A NEW APPROACH FOR MODELING GRAVITATIONAL RADIATION FROM 
THE INSPIRAL OF TWO NEUTRON STARS

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Abstract

by

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In this dissertation, a new method of applying the ADM formalism of general relativity to model the gravitational radiation emitted from the realistic inspiral of a neutron star binary is described. A description of the conformally flat condition (CFC) is summarized, and the ADM equations are solved by use of the CFC approach for a neutron star binary. The advantages and limitations of this approach are discussed, and the need for a more accurate improvement to this approach is described.

To address this need, a linearized perturbation of the CFC spatial three metric is then introduced. The general relativistic hydrodynamic equations are then allowed to evolve against this basis under the assumption that the first-order corrections to the hydrodynamic variables are negligible compared to their CFC values. As a first approximation, the linear corrections to the conformal factor, lapse function, and shift vector are also assumed to be small compared to the extrinsic curvature and the three metric.
A boundary matching method is then introduced as a way of computing the gravitational radiation of this relativistic system without use of the multipole expansion as employed by earlier applications of the CFC approach. It is assumed that at a location far from the source, the three metric is accurately described by a linear correction to Minkowski spacetime. The two polarizations of gravitational radiation can then be computed at that point in terms of the linearized correction to the metric.

The evolution equations obtained from the linearized perturbative correction to the CFC approach and the method for recovery of the gravity wave signal are then tested by use of a three-dimensional numerical simulation. This code is used to compute the gravity wave signal emitted a pair of equal mass neutron stars in quasi-stable circular orbits at a point early in their inspiral phase. From this simple numerical analysis, the correct general trend of gravitational radiation is recovered. Comparisons with \( \frac{5}{2} \) post-Newtonian solutions show a similar gravitational waveform, although inaccuracies are still found to exist from this computation. Finally, several areas for improvement and potential future applications of this technique are discussed.
For Tanu, whose patience and faith in me I will never be able to repay.
CONTENTS

Figures.......................................................................................................................................................... v

Tables ............................................................................................................................................................vi

Acknowledgments........................................................................................................................................vii

Chapter 1: Introduction and Overview ................................................................................................. 1
  1.1. Motivation ........................................................................................................................................ 1
  1.2. Neutron Star Binary Model ............................................................................................................. 6
  1.3. Perturbation Theory ........................................................................................................................ 10
  1.4. Recovery of the Gravity Wave Signal .............................................................................................. 11
  1.5. A Simple Test of the Model ............................................................................................................ 12
  1.6. Conclusions and Future Applications ............................................................................................. 14
  1.7. References ..................................................................................................................................... 14

Chapter 2: Neutron Star Binary Model ................................................................................................. 16
  2.1. Introduction ..................................................................................................................................... 16
  2.2. Basic Framework and Assumptions ............................................................................................... 18
    2.2.1. ADM Formalism ..................................................................................................................... 18
    2.2.2. Conformally Flat Condition ................................................................................................. 23
    2.2.3. Hydrodynamic Variables ....................................................................................................... 26
    2.2.4. Multipole Expansion ............................................................................................................. 29
  2.3. Relativistic Field Equations ........................................................................................................... 30
    2.3.1. Conformal Factor .................................................................................................................. 30
    2.3.2. Lapse Function ....................................................................................................................... 31
    2.3.3. Shift Vector ............................................................................................................................ 33
  2.4. Conclusion ...................................................................................................................................... 34
  2.5. References ..................................................................................................................................... 35

Chapter 3: Perturbation Theory ................................................................................................................. 37
  3.1. Introduction ..................................................................................................................................... 37
  3.2. Assumptions ..................................................................................................................................... 39
    3.2.1. Gauge Choices .......................................................................................................................... 39
    3.2.2. Simplifying Assumptions ......................................................................................................... 41
  3.3. Evolution Equations ......................................................................................................................... 42
    3.3.1. Correction to the Inverse Metric ............................................................................................ 42
3.3.2. Evolution of the Three Metric ............................................................... 44
3.3.3. Evolution of the Extrinsic Curvature .................................................... 45
3.4. Conclusion ...................................................................................................... 49
3.5. References ...................................................................................................... 50

Chapter 4: Recovery of the Gravity Wave Signal ..................................................... 51
4.1. Introduction .................................................................................................... 51
4.2. The Weak Field Approximation ................................................................... 53
4.2.1. Coordinate Transformations .................................................................. 54
4.2.2. Weak Field Einstein Equations ............................................................. 55
4.3. The Plane Wave through Vacuum ................................................................. 57
4.3.1. General Form of the Wave ................................................................. 57
4.3.2. The Transverse-Traceless (TT) Gauge .................................................. 57
4.4. The Spherical Wave from a Source ............................................................... 60
4.4.1. Simplifying Assumptions ...................................................................... 60
4.4.2. Definition of the Wave Amplitude ........................................................ 63
4.4.3. The TT Gauge for Spherical Waves ...................................................... 66
4.5. Extracting the Wave in the Model ................................................................. 67
4.5.1. Boundary Matching Condition .............................................................. 68
4.5.2. Gauge Transformations ......................................................................... 69
4.6. Conclusion ...................................................................................................... 70
4.7. References ...................................................................................................... 73

Chapter 5: A Simple Test of the Theory .................................................................... 74
5.1. Introduction .................................................................................................... 74
5.2. The Numerical Code ...................................................................................... 77
5.2.1. Initial Parameters ................................................................................... 78
5.2.2. The Subroutine ...................................................................................... 82
5.2.3. The Main Program ................................................................................ 85
5.3. Numerical Simulation .................................................................................... 87
5.3.1. The Search for Stability ......................................................................... 87
5.3.2. Results ................................................................................................... 90
5.4. Conclusion ...................................................................................................... 96
5.5. References .................................................................................................... 98

Chapter 6: Conclusions and Future Applications ..................................................... 99
6.1. Summary ........................................................................................................ 99
6.2. Continuing Work .......................................................................................... 101
6.3. Future Applications ...................................................................................... 102
6.4. References .................................................................................................... 103
FIGURES

Figure 1.1: Shift of periastron of the Hulse-Taylor pulsar due to decrease in period, and comparison with general relativity ................................................................. 2

Figure 1.2: Schematic drawing of LIGO gravitational wave detector .......................... 3

Figure 2.1: Schematic of ADM foliation of space and time ......................................... 19

Figure 2.2: Demonstration of the stability of the CFC approach. The system modeled is that of a circular binary system with negligible energy loss due to gravitational radiation. The numerical convergence to a stable orbital velocity is apparent .... 24

Figure 2.3: The ratio of energy lost to total mass for binary systems of varying orbital separations for several equations of state. The CFC solution converges with PN solutions at large separation, but shows a more significant energy loss than PN calculations as the stars approach ........................................................................ 25

Figure 5.1: Schematic diagram of the plane of the orbit of two neutron stars. The initial zones where the stars are located are indicated in this diagram, and the location of the gravity wave extraction point is indicated. The spatial resolution in the z-direction is 200 times less than that in the x- or y-directions, to accommodate the need for gravity wave extraction at a location far from the source ....................... 79

Figure 5.2: The plus polarization of gravitational radiation obtained via the post-CFC approach, and the corresponding waveform from the 2.5PN scheme............... 93

Figure 5.3: The cross polarization of gravitational radiation obtained via the post-CFC approach, and the corresponding waveform from the 2.5PN scheme............... 93
TABLES

Table 5.1: List of Parameters Used in the 2.5PN Calculations ........................................ 91
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CHAPTER 1:  
INTRODUCTION AND OVERVIEW

1.1. Motivation

The existence of gravitational waves has been predicted almost since the birth of General Relativity (GR) in 1916. However, the magnitude of any gravitational wave arriving here on Earth is small and could not be detected directly until recently. As with virtually every prediction specific to general relativity, the early proofs of the validity of the theory were indirect in nature. More than a dozen pulsar binaries have been observed.\(^1,2,3\) In particular the observed\(^4\) energy loss of the Hulse-Taylor Pulsar (PSR B1913+16) appears to match within errors the expected\(^5\) energy loss from gravitational waves. Figure 1.1 shows the remarkable agreement of the observed gravitational wave effect on this pulsar with that predicted by GR. Recently, the first ever double-pulsar binary (J0737-3039) has been observed,\(^6\) verifying beyond doubt the existence of double-neutron-star binary systems. These observations and calculations have led us to believe that GR is indeed the correct theory of gravity (at least to current measurement accuracy capabilities), and that gravitational waves are emitted by neutron-star binary systems in our Galaxy and beyond.

Recently, a group of ground based interferometers has begun taking data (e.g. LIGO, GEO 600, VIRGO, and TAMA 300).\(^7,8\) A schematic drawing of the LIGO detector is shown in Figure 1.2. These gravitational detectors, although not the first ever
created, go to great lengths to reduce the random seismic and thermal variations inherent with any ground-based gravitational wave detector. These detectors carry high expectations that they will provide the first direct observations of gravitational waves. A new series of ground-based detectors is currently being developed, headed by Advanced LIGO, which will be able to probe a different frequency spectrum and carry greater sensitivity to a signal than the first set of detectors. In addition, proposals for space-based detectors (e.g. LISA) have already passed initial approval and are currently in the design phase. Needless to say, this is an exciting time for the field of gravitational wave astronomy.
More importantly, gravitational wave astronomy has the potential to lead to new discoveries in astrophysics and cosmology. Historically speaking, virtually every major breakthrough in discovery methods in astronomy and science has led to observations of new physical phenomena and posed new questions about previously observed phenomena. This new branch of astronomy should be the same. Furthermore, gravitational radiation is expected to be significant in a very different frequency band than electromagnetic radiation ($10^{-18}$ Hz to $10^4$ Hz, as opposed to $10^7$ Hz to $10^{24}$ Hz for electromagnetic radiation). Gravitational radiation, like neutrino emission, interacts very weakly with matter, and can therefore be expected to reach us virtually unmodified from the source. As a result, we can expect gravitational waves to carry very different information about astrophysical sources of strong field gravity.

The potential for new discoveries and new questions is, of course, exciting for any theorist, but gravitational wave astronomy is somewhat unique in that the very methods
of detection require theoretical predictions. Any ground-based observation of gravity waves is a monumental task. This is in large part due to one of the very aspects of gravitational radiation that makes it so exciting; namely, the fact that gravity waves interact so very weakly with matter. Furthermore, it would take immense amplitudes of radiation to produce a signal large enough to be detected above the noise of even the best engineered detector. To produce an effective signal-to-noise ratio, these ground-based detectors must employ matched filtering techniques\textsuperscript{15} to amplify the signal received to detectible levels. With a sufficiently accurate theoretical template, observers should be able to extract the signal matching this template out of the noise by using fast Fourier transform techniques.

Of course, in astrophysics, one cannot craft well-controlled experiments of astrophysical phenomena to test the validity of a specific hypothesis; instead, one must rely on observations of naturally occurring phenomena and apply theoretical knowledge and understanding to explain them. The necessity of a solid theoretical prediction to detect gravity waves via ground based detectors is, to me, one of the most exciting aspects of gravitational wave astronomy and the main motivation of this thesis work.

Furthermore, if the signal is significant enough, an inaccurate template is unlikely to identify a spurious signal. Combining a template with an observed data set is more like multiplying than adding; if a signal exists in the data which matches the template, the two will constructively interfere, and the signal will be drawn out of the noise. If the accepted theoretical template does not match a signal in the data, though, one would see minor random correlations, but no clear signal.\textsuperscript{12} Therefore, if the theoretical template used is inaccurate, it will be difficult to see any gravitational wave effect, even if one is
present in the data. Unfortunately (or fortunately, depending on your viewpoint), this
puts a lot of pressure on theoretical astrophysicists; we need to make sure we have our
templates right! This thesis outlines an effort by the author to rise to this challenge by
improving upon the accuracy and applicability of existing theoretical templates
describing one specific source for gravitational radiation. Ultimately this approach could
provide a solid framework for application to various other sources as well.

The first chapter of this thesis is intended to provide motivation for this field and
to summarize briefly the topics included in each of the subsequent chapters. Chapter 2
includes a brief description of a model for numerical hydrodynamic relativity and the
assumptions used to apply this model deep into the strong-field relativistic environment
encountered by two neutron stars as they approach their merger. Chapter 3 outlines the
primary efforts of this thesis work: to describe an accurate extension to this model based
upon linearized gravity, which should provide a more stable numerical approach than
others in the field. Here the basic dynamical equations of the binary system are derived.
Chapter 4 illustrates a simple method of obtaining the gravity wave template from a
strong-field relativistic source without employing an *ad hoc* multipole expansion as has
been applied to the approach summarized in the second chapter. Chapter 5 describes the
design of a simple three-dimensional numerical model of a system of two neutron stars.
The perturbative model outlined in Chapter 3 is applied to this system, and the
gravitational radiation emerging from the system is recovered by the procedures outlined
in Chapter 4. The results of this numerical model are then compared with results
produced by a quadrupole formula or the post-Newtonian (PN) approximation. Chapter 6
summarizes the conclusions of this work. An outline is also given of some proposed
methods by which this approach can be adjusted and applied to other sources of gravitational radiation.

1.2. Neutron Star Binary Model

As discussed above, the primary purpose of this thesis is to provide an accurate and stable theoretical solution to Einstein’s equations and the hydrodynamic evolution of a source of gravitational wave radiation. In principle, we could begin with any source of gravity waves, in particular one within the expected frequency sensitivity range of LIGO. In this thesis work, the most conservative candidate for gravity waves is chosen, i.e. a neutron star binary system. Several pulsars in close binary systems have been observed in association with other compact objects. Recently a binary system (J0737-3039) has been discovered in which BOTH stars are pulsars. Also, a neutron star binary system has been a popular choice in the literature. Since we, as of yet, have no direct measurement of gravitational radiation frequencies or amplitudes for close binary systems, one goal at the present time is to compare the method developed herein to other approaches.

Gravitational radiation, as any other source of radiation, must necessarily remove energy and angular momentum from the system. Therefore, as gravity waves are emitted, the two neutrons stars must come slowly together as their binding energy is decreased until they eventually merge into a single object. It is expected, then, that any neutron star binary should undergo three distinct stages of evolution. The first of these is the inspiral phase, where the orbits of the two neutron stars slowly decay due to this energy loss and the two stars approach one another. Early on, this phase should be very amenable to a
PN approximation, as the energy lost from gravitational radiation (which is purely a result of general relativity) is still negligible compared to the characteristic orbital binding energy of the system.\textsuperscript{14} In this phase, the two stars, although compact and comprised of high density nuclear matter, should evolve in quasi-Keplerian orbits, and the minor corrections to the orbital parameters due to the energy radiated away calculated as a perturbation of the Newtonian system. Ultimately, though, these two stars will come close enough together that strong field general relativity must be employed to accurately model the environment in which the binary evolves.

The effects of strong field general relativity eventually culminate in a violent and volatile end to the inspiral phase. The second phase of evolution of a neutron star binary is called the merger phase, where the two stars plunge together into a single massive black hole. The merger phase begins, appropriately enough, with the end of the inspiral phase, which is known as the innermost stable circular orbit (or ISCO). At this point, the two neutron stars are so affected by each other’s gravitational field that they no longer orbit around one another in a quasi-stable fashion; instead, they plunge together in a sudden, swift catastrophic spiral. The gravitational radiation emitted during the approach to the last stable orbit is characterized by a sudden increase in frequency and amplitude called the “chirp.” Gravitational fields are so strong during this phase that it has been difficult to model them using numerical relativity. Instability in the solution to the time evolution of the energy in the gravity fields (e.g. the extrinsic curvature, see below) has limited the viability of current models of this phase.\textsuperscript{19,20}

The third phase, known as the ringdown phase, is the calming phase of the evolution after the violent merger. Here, the spinning black hole, formed from the
merger of two compact objects, still has two lobes on either side of the black hole. Since gravitational radiation is expected from any time-evolving quadrupole moment (such as spinning non-spherically symmetric objects), one expects this black hole to radiate energy as well. This loss of energy causes the black hole to quickly settle into a simple, axisymmetric Kerr black hole, at which point the radiation from the system comes to an end. As may be expected, this calmer part of the evolution is much easier to model; one can use a perturbation of a Kerr metric to model the ringdown phase to its completion.

As discussed above, an accurate model exists for the early parts of the inspiral phase using post-Newtonian techniques, and an accurate model using a perturbation of a Kerr metric also exists for much of the ringdown phase. However, the end of the inspiral phase and the merger phase requires new techniques in order to completely model the evolution of the system. Attempting to tackle the nonlinear relativistic merger phase with PN techniques, however, is certain to be inaccurate. The simplifying assumptions inherent to a PN approach are not applicable to this phase. It seems more effective at the present time, therefore, to focus on producing an accurate template for the inspiral phase as close to the point of the last stable orbit as possible.

In order to effectively model the system until near the end of the inspiral phase, it is necessary to describe the system using techniques of strong field general relativity as a basis for the model. In order for this model to be effective past the point where PN models fail, though, one must choose from one of two different avenues of attack. The first involves using a scheme for numerically modeling the full relativistic Einstein equations, such as the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) method. Applications of this method, however, have met with limited success in producing a
completely accurate and stable model of the inspiral stage, becoming unstable beyond only a few orbits.\textsuperscript{20}

The second avenue of attack involves using effective techniques to permit the equations to remain linear and stable further into the regime of strong field gravity than that of the PN approach. It is this approach that this thesis will take. An initial model based upon a conformally flat condition (CFC)\textsuperscript{25} on the three metric as described in the Arnowitt-Deser-Misner (ADM) formalism\textsuperscript{26,27} and employing the same types of techniques as proposed by this thesis has already been created and has proven to be very stable up to near the last stable orbit.\textsuperscript{12}

The introduction of simplifications like the CFC (although numerically stable and accurate solutions to the Einstein equations) raises concerns that the technique may not describe an actual physical system. It becomes necessary then to take greater care in comparing the results obtained by a simplifying model with other results. Such a comparison is the subject of Chapter 5, where the results obtained by a simple test of this thesis work are compared with calculations from the PN approximation in a regime where this approximation should be accurate.

The second chapter of this thesis is devoted to a description of a three-dimensional relativistic hydrodynamic model employing the CFC approximation, as introduced by the late James R. Wilson from Lawrence Livermore National Lab along with Grant Mathews and Pedro Marronetti from the University of Notre Dame as the basis for this model.\textsuperscript{25,28} The present thesis work is an improvement on this model. The CFC approximation assumes that the spatial three-metric describing a binary system is everywhere proportional to a flat metric, although the proportionality constant is free to
assume different values at each point in the system. This leads to a considerable simplification in the relativistic field equations, while recovering most of the strong field physics, and allows for remarkable stability throughout the evolution of the system. It is this model that is extended in this thesis work by the introduction of a linearized correction to the aspects of the Einstein equations removed by the CFC approach.

1.3. Perturbation Theory

With a model that has been shown to be effective numerically over thousands of orbits of a double neutron star binary, the initial goal of producing a gravitational wave template for the inspiral of this system was achieved. Moreover, the results appeared to match well with Post-Newtonian approximations during the early part of the evolution. However, as with any model based upon a simplifying assumption, the question remains: How much accuracy was lost by employing this assumption, and is the template achieved still viable as an effective simulation of this system?

The biggest concern with employing the CFC is that all off-diagonal elements of the three metric are forced to remain identically zero. This assumption then necessarily obscures the gravity wave signal from the system, which derives from the transverse traceless components of the metric. To recover the gravity wave signal, the model described in the second chapter employed an ad hoc multipole expansion of the field equations. The radiation reaction potential and gravity wave amplitudes were then recovered to hexadecipole order according to the recipe proposed by Thorne.

The third chapter of this dissertation introduces a post-CFC method of recovering the off-diagonal elements of the three-metric by employing a linear perturbation of the
three-metric and the extrinsic curvature as defined within the ADM formalism. As we shall see, the evolution equations for these variables can be tedious. However, with the first order correction added in, the differential equations governing the evolution of the variables are manifestly stable, accurate, and straightforward to solve numerically.

1.4. Recovery of the Gravity Wave Signal

The original prediction\textsuperscript{14} of gravitational radiation was derived from a weak-field or linearized theory of general relativity. In this theory, and with a clever choice of coordinate system, the Einstein field equations, which dictate the interplay between matter and spacetime curvature, reduce to a second-order differential equation in both space and time for the first-order correction to the spacetime metric. In vacuum, where the stress-energy tensor of general relativity vanishes, one possible solution for this differential equation is that of a propagating plane wave. With another clever shift in coordinate system, this plane wave can be completely described by defining two potential polarizations oriented at an angle of 45° apart from each other.

The problem, though, is that these clever shifts in coordinate system mean that in the ADM formalism employed by the formulation of Chapters 2 and 3 will produce gravitational radiation with different components than the ones we expect to find in linearized general relativity. Therefore, once the ADM variables are found in terms of the perturbation theory of Chapter 3, it becomes necessary to work out a formalism to recover the gravity wave signal from the binary system as described by a different coordinate system than the one typically used to describe the radiation as it reaches a ground based detector.
The purpose of the fourth chapter of this dissertation is to derive such a formalism. It begins with a derivation of the gravitational wave solution in the transverse-traceless gauge, in which coordinate system the wave solution takes on the simple form described in the previous paragraphs. The chapter continues by describing the weak-field solution of gravitational radiation emitted as an outgoing spherical wave from a periodic and spatially confined source. It is possible to show that this solution can also be written in a transverse-traceless gauge and matched with the solution for gravity waves traveling in vacuum.

The final section of this chapter describes a boundary-matching condition, whereby the four metric obtained from the perturbation theory of the third chapter can be matched at the boundary of the calculations, where the weak field solution should start to become valid. By performing a simple one-to-one matching, we can obtain an estimate of the gravitational radiation from a neutron star binary without employing the ad hoc multipole expansion described in the second chapter.

1.5. A Simple Test of the Model

Once the theoretical framework is in place to describe a strongly relativistic system, it remains only to incorporate this theory into a numerical code to obtain the gravitational wave template from this system, and this template should then be compared with the PN solution and with calculations of the template based upon other techniques. Before jumping into a strenuous and robust numerical application of this technique, though, it is wise to test the subroutine in a simple code to determine the accuracy of the gravity wave template obtained in a simple equilibrium state and compare these results
with solutions of the template from such a system in a regime where other techniques are valid.

The fifth chapter of this thesis work describes a first test of the veracity of this theory where the evolution equations of Chapter 3 are incorporated in a subroutine capable of running in tandem with the three-dimensional hydrodynamics code outlined by the original CFC formulation of Chapter 2. This subroutine independently calculates the corrections to the three-metric and extrinsic curvature once the ADM variables and hydrodynamic source variables are determined at a given time.

This subroutine, in turn, is called by a main program which runs the time evolution of the system. After receiving the first-order corrections to the three-metric and the extrinsic curvature, the program then updates these variables. The ADM variables are not evolved in time, but are instead newly calculated from the updated three-metric and extrinsic curvature by virtue of a subroutine from the original code designed to solve for the ADM variables at each time slice.

This numerical framework is applied to the simple model of two rigid, nearly point-mass neutron stars of equal mass in a circular binary orbit. The initial parameters of the system are defined to match those of the observed Hulse-Taylor binary system, which is early enough in its evolution to ensure an accurate description in the PN approximation. The results of this numerical model are then presented and compared to observations and the predicted amplitude and frequency of radiation theoretically expected from this binary system.
1.6. Conclusions and Future Applications

The inspiral and collapse of a neutron star binary to a single black hole is an important source of gravitational waves, since it is one of the few expected gravitational wave sources which have been indirectly observed.\(^1,2,3,4\) In this final chapter, the results and conclusions of the previous chapters are summarized, and some caveats and areas for improvement in both the numerical model and the theory are discussed. This thesis work concludes with a discussion of areas where adaptations of this technique may be successfully applied in the future to the field of numerical relativity.

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CHAPTER 2:

NEUTRON STAR BINARY MODEL

2.1. Introduction

The field of gravitational wave astronomy has recently undergone rapid development. The detection of several compact object binaries has shown an observed energy loss that appears to match within errors the expected energy loss from gravitational waves. A recent discovery of a rare double pulsar binary has also been made, which allows for some of the most effective tests of general relativity (GR) in strong-field gravity. Ground-based gravitational wave detectors are now entering their operational phases and are providing the first real opportunities to directly measure gravitational radiation. Advanced designs of these detectors are in production, which will include in their increased frequency sensitivity spectrum the gravity wave signal emitted from a neutron star binary during the early inspiral phase. In addition, proposals for space-based detectors (e.g. LISA), capable of detecting much fainter signals, have already passed initial approval and are currently in the design phase. The possibilities for this new branch of astronomy are truly exciting.

The data analysis techniques used to extract a gravity wave signal by ground-based detectors rely heavily on the application of an accurate template to extract the signal from the noise. At present, templates exist for the early stages of the inspiral phase based upon post-Newtonian (PN) techniques and for the ringdown phase using
a perturbation of a Kerr metric.\textsuperscript{11} However, an accurate numerical model for a binary system between these two phases has not yet achieved sufficient stability.\textsuperscript{12,13} It becomes necessary in this regime to evolve the system using full strong field general relativity in a numerically stable scheme.

An approach by which the 3+1 formalism of numerical relativity can be made into a series of elliptical evolution equations has been created\textsuperscript{14} by the late James R. Wilson and Grant Mathews (hereafter referred to as WM95). This approach was then successfully applied\textsuperscript{15,16} (hereafter WMM96) to model the inspiral of a neutron star binary system to near the innermost stable circular orbit (ISCO). This approach, known as the conformally flat condition (CFC), was used to model a neutron star binary system in the strong field regime of the binary system evolution, past the ISCO where the PN approximation breaks down.\textsuperscript{15} In this formalism, gravitational radiation emission is approximated using an \textit{ad hoc} multipole expansion of a solution to Einstein’s equations based upon imposing the CFC on the three metric. This approach has recently been employed\textsuperscript{17} to effectively model this binary system through dozens of orbits, through the point of last stable orbit. This chapter is a description of the framework that formed the basis of both this successful simulation and this thesis work.
2.2. Basic Framework and Assumptions

2.2.1. ADM Formalism

When modeling a relativistic system, it is problematic to use time as an evolution parameter, since the time interval between two events is not measured to be the same by every observer. Therefore, the three-dimensional relativistic hydrodynamic codes developed to model relativistic binary systems solve the relativistic field equations using the (3+1) numerical relativistic foliation of space and time as originally outlined by Arnowitt, Deser, and Misner (hereafter referred to as the ADM formalism) and later expanded on by York. In this formalism, one slices spacetime into a series of three-dimensional spacelike hypersurfaces everywhere separated by a timelike evolution parameter.

Figure 2.1 shows an example of two hypersurfaces (or foliations) $\Sigma(1)$ and $\Sigma(2)$, where any interval between two points on the same hypersurface is spacelike. The value of the time-like evolution parameter on $\Sigma(1)$ is designated by $t$, and the evolution parameter on $\Sigma(2)$ has the value $t + dt$. We define two variables, the lapse function $\alpha$ and the shift vector $\beta^i$, to relate the proper time and coordinate labels to the evolution parameter $t$ by the following equations:

\[
d\tau = \alpha dt, \quad \quad \quad (2.1)
\]

\[
dx^i = \beta^i dt. \quad \quad \quad (2.2)
\]
The two hypersurfaces are free to take completely different shapes from each other, since the scalar $\alpha$ and the vector $\beta^i$ are can have different values at different points on the hypersurfaces. The values chosen for $\alpha$ and $\beta^i$ are gauge choices. That is, they do not specify anything physical about the system; just the slicing condition one chooses to apply when allowing the system to evolve. Ultimately, one must extract the physical observables (e.g. gravitational radiation) from these gauge dependent quantities, as will be demonstrated in the subsequent chapters of this work.

The four-metric can be expressed in terms of the lapse function and the shift vector as follows:

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j.$$  \hspace{1cm} (2.3)
Here the quantity $\gamma_{ij}$ denotes a purely spatial three metric defining the distance between two points on a given hypersurface.

In terms of these gauge choices, we can now express the vector normal to the foliation, which describes the four-velocity of an observer momentarily at rest on a hypersurface. This vector, unsurprisingly, is purely timelike:

$$n_{\mu} = (-\alpha, 0) .$$

(2.4)

The contravariant form of this vector can be found in the usual way by applying the inverse metric:

$$n^{\nu} = (1/\alpha, -\beta^i/\alpha) ,$$

(2.5)

These vectors will be used later to transform the Einstein equations into the ADM formalism.

The two gauge parameters $\alpha$ and $\beta^i$ and the three-metric $\gamma_{ij}$ are all necessary to define a hypersurface, but they are not sufficient. In order to completely define a hypersurface, one requires some description of how the foliation is embedded in four dimensional spacetime. To accomplish this, a new parameter called the extrinsic curvature ($K_{ij}$) is defined, which describes the change of a vector initially normal to the hypersurface as it is parallel transported to another point on the hypersurface. After some effort, an expression can be obtained which relates $K_{ij}$ to a combination of the partial derivative with respect to $t$ and the Lie derivative along the shift vector of the three-metric:

$$-2\alpha K_{ij} = (\partial_t - \mathcal{L}_{\beta^i})\gamma_{ij} .$$

20
This expression can be rearranged in a more useful fashion to give an exact evolution equation for the three-metric in terms of the gauge variables and the extrinsic curvature:

$$\partial_{\gamma} \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i.$$  

(2.7)

Here $D_i$ is the normal covariant derivative defined in terms of the spatial three-metric according to:

$$D_i V^j = \partial_i V^j + \Gamma_{ia}^j V^a,$$  

(2.8)

with $\Gamma_{ia}^j$ the Christoffel connections defined in the usual way by the three metric.

To find the evolution equation for the extrinsic curvature, one needs to apply the ten independent Einstein equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu},$$  

(2.9)

within the ADM formalism. To do this, three different projections of these equations are considered. The first equation arises from contracting both indices with the normal vector, and defining the relativistic enthalpy density $\rho_H$ as defined by an observer initially at rest on the hypersurface (i.e. whose four velocity is the normal vector). After appropriate contraction of the four metric and the Ricci tensor, one arrives at the following equation:

$$R^{(3)} + (\text{tr } K)^2 - K_{ij} K^{ij} = 16\pi \rho_H,$$  

(2.10)
where $R^{(3)}$ represents the Ricci scalar as calculated from the three-metric on the hypersurface and $\rho_H$ is the relativistic enthalpy density as defined below in Eqs. (2.21-2.22). This equation does not include a time derivative, and is therefore not an evolution equation, but rather a constraint condition. As it is associated with the enthalpy energy density $\rho_H$, it is referred to as the Hamiltonian constraint.

The second group of equations comes from contracting one index with a normal vector, and projecting the other index onto the hypersurface. The momentum density $S^i$ is defined as the projection of one index of the energy-momentum tensor onto the hypersurface as noted by an observer momentarily at rest on the hypersurface. In terms of this momentum density, one arrives at three independent equations:

$$D_i(K^{ij} - \gamma^{ij} tr K) = 8\pi S^j.$$ (2.11)

Again, these equations do not include a time derivative, and are not therefore evolution equations. Since they include the momentum density, these equations are commonly referred to as the momentum constraints.

The final six independent equations arise from projecting both indices onto the hypersurface. After a lengthy derivation, one arrives at the evolution equation for the components of the extrinsic curvature tensor:

$$\partial_t K_{ij} = \beta^a D_a K_{ij} + K_{ia} D_j \beta^a + K_{ja} D_i \beta^a - D_i D_j \alpha + \alpha[R_{ij}^{(3)} - 2K_{ia} K^a_j + K_{ij} tr K] + 4\pi \alpha [\gamma_{ij}(tr S - \rho_H) - 2S_{ij}].$$ (2.12)

The two tensor evolution equations, Eq. (2.7) and Eq. (2.11), give the progression of the two variables $\gamma_{ij}$ and $K_{ij}$ from hypersurface to hypersurface. Combined with the
choices of the gauge variables $\alpha$ and $\beta^i$, one now has a completely defined and evolvable foliation of spacetime.

### 2.2.2. Conformally Flat Condition

The starting point of this thesis work is the original CFC formulation of WM95, which makes the simplifying assumption that the three-metric is everywhere proportional to a flat metric and is characterized by a scalar conformal factor denoted by $\phi$. This conformal factor is everywhere positive and definite and is free to assume different values throughout a hypersurface. This condition is referred to here as the conformally flat condition (CFC).\(^\dagger\) The three-metric under this assumption is written, in Cartesian coordinates as:

$$\gamma_{ij} = \phi^4 \delta_{ij}.$$  \hspace{1cm} (2.13)

For one to demand that a three-metric initially conformal and flat remain that way after being evolved to the next foliation, the trace free part of the right-hand side of Eq. (2.5) must vanish.\(^{15}\) By employing the maximal slicing condition, where the trace of $K_{ij}$ is set to zero (the justification for this gauge choice will be explained in Section 2.2.4) and rearranging terms, one finds:

$$2\alpha K_{ij} = D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k.$$  \hspace{1cm} (2.14)

\(^\dagger\) This approximation is also sometimes referred to as the “thin-sandwich” approach in the literature.
This equation was used\textsuperscript{15} in the CFC approach to calculate the extrinsic curvature on each foliation of spacetime.

The CFC is an attractive choice for modeling gravitational radiation since the simplification inherent in this assumption allows for the reformulation of the ADM equations into a set of elliptic equations that keep the numerical model stable much further into the evolution of the system. It is expected that this reformulation is a reasonable approximation due to the extremely small amplitude\textsuperscript{20} of the expected off-diagonal metric components. It has already been shown\textsuperscript{17} that, using this approach, it is possible to achieve a template up to near the ISCO (a feat which has thus far eluded numerical relativists adapting other techniques to this problem\textsuperscript{12,13}). Figure 2.2 and Figure 2.3 show some of the results demonstrating the effectiveness of this approach.

![Figure 2.2](image.png)

**Figure 2.2:** Demonstration of the stability of the CFC approach. The system modeled is that of a circular binary system with negligible energy loss due to gravitational radiation. The numerical convergence to a stable orbital velocity is apparent.
Figure 2.3: The ratio of energy lost to total mass for binary systems of varying orbital separations for several equations of state. The CFC solution converges with PN solutions at large separation, but shows a more significant energy loss than PN calculations as the stars approach.

The CFC does have certain shortcomings, as any simplifying assumption must. The first concern is that, by forcing the local three-metric to be proportional to a flat metric, we force the off-diagonal elements of the three-metric to remain zero. Under this assumption, the system can produce no manifest gravitational radiation! The template produced by WMM96 reproduces the off-diagonal elements of the three-metric by using an *ad hoc* multipole expansion, based upon the deduced density and field evolution from the CFC simulation.

Furthermore, as the system progresses further into the region of strong field gravity shortly before the merger phase, we expect the CFC approximation to become increasingly inaccurate as the metric transitions to a Kerr solution. To provide an estimate of the deviation of a real physical system from conformal flatness, one can examine the Cotton-York tensor for a maximally rotating Kerr black hole. The Kerr
metric is not conformally flat, and so the value of the Cotton-York tensor from this metric at a distance equal to the orbital separation provides a measure of the error introduced by the CFC. An analysis of this type has shown\textsuperscript{20} that the value of the Cotton-York tensor at this orbital separation is on the order of $10^{-3}$, and so the CFC should be accurate to within 1% error throughout the inspiral phase and early in the merger phase.

In the absence of observational results with which to compare, one is faced with the question of evaluating the accuracy of the template produced by the CFC. The motivation for this thesis is to provide a preliminary answer to both of these concerns. These will be discussed further in Chapter 3, where a linearized correction to the CFC metric is introduced and new equations for the field equations are derived.

2.2.3. Hydrodynamic Variables

In order to model a system of two neutron stars, one must define the stress-energy tensor for the system. To do this, a perfect fluid is assumed so that the stress-energy tensor takes the form:

$$T^{\mu\nu} = (\rho + \rho\epsilon + P)U^\mu U^\nu + P g^{\mu\nu}. \quad (2.15)$$

In this expression, the total mass energy density is separated into two parts: $\rho$ represents the proper baryon rest mass density of the fluid and $\rho\epsilon$ denotes the internal energy density beyond the rest mass density due to nuclear or thermal effects. The quantity $P$ here represents the isotropic pressure in the fluid.
The application of the CFC approach to this system by WMM96 now defines a set of hydrodynamic variables to draw analogies to special relativity. It begins by defining $W$, a generalization of the Lorentz factor “$\gamma$”:

$$W = \alpha U^t.$$  \hspace{1cm} (2.16)

Here, $W$ is used to define a coordinate invariant version of the state variables: i.e. baryon mass density, internal energy density, and three-velocity as follows:

$$D = \rho W,$$  \hspace{1cm} (2.17)

$$E = \rho \varepsilon W,$$  \hspace{1cm} (2.18)

$$V^i = \alpha \phi^{-4} U^i/W - \beta^i.$$  \hspace{1cm} (2.19)

It is convenient to also introduce an equation of state adiabatic index $\Gamma$ that relates the pressure to the internal energy density via the following equation:

$$P = (\Gamma - 1)E/W = (\Gamma - 1)\rho \varepsilon.$$  \hspace{1cm} (2.20)

Now that one has the hydrodynamic state variables in place, an expression for the energy and momentum densities consistent with the ADM formalism must be derived. To do this, the enthalpy energy density $\rho_H$ is defined to be the energy density as measured by an observer momentarily at rest on a foliation, and can therefore be written:

$$\rho_H = n_\mu n_\nu T^{\mu\nu}.$$  \hspace{1cm} (2.21)

Inserting the expression for the perfect-fluid stress energy tensor as given above and the expression for the covariant version of the normal vector from Eq. (2.4), one finds:
\[ \rho_H = (\rho + \rho c + P)(aU^a)^2 - P. \quad (2.22) \]

By inserting the expressions given above for the hydrodynamic variables and the equation of state, one arrives at the following expression for the ADM enthalpy energy density:

\[ \rho_H = DW + EW[\Gamma - (\Gamma - 1)/W^2]. \quad (2.23) \]

In similar fashion, the ADM momentum density is found by contracting one index of the stress energy tensor and projecting the other index onto the hypersurface:

\[ S^i = n_{\mu}T^i{}_{\nu}, \quad (2.24) \]

where represents the projection operator onto the hypersurface and is defined as:

\[ II^i_0 = 0; \quad II^i_j = \delta^i_j + n^i n_j. \quad (2.25) \]

After inserting the hydrodynamic variables, the following covariant expression for the ADM three-momentum density is obtained:

\[ S_i = (D + E\Gamma)U_i. \quad (2.26) \]

With the ADM hydrodynamic variables defined, one still needs to deduce how those hydrodynamic variables evolve from hypersurface to hypersurface. The evolution equation for \( D \) arises from the conservation equation for baryon number \((\rho U^a)_{;a} = 0\), and can be written:

\[ \partial_c D = -6\partial_c(\log \phi) - \phi\partial_a(\phi^b D^b). \quad (2.27) \]
An evolution equation for internal energy is found from the conservation equation for energy \(((U_\mu T^{\mu\nu})_{\nu} = 0)\):

\[
\partial_t E = -6\Gamma E \partial_t (\log \phi) – \phi^{-6} \partial_a (\phi^6 EV^a) – P[\partial_i W + \phi^{-6} \partial_a (\phi^6 V^a)] .
\] (2.28)

Finally, the momentum evolution equation comes from the three spatial components of the conservation law for momentum density \((T^{\mu\nu})_{\mu} = 0)\):

\[
\partial_t S_i = -6S_i \partial_t (\log \phi) – \phi^{-6} \partial_a (\phi^6 S_i V^a) – \alpha \partial_i P + 2\alpha(D + \Gamma E)(W – W^{-1}) \partial_i (\log \phi) + S_i \partial_i \beta^2 – W(D + \Gamma E) \partial_i \alpha – \alpha W(D + \Gamma E) \partial_i \chi .
\] (2.29)

where \(\chi\) represents the radiation reaction potential. This quantity was deduced in WMM96 in terms of the mass and current multipole moments.

Numerically, these evolution equations, coupled with the equation of state, allow the hydrodynamic variables to be propagated from each foliation to the next. These variables then, in turn, define the ADM source terms for the new foliation.

2.2.4. Multipole Expansion

As has already been stated in Section 2.2.2, the use of the CFC necessarily obscures the existence of gravity waves. The original CFC formulation employed an ad hoc multipole expansion to recover this signal. Rather than rewrite the equations here, which are quite long and not relevant to the extension of this approach described by this thesis work, I simply refer the reader to WMM96, where the relevant equations can be found in Section IIE.
2.3. Relativistic Field Equations

2.3.1. Conformal Factor

The original justification for introducing the conformal factor $\varphi$ into the ADM formalism was to quantify a set of variables could be solved using the ADM constraint equations alone (Eqs. 2.10, 2.11).\textsuperscript{19} This avoids the instability associated with the time evolution of the 3-metric and extrinsic curvature as described below. One can use the Hamiltonian constraint (Eq. 2.10) to find an elliptic equation for $\varphi$ that is amenable to fast numerical techniques.

The introduction of the conformal factor, in effect, creates a parallel manifold to the original ADM manifold, where the parallel manifold is freely specifiable. Within the framework of the CFC, this parallel manifold can be chosen to coincide with a flat manifold, so that the Christoffel connections, Riemann tensor, and Ricci tensor and scalar of that parallel manifold are identically zero. By calculating the three-dimensional Ricci scalar in the usual way, and by setting the Christoffel connections and Ricci scalar from the flat metric to zero, one finds the following expression relating the flat space Laplacian of the conformal factor to the three-dimensional Ricci tensor of the more general ADM metric:

$$R^{(3)} = -8\varphi^{-2}\Delta\varphi,$$  \hspace{1cm} (2.30)

where $\Delta$ denotes the flat space Laplacian operator. One can then replace $R^{(3)}$ with the equivalent terms from the equation for the Hamiltonian constraint (Eq. 2.10). In this way, one finds:
\[ \Delta \phi = -\frac{1}{8} \phi^5 [16 \pi \rho_{H} + K^{ij} - (tr K)^2] . \]  \hspace{1cm} (2.31)

By employing the maximal slicing condition \((tr K = 0)\), as will be discussed in Section 2.3.2) and inserting the expression for the ADM enthalpy energy density \(\rho_{H}\), one arrives at an elliptic equation for \(\phi\) in terms of the hydrodynamic variables defined above:

\[ \Delta \phi = -2 \pi \phi^5 [DW + E(\Gamma W - (\Gamma - 1)/W)] - \frac{1}{8} \phi^5 K^{ij} . \]  \hspace{1cm} (2.32)

In this expression, the following conformal scalings relate \(D\), \(E\), and \(K^{ij}\) to the corresponding quantities on the parallel flat manifold:

\[ D = \phi^{-6} D_{\text{flat}} , \]  \hspace{1cm} (2.33)

\[ E = \phi^{-6 \Gamma} E_{\text{flat}} , \]  \hspace{1cm} (2.34)

\[ K^{ij} = \phi^{-10} K^{ij}_{\text{flat}} . \]  \hspace{1cm} (2.35)

This choice of conformal scaling for \(D\) and \(E\) in WMM96 are motivated by the terms appearing in the evolution equations for each variable (Eqs. 2.27 2.28). These scalings allow \(D_{\text{flat}}\) and \(E_{\text{flat}}\) to remain constant on the parallel flat manifold. The scaling for \(K^{ij}\) follows that originally proposed by York in his reformulation of the ADM formalism.\(^{19}\)

2.3.2. Lapse Function

As has already been stated, this approach employs the maximal slicing gauge condition, whereby the trace of \(K_{ij}\) is set to zero. This gauge choice leads to a solvable constraint equation for \(\alpha\).
To see this, one can begin by taking the time derivative of the trace of $K_{ij}$. Using the chain rule, one can write this as:

$$ \partial_t (\text{tr} K) = \partial_t (\gamma^{ij} K_{ij}) = \gamma^{ij} (\partial_t K_{ij}) + K_{ij} (\partial_t \gamma^{ij}) . \quad (2.36) $$

By inserting the two evolution equations for the three-metric and the extrinsic curvature (Eqs. 2.7, 2.12), and using the Hamiltonian constraint to remove the Ricci scalar from the resulting equation, one arrives at the following expression:

$$ \partial_t (\text{tr} K) = \beta^a D_a (\text{tr} K) - D_i D_i \alpha + \alpha [4\pi (\rho_H + \text{tr} S) + K_{ij} K_{ij}] . \quad (2.37) $$

From this expression, it is easy to see that if we set the trace of $K_{ij}$ equal to zero (and the time derivative equal to zero as well so it stays that way!), we can rearrange the equation to deduce an elliptical equation for $\alpha$ in terms of the ADM source variables and the extrinsic curvature:

$$ D_i D^i \alpha = \alpha [4\pi (\rho_H + \text{tr} S) + K_{ij} K_{ij}] . \quad (2.38) $$

Thus, the choice of setting the trace of the extrinsic curvature equal to zero amounts to a gauge choice defining the value of $\alpha$ on a given foliation. The practical effects and merits of this gauge choice will be discussed further in Chapter 3.

The lapse function is determined on a given foliation by way of an elliptic equation for the quantity $(\alpha \phi)$. One can begin by using the identity:

$$ D_i D^i \alpha = D_i D^i [\phi^{-1} (\alpha \phi)] = \phi^{-1} D_i D^i (\alpha \phi) - 2 \phi^{-1} \delta^{ij} D_i \phi D_j (\alpha \phi) + \alpha \phi D_i D^i (\phi^{-1}) . \quad (2.39) $$

Within the CFC, one can also write:
\[ D_i D^j (\alpha \phi) = \phi^4 \Delta (\alpha \phi) + 2\phi^5 \delta^{ij} D_i \phi D_j (\alpha \phi) . \] (2.40)

Inserting Eq. (2.40) into Eq. (2.39), one arrives at:

\[ D_i D^i (\alpha \phi) = \phi^5 \Delta (\alpha \phi) - \alpha \phi D_i D^i (\phi^{-1}) . \] (2.41)

By inserting the expression for the three-dimensional Ricci scalar (Eq. 2.30), and then inserting the Hamiltonian constraint (Eq. 2.10) and the expression for the maximal slicing condition (Eq. 2.38), one finally arrives at an elliptical equation for the scalar quantity \((\alpha \phi)\):

\[ \Delta (\alpha \phi) = 2\pi \alpha \phi^5 \left\{ \left[ D(3W^2 - 2) + E(3\Gamma(W^2 + 1) - 5) \right]/W \right\} + \frac{7}{8} \alpha \phi^5 K_{ij} K^{ij} . \] (2.42)

Given \(\phi\) from Eq. (2.32), it is straightforward to obtain \(\alpha\) from the result of Eq. (2.42).

### 2.3.3. Shift Vector

The second gauge choice, \(\beta^i\), is obtained by incorporating the momentum constraints (Eq. 2.11).  Imposing the maximal slicing condition, these constraints take the simple form:

\[ D_i K^{ij} = 8\pi S^i . \] (2.43)

By inserting the conformal scalings (Eqs. 2.33-2.35) and employing the CFC, one arrives at the following equation to replace the general three-dimensional covariant derivative with the corresponding flat space derivatives:

\[ D_i K^{ij} = \frac{5}{2} \phi^{-10} \partial_i [\phi^6/\alpha(\partial^j \beta^i + \partial_i \beta^j - \gamma^j_\ell \partial_\ell \beta^\rho)] . \] (2.44)
One can then replace the derivative in Eq. (2.43) with the expression from Eq. (2.44) and reduce and rearrange terms. After inserting the expression for the ADM momentum density given by Eq. (2.24), the following elliptic equation for the shift vector $\beta^i$ is found to be:

$$\Delta \beta^i = \delta^{ij} \left[ 16\pi \alpha S_j - \frac{1}{3} (\partial_j \zeta) (\partial_k \beta^k) - \frac{1}{3} \partial_j (\partial_k \beta^k) \right] + (\partial_j \zeta) [\partial^j \beta^i + \partial^i \beta^j] ,$$

(2.45)

where $\zeta \equiv \ln(\alpha \varphi^{-6})$.

Thus, by utilizing the momentum constraints in the choice of the shift vector, these conditions are manifestly satisfied in the calculations. As discussed further in Chapter 3, this choice of the shift vector can be further broken down to allow for some simplifying assumptions in the linearized perturbative approach.

2.4. Conclusion

The inspiral and collapse of a neutron star binary to a single black hole is an important source of gravitational waves, since it is one of the few expected gravitational wave sources which have been indirectly observed. In this chapter, I have summarized an approach to the ADM formalism to produce a solvable system of numerically relativistic equations.

This chapter has also discussed the parameters of a numerical model used to evolve a neutron star binary system under the assumption that the ADM three metric is everywhere conformal and flat. This approach should be acceptable through most of the evolution of the system and considerably simplifies and stabilizes the system of
equations. Hydrodynamic variables and ADM parameters and gauge variables were defined and the framework by which these variables are evolved forward in time was explained. The gravity wave signal for this model is recovered by means of a multipole expansion for the three-metric. This model is completely self-sufficient, and has been able to produce an effective gravity wave signal through several orbits and was able to remain stable into the beginnings of the merger phase.

Without the existence of observations or reliable templates using other techniques, however, it is difficult to have confidence in the veracity of the template achieved at the late stages of the inspiral, particularly since the CFC approach is expected to become inaccurate as the field becomes more strongly relativistic and the spacetime approaches that of a Kerr solution. The subsequent chapters of this thesis will describe new work on a post-CFC correction using a linear perturbation of the relevant ADM variables. This set of equations is then applied to a simple numerical test of this new technique.

2.5. References

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7 C. Cutler, notes from Center for Gravitational Wave Astronomy (CGWA), UT-Brownsville (2005).


3.1. Introduction

It was observed\textsuperscript{1} during early simulations using the CFC model as originally proposed by Wilson and Mathews (WM95)\textsuperscript{2} and applied by Wilson, Mathews, and Marronetti to a neutron star binary (WMM96)\textsuperscript{1,3} that the neutron stars underwent a unique compression effect. That is, the neutron stars experienced an increase in central pressure due to induced self gravity from higher order gravitational effects.\textsuperscript{4} Since this effect has not yet been reported in other simulations, it remains an open question whether this effect is an artifact of the use of the CFC metric. This concern has called into question the accuracy of the gravity wave template obtained within the CFC framework.\textsuperscript{5,6} Furthermore, although this approach has been shown to provide a time sequence of solutions to the Einstein equations, there is no guarantee that this sequence is an accurate depiction of the one that a physical binary follows.

In an effort to improve the veracity and accuracy of the gravity wave template created\textsuperscript{7} by use of the CFC approach in the regime of strong field relativity, this chapter will introduce a model employing a perturbative correction to the CFC. This approach avoids the imposition of vanishing off diagonal metric components and eliminates the need for an \textit{ad hoc} multipole expansion to describe the radiation reaction potential. This approach is based upon a linearized perturbation of the ADM variables and
hydrodynamic state variables. By reworking the evolution equations to include this perturbation, it should be possible, in principle, to correct for inaccuracies that the CFC might introduce.

The danger of this approach is the possibility of losing the simplicity that makes the CFC so appealing in the first place. Even at first order, the evolution equations, although still linear differential equations, become complicated. Any higher order perturbation would introduce evolution equations that are no longer linear, and thus numerically unstable.

However, one expects that a linear perturbation will be sufficient. Due to the small measure of gravitational wave effects, it is highly unlikely that any higher order corrections to the CFC will be significant until very near the onset of the merger phase. By reworking the evolution equations themselves, it is possible to remove the multipole expansion entirely and thus to provide a way of testing the template produced by the original model without any errors that may have been introduced by the multipole expansion.

The model employing this theory makes use of two simplifying but realistic assumptions. The gauge variables from the ADM formalism are freely specifiable, and as such can be redefined so that the first order correction to these variables is negligible. Also, the gauge choice of the shift vector employs what we call “grid matching coordinates.” This ensures that the stars retain coordinate values as they evolve. The fluid velocities then remain small, and therefore the hydrodynamic variables based upon this fluid motion can be assumed to contribute only at higher-order. These assumptions
reduce the theory to a simple problem (to first order accuracy): determining dynamical equations to describe the correction to the three-metric and the extrinsic curvature.

3.2. Assumptions

3.2.1. Gauge Choices

As noted in Chapter 2, the choice of the lapse function $\alpha$ is determined by the maximal-slicing condition (Section 2.3.2), which involves setting the trace of the extrinsic curvature equal to zero. This choice for the slicing condition is one of the most popular, since it has the effect of keeping a volume element in space equal from time slice to time slice. Thus, if a singularity exists within a spacetime, observers initially at different spatial coordinates would find their surroundings increasingly pinched as they are focused towards the singularity. The maximal slicing condition forces the lapse function to asymptotically “collapse” to zero near the singularity so as to keep a volume element around a particular coordinate constant. This removes the problem of modeling a singularity. This condition can be maintained in the perturbed equations without loss of generality.

The neutron star binary model of WMM96 makes use of the momentum constraint to calculate the shift vector as summarized in Section 2.3.3. Aside from automatically incorporating momentum conservation into the formulation, this has important implications for the perturbative model. This will be discussed in Section 3.2.2.
To see this, it is possible to decompose the shift vector into three separate components:

\[ \beta^i = G^i + (\omega \times r)^i - \frac{1}{4} \partial^i \chi, \quad (3.1) \]

where \((\omega \times r)^i\) represents the orbital motion of the binary and \(G^i\) represents a catch-all term including the relativistic components of the momentum density, such as the effects due to frame-drag. Since \(\Delta (\omega \times r)^i = 0\), one can take the Laplacian of Eq. (3.1) and plug in the expression for the elliptic equation for \(\beta^i\) (Eq. 2.45). This results in two separate elliptic equations for the non-orbital components of the shift vector:

\[ \Delta G^i = \phi ^{\gamma j} \left[ 16 \pi \alpha S_j - \frac{2}{3} (\partial_j \rho)(\partial^j \beta^i) \right] + (\partial_j \rho)[\partial^j \beta^i + \partial^i \beta^j], \quad (3.2) \]

\[ \Delta \chi = 4/3 (\partial_i \beta^i). \quad (3.3) \]

By splitting the shift vector into several terms, one can remove the orbital motion term \((\omega \times r)^i\) from the shift vector and study the frame-drag term \(G^i\) independently from the rest of the shift vector. This removal of the orbital motion term has no effect on the functional relation of the shift vector on the momentum constraints, because the Laplacian of a cross-product is necessarily zero. By doing this, however, one effectively demands that the shift vector move the coordinate labels to match the motion of the neutron stars. For this reason, this gauge choice is referred to in WMM96 as “grid matching coordinates.”
3.2.2. Simplifying Assumptions

With the gauge choices defined as above, one can begin to rework the evolution equations to incorporate a perturbative linearized approach. The original ADM plus CFC equations are listed again here:

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i , \]  
\[ \partial_t K_{ij} = \beta^a D_a K_{ij} + K_{ia} D_j \beta^a + K_{ja} D_i \beta^a - D_i D_j \alpha + \alpha [R_{ij}^{(3)} - 2K_{ia} K_{aj} + K_{ij} \text{tr } K] + 4\pi \alpha [\gamma_{ij} (\text{tr } S - \rho H) - 2S_{ij}] , \]  
\[ \partial_t D = -6D \partial_t (\log \phi) - \phi^6 \partial_a (\phi^6 V^a) , \]  
\[ \partial_t E = -6\Gamma E \partial_t (\log \phi) - \phi^6 \partial_a (\phi^6 V^a) - P [\partial_t W + \phi^6 \partial_a (\phi^6 V^a)] , \]  
\[ \partial_t S_i = -6S_i \partial_t (\log \phi) - \phi^6 \partial_a (\phi^6 S_i V^a) - \alpha \partial_i P + 2\alpha (D + \Gamma E) (W - W^f) \partial_t (\log \phi) + S_a \partial_i \beta^a - W (D + \Gamma E) \partial_t \alpha - \alpha W (D + \Gamma E) \partial_t \chi . \]  

As previously stated, the removal of the orbital motion term from the formula for the shift vector creates a grid matching coordinate system, where the coordinate values inside of the stars move with the bulk fluid elements. These small changes in the coordinates of the fluid elements imply that the fluid velocities are small with respect to the moving object. It is logical, then, to assume that the hydrodynamic variables in this coordinate system are slowly-varying as compared with the other evolution parameters.

Consequently, the corrections to the hydrodynamic variables describing these fluid elements likely contribute only higher order perturbative corrections. Under this assumption, one can ignore any corrections to the hydrodynamic variables to first order in the perturbative model. Also, it is no longer necessary to rework the evolution equations for the hydrodynamic variables, as those equations remain unchanged from the original
model to first order under this assumption. The only difference will be a more realistic
evaluation of the radiation reaction term in Eq. (3.8).

The framework developed in this work makes one further preliminary
approximation before reworking the equations. In this model, it is assumed that the
corrections to the conformal factor $\varphi$, lapse function $\alpha$, and shift vector $\beta^i$ are all small as
compared to the CFC zeroth order solution for the extrinsic curvature and the three
metric. The justification for this assumption is that the extrinsic curvature only
contributes a small source term for $\alpha$ and $\varphi$. Moreover, the form for the lapse function
and shift vector can be taken as gauge choices, and are therefore freely specifiable. To
the extent, therefore, that the zeroth-order values for the lapse function, shift vector, and
conformal factor remain accurate solutions to the Hamiltonian constraint and momentum
constraints, neglecting the first-order corrections to these variables should be acceptable.

These assumptions considerably simplify the first-order evolution equations and
reduce the number of evolution equations required to two tensor equations: one for the
three metric ($\gamma_{ij}$), and one for the extrinsic curvature ($K_{ij}$).

3.3. Evolution Equations

3.3.1. Correction to the Inverse Metric

To begin the discussion of how this perturbation is to be carried out, the linearized
corrections to the three metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ are defined as the
working first-order variables within this framework. As stated in the previous section,
the only source or ADM variables considered to be significant to first order are those
variables directly derived from these working variables. In order to retain the
functionality of the metric to first order, i.e. to enable raising and lowering indices, it is
necessary to define the first order correction to the inverse metric in terms of the
correction to the three metric.

One requires that, in order to maintain this functionality, the perturbed metric
satisfy the standard condition for the contraction of one index of the metric and the
inverse metric, to both zeroth order and first order accuracy:

$$\gamma_{ja}^{\prime \prime} = \delta_j \delta_a \ .$$

(3.9)

Writing out the full expressions for the three-metric and inverse three metric to first
order, and substituting the CFC expression for the zeroth-order metric and three metric
(defined from Eq. 2.13), one arrives at:

$$\delta_{ja} \delta_{ai} + \phi \delta_{ja} \gamma_{ai} + \phi \delta_{ja} \gamma_{ai} + \gamma_{ja} \gamma_{ai} = \delta_j \delta_a .$$

(3.10)

It is trivial to see that this first term equals the right-hand side of the equation and that the
fourth term is second-order in nature and can be ignored. It follows, then, that:

$$\phi \delta_{ja} \gamma_{ai} = -\phi \delta_{ja} \gamma_{ai} .$$

(3.11)

By performing a little algebra and relabeling indices, the perturbation of the inverse
metric can be written:

$$\gamma_{ji} = -\phi \delta_{ja} \gamma_{ai} \gamma_{mn} .$$

(3.12)
This expression for the first-order inverse metric can be used to raise and lower indices within this linearized post-CFC framework, and is now defined in terms of the base variables.

3.3.2. Evolution of the Three Metric

To evaluate the time evolution equation for the first-order correction to the three metric, it is possible to expand the last two terms in Eq. (3.4) to explicitly include the three-metric and the contravariant shift vector:

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \gamma_{aj} D_i \beta^a + \gamma_{ai} D_j \beta^a + \beta^a D_i \gamma_{aj} + \beta^a D_j \gamma_{ai}.
\] (3.13)

The final two terms of Eq. (3.13) vanish, since the covariant derivative of any metric is by definition zero. Expanding the remaining covariant derivatives in terms of the Christoffel connections as defined from the three-metric, Eq. (3.13) becomes:

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \gamma_{bj} \Gamma^b_{ia} \beta^a + \gamma_{bi} \Gamma^b_{ja} \beta^a.
\] (3.14)

By replacing the Christoffel connections in the final two terms of Eq. (3.14) and simplifying and re-labeling indices, one can write these two terms in the following way:

\[
\gamma_{bj} \Gamma^b_{ia} \beta^a = \frac{1}{2} \left[ \partial_a \gamma_{ij} + \partial_j \gamma_{ai} - \partial_i \gamma_{ja} \right] \beta^a,
\] (3.15)

\[
\gamma_{bi} \Gamma^b_{ja} \beta^a = \frac{1}{2} \left[ \partial_a \gamma_{ij} + \partial_j \gamma_{ia} - \partial_i \gamma_{ja} \right] \beta^a.
\] (3.16)

From this expression, it is easy to see that the overall expression for the evolution of the three-metric can be written simply as:

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \gamma_{ia} (\partial_j \beta^a) + \gamma_{aj} (\partial_i \beta^a) + (\partial_a \gamma_{ij}) \beta^a.
\] (3.17)
Equation (3.17) is completely general, and as of yet has employed no assumptions. In order to find a linearized evolution equation for the three metric, one can now assume that the three metric and extrinsic curvature are both well approximated by a first order correction term. The contravariant shift vector in Eq. (3.17) is assumed to be well approximated to first order accuracy by only a zeroth order term, as described in Section 3.2.3. Excluding those terms involving a zeroth order correction, the evolution equation for the first order three metric takes the form:

\[
\partial_t \tilde{\gamma}_{ij} = -2\alpha K_{ij} + \tilde{\gamma}_{ia}(\partial_j \beta^a) + \tilde{\gamma}_{aj}(\partial_i \beta^a) + (\partial_a \tilde{\gamma}_{ij})\beta^a.
\]

(3.18)

In Eq. (3.18), tildes represent the linearized corrections to a variable, and all variables without tilde are designated as zeroth order variables only.

Eq. (3.18) forms the first set of evolution equations within this perturbative framework. These equations do not have an explicit dependence on source variables; so it is expected that the term that dominates the production of the gravity wave signal from a relativistic source should come from the correction to the extrinsic curvature, \( \tilde{K}_{ij} \). The evolution of the correction to this tensor is described in the next section.

**3.3.3. Evolution of the Extrinsic Curvature**

The approach to finding the second set of evolution equations for this framework, namely, the evolution equations for the correction to the extrinsic curvature, is slightly different than the one considered above. The zeroth-order value for the extrinsic curvature \( \tilde{K}_{ij} \) is calculated in the original CFC framework not by the exact evolution...
equation for $K_{ij}$ (Eq. 3.4), but from Eq. (2.14), where it is assumed that the trace free part of Eq. (3.3) must be zero to maintain the CFC approximation. The expression by which $K_{ij}$ is calculated in the model of WMM96 can then be written:

$$2\alpha K_{ij} = D_i\beta_j + D_j\beta_i - \frac{4}{3}\phi \delta_{ij} D_k \beta^k.$$  

(3.19)

By inspection of Eq. (3.19), it is clear that there is no direct dependence on any source variable; the only dependence on matter and source terms comes implicitly from the ADM variables present in the expression. As these variables should carry the gravity wave signal to the three-metric equations, it may be significant that the zeroth order expression does not explicitly depend on the source terms.

In an effort to explicitly include the contribution from the gravitational wave source in the evolution of the extrinsic curvature, but still keeping the equations as simple as possible, this work calculates what would usually be considered the zeroth order value of the extrinsic curvature from the evolution equation. Ultimately, the goal is to directly include the effect of the source terms in the linear correction for $K_{ij}$, and the source terms are assumed to be zeroth order significant only (as described in Section 3.2.3).

The correction to the extrinsic curvature, therefore, is assumed to be given by the difference between the extrinsic curvature as calculated from the exact evolution equation (Eq. 3.5) and the extrinsic curvature employed by the original CFC framework, given by Eq. (3.19). As mentioned above, the terms in Eq. (3.5) are taken to be significant only to zeroth-order, and so all variables are as defined in the original CFC approach. The lone exception to this is the extrinsic curvature itself; due to the small magnitude of the zeroth-
order term, it is necessary to include and consider the whole extrinsic curvature in the evolution equation.

Including the full evolution equation for $K_{ij}$, however, carries its own risks. The evolution equation for $K_{ij}$ as originally proposed in the ADM formalism is non-linear and therefore is not stable when solved numerically. This instability was found in early tests\(^8\) of this theory, and corrections for this instability will be discussed in Chapter 5.

In order to apply Eq. (3.5), it is necessary to explicitly describe the Christoffel connections of the CFC metric, due to the appearance of the covariant derivative $D_i$ and the Ricci tensor $R_{ij}$. To simplify the expression for these connections, it is possible to introduce a new variable $G^0$, which replaces $\varphi$ according to the following relation:

$$G = 2(ln \varphi) \, .$$

(3.20)

In terms of this variable $G$, the Christoffel connections, being defined from the three-metric in the usual fashion, take the simple form:

$$\Gamma^x_{xx} = \Gamma^x_{xy} = \Gamma^x_{xz} = \partial_x G \, ,$$

(3.21)

$$\Gamma^x_{xy} = \Gamma^x_{yy} = \Gamma^x_{yz} = \partial_y G \, ,$$

(3.22)

$$\Gamma^x_{xz} = \Gamma^x_{yz} = \Gamma^x_{zz} = \partial_z G \, ,$$

(3.23)

$$\Gamma^x_{yy} = \Gamma^x_{zz} = -\partial_y G \, ,$$

(3.24)

$$\Gamma^y_{xx} = \Gamma^y_{zz} = -\partial_y G \, ,$$

(3.25)

$$\Gamma^y_{xx} = \Gamma^y_{xy} = -\partial_z G \, ,$$

(3.26)

$$\Gamma^y_{yz} = \Gamma^y_{xz} = \Gamma^y_{xy} = 0 \, .$$

(3.27)
The zeroth order Ricci tensor $R_{ij}$ can then be constructed from the Christoffel connections according to the usual definition (the sign convention used here is the same as that adopted by Misner, Thorne, and Wheeler):

\begin{align}
R_{xx} &= -(2\partial^2_x G) - (\partial^2_y G) - (\partial^2_z G) - (\partial_x G)^2 - (\partial_y G)^2, \\
R_{yy} &= -(2\partial^2_y G) - (\partial^2_x G) - (\partial^2_z G) - (\partial_y G)^2 - (\partial_x G)^2, \\
R_{zz} &= -(2\partial^2_z G) - (\partial^2_x G) - (\partial^2_y G) - (\partial_x G)^2 - (\partial_y G)^2, \\
R_{xy} &= -\partial_x \partial_y G + (\partial_x G)(\partial_y G), \\
R_{yz} &= -\partial_y \partial_z G + (\partial_y G)(\partial_z G), \\
R_{zx} &= -\partial_z \partial_x G + (\partial_z G)(\partial_x G).
\end{align}

With these definitions in place, Eq. (3.5) can be used to evolve the full extrinsic curvature. The correction to the extrinsic curvature can then be found by subtracting from this value the CFC expression for the extrinsic curvature.

It is important to note that this correction does not correspond to the first order correction as calculated for the three metric. Thus, some accuracy may be sacrificed by not considering all of the first order corrections to the evolution equations. However, the values of extrinsic curvature employed in Eq. (3.5) include the entire updated value and not just the zeroth order term, and the only other term assumed to have significant first order corrections in this equation is the three metric. Therefore, to the degree that the simplifying assumptions outlined in this chapter are valid and the three metric correction is small as compared to the zeroth order value, the correction calculated from Eq. (3.5) should be sufficiently accurate.
3.4. Conclusion

In an effort to improve upon the accuracy and veracity of results garnered from the CFC approximation as applied to the model of an inspiraling and eventually merging neutron star binary system, this chapter introduced a linear post-CFC correction to the framework summarized in Chapter 2. For simplicity, the CFC approach is maintained as a basic zeroth order model of the system, and a first order correction term is then added to the field and source variables in the ADM formalism. By employing appropriate gauge choices, a formulation can be made in which the only variables contributing non-negligible first order corrections are the off diagonal components of the three metric and the extrinsic curvature. By including a perturbation in each term of the evolution equations (Eq. 3.18) for the components of the three metric, a set of evolution equations for the first order correction to the three metric was derived in this chapter in terms of the correction to the extrinsic curvature. In order to determine this correction, the complete evolution equation for the extrinsic curvature was employed, keeping most terms accurate only to zeroth order (with an exception of the extrinsic curvature itself). Once the full extrinsic curvature has been updated, the first order correction to the extrinsic curvature is defined by subtracting from the full value the value defined by the CFC equations of Chapter 2.

The evolution equations described in this chapter constitute a framework in which corrections to the two most significant tensors employed by the CFC approximation may be independently updated. These corrections can then be applied to the CFC solution of the model. As the CFC approximation tends to hide the gravitational wave signal, this perturbation provides an opportunity to make the gravitational wave signal explicitly
manifest in the solution to the system without employing a multipole expansion to recover the signal. The method by which the gravitational wave signal is recovered is the subject of the next chapter.

3.5. References


9 J. R. Wilson, private communication (2006).

4.1. Introduction

The purpose of this dissertation is to describe the framework by which an accurate gravitational wave template can be produced from a strong-field general relativistic source. In Chapter 2, the equations and assumptions underlying a three-dimensional fully relativistic hydrodynamic code, originally proposed by Wilson, Mathews, and Marronetti (WMM96), were described. This code has been used to create a template for gravity waves from a binary source well into the regime of strong-field general relativity. This template, though, is based upon several assumptions which must be tested.

Chapter 3 outlined a new method for numerically solving for the evolution parameters describing the curvature of spacetime around a strongly gravitating source: i.e. the three metric and the extrinsic curvature. Within this method, both the three metric and the extrinsic curvature of a strong-field gravitational system are considered well approximated as the sum of a zeroth order term and a first order correction. The zeroth order term is described by the conformally flat condition (CFC), which is also the foundation for the model proposed in WMM96. All other variables describing the curvature of spacetime and the source are assumed to be slowly-varying with respect to the three-metric and extrinsic curvature, and therefore negligible to first order. The essence of the post-CFC method is the derivation of a pair of tensor evolution equations;
one for the first order correction to the extrinsic curvature, and one for the first order correction to the three-metric. Although algebraically complex, these linked evolution equations are either linear differential equations or can be made linear by use of an approximation, to be discussed in the next chapter. As such, these evolution equations can be stably numerically integrated.

This chapter describes the development of a framework by which the gravity wave template can be recovered from the three-metric evolved in this new scheme. To accomplish this, a boundary matching condition is used. Here it is assumed that far away from the source, the spacetime metric should resemble a standard spherical outgoing gravity wave in the weak field gravity approximation, as described in Misner, Thorne, and Wheeler. It is possible to match conditions at the boundary of the numerical simulation with that of a theoretically predicted outgoing wave to obtain the essential elements of this wave in terms of our numerically computed boundary values of the lapse function, conformal factor, shift vector, three-metric, and extrinsic curvature.

Once the elements of the outgoing spherical wave are found, one can perform a series of coordinate transformations that lead to the transverse-traceless (TT) gauge. This gauge makes use of various coordinate freedoms to describe a gravity wave as being trace free and having an amplitude perpendicular to the direction of propagation. In this gauge, the expression for the weak-field correction within a gravity wave signal simplifies to the expressions:

\begin{align}
  h_{xx} &= -h_{yy} = h_+ , \\
  h_{xy} &= h_{yx} = h_\times ,
\end{align}

(4.1)

(4.2)
where the direction of propagation is assumed to be parallel to the z-axis, while the orbit is in the x-y plane. The variables $h_+$ and $h_\times$, then, represent the two independent polarizations of a gravitational wave, and all other $h_{\mu\nu}$ vanish.

Assuming a source sufficiently far from the detector, the gravity wave should appear to a terrestrial detector as a plane wave propagating along a straight line connecting the detector with the source. By choosing a local coordinate system in the TT gauge so that the z-axis points in the direction of propagation, the amplitudes should be the same as those calculated from the boundary conditions of the source. This distortion in the spatial components of the metric causes a slight shift in proper distance; immeasurable by normal standards, but enough to cause a detectable phase shift in a laser beam traveling perpendicular to the direction of propagation of the wave. This is the phenomenon measured by LIGO and other ground based interferometric detectors.\(^5\)

4.2. The Weak Field Approximation

Far from a strongly relativistic region of spacetime, one expects the field to become increasingly better described by a weak field approximation. In that regime, the fully general relativistic metric in some simple coordinate system is effectively described by a small correction to Minkowski spacetime ($\eta_{\mu\nu}$), as in the following expression:

$$ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. $$ \hspace{1cm} (4.3)

Here, the weak field correction $h_{\mu\nu}$ is assumed to be small ($h_{\mu\nu} \ll 1$). The discussion of this section and the subsequent sections follows the standard derivation of the weak-field equations.\(^6\)
4.2.1. Coordinate Transformations

Eq. (4.3) relates the components of three second rank tensors, and is therefore a coordinate dependent expression. As such, one has the freedom to define coordinate transformations that maintain the assumptions of weak-field gravity. In particular, one can write:

\[ h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} , \] (4.4)

where superscripted or subscripted commas represent partial differentiation and \( \xi^\alpha \) is a function of position and represents a small (\( \xi^\alpha_{\alpha,\beta} \ll 1 \)) coordinate transformation defined by the expression:

\[ x'^\alpha = x^\alpha + \xi^\alpha . \] (4.5)

Note that it is possible in the weak field approximation to raise and lower free indices by use of the Minkowski metric. Hence, the covariant and contravariant coordinate transformations are related by:

\[ \xi^\beta = \eta_{\alpha\beta} \xi^\alpha . \] (4.6)

Furthermore, all derivatives in the weak-field approximation are given by simple partial derivatives, since Christoffel connections would be defined in terms of the Minkowski metric and are therefore identically zero in a Cartesian coordinate system.

Eq. (4.4) represents a gauge freedom in the choice of coordinates. This choice is valid for any small \( \xi^\alpha \). This freedom can be used to considerably simplify the Einstein equations, as described in the next section.
4.2.2. Weak Field Einstein Equations

To derive the Einstein field equations in the weak field approximation, one can begin with the expression for the fully covariant Riemann curvature tensor in a locally inertial point at which the Christoffel connections are identically zero:

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}) . \] \hspace{1cm} (4.7)

This expression holds in the weak field approximation since the weak-field correction can be treated as a metric in Minkowski spacetime, where (with appropriate choice of coordinate system) the Christoffel connections are always zero.

By inserting the expression for the general spacetime metric in terms of the flat-space metric and the weak field correction, one finds that, to first order in \( h_{\mu\nu} \), the Riemann tensor can be written in terms of the weak field correction:

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) . \] \hspace{1cm} (4.8)

The “trace-reversed” weak field correction \( h_{\mu\nu} \) is now defined, given in terms of the original weak field correction by the expression:

\[ h^{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h , \] \hspace{1cm} (4.9)

where \( h = h^{\lambda}_{\lambda} \) is the trace of the weak field correction. By inserting this expression in the Riemann tensor and calculating the Ricci tensor and Ricci scalar (raising and lowering indices with \( \eta_{\mu\nu} \) where appropriate), one finds the following expression (correct to first order in \( h_{\mu\nu} \)) for the Einstein tensor \( G_{\mu\nu} \):

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{2}[h_{\mu\nu,\lambda}^{\lambda} + \eta_{\mu\nu}h_{\lambda\lambda}^{\lambda} - h_{\mu\lambda,\lambda}^{\lambda} - h_{\lambda\lambda,\mu}^{\lambda}] . \] \hspace{1cm} (4.10)
The motivation behind introducing the trace-reversed metric $h_{\mu\nu}$ is more clear if it is noted that $\hat{h}_{\alpha\lambda}^{,\lambda}$ appears several times (with variations in index labels) in this expression. By choosing a coordinate transformation which forces the expression $\hat{h}_{\alpha\lambda}^{,\lambda}$ to be equal to zero, only one term in Eq. (4.10) survives, so that this equation reduces to the simple form:

$$G_{\alpha\beta} = \frac{-1}{2} \hat{h}_{\alpha\beta}^{,\lambda}.$$  \hspace{1cm} (4.11)

This choice of coordinate transformations, known as the harmonic or de Donder gauge, is valid for a coordinate transformation satisfying:

$$\xi_{\mu,\lambda}^{,\lambda} = \hat{h}_{\mu\lambda}^{,\lambda}.$$  \hspace{1cm} (4.12)

Here $\hat{h}_{\mu\lambda}^{,\lambda}$ represents the value of that tensor before the coordinate transformation is employed.

By inserting our expression for the Einstein tensor into the Einstein field equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$  \hspace{1cm} (4.13)

one can arrive at an equation defining the weak-field correction in terms of the source:

$$\hat{h}_{\mu\nu,\lambda}^{,\lambda} = (-\partial_t^2 + \Delta) \hat{h}_{\mu\nu} = -16\pi T_{\mu\nu},$$  \hspace{1cm} (4.14)

Here, as earlier, the $\Delta$ symbol represents the flat-space Laplacian operator.
4.3. The Plane Wave through Vacuum

4.3.1. General Form of the Wave

Ultimately, the goal is to apply this weak-field approximation to the case of a gravity wave propagating through empty space, where \( T_{\mu\nu} = 0 \), so that Eq. (4.14) takes the form:

\[
(-\partial_t^2 + \Delta)\h_{\mu\nu} = 0 .
\]  

(4.15)

Eq. (4.15) looks suspiciously like a wave equation, and indeed it can be shown that in an area of space where the time variation of the weak-field correction is non-negligible, the solution of this equation is a plane wave described by the real part of:

\[
\h_{\mu\nu} = A_{\mu\nu} \exp(ik_{\lambda}x^\lambda) ,
\]  

(4.16)

where \( A_{\mu\nu} \) is a constant tensor describing the amplitude of the wave, and \( k_{\lambda} \) is a constant null one-form representing the frequency and wave number of the wave.

4.3.2. The Transverse-Traceless (TT) Gauge

In Section 4.2.2, the gauge freedom of the weak-field correction term is described; namely, that any one-form \( \zeta_a \) whose derivative is small in magnitude can be used to effect a coordinate transformation on the system provided it satisfies Eq. (4.12). With a sinusoidal expression for the weak-field correction with a four-dimensional wave number \( k_\alpha \), it makes sense to choose a coordinate transformation which has the same general sinusoidal format with constant amplitude \( B_a \):
$$\tilde{\xi}_\alpha = B_\alpha \exp(i k \lambda x^\lambda) .$$

(4.17)

One can insert the expressions for the trace-reversed weak-field correction and the coordinate transform one-form to create a new coordinate transformation by plugging into the right hand side of Eq. (4.4). The new weak-field correction is therefore:

$$h_{\mu\nu}^{(new)} = h_{\mu\nu}^{(old)} - \tilde{\xi}_{\mu,\nu} - \tilde{\xi}_{\nu,\mu} .$$

(4.18)

Plugging in Eq. (4.16) and Eq. (4.17) one finds, after canceling common terms:

$$A_{\mu\nu}^{(new)} = A_{\mu\nu}^{(old)} - i B_{\mu} k_{\nu} - i B_{\nu} k_{\mu} + i \eta_{\mu\nu} B^i k_\lambda .$$

(4.19)

There are four degrees of freedom in Eq. (4.19) with which to specify conditions on the amplitude of the gravity wave in this new coordinate system. One can use these conditions to specify what is known as the transverse-traceless (TT) gauge. This gauge consists of two separate conditions: 1) that the new amplitude of the wave is traceless ($A_{\lambda}^\lambda = 0$); and 2) that the amplitude is orthogonal to some arbitrary timelike constant four vector ($A_{\alpha\nu} U^\nu = 0$). By setting the trace of Eq. (4.19) equal to zero, one can arrive at the first condition on the amplitude of the coordinate transform:

$$A_{\lambda}^\lambda^{(old)} = -2i B^i k_\lambda = -2 B^i_{,\lambda} .$$

(4.20)

This is a scalar equation, so only one gauge degree of freedom has been utilized here. The other three are found from applying the transverse condition†

† The transverse condition only amounts to three degrees of freedom because the fourth component of the condition can be found from the contraction $k^\mu A_{\mu\nu} U^\nu$, which vanishes for any $B^\nu$. 

58
solution. The arbitrary timelike four vector is chosen to be the simplest possible choice, \( U^\alpha = \delta^\alpha_0 \). This corresponds to doing a Lorentz transformation to a momentarily comoving reference frame.

By applying this choice for \( U^\alpha \) in the transverse condition and combining with the traceless condition, Eq. (4.20), one can, after some algebra, arrive at a general expression for the components of \( B^\mu \) in this coordinate system:

\[
B^\mu = [\frac{1}{2}A^\gamma_2 \delta^\mu_0 - A^\mu_0] / i\omega .
\] (4.21)

Here, the amplitudes of the weak-field correction are the values before the coordinate transformation, and \( \omega \) represents the frequency of the wave solution.

This choice of the TT gauge has several advantages. First, the condition that the trace of \( A_{\alpha\beta} \) is zero implies that the trace-reversed weak field correction is equal to the original weak field correction. Therefore, in the TT gauge (which is itself a subset of the de Donder gauge), the Einstein equations retain the simple form of Eq. (4.15) with the original weak-field corrections substituted for \( h_{\alpha\beta} \). Further, the transverse condition implies that in a momentarily comoving reference frame, all of the \( h_{\alpha0} \) are equal to zero.

Our expression for the correction to the metric becomes even simpler if we reorient our axes so that the direction of propagation of the wave is in the \( z \)-direction. Since the amplitude of the wave is necessarily orthogonal to the direction of propagation, this implies that all \( h_{jz} = 0 \). Finally, the condition that the trace of the amplitude must vanish implies that \( h_{xx} = -h_{yy} \) in this coordinate system. Combined with the requirement that \( h_{\mu\nu} \) (like \( g_{\mu\nu} \)) must be symmetric, one is left with just two independent components...
for the weak-field correction due to a gravity wave propagating through a vacuum, which can be written:

\[ h_{xx} = -h_{yy}, \quad h_{xy} = h_{yx}, \quad \text{all other } h_{\mu\nu} = 0. \quad (4.22) \]

These independent components represent two possible polarizations for a gravity wave, rotated 45° from each other in the x-y plane, and are sometimes denoted as to as \( h_+ \) and \( h_\times \). They are related to the independent \( h_{\mu\nu} \) by Eqs. (4.1) and (4.2).

4.4. The Spherical Wave from a Source

4.4.1. Simplifying Assumptions

Unfortunately, the gravity wave emitted by a strongly gravitating source does not produce such a simple and neat solution. One of the biggest challenges in determining the waveform of a gravity wave near a source is that the wave no longer propagates through vacuum. Therefore, it becomes necessary to consider the nature of the source and the effect the source has on spacetime while producing radiation. One can, however, arrive at a simplified but fairly accurate analytic solution for a spherical gravity wave emitted by a source in strong field gravity by making certain realistic assumptions.

First, it is assumed that the time dependence of the source is periodic and sinusoidal, and can therefore be approximated in a wave form as the real part of the function:

\[ T_{\mu\nu} = S_{\mu\nu}(\chi^i) e^{-i\Omega t}, \quad (4.23) \]
where the amplitude of this oscillation, $S_{\mu\nu}$, is solely a function of position. This first assumption is not entirely accurate, of course, but one can typically describe a non-periodic source with a Fourier transform, and therefore describe the source as a superposition of sinusoidal solutions of the form in Eq. (4.23). In the case of a neutron star binary system, which is inherently periodic, one does not expect a significant deviation from the actual source due to this assumption.

One can further simplify the expression if what is commonly referred to as the “slow-motion” approximation is invoked. Under this assumption, the source is assumed to be confined to a region of space that is much smaller than the wavelength of the gravity wave emitted by the source. The wavelength of this wave is equal to

$$\lambda = 2\pi/\Omega,$$  \hspace{1cm} (4.24)

where $\Omega$ is the frequency of the source appearing in Eq. (4.23). In order to apply this assumption to a neutron star binary system, the wavelength of the gravity wave emitted by the source must be much larger than the orbital separation of the stars.

As a simple determination of the regime in which this second approximation is valid, one can perform a back-of-the-envelope estimate for the binary separation when the above condition is no longer satisfied. One can assume the slow motion approximation is valid in the regime where the wavelength of gravitational radiation produced is at least factor of 10 greater than the orbital separation ($\lambda = 20r$). The orbital separation can be written in terms of the angular frequency $\omega$ of an equal mass binary system according to Kepler’s Third Law as:

$$r^3 = M_{\text{tot}}/\omega^2.$$  \hspace{1cm} (4.25)
The frequency of radiation $\Omega$ is twice the angular frequency $\omega$ of the binary, so the angular frequency of the binary becomes:

$$\omega = \pi/\lambda = \pi/(20r), \quad (4.26)$$

when the condition for the slow motion approximation is included. Inserting Eq. (4.26) into Eq. (4.25), one can write an expression for the minimum orbital radius for which the slow motion approximation is valid in terms of the gravitational mass:

$$r = (400/\pi^2) M_{\text{tot}} \approx 40 M_{\text{tot}}. \quad (4.27)$$

For a system of two neutron stars with a total gravitational mass of 2.8 solar masses, the minimum orbital radius is approximately 170 km. Thus, one can expect the slow-motion approximation to remain accurate for much of the inspiral stage. It breaks down, however, when the strong field regime is entered.

The gravity wave signal emitted by a periodic source must be quasi-periodic with a frequency twice that of the source. Therefore, one can look for a solution of the weak-field corrections of the form:

$$h_{\mu\nu} = B_{\mu\nu}(x') e^{i\omega t}, \quad (4.28)$$

where $\Omega = 2\omega$ is the angular frequency of radiation, and $B_{\mu\nu}$ is considered solely a function of position, as was the case with the source amplitude $S_{\mu\nu}$. 
Inserting Eqs. (4.23) and (4.28) into the weak-field equation, Eq. (4.14), one finds that the amplitude of the gravity wave must be related to the amplitude of the source by the following equation:

\[(\Delta + \Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu}.\]  

(4.29)

4.4.2. Definition of the Wave Amplitude

The spatially dependent amplitude of the gravity wave should reduce to a simple outgoing spherical wave far away from the source. The simplest solution for such a wave can be written:

\[B_{\mu\nu} = A_{\mu\nu} e^{i\Omega r/r},\]  

(4.30)

with \(\Omega\) the same source frequency as noted above. It is then necessary to define the amplitude \(A_{\mu\nu}\) in terms of the source amplitude \(S_{\mu\nu}\).

To do this, one can integrate Eq. (4.29) over three-dimensions making use of the simplifying assumptions outlined in Section 4.4.2. The slow motion approximation implies that the integral of the \(\Omega^2\) term should be negligible, since \(\Omega\) is assumed to be small in comparison with the volume of the source region. The second integral on the right hand side can be reduced to an integral over a surface by Gauss’ theorem, where the amplitude \(B_{\mu\nu}\) takes the form of Eq. (4.30). Thus, one can write:

\[\int [\Delta B_{\mu\nu} = \int e^{i\Omega r/r} dS .\]  

(4.31)
Again, assuming that the source is only non-zero in a small area of space, one can integrate Eq. (4.31) on the surface of a sphere of small radius \( \varepsilon \). The result is:

\[
\int \Delta B_{\mu \nu} = 4\pi \varepsilon^2 \left[ (\partial_r B_{\mu \nu})_{r=\varepsilon} \right] = -4\pi A_{\mu \nu}. \tag{4.32}
\]

The right hand side of the integration of Eq. (4.29) cannot be simplified further without making assumptions about the nature of the source. One can, however, define a new variable \( J_{\mu \nu} \) such that:

\[
J_{\mu \nu} = \int S_{\mu \nu} \, dx^3, \tag{4.33}
\]

where \( S_{\mu \nu} \) is the source amplitude from before. In terms of this new variable describing the source, the weak-field correction can be written as:

\[
\hat{h}_{\mu \nu} = 4J_{\mu \nu} e^{i\Omega(r-t)/r}. \tag{4.34}
\]

One can now restrict the possible values of the weak-field corrections by recognizing that one can rewrite Eq. (4.33), after taking a time derivative and rearranging, so that it appears in the form:

\[
-i\Omega J_{\mu 0} e^{-i\Omega t} = \int T^{\mu 0}_{\ ,0} \, dx^3. \tag{4.35}
\]

The conservation law for the stress-energy tensor implies that:

\[
T^{\mu 0}_{\ ,0} = -T^{\mu a}_{\ ,a}, \tag{4.36}
\]

which can be inserted into Eq. (4.35). The right hand side of Eq. (4.35) then again reduces to a surface integral by Gauss’ theorem. Since the expression for the weak-field
correction is only valid outside the source (where $T^{\mu\nu} = 0$), the right hand side of Eq. (4.35) also reduces to zero. Therefore, outside of the source:

$$J^\mu_0 = \mathcal{I}^\mu_0 = 0.$$  \hspace{1cm} (4.37)

Just as was the case with plane waves propagating through vacuum, any component of the weak-field correction tensor with a time component must vanish.

To simplify the spatial components of the weak-field correction, one can use the tensor virial theorem to write the right hand side of Eq. (4.35) in the form:

$$\int T_{jk} \, dx^3 = \frac{1}{2} \partial_t^2 \left[ \int T^{00} x^j x^k \, dx^3 \right] = \frac{1}{2} \partial_t^2 (I^{jk}),$$  \hspace{1cm} (4.38)

where the integral of the zero-zero component of the stress-energy tensor has been replaced by the quadrupole moment of the system. We can further break the quadrupole moment into a spatially dependent amplitude $D_{jk}$ and a time-varying sinusoidal component as follows:

$$I^{jk} = D^{jk} e^{-i\Omega t}.$$  \hspace{1cm} (4.39)

In terms of $D^{jk}$, we can then write the spatial components of Eq. (4.34) as:

$$h_{jk} = -2\Omega^2 D_{jk} e^{i\Omega (r - t)/r}.$$  \hspace{1cm} (4.40)
4.4.3. The TT Gauge for Spherical Waves

It is possible now to employ a gauge freedom to ensure that \( h_{jk} \) for spherical waves is traceless and has an amplitude perpendicular to the direction of propagation. In general, this can be accomplished by employing a similar coordinate transformation as in Section 4.3.2, where we define a vector \( \zeta^\alpha \) such that:

\[
\zeta^\alpha = B^\alpha e^{i(\Omega r - t)/r} .
\]  

(4.41)

Here the only restriction that needs to be made on \( \zeta^\alpha \) is that \( B^\alpha \) is slowly varying (i.e. \( B^\alpha,\beta \ll 1 \)).

By making the coordinate transformation defined by Eq. (4.4) and employing the definition of \( \zeta^\alpha \) given by Eq. (4.41), a description of the gravity wave polarizations in terms of the quadrupole moments of the source in the TT gauge can be found (written here with coordinate axes oriented so that the direction of propagation at that point is in the z-direction):

\[
h_{xx} = -h_{yy} = -\Omega^2 (\tilde{I}_{xx} - \tilde{I}_{yy}) e^{i\Omega r/r} ,
\]  

(4.42)

\[
h_{xy} = h_{yx} = -2\Omega^2 \tilde{I}_{xy} e^{i\Omega r/r} ,
\]  

(4.43)

where \( \tilde{I}_{ij} \) represents the trace-free part of the quadrupole moment tensor, given by the equation:

\[
\tilde{I}_{jk} = I_{jk} - \frac{1}{3} \delta_{jk} I^a_a .
\]  

(4.44)
By employing the expressions given in Eqs. (4.42) and (4.43), one can write the spherical wave solution emitted from the source in the same TT gauge as used in plane waves. It is straightforward to then identify the amplitudes of the two polarizations of a gravity wave signal from these expressions and thereby deduce the gravity wave signal from a source. How, though, can one arrive at these simple expressions from a strong field gravitational wave source?

4.5. Extracting the Wave in the Model

In general, the gravity wave production in an area of spacetime near a relativistic source should be extremely violent and non-linear, and thus difficult to calculate numerically (and impossible to calculate analytically!) On the other hand, the gravity wave template inside of a source is not particularly important; the purpose of the simulation is to deduce what the gravity wave template is far away from the source. As the gravity wave signal moves further from the source, the gravitational field it propagates through becomes asymptotically Minkowski, and should then be more amenable to analysis in terms of the weak field approximations described in the previous sections of this chapter. Our goal here is to determine the amplitude and frequency of this oscillation as defined in the previous sections in terms of the ADM variables solved in the framework of Chapter 3.
4.5.1. Boundary Matching Condition

In Chapter 3, a perturbative theory was derived to describe the time evolution of the three metric and extrinsic curvature. These updated values can then be used to determine new values for the hydrodynamic variables of the system in one time step, which can then be used to update the three metric and extrinsic curvature for the next time step. If this integration is performed far enough away from the source, the three metric should oscillate as the gravity wave travels out from the source.

To solve for the weak-field correction far from the source, we start with the ADM four metric:

\[ ds^2 = -(\alpha^2 - \beta^a \beta_a)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j. \] (4.47)

This four metric is thus fully described at any point in the evolution by the values of \( \alpha \), \( \beta^a \), and \( \gamma_{ij} \) at that point. At a point which is far enough from the source, the weak-field approximation becomes valid, and this metric becomes equivalent to a flat-space metric plus a weak field gravity term:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \] (4.48)

In terms of the ADM variables, the weak-field correction at this boundary point far from the source should then be accurately described by the following equations:

\[ h_{00} = \beta^a \beta_a - \alpha^2 + 1, \] (4.49)

\[ h_{0i} = \beta_i, \] (4.50)

\[ h_{jk} = \gamma_{jk} - \delta_{jk}. \] (4.51)
The trace of this correction is given by the expression:

\[ h = \alpha^2 - \beta^a \beta_a + \delta^{ab} \gamma_{ab} - 4 , \quad (4.52) \]

so that the trace-reversed components of this correction can be written:

\[ \hat{h}_{00} = \frac{1}{2} \left[ \beta^a \beta_a - \frac{1}{2} \alpha^2 + \frac{1}{2} \delta^{ab} \gamma_{ab} - 2 \right] , \quad (4.53) \]

\[ \hat{h}_{0i} = \beta_i , \quad (4.54) \]

\[ \hat{h}_{jk} = \gamma_{ab} + \frac{1}{2} \delta_{jk} \left[ \beta^a \beta_a - \alpha^2 - \delta^{ab} \gamma_{ab} + 2 \right] . \quad (4.55) \]

### 4.5.2. Gauge Transformations

Far from the source, the outgoing gravity wave should resemble to good accuracy a plane wave propagating in the radial direction. Here, the components \( h_{jk} \) are related to the amplitudes of the gravitational waves emitted by the source. In general, though, one must perform a coordinate transformation on these components to move them to the TT and de Donder gauges before these equations can truly represent the polarizations of the gravity wave.

To accomplish this, one would need to employ coordinate transformations given by the coordinate transformation vector \( \zeta^a \) defined in Eq. (4.41). It can be shown that this coordinate transformation also satisfies the de Donder condition (Eq. 4.12) to first order in \( 1/r \), so that this transformation also ensures the validity of Eq. (4.14) for these values.

Computing the coordinate transformation for the de Donder gauge (Eq. 4.12), however, involves numerically solving a second order differential equation in both space
and time, and is therefore difficult to accomplish. For this thesis work, a simplification is achieved by removing the common derivative term from either side of Eq. (4.12), so that:

$$\xi_{\mu,\nu} = h_{\mu\nu}. \quad (4.56)$$

The value of $h_{\mu\nu}$ in the de Donder gauge can then be found by employing Eq. (4.4) with the coordinate transformation given by Eq. (4.56).

The transformation to the standard TT gauge is simpler. It is possible to split any second rank tensor into three terms: the trace, a longitudinal traceless term, and a transverse traceless term.\textsuperscript{4} The TT gauge simply retains the transverse-traceless term from this type of expansion of the metric. If the location of the recovery of the gravity wave signal is chosen to lie on the z-axis above the center of mass of the binary, then the weak field corrections in the TT gauge (under these assumptions) equals:

$$h_{xx}^{TT} = -h_{yy}^{TT} = \frac{1}{2}(\gamma_{xx} - \gamma_{yy}), \quad (4.57)$$

$$h_{xy}^{TT} = \gamma_{xy}. \quad (4.58)$$

4.6. Conclusion

The purpose of this chapter has been to develop a framework for defining the gravity wave signal emitted by a system that is numerically evolved according to the perturbation theory outlined in Chapter 3. This definition is not trivial, due to the strong-field nature of the source as the point of last stable orbit is approached. It is possible, however, to employ a boundary-matching condition at the fringe of our numerical calculations, where the weak-field approximation is valid.
In this chapter, the basic assumptions of the weak-field or linearized solutions to Einstein’s equations have been described. In this approximation, the corrections to the spacetime metric are assumed to be small and therefore any terms higher than first order are neglected. Within this assumption, the weak-field correction acts as a second rank tensor on a background Minkowski spacetime, and as such, can be treated as a tensor in special relativity to first order accuracy. Specifically, the tensor can be boosted by any Lorentz transformation according to special relativistic rules, and each index of the tensor and the subsequent equations derived from this tensor can be raised and lowered using the flat space metric. One has a further gauge freedom by employing a translation of the coordinate system, provided that the shift is slowly varying. This coordinate transformation is employed to transform to the de Donder gauge, thereby simplifying the Einstein equations.

The linearized Einstein equations in this gauge were then applied to obtain the form of gravitational radiation through a vacuum (i.e. far away from the source of the wave). The simplest wave solution to the Einstein equations is that of a plane wave traveling at the speed of light. It is possible to exploit further gauge freedom by transforming again, this time to the TT gauge. In this gauge, the weak field correction is forced to be orthogonal to the direction of propagation of the wave and traceless, and the possible degrees of freedom for the gravity wave reduce to two independent polarizations, oriented 45° apart from each other.

While the solution of a plane wave is a possible solution within the linearized Einstein equations, it is not the only such solution. It becomes necessary, therefore, to consider what signal a physical source of gravitational radiation would emit. To
accomplish this, the solution of a spherically outgoing wave is invoked from a source assumed to be quasi-periodic and confined to a region of space that is small compared to the wavelength of the gravity wave signal emitted. It is possible, within this solution, to perform a coordinate transformation where the transverse-traceless gauge is recovered by employing a different coordinate transformation on the outside of the source.

A method is then described by which one can convert the spacetime metric (in terms of the ADM variables employed in the perturbative model) to a gravity wave template as expected within the TT gauge. This is achieved by assuming that the metric could be well approximated by a Minkowski metric and a linear correction at the boundary of the numerical calculation. The amplitude of each polarization of radiation is then calculated in terms of the ADM variables at a location on the boundary.

However, one must ensure that, when employing a boundary-matching scheme such as this one, the boundary is far enough from the source that the linearized weak-field correction is sufficient to accurately describe the metric. This is always feasible simply by extending the numerical solution over a larger area of space, but getting around the problem this way increases the risk of numerical noise causing inaccuracies in the solution. An application of this perturbative model and the determination of the gravity wave signal is the subject of Chapter 5 of this thesis.
4.7. References


CHAPTER 5:
A SIMPLE TEST OF THE THEORY

5.1. Introduction

The second chapter of this thesis described the equations for a framework of a numerical general relativistic hydrodynamic approach. This approach incorporates the ADM formalism and extends the conformally-flat-condition (CFC), as was originally proposed by Wilson and Mathews (hereafter referred to by WM95).\(^1\) A code based upon this framework has been shown to remain linear through dozens of orbits when applied to a binary system of two neutron stars in strong field gravity.\(^2\) This result is an improvement in stability over existing models using the BSSN approximation, which (though incorporating full general relativity) have become unstable after only a few orbits in this same regime.\(^3,4\)

The purpose of these simulations has been to find accurate templates of gravitational radiation emitted by a neutron star binary near the end of the inspiral phase and the merger phase. Such templates are exceedingly important to ground based gravitational wave detectors (such as LIGO and VIRGO), since accurate templates must be employed by these detectors to increase the signal to noise ratio of gravity wave signals.\(^5\) The gravitational waveform for such a binary computed in Wilson, Mathews, and Marronetti (hereafter referred to as WMM96)\(^6\) and Haywood (hereafter H07)\(^2\) employed a multipole expansion to obtain the previously excluded off-diagonal elements
of the three-metric in the TT gauge. In order for the gravitational wave template to be effective, however, it must be accurate. If the gravity wave phase is even a few percent removed from the expected template, then the signal to noise boost required to extract the signal is not realized. Thus, although we have a model in place to produce a template well into the inspiral phase of the binary, there are not at present any other computational calculations (or observations, for that matter) with which to compare the accuracy of the gravity wave template so obtained.

To help address this concern, a new technique was described in Chapter 3 by which the ADM evolution equations may be evolved directly, without employing an *ad hoc* multipole expansion or the BSSN approach. This new numerical scheme is based upon a linearized perturbation to the CFC framework as employed by WM95. That is, each relevant variable is represented by the sum of a zeroth order (CFC) term and a first order perturbation. By maintaining the perturbation only to first order, a new set of evolution equations were derived that are either linear differential equations or capable of achieving linearity through a reasonable assumption (to be discussed in Section 5.3.1). Therefore, these evolution equations should be well suited for numerical analysis. The advantage is that both the zeroth order solution obtained in the framework of the robust CFC metric as employed in WM95 and H07 and the linearized post-CFC correction are numerically well-posed solutions. Therefore, the linear correction to the CFC can be effectively employed to increase the accuracy of the CFC solution.

Chapter 4 consisted of a discussion of an approach to extract wave solutions to the Einstein equations in this scheme without resorting to the *ad hoc* multipole expansion. This approach arose from invoking the weak field approximation in vacuum far from a
strongly gravitating source. The effect of the orbiting binary on the three metric and ADM variables is a gravitational wave propagating radially outward. By calculating the gravity wave at a location far from the source, one can employ a boundary matching condition to the weak field correction and thus recover the gravity wave signal at that location.

The most significant inaccuracies in the perturbative model lie in assuming that the first-order corrections are small enough to render any higher order corrections negligible. Nevertheless, this assumption should be valid since the gravity wave amplitude is extremely small compared with any measure of the CFC metric throughout most of the evolution of the two neutron stars. However, it is important to test this model to ensure that it is numerically stable and that it does produce an accurate gravitational wave.

In this chapter, I employ the perturbative approach in a program to simulate a gravity wave signal from a binary system of two orbiting rigid non-rotating equal mass neutron stars. The stars are moved through a coarse grid, and the subsequent changes to the three-metric and the extrinsic curvature are calculated. I then describe the resulting gravity wave obtained from the evolution of the model and compare the results obtained from this model with results from PN simulations of similar binary systems in a regime where the PN scheme should be valid.
5.2. The Numerical Code

The perturbative approach of Chapter 3 was developed by adding a first order correction to the variables describing the curvature of spacetime and the stress-energy tensor of the system. It was assumed that the conformal factor $\varphi$, lapse function $\alpha$, and shift vector $\beta^i$ all contribute only higher order corrections. This is a gauge freedom associated with $\alpha$, $\beta^i$, and $\varphi$, as long as a well-posed Cauchy problem is maintained. Further, I assume that the hydrodynamic variables contribute only corrections to higher than first order as well, due to the fact that the grid matching coordinate system and choice of the shift vector keep fluid elements near their original coordinates, thus maintaining low fluid velocities and associated changes in density, pressure, internal energy, and momentum density.

The equations of this model are programmed in a FORTRAN subroutine designed to read in the ADM and hydrodynamic variables at a given iteration and return an updated value of the linear correction to the three-metric and the extrinsic curvature. This subroutine is called by a FORTRAN program controlling the movement of the stars and the evolution of the source variables. The main program is similar to the model employed by WMM96 and H07. However, the program developed employs certain additional simplifications and is therefore independent of the original CFC simulations described in those publications.
5.2.1. Initial Parameters

As discussed in Chapter 2, the usual ADM (or 3+1 formalism) achieves a separation of spacetime into purely spacelike hypersurfaces and a timelike evolution parameter $t$, which is related to the proper time of a coordinate observer at any point on the hypersurface by the lapse function $\alpha$. I then subdivide the system into a $20 \times 20 \times 25$ grid of space on a given hypersurface. The coarseness of this grid is necessary to ensure that the program has a viable duration. Of course, the choice of a spatial resolution this low introduces concerns about the effect of numerical finite differencing error entering into the solution. However, an analysis using a coarse grid of this type will be sufficient to evaluate the numerical stability of this technique. Moreover, studies of test problems solved over grids of different resolutions have shown no significant alteration of the solution due to the reduction of error by the higher resolution grid.\textsuperscript{8} Furthermore, by modeling the stars as non-rotating point masses, the hydrodynamics of the solution are negligible, and it becomes less important to incorporate a fine grid to measure individual fluid flows.

The center-of-mass centered coordinate system is chosen so that the stars follow circular orbits on an x-y plane at a z-array in the 3\textsuperscript{rd} zone, and are placed in array zones (3,10,3) and (18,11,3), respectively, as shown in Figure 5.1. The zone widths in the x and y directions are equal in size, and are defined such that the stars have an orbital separation of $1.9 \times 10^8$ m. This distance is chosen to ensure that the calculation is made in a regime where PN calculations are accurate, while keeping the spatial separation between zones (and therefore the associated finite differencing errors) as small as possible within this coarse grid. The zone widths in the z direction are chosen to be 200
times the x and y separations, so that one can recover the gravity wave signal far away from the source without sacrificing resolution in the plane of the orbit.

Figure 5.1: Schematic diagram of the plane of the orbit of two neutron stars. The initial zones where the stars are located are indicated in this diagram, and the location of the gravity wave extraction point is indicated. The spatial resolution in the z-direction is 200 times less than that in the x- or y-directions, to accommodate the need for gravity wave extraction at a location far from the source.
Due to the coarsity of the grid, the initial values of the conformal factor, lapse function, and shift vector are all set by their asymptotic values at far zones. It is found by inspection that even in adjacent zones in the orbital plane, the values obtained for the conformal factor $\phi$ and lapse function $\alpha$ are near their Newtonian limits ($\phi = 1 + \Phi_N$, $\alpha = 1 - 2\Phi_N$, where $\Phi_N$ is the Newtonian gravitational potential). The shift vector is taken to be equal to $(\omega \times r)$, where $r$ and $\omega$ are the coordinate orbital radius and the orbital angular velocity, respectively. The grid rotates along with the stars and therefore justifies the assumptions of Chapter 3 based upon low hydrodynamic fluid velocities. As an initial guess for the three-metric perturbation, the initial values of each component of the three-metric correction are zero. With these initial parameters defined, the three metric correction components are free to evolve toward a numerical solution.

The hydrodynamic variables are approximated to simulate a system of two rigid non-rotating orbiting equal-mass neutron stars. The density profile on the grid was chosen to be a Gaussian surrounding the grid location of each star. The reason for this choice is to prevent discontinuities between the zones containing the point mass stars and the adjacent zones, which can promote instability in the numerical solution. Since these stars are rigidly rotating and are in orbit around each other on a grid which itself rotates, the coordinate fluid velocities $V^i$ of the system are identically zero. The coordinate-independent value of internal energy ($E$) is also set to zero, as is the pressure (by setting $\Gamma$, the equation of state parameter, equal to one).

The only other basic hydrodynamic variable that must be specified initially, then, is the mass density $D$, defined by:

$$D = \rho W = 1.4(M_\odot) (\delta(r - r_1) + \delta(r - r_2)) W.$$

(5.1)
Here, $W$ is a Lorentz-like factor determined from the normalization of the four-velocity $U^i$, and is given by the zeroth order equation:\(^6\)

$$W^2 = 1 + \varphi^{-4} \delta^{ij} U_i U_j .$$

(5.2)

where $U_i$ is the covariant spatial component of the four-velocity of the stars. This is given in Chapter 2 as:

$$U_i = \varphi^4 \delta_{ij} (V^j + \beta^j) W/\alpha .$$

(5.3)

The ADM momentum density $S_i$ can then be defined as:

$$S_i = (D + E \Gamma) U_i .$$

(5.4)

The gravitational matter density $D$, Lorentz-like factor $W$, and the momentum density $S_i$ determine the ADM enthalpy density $\rho_H$ and the stress tensor $S_{ij}$ that appear in the evolution equations for the perturbed variables and are defined in Section 5.2.2.

The purpose of this study is primarily to test the stability of this post-CFC technique in modeling the idealized binary star system described above, and not to model the evolution of this system as they inspiral at the present time. Rather than hydrodynamically evolving these variables, it is assumed for this study that the gravitational energy radiated by this system is small enough that the orbit is in essence a simple Keplerian circular orbit. This assumption should be valid for the parameters chosen during this application and should not adversely affect the amplitude, frequency,
or phase difference of the two polarizations of gravitational radiation obtained in this study.

5.2.2. The Subroutine

The subroutine is designed to be completely independent of the assumptions employed by the main program, and uses the perturbative model of Chapter 3 to obtain updated values of the first order three metric from the source, curvature, and first order variables describing the model. To begin with, the subroutine receives values for the conformal factor $\varphi$, the lapse function $\alpha$, the shift vector $\beta^i$, and the coordinate invariant baryon density $D$, internal energy density $E$, equation of state parameter $\Gamma$, and momentum density $S_i$ as determined in the previous section. The ADM source terms $\rho_H$ and $S_{ij}$ are determined from this initial data according to the equations given in Chapter 2:

$$
\rho_H = DW + EW[\Gamma - (\Gamma - 1)/W^2],
$$

$$
S_{ij} = \varphi^4 \delta_{ij}(\Gamma - 1)E/W + S_iS_j/[(D + E\Gamma)W].
$$

The three-metric and extrinsic curvature are updated through the evolution equations from the perturbative model under the simplifying assumptions outlined in Chapter 3.

Spatial first and second derivatives of the ADM variables and existing first-order variables at interior grid points are computed by use of a three-point finite differencing stencil. The boundary values of first derivatives are determined by either a forward or backward derivative depending upon the location of the grid point. This method of derivative calculation leaves the boundary values of the derivative centered on the interface between the boundary point and the next interior grid point, and not on the
boundary itself. To correct this, a linear interpolation is used to re-center the boundary
derivatives using the existing value and the value at the next interior grid point. Since in
the three-point scheme, second derivatives are calculated identically to those at the first
interior grid point, it is assumed that boundary values of second derivatives are again
given by a linear interpolation from the two interior values adjacent to the relevant
boundary grid point.

The only time derivatives that appear in the subroutine are those used to evolve
the first order corrections to the three-metric and the extrinsic curvature:

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \gamma_{ia}(\partial_j \beta^a) + (\partial_a \gamma_{ij})\beta^a,
\]

\[
\partial_t K_{ij} = \beta^a D_a K_{ij} + K_{ia}D_j \beta^a - D_i D_j \alpha + \alpha[R_{ij}^{(3)} - 2K_{ia}K_{aj} + K_{ij}\text{tr } K] + \\
4\pi\alpha[\gamma_{ij}(\text{tr } S - \rho H) - 2S_{ij}],
\]

where all variables are as defined in Chapter 3. The updated variables are obtained at the
end of the subroutine by employing the Lax scheme\textsuperscript{9} of time-evolution:

\[
\tilde{\gamma}_{ij} \rightarrow \tilde{\gamma}_{ij, \text{avg}} + (\partial_t \tilde{\gamma}_{ij})dt \text{,} \tag{5.9}
\]

\[
K_{ij} \rightarrow K_{ij, \text{avg}} + (\partial_t K_{ij})dt \text{.} \tag{5.10}
\]

Here, \(\tilde{\gamma}_{ij, \text{avg}}\) and \(K_{ij, \text{avg}}\) represent the values of the linearized correction to the three metric
and the full extrinsic curvature as averaged over all the surrounding grid points in the xy
plane (parallel to the orbital plane).

The full extrinsic curvature is updated first in the subroutine by updating the full
extrinsic curvature according to Eqs. (5.8) and (5.10). The linearized extrinsic curvature
correction \(\tilde{K}_{ij}\) is then computed by subtracting from the full extrinsic curvature the CFC
value of the extrinsic curvature as defined by:
\[ 2\alpha K^{(0)}_{ij} = D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k. \] (5.11)

This updated value of \( \vec{K}_{ij} \) is then used to evolve the linearized correction to the three-metric through Eqs. (5.7) and (5.9).

To ensure that the Lax scheme is stable, it is necessary to choose a time step small enough to satisfy the Courant condition\(^\text{10} \). That is, the update must not require information from adjacent zones to reach the zone being updated faster than it is possible for that information to travel. Since information from two spatial dimensions is included in the evolution equations (Eqs. 5.9, 5.10), the time interval must satisfy the condition:

\[ \frac{(2cdt)}{dx} < C, \] (5.11)

where \( c \) is the speed of light, and \( C \) is a constant that must be chosen to be less than one in order for stability to be achieved. For this work, a value of \( C = 0.2 \) is chosen as suggested in the literature\(^\text{9} \) to promote accuracy.

At the end of the subroutine, the off-diagonal elements are compared with the diagonal components of the CFC zeroth order three-metric to ensure that the code is producing corrections to the three-metric that are still insignificant as compared to the zeroth order conformally flat solution. The updated values of the corrections to the three-metric and extrinsic curvature are then returned to the main program.
5.2.3. The Main Program

The main program employing this subroutine contains the time evolution of the system. After declaring initial values as defined in Section 5.2.1, the program begins an iteration in time by calling the perturbation subroutine outlined in Section 5.2.2 and obtaining the properly updated correction to the three metric and the extrinsic curvature. These values are then used to calculate the full three metric, inverse three metric, and extrinsic curvature as defined by the linearized perturbation scheme outlined in Chapter 3.

The zeroth order CFC model employs a subroutine to update the ADM field variables $\phi$, $\alpha$, and $\beta^i$ by use of the conjugate gradient method for solving elliptical differential equations. The updated values of the conformal factor, lapse function, and shift vector were then given from solutions to the elliptical equations:

\[
\Delta \phi = -2\pi \phi^5 \left[ DW + E(\Gamma W - (\Gamma - 1)/W) \right] - \frac{5}{8} \phi^5 K_{ij}K^{ij},
\]

\[
\Delta (\alpha \phi) = 2\pi \alpha \phi^5 \left\{ D(3W^2 - 2) + E(3\Gamma(W^2 + 1) - 5) \right\}/W + \frac{5}{8} \alpha \phi^5 K_{ij}K^{ij},
\]

\[
\Delta \beta^i = \delta^i_{jl} \left[ 16\pi \alpha S_j - \frac{1}{2} (\partial_j \zeta)(\partial_k \beta^k) - \frac{1}{2} (\partial_j \zeta)(\partial_k \beta^k) \right] + (\partial_j \zeta)[\partial^i \beta^j + \partial^j \beta^i],
\]

where all variables are as defined in Chapter 2. Thus, the ADM variables are not directly evolved in time; rather, they are calculated at each time step from the new values of extrinsic curvature and the three-metric.

In general, the program then updates the hydrodynamic variables $D$, $E$ and $S^i$ using the updated field variables at the new time step. These values are updated via the evolution equations:
\[ \partial_t D = -6D \partial_t (\log \varphi) - \varphi^a \partial_a (\varphi^a DV^a), \tag{5.16} \]
\[ \partial_t E = -6\Gamma E \partial_t (\log \varphi) - \varphi^a \partial_a (\varphi^a EV^a) - P[\partial_t W + \varphi^a \partial_a (\varphi^a V^a)], \tag{5.17} \]
\[ \partial_t S_i = -6S_i \partial_t (\log \varphi) - \varphi^a \partial_a (\varphi^a S_i V^a) - \alpha \partial_i P + 2\alpha(D + \Gamma E) (W - W^i) \partial_i (\log \varphi) + S_a \partial_i \beta^a - W(D + \Gamma E) \partial_i \alpha - \alpha W(D + \Gamma E) \partial_i \chi, \tag{5.18} \]

where all variables are as defined in Chapter 2.

The simple model used here to update the motion of the stars, however, includes very little hydrodynamics due to the assumptions of non-rotating rigid neutron stars, and each array element has a resolution approximately 100 times larger than the characteristic radius of a neutron star. Therefore, within the parameters of this simple test, the field variables and hydrodynamic variables are assumed to continue to evolve in the simple manner defined in Section 5.2.1, where the conformal factor, lapse function, and shift vector all take their asymptotic values at each grid point, and the matter density and coordinate three-velocity of the stars are evolved assuming a Keplerian circular orbit.

The final operation of each update is the calculation of the gravity wave signal from the updated three-metric and field variables. The weak-field correction to the four-metric is found as described in Chapter 4; i.e. the value of the four-metric at a boundary is assumed to be well-approximated by a linear weak field correction to Minkowski spacetime. In this case, the boundary element chosen is the array element located near the boundary in the z-direction directly above the center of mass of the two neutron stars. Assuming a simple spherical outgoing wave, the direction of propagation at this point is parallel to the z-axis. Therefore, I calculate the first TT polarization, \( h_+ \), by taking the difference between \( \gamma_{xx} \) and \( \gamma_{yy} \) and dividing by 2, and the second TT polarization, \( h_\times \), is
simply equal to $\gamma_{xy}$. These amplitudes are then printed to a data file, along with the time parameter $t$ at each time step.

Once the update of the code parameters and calculation of the gravity wave signal is completed for a time step, the program begins the next iteration in time. For the simple test described here, the code was intended to run over one full orbit of the neutron stars, where each orbit requires 75,000 cycles to be completed. The number of cycles is chosen to ensure that the time step is small enough to satisfy the Courant condition (Eq. 5.11).

5.3. Numerical Simulation

The goal of this work is to test this linearized perturbative approach first in a system well within the regime where PN calculations are valid, so that the accuracy of this new theoretical gravity wave template can be evaluated. Once the effectiveness of this new approach and the programming technique by which it is applied has been verified, it will be possible to apply the subroutine to a numerical model describing a more physically realistic close binary system in the appropriate sensitivity range of ground-based observatories. Indeed, the choice of initial parameters employed by this simulation, although within the regime of accuracy of PN calculations, does not fall within the current sensitivity range of LIGO or other ground-based observatories.

5.3.1. The Search for Stability

Upon first running this program according to the assumptions and equations outlined in this chapter and previous chapters, an unanticipated non-linear and unstable behavior occurred that relates to the coupling between the equations for the extrinsic
curvature and the three-metric. This led to an unphysical a three metric correction greater than 20% of the zeroth order three metric after only about 20 cycles at very specific and localized grid points.

After a lengthy period of testing each individual term in the evolution equations to determine which term(s) were most directly responsible for this observed instability, it was eventually deduced that the non-linearity could be traced to one major source term. Specifically, the term proportional to $K_{ij}^2$ squared in the evolution equation for the extrinsic curvature (Eq. 3.8) is the likely source of the instability. A similar instability has been observed in previous attempts to employ the ADM formalism to problems in numerical relativity.\textsuperscript{12} The ADM formalism was originally intended as a framework for quantum gravity, and not as a system of dynamical equations for numerical relativity.\textsuperscript{13} As such, it has been shown\textsuperscript{14} that the formalism in its purest form is only weakly hyperbolic, and is therefore unstable with respect to small numerical errors. It is this weak hyperbolicity that has helped to promote alternate formalisms of space and time, in particular the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism.\textsuperscript{15,16} The use of the full evolution equation for the extrinsic curvature is therefore a likely suspect in promoting this originally observed instability.

Eq. (5.8) is clearly non linear due to the existence of the two terms proportional to $K_{ij}^2$ squared. The second of these terms involves the trace of the extrinsic curvature, and is therefore equal to zero to the extent that corrections to the lapse function are insignificant. The first of these terms, however, is not identically zero and is therefore most likely driving the instability of the extrinsic curvature evolution.
The simplest way to correct the instability arising from this term is to simply remove this contribution from the evolution equation. It was found in the use of this original model\textsuperscript{6} that the value of $K_{ij}$, and hence the value of $K_{ij}K^{ij}$ used in the original elliptic equations for the conformal factor, lapse function, and shift vector was much smaller than the hydrodynamic terms in those expressions. This is as expected, since the extrinsic curvature represents the deviation of the vector normal to a hypersurface as it is parallel transported across a surface, and thus the deformation of a hypersurface as it is embedded in a higher dimensional space. The gauge choice of maximal slicing implies that significant deformation in a hypersurface should only occur in the region of a singularity, and this phenomenon should only occur well beyond the ISCO. Therefore, the value of the extrinsic curvature should remain small throughout the epoch of interest in the binary system evolution. Simulations in the CFC approach showed that the contribution of terms proportional to the square of the extrinsic curvature contributed only $10^{-4}$ solar masses to the gravitational mass of a neutron star.\textsuperscript{6} With such a small contribution to relativistic effects, it is therefore unlikely that terms proportional to the square of the extrinsic curvature drive either the elliptical equations for the field variables or the full evolution equation for the extrinsic curvature.

By employing this assumption, a marked increase in stability was achieved. The code was able to run for ten full orbits without the off-diagonal corrections to the CFC solution becoming too significant to call into question the post-CFC approach of Chapter 3.
5.3.2. Results

With the primary concern of stability answered to a great degree, the next concern becomes the recovery of the gravitational radiation signal from this system. This signal should be that of a simple sine wave of constant amplitude $\Omega$ and with a constant frequency $\omega$ equal to twice the orbital frequency of the system ($\Omega = 2\omega$). The amplitude is anticipated to be constant since the energy lost to the system by the stars has not been included to adjust the orbital parameters of the stars. Therefore, any increase or decrease in amplitude must necessarily be due to inaccuracies or instability in the numerical solution. Further, the two polarizations of radiation should be shifted by a constant phase difference of 90°, as expected for the circularly polarized radiation emanating from this system.

To compare with the predicted output signal, a Post-Newtonian approximation is calculated to order $(v/c)^{5/2}$ (2.5PN) for an equal mass binary system of the same parameters as the model. When calculated at a point on the z-axis at a distance $D$ from the center of mass of a point-mass binary system, the plus polarization of gravitational radiation can be written in this approximation\(^\text{17}\) as:

$$h_+ = 2\eta M v^2/D \left[ 2H_{22} + H_{32} + 2H_{42} + 2H_{52} + 32H_{62} \right] \cos(2\omega t) \ .$$

(5.19)

Here $\eta$ represents the ratio of the reduced mass to the total ADM mass ($M_t$) of the binary system ($\eta = \frac{1}{3}$ for an equal mass binary), $M$ is the total gravitational mass of the binary system (including the orbital binding energy), $\omega$ is the orbital angular frequency of the binary, and $v = (M_\omega)^{1/3}$ is the characteristic orbital velocity of the stars. The coefficients
$H_{22}$-$H_{62}$ denote the contributions to the amplitude of successively higher order powers of $(v/c)$. For a point mass binary system these are defined by:

$$H_{22} = -\{1 - [(107 - 55\eta)/42]v^2 + 2\pi v^3 - [(2173 + 7483\eta - 2047\eta^2)/1512]v^4 - [(107 - 55\eta)/21]2\pi v^5\}, \quad (5.20)$$

$$H_{32} = -2/3\{[1 - 3\eta]v^2 - [(193 - 725\eta + 365\eta^2)/90]v^4 + [1 - 3\eta]2\pi v^5\}, \quad (5.21)$$

$$H_{42} = -1/21\{[1 - 3\eta]v^2 - [(1311 - 4025\eta + 285\eta^2)/330]v^4 + [1 - 3\eta]2\pi v^5\}, \quad (5.22)$$

$$H_{52} = -2/135\{[1 - 5\eta + 5\eta^2]v^4\}, \quad (5.23)$$

$$H_{62} = -1/11880\{[1 - 5\eta + 5\eta^2]v^4\}. \quad (5.24)$$

Table 5.1 lists a summary of the values used for each variable to calculate the 2.5PN wave signal. The amplitude and frequency of radiation for the cross polarization is analogous to the plus polarization; the only difference between the polarizations is a phase shift of 90°, as is expected for circularly polarized radiation.

**TABLE 5.1:**

**LIST OF PARAMETERS**

**USED IN THE 2.5PN CALCULATIONS**

<table>
<thead>
<tr>
<th>$H$</th>
<th>$M$ (m)</th>
<th>$M_\ell$ (m)</th>
<th>$\omega$ (rad/m)</th>
<th>$v$</th>
<th>$D$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>4135.5775</td>
<td>4135.6</td>
<td>$6.938 \times 10^{-11}$</td>
<td>$6.5957 \times 10^{-3}$</td>
<td>$5.055 \times 10^{10}$</td>
</tr>
<tr>
<td>$H_{22}$</td>
<td>$H_{32}$</td>
<td>$H_{42}$</td>
<td>$H_{52}$</td>
<td>$H_{62}$</td>
<td></td>
</tr>
<tr>
<td>-0.99990521</td>
<td>-7.250072 \times 10^{-6}</td>
<td>-5.178088 \times 10^{-7}</td>
<td>-1.75235 \times 10^{-12}</td>
<td>-9.95651 \times 10^{-15}</td>
<td></td>
</tr>
</tbody>
</table>
The results obtained from this application of the post-CFC method are displayed in Figure 5.2 and Figure 5.3 in blue, and the predicted signal from the (5/2) post-Newtonian approach are shown in magenta. It is clear that the results obtained do not match the 2.5PN solution, although they do share many of the same trends. The post-CFC polarizations do appear to change in phase with the 2.5PN predictions, and have a proper phase shift with respect to each other. Despite some phase inaccuracy present in the post-CFC solutions, the general radiation frequency appears to consistent with that expected from the 2.5PN predictions. Of the two polarizations, it appears that the plus polarization is in closer agreement with the 2.5PN predictions, although instability seems to be affecting the amplitude over time as well. It appears that there is a vertical offset in the post-CFC solution for the cross polarization. A possible cause for this discrepancy is the initial definition of the three metric correction components. All of these were given zero initial values, and were therefore perfectly in phase to begin the simulation. The most likely cause of these phase and amplitude inaccuracies in this simplification are due to the large spatial resolution of zones in the z-direction, where the solution is calculated. Since the error in a three-point stencil is proportional to the spatial resolution squared, a relatively small decrease in resolution can cause significant errors to develop far from the orbital plane.
Figure 5.2: The plus polarization of gravitational radiation obtained via the post-CFC approach, and the corresponding waveform from the 2.5PN scheme.

Figure 5.3: The cross polarization of gravitational radiation obtained via the post-CFC approach, and the corresponding waveform from the 2.5PN scheme.
In support of this explanation for the discrepancy observed between the 2.5PN results and those obtained from this application of the post-CFC approach, it is instructive to consider the time necessary for light to travel across a zone in the z-direction. The zone width in the z-direction for this simulation is approximately \(3 \times 10^9\) and so the time necessary for a light ray propagating in the z-direction to traverse a zone is approximately 10 seconds. By inspection of Figure 5.2, it is clear that the observed phase discrepancies between the 2.5PN and post-CFC solutions are on the order of the light-crossing time for this grid. This suggests that the spatial resolution employed here is simply too low for the post-CFC solution to self-correct and thereby reach a stable value numerically. Simulations employing a higher resolution grid are currently underway, and will ultimately provide a definitive answer to the veracity of this particular explanation.

One final area in which the 2.5PN solution can be compared with the post-CFC approximation is via the expected energy loss from the system. Although it is assumed that the stars do not radiate enough energy to appreciably change the shape of their orbit, it is still possible to estimate the energy lost by this radiation. The 2.5PN expression for the power of gravitational radiation from this system can be written as:

\[
s\frac{dE}{dt} = \frac{32}{5} \eta^2 v^{10} \left\{ 1 - \left[ \frac{1247}{336} + \frac{35}{12} \eta \right] v^2 + 4\pi v^3 + \left[ -\frac{44711}{9072} + \frac{9271}{504} \eta + \frac{65}{18} \eta^2 \right] v^4 - \left[ \frac{8191}{672} + \frac{535}{24} \eta \right] \pi v^5 \right\}.
\] (5.25)

For an equal point mass binary with orbital velocity given in Table 5.1, we find a luminosity of \(6.23 \times 10^{-23}\) in geometrized units.

The power loss from the post-CFC solution can be estimated by use of the quadrupole expression in the slow motion approximation.\(^{18}\)
\*\*dE/dt = (1/5)Ω^6<\hat{I}_{ij}>\*\* 

where \( \hat{I}_{ij} \) represents the reduced quadrupole moment, and the brackets represent the time averaged quantity. From the solution of the wave equation at a point \( D \) far from the source, the amplitudes of polarization can be written in terms of the reduced quadrupole moment as:

\[
h_+ = (\Omega^2/D)[\hat{I}_{xx} - \hat{I}_{yy}] ; \quad h_\times = (2\Omega^2/D)\hat{I}_{xy}
\]  

(5.27)

After a little algebra, one can insert Eq. (5.27) into Eq. (5.28) to obtain:

\[
dE/dt = [(\Omega D)^2/5] <h_+^2 + h_\times^2>
\]  

(5.28)

Since the two amplitudes should be equal, and because of the offset present in the \( h_\times \) figure, I assume that the time average of the sum of the squared amplitudes is simply equal to:

\[
<h_+^2 + h_\times^2> = 2(\frac{1}{2}h_{+, \text{max}})^2 = h_{+, \text{max}}^2.
\]  

(5.27)

Here, the value of \( h_{+, \text{max}}^2 \) is taken to be the square of the average of each peak represented in Figure 1, which gives a value of \( 2.1 \times 10^{-23} \). Using values for \( \Omega \) and \( D \) from Table 1, the power lost can be estimated as \( 2.2 \times 10^{-22} \). The estimate is higher than that calculated from the 2.5PN solution, as expected for a solution of larger wave amplitude.
5.4. Conclusion

In this chapter, the parameters of a simple numerical code have been described. This code was intended to demonstrate the feasibility of incorporating the perturbative equations outlined in Chapter 3 as a correction to the computationally simple CFC approach and as a method to independently recover the gravity wave signal from a relativistic binary in close orbit. The code simulated the deformation of space and time surrounding a neutron star double binary, and was designed to calculate the gravity wave signal emitted by such a binary.

For the purposes of producing a simple but verifiable simulation of the gravity wave signal emanating from a double neutron star binary, these neutron stars were assumed to be equal in mass, rigid, non-rotating, and moving in circular Keplerian orbits with negligible binding energy carried away by gravitational waves. This quasi-stable equilibrium orbit assumption will only be valid when the stars are early in their inspiral stage. Therefore, the neutron stars were chosen to have an orbital separation of $1.9 \times 10^8 m$ and an orbital angular frequency of $0.0208 \text{ rad/s}$. These simulation parameters were chosen to ensure a simulation where the quasi-stable equilibrium condition and the predictions from PN simulations are valid, but also where the stars are close enough together to minimize error caused by the coarse grid solution. The perturbed values for the three-metric and extrinsic curvature were then calculated using the assumptions that the field variables take their asymptotic values everywhere on the grid, and that the hydrodynamic variables are solely represented by the baryon mass density and momentum density of the rigid non-rotating stars.
However, this simple numerical application of the technique outlined in Chapter 3 did not immediately produce a stable gravity wave signal. The level of stability in the program was significantly increased by incorporating a Lax scheme and by isolating and removing a term in the evolution equation for the extrinsic curvature (Eq. 5.8) that caused a non-linearity in this equation. These modifications greatly increased the level of stability achieved, allowing for a total of 10 orbits to be stably computed.

The two polarizations of gravitational radiation were then calculated for a single orbit and compared with the post-Newtonian solution for the same binary system, valid to order \((v/c)^{5/2}\). The expected power lost via gravitational radiation was also calculated from both the post-CFC and 2.5PN solutions. It was found that the solution obtained by the framework of Chapter 3 followed many of the general trends of the expected 2.5PN solution. Inaccuracies in both phase and amplitude were apparent, though, and so it must be concluded that this method did not produce an accurate gravitational wave template in this application. The final chapter will discuss some additional tests and modifications that can be performed on this technique, and some potential improvements to both the modeling and theory, which will define the author’s direction of future postdoctoral research.
5.5. References

5. C. Cutler, notes from Center for Gravitational Wave Astronomy (CGWA), UT-Brownsville (2005).
CHAPTER 6:
CONCLUSIONS AND FUTURE APPLICATIONS

6.1. Summary

Gravitational wave astronomy is one of the newest and most exciting areas of science. The existence of gravity waves is already well-accepted, and, as with the advent of any technology that allows us to probe a new type of scientific phenomena, it is widely expected to not only reveal imperfections in our understanding of known problems but to discover a whole new class of phenomena that were never known to exist. This field is on the verge of taking off with the production and initial data runs of multiple ground-based detectors taking place\(^1,2\). In addition, an improved design of a ground based detector\(^3\) and a new space based detector\(^4\) are already in the design phase, both of which represent dramatic improvements in frequency range and signal sensitivity.

The inspiral and collapse of a neutron star binary to a single black hole is an important source of gravitational waves, since it is one of the few expected gravitational wave sources which have been observed\(^5,6,7,8\). However, the ground based detectors already in use need accurate templates in order to boost the signal to noise ratios to adequate levels for detection.\(^9\) Further, the only parts of the evolution of a neutron star binary which lies in the frequency sensitivity range of current ground-based detectors are the final orbits of the inspiral phase, which are the most difficult to model due to the existence of strong non-linear gravitational dynamics.\(^1\) The purpose of this dissertation
has been to develop a new numerical modeling technique to be applied to this sensitive region, and by which an accurate and complete template of the gravitational radiation produced by this system can be more effectively modeled.

The success\textsuperscript{10} of the theoretical framework outlined in Chapter 2 (as proposed by WM95 and WMM96)\textsuperscript{11,12,13} shows promise in extending the accepted theoretical template for gravitational waves to near the innermost stable circular orbit (ISCO). As it is important to improve the accuracy of this template, an extension of the conformally flat condition (CFC) approach was proposed based upon of a linearized perturbation to the three metric. Here, the gravitational wave amplitude and frequency was extracted by means of a boundary matching condition to the anticipated weak field emitted signal as captured numerically in vacuum far from the source.

This model was then tested in a code which models the simple system of two point-mass neutron stars of equal mass. The gravity wave signal in the transverse-traceless gauge was recovered by the framework outlined in Chapter 4. After comparing to post-Newtonian (PN) simulations of similar systems, it was clear that the general trend of expected radiation was recovered by use of this model, but significant amplitude and phase errors were inherent in the post-CFC solution recovered. For the most part, these can be attributed to the large numerical error associated in extracting the radiation at a point far removed from the source.
6.2. Continuing Work

It is important to emphasize that this thesis was a first step toward the development of a comprehensive, stable, algorithm for the extraction of gravity waves in the strong-field relativistic regime. Although considerable progress was made in this thesis work, the method by which the gravitational radiation template was extracted contains several assumptions that may point out areas in which the method proposed in this thesis can be improved. First and foremost is the assumption that the point at which the numerically computed spacetime metric is well approximated by the weak-field approximation. Of course, it is always possible to extend the code boundaries to guarantee the effectiveness of the weak-field approximation; however, the possibility of numerical error swamping the already weak gravity wave signal then becomes a problem. A potential solution, though computationally expensive, is to compute several “empty” zones in the z-direction. By doing this, it would be possible to extend the point of wave extraction without significantly decreasing the resolution of the grid along the z-axis.

It is also possible to model this system using a grid resolution that varies with grid location. By inflating the size of $dz$ as one moves further from the plane of the orbit, one can keep errors low in the area where the source terms are significant and still achieve extraction at a far zone without paying the high computational price for the empty zones.

Several opportunities for improvement exist for this method. A full scale incorporation of the subroutine with the three-dimensional hydrodynamic code is an important next step toward producing a realistic gravity wave template from a neutron star binary. The inclusion of a finer mesh would also help to decrease the expected
numerical noise from the numerical model. It is also possible that the inclusion of higher order schemes for updating the initial value equations in this framework will promote stability in the solution and decrease the errors observed in this thesis work.

Analytically speaking, the possibility also exists that the residual instability and error observed in the post-CFC solution is the result of the instability inherent in the ADM formalism. Although removal of the term proportional to $K^2$ removed the non-linearity of the differential equation and thus the most egregious source of the instability, the possibility exists that the ADM formalism remains weakly hyperbolic and therefore unstable, even under this assumption. A potential solution lies in perturbing the extrinsic curvature equations as was performed for the three metric equation in this thesis work, thereby creating a linear ODE for the linear correction to the extrinsic curvature. However, this approach is not straightforward, since all terms in the ADM evolution equation of $K_{ij}$ must be reduced to first and second spatial derivatives of the fundamental source variables (as was accomplished in this work by the introduction of the variable $G$, where $G = 2\ln \phi$. The formulation of such an approach, however, is a promising next step in the development of this technique.

6.3. Future Applications

This dissertation was intended to be a demonstration of the usefulness and accuracy of the linearized post-CFC method to extract an accurate gravity wave template from relativistic binary systems. Although this method has not yet proven to be a viable alternative, it is clear that the correct general trend of the gravitational radiation signal was achieved, and therefore the post-CFC method should not yet be abandoned. An
obvious first step in continuing research is to employ more accurate numerical schemes, so as to establish the viability of applying this technique to areas of strong-field gravity. The possibility also exists of applying this technique to other binary merger sources of gravity wave signals, as well as to asymmetric strong field astronomical systems such as accreting neutron stars or black holes, active galactic nuclei, or certain cosmological models. The increase in sensitivity in gravitational wave frequency instituted by the next generation of ground based gravitational interferometers will create opportunities to model a whole new group of interesting and relevant phenomena. This thesis has been a first step toward developing a theory fully capable of meeting the required numerical precision for this next generation of detectors.

6.4. References

4. K. Danzmann et al., Class. Quantum Grav. 13 (1996).

