \cdot p \mid p \in M\}$$
The projection map $\pi : M \to M/G$ gives a natural topology on $M/G$ by declaring a set to be open in $M/G$ if its preimage is open in $M$. Moreover, given a metric on $M$, there is a natural orbit metric on $M/G$ where one declares the distance between two points of $M/G$ to be the distance between the corresponding orbits in $M$. With this, the projection map $\pi$ is an example of a submetry:

$$\pi(B_r(p)_M) = B_r(\pi(p))_{M/G}$$

(i.e. a map which takes metric balls to metric balls of the same radius). We will see later that this metric space generalization of a Riemannian submersion preserves a metric space generalization of lower curvature bounds.

Examples.

(i) Let $G$ be the group of rotations, $SO(3)$ of the sphere $S^2 = M$. This action is transitive, but fails to be free because every group element fixes the pair of antipodal points along the axis of rotation. In fact, the isotropy group of any point $p \in M$ is isomorphic to $SO(2)$. This example gives a realization of the diffeomorphism $S^3 \cong SO(3)/SO(2)$, and more generally $S^n \cong SO(n + 1)/SO(n)$.

(ii) Let $G = SO(2) \times \mathbb{R}$ act on $\mathbb{R}^3$ by $\mu((A, b), x) = (A(x_1, x_2), b + x_3)$. Points off the $z$-axis have trivial isotropy and cylinders as orbits and those on the $z$-axis have isotropy isomorphic to $SO(2)$ and the $z$-axis as the orbit. Moreover, the quotient $\mathbb{R}^3/G$ is a ray $\mathbb{R}_{\geq 0}$, since each orbit is represented by exactly one point on the nonnegative real axis.

(iii) Let $\theta : [0, \infty) \to \mathbb{R}$ be a smooth function. Then $\mu(t, z) = e^{it\theta(|z|)}z$ defines an $\mathbb{R}$-action on $\mathbb{C}$. The orbits are either circles (when $\theta(|z|) \neq 0$) or fixed points (when $\theta(|z|) = 0$). In the latter case, the isotropy group is $\mathbb{R}$, so the action fails to be proper.

(iv) Let $G = S^1 \subseteq \mathbb{C}$ and $M = S^3 \subseteq \mathbb{C}^2$. Let $k, \ell \in \mathbb{Z}$ and set $\mu(z, (z_1, z_2)) = (z^k z_1, z^\ell z_2)$. This action fails to be effective when $\gcd(k, \ell) > 1$. If $\gcd(k, \ell) = 1$, the action is effective, but not free (unless $k = \pm 1$ and $\ell = \pm 1$). When $k = \pm 1$ and $\ell = \pm 1$, we get a free circle action on $S^3$ (Hopf action). In the case that $k = \ell = 1$, the map from $S^3 \to S^3/S^1 \cong \mathbb{CP}^1 \approx S^2$ is known as the Hopf fibration.
2.1.2. Orbit Types

Given an action $G$ on a manifold $M$, each point $p \in M$ comes with an orbit $G \cdot p$ and an isotropy subgroup $G_p$. As stated above, points belonging to the same orbit have conjugate isotropy groups. In general, points (not necessarily belonging to the same orbit) with isomorphic isotropy groups will belong to diffeomorphic orbits. This leads to the following

**Definition.** Let $\{G_p \mid p \in M\}$ be the set of isotropy subgroups of $G$. Two such subgroups $G_p, G_q$ are said to be equivalent if they are conjugate in $G$. The equivalence classes of this relation are called *isotropy types*. An isotropy type is sometimes written as $(H)$, where $H$ is some representative from the class.

Similarly, two orbits $G \cdot p, G \cdot q$ are said to be equivalent if the isotropy groups at $p$ and $q$ are the same type. The equivalence classes of orbits are called the *orbit types*.

These isotropy and orbit types can be partially ordered. Given two isotropy types $(H), (K)$, we say $(H) \leq (K)$ if $H$ is conjugate to a subgroup of $K$. Since each isotropy type is associated to an orbit type, this also gives a partial ordering of the orbit types. That is, we say one orbit type is larger than another if its isotropy type is smaller.

An orbit is called *principal* if its type is locally maximal, i.e. there is a neighborhood of the orbit such that each other orbit in the neighborhood has smaller (or equal) orbit type (the isotropy group of this orbit is also called *principal*).

**Theorem** (Principal Orbit Theorem). Let $G$ be a compact Lie group acting on a connected Riemannian manifold $M$. Then

1. There exists a unique maximal orbit type.
2. The union $M_0$ of maximal orbit types is open and dense in $M$.
3. The $G$-action restricts to $M_0$ and the map $M_0 \to M_0/G$ is a Riemannian submersion with fiber $G/H$, where $H$ is a principal isotropy group.
4. The quotient $M_0/G$ is connected, open, and dense in $M/G$. 

Let \( G \cdot p \) be a principal orbit. An orbit \( G \cdot q \) is said to be singular if \( \dim(G \cdot q) < \dim(G \cdot p) \). The orbit \( G \cdot q \) is said to be exceptional if \( \dim(G \cdot q) = \dim(G \cdot p) \), but the orbits are not of the same type. Equivalently, \( G \cdot q \) is exceptional if \( G_q / G_p \) is a nontrivial finite group, i.e. \( G \cdot q \) is finitely covered by \( G \cdot p \).

**Examples.**

(i) In the action of \( SO(3) \) on \( S^2 \) described above, there is only one orbit, which must be principal. Its isotropy group is \( SO(2) \).

(ii) In the action of \( SO(2) \times \mathbb{R} \) on \( \mathbb{R}^3 \) described above, the cylinder orbits are the principal orbits and they have trivial isotropy. The \( z \)-axis is a singular orbit.

(iii) [Weighted Hopf Action] Let \( G = S^1 \) and \( M = S^3 \subset \mathbb{C}^2 \) and set \( z \cdot (z_1, z_2) = (z^k z_1, z^\ell z_2) \) where \( \gcd(k, \ell) = 1 \) and \( k, \ell \neq \pm 1 \). Points of the form \( q = (0, z_2) \) have isotropy \( G_q = \mathbb{Z}_\ell \) and orbits \( G \cdot q = S^1 / \mathbb{Z}_\ell \cong S^1 \). Similarly points of the form \( p = (z_1, 0) \) have isotropy \( G_p = \mathbb{Z}_k \) and orbits \( G \cdot p = S^1 / \mathbb{Z}_k \cong S^1 \). Both types of points belong to exceptional circle orbits, and all other orbits are principal with trivial isotropy. The fact that all orbits are either principal or exceptional gives the quotient \( M/G \) a natural orbifold structure.

The proof of the following well known fact uses concepts from the next section, but since we omit the proof, we state it here instead, where it fits nicely.

**Theorem.** If \( G \) acts on \( M \) with a single isotropy type, then \( M/G \) is a smooth manifold.

2.1.3. The Slice Theorem and Isotropy Representation

Let \( G \) be a lie group acting isometrically on a Riemannian manifold \( M \). In this section, we see how such an isometric group action endows small “tubes” around a given orbit with a nice structure, described by the isotropy.

**Definition.** Let \( \mu : G \times M \to M \) be an isometric action and \( G \cdot p \) be an orbit. A slice at \( p \in M \) is an embedded submanifold \( S_p \) containing \( p \) which satisfies

(i) \( T_p M = T_p (G \cdot p) \oplus T_p S_p \) and \( T_q M = T_q (G \cdot q) + T_q S_p \) for all \( q \in S_p \).
(ii) $S_p$ is invariant under $G_p$.

(iii) If $q \in S_p$ and $g \in G$ are such that $g \cdot q \in S_p$, then $g \in G_p$.

Note that condition (iii) implies that $G_q \subseteq G_p$ for all $q \in S_p$. Moreover, by part (i), we can identify $T_p(S_p)$ with the normal space $\nu_p(G \cdot p)$.

**Remark.** Slices exist if the $G$-action is proper, though this is not trivially true. A proof can be found in [1] (Theorem 3.49 pg 65)

**Definition.** Let $G$ act properly on $M$, $p \in M$, and $S_p$ be a slice at $p$. The *tubular neighborhood* of the orbit $G \cdot p$ is

$$\text{Tub}(G \cdot p) := \mu(G, S_p) = G \cdot S_p$$

**Proposition.** For an isometric $G$-action on $M$, let $\nu^\epsilon(G \cdot p) \subseteq TM$ represent the normal $\epsilon$-disc bundle (to the orbit). For sufficiently small $\epsilon > 0$, the map $\exp : \nu^\epsilon(G \cdot p) \to \text{Tub}^\epsilon(G \cdot p)$ given by

$$(p, v) \mapsto \exp_p(v)$$

is a $G$-equivariant diffeomorphism.

**Corollary.** If $G$ and $M$ are compact, then any action of $G$ on $M$ has only finitely many isotropy types.

The next theorem provides a concrete description of tubular neighborhoods as ‘slice bundles’ over the central orbit.

**Theorem** (The Slice Theorem). Let $\mu : G \times M \to M$ be a proper isometric action. For every $p \in M$, the map $[(g, v)] \mapsto \exp_{g \cdot p}(g \cdot v)$ is a diffeomorphism

$$G \times_{G_p} \nu^\epsilon_p(G \cdot p) \to \text{Tub}(G \cdot p)$$
where $G \times_{G_p} \nu^\epsilon_p (G \cdot p)$ is the orbit space $(G \times \nu^\epsilon_p (G \cdot p)) / G_p$ (called twisted space) of the action $h \cdot (g, v) = (gh^{-1}, h_* v)$. Moreover, this diffeomorphism is $G$-equivariant.

We conclude this section by describing an important representation for a given isometric group action, namely the slice representation.

Each element $g \in G$ is a diffeomorphism of $M$, hence has a differential (a map $TM \to TM$) at $p$. If $g \in G_p$, then $g$ fixes $p$, and the differential of $g$ is a linear map $T_p M \to T_p M$. In this way, the isotropy subgroup $G_p$ can be seen to act on $T_p M$ (the isotropy representation). Because $G$ acts by isometries, the isotropy representation preserves the normal (and tangent) space to the orbit, $\nu_p (G \cdot p) \subset T_p M$. The action on the normal space is a subrepresentation known as the slice representation (so-called because we identify $\nu_p (G \cdot p) = T_p (S_p)$). This is of particular importance because it encodes the local horizontal geometry in $M$ at $p$, hence the local geometry of $M/G$ at $\pi(p)$. Moreover, this action is linear and isometric, so the orbits of this infinitesimal action are all represented in the (unit) normal sphere to the orbit $G \cdot p$ at $p$. Given the corollary above, we see that the local geometry of $M/G$ can thus be understood through finitely many linear actions on spheres.

2.2. Singular Riemannian Foliations

2.2.1. Basic Concepts and Examples

Here we provide a rundown of the material used throughout. Everything in this section can be found with proofs in [27] (see chapter 6) and also in the unpublished lecture notes of Marco Radeschi [30]. See either source for a thorough treatment of the topics, as some more involved definitions have been omitted here for brevity.

A singular Riemannian foliation is a partition $\mathcal{F} = \{L_p\}_{p \in M}$ of a Riemannian manifold $(M, g)$ into smooth, connected, injectively immersed manifolds (called leaves) with two properties: (1) $\mathcal{F}$ is a transnormal system - i.e. any geodesic that emanates
perpendicular to a leaf is perpendicular to all leaves it intersects and (2) $\mathcal{F}$ is a singular foliation - i.e. there exists a collection of vector fields on $M$ that span the tangent space to the leaves at all points. We refer to the triple $(M, g, \mathcal{F})$ as a singular Riemannian foliation (SRF) and simply write $(M, \mathcal{F})$ when the metric on $M$ is understood. For our purposes, we will only consider singular Riemannian foliations (SRFs) whose leaves are closed (this guarantees the leaf space $M/\mathcal{F}$ will be Hausdorff and that the metric of $M$ descends to an honest metric on $M/\mathcal{F}$).

Examples.

(i) There are two trivial foliations that can be placed on any manifold. Namely, the point foliation where each point $p \in M$ is a leaf, and the single leaf foliation where all of $M$ is the lone leaf. In the former case, $M/\mathcal{F} = M$ and in the latter case, $M/\mathcal{F} = \{pt\}$. The latter is the SRF analogue of a transitive group action.

(ii) The orbit decomposition of any isometric group action by a connected group induces an SRF, according to our definition. One must allow for disconnected leaves to generalize actions by groups with multiple components.

(iii) The Hopf fibration $S^{15} \to S^8$ is a famous nonhomogeneous SRF (i.e. not induced by a group action). The leaves are all diffeomorphic to $S^7$, making this a Riemannian foliation.

(iv) Let $M$ be the Möbius band, foliated by ‘lateral’ circles. All leaves in this case are circles, but there is a special ‘middle’ leaf, which is double covered by the nearby leaves. This property cannot be detected locally. What distinguishes the central leaf from the others will be discussed in the coming sections.

(vi) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the flat 2-torus. Consider the foliation whose leaves are lines with irrational slope $\alpha$:

$$L_{(x,y)} = \{(x + t, y + \alpha t)\}$$

This forms a Riemannian foliation all of whose leaves are copies of $\mathbb{R}$ which are dense in $M$. All leaves have distance 0 from each other, making $M/\mathcal{F}$ a ‘very’ non-Hausdorff space (none of its points can be separated).

2.2.2. Infinitesimal Foliations and Holonomy

As in the case of isometric group actions, it is important to study the local, infinitesimal behavior about the leaves. For this, fix $p \in M$ and let $P$ be a connected
open subset of $L_p$ (as a manifold in its own right, i.e. not necessarily open in the subspace topology of $M$) containing $p$ for which the inclusion $P \subset M$ is an embedding, (sometimes called a plaque) and take $\varepsilon > 0$ sufficiently small so that $\exp : \nu^\varepsilon P \to M$ is a diffeomorphism onto its image (here, $\nu^\varepsilon P$ denotes the normal bundle to $P$ at $p$). A distinguished tubular neighborhood is an open set $O \subset M$ of the form $O = O_\varepsilon(P) := \exp(\nu^\varepsilon P)$. There is a partition $\mathcal{F}_O$ of $O$ by the connected components of the intersections $L' \cap O$, for $L' \in \mathcal{F}$. We use $\mathcal{F}_O$ to form a singular Riemannian foliation on $T_p M$ with respect to the Euclidean metric. To do so, we will use the following

**Definition.** Let $X$ be a vector field on $O_\varepsilon(P)$ that is tangent to $P$ along $P$. The linearization of $X$ around $P$ is the vector field

$$X^t(p) := \lim_{\lambda \to 0} (h_{1/\lambda})_* X(h_\lambda(p))$$

where $h_t$ is the homothetic transformation:

$$h_t : O_{\varepsilon_1}(P) \to O_{\varepsilon_2}(P)$$

$$\exp(v) \mapsto \exp(tv), \quad v \in \nu^{\varepsilon_1}(P)$$

where $\varepsilon_1, \varepsilon_2 < \varepsilon$ and $\varepsilon_2 = t\varepsilon_1$.

**Fact.** Any vector field on $P$ can be extended to vector field on $O_\varepsilon(P)$ which is linearized with respect to $P$. In particular, coordinate vector fields on $P$ have linearized extensions to $O_\varepsilon(P)$.

Now let $U \subset T_p P$ be a coordinate chart, $X_i$ be the linearized extension of the coordinate vector field $\frac{\partial}{\partial x_i} (i = 1, \ldots, k)$ and $(\nu^\varepsilon P, \mathcal{F}_p)$ be the foliation defined by
pulling back the leaves of $(O, \mathcal{F}_O)$ via \(\exp : \nu_p^c O \to O\). The map

\[
\Phi_O : (U \times \nu_p^c O, U \times \mathcal{F}_p) \to (O, \mathcal{F}_O)
\]

\[
((x_1, \ldots, x_k), v) \mapsto \Phi_{X_k}^t \circ \cdots \circ \Phi_{X_1}^t (\exp_p (v))
\]

where \(\Phi_X^t\) denotes the flow of \(X\) for time \(t\), is a foliated diffeomorphism (see [30], Chapter 2). Moreover, the rescaling map given by \(r_\lambda (x, y) = (\lambda x, \lambda y)\) for \(\lambda \in (0, 1)\) is a foliated diffeomorphism of \((U \times \nu_p^c P, U \times \mathcal{F}_p)\). Thus, we can pull back the metric from \(O \subset M\) via the composition

\[
U \times \nu_p^c P \xrightarrow{r_\lambda} U \times \nu_p^c P \xrightarrow{\Phi_O} O
\]

and obtain a singular Riemannian foliation \((U \times \nu_p^c P, g', U \times \mathcal{F}_p)\), where \(g' = (\Phi_O \circ r_\lambda)^* g\).

The same leaves also form a singular Riemannian foliation with the metric \(g^\lambda := \frac{1}{\lambda^2} g' = \frac{1}{\lambda^2} (\Phi_O \circ r_\lambda)^* g\). As \(\lambda \to 0\), the metrics \(g^\lambda\) converge pointwise to the flat metric \(g_p\) on \(U \times \nu_p^c P\). This SRF extends to an SRF on \(T_p M\) which is invariant under rescalings and splits as \((T_p L_p \times \nu_p L_p, T_p L_p \times \mathcal{F}_p)\). We will refer \((\nu_p L_p, g_p, \mathcal{F}_p)\) as the (normal) infinitesimal foliation. Note that the origin \(\{0\} \in \nu_p L_p\) is always a leaf of \(\mathcal{F}_p\), and thus by equidistance, the other leaves of \(\mathcal{F}_p\) all lie in distance spheres about the origin. Because of this, and the fact that rescalings preserve \(\mathcal{F}_p\), we will from now on use “infinitesimal foliation” to refer to \((S_p^\perp, \mathcal{F}_p)\), the restriction of \(\mathcal{F}_p\) to the unit normal sphere \(S_p^\perp\) to \(L_p\) at \(p\).

The leaves of \((\nu_p L_p, \mathcal{F}_p)\) are diffeomorphic to the intersections of the global leaves of \(\mathcal{F}\) with the normal slice \(\exp_p (\nu_p L)\). In general, some of these ‘local’ leaves will belong to the same global leaf (e.g. example (iv) in the previous section) and should be identified in \(M / \mathcal{F}\). This is done via “linearized lifts of loops.”

**Definition.** Let \(\gamma : [0, 1] \to L_p\) be an integral curve for some vector field \(X\) along \(L_p\) and \(\hat{X}^\ell\) be a linearized extension of \(X\) around a plaque \(P \ni \gamma\). A linearized lift of \(\gamma\)
is the map
\[ \Gamma : [0, 1] \times \nu_p L \to \nu L, \quad (t, v) \mapsto \Gamma_t(v) := \Phi_t^t(v) \]
where \( \Phi_t^t \) denotes the flow of \( \hat{X}^t \).

Of particular importance are lifts of \( p \)-loops (i.e. piecewise smooth curves \( \gamma : [0, 1] \to L_p \) with \( \gamma(0) = p = \gamma(1) \)).

**Definition.** The holonomy group of \( F \) at \( p \) is the set
\[ G_p = \{ \Gamma_1 : \nu_p L_p \to \nu_p L_p \mid \Gamma \text{ a linearized lift of a } p \text{-loop } \gamma : [0, 1] \to L_p \} \]

It is a fact (Lemma 2.36 in [30]) that \( G_p \subset O(\nu_p L_p) \) is a Lie subgroup whose elements maps leaves of \( F_p \) onto leaves of \( F_p \) (i.e. acting by foliated isometries). Moreover, elements of \( G_p \) in the identity path component \( G_p^0 \) will take a point in a local leaf (a leaf of \( F_p \)) to a point in the same local leaf. For such an element \( g \), the existence of a continuous path \( g(s) : [0, 1] \to G_p \) with \( g(0) = g \) and \( g(1) = e \) means that for \( v \in \nu_p L_p \), \( g_s(v) \) traces out a continuous path in a local leaf (it’s in a constant leaf of \( F \) and stays in the slice through \( p \in L_p \)). Thus, at the level of leaves, we see that the group \( G_p^0 \) is the ineffective kernel of the action of \( G_p \) on the local quotient \( O/F_O \). We will refer to the finite group \( \Gamma_p := G_p/G_p^0 \) as the leaf holonomy group of \( L_p \), which acts effectively on the leaf space \( O/F_O \).

**Remark.** The map \( \pi_1(L_p) \to \Gamma_p \) sending a homotopy class of loops to its path component in \( G_p \) is a surjection. In particular, when \( L_p \) is simply connected, the leaf holonomy is trivial.

Though we will not often use it explicitly, it is worth stating that there is a version of the Slice Theorem for singular Riemannian foliations with closed leaves (see [26]):

**Theorem** (Foliated Slice Theorem). Let \((M, F)\) be a singular Riemannian foliation and \( L_p \) be a closed leaf with infinitesimal foliation \((\nu_p L_p, F_p)\) at \( p \). Then there is a
group \( G_p \subset \mathbf{O}(v_p L_p) \) of foliated isometries of \((v_p L_p, \mathcal{F}_p)\) and a principal \( G_p \)-bundle \( E \) over \( L_p \), such that for small enough \( \varepsilon > 0 \), the \( \varepsilon \)-tube around \( L_p \) is foliated diffeomorphic to \((E \times_{G_p} v_p L_p, E \times_{G_p} \mathcal{F}_p)\).

2.2.3. Leaf Types and Stratification

A singular Riemannian foliation can be naturally stratified by the dimension of the leaves. We say \( \mathcal{F} \) has dimension \( k \) (\( \dim(\mathcal{F}) = k \)) if the maximal dimension among the leaves of \( \mathcal{F} \) is \( k \). For \( d \leq k \), define

\[
M_d = \{ p \in M \mid \dim(L_p) = d \}
\]

The connected components of \( M_d \) are called the \( d \)-strata and each such component is called a \( d \)-stratum. Each stratum is a submanifold of \( M \) and the restriction of \( \mathcal{F} \) to a \( d \)-stratum is a regular Riemannian foliation (possibly with leaf holonomy). This stratification is known as the canonical (or coarse) stratification.

Leaves of this maximal dimension \( k \) are called regular leaves. A regular leaf with trivial leaf holonomy is called principal. All other regular leaves are exceptional and all non-regular leaves are called singular. The restriction of \( \pi \) to the regular stratum \( M_{\text{reg}} \) is a Riemannian submersion when there are no exceptional leaves, but in the presence of such leaves, \( \pi(M_{\text{reg}}) = \overline{M}_{\text{reg}} \) will be a Riemannian orbifold. In this case, the image \( \pi(M_{\text{princ}}) = \overline{M}_{\text{princ}} \) of the collection of principal leaves is the manifold part of the orbifold \( \overline{M}_{\text{reg}} \), hence is open and dense in \( \overline{M}_{\text{reg}} \). As with principal orbits of group actions, \( \overline{M}_{\text{princ}} \) is connected, open and dense in \( M/\mathcal{F} \).

Similar things can be said about any \( d \)-stratum. That is, \( \overline{M}_d \) is a Riemannian orbifold whose manifold part is the image of the collection of leaves of dimension \( d \) with trivial leaf holonomy (when restricted to \( T_p(M_d) \)). We will sometimes refer to such leaves as stratum regular and to leaves finitely covered by nearby leaves in the
same $d$-stratum as *stratum exceptional*. In this language, each $\overline{M}_d$ is a Riemannian orbifold whose (dense) manifold part consists of stratum regular leaves.

Now we can further stratify $M/\mathcal{F}$ (hence $M$) by the orbifold singularities within each $\overline{M}_d$, which is equivalent to stratifying $M_d \subset M$ by the leaf holonomy groups. We call strata of this finer stratification the *fine* strata. This is the implicit stratification used in Chapter 4 when we distinguish points of $M/\mathcal{F}$ by their spaces of directions. Note that because the fine strata are fixed point sets of the leaf holonomy actions, their images in $M/\mathcal{F}$ are (locally) totally geodesic submanifolds.

Abusing the language a bit, we will also refer to the image of components of $M_d$ under the quotient map $\pi : M \to M/\mathcal{F}$ as the $d$-strata and denote them by $\overline{M}_d$. 


CHAPTER 3
ALEXANDROV GEOMETRY

When the leaves of an SRF are closed, the Hausdorff distance between the leaves induces an (inner) metric on the leaf space $M/\mathcal{F}$. This quotient is rarely a manifold, but when $\sec(M) \geq k$, it will be an *Alexandrov space with lower curvature bound* $k$. These spaces generalize Riemannian manifolds with lower curvature bound to the realm of metric spaces. In this chapter, we will introduce some basic concepts from Alexandrov geometry and show how these concepts are interpreted for leaf spaces $M/\mathcal{F}$. For a thorough treatment, see the seminal work of Burago, Gromov, and Perelman [7].

3.1. Definitions and Examples

A metric space $X$ is called a *length space* if the distance between points is the infimum of lengths of paths between them in $X$. A length space in which this infimum is attained by a path is called a *geodesic space*.

The curvature bounds we discuss here are synthetic in that they are defined by comparison (as in Topogonov’s Theorem). For this, define the *model $k$-plane*, $\mathbb{M}^2_k$, to be the 2-dimensional constant curvature manifold (a sphere, Euclidean plane, or hyperbolic plane). We wish to capture the idea that triangles in a given space are ‘fatter’ than those in the model $k$-plane. For a triple of points $p, q, r$ in a length space $X$, a *comparison triangle* $\triangle \bar{p}\bar{q}\bar{r}$ in $\mathbb{M}^2_k$ is a triangle with side lengths $|\bar{p}\bar{q}| = |pq|$, $|\bar{p}\bar{r}| = |pr|$, and $|\bar{q}\bar{r}| = |qr|$. For $k \leq 0$, such a triangle is unique up to rigid motion. For $k > 0$, we require in addition that
\[ |pq| + |pr| + |qr| \leq 2\pi/\sqrt{k} \] (so that such a triangle exists in \( \mathbb{M}_k^2 \)). Denote by \( \tilde{\angle}pqr \) the comparison angle at \( \tilde{p} \) in the comparison triangle \( \triangle\tilde{p}\tilde{q}\tilde{r} \subset \mathbb{M}_k^2 \) (we have not defined a notion of angle in \( X \) yet).

Length spaces \( X \) where, locally, any two points can be joined by a geodesic (e.g. locally compact spaces) allow one to discuss triangles in \( X \). Moreover, if such a space satisfies the first definition below, then one has a well defined notion of angle: For two curves \( \alpha, \beta \) emanating from the same point \( \alpha(0) = p = \beta(0) \), the angle between \( \alpha \) and \( \beta \) is

\[
\angle(\alpha, \beta) := \lim_{s,t \to 0} \tilde{\angle}(\alpha(s), p, \beta(t))
\]

where \( \tilde{\angle}(\alpha(s), p, \beta(t)) \) is the angle at \( \tilde{p} \) in the comparison triangle with side lengths \(|p\alpha(s)|, |p\beta(t)|, \text{ and } |\alpha(s)\beta(t)|\).

We can now state a few equivalent definitions for an Alexandrov space given the local existence of geodesics. First, a hinge in \( X \) is two geodesic segments emanating from the same point with an angle between them; the comparison hinge in \( \mathbb{M}_k^2 \) is a hinge with equal side lengths and equal angle to a hinge in \( X \).

**Definition.** A finite (Hausdorff) dimensional length space \( X \) is said to be an Alexandrov space with lower curvature bound \( k \) if for any \( x \in X \), one of the following hold in a neighborhood \( U_x \ni x \):

- **Quadruple:** for any four distinct points \( a, b, c, \text{ and } d \in U_x \), we have
  \[
  \tilde{\angle}bac + \tilde{\angle}cad + \tilde{\angle}bad \leq 2\pi.
  \]
- **Distance:** for a triangle \( \triangle pqr \in U_x \) and \( s \) a point on the side \([qr]\),
  \[
  |ps| \geq |\tilde{p}s|,
  \]
  where \( \tilde{s} \) is the corresponding point on side \([\tilde{q}\tilde{r}]\) with \(|\tilde{q}\tilde{s}| = |qs|\) and \(|\tilde{s}\tilde{r}| = |sr|\).
- **Angle:** for a triangle in \( U_x \), each angle is at least as large as the corresponding angles in the comparison triangle in \( \mathbb{M}_k^2 \).
- **Hinge:** for a hinge in \( U_x \), the distance between the endpoints of the hinge is smaller than the distance between the endpoints of the comparison hinge in \( \mathbb{M}_k^2 \).
A few remarks:

1. The quadruple definition above is the most general in that it assumes nothing more than the metric (i.e. there is no need for angles, or even geodesics).

2. It is shown in [7] that the (finite) Hausdorff dimension of such a space is necessarily an integer and coincides with the topological dimension of the space. So we can discuss dimension of Alexandrov spaces without ambiguity.

3. We will also use $X \in Alex(k)$ or $curv(X) \geq k$ to denote and Alexandrov space with lower curvature bound $k$.

There are a number of ways to construct Alexandrov spaces from old (see [7] pg. 13), but the following is the only one which will appear throughout.

**Definition.** Given a metric space $X$ with $\text{diam}(X) \leq \pi$, the (Euclidean) cone on $X$, denoted $C(X)$, is the topological cone on $X$ equipped with the metric from the law of cosines:

$$d_C(p, q) = \sqrt{t^2 + s^2 - 2ts \cos(d_X(x, y))}$$

where $p = (x, t), q = (y, s)$.

**Examples.**

- Riemannian manifolds with $\text{sec}(M) \geq k$ are $Alex(k)$ spaces.
- Convex subsets of $Alex(k)$ spaces are themselves $Alex(k)$
- If $X$ is positively curved (i.e. $Alex(1)$), then the Euclidean cone $C(X)$ on $X$ is nonnegatively curved (i.e. $Alex(0)$).
- Orbit spaces $M/G$ where $\text{sec}(M) \geq k$ and $G$ is compact are $Alex(k)$ spaces. (This will be shown in the next section in more generality)
- (Non-example) Take the union of the $x$ and $y$ axes in $\mathbb{R}^2$ with the induced intrinsic metric. This fails to be a Alexandrov space for any $k$. To see this, consider the 4 points $a = (0,0)$ $b = (0,1)$, $c = (1,0)$ and $d = (-1,0)$. With the induced metric from $\mathbb{R}^2$, $|ab| = 1 = |ac|$ and $|bc| = 2$, so by the triangle inequality, any comparison triangle for the points $a, b, c$ (in any model $\mathbb{M}^2_k$) will be degenerate. The same is true for the comparison triangles for $a, b, d$ and $a, c, d$. Thus

$$\tilde{Z}bac + \tilde{Z}cad + \tilde{Z}bad = \pi + \pi + \pi = 3\pi \not\leq 2\pi.$$
**Remark.** Suppose that in an $Alex(k)$ space $X$, a geodesic emanating from $b$ bifurcates at $a$ and take $c$ and $d$ to be points on the respective “arms” of the split. Then, similar to above, $\triangle ab\hat{c}$ is degenerate since $|bc| = |ba| + |ac|$ and we arrive at the same failure of the quadruple condition. Thus for Alexandrov spaces with any lower curvature bound, geodesics do not bifurcate:

A fundamental idea in Alexandrov geometry is the quite natural concept of the *space of directions*, which generalizes the ‘tangent sphere’ of a Riemannian manifold. Consider the collection of all geodesics emanating from the same point $p \in X$. Consider two geodesics emanating from equivalent if they form a zero angle with each other (one continues the other). The set of equivalences classes $\Sigma'_p$ is a metric space (the *space of geodesic directions*) where we define the distance between two equivalence classes to be the angle formed by any two representatives from each. The metric completion of this space is the *space of directions at $p$* and is denoted $\Sigma_p$.

**Example.** One can see the difference between the spaces $\Sigma'_p$ and $\Sigma_p$ by considering the closed 2-disc in $\mathbb{R}^2$ (an $Alex(0)$ space). For a point on the boundary circle, the space of directions is an interval $[0, \pi]$. However, the directions corresponding to the points $\{0\}$ and $\{\pi\}$ do not represent geodesics in the disc (the boundary circle is not a geodesic).

The *tangent cone* $T_pX$ of an Alexandrov space at a point $p \in X$ is the Euclidean cone on the space of directions $\Sigma_p$. This can also be constructed intrinsically from $X$ in the following way: Let $p \in X$ and consider the pointed *blow-up* of $(X, p)$ defined to be the Gromov-Hausdorff limit of $(\lambda X, p)$ as $\lambda \to \infty$ ("dilating $X$ at $p$”). This limit is $Alex(0)$ and can be identified with $T_pX$ (Theorem 7.8.1 in [7]). The unit “sphere” in $T_pX$ represents all unit speed geodesics which emanate from $p$ as well as all limits of sequences of such geodesics. With its induced intrinsic metric, is isometric to $\Sigma_p$.

The following is a summary of important nontrivial facts regarding the space of directions that will be used often (see [7], section 7):
Theorem 3.1.1. For a point \( p \) in an \( \text{Alex}(k) \) space \( X^n \), the space of directions \( \Sigma_p \) is a compact \( \text{Alex}(1) \) space of dimension \( n - 1 \).

We use the of directions to inductively define the boundary \( \partial X \) of an Alexandrov space:

\[
\partial X = \{ p \in X : \partial \Sigma_p \neq \emptyset \}
\]

Such an inductive definition is anchored by the theorem above and the fact that a compact 1-dimensional Alexandrov spaces is a manifold - either a circle, with no boundary, or an interval, with two boundary points.

3.2. Leaf Spaces

Recall that when the leaves of a singular Riemannian foliation \( (M, \mathcal{F}) \) are closed, the quotient \( M/\mathcal{F} \) is Hausdorff and thus the distance between the leaves of \( \mathcal{F} \) forms a metric on the leaf space. The following was stated as fact for orbit spaces of isometric group actions, but we prove it here for leaf spaces.

Proposition 3.2.1. For a singular Riemannian foliation with closed leaves \( (M, \mathcal{F}) \), the projection map \( \pi : M \to M/\mathcal{F} \) is a submetry (i.e. \( \pi(B_r(p)) = B_r(\pi(p)) \)).

Proof. With the natural metric on \( M/\mathcal{F} \), the projection map is clearly distance nonincreasing. Thus, \( \pi(B_r(p)) \subset B_r(\pi(p)) \). For the reverse containment, let \( \bar{x} \in B_r(\pi(p)) \). By equidistance of leaves, there exist \( x \in \pi^{-1}(\bar{x}) \) such that \( |px| < r \). Thus, \( \pi(x) = \bar{x} \in \pi(B_r(p)) \). \( \Box \)

Proposition 3.2.2. If \( (M, \mathcal{F}) \) is a singular Riemannian foliation of \( M \) with closed leaves and \( \sec(M) \geq k \), then \( M/\mathcal{F} \in \text{Alex}(k) \).

Proof. Let \( a \in M \) and \( \varepsilon > 0 \) be sufficiently small so that the quadruple condition holds in \( B_\varepsilon(a)_M \subset M \). Also let \( \bar{a} = \pi(a) \) and take distinct points \( \bar{b}, \bar{c}, \bar{d} \in B_\varepsilon(\bar{a})_{M/\mathcal{F}} \). By equidistance of leaves, there exist corresponding preimages \( b, c, d \in B_\varepsilon(a)_M \) such
that $|ab| = |\bar{a}b|$, $|ac| = |\bar{a}c|$ and $|ad| = |\bar{a}d|$. Since $\pi$ is distance non-increasing, so we have that $|\bar{b}c| \leq |bc|, |\bar{c}d| \leq |cd|$ and $|\bar{b}d| \leq |bd|$. It follows that, $\bar{c}\bar{d}a \leq \bar{c}d$, $\bar{b}\bar{a}d \leq \bar{b}d$, and thus, the quadruple condition is satisfied in $B_{\varepsilon}(\bar{a})_{M/F}$.

The following shows that the space of directions at a point $\bar{p} \in M/F$ is encoded in the infinitesimal foliation and the leaf holonomy at $\bar{p}$.

**Proposition 3.2.3.** The space of directions $\Sigma_{\bar{p}}$ at $\bar{p} = \pi(p) \in M/F$ consists of geodesic directions and is isometric to $(S^+_p / F_p) / \Gamma_p$, where $F_p$ is the restriction of the normal infinitesimal foliation to $S^+_p$.

**Proof.** Identify $(O, g, F_O)$ with $(U \times \nu_p P, \Phi^*_O g, F_p)$ by pulling back both the foliation and the metric via $\Phi_O : U \times \nu_p P \to O$. Consider the action of $\Gamma_p = G_p/G^0_p$ on $O/F_O$ described as follows: take an element $g_\gamma \in \Gamma_p$ (represented by the ‘end-point map’ of linearized lifts of normal vectors along the $p$-loop $\gamma$) and a leaf $L$ of $F_O$ (i.e. an element of $O/F_O$). Choose a point $q = \exp_p(v)$ in this leaf, so that $L = L_{\exp_p(v)}$. Then $L_{\exp_p(V(1))}$, where $V(t)$ is a linearized lift of $v$ along $\gamma$, is also a leaf of $F_O$, hence an element of $O/F_O$. So define $g_\gamma \cdot L_{\exp_p(v)} := L_{\exp_p(V(1))}$

Since linearized lifts along loops carry leaves to leaves, our choice of $q \in L_{\exp_p(v)}$ is irrelevant at the level of leaves (provided $q$ is in the slice through $p$). So the above action is well defined (and effective, as we saw earlier). By equidistance of leaves of $F$ (and $F_O$) it follows that this action is by isometries.

Note that if $O$ is a radius $\varepsilon$ tubular neighborhood of $P \subset L_p$, then the orbit space $(O/F_O)/\Gamma_p$ is isometric to an $\varepsilon$-neighborhood of $\bar{p} \in M/F$. Moreover, since linearized lifts commute with rescalings $r^*_\lambda : O_\varepsilon(P) \to O_\varepsilon(P)$, it follows that $\Gamma_p$ acts by isometries on the leaves of $(O, r^*_\lambda g, F_O)$ with quotient isometric to a $\lambda$-rescaling of the $\varepsilon$-neighborhood above. Moreover, $\Gamma_p$ acts on the leaves $(O, g^\lambda, F_O)$, where $g^\lambda$ is as defined in section 2.2.2. In the limit as $\lambda \to 0$, we obtain an isometric action.
of $\Gamma_p$ on the tangent cone $T_{\bar{p}}(O/F_O)$ with quotient isometric to the tangent cone $T_{\bar{p}}(M/F)$. Because the action is by isometries, $\gamma_p$ acts on the unit “sphere” $(S_p^\perp/F_p)$ of $T_{\bar{p}}(O/F_O)$, with quotient isometric to the unit sphere of $T_{\bar{p}}$, which is $\Sigma_{\bar{p}}$. That all these directions are geodesic follows from the slice theorem.

The quotient $(S_p^\perp/F_p)/\Gamma_p$ carries a stratification induced by the (fine) stratification of $S_p^\perp/F_p$. With the proposition above, we have a stratification of $\Sigma_{\bar{p}}$ by the images of the strata of $(S_p^\perp/F_p)/\Gamma_p$ (and these images are also locally totally geodesic). This highlights another geometric aspect of leaf spaces of SRFs with closed leaves:

**Lemma 3.2.4.** For $\bar{p} \in M/F$, there is a natural stratification of $\Sigma_{\bar{p}}$ whose strata correspond bijectively with the (fine) strata of $M/F$ whose closure contains $\bar{p}$.

**Proof.** As mentioned at the end of section 2.2.3, there is a stratification of $(S_p^\perp/F_p)$ whose strata correspond bijectively to the dimensional strata of $(S_p^\perp, F_p)$. Because (normal) homotheties out of $\nu^\varepsilon_p P$ for a plaque $P \ni p$ take leaves of $(\nu^\varepsilon_p P, F_p)$ to leaves of $(\nu^\varepsilon_p P, F_p)$ of the same dimension, it follows that strata which appear in $(S_p^\perp, F_p)$ correspond bijectively to (coarse) strata of local leaves of $(\nu^\varepsilon_p P, F_p)$ whose closure contains $p$ (the homotheties show that leaves of these strata appear arbitrarily close to $p$). Including the action of $\Gamma_p$ gives a bijective correspondence between strata of $(S_p^\perp/F_p)/\Gamma_p$ and (fine) strata of local leaves of $(\nu^\varepsilon_p P, F_p)$ whose closure contains $p$. By the proposition above and the fact that (local) leaves of $(S_p^\perp/F_p)$ of the same dimension belong to global leaves of $(M, F)$ of the same dimension, the result follows.

**Remark.** In his thesis, Marco Radeschi proved that Singular Riemannian foliations on round spheres decompose as joins $(S_p^\perp, F_p) \cong (SF_p, F_0) \ast (SF_p^\perp, F_1)$, where $SF_p$ is a (totally geodesic) subsphere foliated by points and $F_1$ contains no point leaves. Both subspheres are invariant under the action of $\Gamma_p$ and in particular, lemma 3.2.4 applies to both $SF_p/\Gamma_p$ and $(SF_p^\perp/F_1)/\Gamma_p$. For $SF_p/\Gamma_p$, the lemma refers to other fine
strata within the coarse stratum of $L_p$ whose closures contain $\bar{p}$. For $(SF_p^\perp/F_1)/\Gamma_p$, the lemma refers to fine strata whose closure contains $\bar{p}$ which are orthogonal to the (coarse) stratum of $\bar{p}$ (these will be fine strata consisting of leaves of higher dimension than $L_p$).

We refer to the quotient $SF_p/\Gamma_p$ as the \textit{space of tangent directions to the (coarse) stratum} of $\bar{p}$ and $(SF_p^\perp/F_1)/\Gamma_p$ as the \textit{space of normal directions to the (coarse) stratum}. We also have a decomposition for tangent and normal spaces to (fine) strata.

Note that $SF_p$ is a subsphere of $S_p^\perp$ consisting of point leaves and is preserved by $\Gamma_p$, which acts by foliated isometries. Hence, a fixed point of $SF_p$ under this action represents a leaf near $L_p$ with the same dimension that does not cover $L_p$ (i.e. belongs to $L_p$’s fine stratum). The non-fixed points of $SF_p$ represent nearby leaves which finitely cover $L_p$ (see Lemma 3.14 of [30]). Let $d_0$ represent the dimension of the (possibly empty) fixed point set (a totally geodesic subsphere) of $SF_p$ and say $\dim(SF_p) = d_1$. Then we have the decomposition

$$
(S_p^\perp, F_p) \cong (SF_p, \{pts\}) \ast (SF_p^\perp, F_1)
$$

$$
\cong (S_{d_0} \ast S_{d_1-d_0-1}, \{pts\}) \ast (SF_p^\perp, F_1)
$$

$$
\cong (S_{d_0}, \{pts\}) \ast (S_{d_1-d_0-1} \ast SF_p^\perp, F'_1)
$$

where the leaves of the \textit{join foliation} $F'_1$ are $\{pt\} \times L$ where $L$ is a leaf of $F_1$. Note that $\Gamma_p$ acts trivially on $S_{d_0}$, giving the following decomposition of the space of directions

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at $\bar{p} \in M/F$

\[\Sigma_{\bar{p}} \cong (S^d_p / F_p) / \Gamma_p \]

\[\cong S^{d_0} \ast ((S^{d_1 - d_0 - 1} \ast SF^\perp_p) / F') / \Gamma_p \]

\[\cong S^{d_0} \ast ((S^{d_0})^\perp / F'_1) / \Gamma_p \]

\[\cong \Sigma^T_{\bar{p}} \ast \Sigma^\perp_{\bar{p}} \]

We say the space of tangent directions to the fine stratum of $\bar{p}$ is $\Sigma^T_{\bar{p}} := S^{d_0}$ (an honest round sphere) and the space of normal directions to the fine stratum of $\bar{p}$ is $\Sigma^\perp_{\bar{p}} := ((S^{d_0})^\perp / F'_1) / \Gamma_p$. This decomposition is important when describing the boundary $\partial (M/F)$.

Recall from the previous section that a point $\bar{p} \in M/F$ belongs to the boundary precisely when its space of directions $\Sigma_{\bar{p}}$ has nonempty boundary. Because each (fine) stratum is a smooth manifold (whose space of tangent directions is a round sphere), the boundary will be detected by looking at the space of normal directions to the fine stratum $((S^{d_0})^\perp / F'_1) / \Gamma_p$ (which we denoted $\Sigma^\perp_{\bar{p}}$). An extreme case is when the space of normal directions to the fine stratum of $\bar{p}$ is a single point. In such a case, the total space of directions is a hemisphere with boundary $S^{d_0}$. This happens in two cases:

1. When the infinitesimal foliation on the normal space to the fine stratum is by concentric spheres (i.e. when the foliation $F'_1$ of $(S^{d_0})^\perp$ is by a single leaf and $\Gamma_p = \{1\}$).

2. When $(SF_{d_0})^\perp = S^0$ and $\Gamma_p = \mathbb{Z}_2$. In which case, the infinitesimal foliation is still by concentric spheres (in a sense).

**Definition.** Let $\bar{p} \in M/F$ and take $p \in \pi^{-1}(\bar{p}) \subset M$. A connected component of the fine stratum of $\bar{p}$ is said to be an (open) boundary face if the infinitesimal foliation of the normal space to the fine stratum of $L_p$ is a foliation by concentric spheres of dimension $> 0$ or by spheres of dimension 0 with $\Gamma_p = \mathbb{Z}_2$.

Intuitively, the boundary faces are those components of a stratum for which the
only direction normal to the fine stratum is toward the interior. One may also define
the open boundary faces as those components of fine strata for which the tangent
cone splits as the tangent space to the stratum and a single normal ray. Thus, open
boundary faces can also be thought of as the codimension 1 strata of a leaf space.

As one might expect, these definitions of boundary face agrees with the usual one
given for orbit spaces. That is, in the case of homogeneous SRFs, the boundary faces
are those components of strata for which the isotropy action on the spheres normal
to the stratum is transitive (see [15]). As in that case, we also have

**Proposition 3.2.5.** The boundary \( \partial(M/F) \) is the union of the closures of the boundary faces.

**Proof.** Because the boundary of a (finite dimensional) Alexandrov space is closed (see
[7], pg. 54), a point in the closure of a boundary face cannot be interior.

On the other hand, let \( \bar{p} \) be an arbitrary point of \( \partial(M/F) \). Given the local
splitting mentioned later in Lemma 4.2.2 (which is not a result of the work here!) if
\( [\gamma_{\bar{q}}] \in \Sigma_{\bar{p}}^\perp \) is a direction into the stratum containing the point \( \bar{q} \), then each nearby
point in the stratum of \( \bar{p} \) will have representatives into the stratum of \( \bar{q} \) as well. In
particular, the stratum of \( \bar{q} \) has strictly higher dimension (hence lower codimension)
than the stratum of \( \bar{p} \). Thus, the space of normal directions to any (fine) stratum
which appears in \( \Sigma_{\bar{p}}^\perp \) will have strictly lower dimension than \( \Sigma_{\bar{p}}^\perp \). By definition of
boundary, we know that \( \partial(\Sigma_{\bar{p}}) \neq \emptyset \), so there are boundary strata (whose closures
contain \( \bar{p} \)) with strictly lower dimensional space of normal directions. If the stratum
of \( \bar{q} \) is a boundary face, we are done. If it is not, then we find a direction \( [\gamma_{\bar{r}}] \in \Sigma_{\bar{q}}^\perp \)
into a stratum with even lower dimensional space of normal directions. This process
necessarily ends in a boundary stratum with 0-dimensional space of normal directions,
say \( \Sigma_{x}^\perp \). But we’ve seen above that \( \Sigma_{x}^\perp \) is a finite quotient of a leaf space of a singular
Riemannian foliation on the sphere normal to the stratum of \( L_x \subset M \). It follows
that it is either one or two points. If \( \Sigma_{x}^\perp \) is two points, then given the decomposition
above, it follows that $\Sigma_{\bar{x}}$ is the suspension of the tangent sphere to the stratum of $\bar{x}$, hence a sphere (which lacks boundary). Thus, it must be that $\Sigma_{\bar{x}} = \{pt\} \text{ and } \bar{x}$ belongs to a boundary face.

For general Alexandrov spaces with boundary, it is unknown if the boundary is itself an Alexandrov space with the same lower curvature bound (with its induced intrinsic metric). However, for leaf spaces $M/\mathcal{F}$, the geometry of $\partial(M/\mathcal{F})$ is more approachable, which allowed us in [22] to prove:

**Theorem 3.2.6 (Boundary Conjecture for Leaf Spaces).** For a leaf space $M/\mathcal{F} \in Alex(k)$ with nonempty boundary, the boundary $\partial(M/\mathcal{F})$, when equipped with its induced intrinsic metric, is $Alex(k)$.

Now consider a point $x \in \partial(M/\mathcal{F})$ (an Alexandrov space, by the theorem). By definition, the unit “sphere” in the tangent cone $T_x(M/\mathcal{F})$ is the space of directions $\Sigma_x$. The boundary of this tangent cone is of course the cone on $\partial(\Sigma_x)$, which by the lemma above, when equipped with the intrinsic metric induced from that on $\Sigma_x$, is itself an $Alex(1)$ space consisting of boundary directions. This intrinsic metric on $\partial\Sigma_x$ measures the angle between boundary geodesics as measured within the boundary. That is:

**Proposition 3.2.7.** For a boundary point in a leaf space $x \in M/\mathcal{F}$, the space of boundary directions is isometric to the boundary of the space of directions. That is, $\Sigma_x(\partial(M/\mathcal{F})) \cong \partial(\Sigma_x(M/\mathcal{F}))$. 
4.1. Strata

We now discuss the structure of positively curved leaf spaces of singular Riemannian foliations. Throughout this section, assume that $M$ is a closed Riemannian manifold with $\text{sec}(M) \geq 1$ carrying a singular Riemannian foliation $\mathcal{F}$ with closed leaves.

It is known that for a positively curved Alexandrov space $X$ with nonempty boundary, the distance function to the boundary is strictly concave (see [29]). Thus, there is a unique point called the soul which is at maximal distance from the boundary. In the case that $X$ is a leaf space $M/\mathcal{F}$, we call this point the soul leaf $\bar{s} \in M/\mathcal{F}$. We also use the term soul leaf when referring the actual leaf $L_s := \pi^{-1}(\bar{s}) \subset M$; what is meant should be apparent given the context.

The following technical lemma is crucial in proving the stratum theorem below.

**Lemma 4.1.1.** Let $M/\mathcal{F}$ be a positively curved leaf space with nonempty boundary. Let $\bar{p} \in M/\mathcal{F}$ be an interior, non-soul point and $\gamma(t)$ be a minimal geodesic from $\bar{p}$ to $\bar{s}$. Also, let $\bar{x} \in \partial M/\mathcal{F}$ be a point realizing $\text{dist}(\partial M/\mathcal{F}, \bar{p})$. Then the angle $\angle \bar{p}_s^\bar{x}$ is strictly greater than $\pi/2$.

**Proof.** Suppose otherwise that $\theta = \angle \bar{p}_s^\bar{x} \leq \pi/2$. Let $\gamma : [0,1] \to M/\mathcal{F}$ be a minimal geodesic from $\gamma(0) = \bar{p}$ to $\gamma(1) = \bar{s}$. Consider the family of comparison hinges at $\bar{p}$ with legs $c = |px|$, $\ell(t) = |p\gamma(t)|$ and common angle $\theta = \angle \bar{p}_s^\bar{x}$. Define $\tilde{d}(t)$ by

$$\tilde{d}(t) := |x\gamma(t)|$$
where \( \tilde{\gamma}(t) \) is the minimal geodesic in \( S^2 \) from \( \tilde{\gamma}(0) = \tilde{p} \) to \( \tilde{\gamma}(1) = \tilde{s} \). The spherical law of cosines gives
\[
\cos(d(t)) = \cos(c) \cos(\ell(t)) + \sin(c) \sin(\ell(t)) \cos(\theta)
\]
from which we find
\[
d'(t) = \frac{\ell'(t) \left[ \cos(c) \sin(\ell(t)) - \sin(c) \cos(\ell(t)) \cos(\theta) \right]}{\sin(d(t))}
\]
At \( t = 0 \), \( \ell(0) = c, d(0) = 0, \ell'(0) > 0 \), and since we’ve assumed \( \theta \leq \pi/2 \), we have,
\[
d'(0) = \frac{\ell'(0) \left[ 0 - \sin(c) \cos(\theta) \right]}{\sin(c)} = -\ell'(0) \cos(\theta) \leq 0
\]
Thus, given any \( \alpha > 0 \), there exists \( \varepsilon > 0 \) such that
\[
\frac{d(\varepsilon) - d(0)}{\varepsilon} < \alpha
\]
In particular, there is \( \varepsilon > 0 \) such that
\[
\frac{d(\varepsilon) - d(0)}{\varepsilon} < d_\partial(1) - d_\partial(0)
\]
where \( d_\partial(t) := \text{dist}(\partial(M/\mathcal{F}), \gamma(t)) \).

Note that \( d_\partial(t) \leq d(t) \) for all \( t \in [0, 1] \) and by the Alexandrov hinge comparison, we have \( |x\gamma(t)| \leq d(t) \). Now since \( d(0) = d_\partial(0) \) and \( d_\partial(t) \leq d(t) \) for all \( t \in [0, 1] \), we have
\[
\frac{d_\partial(\varepsilon) - d_\partial(0)}{\varepsilon} \leq \frac{d(\varepsilon) - d(0)}{\varepsilon} < \frac{d_\partial(1) - d_\partial(0)}{1}
\]
which contradicts the strict concavity of \( d_\partial \) along \( \gamma \). \( \square \)

**Theorem 4.1.2** (Stratum Theorem). Let \((M, \mathcal{F})\) be a singular Riemannian foliation
with closed leaves such that \( \text{curv}(M/F) \geq 1 \) and \( \partial(M/F) \neq \emptyset \). Then the closure of any interior (fine) stratum contains both the soul point and points on the boundary, or is the soul point alone. In particular, if there are any interior singular strata, then the soul is singular.

**Proof.** The lemma above implies that the space of directions \( \Sigma_p \) has diameter \( > \pi/2 \). Since \( \Sigma_p \) is itself isometric to the quotient of an SRF on a sphere (the infinitesimal foliation on normal sphere to \( L_p \) at a point \( p \in \pi^{-1}(\bar{p}) \)), it follows that this diameter is in fact equal to \( \pi \) and hence the infinitesimal foliation is a suspension (see [22], Lemma 1.1). In particular, \( \Sigma_p \) contains points at distance \( \pi \), which represent leaves in the same (fine) stratum as \( L_p \).

Now we proceed inductively. Consider the most singular interior stratum (that with the lowest dimensional leaves). Because the closure of a stratum contains only more singular strata, this (most singular, interior) stratum is either closed, or contains points of \( \partial(M/F) \).

Suppose that this stratum is closed (hence compact) and let \( \bar{p} \) be a point which minimizes dist(\( \bar{s}, \cdot \)) in the stratum. Suppose further that this minimum is nonzero. By the lemma, any geodesic from \( \bar{p} \) to \( \bar{s} \) makes an angle greater than \( \pi/2 \) with any minimal geodesic from \( \bar{p} \) to \( \partial(M/F) \). Let \( v_1 \) and \( v_2 \) represent a pair of these minimizing geodesics (to \( \bar{s} \) and \( \partial(M/F) \) respectively). That the angle between \( v_1 \) and \( v_2 \) is greater than \( \pi/2 \) means that dist\( \Sigma_p(v_1, v_2) \) > \( \pi/2 \). By the decomposition of \( \Sigma_p \) given above, it follows that neither \( v_1 \) nor \( v_2 \) can be contained in \( ((S^{d_0})^\perp/F'_1) / \Gamma_p \) (this proves immediately that there can be no isolated interior non-soul strata). Thus, there must exist (fine) stratum directions (points in \( S^{d_0} \)), one of which makes angle < \( \pi/2 \) with \( v_1 \) and the other making angle < \( \pi/2 \) with \( v_2 \). That is, there is a stratum curve through \( \bar{p} \) which along one direction from \( \bar{p} \), decreases the distance from \( \bar{s} \) and along the other, decreases the distance from \( \partial(M/F) \). However, this contradicts that \( \bar{p} \) realized the minimum distance to \( \bar{s} \). Therefore, if the most singular interior
stratum is closed, it contains the soul. If the most singular interior stratum is not closed, its closure cannot contain a more singular interior stratum, yet will achieve a minimum for \( \text{dist}(\bar{s}, \cdot) \). So the stratum must contain a point minimizing the distance to \( \bar{s} \). As above, if this distance is nonzero, it can be decreased within the stratum - a contradiction. Thus, the most singular interior stratum must contain the soul.

In fact, if any interior singular stratum is closed, then \( \text{dist}(\partial(M/F), \cdot) \) achieves a nonzero minimum within this stratum. If this stratum has positive dimension, we achieve a contradiction as above. Therefore, a closed interior singular stratum consists of the soul alone. This proves the ‘base case.’

Now suppose statement \((ii)\) holds for all interior singular stratum up to dimension \( n - 1 \) and let \( Y \) be an interior singular stratum of dimension \( n \). The closure \( \bar{Y} \) of this stratum is compact, so achieves a minimum for \( \text{dist}(\bar{s}, \cdot) \). Suppose this minimum is nonzero and let \( \bar{p} \in \bar{Y} \) realize this minimum. If \( \bar{p} \) is in a more singular interior stratum than \( Y \), then by induction, we know \( \bar{s} \) is in the closure of this more singular stratum, hence in \( \bar{Y} \) - a contradiction. On the other hand if \( \bar{p} \in Y \), then we derive a contradiction with the same space of directions argument above. So in either case, we see that this minimum distance must be zero (i.e. \( \bar{s} \in \bar{Y} \)). The same style argument also shows that \( \bar{Y} \) contains points of \( \partial(M/F) \). This completes the proof.

\[ \square \]

**Corollary 4.1.3.** A 2-dimensional positively curved leaf space with boundary can have no interior singularities other than possibly the soul.

**Proof.** By Theorem 4.1.2 any non-soul interior singularity will belong to a stratum whose closure contains the soul. By lemma 3.2.4, the space of directions at the soul will have a stratum for each such interior stratum of \( M/F \). However, this space of directions is 1-dimensional, positively curved, and without boundary, hence it is a circle of length \( \leq 2\pi \) and has just one stratum.

\[ \square \]

**Remark.** The corollary above can also be proved by observing that the closure of any interior singular strata will contain points in the boundary (by Theorem 4.1.2).
Since the space of directions at such a boundary point must be an interval of length \( \leq \pi \), which has no interior singular strata itself, the result follows.

4.2. Soul Theorems

In this section, we begin with some geometric facts about positively curved leaf spaces regarding the soul leaf. We then discuss the topology of such leaf spaces regarding the soul leaf by generalizing the soul theorems of [17] and [34] to the setting of singular Riemannian foliations.

**Lemma 4.2.1.** Let \( M/\mathcal{F} \) be a positively curved leaf space with nonempty boundary \( \partial(M/\mathcal{F}) \) and soul leaf \( \bar{s} \in M/\mathcal{F} \). Then

(i) The distance from \( \bar{s} \) to \( \partial(M/\mathcal{F}) \) is at most \( \pi/2 \).

(ii) The distance from \( \bar{s} \) to any point of \( M/\mathcal{F} \) is at most \( \pi/2 \).

(iii) For a non-boundary point \( p \in M/\mathcal{F} \), any geodesic \( \gamma \) to a nearest point boundary point \( x \in \partial(M/\mathcal{F}) \) makes angle \( \pi/2 \) with all boundary geodesics emanating from \( x \).

(iv) If \([\gamma]\) is the direction in \( \Sigma_x \) corresponding to \( \gamma \), it is the soul of \( \Sigma_x \). In particular, such a minimizing geodesic is unique and \( \Sigma_x \cong C(\partial\Sigma_x) \).

**Proof.** We prove (iii) first, which is used to prove (i), which is used to prove (iv), which is used to prove (ii).

(iii) A simple application of the Alexandrov hinge comparison shows that \( \gamma \) cannot make angle \( \leq \pi/2 \) with any boundary geodesic (for along such a boundary geodesic, we’d find points closer to \( p \)). On the other hand, suppose that there is a boundary geodesic \( \gamma' \) making angle \( \geq \pi/2 \) with \( \gamma \). Recall the space of directions decomposition from the previous section:

\[
\Sigma_x \cong S^{d_0} \ast ((S^{d_0})^\perp / \mathcal{F}_x^\prime)/\Gamma_x
\]
where $S^{d_0}$ is the (possibly empty) tangent sphere to the fine stratum containing $x$ (on which $\Gamma_x$ acts trivially). In particular, $\text{diam}(S^{d_0}) = \pi$ and $\text{diam}\left(\left((S^{d_0})\perp/F_1\right)/\Gamma_x\right) \leq \pi/2$. This means that if $S^{d_0} = \emptyset$, then our proposed scenario is impossible. Moreover, when it is possible, neither of the directions $[\gamma], [\gamma'] \in \Sigma_x$ can belong to the space of normal directions to the fine stratum of $x$. This means there are antipodal directions (into the same fine stratum as $x$), one of which must make angle $< \pi/2$ with $\gamma$. The same hinge argument as above again produces a contradiction. Thus, $\gamma$ makes angle $\pi/2$ with all boundary geodesics.

(i) Let $x \in \partial(M/F)$ be a point which realizes the distance from $\bar{s}$ to the boundary. Let $\gamma$ be any minimal geodesic from $\bar{s}$ to $x$ with length $r_x$ and suppose $r_x > \pi/2$. Along any boundary geodesic emanating from $x$, there is, by continuity, a point $y \in \partial(M/F)$ and $\varepsilon > 0$ such that $|xy| = \varepsilon$ and $|\bar{s}y| = r_y > \pi/2$. Because $r_x$ is minimal, we have

$$r_y \geq r_x > \pi/2 \quad \implies \quad \cos(r_y) \leq \cos(r_x) < 0$$

Consider the comparison triangle $\bar{x}sxy \subset S^2$. By the spherical law of cosines, we have

$$\cos(r_y) = \cos(\varepsilon) \cos(r_x) + \sin(\varepsilon) \sin(r_x) \cos(\bar{x}sxy)$$

where $\bar{x}sxy$ is the comparison angle to $\bar{x}sxy$. Rewriting the law of cosines, we have

$$\cos(\bar{x}sxy) = \frac{\cos(r_y) - \cos(\varepsilon) \cos(r_x)}{\sin(\varepsilon) \sin(r_x)}$$

Note that because the denominator on the right hand side is positive, the sign of $\cos(\bar{x}sxy)$ depends on the numerator. Since $0 < \cos(\varepsilon) < 1$, it is easy to see that we must have $\cos(\bar{x}sxy) < 0$, which implies $\pi/2 < \bar{x}sxy < \angle x_y$, a contradiction (we know $\angle x_y = \pi/2$ by statement (iii)). Therefore, $|\bar{s}\partial(M/F)| \leq \pi/2$
(iv) Now suppose there were two such directions \( [\gamma_1], [\gamma_2] \in \Sigma_x \) and consider a geodesic \( c \) in \( \Sigma_x \) between the two directions. By the strict concavity of \( \text{dist}(\partial(\Sigma_x), \cdot) \) along geodesics (see [29]), we’d find interior directions along \( c \) at distance \( > \pi/2 \) from \( \partial(\Sigma_x) \), which is impossible (the soul is at maximal distance \( \leq \pi/2 \) from the boundary, by statement (i)).

(ii) To show that the maximal distance from \( \bar{s} \) to any point of \( M/\mathcal{F} \) is at most \( \pi/2 \), we induct on the dimension of \( M/\mathcal{F} \). First, a one-dimensional positively curved leaf space with boundary is an interval of length at most \( \pi \), so here the claim is obvious. Assume the claim holds for any leaf space of dimension less than \( n \) and let \( M/\mathcal{F} \) be an \( n \)-dimensional, positively curved leaf space with boundary. Let \( p \in M/\mathcal{F} \) be any point and \( c \) be a minimal geodesic from \( \bar{s} \) to \( p \). Because \( \bar{s} \) is a critical point for \( \text{dist}(\partial(M/\mathcal{F}), \cdot) \), there is a minimal geodesic \( d \) (unique, by part (iv)) of length \( \leq \pi/2 \) from \( \bar{s} \) to a nearest point \( q \in \partial(M/\mathcal{F}) \) making angle at most \( \pi/2 \) with \( c \). Moreover, if \( e \) is the minimal geodesic from \( q \) to \( p \), we know that \( e \) and \( d \) make angle \( \leq \pi/2 \) at \( y \) (induction hypothesis; we are measuring distance in \( \Sigma_y \), a finite quotient of an \( (n - 1) \)-dimensional leaf space). Gathering all this information, we see that the comparison triangle \( \tilde{\triangle} \bar{s}pq \) lies entirely in a quarter-sphere of \( S^2 \). Hence, the geodesic \( c \) (and also \( e \)) has length at most \( \pi/2 \).

The following lemma is a consequence of Theorem 6.1 in [2] which describes the structure of small tubular neighborhoods of strata of \( M/\mathcal{F} \). We use this structure to extend flows induced by vector fields on strata of \( M/\mathcal{F} \) (which are honest manifolds) to these small tubes.

**Lemma 4.2.2.** Let \( Y \subset M/\mathcal{F} \) be a \( k \)-dimensional fine stratum and let \( \bar{y} \in Y \). There exists a ‘tubular’ neighborhood of \( Y \) which is locally homeomorphic to \( \mathbb{D}^k \times C^\epsilon(\Sigma_{\bar{y}}) \), where \( C^\epsilon(\Sigma_{\bar{y}}) \) denotes the (open) cone on the space of directions at \( \bar{y} \) normal to \( Y \).

**Proof.** Let \( \hat{Y} := \pi^{-1}(Y) \). By Theorem 6.1 of [2], there is a tubular neighborhood of
\( \hat{Y} \) locally foliated diffeomorphic to

\[
\left( P \times (\nu_p L_p \cap T_p \hat{Y}) \times \nu_p \hat{Y}, P \times \{\text{pts}\} \times \mathcal{F}|_{\nu_p \hat{Y}} \right)
\]

where \( P \ni p \) is a plaque of a leaf \( L_p \subset \hat{Y} \). Moreover, this \( \left( \nu_p L_p \cap T_p \hat{Y}, \{\text{pts}\} \right) \) is locally foliated diffeomorphic to \( (\mathbb{D}^k, \{\text{pts}\}) \) via the restriction of \( \exp_p : \nu_p L_p \to M \). The quotient of

\[
\left( P \times \mathbb{D}^k \times \nu_p \hat{Y}, P \times \{\text{pts}\} \times \mathcal{F}|_{\nu_p \hat{Y}} \right)
\]

then locally models a tubular neighborhood of \( Y \) when there is no leaf holonomy for \( L_p \). In the presence of holonomy, it acts only on \( \nu_p \hat{Y} \), since \( \hat{Y} \) is a fine stratum. Thus, a tubular neighborhood in \( M/\mathcal{F} \) about \( Y \) locally has the form \( \mathbb{D}^k \times \left( \left( \nu^\varepsilon_p(\hat{Y})/\mathcal{F}|_{\nu^\varepsilon_p(\hat{Y})} \right)/\Gamma_p \right) \). As in the proof of Proposition 3.2.3, the second factor is isometric to the open cone \( C^\varepsilon(\Sigma^+_\hat{s}) \).

**Theorem 4.2.3** (Boundary Soul Theorem). Let \( M \) be a compact Riemannian manifold, and \( \mathcal{F} \) be a singular Riemannian foliation with closed leaves such that \( \text{curv}(M/\mathcal{F}) \geq 1 \) and \( \partial(M/\mathcal{F}) \neq \emptyset \). Then

1. \( M - \pi^{-1}(\partial(M/\mathcal{F})) \) is foliated diffeomorphic to \( \nu(L_s) \), the normal bundle of the soul leaf.

2. \( M/\mathcal{F} - \partial(M/\mathcal{F}) \) is stratified homeomorphic to \( C(\Sigma_s) \), the (open) cone on the space of directions at the soul \( s \).

3. Any stratum of \( M/\mathcal{F} - \partial(M/\mathcal{F}) \) is either the soul alone or is diffeomorphic to a subcone (possibly minus cone point) of \( C(\Sigma_s) \) over some stratum of \( \Sigma_s \).

**Proof.** For \( \varepsilon > 0 \), define

\[
M^\varepsilon := M - (B_\varepsilon(L_s) \cup B_\varepsilon(\pi^{-1}(\partial(M/\mathcal{F}))))
\]

and

\[
M/\mathcal{F}^\varepsilon := \pi(M^\varepsilon) = M/\mathcal{F} - (B_\varepsilon(s) \cup B_\varepsilon(\partial(M/\mathcal{F})))
\]
Let $Y \subset M/\mathcal{F}$ and $\hat{Y} \subset M$ be a fine stratum of $M/\mathcal{F}$ and its preimage (also called the fine stratum) in $M$, respectively. Define $Y_\varepsilon^\varepsilon := Y \cap M/\mathcal{F}_\varepsilon$ and let $\hat{y} \in Y_\varepsilon^\varepsilon$. Because $\hat{y}$ is noncritical for $\text{dist}(\partial(M/\mathcal{F}), \cdot)$, there exists a stratum direction $v$ (an honest vector since $Y$ is a manifold) along which $\text{dist}(\partial(M/\mathcal{F}), \cdot)$ is strictly decreasing. We will extend this vector field $V$ to a \textit{basic} vector field (i.e. horizontal and taking leaves to leaves) in a tubular neighborhood of $\hat{Y}$.

First note that (downstairs in $M/\mathcal{F}$) since $Y$ is a smooth manifold, in a neighborhood of $\hat{y}$, we can smoothly extend $v$ to a vector field whose flow, by continuity, also strictly decreases $\text{dist}(\partial(M/\mathcal{F}), \cdot)$. Doing this in a neighborhood (of $Y$) around each point of $Y$ and using a partition of unity, we arrive at a vector field $\hat{V}$ on $\hat{Y}_\varepsilon^\varepsilon$ also with the property that its flow strictly decreases $\text{dist}(\partial(M/\mathcal{F}), \cdot)$. Since $Y_\varepsilon^\varepsilon$ is the base of a Riemannian submersion $\pi| : \hat{Y}_\varepsilon^\varepsilon \to Y_\varepsilon^\varepsilon$, there is a unique horizontal lift $\hat{V}$ of $V$ to $\hat{Y}_\varepsilon^\varepsilon$.

As above, there is a neighborhood $W \subset \hat{Y}$ containing a plaque $P \in y$ of $L_y$ such that the preimage (under the projection to $Tub(\hat{Y}) \to \hat{Y}$) of $W$ in a small enough tubular neighborhood of $\hat{Y}$ is locally foliated diffeomorphic to

$$(P \times \mathbb{D}^k \times \nu_y(\hat{Y}), P \times \{\text{pts}\} \times \mathcal{F}_p|_{\nu_y(\hat{Y})})$$

Let $x_1, \ldots, x_\ell$ be coordinates for $P$, $y_1, \ldots, y_k$ be coordinates for $\mathbb{D}^k$ and $z_1, \ldots, z_{n-k-\ell}$ be coordinates for $\nu_y(\hat{Y})$. Because $\hat{V}$ is normal to $P$ yet tangent to $\hat{Y}$ and projects to $V$, in these coordinates, we can express $\hat{V}$ by

$$\hat{V}(x, y, 0) = \sum_{j=1}^{k} f_j(y) \frac{\partial}{\partial y_j}$$

where the $f_j(y)$ are smooth functions depending only on $y$ (i.e. the location within the
stratum). We then smoothly extend to the tubular neighborhood by locally declaring

\[
\hat{V}(x, y, z) = \sum_{j=1}^{k} f_j(y) \frac{\partial}{\partial y_j}
\]

By construction, this vector field is tangent to fine strata and basic. Moreover, we may adjust the size of both the neighborhood \(W \subset \hat{Y}\) and the radius of the tubular neighborhood of \(\hat{Y}\) so that by continuity, the flow of \(\hat{V}\) strictly decreases \(\text{dist}(\pi^{-1}(\partial(M/\mathcal{F})), \cdot)\) in the new smaller neighborhood. One then constructs such vector fields for such tubular neighborhoods about each point in \(M_\varepsilon\), and with a partition of unity, forms a vector field \(\tilde{V}\) on \(M_\varepsilon\) with these same properties (here it is crucial that the flows strictly decrease \(\text{dist}(\pi^{-1}(\partial(M/\mathcal{F})), \cdot)\)). The length of the integral curves of \(\tilde{V}\) vary continuously over \(\partial B_{\varepsilon}(L_s)\), thus there is a continuous (basic) function \(f : \partial B_{\varepsilon}(L_s) \to \mathbb{R}\) such that the integral curves of \(f\tilde{V}\) all have equal length. Now for \(\varepsilon_1 > \varepsilon\), there is a diffeomorphism

\[
\nu(L_s) \to B_{\varepsilon_1}(L_s)
\]

\[
tv \mapsto \exp \left( \frac{\varepsilon_1}{\varepsilon_{1/\varepsilon}} v \right)
\]

where \(v\) is a unit vector with footpoint in \(L_s\). The restriction of this diffeomorphism to those vectors in \(\nu(L_s)\) with length \(\leq \frac{1}{\ln(\varepsilon_1/\varepsilon)}\) is a diffeomorphism onto its image \(B_{\varepsilon}(L_s)\) and thus further restricts to a diffeomorphism from vectors of length exactly \(\frac{1}{\ln(\varepsilon_1/\varepsilon)}\) to \(\partial(B_{\varepsilon}(L_s))\). Composing with the flow of \(g\tilde{V}\) above gives a foliated diffeomorphism

\[
\nu(L_s) \to M - B_{\varepsilon}(\pi^{-1}(\partial(M/\mathcal{F})))
\]

for arbitrarily small \(\varepsilon\). This proves the first statement.

Because vector field \(f\tilde{V}\) constructed above is basic, it projects to a (continuous) vector field \(fV\) on \(M/\mathcal{F}_\varepsilon\) that is tangent to fine strata of \(M/\mathcal{F}\) and whose flow
strictly decreases dist(∂(M/F), ·). The flow of fV now provides a homeomorphism from ∂B_ε(¯s) to ∂(B_ε(∂(M/F))) with integral curves all of equal length. By [29], there is a homeomorphism from C(Σ_¯s) to B_ε_1(¯s), which is stratified by Lemma 3.2.4 (the cone point potentially belongs to an isolated stratum). Thus we arrive at a stratified homeomorphism

\[ C(Σ_¯s) \rightarrow M/F - B_ε(∂(M/F)) \]

for arbitrarily small ε, proving the second statement.

The third statement follows from the fact that the restriction of fV to any fine stratum is smooth (by construction) and the fact that all strata near the soul point ¯s (though possibly not that of the soul, if it is isolated) are represented as strata of Σ_¯s.

As Wilking [34] observed for quotients of isometric group actions, the distance function to a (closed) boundary face F (i.e. the closure of a component of a codimension 1 stratum) of a positively curved leaf space M/F is also strictly concave. A proof of this fact for general Alexandrov spaces and codimension 1 extremal sets can be found in Andreas Wörner’s dissertation [36].

Thus, similar statements as in the results above can be made in reference to each boundary face F for leaf spaces whose boundary have multiple strata. The proof is the same as above modulo replacing “∂(M/F)” with “F” and “non-soul interior point” with “non-face soul point of M/F − F.”

**Theorem 4.2.4** (Face Soul Theorem). Let M/F be a positively curved leaf space and F be a boundary face and F its closure in M/F. Then

1. There exists a unique “face soul” s_F at maximal distance from F.
2. The distance from s_F to any point of F is at most π/2.
3. The closure of any stratum of M/F − F contains both s_F and points of F, or is s_F alone.
4. $M/\mathcal{F} - \overline{\mathcal{F}}$ is stratified homeomorphic to the (open) cone $C(\Sigma_{s_F})$.

5. $M - \pi^{-1}(\overline{\mathcal{F}})$ is diffeomorphic to $\nu(\pi^{-1}(s_F))$, the normal bundle of the face soul leaf.

4.3. Leaf Spaces with One Boundary Stratum

The case when $\partial(M/\mathcal{F})$ consists of a single stratum is particularly nice:

**Theorem 4.3.1.** If $M/\mathcal{F}$ is a positively curved leaf space with boundary consisting of a single stratum, then

1. $M/\mathcal{F}$ contains at most 3 strata: the lone boundary stratum, the principal stratum, and possibly an isolated soul stratum.

2. $M/\mathcal{F}$ is stratified homeomorphic to the (closed) cone $C(\overline{\Sigma_s})$.

3. $\Sigma_s$ is diffeomorphic to $\partial(M/\mathcal{F})$.

4. The infinitesimal foliation normal to the soul leaf $L_s$ is a (principal) Riemannian foliation. In particular, $\partial(M/\mathcal{F})$ is diffeomorphic to a CROSS, a space form $S^\ell/\Gamma$, or a $\mathbb{Z}_2$-quotient of an odd-dimensional complex projective space.

5. The leaves of $M - (L_s \cup \pi^{-1}(\partial(M/\mathcal{F})))$ are principal and diffeomorphic to some fixed sphere bundle with base diffeomorphic to a boundary leaf.

6. $M$ is foliated diffeomorphic to double disc bundle:

$$M \cong D(L_s) \cup D(\pi^{-1}(\partial(M/\mathcal{F})))$$

glued along their common boundary.

**Proof.** The first statement follows from Lemma 3.2.4 and the fact that only the principal stratum appears in the space of directions normal to $\partial(M/\mathcal{F})$, since it is the lone codimension 1 boundary stratum.

In the case that $\partial(M/\mathcal{F})$ consists of a single stratum, then there is exactly one normal direction to the boundary. Thus, by lemma 4.2.2, there is a $\varepsilon''$-neighborhood of $\partial(M/\mathcal{F})$ locally modeled by $\partial(M/\mathcal{F}) \times [0, \varepsilon'')$. Because there are no critical points for $\text{dist}(\partial(M/\mathcal{F}), \cdot)$ away from the boundary and $\overline{s}$, the gradient flow of the (normal)
Riemannian exponential $\exp^\perp : \partial(M/F) \times [0, \varepsilon'') \rightarrow M/F$ provides diffeomorphisms from $\partial(M/F)$ to $\partial(B_t(\partial(M/F)))$ for all $t \in (0, \varepsilon'')$.

Let $fV$ be the vector field on $M/F_\varepsilon$ constructed as in the proof of Theorem 4.2.3. The flow of $fV$ together with a conic neighborhood of $\bar{s}$ and the diffeomorphisms mentioned above proves the second and third statements.

For the fourth statement, recall that the leaves of the infinitesimal foliation $\mathcal{F}_s$ on the normal sphere to $s \in L_s$, say $S^\ell$, are (diffeomorphic to) the intersections of the global leaves of $\mathcal{F}$ with $\exp_s(S^\ell)$. Explicitly, they are obtained by restricting to $S^\ell$ the pullback foliation by the map $\exp : \nu_sL_s \rightarrow O$, where $O$ is a distinguished tubular neighborhood of $L_s$. In our case, we know only principal leaves appear near $L_s$, so these infinitesimal leaves are all principal and diffeomorphic to $\mathcal{L} = \exp^{-1}_s((\exp_s(S^\ell_\varepsilon) \cap L_p)$, where $L_p$ is a nearby leaf at fixed distance $\varepsilon$ from $L_s$. Thus, the local quotient $S^\ell/F_s$ is a manifold with quotient map a Riemannian submersion, giving the following fibration

$$\mathcal{L} \rightarrow S^\ell \rightarrow S^\ell/F_s$$

Riemannian submersions (with connected fiber) from Euclidean spheres are classified up to metric congruence: they are exactly the Hopf fibrations (see [14], [33]). In particular, for nontrivial such submersions, $\mathcal{L}$ is a round 1-, 3-, or 7-sphere and $S^\ell/F_s$ is $\mathbb{CP}^n, \mathbb{HP}^n$ or $\mathbb{OP}^2$ with their “round” metrics.

We’ve seen before that $\Sigma_{\bar{s}}$ is isometric to $(S^\ell/F_s)/\Gamma_s$ (see Proposition 3.2.3). Moreover, the action of $\Gamma_s$ is free: if not, then since there is only one orbit type, the principal isotropy must be nontrivial and it follows that $\Sigma_{\bar{s}}$ has boundary (see [34]), which is absurd. Thus, the fourth statement follows. In addition, we only see non-simply connected $\Sigma_{\bar{s}}$ (and thus $\partial(M/F)$) when the leaf holonomy $\Gamma_s$ is non-trivial, which in turn is only possible when $L_s$ is non-simply connected. That is, if the soul
leaf $L_s$ is simply connected, then the boundary of the leaf space $M/\mathcal{F}$ is a CROSS.

Consider a point $p \in M/\mathcal{F}$ along the (unique) geodesic normal from $x \in \partial(M/\mathcal{F})$ at distance $\varepsilon$ from $\partial(M/\mathcal{F})$. The leaf $L_p := \pi^{-1}(p) \subset M$ is thus at distance $\varepsilon$ from $L_x := \pi^{-1}(x)$ and by local equidistance, there are points of $L_p$ at distance $\varepsilon$ from a fixed point $x \in L_x$. These are actually diffeomorphic to the leaves of the infinitesimal foliation of the sphere normal to the stratum of $L_x$. Because there is just one normal direction to the stratum of $x \in M/\mathcal{F}$, it follows that this ‘infinitesimal leaf’ is the entire normal sphere, say $S^k$, at $x \in L_x$. Thus, $L_p$ is a union of spheres parametrized by points of $L_x$, and the closest point map $L_p \to L_x$ is a $k$-sphere bundle. This proves the fifth statement.

The final statement follows similarly to statement (i) of Theorem 4.2.3 with the added observation that $B_\varepsilon(\pi^{-1}(\partial(M/\mathcal{F})))$ is diffeomorphic to a $(k+1)$-disc bundle over $\pi^{-1}(\partial(M/\mathcal{F})))$ via the restriction of exp to the normal space to the stratum $\pi^{-1}(\partial(M/\mathcal{F})))$.

Example. It is easy to construct such SRFs on round spheres. Begin by taking any principal foliation (i.e. one with all leaves diffeomorphic and without leaf holonomy). As mentioned above, the only nontrivial such foliations are those given by the Hopf fibrations, so take, say $(S^{4n+3}, \text{Hopf } S^3')s)$. Joining this foliation with the single leaf foliation $(S^m, S^m)$ gives a foliation $(S^{4n+m+4}, \mathcal{F})$ with exactly three types of leaves: $S^m$, $S^m \times S^3$ and $S^3$. There is an $S^{4n+3}$’s worth of leaves of type $S^3$ forming a stratum for which the normal infinitesimal foliation is $(S^m, S^m)$. Hence, the image of this $S^{4n+3}$ has a single point as its space of direction normal to its stratum and thus forms a boundary face $F$ of the leaf space $S^{4n+m+4}/\mathcal{F}$. The lone $S^m$ is the soul leaf, with infinitesimal foliation exactly $(S^{4n+3}, \text{Hopf } S^3's)$. Thus, from above, we know $F$ is diffeomorphic to $\mathbb{H}P^n$. The remaining leaves form the principal stratum. Also from above, we know the leaf space $S^{4n+m+4}/\mathcal{F}$ is homeomorphic to a closed cone $C(\mathbb{H}P^n)$. One can construct such examples with any of the Hopf fibrations and arbitrary $S^m$. 44
CHAPTER 5

POINT LEAF MAXIMALS SRFS

We will now generalize results from [18] and [19] to the setting of singular Riemannian foliations, guided by the ideas and tools developed there. We analyze a special case satisfying the hypothesis of Corollary 4.3.1. Namely, when a positively curved leaf space has just one boundary strata which consists of point leaves (0-dimensional leaves).

Definition. A singular Riemannian foliation with closed leaves is said to be point leaf maximal if the infinitesimal foliation on the normal spheres to (a component of) the point leaf stratum are trivial, single leaf foliations.

We denote by $\Sigma_0$ the set of point leaves of $(M, \mathcal{F})$ and denote by $F_0 \subset \Sigma_0$, a component of $\Sigma_0$ which meets the condition of the definition above.

Remark. By definition, $F_0$ (as a subset of $M/\mathcal{F}$, into which it embeds isometrically) is a codimension 1 stratum of the Alexandrov space $M/\mathcal{F}$. Since $\partial(M/\mathcal{F})$ is the closure of the codimension 1 strata, it follows that $F_0$ is a boundary component. Also, when $\dim(M/\mathcal{F}) > 1$ and $M/\mathcal{F}$ is positively curved, the boundary is connected. So, with the exception of codimension 1 SRFs, it is convenient to think of point leaf maximal foliations as those for which $F_0 = \partial M/\mathcal{F}$. In the case that $\dim(M/\mathcal{F}) = 1$, $M/\mathcal{F}$ is an interval and $F_0$ is one of the two endpoints.

Theorem 5.0.1 (Structure of PLM SRFs). Let $(M, \mathcal{F})$ be a point leaf maximal singular Riemannian foliation such that $M/\mathcal{F}$ is positively curved. If $F_0$ is a component of $\Sigma_0$ with maximal dimension, then the following hold:
There is a unique ‘soul’ leaf, \( L_s \), at maximal distance to \( F_0 \).

All leaves in \( M - (F_0 \cup L_s) \) are principal and diffeomorphic to \( S^k \), the normal sphere to \( F_0 \).

The infinitesimal foliation \((S^\ell, F_s)\) of the normal sphere \( S^\ell \) at \( s \) is a (principal) Riemannian foliation. Moreover, \( F_0 \) is diffeomorphic to \((S^\ell/F_s)/\Gamma_s\) the space of directions at \( \bar{s} \).

\( M/\mathcal{F} \) is homeomorphic to a (closed) cone on \( F_0 \).

\( M \) is foliated diffeomorphic to a double disc bundle:

\[
M \cong D(L_s) \cup D(F_0)
\]

Proof. The theorem is a special case of Theorem 4.3.1.

Remark. We assume above that \( M/\mathcal{F} \) is positively curved (which guarantees the soul and that \( F_0 \cong \partial M/\mathcal{F} \)). However, this doesn’t make sense when \( \dim(M/\mathcal{F}) = 1 \).
In that case, one must decide what is meant by the “soul.” Certainly, \( F_0 \) will be one of the endpoints of the interval \( M/\mathcal{F} \). If we call the other endpoint the soul, then everything follows as above.

5.1. The Submanifolds \( L_s \) and \( F_0 \)

For a point leaf maximal SRF \((M, \mathcal{F})\), the submanifolds \( L_s \) and \( F_0 \) are in some sense “dual” in \( M \). We wish to classify what these two submanifolds can be, then, given the decomposition \( M = D(L_s) \cup D(F_0) \), determine which positively curved manifolds \( M \) can admit such foliations. To begin, we exhibit fiber sequences in which \( L_s \) and \( F_0 \) appear.

The quotient of the local quotient \( S^\ell/F_s \) by the holonomy action of \( \Gamma_s \) identifies the leaves of \( \mathcal{F}_s \) belonging to the same global leaf, and is isometric to the space of directions at \( \pi(L_s) \in M/\mathcal{F} \) (which is in turn diffeomorphic to \( F_0 \)). So in the case
that $\Gamma_s$ acts trivially, we have

$$\mathcal{L} \longrightarrow S^\ell \longrightarrow F_0 \tag{\ell}$$

We address the case when $\Gamma_s$ acts nontrivially in section 5.3.

On the other hand, the closest point projection map from a leaf $L_p$ (at distance $\varepsilon$ from $L_s$) to $L_s$ is a submersion (see [30]). The fiber of this map is clearly $\exp_s(S^\ell_\varepsilon) \cap L_p \cong \mathcal{L}$. This gives a fibration

$$\mathcal{L} \longrightarrow S^k \longrightarrow L_s \tag{k}$$

**Remark.** It would be nice to know what conditions imply trivial leaf holonomy about $L_s$. We saw in Section 2.2.2 that this is guaranteed when $L_s$ is simply connected. If we insist that the ambient manifold $M$ is also simply connected, then by transversality, $\text{codim}(F_0) \geq 3$ implies $L_s$ is simply connected. But $\text{codim}(F_0) = 0$ and $\text{codim}(F_0) = 1$ imply that $\mathcal{F}$ is a trivial (by points) foliation. So if $M$ is simply connected, nontrivial leaf holonomy about $L_s$ is only possible when $\text{codim}(F_0) = 2$, in which case the principal leaves are necessarily circles. This forces $\text{dim}(L_s)$ to be either 0 or 1. If $\text{dim}(L_s) = 0$, then the leaves of the foliation on $S^\ell$ are diffeomorphic to the global leaves of $\mathcal{F}$ and the holonomy action is necessarily trivial. Thus, the only possible case where nontrivial leaf holonomy about $L_s$ may occur (while $M$ is simply connected) is when $L_s$ is an exceptional circle leaf. In such a point leaf maximal SRF, the infinitesimal foliations of both the principal leaves and $L_s$ are trivial (by points), and the infinitesimal foliations of the point leaves are products of a point foliation and a concentric sphere foliation. In particular $\mathcal{F}$ is *infinitesimally polar*. Thus, by Theorem 1.8 in [25], we see that $L_s$ cannot possibly be exceptional. In short, if $(M, \mathcal{F})$ is point leaf maximal and $M$ is simply connected, there is no leaf holonomy.
5.2. The Simply Connected Case

We assume now that \((M, \mathcal{F})\) is a point leaf maximal SRF and \((M, g)\) is a simply connected, positively curved Riemannian manifold. As mentioned at the end of the previous section, this guarantees that \(L_s\) is simply connected unless \(\text{codim}(F_0) = 2\), in which case the generic leaves are circles. In this case, we have the following reformulation of Thm 1.2 in [18]:

**Theorem 5.2.1.** Let \(M\) be a simply connected positively curved manifold. If \(M\) admits a 1 dimensional point leaf maximal singular Riemannian foliation, then \(M\) is foliated diffeomorphic to a sphere \(S^n\) or a complex projective space \(\mathbb{C}P^m = S^{2m+1}/S^1\).

**Proof.** First note that such foliations are homogeneous (Thm 3.11 in [11]), but possibly with respect to a different metric. That is, there is a metric \(g'\) (not necessarily of positive curvature) and an isometric \(S^1\) action on \((M, g')\) such that the orbits coincide with the leaves of \(\mathcal{F}\). In particular, \(M/S^1\) and \(M/\mathcal{F}\) have diffeomorphic strata and isometric spaces of directions.

Now the dimension of \(L_s\) is either 0 or 1, and since there is no leaf holonomy when \(M\) is simply connected, it follows that \(L_s\) is either a principal circle leaf or a point leaf. Fix a point \(s \in L_s\) and a small metric normal sphere \(S^\ell\) centered at \(s\).

Suppose \(L_s\) is a principal circle leaf. Then \(M/\mathcal{F}\) is a \((\ell + 1)\)-disc and \(F_0 \cong S^\ell\). Flowing the normal sphere above with the horizontal lift of a gradient-like vector field for \(\text{dist}(\bar{s}, \cdot)\), \(\bar{s} \in M/\mathcal{F}\), will cut out a smooth, nonvanishing section of \(D(F_0)\), viewed as an \(S^1\)-bundle over \(F_0\). The isometric action (with respect to \(g'\)) of \(S^1\) on \(M\) orients this bundle fiberwise, making it a principal \(S^1\)-bundle with a nowhere vanishing section, hence trivial. Flowing the normal spheres over all points of \(L_s\) simultaneously then defines a diffeomorphism between \(M - (\text{int}D(L_s) \cup \text{int}D(F_0))\) and \(S^1 \times S^\ell \times (\delta, \pi/2 - \delta)\) for some small \(\delta > 0\) (i.e. the “interior” of the join \(S^1* S^\ell \cong S^{\ell+2}\)). Since both \(D(L_s)\) and \(D(F_0)\) are trivial bundles, both normal projections are trivial.
(as are those to the focal spheres in the join) and we see that $M$ is diffeomorphic to $S^{\ell+2}$. This diffeomorphism is foliated if we foliate the sphere by the $S^1$’s from the first factor.

Suppose on the other hand that $L_s$ is a point leaf. Then $S^\ell = S^{2m+1}$ and $M/F$ is a cone on $F_0 \cong \mathbb{C}P^m$. As above, flowing the horizontal lift of a gradient-like vector field for $\text{dist}(\tilde{s}, \cdot)$ will give a diffeomorphism between $M - (\text{int}D(L_s) \cup \text{int}D(F_0))$ and $S^{2m+1} \times \{pt\} \times (\delta, \pi/2 - \delta)$. Invoking the group action, which must be free on $M - (\text{int}D(L_s) \cup \text{int}D(F_0))$, we see that the restriction $\pi : \partial D(F_0) \to \partial D(\pi(F_0))$ is a Riemannian submersion from $S^{2m+1}$ to $\mathbb{C}P^m$, i.e. it is the Hopf map. This is exactly the normal projection from $\partial D(F_0)$ to $F_0$. Since the normal projection from $\partial D(L_s) \cong S^{2m+1}$ to $L_s = \{pt\}$ is necessarily trivial, we see that $M$ is diffeomorphic to $\mathbb{C}P^m$. This diffeomorphism is foliated if we foliate $\mathbb{C}P^m$ by distance spheres around the fixed point corresponding to $L_s$. \hfill \Box

Now we will assume that $\pi_1(M) = 0$ and $\text{codim}(F_0) \geq 3$. Recall that in these cases, we have the fiber sequences (l) and (k):

\[ \mathcal{L} \to S^\ell \to F_0 \quad (\ell) \]

\[ \mathcal{L} \to S^k \to L_s \quad (k) \]

As we mentioned previously, as long as (l) is a nontrivial fibration (i.e. $\mathcal{L} \neq \{pt\}$ and $\mathcal{L} \neq S^\ell$), then the fiber $\mathcal{L}$ is isometric to $S^1$, $S^3$ or $S^7$ (and $S^7$ can only occur when $\ell = 15$). Moreover, the space of directions at the soul of $M/F$ (call it $\Sigma$) is isometric to a compact rank one symmetric space (CROSS). Thus, $F_0$ is diffeomorphic to a CROSS unless $\mathcal{L} = S^\ell$, in which case $F_0$ is a point. On the other hand, because the sphere $S^k$ is NOT a standard round sphere, the possibilities for $L_s$ are the same, but only up
to cohomology ring (this can be seen via the Gysin sequence, for example). This is a key difference between point leaf maximal SRF’s and fixed point homogeneous group actions (where both submanifolds are known up to diffeomorphism).

Cases can now be generated by the topological restrictions imposed by the fibrations \((\ell)\) and \((k)\). Based on the dimension of \(L\), we get the following possibilities for \(F_0\) (up to diffeomorphism) and \(L_s\) (up to cohomology ring):

**TABLE 5.1**

POSSIBILITIES FOR \(F_0\) AND \(L_s\)

<table>
<thead>
<tr>
<th>Case</th>
<th>(\text{dim}(L))</th>
<th>(S^\ell)</th>
<th>(F_0)</th>
<th>(S^k)</th>
<th>(L_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(l = k)</td>
<td>(S^k)</td>
<td>({pt})</td>
<td>(S^k)</td>
<td>({pt})</td>
</tr>
<tr>
<td>(B)</td>
<td>0</td>
<td>(S^\ell)</td>
<td>(S^\ell)</td>
<td>(S^k)</td>
<td>(S^k)</td>
</tr>
<tr>
<td>(C)</td>
<td>1</td>
<td>(S^{2m+1})</td>
<td>(\mathbb{C}P^m)</td>
<td>(S^{2j+1})</td>
<td>(\mathbb{C}P^j)</td>
</tr>
<tr>
<td>(D)</td>
<td>3</td>
<td>(S^{4m+3})</td>
<td>(\mathbb{H}P^m)</td>
<td>(S^{4j+3})</td>
<td>(\mathbb{H}P^j)</td>
</tr>
<tr>
<td>(E)</td>
<td>7</td>
<td>(S^7, S^{15})</td>
<td>({pt}, S^8)</td>
<td>(S^{15}, S^7)</td>
<td>(S^8, {pt})</td>
</tr>
<tr>
<td>(X)</td>
<td>7</td>
<td>(S^{15})</td>
<td>(S^8)</td>
<td>(S^{15})</td>
<td>(S^8)</td>
</tr>
</tbody>
</table>

Evidently, the larger \(F_0\) is relative to \(L_s\), the more we can say about \(M\). This is particularly interesting when \(L_s = \{pt\}\), where we get a homeomorphism classification of \(M\), by part (v) of Theorem 5.0.1:

**Theorem 5.2.2** (PLM with point soul). Let \(M\) be a simply connected positively curved manifold. If \(M\) admits a point leaf maximal singular Riemannian foliation whose soul leaf is a point, then \(M\) is foliated homeomorphic to a CROSS.

Cases (A) and (B) correspond to the two “trivial” fibrations \((\ell)\) and \((k)\). We address
Theorem 5.2.3. Let $M$ be a simply connected positively curved manifold. If $M$ admits a codimension 1 point leaf maximal SRF such that the soul leaf is a point, then $M$ is foliated homeomorphic to a sphere.

Proof. In this case, we have that the endpoints of $M/\mathcal{F}$ correspond to two isolated point leaves of $\mathcal{F}$. Thus, $M$ is two $(k+1)$-discs foliated by concentric $S^k$ glued along their common boundary $S^k$. This is clearly foliated homeomorphic to $S^{k+1}$ (foliated by ‘lateral’ subspheres). Depending on the gluing map $M$ may potentially be an exotic sphere. It would be interesting to know under what conditions exotic spheres arises.

Next up is case (B).

Theorem 5.2.4. Let $M$ be a simply connected positively curved manifold. If $M$ admits a point leaf maximal SRF such that the soul leaf is a principal leaf, then $M$ is foliated homeomorphic to a sphere.

Proof. Although we do not use it, note that in this case $M/\mathcal{F}$ is diffeomorphic to an $(\ell + 1)$-disc. This is because $L_s$ in this case is itself a principal leaf (diffeomorphic to $S^k$), so $M/\mathcal{F}$ is a smooth manifold (with boundary) whose boundary is isometric to a sphere $F_0 \cong S^\ell$.

By Theorem [4.3.1], we know that $M$ has a foliated diffeomorphism double disk decomposition: $M \cong D(L_s) \cup D(F_0)$. We know the diffeomorphism type of both $L_s$ and $F_0$ in this case as well ($L_s$ is principal, hence diffeomorphic to $S^k$). Moreover, we know $D_\varepsilon(L_s) \cong S^k \times D^{\ell+1}$, foliated by $S^k$’s from the first factor. As a model space, consider the join foliation $(S^\ell, \{pts\}) \ast (S^k, S^k)$ on $\hat{M} = S^{\ell+k+1}$. This is a point leaf maximal SRF, with $\hat{L}_s = S^k$ and $\hat{F}_0 = S^\ell$. The tubular neighborhood about $\hat{L}_s$ is $S^k \times D^{\ell+1}$, as in our case. Clearly, this is diffeomorphic to $D(L_s)$, and in particular, the restriction to the boundary of this tubular neighborhood $S^k \times S^\ell$ is
also a diffeomorphism. We can then use the flows on \( \widehat{M} \) and \( M \) of the gradient-like vector fields for \( \text{dist}(L_s, \cdot) \) and \( \text{dist}(L_s, \cdot) \), respectively, to uniquely extend this to a homeomorphism from \( \widehat{M} \) to \( M \). This homeomorphism a foliated one if one foliates \( \widehat{M} = S^\ell \times S^k \) by the \( S^k \)'s from the second factor.

The following more general statement has a weaker conclusion than the results above, so it suffices to prove it for the remaining cases. The arguments below are purely algebraic topological and “symmetric” in \( F_0 \) and \( L_s \), so we will assume without loss of generality that \( \dim(F_0) \leq \dim(L_s) \) (or equivalently, \( \ell \leq k \)).

**Theorem 5.2.5.** Let \( M^n \) be a simply connected positively curved manifold. If \( M \) admits a point leaf maximal SRF, then \( M \) has the cohomology ring of a CROSS.

**Proof.** Consider the following long exact sequence in relative cohomology:

\[
\cdots \to H^{i-1}(D(L_s)) \to H^i(M, D(L_s)) \to H^i(M) \to H^i(D(L_s)) \to H^{i+1}(M, D(L_s)) \to \cdots
\]

Now \( H^i(M, D(L_s)) \cong \widetilde{H}^i(M/D(L_s)) \), and given that \( M = D(F_0) \cup E D(L_s) \), we see that \( M/D(L_s) \cong D(F_0)/S(F_0) \), where \( S(F_0) \) denotes the \( k \)-sphere bundle over \( F_0 \). This is precisely the Thom space of the rank \( k+1 \) vector bundle over \( F_0 \) which induces \( D(F_0) \). By the Thom isomorphism theorem, we have that

\[
H^i(M, D(L_s)) \cong H^{i-(k+1)}(F_0)
\]

and since \( D(L_s) \) deformation retracts to \( L_s \), we have the long exact sequence

\[
\cdots \to H^{i-1}(L_s) \to H^{i-(k+1)}(F_0) \to H^i(M) \to H^i(L_s) \to H^{i+1-(k+1)}(F_0) \to \cdots
\]

For dimension reasons alone, we recover the following cohomology groups:

(C) \( M \) has additive cohomology of \( \mathbb{C}P^{3(\ell+k)} \)}
(D) $M$ has additive cohomology of $\mathbb{H} \mathbb{P}^{1(l+k-2)}$

(E) $M$ has additive cohomology of $\mathbb{O} \mathbb{P}^2$

(X) $H^i(M) = \begin{cases} \mathbb{Z} & i = 0, 8, 16, 24 \\ 0 & \text{otherwise} \end{cases}$

In the cases where $l < k$, the ring structure is recovered via Poincaré duality: the isomorphisms $H^i(M) \cong H^i(L_s)$ for $i \leq k - 1$ induced by the inclusion $L_s \hookrightarrow M$ induce a ring isomorphism up to $i = \dim(L_s)$. In cases (C)-(E), $\dim(L_s) \geq \frac{1}{2} \dim(M)$, so all cup products are determined.

For example, in case (C), we know that up to $i = k - 1$, the cohomology ring is generated by an element $[\alpha] \in H^2(M)$. Since $k - 1 \geq \frac{1}{2} \dim(M)$, Poincaré duality gives that the top cohomology $H^n(M)$ can be generated by cup products of generators below dimension $k - 1$, which are powers of $[\alpha]$. It necessarily follows that all cohomology groups are generated by powers of $[\alpha]$.

Remark. At this point, most cases are settled, but special care needs to be taken when $l = k$ since Poincaré duality fails to determine all cup products. We will exhibit the specifics in case (C) when $\ell = k$. The proof works the same way in the other cases, with minor adjustments.

So we have $l = k = 2m + 1$, $F_0 = \mathbb{C} \mathbb{P}^m$, $L_s \sim \mathbb{C} \mathbb{P}^m$, and we know $M$ has the additive cohomology of $\mathbb{C} \mathbb{P}^k$. As we saw above, we have a ring isomorphism between $H^*(L_s)$ and $H^*(M)$ up to dimension $i = k - 1 = 2m$. So we have an element $[\alpha] \in H^2(M)$ for which $[\alpha]^m$ generates $H^{k-1}(M)$. Poincaré duality only tells us that there exists a generator of $H^{k+1}(M)$ whose cup product with $[\alpha]^m$ generates the top cohomology, but it does not guarantee that this generator is equal to $[\alpha]^{m+1}$. For this, we will exhibit a topologically embedded submanifold $N \cong \mathbb{C} \mathbb{P}^{m+1}$ whose inclusion induces a ring isomorphism with $H^*(M)$ up to dimension $i = 2m + 2 = k + 1 \geq \frac{1}{2} \dim(M)$. Then apply Poincaré duality as before.
Consider a small minimal geodesic segment emanating from \( \pi(L_s) \in M/F \). The other endpoint of this segment can be flowed along the gradient-like vector field for \( \text{dist}(F_0, \cdot) \) (which is radial near \( F_0 \)) to a nearest point \( p \in F_0 \). This gives an integral curve \( \gamma \) in \( M/F \) which can be ‘covered’ by two half curves \((\gamma_1 \text{ from } p \text{ to the midpoint } x \in \gamma, \text{ and } \gamma_2 \text{ from } x \text{ to } \pi(L_s)) \). Thus \( \pi^{-1}(\gamma) \) is covered by the preimages of these two half segments. The preimage \( \pi^{-1}(\gamma_1) \) is a \((2m+2)\)-disc centered at \( p \) and normal to \( F_0 \), whereas \( \pi^{-1}(\gamma_2) \) is a subbundle of \( D(L_s) \) (the fiber being the 2-disc bounded by a smoothly varying \( L \) at each point). In fact, the boundary of this subbundle is simply a global leaf \( S^k \) (it is the preimage of the midpoint \( x \in M/F \)). This gives a submanifold \( N \subset M \) whose cohomology can be attained from the Mayer-Vietoris sequence as

\[
\cdots \to H^{i-1}(S^k) \to H^i(N) \to H^i(\{pt\}) \oplus H^i(L_s) \to H^i(S^k) \to \cdots
\]

from which we see that \( N \) has the cohomology ring of \( L_s \sim \mathbb{C}P^m \) up to \( i = k - 1 \), for dimension reasons. Since \( \dim(N) = k + 1 = 2m + 2 \), it follows from Poincaré duality that \( N \) has the cohomology ring of \( \mathbb{C}P^{m+1} \). We have inclusions \( N \hookrightarrow M \) and \((N, D(L_s) \cap N) \hookrightarrow (M, D(L_s)) \) and thus a sequence of maps

\[
\begin{array}{ccccccccc}
& & & & & & & & \downarrow j^* & \\
\to & H^{i-1}(L_s) & \to & H^i(M, D(L_s)) & \to & H^i(M) & \to & H^i(L_s) & \to & H^{i+1}(M, D(L_s)) & \\
& & & & & & & & \downarrow \cong & \\
\to & H^{i-1}(L_s) & \to & H^i(N, D(L_s) \cap N) & \to & H^i(N) & \to & H^i(L_s) & \to & H^{i+1}(N, D(L_s) \cap N) & \\
\end{array}
\]

If we can show that the induced map \( H^*(M, D(L_s)) \xrightarrow{j^*} H^*(N, D(L_s) \cap N) \) is a ring isomorphism up to dimension \( i = k + 2 \), we are done, by the five lemma. Now the map \( j^* \) fits in the following diagram (“unpacking” the Thom isomorphism)

\[
\begin{array}{ccccccccc}
H^i(M, D(L_s)) & \xleftarrow{\text{exc}} & H^i(D(F_0), \partial D(F_0)) & \xrightarrow{\text{Thom}} & H^{i-(k+1)}(D(F_0)) \\
\downarrow j^* & & \downarrow \eta^* & & \downarrow \eta^* \\
H^i(N, D(L_s) \cap N) & \xleftarrow{\text{exc}} & H^i(D^{k+1}, S^k) & \xrightarrow{\text{Thom}} & H^{i-(k+1)}(D^{k+1}(\{pt\}))
\end{array}
\]

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where the rightmost $\eta^*$ is technically a restriction of the middle $\eta^*$, both of which are induced by the inclusion $(D^{k+1}(\{pt\}), S^k) \hookrightarrow (D(F_0), \partial D(F_0))$. Now since $H^{i-(k+1)}(F_0) = 0 = H^{i-(k+1)}(\{pt\})$ for $i \leq k$ and $i = k + 2$, we need only check that the diagram above commutes for $i = k+1$ ($j^*$ is certainly an isomorphism when its domain and target are both 0). The rightmost $\eta^*$ is an isomorphism between $H^0(D(F_0))$ and $H^0(D^{k+1}(\{pt\}))$, since both are path connected. Now if $\tau \in H^{k+1}(D(F_0), \partial D(F_0))$ is the Thom class of $D(F_0)$, its restriction $\eta^*(\tau) \in H^{k+1}(D^{k+1}, S^k)$ is the Thom class of $D^{k+1}(\{pt\})$ and since the Thom isomorphism is the cup product with the Thom class, the right square commutes by naturality of the cup product.

Thus, we conclude that the induced map $H^*(M, D(L_s)) \xrightarrow{j^*} H^*(N, D(L_s) \cap N)$ is a ring isomorphism up to dimension $i = k+2$, as desired. Now Poincaré duality gives the correct cohomology ring $H^*(M) \cong \mathbb{Z}[x]/x^{k+1}$ in case (C) when $\ell = k$. Similarly for case (D) when $\ell = k$. This also shows that in case (X), $H^*(M) \cong \mathbb{Z}[x]/x^4$. But in this case, $|x| = 8$, which is impossible (see Corollary 4.L.10 in [23]). So case (X) can be thrown out. This proves the first statement.

In the special case that $L_s = \{pt\}$, then $D(L_s)$ is diffeomorphic to a disc foliated by concentric $S^k$‘s. The boundary of this disc is diffeomorphic to a regular leaf $S^k$. This diffeomorphism has a unique radial extension “down to” $F_0$, which is foliated as well, by the homothetic transformation lemma and the definition of point leaf maximal. This describes a foliated homeomorphism from $M$ to a CROSS of the same ‘type’ as $F_0$ (of one higher dimension).

An important aspect of point leaf maximal SRFs is that they ensure a double disk bundle decomposition of the manifold $M$. In low dimensions, the work of Ge and Radeschi in [12] gives us the following stronger result for 4-manifolds.

**Corollary 5.2.6.** If $M^4$ is a simply connected, positively curved manifold admitting a point leaf maximal singular Riemannain foliation, then $M^4$ is diffeomorphic to $S^4$ or $\mathbb{CP}^2$. 

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5.3. The Non-simply Connected Case

If \((M, \mathcal{F})\) is a point leaf maximal SRF and \(M\) is not simply connected, we can pullback the foliation to the universal cover \(\tilde{M}\) and use the results above. We have

**Theorem 5.3.1.** If \(M\) is a non-simply connected, positively curved manifold which admits a point leaf maximal SRF, then \(M\) is a \(\mathbb{Z}_2\)-quotient of an odd dimensional cohomology complex projective space or is a topological space form \(S^n/\Gamma\). Moreover, both \(F_0\) and \(L_s\) have the same fundamental group as \(M\).

**Proof.** Let \((M, \mathcal{F})\) be our point leaf maximal SRF and consider its pullback foliation to the universal cover \((\tilde{M}, \tilde{\mathcal{F}})\). The condition of being point leaf maximal is a local one, hence holds for \((\tilde{M}, \tilde{\mathcal{F}})\). So we know from the work above that \(\tilde{M}\) is homeomorphic to a sphere or has the cohomology ring of a CROSS.

Denote by \(\tilde{L}_s\) and \(\tilde{F}_0\) the soul leaf and point leaf component (that fits the definition) for \(\tilde{\mathcal{F}}\). Because \(\tilde{\mathcal{F}}\) is the pullback foliation of \(\mathcal{F}\), its leaves are preserved by the action of \(\pi_1(M)\) on \(\tilde{M}\) and thus

\[
(\tilde{M}/\tilde{\mathcal{F}})/\pi_1(M) = M/\mathcal{F}
\]

and in particular, \(\tilde{F}_0/\pi_1(M) = F_0\) and \(\tilde{L}_s/\pi_1(M) = L_s\) (neither \(F_0\) nor \(L_s\) is simply connected). Moreover, since \(\pi_1(M)\) is finite (Bonnet-Myers) and acts freely by isometries on \(\tilde{M}\), we must have that \(\tilde{M}\) is homeomorphic to a sphere or is an odd-dimensional cohomology complex projective space. We also know that \(\tilde{F}_0\) and \(\tilde{L}_s\) admit free finite actions by \(\pi_1(M)\) as well. It follows that \(\tilde{F}_0\) (resp. \(\tilde{L}_s\)) is diffeomorphic to (resp. has the cohomology ring of) a sphere or an odd dimensional complex projective space. In the case that \(\tilde{M}\) is homeomorphic to a sphere, \(M\) is homeomorphic to a space form, \(F_0\) is diffeomorphic to a space form, and \(L_s\) has the cohomology ring of a space form. On the other hand, if \(\tilde{M}\) has the cohomology of an odd dimensional complex projective space, then so does \(\tilde{L}_s\), and \(F_0\) is diffeomorphic to an
odd dimensional complex projective space and $\pi_1(M) = \mathbb{Z}_2$. Finally, since the $\pi(M)$ action is a covering action, we have that $\pi_1(F_0) \cong \pi_1(\tilde{F}_0/\pi_1(M)) \cong \pi_1(M)$, and similarly, $\pi_1(L_\delta) \cong \pi_1(M)$.

The following special results are proved the same way, but have stronger conclusions coming from the stronger conclusions in the simply connected case:

**Corollary 5.3.2.** If $M$ is a non-simply connected, positively curved manifold admitting a 1-dimensional point leaf maximal SRF, then $M$ is foliated diffeomorphic to a space form $S^n/\Gamma$ or a $\mathbb{Z}_2$-quotient of an odd dimensional complex projective space.

**Corollary 5.3.3.** If $M$ is a non-simply connected, positively curved manifold admitting a point leaf maximal SRF such that the soul leaf is either a point or a principal leaf, then $M$ is foliated homeomorphic to a space form $S^n/\Gamma$.

**Remark.** The only finite quotients of spheres in the classification of fixed point homogeneous manifolds are quotients by cyclic groups $\mathbb{Z}_q$, or finite subgroups of $SU(2)$ or $Sp(1)$ (see [19]). However, any finite group $\Gamma$ which acts freely on some $S^k$ may appear in the foliation setting. That is, there are point leaf maximal singular Riemannian foliations that are not induced by a fixed point homogeneous group action. To see this, let $\mathcal{F}$ be the SRF given by the join $(S^{2k+1}, \mathcal{F}) \cong (S^k, \{pts\}) \ast (S^k, S^k)$. This foliation is point leaf maximal - there is a stratum of point leaves (the ‘left’ $S^k$) and the infinitesimal foliation normal to this stratum is the trivial $(S^k, S^k)$. Suppose $\Gamma$ acts freely on $S^k$. Then the diagonal action of $\Gamma$ on $S^{2k+1}$ is free and preserves $\mathcal{F}$, and hence $\mathcal{F}$ descends to a point leaf maximal foliation on the space form $S^{2k+1}/\Gamma$. 

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6.1. Hsiang-Kleiner for SRF’s

Given the theory developed so far, the theorem of Hsiang-Kleiner [24] generalizes easily to Singular Riemannian foliations. That is,

**Theorem 6.1.1** (Hsiang-Kleiner for SRFs). A closed, simply connected, positively curved 4-manifold admitting a 1-dimensional singular Riemannian foliation with closed leaves is foliated diffeomorphic to $S^4$ and $\mathbb{CP}^2$, foliated by the orbits of a linear circle action.

**Remark.** This theorem just a slight strengthening of Theorem H in [11]. Namely, the theorem here classifies such manifolds up to foliated diffeomorphism.

**Proof.** The proof in this setting is virtually identical to the original proof of Hsiang-Kleiner in [24]. Their proof in the group action setting relies on analyzing the fixed point set of the action and, in this regard, the settings are interchangeable. This is because such a foliation $(M, F)$ is homogenous (cf. Thm 3.11 in [11]), though possibly with respect to a different metric. So even though the theorem above presumes no isometric group action, there exists an $S^1$ action and an invariant metric (not necessarily positively curved) such that the orbits coincide with the leaves of $F$ and the strata are diffeomorphic. In particular, any topological facts about fixed point sets will hold for the point leaf set of $F$ (e.g. those concerning Euler characteristic, codimension, orientability, etc.). Moreover, the point leaf set of an SRF is a closed,
totally geodesic submanifold. Thus, the point leaf set of $\mathcal{F}$ behaves like the fixed point set of an isometric group action for all purposes necessarily to borrow the same proof. As in [24], we arrive at two scenarios: either the point leaf set of $\mathcal{F}$ (call it $F$) contains exactly one $S^2$ and some isolated point leaves, or $F$ consists of isolated point leaves:

Suppose $F$ contains an $S^2$. Then the normal sphere to this stratum is $S^1$, infinitesimally foliated either by points or by a single leaf. If the infinitesimal foliation were by points, then the entire normal space of $S^2$ would be foliated by leaves of the same dimension as those in $S^2 \subset F$, a contradiction (this would extend the alleged strata of $F$). So the normal $S^1$ to a component of the point leaf set is foliated by a single leaf (i.e. the foliation is point leaf maximal). Applying Corollary 5.2.6 gives the result in this case.

Now suppose the point leaf set consists of isolated point leaves. The infinitesimal foliation $(S^3, \mathcal{F})$ near each such point leaf is a 1-dimensional Riemannian foliation of the round sphere $S^3$. In fact, since the foliation $(M, \mathcal{F})$ is homogenous, all infinitesimal foliations are homogeneous, and in particular, induced by linear actions on round spheres. Thus, the work of Hsiang and Kleiner in analyzing the local geometry near the fixed points carries over directly to the foliation setting. In particular, Lemma 4 of [24], which implies that the perimeter of any geodesic triangle in the space of directions at an isolated fixed point is at most $\pi$. For a homeomorphism classification, the remainder of the theorem then boils down to Alexandrov geometry.

In $M/\mathcal{F}$, the space of directions $\Sigma_p$ at an isolated point leaf is ‘smaller than’ $S^2(\frac{1}{2}) = S^3/$Hopf (in the sense that there is a distance nonincreasing bijection $S^2(\frac{1}{2}) \rightarrow \Sigma_p$). If there were four isolated point leaves, then by looking at the angles in the four triangles formed by these points, we will reach a contradiction. On one hand, each triangle has a angle sum strictly greater than $\pi$, since $M/\mathcal{F}$ is a positively curved Alexandrov space, giving a total angle sum (among all four triangles)
strictly greater than $4\pi$.

But this sum can also be counted by adding the angles at each isolated point leaf. Here, the angle between two geodesics is precisely the distance in the space of directions $\Sigma_p$ between representatives for the geodesics. Thus, the sum of the angles at a point corresponds to the perimeter of a geodesic triangle in $\Sigma_p$, which as mentioned above, is at most $\pi$. Summing over the four isolated point leaves gives a total sum of at most $\pi$, contradicting the sum above. So there can be at most 3 isolated fixed points. So the Euler characteristic of $F$ is either 2 or 3. Now apply Freedman’s work as in [24] to get the homeomorphism classification.

We achieve a diffeomorphism classification as Grove and Wilking did in [20]. We already have the result up to diffeomorphism in the case where the point leaf set contains an $S^2$. In the case when the point leaf set consists of isolated fixed points, we look at the infinitesimal foliation at each. Because the ambient foliation is homogeneous, the infinitesimal foliations are as well, and in particular induced by linear, almost free actions $S^1$ actions on $S^3$. So $M/F$ is topologically a 3-sphere (Lemma 2-1 of [20]). These infinitesimal foliations can have at most 2 isolated exceptional circle leaves, and they correspond to 1-dimensional strata of $M/F$ joining distinct isolated point leaves. The union of closures of such curves forms a circle in $S^3$ which is unknotted. From this it follows that $M^4$, with the $S^1$ action which induces $F$, is equivariantly diffeomorphic to a linear action on $S^4$ or $\mathbb{C}P^2$, and hence $(M,F)$ is foliated diffeomorphic to $S^4$ or $\mathbb{C}P^2$ foliated by the orbits of such actions.
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