A PHILOSOPHICAL INQUIRY INTO THE CONCEPT OF NUMBER

Abstract

by

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The dissertation is an inquiry into the ontology and epistemology of numbers. As regards the former, the Fregean conception of numbers as objects and the Russelian conception of numbers as higher-level entities are both critically examined. A conception of numbers as modes of existence, that is, ways or manners in which things exist, is introduced and defended instead. As regards the latter, the basic concepts of arithmetic are explicated in terms of pure logic alone, and all the truths of arithmetic are shown to follow from those explications solely by logical means. A new version of logicism in the philosophy of arithmetic is thereby established.
CHAPTER 1

INTRODUCTION

The chief aim of this entire dissertation is to establish a version of logicism in the philosophy of arithmetic, namely the view that arithmetic is nothing but logic, or more precisely, every truth of arithmetic is a truth of logic.

Chapter 2 begins by raising a hard problem that a logicist project like this has to face, namely why does our arithmetical knowledge continue to grow without showing its limits while our logical knowledge is essentially at an end? To resolve the problem we make a detour around the notion of analyticity. We present and defend a rigorous reconstruction of the intuitive conception of an analytic sentence as one which we can recognize as true by a mere grasp of the concepts involved. More specifically, we introduce the notion of a possession condition of a concept and argue that analytic sentences are those that express possession conditions of concepts. Then we examine a type of definition we shall call *explication*, and argue that explications are analytic yet informative. The reason that arithmetic is fertile as opposed to logic is that arithmetic is based upon the explications of its basic concepts.

To obtain explications of the basic concepts of arithmetic, it is crucial to find out what numbers are. Chapters 3 and 4 critically discuss Frege’s arguments for the conception of numbers as objects. The gist of the first argument, which is the topic for Chapter 3, is that given that numerals behave in arithmetic as singular terms
and singular terms refer to objects, numbers must be objects. After introducing a difficulty with the premise that singular terms refer to objects, namely what is known as the paradox of the concept *horse*, we explain Frege’s own resolution of the paradox, which anticipates much of Wittgenstein’s distinction between *saying* and *showing* in the *Tractatus*. Then we examine why Frege, unlike Wittgenstein, thought it possible to say of an object that it is an object, while denying that it is possible to say of a concept that it is a concept. We argue that the reason is that Frege believed that certain object-expressions can have reference independently of the context of a sentence. However, we argue against Frege that certain concept-expressions can also have reference in themselves in isolation from the context of a sentence. Consequently, we can truly say of a concept, using a singular term, that it is a concept. The upshot is that not all proper names refer to objects. The logical behavior of numerals as singular terms cannot be a sign of the status of numbers as objects.

Underlying the second argument for the conception of numbers as objects is the assumption that the adjectival use of number words as in sentences of the form ‘there are $n$ Fs’ can be explained away in terms of the substantival use of number words as in sentences of the form ‘the number of Fs is $n$’. Chapter 4 examines the Fregean attempt to introduce the concept of number by means of what is commonly called *Hume’s Principle*, and argues that an understanding of the right-hand side of the principle requires a prior understanding of the adjectival use of the number word ‘one’ as in sentences of the form ‘there is (at least) one $F$’ and that this use of the word ‘one’ is inevitable. So by the Fregean standards themselves, numbers cannot be conceived as objects.

Chapter 5 specifically addresses the neo-Fregean claim that Hume’s Principle can serve as a foundation of arithmetic. We argue against the neo-Fregeans that Hume’s
Principle does not give a definition of the concept of number, not even a partial one. First we construct a variant of Frege’s so-called Caesar objection and thereby show that Hume’s Principle does not explain the use of the concept of number in non-arithmetical contexts. More crucially, however, we argue that Hume’s Principle fails to explain the use of the concept of number in arithmetical contexts by showing that the principle can be satisfied by concepts other than the concept of number.

It is widely assumed that the major obstacle to accepting the broadly Russellian conception of numbers as higher-level entities is its need for an axiom of infinity. Chapter 6 argues that the real difficulty with it is its inability to explain the fact that numbers are *determinate* in the sense that if there are \( m \) Fs and there are \( n \) Fs, then \( m \) must be identical with \( n \). A criterion of numerical identity is necessary to explain the determinacy of numbers, but we argue that no adequate criterion for the identity of numbers is available to the proponent of the conception of numbers as higher-level entities.

The penultimate chapter presents our own logicist theory of numbers. We argue that the number word ‘\( n \)’ in ‘there are \( n \) Fs’ plays the role of an adverbial modifier of the verb phrase ‘there are’, and then develop a corresponding conception of numbers as *modes of existence*, that is, ways or manners in which things exist. This new conception of numbers as modes of existence leads to a series of intuitive explications of the basic concepts of arithmetic in terms of pure logic, and it is shown that the Peano Axioms can be derived from those explications solely by logical means.

The derivation of the Peano Axioms from our explications of the basic concepts of arithmetic has two immediate consequences: first, every truth of arithmetic is analytic and, secondly, every truth of arithmetic is a truth of logic. A new version of logicism in the philosophy of arithmetic is established.

The last chapter summarizes the central ideas and theses of the dissertation.
and presents some major epistemological and ontological implications of our logicist philosophy of arithmetic.
CHAPTER 2

ON THE FERTILITY OF ARITHMETIC

Any logicist project like ours must face the hard problem of explaining why, if logicism were true, there should be differences between the epistemic status of logic and that of arithmetic. If truths of arithmetic could be in principle turned into truths of logic, how could it possibly be that truths of logic are all more or less banal but truths of arithmetic often take the best minds to discover? And if arithmetic were nothing but logic, how could our arithmetical knowledge continue to grow without showing its limits while our logical knowledge was essentially at an end? In this chapter, we shall address these questions by making a necessary detour around the notion of analyticity. We shall argue that there is a special type of analytic sentence that can contribute to the growth of our knowledge and that the reason that arithmetic is fruitful as opposed to logic is because it is underpinned by analytic sentences of that type.

It would seem that there are sentences which we should be able to recognize as true merely by virtue of our grasp of the concepts involved. A typical example is ‘bachelors are unmarried’. What is interesting about this sentence is that if you have a grasp of the concept of bachelor, then you should be able to recognize it as true. Such sentences are called analytic. However, to say that analytic sentences are those that can be recognized as true merely by virtue of a grasp of the concepts involved is not quite helpful. A grasp of a concept usually comes in varying degrees
and so, depending upon how much one is required to know in order to have a grasp of a concept, any true sentence could count as analytic. Leibniz famously held that “in all true affirmative propositions, necessary or contingent, universal or singular, the notion of the predicate is always in some way included in that of the subject—the predicate is present in the subject—or I do not know what truth is” (Leibniz, 1998), which, at least in one way of reading it, would amount to saying that every true affirmative sentence is analytic, given the plausible assumption that whoever has a grasp of a concept should also have a grasp of each of the concepts included in it. So we need a precise characterization of how good a grasp of a concept one has to have in order to be able to recognize an analytic sentence as true solely by virtue of a grasp of the concepts involved.

But before turning to that, let me explain our terminology. By a concept $C$ we shall mean that which you entertain in your mind when you ask, “what is it to be $C$?” (or any appropriate question of that sort). Among concepts are thus included not only general concepts such as BACHELOR\(^1\) (“what is it to be a bachelor?”) but also singular concepts such as SOCRATES (“what is it to be Socrates?”), logical concepts such as ALL (“what is it for all things to be $F$?”), modal concepts such as MUST (“what is it for one to have to do something?”), and so on. We shall take it for granted that there is a certain thing you entertain in your mind when you raise a question like these.\(^2\) It is fair indeed to ask exactly what sort of thing it is, but for present purposes, we can set that aside. Suffice it to say that it is a certain kind of representation of an entity, be it a property such as bachelorhood, an external object such as Socrates, a logical property such as all-ness, a modality such as necessity, or any other appropriate thing.

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\(^1\)From now on we shall write names of concepts in capitals.

\(^2\)If you do not believe in such a thing, you can still follow our discussion by converting talk of concepts into talk of linguistic expressions.
Another notion that will figure significantly in what follows is that of possession of a concept: we shall say that one possesses a concept $C$ if and only if one’s own concept of $C$ represents (that is, is about) what the concept $C$ normally represents. It needs to be noted that possessing a concept $C$, thus understood, is not necessarily the same as understanding it, viz. knowing what it is to be $C$.\footnote{For some notable attempts at developing a theory of concept possession along these alternative lines, see Peacocke 1992 and Bealer 1998.} For instance, your own concept of gold could represent what GOLD normally represents, namely the chemical element of atomic number 79, even if you were ignorant of the atomic number of gold and even your commonsensical conceptions of gold as being yellow, being malleable and being a precious metal were to turn out all false (Kripke 1972, 116–23). That is, even if you did not know what it is to be gold, still you could possess the concept GOLD.

Let us call any necessary condition of possessing a concept $C$ a possession condition of $C$. By an external possession condition of $C$, we shall mean any possession condition of $C$ that dictates a certain relationship between the thinker and the external environment. The concept GOLD is a concept that has an external possession condition: in order to possess the concept GOLD, the thinker’s own concept of gold must be related in a relevant, causal-historical way to the natural kind gold. Kripke’s and Putnam’s work in the 1970s indicates that natural kind concepts and singular concepts have external possession conditions.

On the other hand, we shall call a possession condition of a concept $C$ internal if it requires the thinker to know something about what it is to be $C$. Consider the concept BACHELOR. Would it be possible for your own concept of bachelor to represent the property bachelorhood if you did not know that bachelors are men who are not married and have never been married? Suppose you had no idea what it is
to be a bachelor or believed, falsely, that bachelors are currently unmarried men. It would seem, then, that your thoughts involving your own concept of bachelor are not really thoughts about bachelorhood. To be sure, you could utter sentences that contain the word ‘bachelor’ and transmit, by means of them, genuine thoughts about bachelorhood. Imagine you saw a man saying that he was over 50 and a bachelor and thereby eliciting a gasp from another. You did not really understand why one would gasp at the idea of being over 50 and a bachelor, but you happened to have a liking for making people gasp. So you tried telling your friend that a man was over 50 and a bachelor, and to your excitement, you got the desired response. Here you clearly succeeded in passing a genuine thought about bachelorhood; otherwise your friend would not have gasped. However, the thought in question was not your own. You merely served as a medium of the transmission of the thought. In fact, a skilled sparrow or a recording device could have done as well. Note, in contrast, that provided your own concept of gold stood in appropriate relation to the natural kind gold, your thoughts involving it would be about gold even if you were ignorant of what it is to be gold or if you had wildly false beliefs about gold. The moral of the story is that in order to possess the concept BACHELOR and to be able to entertain thoughts about bachelorhood, you must know what it is to be a bachelor. So the concept BACHELOR has an internal possession condition.

Many questions could be raised about external and internal possession conditions of concepts, especially about their relationship. Is it possible to possess concepts like GOLD merely by meeting their external possession conditions, that is, without meeting any internal ones? To put it differently, are there concepts such that meeting their external possession conditions is sufficient to possess them? And, conversely, is it possible to possess concepts like BACHELOR by meeting their internal possession conditions alone? If not, then in what would their external possession conditions
consist? These are all interesting questions we shall have to leave unanswered here.
And, for our present purposes, we shall be henceforth concerned only with internal possession conditions of concepts and refer to them simply as possession conditions of concepts unless otherwise noted.

We are now in a position to characterize analyticity in precise terms: an analytic sentence is one which expresses anything one has to know in order to possess a concept involved. That is, an analytic sentence is one which expresses an (internal) possession condition of a concept. The sentence ‘bachelors are men’ is analytic, since it expresses a possession condition of the concept BACHELOR. One has to know that bachelors are men in order to entertain thoughts about bachelorhood. For a similar reason, ‘bachelors are unmarried’ is also analytic. Indeed, the compound sentence ‘bachelors are unmarried men’ expresses a possession condition of the concept BACHELOR. Unless you know that much, your own concept of bachelor cannot represent what BACHELOR normally does represent.

Let us pause to consider how well our view of analyticity fares compared to some classic and/or popular alternatives. Analytic sentences are quite often characterized as those which are true solely by virtue of meaning. A main difficulty here lies with the controversial notion of meaning. In one important sense of the term, the meaning of an expression could be identified with a Fregean sense of the expression, that is, a mode in which the referent of the expression is given to us. But then anyone who attaches to the proper name ‘Aristotle’ the sense of being the teacher of Alexander the Great who was born in Stagira will recognize the sentence ‘Aristotle was born in Stagira’ as true by virtue of meaning\(^4\) and hence as analytic, which is not a desired result. It is not analytic, because, intuitively, we cannot recognize it as true merely by virtue of a grasp of the concepts involved. Or, to put it more precisely, it does

\(^4\)Frege himself all but acknowledges this point (1892, 158n4).
not express a possession condition of any concepts involved. You can possess the concept Aristotle without knowing where Aristotle was born; you can possess the concept STAGIRA without knowing who was born in Stagira; and you can possess the concept BIRTH without knowing who was born where. In another important sense of the term ‘meaning’, the meaning of an expression could mean a Fregean meaning of the expression, namely its referent. But then the identity ‘the morning star is the evening star’ would be true by virtue of meaning, for the referent of the phrase ‘the morning star’ is the same as the referent of the phrase ‘the evening star’. One might retort that all this only shows that a Fregean sense or meaning of an expression should not be confused with its true meaning. But it is highly doubtful whether there is any such thing. Quite possibly, meanings do not form a single scientific kind but a motley of interrelated kinds. If this is true, then the notion of meaning is unsuitable for scientific purposes and should be best left out in favor of a notion that picks out a definite kind.

Another popular characterization of analyticity is the one Frege gives in the *Grundlagen*: a sentence is analytic if it can be derived from definitions by means of logical rules (1884, §3)\(^5\) or as Quine put it, if “it can be turned into a logical truth by putting synonyms for synonyms” (1951, 23). A precursor of Frege’s conception of analyticity may be found in Kant’s remark that the principle of contradiction is “the universal and completely sufficient principle of all analytic cognition” (1781, A151/B191).\(^6\) There are two problems with the present conception of analyticity. One of them, which Quine dissected relentlessly in his “Two Dogmas”, is that the notion of definition or synonymy is as difficult to make clear as the notion

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\(^5\)Strictly speaking, for Frege, a sentence is analytic if it can be derived from definitions and logical axioms by means of inference rules. But as is customary nowadays, we assume a formalization of logic which consists of inference rules and requires no logical axioms.

\(^6\)Frege himself admits that Kant “did have some inkling of” his notion of analyticity (1884, §88).
of analyticity itself. But more importantly, Frege took the analyticity of logic itself for granted, just as Kant apparently assumed the analyticity of the principle of contradiction.

Our own conception of analyticity provides an explanation of why logic is analytic. First, we need to extend our definition of analyticity so as to make it apply not only to sentences but also to rules of inference. We can say that a rule of inference from sentences $S_1, \ldots, S_n$ to a sentence $T$ is analytic just in case the conditional, ‘if $S_1, \ldots, S_n$ are all true, then $T$ must be true’, is analytic, that is, expresses a possession condition of a concept. We can easily see from this that every single deductive rule of inference must be analytic. Take, for instance, the introduction rule for $\text{and}$. It is analytic, because the sentence ‘if $S_1$ and $S_2$ are all true, then $S_1$ and $S_2$ must be true’ expresses a possession condition of the logical concept AND. Unless you know that if $S_1$ and $S_2$ are all true, then $S_1$ and $S_2$ must be true, your own concept of $\text{and}$ does not represent the truth-function conjunction and so you cannot entertain genuine conjunctive thoughts.

One might question whether our account of analyticity is inclusive enough. Take any complicated truth of logic, say, ‘if $P$ then [if $Q$ then (if $P$ then $Q$)]’. Intuitively, anyone who has a grasp of the concept IF-THEN should be able to recognize it as true and so it must be analytic. But do we have to know that that is true in order to possess the concept IF-THEN? It seems not. A grasp of the introduction and elimination rule for the ‘if-then’ operator should be sufficient to possess the logical concept IF-THEN. And that would mean that the sentence ‘if $P$ then [if $Q$ then (if $P$ then $Q$)]’ does not express a possession condition of a concept and hence is not analytic. Apparently, then, our account of analyticity falls short. It robs some sentences of their deserved analyticity.

However, notice that if you possess the concept IF-THEN, then you must be able
to see, given some thinking, that it is indeed the case that if $P$ then [if $Q$ then (if $P$ then $Q$)]. In fact, you should be able to prove that from a mere grasp of the rules of inference involving the ‘if-then’ operator. Therefore, should you fail to accept in time that that must indeed be the case, then you must be denied a possession of the concept IF-THEN even if you initially seemed to have a perfect grasp of the logical rules governing the ‘if-then’ operator. In other words, your failure to accept the sentence ‘if $P$ then [if $Q$ then (if $P$ then $Q$)]’ should defeat your *prima facie* possession of IF-THEN.

These considerations suggest an important distinction to be drawn between possession conditions of a concept. On the one hand, there are such things that without knowing them, it would be out of the question for you to have even a *prima facie* possession of a concept in question. We can call them *primary* possession conditions of the concept. The introduction and elimination rule of the ‘if-then’ operator express primary possession conditions of the concept IF-THEN. On the other hand, there are also such things that although you can have a *prima facie* possession of a given concept without knowing them, you must accept them, or at least cannot possibly deny them, on pain of losing your *prima facie* possession of the concept. And we can call them *collateral* possession conditions of the concept. The sentence ‘if $P$ then [if $Q$ then (if $P$ then $Q$)]’ expresses a collateral possession condition of the concept IF-THEN. Logical truths in general express either primary or collateral possession conditions of logical concepts. Therefore, every logical truth is analytic. Our account of analyticity is saved.

Kant originally introduced the notion of analyticity in terms of containment relation between concepts (1781, A6/B10–A7/B11). Besides the fact that it is hard to pin down exactly what it is for one concept to contain another, one major source of difficulty with Kant’s conception of analyticity is the existence of sentences which
state facts about interrelationships among *cluster concepts*, that is, those which are closely related to one another such as color concepts and kinship concepts. Consider the sentence ‘navy blue is darker than sky blue’. Intuitively, anyone who has a grasp of the concepts involved must recognize it as true. But the concept DARKER THAN SKY BLUE is not contained in the concept NAVY BLUE. Consider another: ‘a brother is a male sibling’. It seems as analytic as it can get, but the subject concept BROTHER does not contain the predicate concept MALE SIBLING. Rather, it would be more plausible to say that the concept SIBLING contains the concept BROTHER: a sibling is a brother or sister. Similarly, the sentence ‘a father is a male parent’ is analytic, but it is the concept PARENT that contains the concept FATHER, not the other way round. You cannot, as it seems, possess the concept PARENT unless you know that a parent is a father or mother. But you can possess the concept FATHER without knowing that a father is a male parent: look at small children.

At first glance our account of analyticity is not any better at dealing with the analyticity of the sentences involving cluster concepts. If you know what it is like to see the color navy blue (or sky blue), then you possess the concept NAVY BLUE (or SKY BLUE). And you could possess the concept DARKER THAN even if you were unable to distinguish any colors at all. So it would seem that without knowing that navy blue is darker than sky blue we can still possess any concepts involved. However, this only shows that the sentence ‘navy blue is darker than sky blue’ does not express a primary possession condition of a concept involved. If you do possess all the concepts involved, then you must be able to recognize that navy blue is darker than sky blue. Should you balk at the notion that navy blue is darker than sky blue, then that would indicate that you lack one or more of the concepts involved. That is, even if you originally appeared to possess all the concepts involved, your failure
to accept that navy blue is darker than sky blue would defeat your *prima facie* possession of one or more of the concepts involved. So the sentence ‘navy blue is darker than sky blue’ expresses a *collateral* possession condition of the concepts involved and hence is analytic.

One major consequence of our account of analyticity is that true numerical identities are analytic. Take, for instance, the identity ‘7 + 5 = 12’. It does not express a *primary* possession condition of any numerical concepts involved. You can be said to possess the individual number concepts 7, 5 and 12, at least *prima facie*, if you can correctly count to 12. And you can possess the concept PLUS, at least *prima facie*, without knowing in particular that 7 + 5 = 12. However, the identity ‘7 + 5 = 12’ expresses a *collateral* possession condition of the concepts involved. To show this, we only need to establish that you must accept that 7 + 5 = 12 on pain of losing your *prima facie* possession of one of the concepts involved.

Note first that it would be impossible for you to possess the concept 5 unless you knew at the very least that 5 = 4 + 1. So by assumption, you know that 5 = 4 + 1. Then you have to acknowledge that

\[ 7 + 5 = 7 + (4 + 1). \]

Otherwise you will lose your *prima facie* possession of the concept 5 (or the concept IDENTITY). Next, if you possess the concept PLUS, then you cannot fail to realize that \( l + (m + n) = (l + m) + n \). You may be said to possess the concept PLUS, at least *prima facie*, if you can add numbers. But if you fail to see that addition is associative, your *prima facie* possession of the concept PLUS should be defeated. That means that the associativity law of addition expresses a collateral possession condition of the concept PLUS. So on pain of losing your *prima facie* possession of the concept PLUS, you must accept that

\[ \text{We shall use italic numerals as names of number concepts.} \]
$7 + 5 = (7 + 4) + 1.$

Repeat the same procedure a few more times, and you will reach the conclusion:

$7 + 5 = (((7 + 1) + 1) + 1) + 1.$

By assumption, you possess the concept $12$, at least *prima facie*, which means that you must know at least that $12 = 11 + 1$. But unless you also possess the concept $11$, at least *prima facie*, you cannot entertain the thought that $12 = 11 + 1$. Hence you must know at least that $11 = 10 + 1$. For similar reasons, you must that $10 = 9 + 1$, $9 = 8 + 1$, and $8 = 7 + 1$. So you would have to accept that

$7 + 5 = 12,$

lest you lose your *prima facie* possession of one or more of those individual number concepts. Therefore, ‘$7 + 5 = 12$’ expresses a collateral possession condition of a cluster of concepts involved, some of which appear on the surface but many of which do not.

It is quite interesting to find Kant denying that the sentence ‘$7 + 5 = 12$’ is analytic for the reason that “in the case of an analytic proposition the question is only whether I actually think the predicate in the representation of the subject” (1781, A164/B205). Kant’s conception of an analytic sentence as one whose predicate concept we *actually think* in the subject concept exhibits some affinity with our notion of a sentence that expresses a *primary* possession condition of a concept. The identity ‘$12 = 11 + 1$’ (or ‘$12 = 10 + 2$’) should be analytic for Kant as it is for us, for in the concept $12$ we do indeed *actually think* the concept $11$ PLUS $1$ (or the concept $10$ PLUS $2$). So it seems that insofar as numerical identities are concerned, our departure from Kant’s conception of analyticity is limited to those sentences that express *collateral* possession conditions of concepts of arithmetic.
Considering that Kant’s analytic sentences are similar to our sentences that express primary possession conditions of concepts, we can now understand why Kant thinks that analytic sentences merely clarify concepts, while sentences like ‘7 + 5 = 12’ amplify concepts in the sense that they “add to the concept of the subject a predicate that was not thought in it at all” (1781, A7/B11). As he rightly points out, “one becomes all the more distinctly aware of that if one takes somewhat larger numbers” (1781, B16). Take, for instance, the identity ‘135664 + 37863 = 173527’. We could never actually think the concept 173527 while thinking the concept 135664 PLUS 37863. So, for Kant, the identity serves to amplify the concept 135664 PLUS 37863 and so is synthetic.

Now, thus understood, Kant’s clarification/amplification distinction should seem rather frivolous, if correct. What interests us is not whether a sentence clarifies or amplifies any concept but whether it is informative in the sense that it adds something new to what we already know. To see that Kant’s clarification/amplification distinction is different from our intuitive distinction between trivial (that is, non-informative) and informative sentences, note that not all amplification-sentences in Kant’s sense are informative. Consider ‘7 + 5 = 12’, again. By Kant’s standards, it is synthetic and an amplification-sentence: we do not think the concept 12 while thinking the concept 7 + 5. But, as seen above, we can derive it from a number of premises in a strictly logical way. Moreover, all of its premises express primary possession conditions of some or other concepts, and hence none of them is informative. Therefore, it cannot be informative by inheriting informativeness from its premises. How can a sentence be informative, then, if it is derived from trivial premises by strictly logical means? The only possible way is for the derivation process itself to give rise to something new, but that is not a real possibility. Logical inference cannot induce anything that is not already contained in the premises.
Some people have held that Frege believed otherwise. Citing his remark in §91 of the *Grundlagen* that “propositions which extend our knowledge can have analytic judgments for their content” and his earlier remark in §17 that each truth of arithmetic “would contain concentrated within it a whole series of deductions for future use, and the use of it would be that we need no longer make the deductions one by one, but can express simultaneously the result of the whole series”, Michael Dummett concludes that for Frege, “The value of analytic propositions and that of deductive inference are essentially the same” (1991, 36). He interprets Frege’s latter remark as follows:

The point of an analytic proposition, in other words, is to encapsulate an inferential subroutine which, once established, may be repeatedly appealed to without itself having to be repeated: it is not the truth of analytic propositions which is in itself important, but their service in easing our deductive transitions from synthetic truths to other synthetic truths. Frege’s contradiction of Kant’s dictum [that analytic propositions do not extend our knowledge] thus represents his acknowledgement of the fruitfulness of deductive inference. (1991, 36)

No doubt Frege’s text leaves enough room for such an interpretation. Right after emphasizing the deduction-facilitating role of truths of arithmetic, he adds that “If this be so, then indeed the prodigious development of arithmetical studies, with their multitudinous applications, will suffice to put an end to the widespread contempt for analytic judgments and to the legend of sterility of pure logic” (1884, §17). Surely it sounds as if he attributes the fruitfulness of pure logic to the role of analytic sentences as deduction-facilitators in logical inference.

There is no denying that if a proof proceeds by appeal to a sentence that encapsulates a series of deductions, that is, by making a jump, then the conclusion will certainly appear to say something new. And quite often, as Frege keenly observed,

the correctness of such a transition is immediately self-evident to us, without our ever becoming conscious of the subordinate steps condensed
within it; whereupon, since it does not obviously conform to any of the recognized types of logical inference, we are prepared to accept its self-evidence forthwith as intuitive, and the conclusion itself as a synthetic truth . . . . (1884, §90)

But that would be a mistake. It is a psychological illusion that we have made an intuitive, non-logical transition and attained a conclusion that contains new information. It was the very purpose of Frege’s invention of a formal language to eliminate any such illusion by making sure that “no step is permitted which does not conform to the rules which are laid down once and for all” (1884, §91). That is, Frege’s formal language is deliberately designed to make sure that every single step in the proof is spelled out in accordance with one of those rules. But if deduction-facilitating sentences serve no essential purposes in Frege’s formal language, how could Frege (be taken to) claim that analytic sentences extend our knowledge because they serve “in easing our deductive transitions from synthetic truths to other synthetic truths”?

Dummett realizes later that “Frege believed . . . that every proof could be broken down into extremely small steps” and “It was therefore necessary to solve the problem of the fruitfulness of deductive inference, not at the level of entire proofs, but at that of the simplest single steps” (1991, 38). According to Dummett, Frege’s solution can be found in his insight that deductive reasoning, at every single step, requires a creative process of discerning a logical pattern (1991, 38–42). “Since it has this creative component”, Dummett claims, “a knowledge of the premisses of an inferential step does not entail a knowledge of the conclusion, even when we attend to them simultaneously” “and so deductive reasoning can yield new knowledge” (1991, 42). One cannot help wondering if he is equivocating. The type of new knowledge Dummett here says deductive reasoning yields concerns a certain logical pattern in reference to which the logical transition from the premises to the
conclusion is justified. But this is definitely not the type of new knowledge that we have in mind when we ask whether logical inference, at the smallest steps, can extend our knowledge.

The question is whether an application of a simple logical rule can result in a conclusion that is richer in content than its premises. Could there be any such rules in deductive logic? We effectively answered ‘no’ when we argued above that logical rules express primary possession conditions of logical concepts. Take the concept AND again. It is impossible to possess it and to entertain thoughts involving it, unless you already know that if $S_1$ and $S_2$ are all true, then $S_1$ and $S_2$ must be true. So, provided that you possessed the concept AND, you could not possibly find the truth of $S_1$ and $S_2$ to be any more revealing than that of $S_1$ and that of $S_2$ taken together. And essentially the same can be said of any other deductive rule of inference.

One might object, as Dummett does, that it would follow, then, that “we should already know all consequences attainable by a sequence of such simple steps, however long” (1991, 36–37). Presumably, this is not desirable because, as a matter fact, we do not know all such consequences. But we need to draw a distinction between being actually committed to a belief and being in principle committed to it. The human mind is not an ideal cognitive system; it is realized in a physical medium with all sorts of consequent compromises and limitations. We are actually committed to no more than a very small number of consequences of our premises. However, all this is a matter of psychology, not logic. We are in principle committed to all consequences of our premises, and that is what matters from the logical point of view.

The upshot is that deductive inference cannot extend our knowledge, and to that extent, pure logic is sterile. Now, recall that for us analytic sentences express possession conditions of concepts, either primary or collateral. Those analytic sentences
which express primary possession conditions of concepts are trivial, for unless we accept them as true, we cannot possess the concepts involved, not even \textit{prima facie}. And since logical inference does not increase our knowledge, all the sentences that can be logically derived from those which express primary possession conditions of concepts must be also trivial. That is, those analytic sentences which express \textit{collateral} possession conditions of concepts are trivial if they are logical consequences of those analytic sentences which express primary possession conditions of concepts.

Does this mean that no analytic sentences whatsoever can be informative and extend our knowledge? My answer is ‘no’. To explain why, we turn to a discussion of definitions. By a definition we shall mean whatever provides a complete answer to the question, “what is it to be \(C?\)” (or any other appropriate what-is-it question) without recourse to the concept \(C\).\footnote{The so-called \textit{implicit} definitions do not provide such an answer and hence are not definitions. For details, see Chapter 5.} In the broadest terms, depending upon whether the answer given can be assessed in terms of correctness, we may divide definitions into \textit{normative} and \textit{stipulatory} definitions. Stipulatory definitions create simple concepts out of compound concepts whose elements are already given, and to that extent they can be compared to Kant’s \textit{mathematical} definitions (1781, A727/B755–A732/B760) and to Frege’s \textit{constructive} definitions (1979, 207–11). Since they arbitrarily introduce concepts, it makes no sense to ask whether they define the concepts correctly or incorrectly, and so the sentences that express them can be neither true nor false. However, stipulatory definitions can still be assessed in terms of usefulness. For instance, you can create the concept \textsc{Biangle} by defining a biangle as a geometrical figure enclosed by two straight lines; you can even introduce the concept \textsc{Uniangle} if you please. But such efforts will be of no use since the concepts pick out no objects. Mathematics proceeds by making stipulatory
definitions out of a handful of primitive, undefined concepts. The chief purpose of stipulatory definition is to streamline a body of truths so as to make it manageable to us. Strictly speaking, concepts introduced by stipulation are superfluous.

On the other hand, normative definitions explain concepts which are already firmly established in use. That makes it possible for us to raise the question of correctness about them. Depending upon whether or not this question can be resolved a priori, normative definitions divide into *a priori* definitions and *a posteriori* definitions. A posteriori definitions coincide with those which identify the inner essence of things found in nature. Examples may include: ‘gold is the chemical element of atomic number 79’, ‘water is H$_2$O’, ‘heat is mean molecular kinetic energy’, and ‘a horse is an *equus caballus*’. To the extent that a posteriori definitions attempt to reduce familiar natural kinds to scientific kinds, they might be called *reductive* definitions. Note that no reductive definitions express possession conditions of concepts and hence none of them can be analytic.

A priori definitions are also subdivided into two groups. Firstly, there are those a priori definitions which report conventional uses of certain concepts and hence might be called *conventional* definitions. We have already seen some of them above: ‘bachelors are unmarried men’ and ‘a parent is a father or mother’. Conventional definitions are trivial, though a priori, because they express *primary* possession conditions of the concepts involved. Secondly, and these are the ones we are chiefly interested in here, there are those which result from a priori analysis of concepts. Kant called them *philosophical* definitions or *expositions* (1781, A727/B755–A732/B760) and Frege *analytic* definitions (1979, 207–11). But we shall call them *explicative* definitions, or *explications*, in short, for a reason that will become clear later. There is widespread doubt about the possibility of conceptual analysis. But for our immediate purposes, what is important is not whether there can be any genuinely successful
cases of explication but, provided there was one, what its epistemic status would be like.

Consider the ancient definition of knowledge as justified true belief. It is now generally acknowledged to be a failure, but let us suppose it was a correct explication of the concept KNOWLEDGE. The sentence ‘knowledge is justified true belief’, which expresses the supposed explication, is not at all trivial. Quite the contrary, it is one of the most informative sentences one could ever imagine. Frege once held that an explication is correct if and only if the definiendum has the same sense as the definiens (1979, 209–10), where the sameness of sense is supposed to be “self-evident” to us (1979, 210). But clearly this is not an adequate criterion of correctness of an explication. Explicative definitions do not simply report the meaning of words as conventional definitions do. It is plausible to say that if you use a word perfectly well, then you know its meaning. However, you can use the verb ‘know’ perfectly well without having an inkling of how knowledge is related to justified true belief. Kant was far closer to the truth of the matter, though a little too skeptical, I think, when he observed that whether an explication is correct “is always doubtful, and by many appropriate examples can only be made probably but never apodictically certain” (1781, A728/B756–A729/B757). Think of how many examples have been devised to test how many different accounts of what knowledge is!

Another crucial point to note about explications is that their consequences are informative, too. The fact that knowledge requires belief, let alone true belief or justification, might never occur to many people who do possess the concept KNOWLEDGE and are fully capable of entertaining thoughts involving it. Note that the

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9 We shall see one below later and have plenty in Chapter 7. In point of fact, we have been involved in giving and defending an explication of the concept of analyticity, though we should await the ruling of the tribunal of reason concerning whether it is ultimately successful.

10 One could argue that it did give a partially correct explication of the concept, but in our sense of the term, an explication, as a definition, has to be complete.
consequences of an explicative definition are informative not because deductive reasoning itself yields new information but because they inherit information from the definition, which is the original source of information. So they do not add anything to what the source sentence says, but they indeed extend our knowledge in that they are informative (considered in and by themselves).

While we have shown that explications and their consequences are informative, we should not conclude from this that there are informative analytic sentences until we have good reason to believe that explications are analytic. In what follows, I shall argue that we do. On our view, analytic sentences are those which express either primary or collateral possession conditions of concepts. Clearly, explication-sentences do not express primary possession conditions of concepts. So in order to be analytic, they must express collateral possession conditions of concepts, that is, those things you cannot possibly fail to accept under pain of losing your prima facie possession of them. The question, then, is this: if you did fail to see the truth of an explication-sentence, would that really be enough to defeat your prima facie possession of the concept to be explicated? Or to put it conversely, would it really be the case that if your prima facie possession of the concept to be explicated were so secure that you could never lose it, then you could not possibly fail to see the correctness of an explication of it?

To answer these questions, it would be necessary to figure out what one would have to know in order to possess, prima facie, a concept to be explicated. In other words, what constitutes a primary possession condition of a concept to be explicated? Take the concept KNOWLEDGE. Under what circumstances could, and should, we say that one possesses, at least prima facie, the concept KNOWLEDGE? It would seem that if you are able to apply the concept KNOWLEDGE consistently to all and only those cases that fall under it, then you possess the
Let us say that one has an *implicit* grasp of a concept $C$ if one can correctly tell, of any given thing, whether it falls under $C$ or not. For instance, if you can correctly tell, of any given piece of information, whether you know it or not, then you have an implicit grasp of the concept KNOWLEDGE. The point is that in order to possess, at least *prima facie*, a concept to be explicated, it would be sufficient to have an implicit grasp of the concept. For example, you can possess the concept NATURAL NUMBER, at least *prima facie*, if you know that 1, 2, 3 and the like, and none others, are natural numbers. Philosophical concepts are almost without exception those of which we have only an implicit grasp: Kant lists SUBSTANCE, CAUSE, RIGHT and EQUITY (1781, A728/B756), but we can add VIRTUE, JUSTICE, OBJECT, PROPERTY, EVENT, NECESSITY, and so on and so forth.

We can now explain what it is to explicate a concept: to *explicate* a concept $C$ is to make an *implicit* grasp of it *explicit* by giving a necessary and sufficient condition of falling under $C$. It is a distinct possibility that there be concepts of which we have an implicit grasp but which cannot be explicated. For, even if we knew what things fall under them and what not, we still might inherently lack the more basic conceptual resources necessary to capture the conditions of falling under them. Such concepts might be called *intuitive* concepts and could include moral concepts such as RIGHT and VIRTUE (that is, MORAL GOODNESS) and metaphysical concepts such as OBJECT and PROPERTY. And recently a suggestion was made to the effect that the concept KNOWLEDGE could not be explicated in terms of the concept BELIEF and must be instead considered an intuitive concept (Williamson 2000).

Put in terms of the distinction between an implicit and explicit grasp of a concept, our original question was this: provided you had an implicit grasp of a concept $C$ that could be explicated (at least in principle), would it be possible for you to
fail to see the truth of a sentence which gives a necessary and sufficient condition of being a $C$? That is, if you already know what things are $C$s and what things not, could you possibly fail to see that $C$s and only $C$s satisfy the condition given by an explication of $C$?

The answer must be ‘no’, and the reason has to do with the fact that explications allow of a priori assessment of their correctness. Consider an a posteriori definition, say, ‘a horse is an *equus caballus*’. It would be totally unreasonable to think that anyone who can tell horses from other animals including asses and zebras must not fail to see that a horse is an *equus caballus*. The definition gives the inner nature of horses, which it takes serious scientific research to penetrate. So no one should be supposed to be able to see the *essence* of horses merely by virtue of knowing their apparent features, however thoroughly one did. In contrast, consider again our supposed explication of the concept KNOWLEDGE. All it takes to see that knowledge is justified true belief is a priori reflection upon what you already know, namely the individual cases you class as your knowledge and the individual cases you do not. You simply ask yourself whether all and only those cases you would class as what you know are justified true beliefs. And provided you know what it is to be a justified true belief, you should be able to reach an answer in time by mere reflection. In other words, you can just think carefully about your implicit grasp of the concept KNOWLEDGE and thereby determine whether knowledge is justified true belief.

Generally speaking, if you have an implicit grasp of a concept $C$, then you must be able to assess the correctness of an explication of $C$, for that implicit grasp of $C$ is all you need for that purpose apart from some thinking. If you fail to see the truth of an explication-sentence which specifies a necessary and sufficient condition of being a $C$, then you must be denied your implicit grasp of $C$. But since anyone
who has an implicit grasp of $C$ possesses $C$, at least \emph{prima facie}, it follows that you cannot fail to see the truth of an explication-sentence of a concept $C$ under pain of losing your \emph{prima facie} possession of $C$. And that means that explication-sentences express \emph{collateral} possession conditions and hence are analytic. The upshot is that explication-sentences are informative yet analytic.

Now we can finally give an answer to the question we raised at the very beginning of this chapter: how can arithmetic be so fertile if logicism is true? Our investigation into analyticity suggests the following answer: unlike logic, arithmetic may be founded upon \emph{explications} of its basic concepts. Logical concepts cannot be explicated, not because they are too basic and fundamental to human reason but because there is nothing about our grasp of them that can be made more explicit. Once you have grasped the rules of inference associated with them, you have learned everything you need to know about them. The sterility of pure logic is a natural result of its being based upon no presuppositions whatsoever including explications, except a set of rules of inference.

But the situation is quite different with arithmetic. Consider the (second-order) Peano Axioms, which have all the truths of arithmetic as logical consequences. For our purposes, they can be presented as follows:

\begin{enumerate}
  \item \textbf{P1} 1 is a natural number;
  \item \textbf{P2} for every $n$, if $n$ is a natural number, then a successor of $n$ is a natural number;
  \item \textbf{P3} for every $m$ and $n$, if $m$ and $n$ are natural numbers, then ($m$ is identical with $n$ if and only if a successor of $m$ is identical with a successor of $n$);
  \item \textbf{P4} for every $n$, if $n$ is a natural number, then 1 is not identical with a successor of $n$;
  \item \textbf{P5} for every $P$, if 1 is a $P$ and (for every $m$, if $m$ is a $P$, then a successor of $m$ is a $P$), then (for every $n$, if $n$ is a natural number, then $n$ is a $P$).
\end{enumerate}
Peano Arithmetic, namely a set of logical consequences of the Peano Axioms, presupposes several concepts: witness ONE, NATURAL NUMBER and SUCCESSOR. It is not that there is anything wrong with Peano Arithmetic. As a mathematical theory, it can be no more perfect. And a mathematician’s job is just to proceed from a set of axioms downwards toward its less abstract, more concrete consequences. But let us remember we are philosophers and as such should never “consider these hypotheses as first principles but truly as hypotheses—but as stepping stones to take off from, enabling [us] to reach the unhypothetical first principle of everything” (Plato 1997, 511b/1132). We philosophers should never content ourselves with the mathematicians’ hypotheses containing concepts that are assumed to be known, however self-evident they might be. A philosopher’s job is to travel up towards the Platonic First Principles, which shall be none other than the explications of all the concepts that the mathematicians had to leave unexplained.

Take P5, for instance. It states the method of proof called mathematical induction: once it is established that a property is possessed by 1 and by every successor of a number possessing it, it is thereby proved that it is possessed by every natural number. The induction principle appears to allow a transition of irreducibly arithmetical nature and was considered for a long time a hallmark of arithmetical, as opposed to logical, reasoning. But consider what it is to be a natural number. Usually, the best answer we can offer to this question is that 1, 2, 3 and the like (and nothing else) are natural numbers. That is, we usually have only an implicit grasp of the concept NATURAL NUMBER. Yet there is a condition which 1, 2, 3, and the like satisfy and nothing else does: a natural number is a number which has every property possessed by 1 and by every successor of a number possessing it. To put it more precisely:

\[ \text{Ind } n \text{ is a natural number } \leftrightarrow \text{ for every } P, \text{ if } 1 \text{ is a } P \text{ and (for every } m, \text{ if } m \text{ is } \]
a $P$, then a successor of $m$ is a $P$), then $n$ is a $P$.$^{11}$

Thus, Ind gives an explication of the concept NATURAL NUMBER and as such is analytic.$^{12}$ Note that $P5$ is an immediate consequence of Ind. That is as it should be, for Ind says in effect that natural numbers are those for which mathematical induction holds. No one could employ induction without being struck by its power; but it turns out to be a trivial consequence of an explication of a concept. In point of fact, Ind is so rich in content that it also directly implies $P1$ and $P2$. So from a single explication-sentence follow no less than three self-evident principles of Peano Arithmetic, which originally appeared to admit of no further proof at all.

However, we have barely begun our job. We have yet to explain the concept ONE and the concept SUCCESSOR. But most importantly, note that in formulating the Peano Axioms and Ind we made use of numerical variables such as $m$ and $n$, and by doing so we assumed the concept NUMBER to be known. It is a different concept than NATURAL NUMBER: the variables are supposed to range over all (cardinal) numbers, not just natural numbers. What is it, then, to be a (cardinal) number? We are totally in the dark here. Indeed, we cannot even seem to answer what sort of thing a number is, let alone what exactly it is. Is it an object of a certain kind? Or is it rather a property of a certain kind? Moreover, by allowing numbers to belong to the range of quantification, we assumed that we could always determine whether a number is identical with another. That is, we assumed that there is a criterion for the identity of numbers. But how could we justify such an assumption? And if we could, what would such a criterion be like?

Welcome to the most difficult conundrums in the philosophy of arithmetic!

$^{11}$This definition of natural number is essentially the same as Frege’s definition of “finite number”. See Frege 1884, §83. It was Russell who explicitly formulated it as presented here. See Russell 1908, 80. Also see Russell 1912, 27.

$^{12}$Ind has been often accused of being impredicative. We shall argue in Chapter 7 that it is impredicative but only in an innocent, harmless way.
CHAPTER 3

ARE NUMBERS OBJECTS? PART I:
AGAINST THE ISOMORPHISM ARGUMENT

According to Frege, numbers are objects of a certain sort, or more precisely, extensions of concepts.¹ But he did not hold the view of numbers as objects because he believed, first, that numbers are extensions of concepts and, second, that extensions must be objects. In his philosophical masterpiece, the *Grundlagen der Arithmetik*, he forcefully concludes that numbers must be objects (1884, §57) well before he identifies them, reluctantly, with extensions of concepts (1884, §69). And the eventual discovery that our intuitive notion of an extension of a concept is inconsistent did not shake his conviction that numbers are objects. On the contrary, there are signs that at the end of his life he turned to geometry for a foundation of arithmetic exactly because he thought he could thereby establish the status of numbers as objects (1979, 278–81). In this chapter we shall examine in detail Frege’s deceptively simple argument for the conception of numbers as objects. It is, as we shall see, extremely sophisticated and strong, yet we shall argue against it by showing that the so-called context principle it crucially relies upon is wrong.

¹For the sake of ease of discussion, we will adopt Frege’s technical use of the word ‘concept’, which should not be confused with its most common use in English. Frege’s concepts are what we normally call *properties*, as can be seen from his own following remark: “I call the concepts under which an object falls its properties” (1892b, 190). We shall use ‘notion’ as a substitute for ‘concept’ in the common sense of the word.
In §57 of the *Grundlagen* Frege says that “we speak of ‘the number 1’, where the definite article serves to class it as an object” and claims that “In arithmetic this self-subsistence comes out at every turn, as for example in the identity $1 + 1 = 2$”. Following Frege’s terminology, let us call any singular term that is not an individual variable a *proper name* and any expression obtained by omitting one or more proper names from a sentence a *predicate* (1979, 187). Thus understood, proper names include definite descriptions of the form ‘the $F$’ and predicates include expressions for relations. His argument, at a first approximation, is simply this: numbers are objects because numerals behave as proper names as manifested in their use in arithmetical contexts, especially in the context of a numerical identity of the form ‘$m = n$’, and proper names refer to objects.

The all-important premise that proper names refer to objects is itself part of a thesis we might call *Isomorphism*: proper names are correlative to objects and predicates to concepts. The thesis finds a clearest expression in the following passage:

> Take the proposition ‘Two is a prime number’. . . . The first constituent, ‘two’, is a proper name of a certain number; it designates a whole that no longer requires completion. The predicative constituent ‘is a prime number’, on the other hand, does require completion and does not designate an object. I also call the first constituent saturated; the second, unsaturated. To this difference in the signs there of course corresponds one in the realm of meanings: to the proper name there corresponds the object; to the predicative part, something I call a concept. (1903a, 281)

The saturated/unsaturated or complete/incomplete distinction is supposed to apply not only to linguistic expressions but also to what they refer to. Proper names are *complete* and predicates *incomplete* in the sense that the latter, but not the former, contain empty places (1891, 146). Similarly, referents of proper names, namely *objects*, are complete wholes themselves and referents of predicates, namely *concepts*,

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2Strictly speaking, it should be called a *first-level* predicate. But we shall use the term ‘predicate’ in this restricted sense throughout this chapter.
are “unsaturated” and “predicative in nature” themselves (1979, 193).\(^3\) No complete expressions can refer to incomplete entities, just as no incomplete expressions can refer to complete entities. Hence the criterion: “the singular definite article always indicates an object, whereas the indefinite article accompanies a concept-word” (1892b, 184).

Is Isomorphism correct? One well-known problem with it is that it immediately leads to the so-called paradox of the concept horse. The sentence ‘the concept horse is a concept’ must be false, for by Isomorphism the proper name ‘the concept horse’, which contains no empty places, has to refer to an object, not a concept. Frege’s immediate response to this paradoxical result was that the expression ‘the concept horse’ “do[es] designate an object” (1892b, 184) which “must go proxy for” what it really purports to refer to, namely, a concept (1892b, 186). But in a paper he composed shortly after “Über Sinn und Bedeutung” and failed to get published, he takes the previous position back and affirms that “objects and concepts are fundamentally different and cannot stand in for one another” (1979, 120). The use of expressions like ‘the concept horse’ only “obscures the real sense” (1979, 120) and—this is crucial—the incomplete or predicative nature of a concept can be “represented [sich darstellen] in the concept-script by leaving at least one empty place in its designation where the name of the object which we are saying falls under the concept is to go” (1979, 120). The German expression ‘sich darstellen’, which means in this context exactly the same as ‘sich zeigen’, might as well be rendered as ‘shown’. Frege’s unmistakable point here, the point Wittgenstein was to make so much of in his Tractatus Logico-Philosophicus, is that a concept can be shown to

\(^3\)Frege also claims that functions in general “must be called incomplete, in need of supplementation, or ‘unsaturated.’” (1891, 140) Given that for him, a concept is “a function whose value is always a truth-value” (1891, 146), it follows that concepts are incomplete by themselves. In fact, he says already in the Grundlagen that a relation concept is “incomplete at two points” and “has always to be completed” (1884, §70).
be a concept through the form of the predicate that designates it. We cannot say of any concept that it is a concept.  

Inasmuch as the concept *concept* cannot be ascribed to any concepts, it is not a genuine concept: it is a *pseudo-concept*. The incomplete expression ‘... is a concept’, which is a linguistic correlative of the concept *concept*, can never yield a true sentence when completed by an expression that designates a concept: it is a *pseudo-predicate*. Unlike a concept proper, which can be represented by a predicate, the concept *concept* can only be represented by the characteristic form of predicates. To borrow a *Tractarian* term, the concept *concept* is a *formal* concept (1921, 4.126).

Interestingly, while Wittgenstein further holds in the *Tractatus* that the concept *object* is a pseudo-concept as well (4.1272), Frege regards the concept *object* as a genuine concept. The corresponding predicate, ‘... is an object’, is a genuine predicate that combines with proper names to yield true sentences. The sentence ‘the planet Venus is an object’ is not only well-formed but true. In what follows, we shall first argue that the so-called context principle, namely that an expression can have reference only in the context of a sentence (Frege 1884, §62; Wittgenstein 1921, 3.3), implies that the concept *object* is indeed a pseudo-concept. Then we shall show that unlike Wittgenstein, Frege did not intend to apply the context principle to expressions for actual concrete objects. Expressions like ‘Venus’ can have reference independently of the context of a sentence by way of the *senses* attached to them. These considerations will provide us with a strategy in our argument against Isomorphism. Frege ultimately holds the concept *concept* to be a pseudo-concept because he believes that the context principle applies to concept-expressions.

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4Peter Geach argued in his 1976 that Frege’s philosophy of logic implies the *Tractarian* distinction between saying and showing.

5In the *Tractatus* Wittgenstein favors the word ‘function’ over ‘concept’ and says that the concept *function* is a formal concept (1921, 4.1272).
in general. We shall argue, however, that just like certain object-expressions, certain concept-expressions can have reference independently of the context of a sentence through the *senses* attached to them. The upshot will be that just like the concept *object*, the concept *concept* is a genuine concept, and sentences like ‘the concept *horse* is a concept’ are not only well-formed but true. Therefore, as we shall conclude, not all proper names refer to objects and hence the Isomorphism argument for the conception of numbers as objects breaks down.

The context principle denies that an expression can have reference in itself outside the context of a sentence. What makes a certain expression an object-expression or a concept-expression is the context of a sentence in which it occurs. We should not take an expression in isolation from the context of a sentence and ask whether it refers to an object or a concept, for otherwise we would be inevitably led to consider the mental images associated with it and thus fall into the error of psychologism (Frege 1884, x). To see whether an expression refers to a concept, we should rather look at the sentences in which it occurs and see whether it contains empty places to be filled in by an expression or expressions that are complete. By the same token, what makes an expression refer to an object is nothing other than the fact that it is used in sentences to fill in an empty place contained in an incomplete expression. Take the sentence ‘Venus is not a planet that can sustain life but Mars is no Venus’. The word ‘Venus’ first fills in an empty place in the predicate ‘… is a planet that can sustain life’. So it is a proper name and refers to the planet Venus. But it plays the role of a predicate in ‘Mars is no Venus’, in which it means more or less the same as ‘planet that cannot sustain life’ and hence refers to a concept.

Now consider the sentence ‘Venus is an object’. It appears to be of the same logical structure as the sentence ‘Venus is bright’. However, there is a crucial difference between the two: to say that Venus is bright is to ascribe a genuine quality to
the object, Venus, but if the context principle is right, that is, if we can determine
whether or not a certain thing is an object only by the logical behavior of an expres-
sion for that thing in the context of a sentence, then to say that Venus is an object
is simply to say that the word ‘Venus’ logically behaves as a proper name in the
sentences in which it occurs. Thus, contrary to appearances, the sentence ‘Venus is
an object’ expresses no genuine thought about the intended object, namely Venus,
and to that extent, it is a pseudo-sentence. The concept object cannot be genuinely
ascribed to any objects themselves and so it is a pseudo-concept. The corresponding
incomplete expression, ‘… is an object’, cannot be genuinely attached to a proper
name and so it is a pseudo-predicate. An object cannot be said to be an object;
it can only be shown to be an object by the logical behavior of an expression for
it in the context of a sentence. In short, the concept object is represented by the
characteristic form of proper names: it is a formal concept.

Given these considerations, we can see the point of Wittgenstein’s claim that the
word ‘object’ yields a pseudo-sentence if it occurs in the place of a genuine predicate
(1921, 4.1272). For instance, if the existential quantifier is attached to it, there arise
unintelligible sentences like ‘there are objects’ (1921, 4.1272). This sentence appears
to be structurally on a par with the sentence ‘there are books’. But notice that the
latter actually means that there are objects that are books. Existential sentences
of the form ‘there is an \(F\)’, where ‘\(F\)’ stands for a genuine predicate, can always
be expanded into those of the form ‘there is an object that is an \(F\)’.
In contrast, the sentence ‘there are objects’ could not be expanded in a similar way: it is plain
nonsense that there are objects that are objects. The word ‘object’, if correctly used,
always functions as a variable (1921, 4.1272). So the logical form of the sentence
‘there are objects’ should be symbolically represented simply as ‘\(\exists x\)’, which is not
a well-formed sentence at all.
But if the context principle leads to the conclusion that the expression ‘object’ cannot be properly used as a predicate but only as a variable, how could Frege, the original proponent of the principle, hold otherwise? To answer this question we need to examine Frege’s celebrated distinction between sense and reference. Consider an identity sentence of the form ‘a is b’. Wittgenstein holds in the *Tractatus* that ‘a is b’ says nothing about the objects a and b (1921, 4.242), just as ‘a is an object’ says nothing really about the object a. Just as ‘a is an object’ rather concerns the expression ‘a’ and says that it behaves as a proper name, so the identity sentence actually concerns the two expressions ‘a’ and ‘b’ and says that they refer to one and the same object (1921, 4.241). This means that identity is not a genuine relation that holds between *objects* (1921, 5.5301) and so it is a pseudo-relation. It is merely a rule of substitution of two distinct expressions (1921, 4.241).

Since an identity does not express a relation between objects, *a fortiori* a self-identity of the form ‘a is a’ cannot be a relation which an object bears to itself. Nor can it serve as a rule of substitution of two distinct expressions, for inasmuch as ‘a’ and ‘a’ refer to the same object, they are the same expression (Wittgenstein 1921, 3.203). The self-identity, ‘a is a’, thus concerns neither the object a nor the expression ‘a’. What it purports to say, namely that a is identical to itself, should rather be *shown* by using the expression ‘a’ consistently as the name of a same object in every sentence in which it occurs (1921, 5.53). Similarly, two objects a and b cannot be *said* to be distinct by means of a non-identity ‘a is not b’. They can only be *shown* to be distinct by employing the names ‘a’ and ‘b’ to designate two distinct objects (1921, 5.53), for instance, by asserting that a is F but b is not.

Frege’s mature view of identity diverges from the *Tractarian* view of identity as a substitution rule governing the use of two distinct *signs*, which he himself
had originally proposed in the *Begriffsschrift*\textsuperscript{6} and continued to hold onto in the *Grundlagen*.\textsuperscript{7} He realizes a serious difficulty with his previous view of identity: if ‘\(a\) is \(b\)’ were a mere license to substitute one name with the other when either was used in a sentence, then it would deliver no information about any *objects*. But obviously that is not the case. The sentence ‘Mark Twain is identical with Samuel Clemens’ does not say that the *expression* ‘Mark Twain’ refers to the same object as the *expression* ‘Samuel Clemens’ but that the *object* called *Mark Twain* is the same as the *object* called Samuel Clemens. It is not about the *signs* ‘Mark Twain’ and ‘Samuel Clemens’ but about the *object* in question, namely the famous American writer himself.

Frege’s new approach to the notion of identity in “Über Sinn und Bedeutung” aims at resolving the puzzle of how an identity can be a genuine relation between *objects*. Unfortunately, this fact remains largely ignored in the literature on Frege, mainly because the above-mentioned paper has been interpreted as an initiative to tackle an epistemic puzzle, namely why ‘\(a\) is \(a\)’ is immediately certain while ‘\(a\) is \(b\)’ is not. If an identity sentence merely said that a relation holds between objects, then there should be no epistemic difference between ‘\(a\) is \(a\)’ and ‘\(a\) is \(b\)’, for both ‘\(a\) is \(a\)’ and ‘\(a\) is \(b\)’ express the same relation that one and the same object bears to itself. By drawing a distinction between sense and reference one can easily answer the epistemic puzzle. An identity sentence is not merely about reference but also about sense: at the level of *reference*, ‘\(a\) is \(a\)’ and ‘\(a\) is \(b\)’ make no difference, but at the level of *sense*, the latter has a cognitive value that the former lacks.

But note that the epistemic puzzle can be also readily solved if an identity is

\textsuperscript{6}Frege defines ‘\(A = B\)’ to mean that “the sign \(A\) and the sign \(B\) have the same conceptual content, so that we can everywhere put \(B\) for \(A\) and conversely” (1879, 21).

\textsuperscript{7}For instance, Frege says that the sentence ‘the number of Jupiter’s moons is four’ states that “the expression ‘the number of Jupiter’s moons’ signifies the same object as the word ‘four’” (1884, §57).
taken to express a relation between *signs* themselves. Frege himself says that that is why he originally came up with his *Begriffsschrift* view of identity (1892a, 157). The problem Frege wanted to address by making a distinction between sense and reference of an expression is *not*, as it is widely believed to be, the epistemic puzzle of why ‘*a* is *a*’ is trivial and ‘*a* is *b*’ is informative. The real problem he wanted to solve is rather that on the *Begriffsschrift* view of identity an identity sentence could not be about the intended *objects* and we could take it as concerning any arbitrary objects we please. That is, the problem is that if identity is a relation between *signs*, then “we would express no proper knowledge by its means” (1892a, 157).

So the real question is this: how is it that an identity of the form ‘*a* is *b*’ expresses a genuine relation between the objects *a* and *b* and, at the same time, can be of epistemic significance? Frege’s answer is that a proper name referring to an object expresses a *sense*, which contains a way or mode in which the object is given to us (1892a, 158). So a sense of an expression *connects* the expression to its referent; an expression is *anchored*, as it were, to its referent via a sense it expresses. An identity sentence, ‘*a* is *b*’, can be genuinely about the objects *a* and *b* because the names ‘*a*’ and ‘*b*’ are connected to those objects by way of their senses, and at the same time it can have cognitive value because those names express different ways or modes in which one and the same object is given to us.

Now, given that a sense of an expression contains a mode of presentation of its referent, an expression can attain its sense independently of the context of a sentence *if* the referent itself can be given to us independently of such a context. An actual object like Venus can be introduced in many different ways. It can be introduced by description, for instance by saying that Venus is the second planet from the sun in the solar system. Introducing an object by description involves using an identity sentence, and so the name of the object thus introduced obtains a
sense in the context of a sentence. But Venus can also be introduced independently of the context of a sentence by directly pointing at it or presenting a picture of it. To be sure, even in these cases some or other linguistic devices would have to be used. We might have to utter the word ‘Venus’ or write it on paper. But it would seem at least that no sentential contexts are involved.

One might object that merely uttering the word ‘Venus’ while pointing to Venus would not be enough to establish it as a proper name for the planet. For one could also utter the word ‘planet’ without thereby intending it as a proper name for the planet. This objection seems fine as long as we consider Venus alone. But imagine a situation where one utters a name of each of the brightest celestial bodies while each time pointing to an appropriate luminous object in the sky. In such a situation words like ‘Venus’, ‘Mars’, ‘Sirius’ and ‘Canopus’ will be introduced as proper names for distinct individual celestial bodies. In fact, there is a way to establish the word ‘Venus’ as a proper name without even that much ado. The trick is to point at Venus and utter, not the word ‘Venus’, but the sentence ‘it is called Venus’ (or any other sentence that does the same job). Here the italicized word ‘Venus’ is not used to refer to any object but mentioned as a linguistic expression itself. To make the point more definite, we can rewrite the sentence ‘it is called Venus’ as ‘it is called ‘Venus’’. Thus, strictly speaking, it is not the proper name ‘Venus’ but a name of that proper name, namely ‘‘Venus’’ that occurs in ‘it is called Venus’. But whoever understands this sentence will attach a certain sense to the expression ‘Venus’, a sense which links it with the object, Venus, and this means that ‘Venus’ can refer to Venus independently of the context of any sentence in which it plays the role of a proper name.

If an expression is considered as such, stripped of any sense attached to it, then there is indeed no way to tell if it refers to an object or a concept independently
of the context of a sentence in which it occurs. For, as illustrated above with the sentence ‘Venus is not a planet that can sustain life but Mars is no Venus’, one and the same expression can occur as a proper name in a context, and as a predicate in another context. I take it that this is at least part of the reason why so many find the context principle plausible. But an actual object like Venus can be presented to us directly, and an expression for such an object can thereby obtain a sense independently of the context of a sentence. If this is true, then such an expression can have reference in and by itself via the sense attached to it.

Frege’s theory of sense thus apparently leads to the denial of the context principle. On the face of it, one might think that Frege abandoned the context principle of his earlier period in favor of his new theory of sense that emerged in his middle period. But Frege continues to make use of the context principle long after he adopted the distinction between sense and reference. Against this background arises an interpretative question about how to reconcile the context principle with Frege’s theory of sense. Michael Dummett once held, following Anscombe (1959, 66), that the point of the context principle is not that no expression can have reference outside the context of a sentence but rather that “To assign a reference to a name or a set of names . . . could only have a significance as a preparation for their use in sentences” (1973, 193), or to borrow the later Wittgenstein’s words, for making a move in the language game. But if we could make a connection between a name and an object independently of the context of a sentence after all, why would Frege insist that we should not ask for the referent of a word in isolation but only in the context of a sentence (1884, x)?

My view is that, unlike Wittgenstein, Frege did not intend the context principle

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\(^8\)For a discussion of the role of the context principle in Frege’s *Grundgesetze*, see Dummett 1995.
as a principle of reference governing all sorts of expressions. Frege first mentions the context principle in the foreword to his *Grundlagen* as one of the three fundamental maxims guiding his work; but it is actually invoked only to deal with what he elsewhere calls the *Urproblem* of arithmetic (1903b, 265), namely, the question “How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?” (1884, §62) The very way the question is formulated suggests that Frege himself agrees that ordinary objects can indeed be given to us through ideas or intuitions. He raises the prime problem of arithmetic in the course of his discussion of an objection to the effect that numbers cannot be objects since we cannot form ideas of them (1884, §58). His reply to the objection is that it is not really necessary to form an idea of an object in order to make judgments about it and “It is enough if the proposition taken as a whole has a sense” (1884, §60). So the purpose of the context principle is to argue that objects can be also given to us in the context of a sentence; it is definitely not to argue that no objects can be given to us directly through ideas or intuitions. Moreover, Frege says in §89 of the *Grundlagen* that “I must also protest against the generality of Kant’s dictum: without sensibility no object would be given to us.” His use of the word ‘generality’ here clearly suggests that as far as ordinary objects are concerned, he was willing to accept Kant’s view that they are given to us through sensibility. The point which Frege wanted to make is that although numbers could not be given to us in the same straightforward way as actual objects, they could still be given to us in the context of a sentence. Therefore, there is no contradiction in holding at once that abstract objects such as numbers can be given to us only in the context of a sentence and that concrete objects such as Venus can be given to us independently of the context of a sentence.

To turn back to our original problem of the concept *object*, our discussions show that Wittgenstein’s argument in the *Tractatus* for the view of the concept *object* as
a pseudo-concept fails. He radicalized Frege’s context principle by converting it into a universal principle of reference. But the fact of the matter is that it does not apply to expressions for ordinary objects. An expression for an ordinary object can have reference in itself, independently of the context of a sentence, via a sense attached to it. If this is true, then the sentence ‘Venus is an object’ is not a pseudo-sentence about the expression ‘Venus’. The proper name ‘Venus’ refers to Venus in and by itself through a sense attached to it and so the sentence ‘Venus is an object’ says of Venus itself that it is an object. It is a genuine sentence about Venus; in fact, it is true. The concept object is a genuine concept that can be truly ascribed to objects.

But the question remains as to whether sentences of the form ‘a is an object’ are not totally trivial. Suppose Isomorphism is true. That is, suppose complete expressions always refer to objects. Then the word ‘a’ carries the message of its being an object-expression on its face, as it were, by virtue of being a complete expression. If so, why should one ever bother to say that a is an object? The original Tractarian argument against Frege’s view of the concept object as a genuine concept was based on a radicalized version of the context principle and outright denied that the sentence ‘a is an object’ talks about any object. But if granted that ‘a’ can have reference in itself and hence ‘a is an object’ is really about the object a, it would seem that the predicate ‘. . . is an object’ plays no substantial role at all. It purports to ascribe the concept object to a, but that is already done by the expression ‘a’ itself. The proper name fills in the empty place of the predicate and thereby shows itself to refer to an object. But if an expression is of no use, then it is not a genuine expression (Wittgenstein 1921, 3.328; 5.47321). Hence Wittgenstein suggests in the Tractatus that we should not add the predicate ‘. . . is an object’ to our vocabulary: once proper names are introduced, there remains no need for it (1921, 4.12721).

This new argument for the Tractarian view of the concept object as a pseudo-
concept is valid: if Isomorphism is correct, then the predicate ‘... is an object’ is not a genuine predicate. The question is whether the premise holds true. Is it really the case that all complete expressions refer to complete entities, namely objects? Is it really totally trivial to attach the predicate ‘... is an object’ to a proper name? Consider the predicate ‘... is a unit’. This predicate is unusual in that every single expression that can properly fill in its empty place can combine with it to yield a true sentence: Socrates is a unit, wisdom is a unit, the word ‘Socrates’ is a unit, the predicate ‘... is wise’ is a unit, the thought that Socrates is wise is a unit, the sentence ‘Socrates is wise’ is a unit, and so on. This means that when we say of something that it is a unit, we actually say nothing about it. Frege himself rightly holds that a concept is not genuine if it is true of everything and for that reason dismisses ‘... is a unit’ as a pseudo-predicate (1884, §29). But the situation is quite different in the case of the predicate ‘... is an object’. It is quite natural to deny that the concept horse is an object; one should be inclined to say that that is a different sort of thing, that that is a concept. So the predicate ‘... is an object’ is not as universally applicable as the pseudo-predicate ‘... is a unit’. And to that extent, it must be a non-vacuous, genuine predicate.

Dummett has claimed that for Frege, the notion of a proper name and that of a predicate have explanatory priority over the notion of an object and that of a concept, respectively, in the sense that the latter notions should be explained or understood in terms of the former ones (1973, 55–57; 1981, 234–35). Isomorphism is not something one could dispute, as we are trying to do here. According to Dummett, the reason that for Frege all proper names refer to objects is simply that he understands by objects whatever things proper names refer to. If an expression perfectly behaves like a proper name, then it refers to an object, for that is what it is to be an object. In a similar spirit, Crispin Wright advanced “the thesis of the
priority of syntactic over ontological categories”, according to which “the question
whether a particular expression is a candidate to refer to an object is entirely a
matter of the sort of syntactic role which it plays in whole sentences” (1983, 51).9
For Dummett and Wright’s Frege, then, a grasp of the notion of a proper name must
be prior to that of the notion of an object. He should start with a characterization of
what a proper name is and then explain an object simply as the referent of a proper
name. This is why it is “essential” that “it should be possible to give clear and
exact criteria” “for discriminating proper names from expressions of other kinds”
(Dummett 1973, 58).

But in his discussion on functions, Frege raises the question “what it is that we
are here calling an object” (1891, 147) and answers as follows:

I regard a regular definition as impossible, since we have here something
too simple to admit of logical analysis. It is only possible to indicate
what is meant. Here I can only say briefly: An object is anything that is
not a function, so that an expression for it does not contain any empty
place. (1891, 147)

An object is not the referent of a complete expression by definition. The notion of
an object is too primitive to define. Similarly, regarding the notion of a concept,
Frege says that it is “logically simple” and thus “cannot have a proper definition”
(1892b, 182). Apparently he is contradicting his former self here: he had defined a
concept as “a function whose value is always a truth-value” (1891, 146). But there is
no real contradiction. For, while the notion of a concept can be explained in terms
of that of a function, Frege indeed takes the notion of a function to admit of no
further analysis.10 For something logically simple, one can only hope to “lead the

9See also Wright 1983, 12–17. There is a substantial disagreement between Dummett and
Wright regarding the strength of the syntactic priority thesis. For detail, see Dummett 1991, Ch.
15.
10Frege himself makes a similar point in a footnote to a posthumous draft of his paper “Über
Begriff und Gegenstand” (1979, 89n).
reader or hearer, by means of hints, to understand the word as is intended” (1892b, 183). And the hint Frege offers for the notion of a function is that a function is something “incomplete, in need of supplementation, or ‘unsaturated’” (1891, 140). Thus, when Frege says that “an object is anything that is not a function, so that an expression for it does not contain any empty place”, he is merely hinting that an object is something antithetical to a function, that is, something complete. He is not proposing to understand by an object the referent of a complete expression. Quite the contrary, it is because an object is a complete thing—note Frege’s use of the expression ‘so that’—that “an expression for it does not contain any empty place”.

It is not true that Frege understood by an object the referent of a proper name. At the very least, he had an intuitive idea of an object as a complete entity. In fact, it is this idea that Frege appeals to when he infers the conception of numbers as objects from the logical behavior of numerical expressions as proper names. Recall Frege’s crucial remark that the definite article in ‘the number 1’ “serves to class it as an object” and “In arithmetic this self-subsistence comes out at every turn, as for example in the identity 1 + 1 = 2” (1884, §57). He is not claiming here that the number 1 is an object because by an object he means the referent of a proper name. Such a claim would be totally uninteresting. One may as well claim to be a millionaire by defining a millionaire as a person whose assets are worth one million or more in any monetary unit including cent. Rather, Frege’s claim is that the use of the definite article in ‘the number 1’ indicates that the referent of the expression is something self-subsistent, or in his middle-period terminology, something complete. And since it is objects that are complete, the very expression ‘the number 1’ shows what the number 1 is, namely an object. Now, this is an interesting claim.

Isomorphism, namely the thesis that an object is the referent of a proper name
and a concept that of a predicate, was not, for Frege, a trivial consequence of his stipulatary understanding of the words ‘object’ and ‘concept’ but a substantial truth based upon the nature of objects and of concepts. But one should wonder why an expression itself must have the same form or structure as its referent. It would seem that what really matters is not the form of the expression but what it expresses, that is, its sense. A sense of an expression contains a way in which its referent is given. If an expression expresses a sense which contains a way in which only a self-subsistent or complete entity can be given, then its referent must be an object. The word ‘Venus’, for instance, may have a sense which contains a way in which Venus is given to us directly through the senses. Since only a self-subsistent or complete thing can be given in such a way, the word ‘Venus’ must refer to an object.

Now consider the expression ‘the concept horse’. Surely, the form of the expression does not show that it is a concept-expression. But it would seem that whoever has a grasp of its sense should be able to see that it refers to a concept. I am not saying that that should be the case because the expression ‘the concept horse’ contains the word ‘concept’. The same point can be made without using the phrase ‘the concept’: it would seem that whoever grasps a sense of the expression ‘horse’ should be able to see that horse is a concept. The expression ‘horse’ is simple as any in that it contains no empty places. So, just like the expression ‘the concept horse’, it must be deemed a Fregean proper name. But anyone who understands it knows that it does not refer to any particular object but rather an entity under which a number of objects fall, that is, a concept.

In a posthumous paper entitled “Booles rechnende Logik und die Begriffsschrift”, Frege makes a crucial observation that we can infer from the fact that a sentence is

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11Dummett raised a controversy by calling such a simple expression a simple predicate as opposed to what he calls a complex predicate, which corresponds to an incomplete expression or predicate of ours (1973, 27–33). But since his simple predicates are as complete as any actual proper names, they should belong among Fregean proper names.
articulated\textsuperscript{12} that those concepts that admit of no further analysis must be designated by simple expressions (1979, 17). Consider the sentence ‘Romeo loves Juliet’. We can obtain various concepts by subtracting various parts of the sentence. If we subtract ‘Romeo’, we obtain the concept loving Juliet; if we subtract ‘Juliet’, we get the concept being loved by Romeo; and if we subtract both ‘Romeo’ and ‘Juliet’, we have the concept loving. And all these concepts are designated by complex or incomplete expressions: ‘. . . loves Juliet’ designates the concept loving Juliet, ‘Romeo loves ___’ designates the concept being loved by Romeo and ‘. . . loves ___’ designates the concept loving.

But the sentence ‘Romeo loves Juliet’ can be analyzed in such ways only because it is originally composed of three elements, ‘Romeo’, ‘loves’ and ‘Juliet’. Apparently, then, there is a simple concept-expression, namely ‘loves’, that contains no empty places and so is complete! And inasmuch as it is a simple or complete sign, it must be called a name (Wittgenstein 1921, 3.202). That is, not only object-expressions are names; concept-expressions are names, too, provided that they designate simple concepts that cannot be analyzed further. From these considerations Wittgenstein derives the logical conclusion that a sentence whose component expressions are not further analyzable just consists of names (1921, 4.22).\textsuperscript{13}

However, neither Frege’s passing acceptance nor Wittgenstein’s enthusiastic adoption of simple expressions or names for concepts in the Tractatus amounts to an admission that proper names can refer to concepts independently of the context

\textsuperscript{12}The German original is ‘gegliedert’. In the Tractatus Wittgenstein expresses the same idea using the word ‘artikuliert’ (1921, 3.141).

\textsuperscript{13}I am consciously departing from the influential reading of 4.22 proposed by Anscombe. She argued that all names occurring in an elementary sentence are those of objects, not of concepts, and that concept-expressions disappear in a fully analyzed sentence (1959, Ch. 7). In order to give a proper evaluation of the argument we would have to engage in many complex interpretive issues of the Tractatus, which is well beyond the scope of our present work. However, we shall shortly provide a more detailed analysis of the notion of a name as it is used in the Tractatus and argue that Tractarian names include not only object-expressions but also concept-expressions.
of a sentence by way of their *sense*, if not their *form*. As far as Wittgenstein is concerned, that is in fact as it should be, for *Tractarian* names are not allowed to have sense (1921, 3.3). But why does Frege, who upholds an explicit distinction between sense and reference for every concept-expression (1979, 118), deny such a possibility?

Frege himself gives an answer to the question right after making his remark on simple expressions for simple concepts. To appreciate it properly, let us recall our argument that the sentence ‘Venus is an object’ expresses a genuine thought about the object, Venus. The reason it does, we argued, is that the expression ‘Venus’ can refer to Venus in itself, by way of a sense attached to it independently of the context of a sentence. The question is, could a simple *concept*-expression such as ‘horse’ have reference independently of the context of a sentence through its sense? Frege’s answer is a resounding ‘no’, and the reason is that unlike objects, concepts cannot be given to us directly but only in the context of a sentence. To put it in his own words, concepts can only “arise simultaneously with the first judgment in which they are ascribed to things” (1979, 17). The sense of a concept-expression cannot be separated from the context of a sentence where it is ascribed to things and so the expression ‘horse’ in ‘horse is a concept’ cannot refer to a concept in itself by way of a sense attached to it.

Is it a coincidence that after identifying simple component expressions of a sentence as names (1921, 3.2–3.202), Wittgenstein also proceeds to declare that names have reference only in the context of a sentence (1921, 3.3)? Further, Frege argues that since concepts arise only within judgments, expressions for concepts must be of a form of a possible judgment (1979, 17). Since only incomplete expressions are of a form of a possible judgment, it follows that only incomplete expressions can refer to concepts. It cannot be a mere coincidence that after declaring the context
principle in 3.3 of the *Tractatus*, Wittgenstein immediately introduces a new notion, namely the notion of an *Ausdruck* or *Symbol* (1921, 3.31), which is to take up the role similar to that of the Fregean notion of an incomplete expression.

Let me elaborate. From the viewpoint of the *Tractatus*, Frege’s identification of complete expressions (that is, proper names) with object-expressions, and of incomplete expressions (that is, predicates) with concept-expressions, is quite mistaken. In a respect, object-expressions and concept-expressions are both complete: as Frege himself admits, the fact that a sentence is articulated implies that there must be both simple object-expressions and simple concept-expressions. In the *Tractatus* Wittgenstein calls these simple expressions *names*. It is because in the final analysis a sentence must turn out to consist of names, which are simple and unstructured, that the question arises as to how they can combine together and become a sentence (Wittgenstein 1921, 4.221). The so-called problem of the unity of a sentence, which had haunted generations of philosophers and which Frege thought he resolved by his analysis of a sentence into the complete and the incomplete part (1892b, 193), comes alive.\(^{14}\)

In another respect, however, object-expressions and concept-expressions are both incomplete. Notice that a mere collection of names cannot express a sense. What makes the sentence ‘Romeo loves Juliet’ express a sense is the determinate relation in which the simple expressions or names ‘Romeo’, ‘loves’ and ‘Juliet’ stand to one another. Names stand for their referents in a sentence and thereby *contribute* to its sense. But, as simple and unstructured expressions, they do not tell us how those referents are related to one another: they do not *characterize* the sense of the sentence. Any part of a sentence that characterizes its sense must rather be a certain formal feature of the sentence which exhibits how that part is structurally

\(^{14}\)For a notable non-Fregean attempt to resolve the problem, see Wiggins 1984.
related to others. This feature, which Wittgenstein calls a *symbol* (1921, 3.31), is incomplete and always of complex, sentential form. This implies that contrary to an influential reading of the *Tractatus* by Anscombe, symbols do not include names (1959, 98). Unlike names, symbols have *form* as well as reference (1921, 3.31).

It would be helpful to consider Wittgenstein’s own example, the sentence ‘Green is green’, where the first word is an actual proper name and the last a concept-expression (1921, 3.323). Wittgenstein says that those two expressions have different referents, and that means that they are different *names*. But more importantly, while they have the same *physical* form, that is, share the same *sign* (1921, 3.32), their true forms, which are not perceptible to the senses, are different: they are different *symbols*. A symbol, as “the common characteristic mark of a class of propositions” (1921, 3.311), can be made perceptible to our senses by being represented as “a variable whose values are the propositions that contain the expression”, that is, a *sentential variable* (1921, 3.313). The expression ‘Green’ in ‘Green is green’ is really a sentential variable, ‘Φ (Green)’, where ‘Φ’ represents a predicate variable. And the expression ‘green’ in ‘Green is green’ is also a sentential variable, ‘green (x)’, where ‘x’ represents an object variable. From the *Tractarian* point of view, even an actual proper name like ‘Green’ is of a *sentential* form. The problem of the unity of a sentence necessarily arises when we mistake the nature of a sentence and think of it as a combination of *names*; however, it vanishes if we take it instead

15 It is an understandable mistake, though, considering that Wittgenstein himself calls names “simple symbols” (1921, 4.24). But, if I am right, Wittgenstein does not use the word ‘symbol’ here as he does in 3.31. Above all, names are simple *signs* (1921, 3.202) and a sign is “what can be perceived of a symbol” (1921, 3.32). A symbol itself is neither perceivable nor simple.

16 Strictly speaking, symbols have *content* as well as form. This is due to the fact that *Tractarian* symbols include sentences, which lack reference and only have sense as their content. But as far as subsentential symbols are concerned, content can be identified with reference on the *Tractarian* view.

17 Provided those referents are simple, that is. But let us set aside the question what amounts to a *simple* thing in the *Tractatus*. 49
as a combination of symbols.\textsuperscript{18} ‘Φ (Green)’ and ‘green (x)’ naturally “fit into one another like the links of a chain” (1921, 2.03).\textsuperscript{19}

The Tractarian distinction between names and symbols enables us to put our objection to Isomorphism succinctly as follows. The expression ‘horse’ occurring in the sentence ‘horse is a concept’ and the same expression occurring in, say, the sentence ‘a stallion is a horse’ are surely different symbols. But they are the same name just the same: they both designate one and the same thing, namely the concept horse.

To restate Frege’s reply, the simple expression ‘horse’ in the sentence ‘horse is a concept’ cannot refer to the concept horse, for concepts can only arise with judgments where they are ascribed to things and hence expressions for concepts should “never occur on their own, but always in combinations which express contents of possible judgment” or in more detail, “A sign for a property never appears without a thing to which it might belong being at least indicated, a designation of a relation never without indication of the things which might stand in it” (Frege 1979, 17). The idea is that inasmuch as a concept arises only in the context where it is ascribed to one or more objects, an expression can refer to a concept only when it is in action, as it were, occurring as a predicate in a sentence, or at least when it is ready for action, carrying with itself one or more empty places to be filled in by object-expressions. The expression ‘horse’ in ‘horse is a concept’ occurs on its own, without indicating that things can fall under the concept horse. Therefore, the sentence ‘horse is a concept’ cannot be about the concept horse, and so it is not a genuine sentence.

Is Frege right in claiming that concepts arise only in the contexts where they are

\textsuperscript{19}The image of chain or concatenation (Wittgenstein 1921, 4.22) is, I think, an appropriation of Frege’s analogy between the behavior of an incomplete expression and that of an atom, which is “never to be found on its own, but only combined with others, moving out of one combination only in order to enter immediately into another” (1979, 17).
ascribed to things? He must not be making a general claim about all concepts, for complex concepts can be introduced without being ascribed to things falling under them. For instance, the concept even number can be introduced by defining an even number as a number divisible by two without a remainder; so it does not arise in the context of a sentence in which it is ascribed directly to an individual even number such as two, eight or sixteen. However, to introduce a simple concept $F$, which does not admit of further analysis in terms of simpler concepts, one would have to illustrate it, that is, appeal to the things that fall under it. And to that end, one would have to employ sentences of the form ‘$a$ is an $F$’, where ‘$a$’ is a name of an $F$.

The concept natural number is a case in point: in order to introduce it we usually illustrate it by saying that 1, 2, 3 and the like are natural numbers. So it would seem the concept natural number does arise in the context of a sentence where it is ascribed to individual natural numbers.

Is it true, then, that so far as simple, indefinable concepts are concerned, they only arise with the judgments where they are ascribed to things? Consider the fact that there are certain concepts which we can introduce by direct illustration of objects falling under them, instead of making judgments of the form ‘$a$ is an $F$’. For instance, we could introduce the concept horse by uttering the word ‘horse’ while pointing at a horse (or at an image of one). Of course, just a single episode of that kind would be insufficient to establish the word ‘horse’ as a concept-expression as opposed to an object-expression. To that end, we would have to engage in a sequence of acts of pointing at a horse and uttering ‘horse’. But the point is that we could thereby introduce the concept horse without recourse to any sentences in which it is ascribed to individual horses.

A natural response would be that the pointing gesture accompanied by an utterance of ‘$F$’ amounts to no less than making a judgment of the form ‘that is an
\( F \). However, while it is certainly the case that the word ‘\( F \)’ occurs as a predicate in ‘\( \text{that is an } F \)’, it would be a mistake to conclude from this that the concept \( F \) thus introduced arises in the context of a sentence in which it is ascribed to any individual \( F \). To see why, we need to examine more carefully how a simple concept is introduced by illustration. Sentences of the form ‘\( a \) is an \( F \)’ or ‘\( \text{that is an } F \)’ do not really concern the particular object in question insofar as they are used to illustrate a concept \( F \). At first glance, the sentence ‘\( \text{that is } F \)’ may appear to pick out an object by means of the demonstrative pronoun and then to attribute the concept \( F \) to that object. But that is an illusion. No concept can be introduced by picking out a particular thing and then saying that that thing falls under the concept. The reason is that there are too many concepts under which such a thing can fall. For instance, each individual horse falls under the concepts \textit{solid-hoofed}, \textit{plant-eating}, \textit{domesticated}, \textit{mammal}, \textit{having a flowing mane and tale} and so on, apart from the concept \textit{horse}. So if the sentence ‘\( \text{that is a horse} \)’ concerned the individual horse picked out by the pronoun ‘\( \text{that} \)’, then there would be no reason to interpret the word ‘horse’ as designating one concept under which it falls over another under which it also does.

On the surface, the sentence ‘\( \text{that is an } F \)’ is a singular predication about an individual thing, but when it is used to introduce the concept \( F \), it really functions as what we might call an \textit{illustrative} predication, namely a sentence of the form ‘\( \text{that kind of thing is an } F \)’. An illustrative predication is not a singular sentence but a universal one: what it actually says is that \textit{anything} like \( \text{that} \) is an \( F \). We can see now, then, that so long as the sentence ‘\( \text{that is an } F \)’ is used to introduce the concept \( F \), the role of the pronoun ‘\( \text{that} \)’ is not to pick out an individual \( F \) but the \textit{characteristic feature} common to every \( F \). This is why the concept \( F \) and only the concept \( F \) could be introduced by means of the sentence ‘\( \text{that is an } F \)’ that plays
the role of an illustrative predication. And the same can be told about sentences of
the form ‘a is an F’. Insofar as they are used to introduce the concept F, they do
not say of a that it is an F but they rather say that that kind of thing is an F, that
is, anything that is characteristically like a is an F.

So Frege is wrong about how simple concepts arise in the first place. An illus-
trative predication of the form ‘that kind of thing is an F’, in the context of which
the concept F first arises, does not ascribe the concept F to any individual Fs but
identifies the concept F as the kind to which all those exhibiting the characteristic
feature of a given F belong. It is crucial to see that an illustrative predication, if
successful, will create a direct link between the expression ‘F’ and the concept F.
It picks out a kind and give it a name, as can be seen from the fact that to say
that that kind of thing is an F is really to say that that kind of thing is called ‘F’.
That is, introducing a concept by way of an illustrative predication is naming a
kind by associating an expression with the characteristic feature of a given F. The
expression ‘F’ for a concept introduced by illustration will thus attain a sense which
contains that characteristic feature common to Fs. Moreover, it will express the
sense in itself, independently of the context of any sentence in which it occurs as a
predicate. And given the fact that it is a sense of an expression that connects the
expression to its referent, it follows that an expression ‘F’, which expresses a sense
that contains the characteristic feature of Fs, can refer to a concept in itself via the
sense attached to it. The sense of such an expression indicates that its referent can-
not be an independent object but rather a property of such an object. For, clearly,
no object-expression can express a sense that contains a feature of things: a feature
is not a sign of a self-subsistent, complete entity but of a dependent, incomplete
entity.

Let us return to the concept horse. The expression ‘horse’ in ‘horse is a concept’
does not play the role of a predicate and so it does not show itself to be a concept-expression. But since the concept horse can be introduced by illustration, the word ‘horse’ can express a sense containing the characteristic feature of horses, and hence refer to the concept horse in itself by way of that sense, in isolation from the context of any sentence in which it occurs as a predicate. And to the extent that the word ‘horse’ can have reference in itself via its sense, the sentence ‘horse is a concept’ is a genuine sentence about the concept horse and indeed true. It follows that the concept concept can be truly ascribed to concepts: we can say of a concept that it is a concept. Hence, the concept concept is a genuine concept.

This completes our refutation of Isomorphism. The referent of a proper name is not necessarily an object. Proper names can refer to concepts. What makes an expression refer to the kind of entity it does is not its form but its sense. And since the form of an expression alone does not tell us whether its referent is an object or not, we can always intelligibly ask whether the referent of an expression is an object or not. We can conclude, then, that the Tractarian view of the concept object as a pseudo-concept is essentially mistaken: the concept object is a genuine concept. We can also conclude that Frege’s Isomorphism Argument for the conception of numbers as objects fails. The mere fact that numerals genuinely behave as proper names does not imply that numbers are objects.

It is an indisputable fact that the logical behavior of proper names in sentences is totally different from that of predicates. Any attempt to substitute one for the other will result in an ungrammatical sentence. Given these facts, it would seem that there must be a certain philosophical distinction underlying the linguistic distinction between proper names and predicates. Frege was convinced that it is the distinction between objects and concepts, “a distinction of the highest importance” (1892b, 193). But if we are right, that is not the distinction in question. What, then, would
be the ontological category of things that corresponds with the linguistic category of proper names (that is, singular terms)? We shall address this question in Chapter 6.
It is one of the fundamental tenets of Frege’s philosophy that the logical behavior of an expression in a sentence mirrors the ontological type of the entity it represents. For instance, he believed that in an elementary sentence compounded of a singular term and a predicate, the singular term represents an object and the predicate a property.\(^1\) But what happens if, as with color words, an expression of a univocal sense at once behaves as a singular term in one sentence and as a predicate in another? Consider the sentences ‘the color of Socrates’ cloak is white’ and ‘Socrates’s cloak is white’. The word ‘white’ plays the role of a singular term in the first sentence and that of a predicate in the second. It would be implausible to hold that it represents an object in the former and a property in the latter. It always represents one and the same thing, namely the color white, and the color white cannot be at once an object and a property.

Frege’s answer to the above question could be put this way: if an expression of a univocal sense plays more than one logical role, then only one of those roles mirrors the type of the entity it represents and those sentences in which it does not play

\(^1\)As noted in the previous chapter, Frege prefers to call the referent of a predicate a \textit{concept}. So his concepts are what we would normally call \textit{properties} (1892b, 190). From this chapter on we shall use the word ‘concept’ in the usual, non-Fregean sense, as we initially did in Chapter 2.
that role must be explained away in terms of those sentences in which it does. This idea can be found in the following passage of his *Grundlagen*:

In the proposition “the number 0 belongs to the concept $F$”, 0 is only an element in the predicate (taking the concept $F$ to be the real subject). For this reason I have avoided calling a number such as 0 or 1 or 2 a property of a concept. Precisely because it forms only an element in what is asserted, the individual number shows itself for what it is, a self-subsistent object. I have already drawn attention above to the fact that we speak of “the number 1”, where the definite article serves to class it as an object. In arithmetic this self-subsistence comes out at every turn, as for example in the identity $1 + 1 = 2$. Now our concern here is to arrive at a concept of number usable for the purposes of science; we should not, therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be got round. For example, the proposition “Jupiter has four moons” can be converted into “the number of Jupiter’s moons is four”. Here the word “is” should not be taken as a mere copula, as in the proposition “the sky is blue”. This is shown by the fact that we can say: “the number of Jupiter’s moons is the number four, or 4”. Here “is” has the sense of “is identical with” or “is the same as”. (1884, §57)

In §§55–56 of *Grundlagen*, which immediately precede this passage, Frege argued in effect that sentences of the form

\[ S \text{ the number of } F \text{s is } n, \]

cannot be explained in terms of sentences of the form,

\[ A \text{ there are } n \text{ } F \text{s.} \]

And now he claims that sentences of form $A$ can be explained in terms of those of form $S$. Let us say that a sentence is *explanatorily prior* to another if the latter can be explained in terms of the former but not *vice versa*. Frege’s claim is that sentences of form $S$ are explanatorily prior to those of form $A$. The explanatory priority of $S$ to $A$ implies not just that the substantival use of number words is more basic than the adjectival use of them but also that the adjectival use of number words can be explained away in terms of the substantival use of them. That is, the sentences in
which numerical expressions appear in adjectival constructions can be replaced with those in which they occur as singular terms, and that means that number words can play the role of singular terms wherever they occur. And given the fact that singular terms typically represent objects, it would be rational, then, to believe that numbers are objects. Let us call this line of reasoning for the conception of numbers as objects the *Explanatory Priority Argument*. In this chapter we shall argue that it fails.

First of all, the idea that the sentences of form $A$ can be understood in terms of those of form $S$ seems implausible. An understanding of $S$ requires more conceptual resources than an understanding of $A$. It requires a grasp of the concept of identity; it also requires an understanding of the function expression ‘the number of’ and hence a grasp of the concept of number. To be sure, the fact that an understanding of $S$ involves more concepts than an understanding of $A$ does not in itself imply that $A$ cannot be understood in terms of $S$. However, the implication should hold if it is impossible to obtain those additional concepts for an understanding of $S$ independently of an understanding of $A$. That is, if it turns out that we have to have a prior understanding of sentences of form $A$ in order to have an understanding of sentences of form $S$, then that should suffice to show that $A$ cannot be explained in terms of $S$.

Is it possible to understand sentences of the form ‘the number of $F$s is $n$’ without already understanding sentences of the form ‘there are $n$ $F$s’? Especially, could we attain a grasp of the *general* concept of number, which is required for an understanding of the former, without already having a grasp of an *individual* number concept such as ONE? Frege thought so. Thus he initially attempted to introduce the general concept of number by giving it a contextual definition by means of what is commonly called *Hume’s Principle*, namely that the number of $F$s is identical
with the number of Gs if and only there is a one-one correlation between Fs and Gs (1884, §63).

Frege had to drop the idea of giving a contextual definition of the concept of number mainly due to what he saw as an insurmountable difficulty with it, which is known in the literature as the Caesar Problem. Instead he resorted to an explicit definition of the concept of number in terms of extensions of concepts. At the time of writing the Grundlagen he had no serious doubt that the notion of an extension is a harmless logical notion; yet it was to turn his life’s work into a contradiction. It is Crispin Wright who resuscitated the Fregean program in the philosophy of arithmetic by drawing attention to the fact that second-order arithmetic is derivable (within a suitable second-order logic) solely from Hume’s Principle (Wright 1983, 154–69). Unlike the inconsistent Axiom V of Frege’s Grundgesetze, Hume’s Principle is consistent (more exactly, equi-consistent with second-order arithmetic).³

Neo-Fregeans found arithmetic on Hume’s Principle and thereby revert to Frege’s original strategy of introducing the concept of number through contextual definition. But it has never been clear how Hume’s Principle is supposed to contextually define the concept of number. Neither Frege’s metaphor of “carv[ing] up content” (1884, §64) nor the neo-Fregean talk of reconceptualization (Wright 1997; Hale 1997) has been too much help. Frege says that the idea of giving a contextual definition of a concept is “admittedly” “very odd” but “is not altogether unheard of” (1884, §63). He claims that the concept of direction can be obtained by taking ‘line a is parallel to line b’ as an identity, for then we should get ‘the direction of line a is the same as the direction of line b’. According to Frege, it would be “to reverse the true order of things” if we defined parallel lines as “lines whose directions are identical” (1884, 2

²We shall briefly discuss the problem in Chapter 5. For a detailed and up-to-date discussion, see Hale and Wright 2001.

³For a nice proof, see Boolos 1987.
§64). That is because, Frege says, the concept of direction has no basis in intuition and hence must result from the concept of parallel lines of which we do have an intuition. Do we? To say that we have an intuition of parallel lines is to say that we are able to distinguish parallel lines from non-parallel lines in intuition. But how can we distinguish in intuition between real parallel lines and only apparently parallel lines, that is, between lines with the same direction and lines with directions that appear the same by all practical standards but are indeed different by exact standards? How can we detect in intuition lines which never meet as opposed to those which ultimately, but not humanly recognizably, meet?

To be sure, we can have an intuition of non-parallel lines, since there are lines that meet visibly. But it would be a mistake to think that the concept of non-parallel lines could be first obtained by appeal to intuition and then the concept of parallel lines could be introduced by defining them as lines which are not non-parallel. That is in effect what Euclid did when he defined parallel lines as “lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction” (Euclid 1956, 154). The problem is that if parallel lines were by definition lines which do not meet, then it would be a tautology that two parallels never meet. It cannot be a tautology, for it is true of Euclidean space alone. It is certainly a self-evident fact, but nonetheless a fact based upon the nature of Euclidean space, not upon the meaning of the term ‘parallel lines’.

The (neo-)Fregean idea of introducing the concept of (cardinal) number by means of Hume’s Principle is no less problematic. Hume’s Principle, conceived as a single, foundational principle from which our entire arithmetical knowledge derives, is supposed to provide a way of attaining a grasp of the concept of number without having a prior grasp of any individual number concepts. But is it possible to have a grasp of the right-hand side of the principle, which states that there is a one-one
correlation between $F$s and $G$s, without having a grasp of any individual number concepts? The answer is ‘no’: look at the expression ‘one-one’!

Frege, and everyone else for that matter, see no real problem here. They believe that the notion of one-one correlation can be explained away in purely logical terms. Let $F$s be correlated to $G$s many-one if and only if there is a relation $R$ such that every $F$ bears $R$ to a $G$ and to no other $G$. One-one correlation is then simply a matter of reciprocity of many-one correlation. Since many-one correlation is defined in terms of pure logic, so is one-one correlation. But what does it mean that every $F$ bears $R$ to a $G$? It must mean that there is (at least) one such $G$. The occurrence of the indefinite article ‘a’ shows that an understanding of a sentence of the form ‘every $F$ bears $R$ to a $G$’ requires a grasp of the concept ONE.

I am not saying that every occurrence of ‘a’ or ‘an’ connotes the concept ONE. In many sentences such as those of the form ‘a is an $F$’, the indefinite article does not occur significantly and does not connote the concept ONE. It occurs in a sentence significantly and connotes the concept ONE if and only if the sentence remains meaningful after it is replaced with number words such as ‘two’, ‘three’, etc. We can see that in sentences of the form ‘there is an $F$’ the indefinite article does occur significantly and connote the concept ONE, for we could also say meaningfully that there are two $F$s, or three $F$s.

One might object that ‘there is an $F$’ need not necessarily mean that there is (at least) one $F$. It might be a mere shorthand for ‘it is not the case that nothing is an $F$’. And if ‘there is an $F$’ should mean that it is not the case that nothing is an $F$, then an understanding of ‘there is an $F$’ would not require a grasp of the concept ONE. Notice that the indefinite article occurring in ‘nothing is an $F$’ is not significant and has no connotation of the concept ONE.

To see why the objection fails, we need to realize first that there are different
types of existential sentences. The expression ‘there is’ expresses one and the same notion of existence both in ‘there is an \(F\)’ (for a count noun ‘\(F\)’) and in ‘there is \(M\)’ (for a mass noun ‘\(M\)’), inasmuch as both assert the existence of something. Hence any adequate account of the notion of existence must provide a uniform analysis of the structure of both. The problem with the suggestion that ‘there is an \(F\)’ should be understood as the negation of a universal sentence is that ‘there is \(M\)’ cannot be explained away in the same way. There is nothing wrong with understanding ‘there is \(M\)’ as ‘it is not the case that nothing is \(M\)’. However, despite appearances, there is a crucial difference in form between ‘it is not the case that nothing is (an \(F\))’ and ‘it is not the case that nothing is (\(M\))’. Whereas in the former the second occurrence of ‘is’ is a copula, in the latter it is an identity sign. This can be seen by the fact that instead of saying that nothing is water, we can also say that nothing is the stuff called water. So the attempt to explain ‘there is an \(F\)’ as ‘it is not the case that nothing is an \(F\)’ cannot be generalized to yield a uniform account of the structure of existential sentences in general. The notion of existence is not a derivative notion; it is a primitive logical notion.

No one who fails to see that ‘there is an \(F\)’ means that there is (at least) one \(F\) could be said to understand it. So an understanding of sentences of that form should require a grasp of the number concept ONE. I have to hasten to add that the upshot to be drawn here is not that logic contains a non-logical, arithmetical element and hence is impure but that the number concept ONE is a purely logical concept. Indeed, it is a primitive concept of logic: it allows of no reduction to other basic notions of logic, inasmuch as ‘there is (at least) one \(F\)’ is a primitive logical form. Frege and his followers claim that the concept of any individual number \(n\), including that of the number one, can be introduced by defining ‘\(n\)’ as ‘the number of \(F\)s’ for a suitable concept expression ‘\(F\)’. We have argued that this claim is ill-
founded: we have good reason to believe that it is impossible to attain a grasp of the
general concept of number without having a prior grasp of the individual number
concept ONE.

Since it is impossible to have a grasp of the concept of number without having
a grasp of the use of the word ‘one’ in sentences of the form ‘there is (at least)
one $F$', it follows that the substantival use of number words is not more basic than
the adjectival use of them and that not all sentences in which number words occur
in attributive constructions can be replaced with sentences in which they occur as
singular terms. Frege assumed that their occurrences in attributive constructions
“can always be got round”, and this naïve assumption has been crucial for his and
neo-Fregeans’ arguments for the conception of numbers as objects. To the extent
that the adjectival use of number words is indispensable, we can conclude, even by
the (neo)Fregean standards themselves, that numbers should not be conceived as
objects.
In the *Grundlagen*, Frege initially proposed to define the concept of (cardinal) number contextually by means of what is nowadays commonly called *Hume’s Principle*: that the number of $F$s is identical with the number of $G$s if and only if the $F$s correspond to the $G$s one-one. It is a well-known story that he eventually dropped the proposal and resorted to an explicit definition of the concept of number in terms of *extensions* of concepts, which turned out a disastrous move. But the question whether Hume’s Principle could serve as a definition of the concept of number has become one of intense debate since Crispin Wright drew attention to the fact that second-order arithmetic is derivable (within a suitable second-order logic) from Hume’s Principle alone (Wright 1983, 154–69). If it gives a definition of the concept of number, then that means that arithmetic is a logical consequence of a definition and so a version of logicism is vindicated. In this chapter we shall argue against the neo-Fregeans that Hume’s Principle gives no definition of the concept of number, not even a *partial* one.

A definition of a concept $C$ should tell us what it is to be a $C$. So if a purported definition of a concept $C$ does not tell us what it is to be a $C$, then it should not be considered a definition of $C$. Does Hume’s Principle tell us what it is to be a number? Suppose one had no idea of what it is to be a number but did have a grasp of the notion of one-one correlation. And suppose one was told that the number of
$F$s is identical with the number of $G$s if and only if there is a one-one correlation between the $F$s and the $G$s. Could one thereby attain a grasp of the concept of number?

Frege’s ultimate answer was ‘no’. For, if Hume’s Principle told us what it is to be a number, then we should be able to determine whether or not any given thing $a$ is a number, provided we knew what kind of thing $a$ was. But we cannot determine, by means of Hume’s principle alone, whether a certain thing, say, Julius Caesar, is a number. This is not because we do not know what Julius Caesar is. We know at least that Julius Caesar is a person. The reason that Hume’s Principle does not help us determine whether Julius Caesar is a number is because it does not tell us what kind of thing a number is. If we knew that, then we could see whether a number could be a person or not. Thus, for instance, if a number is an abstract, non-physical object, then it cannot be a person and hence Julius Caesar cannot be a number.

But Hume’s Principle merely introduces the concept of number in the context of a numerical identity of the form ‘the number of $F$s is identical with the number of $G$s’ and says nothing about what kind of thing a number is. Of course, everyone knows that Julius Caesar is not a number, but that is not due to Hume’s Principle. This is what has been known in the literature as the Caesar Problem. It poses a genuine threat to the neo-Fregean foundation of arithmetic, since it undermines the foundational status of Hume’s principle as a definition of the concept of number.

Unlike Frege, who decided in the end that the Caesar Problem could not be got around, the neo-Fregeans hold a rather sanguine prospect of the solvability of the problem. Indeed, Wright has proposed a solution to it.¹ His main idea is that Hume’s Principle does tell us what it is to be a number: it tells us that numbers are things

¹The original proposal was made in his 1983, 107–17. Recently, with Bob Hale, he offered a sustained defense of an improved version of the proposal. See Hale and Wright 2001, 335–96.
such that their identity is necessarily assessed in terms of one-one correspondence between concepts. Inasmuch as personal identity is not necessarily assessed in terms of one-one correspondence between concepts, persons and numbers are individuated by different criteria of identity. Therefore, no persons can be numbers and so Julius Caesar cannot be a number.

There are many difficulties with the neo-Fregean proposal, including its commitment to an essentialist metaphysic. But even without these difficulties, I believe that it falls short of being a complete solution to the Caesar Problem. For Frege’s Caesar objection is not about Julius Caesar or any particular thing. It is rather about singular terms: the real point of the objection is that if Hume’s Principle told us what it is to be a number, then we should be able to determine, for any properly functioning singular term ‘a’, whether a is a number (provided we already had an understanding of the term ‘a’). Now, if ‘a’ represents something of a kind whose criterion of identity is clearly different from the criterion for the identity of numbers, as the proper name ‘Julius Caesar’ does, then we could, following the neo-Fregean proposal, conclude that what ‘a’ represents is not a number. But what if ‘a’ does represent a number and yet appears in disguise? That is, what if ‘a’ represents a number but is neither of the form ‘the number of Fs’ nor a definitional abbreviation of an expression of that form? In such a case, could Hume’s Principle help us recognize ‘a’ as representing a number?

To illustrate the point, suppose Julius Caesar favored the number 1. Could we determine, by means of Hume’s Principle, whether Julius Caesar’s favorite number is a number? One might think that regardless of what Hume’s Principle says, we can easily decide that Julius Caesar’s favorite number must be a number, for what else could Julius Caesar’s favorite number be? But how could we know that Julius Caesar’s favorite number is a number in the same sense as the number of Fs is
a number? The problem is this. Hume’s Principle defines the concept of number within the context of a numerical identity of the form ‘the number of Fs is identical with the number of Gs’ and hence one’s grasp of the concept of number via Hume’s Principle is inseparably tied to one’s understanding of the use of an expression of the form ‘the number of Fs’ in such a context. To put it another way, Hume’s Principle says that numbers are essentially those things whose identity can be assessed by considerations of one-one correlation between concepts. Therefore, nothing can be recognized as a number unless it is associated with a concept of which it is the number. The phrase ‘Julius Caesar’s favorite number’ is not associated with any concept expression. So even if we could decide by syntactic considerations alone that Julius Caesar’s favorite number is a number, we cannot determine whether it is one of those things whose identity can be assessed in terms of one-one correlation between concepts. So Hume’s Principle fails to determine whether Julius Caesar’s favorite number is a number in the relevant sense of the term ‘number’.

One might bite the bullet and respond that Julius Caesar’s favorite number is actually no number, at least in the Fregean sense of the term. Frege’s own word for ‘number’ in the phrase ‘the number of Fs’ was ‘Anzahl’, which roughly means cardinal number. I say ‘roughly’, because the usage of the word ‘Anzahl’ in German is quite different from that of ‘cardinal number’ in English. Not only is it a perfectly vernacular term, but more importantly, it can never be used in the way ‘number’ is used in such phrases as ‘Julius Caesar’s favorite number’. For all that, however, it would be a mistake to claim that Julius Caesar’s favorite number is not an Anzahl, that is, that there is no concept F of which Julius Caesar’s favorite number is the Anzahl. For, the mere fact that Julius Caesar’s favorite number is not introduced as the Anzahl of Fs for a concept F does not imply that it could not have been introduced as such. After all, Julius Caesar’s favorite number, namely the number
one, is the \textit{Anzahl} of Roman dictators who conquered Gaul.

Now, one might be tempted to dismiss the Caesar Problem as wholly irrelevant to the main issues in the foundations of arithmetic. Just because we cannot determine by Hume’s Principle whether Julius Caesar’s favorite number is a number, which has nothing to do with arithmetic, why should it constitute a failure of the neo-Fregean foundation of arithmetic as a philosophy of arithmetic? That is, why should the philosopher of \textit{arithmetic} provide an account of the use of singular terms for numbers in \textit{non-arithmetical} contexts? Having these considerations in mind, one might sympathize with the following remarks Wright once made:

\begin{quote}
[the vulnerability of Hume’s Principle to the Caesar Problem] is at worst a shortcoming of \textit{insufficiency} rather than incorrectness. Hume’s Principle, for all Frege says to the contrary in \textit{Grundlagen}, might therefore be conceived as a correct \textit{partial} definition of the use of certain numerical expressions. And if it is at least a correct partial definition, then \textit{[the fact that second-order arithmetic can be derived solely from Hume’s Principle]} shows that the fundamental laws of arithmetic \ldots are logical consequences of a definition. Would not that be a vindication of a species of \textit{logicism} about number theory? (Wright 1997, 203)
\end{quote}

Does Hume’s Principle give at least a correct \textit{partial}, if not sufficient, definition of the concept of number? In other words, does it correctly define the use of the word ‘number’ at least \textit{in the proper contexts of arithmetic}, if not in such unusual phrases as ‘Julius Caesar’s favorite number’? To answer this question properly we would need a criterion of a correct definition of a concept, which is beyond the scope of our present inquiry. However, this much should be uncontroversial: if a purported definition \(D\) of a concept \(A\) fails to distinguish \(A\) from a distinct concept \(B\), then \(D\) should not be considered a correct definition of \(A\). My view is that Hume’s Principle does fail to distinguish the concept of number from other distinct concepts.

To facilitate our discussion of the point, let us first consider Frege’s contextual definition of the concept of direction. He claims that the principle that the direction
of line \(a\) is identical with the direction of line \(b\) if and only if line \(a\) is parallel to line \(b\) (call it the Direction Principle) gives a definition of the concept of direction. More specifically, he says that if we take the right-hand side of the Direction Principle as an identity, then we obtain the concept of direction (Frege 1884, §64).

Frege is making two crucial assumptions here. The first is that there is indeed something such that \(that\) of line \(a\) is identical with \(that\) of line \(b\) if and only if line \(a\) is parallel to line \(b\). Otherwise the parallelism of line \(a\) and line \(b\) could not be taken as an identity. The second is that there is just one such thing. For if there were two or more distinct things each of which might serve as such a thing, how could we say that the Direction Principle explains the concept of direction and not some other concept? Thus, to put it roughly, Frege’s idea is that the Direction Principle defines direction as \(that\) thing such that \(that\) of line \(a\) is identical with \(that\) of line \(b\) if and only if line \(a\) is parallel to line \(b\).

In general, a contextual definition of a concept has to satisfy two conditions. The one is that there be something such that \(that\) of an entity \(e_i\) is identical with \(that\) of an entity \(e_j\) if and only if an equivalence relation holds between \(e_i\) and \(e_j\). To make an algebraic analogy, we may call it the condition of existence of solution. The other is that there be only one such thing, which we may call the condition of uniqueness of solution. Now, obviously, the Direction Principle satisfies the first condition. There is the thing called direction which is such that \(that\) of line \(a\) is identical with \(that\) of line \(b\) if and only if \(a\) is parallel to \(b\). The important question, though, is whether it also satisfies the second condition. Is direction the only such thing that \(that\) of line \(a\) is identical with \(that\) of line \(b\) if and only if \(a\) is parallel to \(b\)? The answer is ‘no’. For instance, the slope of line \(a\) is identical with the slope of line \(b\) if and only if \(a\) is parallel to \(b\). Clearly, the concept of slope is not the same as the concept of direction. Slopes are a certain sort of degrees some of
which are greater or less than others, whereas directions are no quantifiable entities that admit of such a comparison. So the Direction Principle fails to distinguish the concept of direction from a distinct concept and, hence, to give a correct definition of the concept of direction.

The situation is no better in the case of Hume’s Principle. We know that number is one such thing that \( that \) of \( F \)s is identical with \( that \) of \( G \)s if and only if there is a one-one correlation between the \( F \)s and the \( G \)s. But number is not the only such thing. Whenever there is a one-one correlation between \( F \)s and \( G \)s, the \textit{count} of the \( F \)s is identical with the \textit{count} of the \( G \)s. The concept of count is not the same as the concept of number. The count of \( F \)s is the result of the counting of \( F \)s, so a grasp of the concept of count presupposes a grasp of the act of counting. On the other hand, the number of \( F \)s is an abstract magnitude which does not depend upon the act of counting them. So, Hume’s Principle fails to satisfy the condition of uniqueness of solution and hence gives no correct definition of the concept of number.

In point of fact, we did not have to argue that the Direction Principle and Hume’s Principle violate the condition of uniqueness of solution in order to establish that they cannot serve as definitions of concepts. We could have instead simply noted that it is \textit{possible} for them to have more than one solution. For, if the proponents of a contextual definition of a concept are unable to show that it has only one solution, then they cannot claim it to provide a correct definition of the concept, and since there are practically infinitely many concepts, it would be practically impossible to prove that \textit{no} concept except the intended one can serve as a solution and hence that it has only one solution. Even if a contextual definition of a concept could be proved to have a unique solution, the very fact that the correctness of the definition would be thereby turned into a matter of proof should testify against its status as a genuine definition. A definition of a concept must serve as its own proof, as it were,
and should not invoke any further support. Otherwise it should be demoted to the status of a sentence to be proved. Considering that no contextual definitions can be free from the problem of uniqueness of solution, it would seem that no contextual definitions can correctly define concepts.\(^2\)

We have shown that Hume’s Principle, conceived as a definition of the concept of number, suffers from two, collectively fatal problems. The problem of indeterminacy, which we illustrated with a variant of Frege’s Caesar objection, shows that Hume’s Principle gives no account of the use of the word ‘number’ in non-arithmetical contexts. The problem of uniqueness of solution further shows that Hume’s Principle does not give a correct account of the use of the word ‘number’ even in the proper contexts of arithmetic. So Hume’s Principle is not a definition of the concept of number, not even a partial one.

\(^2\)This indeed seems to have been Frege’s own ultimate reason for rejecting contextual definitions. See Frege 1903b, §66.
A striking fact about our statements of number of the form ‘there are $n$ Fs’ is that the number varies depending upon what we are talking about: for example, there are 0 moons of Venus, there is 1 moon of Earth, there are 2 moons of Mars, there are 61 moons of Jupiter, and so on. It would seem that the numbers 0, 1, 2, 61 are being ascribed to certain things, that is, they are being taken as features or properties of certain things. What exactly are those things to which the numbers are ascribed? Apparently, they cannot be objects or aggregates of objects. For, since there are 0 moons of Venus, there are no moons of Venus, and a fortiori no aggregates of moons of Venus, that can possess the property 0. It is rather the property moon of Venus that owns the property 0: to say that there are 0 moons of Venus is to say that the property moon of Venus is not instantiated. These considerations suggest that numbers are properties of properties, that is, second-level properties.\footnote{Frege initially reached the same conclusion from similar considerations in his 1884, §46, although, as is well known, he eventually proclaimed numbers to be objects. Since the collapse of Frege’s logicist program, the conception of numbers as higher-level properties has been the received view among the logicists who followed Russell. See, for instance, Carnap 1931, 41–52.}

Given the fact that on the conception of numbers as higher-level properties, the basic role of number words is adjectival as in sentences of the form ‘there are $n$ Fs’, it would be natural for the proponent of that conception of numbers to introduce numbers by the following definitions:
1 there are 0 Fs ↔ for every x, x is not an F;

2 there is 1 F ↔ there is an x such that x is an F and for every y, if y is an F, then y is identical with x;

3 there are n + 1 Fs ↔ there is an x such that x is an F and there are n y’s such that y is an F and y is not identical with x.

Let us call the suggested analysis of statements of number an *adjectival approach*. The adjectival approach takes the adjectival use of number words in sentences of the form ‘there are n Fs’ as more basic than the substantival use of them in sentences of the form ‘the number of Fs is n’, and it makes a crucial assumption that ‘the number of Fs is n’ expresses the same thought as ‘there are n Fs’ and hence can be safely explained away in terms of it. But that is an error. As can be seen from the use of the definite article ‘the’, ‘the number of Fs is n’ says not only that there are n Fs but also that if there are also m Fs, then m must be identical with n. So in order to explain away sentences of the form ‘the number of Fs is n’, the proponent of the adjectival approach must be able to establish a fundamental fact about numbers, namely that if there are m Fs and there are n Fs, then m must be identical with n, which we shall refer to henceforth as the determinacy of numbers. The aim of this chapter is to argue that the adjectivalist cannot establish the determinacy of numbers and hence is unable to account for the substantival use of number words, including our arithmetic practice where number words conspicuously behave as singular terms.

One might think that it is unreasonable to demand a proof of a numerical identity of the adjectivalist. Presumably, for the adjectivalist, identity is not a relation that can hold for higher-level entities but only for the entities of the lowest level, that is, what we may call, following Frege, *objects*. Since the adjectival approach is based upon the conception of numbers as higher-level properties, it would seem that to ask

2It might be said that §56 of Frege’s *Grundlagen* contains a precursor of the argument. However, not only is his argument barely sketchy but, more importantly, it is based upon a view of identity we shall have to reject.
for a proof of a numerical identity of the form ‘\(m = n\)’ is to assume that numbers are not higher-level entities but objects and hence to beg the question.\(^3\)

However, our demand for a proof of ‘\(m = n\)’ from ‘there are \(m\) Fs’ and ‘there are \(n\) Fs’ presupposes no assumption about what type of entities numbers are. Our claim is not that if there are \(m\) Fs and there are \(n\) Fs, then \(m\) must be the same object as \(n\) and hence \(m\) must be identical with \(n\). Rather, it is simply that given that there are \(m\) Fs and there are \(n\) Fs, \(m\) must be recognized as the same number as \(n\), and hence, \(m\) must be identical with \(n\). The crucial point here is that without being able to decide whether numbers are objects or second-level properties or whatever, we should still be able to recognize \(m\) as the same number as \(n\) if there are \(m\) Fs and there are \(n\) Fs. And if we recognize \(m\) as the same number as \(n\), then we should be able to say that \(m\) is identical with \(n\).

From these considerations arises a view of identity that diverges from the dominant, Fregean view of identity as a relation between objects. For us, identity is a relation between whatever can be recognized as the same again. Objects are not the only things that can be recognized as the same again. For instance, the property horse has been recognized by zoologists as the same as the species Equus caballus, and so there is no reason not to conclude that the property horse is identical with the species Equus caballus.\(^4\) To give another example, the state of affairs that Hesperus is in orbit around the Sun is the same as the state of affairs that Venus is in orbit around the Sun. That is, the former state of affairs is identical with the latter.

In general, things can be recognized as the same again, and hence can bear

\(^3\)Michael Dummett has made essentially the same charge against Frege in his discussion of §56 of the Grundlagen. See his 1991, 106.

\(^4\)An obvious complaint might be that for Frege, ‘the property horse’ would be the name of an object, not a property. Some considerations are given below which should suggest that it is indeed the name of a property, just as it appears to be. For a thorough discussion of the so-called paradox of the concept horse, see Chapter 3.
the relation of identity to one another, if and only if there is a criterion by which we can decide whether they are the same, that is, if and only if a criterion of identity is associated with them. It is not just objects that can have a criterion of identity. Properties, states of affairs, sentences, ideas and even fictional characters admit of identity criteria. Let us call those things in general with which a criterion of identity is associated, *individuals*, with a strong caveat that we are not using the word ‘individual’ here in the way most philosophers do, namely, in the sense of ‘particular’ as opposed to ‘universal’. In our usage of the word, properties are universals *and* individuals, however paradoxical it may sound. So the point is that identity is a relation between any individuals, not just objects. Inasmuch as numbers are individuals, they should be capable of bearing the relation of identity to one another, regardless of the particular type of individual to which numbers belong.

Another way to put it is to say that identity is a relation between any countable things. Things are countable if and only if we can determine in principle, of any two of them, whether one is identical with the other, that is to say, if and only if an identity criterion is associated with them. So it is not just objects that are countable. We can also count properties, states of affairs, sentences, ideas, fictional characters and in fact any individuals whatsoever in our sense of the term. Thus the category of individuals coincides in scope with the category of countable things. Now, Fregean concepts, or functions in general, are themselves individuals inasmuch as they can be distinguished from one another, and so they must be countable. But on Frege’s view, identity “can only be thought of as holding for objects, not concepts” (1979, 120), despite the fact that Fregean concepts are countable as well as objects. We reject this view. Identity is a relation between any countable things, for there must be a criterion of identity associated with them.

To put it yet another way, Frege’s view of singular terms is mistaken. The role of
a singular term is not to represent an object, as he believed. It is rather to represent a certain thing that can be distinguished from everything else, that is, an individual in our sense of the term. Thus, the singular term ‘the property horse’ represents not an object but a property, namely the property horse, which is an individual with a clear boundary that demarcates it from all other things. Likewise, the mere fact that in arithmetic number expressions occur as singular terms does not imply that numbers are objects. It only implies that numbers are things that can be clearly distinguished from one another, which is to say that they are individuals. Therefore, there is no avoiding our demand of a proof of the determinacy of numbers by arguing that the use of numerical singular terms presupposes the conception of numbers as objects.

Now, it might seem that if identity holds for any individuals including higher-level entities, then the proponent of the adjectival approach would have no difficulty introducing numerical identity and thereby establishing what we have called the determinacy of numbers. If numbers are higher-level properties, then since properties in general should have a criterion of identity, there would be a criterion for the identity of numbers. Thus, for instance, suppose that properties in general are the same if and only if they are instantiated by the same things. Then, numbers, taken as second-level properties, would be the same if and only if they are instantiated by the same first-level properties. So, as it seems, the adjectivalist could easily introduce numerical identity by the following definition:

\[ 4 \; m = n \iff (\text{for every } F, \text{ there are } m \; Fs \iff \text{there are } n \; Fs). \]

Given a definition of numerical identity such as 4, the adjectivalist can prove the determinacy of numbers. To see how, suppose there are \( m \) Fs and there are \( n \)

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5For an elaborate discussion of this point, see Chapter 3.
For reductio, suppose there is a $G$ such that there are $m$ $Gs$ yet there are not $n$ $Gs$. Since there are $n$ $Fs$ and there are not $n$ $Gs$, there is no one-one correlation between $Fs$ and $Gs$. On the other hand, there are $m$ $Fs$ and there are $m$ $Gs$, and so there must be a one-one correlation between $Fs$ and $Gs$ after all. Therefore, there can be no such $G$. Parallel reasoning should show that if there are $m$ $Fs$ and there are $n$ $Fs$, then there can be no $G$ such that there are $n$ $Gs$ and there are not $m$ $Gs$. So, from the premise that there are $m$ $Fs$ and there are $n$ $Fs$, the adjectivalist can prove that for every $G$, there are $m$ $Gs$ if and only if there are $n$ $Gs$, and hence, by 4, that $m = n$. It would seem that the determinacy of numbers poses no threat to the adjectival approach.

However, the adjectivalist cannot adopt 4 as a definition of numerical identity, for, in the first place, it is not a correct definition of numerical identity. This can be seen by the fact that it leads directly to a well-known difficulty, namely its need for an axiom of infinity. Unless there are infinitely many entities of the lowest level, there will be two distinct and sufficiently large numbers $m$ and $n$ such that for every $F$, there are neither $m$ $Fs$ nor $n$ $Fs$, and hence, trivially, there are $m$ $Fs$ if and only if there are $n$ $Fs$. The problem is not that the epistemic status of an axiom of infinity as a self-evident principle, let alone as a logical one, is controversial. It is rather that the fact that 4 requires such an axiom indicates that it is not a correct criterion for the identity of numbers. According to 4, two numbers $m$ and $n$ are distinct if and only if there is an $F$ such that either there are $m$ $Fs$ but not $n$ $Fs$ or there are $n$ $Fs$ but not $m$ $Fs$. Let us call such an $F$ a non-identity marker. The question is, why should the non-identity of $m$ and $n$ depend upon the existence of a non-identity marker? To be sure, the existence of a non-identity marker would give sufficient evidence that $m$ is not identical with $n$. However, if 4 gave a correct criterion of numerical identity, then that would be not just a sufficient but also a
necessary condition for the non-identity of \( m \) and \( n \). This cannot be correct, for such considerations as whether there is a non-identity marker for \( m \) and \( n \) are certainly external to the question whether they are identical. If you have a grasp of what it is for there to be \( m \) \( F \)'s and what it is for there to be \( n \) \( F \)'s, then that must be enough for you to decide whether \( m \) and \( n \) are identical. That is, the identity or non-identity between numbers must be determined by their own nature, not by any extraneous circumstances like the existence of a certain property.

One might think that the problem with 4 lies in its requiring the actual existence of a non-identity marker for any two distinct numbers. Perhaps the mere possibility of such a marker is all we need to distinguish a number from another. The suggestion is that two numbers \( m \) and \( n \) are different if and only if possibly, there is an \( F \) such that either there are \( m \) \( F \)'s but not \( n \) \( F \)'s or there are \( n \) \( F \)'s but not \( m \) \( F \)'s. Accordingly, numerical identity might be defined as follows:

\[
5 \quad m = n \leftrightarrow \text{necessarily, (for every } F, \text{ there are } m \ F \text{'s } \leftrightarrow \text{ there are } n \ F \text{'s).}
\]

However, while this modalized version of 4 gets around the axiom of actual infinity, it still requires an axiom of possible infinity. For, suppose it is impossible that there be infinitely many entities of the lowest level. Then there will be two sufficiently large and distinct numbers \( m \) and \( n \) such that it is impossible for there to be a non-identity marker for them, that is, it is necessary that for every \( F \), there are \( m \) \( F \)'s \( \leftrightarrow \) there are \( n \) \( F \)'s. As opposed to the axiom of actual infinity, the axiom of possible infinity seems indisputable and, indeed, an a priori truth. Yet, all the same, the fact that 5 requires such an axiom shows that it is not a correct criterion of numerical identity. For, according to 5, two numbers would count as different only if it is possible for there to be a non-identity marker for them. Given the axiom of possible infinity, this necessary condition will indeed be satisfied for any two distinct numbers. However, the question remains why the possibility of a non-identity marker should be not
just confirmative, but constitutive of numerical non-identity. A proper criterion of numerical identity must account for the non-identity of two numbers in terms of a difference between their own individual natures, not in terms of any extraneous property which can play no role whatsoever in our understanding of them.

It is plausible indeed to say that two properties are identical if and only if they are (actually or possibly) instantiated by the same things. So the fact that it is not plausible to say that two numbers are identical if and only if they are (actually or possibly) instantiated by the same first-level properties implies that numbers are no properties. Both 4 and 5 fail to yield a proper criterion of numerical identity, because they are based upon the mistaken conception of numbers as higher-level properties, actual or possible. Thus, the adjectivalist, who advocates the conception of numbers as higher-level properties, is unable to give a proper criterion of numerical identity and hence unable to introduce a numerical identity of the form ‘m = n’. Consequently, the adjectivalist cannot explain the determinacy of numbers.

In point of fact, 4 and 5 not only do not yield a correct criterion of numerical identity but do not even make sense. The right-hand sides of 4 and 5 contain the phrases ‘there are m’ and ‘there are n’, where ‘m’ and ‘n’ occur as variables. So we can understand 4 and 5 only if we have a prior understanding of a numerical variable. An understanding of a variable involves an understanding of what things it is supposed to range over. So in order to understand a numerical variable, one has to realize that it is supposed to range over 0, 1, and the like. It is crucial to note that on the adjectival approach, an understanding of 0, 1, and the like is supposed to come about by way of the initial definitions 1–3. However, it would be a mistake to think that those definitions explain the individual number words ‘0’, ‘1’, and the like. Recall that on the adjectival approach, numbers are conceived to be second-level properties. In other words, number words are supposed to serve as
second-level predicates. But consider a sentence of the form ‘there is 1 \( F \)’. Which part of the sentence is attached as a higher-level predicate to the first-level predicate ‘\( F \)’? Clearly, it is the entire phrase ‘there is 1’, not the individual number word ‘1’. If so, then it must be the entire phrase ‘there is 1’, not the word ‘1’, that represents a second-level property. This means that when we say that there is 1 \( F \), what we attribute to the property \( F \) is not the number 1 itself but the higher-level property there being 1. The number 1 is not a second-level property itself but merely a proper part of it. Given that the initial definitions 1–3 were given to introduce second-level properties, it follows that the adjectivalist has only defined the phrases of the form ‘there are \( n \) as wholes, not the individual number words ‘0’, ‘1’, etc. That is, we have only been provided with an explanation of the phrases ‘there are 0’ and ‘there is 1’ as wholes, not any explanation of individual number words ‘0’ and ‘1’ as proper parts of them. But unless we can make sense of ‘0’ and ‘1’ as proper parts of the whole phrases ‘there are 0’ and ‘there is 1’, we can attain no understanding of a numerical variable ‘\( n \)’ in the context ‘there are \( n \) \( F \)s’. Therefore, sentences like 4 and 5 are unintelligible on the adjectival approach.

In the light of these considerations we can now make sense of Frege’s cryptic remarks at the beginning of §57 of the Grundlagen, which immediately follow his critical review of an adjectival approach in §§55–56. He says that “In the proposition ‘the number 0 belongs to the concept \( F \)’, 0 is only an element in the predicate (taking the concept \( F \) to be the real subject).” Stripped of his peculiar jargon, this means that in the sentence ‘there are 0 \( F \)s’, ‘0’ is only an element of the higher-level predicate ‘there are 0’ which is attached as a whole to the first-level predicate ‘\( F \)’. He goes on to say that “For this reason I have avoided calling a number such as 0 or 1 or 2 a property of a [property].”6 That is, the crucial reason that Frege rejects the

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6Frege’s technical term ‘concept’ has been replaced with ‘property’. See Frege 1892b, 190.
conception of numbers as higher-level properties is that number words never behave by themselves like higher-level predicates, although they can indeed combine with the phrase ‘there are’ to form such predicates. Numbers can only occur as elements in higher-level properties; they can never be higher-level properties themselves.

In a set of notes written in 1919, Frege summarizes the difficulty as follows:

Since a statement of number based on counting contains an assertion about a [property], in a logically perfect language a sentence used to make such a statement must contain two parts, first a sign for the [property] about which the statement is made, and secondly, a sign for a second level [property]. These second level [properties] form a series and there is a rule in accordance with which, if one of these [properties] is given, we can specify the next. But still we do not have in them the numbers of arithmetic; we do not have objects, but [properties]. How can we get from these [properties] to the numbers of arithmetic in a way that cannot be faulted? Or are there simply no numbers in arithmetic? Could the numerals help to form signs for these second level [properties], and yet not be signs in their own right?7 (1979, 256–57)

The main problem with the adjectival approach is that while we can define phrases of the form ‘there are n’ along the lines of 1–3 and thereby introduce a series of second-level properties there being n, that gives us no numbers, namely the things we apparently talk about in arithmetic. For instance, we apparently use the sentence

\[ 6 \quad 7 + 5 = 12 \]

to make a statement about the numbers 7, 5 and 12, but the adjectival approach gives us only the properties there being 7, there being 5, and there being 12. How can we obtain the individual numbers from the second-level properties? Or would it be rather the case that despite appearances, 6 is not about the numbers 7, 5 and 12 at all? Should we take numerals not to “be signs in their own right”, that is, to stand for numbers on their own, but to merely “help form signs for” higher-level properties?

7Again, Frege’s term ‘concept’ has been replaced with ‘property’.
To me it sounds as if Frege is asking these questions merely rhetorically in a tone of total despair, but some people have hailed them as pointing to a viable alternative in the philosophy of arithmetic. Thus, for instance, Harold Hodes claims that at the end of the above quote “Frege has . . . stumbled onto the right track” (1984, 140). Frege’s chief complaint about the adjectival approach, namely that the ‘n’ in ‘there are n’ is not itself a higher-level predicate but only an element in such a predicate, was based on the assumption that it is a sign for a number. The reason that numbers cannot be higher-level properties is that numerals, which are signs for numbers, are not higher-level predicates. But if the ‘n’ in ‘there are n’ is not a sign for a number and merely helps form a higher-level predicate, and if there are no such things as numbers and all we need is to introduce a series of higher-level properties, then the initial definitions 1–3 will do.

It would be helpful to use simple expressions for numerical quantifiers, that is, higher-level predicates of the form ‘there are n’, in order to emphasize the current point that they are supposed to form single syntactic units which do not contain any expressions for numbers as parts. So let us use a bold-face numeral ‘N’ as a shorthand for a numerical quantifier ‘there are n’. Then the original definitions 1–3 could be modified as follows:

7 0x (Fx) ↔ for every x, x is not an F;

8 1x (Fx) ↔ there is an x such that x is an F and for every y, if y is an F, then y is identical with x;

9 Nx (Fx) ↔ there is an x such that x is an F and Ny (y is an F and y ≠ x).

In order to distinguish the quantifier ‘N’ from the numerical quantifier it abbreviates, namely, ‘there are n’, we shall call it a number-quantifier. Number-quantifiers presumably represent simple higher-level properties, not complex ones in which numbers occur as parts.
This modified approach to the foundations of arithmetic, which explicitly aims to define a series of number-quantifiers, rather than number words, might be called a quantifier approach. For the advocate of the quantifier approach, arithmetic is not about numbers but about higher-level properties. As Hodes puts it, “In making what appears to be a statement about numbers one is really making a statement primarily about [higher-level properties]” (1984, 143).\(^8\) The idea is that sentences of arithmetic in which numerals occur as singular terms must not be construed literally as saying anything about numbers but rather as encoding sentences in which they merely help form expressions for higher-level properties. Thus, 6 appears to talk about the result of adding a number to another number, but it is really a misleading, if convenient, way of writing the following second-order sentence that mentions no numbers, let alone addition:

\[
\text{for any } F \text{ and } G, \text{ if } 7x \ (Fx) \text{ and } 5x \ (Gx) \text{ and nothing is both } F \text{ and } G, \ \text{then } 12x \ (Fx \text{ or } Gx).
\]

While 6 and 10 express exactly the same thought, the former “merely abbreviates” the latter, “which represents the sole analysis of that thought” (Hodes 1984, 140).

Similarly, it would seem that the sentence

\[
7 = 6'
\]

says that the number 7 is a successor of the number 6, but in fact, it merely encodes the following higher-order sentence:

\[
\text{for any } F, \text{ if } 7x \ (Fx), \ \text{then there is an } x \ \text{such that } Fx \text{ and } 6y \ (Fy \text{ and } y \neq x).
\]

\(^8\)Hodes’s own preferred expression for ‘higher-level properties’ is ‘cardinality object-quantifiers’. He uses the word ‘quantifier’ to mean our higher-level properties and the phrase ‘quantifier expression’ to mean our quantifiers.
And, again, 12 says nothing about a number’s succeeding another; it is all about certain higher-level properties and their logical relations.

The fact that arithmetic can be paraphrased away in terms of number-quantifiers is no news since that is basically what Russell’s theory of types showed. The difference between Russell’s view and Hodes’s view is that while Russell allows numerical singular terms and predicates to represent higher-level entities of appropriate types, Hodes adheres to Frege’s theory of singular terms and holds that no singular terms can represent higher-level entities (1984, 140). Thus, on Hodes’s view, the surface grammar of sentences of arithmetic such as 6 and 11 does suggest that they purport to be about certain objects, although in reality there are no such objects. In engaging in arithmetical discourse, we merely talk as if there were such objects. Arithmetical discourse is “a special sort of fictional discourse: numbers are fictions ‘created’ with a special purpose” (Hodes 1984, 144), namely to encode facts about higher-level properties. Or as he later puts it, “the linguistic apparatus of a branch of mathematics is a package built to allow certain higher-order statements to be encoded ‘down’ into a more familiar and tractable first-order form” (1990, 237).

Are sentences of arithmetic mere encoding devices for higher-order sentences? I think not. First of all, it is highly doubtful whether higher-order sentences like 10 and 12 truly capture what sentences of arithmetic like 6 and 11 express. Intuitively, whether 6 is true or not has nothing to do with how many objects there are in the world. But the sentence which it allegedly encodes, namely, 10, would be trivially true if there were fewer than 12 objects in the world, regardless of whether 7 + 5 is the same number as 12. Also, it would seem that 11 is true solely because of the relation 7 bears to 6, but 12 would be true if there were fewer than 7 objects in the world. In fact, the situation is worse. If there were only a finite number of objects in the world, then there would be two sufficiently large numbers m and n such that
for no properties $F$ and $G$, there are $m$ Fs and there are $n$ Gs and nothing is both $F$ and $G$. If so, then for any arbitrary number $l$, $m + n$ would be identical with $l$, for it would be trivially true that for any $F$ and $G$, if there are $m$ Fs and there are $n$ Gs and nothing is both $F$ and $G$, then there are $l$ things that are either $F$ or $G$. Likewise, if there were only finitely many objects in the world, then there would be a sufficiently large number $n$ such that $n$ is a successor of any number whatsoever.

To get around the difficulty, Hodes ultimately bites the bullet and accepts an axiom of (actual) infinity (1990, 247–48). However, this is clearly not an acceptable way out. It is not that it is hard to believe in such an axiom. Instead of positing an axiom of infinity, one might try to introduce a modal operator, as Hodes initially did (1984, 148–49). One would still need an axiom of possible infinity, or at least what might be called an axiom of indefinite extensibility, namely that for any natural number $n$, it is possible that there be a property $F$ such that there are $n$ Fs. But unlike the axiom of actual infinity, either of these alternatives would seem true, and a priori true at that. In fact, one might even insist that the advocate of the quantifier approach need not accept an axiom of infinity of any kind, including the axiom of indefinite extensibility. To illustrate the point, suppose you say that Juliet kills herself with Romeo’s dagger. Clearly you are making a true claim, but that does not make you committed to the existence of Juliet or Romeo’s dagger. This is because your claim is made against a background of Shakespeare’s romantic tragedy, *Romeo and Juliet*. That is, what you really claim is that in *Romeo and Juliet*, Juliet kills herself with Romeo’s dagger. That background assumption exempts you from what otherwise would have been your epistemic duties. Now, similarly, one might construe arithmetical discourse as hypothetical discourse made against a background of an axiom of infinity. For instance, one might read $6$ as expressing the following:

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9 Thanks to Michael Kremer for bringing the latter to my notice.
provided that there are infinitely many objects, for any \( F \) and \( G \), if \( 7x (Fx) \) and \( 5x (Gx) \) and nothing is both \( F \) and \( G \), then \( 12x (Fx \text{ or } Gx) \).

Of course, the price of avoiding commitment to an axiom of infinity is high: unlike 10, 13 is not true but merely hypothetical. Thus, under the current suggestion, not only is arithmetic a fiction in that numbers do not exist, but also what it supposedly encodes is a body of mere hypotheticals. Yet, despite all that, many would be willing to pay the price if they could be exempted from believing in an axiom of infinity.

The real reason that positing an axiom of infinity cannot be an acceptable solution is that numerical identities such as 6 and 11 should not be construed as making any hidden assumptions, including hypothetical ones, about the cardinality of the objects in the world. There may be (actually or possibly) infinitely many objects in the world, but even if the Eleatics were right about the One being the only object there is in the world, which would render even the axiom of indefinite extensibility false, 7 + 5 could not be identical with 11, or 7 with 5′. That is, 7 + 5 must be identical with 12 and nothing else, regardless of the number of objects there are in the world. Any philosophy of arithmetic that fails to acknowledge this fact must be misguided.

The need for an axiom of infinity is actually a mere symptom of a fundamental confusion present in both the adjectival and the quantifier approach. Recall Frege’s complaint about the adjectival approach again, namely that the sign for a number is not a numerical quantifier of the form ‘there are \( n \)’ but merely an element of such a quantifier, namely, the ‘\( n \)’. The quantifier approach is an attempt to bypass Frege’s complaint by insisting that the ‘\( n \)’ in ‘there are \( n \)’ does not stand for a number at all and all we need to make sense of arithmetical discourse is not numbers but only a series of higher-level properties there being \( n \). But the fact that it suffers from the need for an axiom of infinity indicates that Frege’s complaint was indeed legitimate:
the sign for a number is the ‘n’ in ‘there are n’, not the entire quantifier phrase ‘there are n’.

The following considerations should help clarify the point. In denying that the ‘n’ in ‘there are n’ plays any signifying role on its own, Hodes believes that wherever ‘there are n’ occurs, it could, and should, be treated as a single syntactic unit that has a signifying role only as a whole, that is, as a number-quantifier ‘N’. But inasmuch as he attempts to explain away various uses of number words in terms of the use of number-quantifiers, he would have to show how to transform sentences of the form ‘the number of Fs is n’ into those sentences in which no numerical terms occur except number-quantifiers. A critical and fatally complacent assumption he makes at this juncture is that ‘the number of Fs is n’ and ‘there are n Fs’ “express the same thought” (1984, 129) and “The sort of transformation exemplified by [them] is ... mere ‘conversion’ of idiom” (1984, 142). As observed before, ‘the number of Fs is n’ says not only that there are n Fs but also that if there are m Fs, then m must be identical with n. So in order to explain the locution ‘the number of Fs is n’ in terms of the locution ‘there are n Fs’, one has to be able to prove the determinacy of numbers, namely that if there are m Fs and there are n Fs, then m = n. To this end, however, it would be imperative to construe ‘there are n’ as a complex quantifier phrase in which the ‘n’ plays a signifying role of its own. The reason is that in order to derive ‘m = n’ from ‘there are m Fs’ and ‘there are n Fs’, we would need a criterion of numerical identity such as 4:

\[ 4 \quad m = n \leftrightarrow (\text{for every } F, \text{ there are } m \text{ Fs } \leftrightarrow \text{ there are } n \text{ Fs}), \]

where the variables m and n on the right-hand side of the biconditional occur in the places of numerical adjectives ‘m’ and ‘n’, not the quantifier phrases ‘there are m’ and ‘there are n’. But no such use of a numerical variable can be understood on the quantifier approach.
It is important to see that the reason that on the quantifier approach we could not define numerical identity along the lines of 4 is not that it does not allow number-quantifiers to flank the identity sign. As said above, identity is not just a relation between entities of the lowest type, that is, objects; rather, it holds for any entities with which a criterion of identity is associated, that is, individuals. Now, there must be a criterion of identity associated with the higher-level properties there being n, and so we should be allowed to put the identity sign between the expressions for them, namely the number-quantifiers. Thus, for instance, suppose that two higher-level properties M and N are the same if and only if they are instantiated by the same first-level properties. Then the following criterion for the identity of higher-level properties would be available:

\[ M = N \iff (\text{for every } F, Mx (Fx) \leftrightarrow Nx (Fx)). \]

However, this is no help. Recall that what needs to be proved is that if there are m Fs and there are n Fs, then \( m = n \). Assume there are m Fs and there are n Fs, that is, \( Mx (Fx) \) and \( Nx (Fx) \). From this it follows that for every G, \( Mx (Gx) \) if and only if \( Nx (Gx) \).\(^{10}\) So by 14, it follows that \( M = N \). But all this shows is that from the assumption that there are m Fs and there are n Fs, we can prove on the quantifier approach that the higher-level property there being m is identical with the higher-level property there being n. But that is not what we wanted. What we wanted is a proof of the fact that if there are m Fs and there are n Fs, then the number m and the number n are identical, that is, \( m = n \), not \( M = N \).

To sum up, both the adjectival approach and the quantifier approach are unable to explain away the substantival use of number words in sentences of the form ‘the number of Fs is n’. It is commonly assumed that these sentences can be simply

\(^{10}\)See 76–77.
converted into sentences of the form ‘there are n Fs’, but that is an error. They say not just that there are n Fs but also that if there are m Fs, then m must be identical with n. Hence the need for a proof of what we have called the determinacy of numbers, namely that if there are m Fs and there are n Fs, then m is identical with n. But in order to establish this fact we would need a criterion of numerical identity in which numerical variables can occupy the places of number words not just in the identities of the form ‘m = n’ but also in the phrases of the form ‘there are n Fs’. The problem with the present approaches is that so long as we construe numerical identities as concerning higher-level properties there being n and thus define numerical identity in terms of higher-level predicates ‘there are n’, we will be unable to provide such a criterion.

The lesson is that we should construe a numerical quantifier of the form ‘there are n’ not as a single syntactic unit but a compound expression one of whose elements, namely the ‘n’, has its own signifying role. That is, we should answer ‘no’ to Frege’s question: “Could the numerals help to form signs for these second level [properties], and yet not be signs in their own right?” Numerals do signify numbers as we know them. Numbers as we know them are indeed the objects of study of arithmetic. The task before us is to figure out a new conception of numbers which will enable us to construe the numeral ‘n’ in ‘there are n’ as a genuine sign for a number. But, as Frege so aptly put it, “How can we get from these [properties] to the numbers of arithmetic in a way that cannot be faulted?” We will turn to this question in the next chapter.
Chapter 7

Numbers as Modes of Existence

We have concluded in the previous chapter that the ‘n’ in ‘there are n Fs’ signifies a number but does not signify a second-level property. It is not itself attached as a predicate to ‘F’ but occurs only as an element of a complex predicate ‘there are n’ that is attached to ‘F’ as a whole. This invites the question: what type of entity does it represent, then? In this chapter we shall argue that the ‘n’ in ‘there are n Fs’ plays the role of an adverbial modifier of the verb phrase ‘there are’, and develop a corresponding conception of numbers as modes of existence, that is, ways or manners in which things exist.

In order to determine what type of entity the ‘n’ represents in ‘there are n Fs’, we need to examine how exactly it behaves in it. A notable fact is that if ‘n’ is dropped from ‘there are n’, the resulting expression, namely ‘there are’, can still play the same logical role as the original expression: both ‘there are n’ and ‘there are’ are second-level predicates that can be attached to a first-level predicate ‘F’. An expression that can be dropped from a phrase without affecting the logical behavior of the remaining part of the phrase plays the role of a modifier. Thus, the words ‘colorless’ and ‘green’ in the phrase ‘colorless green ideas’ are modifiers: if we drop ‘colorless’ or ‘green’ or both from the noun phrase, the resulting expression will still be a noun phrase. Modifiers also occur in other types of phrases, including verb phrases. For instance, we can drop the word ‘furiously’ from ‘sleep furiously’.
without affecting the logical behavior of the remaining expression. So the adverb ‘furiously’ in the verb phrase ‘sleep furiously’ plays the role of modifying the verb ‘sleep’. Similar considerations should suggest, then, that the ‘\(n\)’ in ‘there are \(n\)’ serves as a modifier of the verb phrase ‘there are’. So it would seem that despite appearances the ‘\(n\)’ in ‘there are \(n\ Fs\)’ does not behave as an attributive adjective but really as an adverb, playing the role of a verb modifier.

Adverbs modifying verbs typically represent ways or manners in which things are done or obtain, and we may broadly call such things *modes*. The adverb ‘furiously’ in the verb phrase ‘sleep furiously’ purports to represent a mode of sleeping. Similarly, then, we could say that the number word ‘\(n\)’ in the verb phrase ‘there are \(n\)’ represents a mode of existence. To say that there are \(n\ Fs\) is really to say that \(F\)s exist in the mode of \(n\). Numbers are *modes of existence*.

Number words quite often appear without being accompanied by any existence verb, but this is not really an obstacle to our construal of number words as modifiers of an existence verb. Appearances notwithstanding, the use of a number word in any context of a sentence can be always reduced to the use of it in the context of a sentence in which it explicitly modifies an existence verb. Consider the sentence ‘the King’s carriage is drawn by four horses’. Apparently it is not an existential sentence, but it really is, as it can be construed as ‘there are four horses by which the King’s carriage is drawn’. Some sentences, especially those where a number word is immediately preceded by a universal quantifier, cannot be construed as existential sentences in a similar way. For example, the sentence ‘any two points determine a line’ is a universal sentence, not an existential one. Yet it can be construed as ‘it is not the case that there are two points that do not determine a line’, where the word ‘two’ does play the role of a modifier of the verb phrase ‘there are’.

Actually there are many types of modes of existence, and numbers form one
of them. For instance, frequency words such as ‘always’ and ‘sometimes’ can be used to answer the question, ‘how often are there Fs?’ and in such a context they represent certain ways in which Fs exist. The way some frequency words behave is indeed strikingly similar to the way number words behave. Just as the ‘n’ occurs deceptively as an attributive adjective in ‘there are n Fs’, so does ‘occasional’ in ‘there are occasional changes in policy’. Unlike the adjective ‘radical’ in ‘there are radical changes in policy’, the adjective ‘occasional’ in ‘there are occasional changes in policy’ does not modify ‘changes’. Radicalness may be a property of a change, but occasionality may not. The real function of the word ‘occasional’ in ‘there are occasional changes in policy’ is to modify the verb phrase ‘there are’: what the sentence really says is that changes in policy exist occasionally. So the adjective ‘occasional’ really behaves as an adverb there and represents a mode of existence. Similar comments can be made about words that represent distribution rates. In ‘Fs are everywhere’, the adverb ‘everywhere’ represents a certain mode of existence of Fs. But despite all the affinities, frequencies and distribution rates are definitely not numbers. Crucially, they cannot serve to answer the question, ‘how many Fs are there?’ Unlike the ‘how often’ question, the ‘how many’ question concerns the quantity of Fs. Numbers are quantitative modes of existence.

But there are quantitative modes of existence that are not numbers. Consider the sentence ‘there is much evidence’. Just like the ‘n’ in ‘there are n Fs’ and the word ‘occasional’ in ‘there are occasional changes in policy’, the word ‘much’ here only apparently modifies the noun ‘evidence’. Compare ‘there is strong evidence’. We can expand this into ‘there is evidence that is strong’, but we cannot expand ‘there is much evidence’ into ‘there is evidence that is much’. The real role ‘much’ plays in ‘there is much evidence’ is to modify the verb phrase ‘there is’: it represents a way in which evidence exists. Moreover, it can be used to answer the question
‘how much $F$ is there?’ The ‘how much’ question is no less about quantity than the ‘how many’ question, for to say that there is much evidence is to say that evidence exists in large quantities. Therefore, much must be a quantitative mode of existence, though clearly not a number.

What distinguishes numbers from quantitative modes of existence like much and little? The characteristic mark of the question, ‘how many $F$s are there?’, is the use of a count noun ‘$F$’. That means that the type of quantity that the ‘how many’ question concerns is the quantity of countable things. Now, things are countable if and only if an identity criterion is associated with them. If we call, as we did in the previous chapter, any entity of a kind with which a criterion of identity is associated an individual, then we can say that the ‘how many’ question specifically concerns the quantity of individuals. So numbers are quantitative modes of existence of individuals. On the other hand, the ‘how much’ question is about the quantity of those things with which no criterion of identity is associated, namely what we called masses. Hence, much, little, and the like are quantitative modes of existence of masses.

However, some quantitative modes of existence of individuals are not numbers. The word ‘many’ in ‘there are many $F$s’ does not modify the count noun ‘$F$’ as it appears to do. It rather modifies the verb phrase ‘there are’. To say that there are many $F$s is really to say that $F$s exist in the mode of many, or to use a neologism, manily. Hence, many, few and the like are quantitative modes of existence. Moreover, they can serve to answer the ‘how many’ question and, to that extent, they are indeed quantitative modes of existence of individuals. But, just the same, they are not numbers. Numbers are definite quantities in that for any $F$ and $G$, if there are $n$ $F$s and there are $n$ $G$s, then there is a one-one correlation between the $F$s and the $G$s. In contrast, many, few and the like are indefinite quantities: many $F$s and
many Gs do not always admit of one-one correlation with one another. Therefore, numbers are definite quantitative modes of existence of individuals.

A consideration of the nature of the number 0 should provide some evidence for our view of numbers as modes of existence. An interesting fact about the number 0 is that its status as a (cardinal) number seems less secure than others. While all other numbers can be used for counting, that is not true of the number 0. Suppose you are asked to count Fs. If there are any Fs, then you will count them by correlating them one to one with the numbers 1 up to \( n \). Notice that you do not start at the number 0. If there are no Fs, then you will likely be baffled at the very idea of counting them and respond that there are no Fs to count. So it would seem that 0 is not at all a counting number. But, on the other hand, suppose you are asked to eliminate Fs one by one and to count the rest each time. You throw away one F at a time, and each time you count the rest and utter an appropriate number word ‘\( n \)’. When you have finally thrown away the last F, you will in all probability shout “0!”, which in an important sense constitutes your counting of (non-existent) Fs. And to that extent, the number 0 perfectly serves as a counting number.

The lesson is that the status of 0 as a number is inherently ambiguous, and this raises the question, what is it about the number 0 that makes it such a peculiar number? Our view of numbers as modes of existence gives a ready answer. If there are \( n \) Fs for any \( n \) greater than 0, we can say with confidence that Fs exist in a certain quantitative way or manner. But if there are no Fs, then a fortiori there can be no way or manner in which Fs exist, for Fs do not exist in the first place. And inasmuch as ‘there are 0 Fs’ expresses the same as ‘there are no Fs’, it would seem that the number 0 is not a mode of existence and hence not a number. This is the one direction in which our intuition pulls us. On the other hand, however, when we say that there are 0 Fs, we are clearly not just denying that there are
any Fs but affirming something. As Frege noticed, “In answering ‘Zero’ [to ‘How many?’], we are not denying that there is such a number: we are naming it” (1894, 206). Frege’s insight is that the answer ‘0’ to the question ‘how many?’ is not quite like the answers ‘nobody’, ‘nowhere’, and ‘never’ to the questions ‘who?’, ‘where?’, and ‘when?’, respectively. ‘Nobody’ does not name any person, ‘nowhere’ does not name any place, and ‘never’ does not name any time. They are all mere fusions of relevant quantifiers and the negation: ‘nobody’ abbreviates ‘not anybody’, ‘nowhere’ ‘not anywhere’ and ‘never’ ‘not ever’. But when we say that there are 0 Fs, we apparently make a positive claim of the form ‘there are n Fs’, as if there were Fs in a particular way or mode. That is, we treat the number 0 as if it were a certain way in which things exist. And insofar as we treat it as a mode of existence, it appears as legitimate a counting number as any others. So on our conception of numbers as modes of existence, our genuinely ambiguous attitude toward the status of the number 0 makes perfect sense.

Most importantly, our conception of numbers as modes of existence enables us to prove what we called the determinacy of numbers, namely the fact that if there are m Fs and there are n Fs, then m must be identical with n. First, recall what went wrong with the conception of numbers as higher-level properties. One could introduce, in terms of pure logic, a series of higher-level predicates of the form ‘there are n’ as follows:

1. there are 0 Fs ↔ for every x, x is not an F;
2. there is 1 F ↔ there is an x such that x is an F and for every y, if y is an F, then y is identical with x;
3. there are n + 1 Fs ↔ there is an x such that x is an F and there are n y’s such that y is an F and y is not identical with x.

1 For detail, see Chapter 6.
But in order to prove ‘\( m = n \)’ from ‘there are \( m \) Fs’ and ‘there are \( n \) Fs’ one would have to also adopt a definition of numerical identity such as the following:

\[ 4 \quad m = n \leftrightarrow (\text{for every } F, \text{there are } m \text{ Fs } \leftrightarrow \text{there are } n \text{ Fs}). \]

We argued in Chapter 6 that 4 does not give a correct criterion of numerical identity, not least because, if number are conceived as higher-level properties and number words as higher-level predicates, then the use of a numerical variable in the place of a numerical adjective ‘\( n \)’ in ‘there are \( n \) Fs’ is simply unintelligible. But if we conceive numbers as modes of existence, then a number \( n \) is not itself a higher-level property but a separate element of a complex higher-level property \( \text{there being } n \). And so we can make sense of the use of numerical variables as on the right-hand side of 4.

But, as we argued in the previous chapter, 4 is not a correct definition of numerical identity. In fact, none of the initial definitions, 1–3, is correct. Consider the definition of the number 1:

\[ 2 \quad \text{there is } 1 \ F \leftrightarrow \text{there is an } x \text{ such that } x \text{ is an } F \text{ and for every } y, \text{if } y \text{ is an } F, \text{then } y \text{ is identical with } x. \]

Strictly speaking, this gives an explanation of what it is for there to be \( \text{exactly } 1 \ F \). The right-hand side of 2 states that there is at least \( 1 \ F \) and that is the only \( F \). So an understanding of what it is for there to be exactly \( 1 \ F \) requires a prior grasp of what it is for there to be at least \( 1 \ F \). Of course, nowhere in the right-hand side of 2 does the numeral ‘1’ or the number word ‘one’ actually occur. But what would it mean that there is an \( x \) such that \( x \) is an \( F \) if it did not mean that there is at least 1 such \( x \)?

\[ ^2 \text{For a detailed argument for this point, see Chapter 4.} \]
In order to explain what the number 1 is, we have to explain what it is for there to be at least 1 \( F \), not what it is for there to be exactly 1 \( F \). But how could we do this, given that existential sentences of the simplest form ‘there is an \( F \)’ mean the same as ‘there is at least 1 \( F \)?’ The answer can be found in the fact that although sentences of the form ‘there is \( M \)’ for a mass noun ‘\( M \)’ contain the same verb phrase ‘there is’ as those of the form ‘there is an \( F \)’ for a count noun ‘\( F \)’, the logical rules governing the use of an existential quantifier apply to the latter but not to the former. For instance, the existential sentence ‘there is water’ cannot be derived by an application of the rule of existential introduction (\( EI \)). For, while ‘there is water’ says that there is a certain lump of matter or mass, any conclusion that can be obtained by an application of the \( EI \) rule says that there is a certain individual, not a mass. These considerations suggest that our grasp of what it is for something to exist is independent of our grasp of the logical rules of existential quantification. Otherwise it would have been impossible for us to attain a grasp of what it is for there to be \( M \) for a mass noun ‘\( M \)’.\(^3\) This means that those quantifier rules do not really concern the bare phrase ‘there is’ or ‘there exists’; they rather govern the use of the complex quantifier phrase ‘there is a(n)’, that is, ‘there is at least 1’. It is by a grasp of the logical rules of existential quantification that we attain a grasp of what the number 1 is.

The upshot is that the number 1 cannot be defined. We have to accept the following as a primitive logical form:

\[
5 \text{ there is at least 1 } F,
\]

\(^3\)The question naturally arises as to what constitutes our grasp of what it is for there to be a thing in general, whether an individual or a mass. A first approximation to a correct answer would be this: we grasp what it is for there to be a thing in general by grasping that if you can say that that (or this) is \( G \) for a count or mass noun ‘\( G \)’, then you can also say that there is \( G \), or alternatively, by grasping that the gesture of pointing at something with an utterance of ‘\( G \)!’ implies that there is \( G \).
which could be symbolized as ‘$\exists_1 x \ (Fx)$’. The quintessential logical form of quantificational logic, namely ‘$\exists x \ (Fx)$’, is nothing more than a convenient abbreviation of the more explicit ‘$\exists_1 x \ (Fx)$’. And now that the number 1 is introduced via a grasp of the logical rules of existential quantification, there is no need for a separate definition such as 1 for the number 0. We can define it in terms of the number 1 simply as follows:

6 there are 0 Fs $\leftrightarrow$ there is not at least 1 F.

For the sake of convenience, however, we shall henceforth assume that the number 1 is the smallest (cardinal) number.

We turn now to the concept of successor. Since we take the context ‘there are at least $n$ Fs’ as more basic than the context ‘there are exactly $n$ Fs’, it would seem natural to introduce the successor operator by modifying 3 as follows:

7 there are at least $n'$ Fs $\leftrightarrow$ there is at least 1 $x$ such that $x$ is an $F$ and there are at least $n$ $y$’s such that $y$ is an $F$ and $y$ is not identical with $x$.

However, 7 is not quite satisfactory. The problem is that the right-hand side of the biconditional sounds much more complicated than what the left-hand side actually says. A notable feature of the context ‘there are at least $n$ Fs’ is that it presupposes that none of the Fs is identical with another. That is, by saying that there are at least $n$ Fs, we automatically commit ourselves to the existence of at least $n$ distinct Fs, and so it would be pointless, if not outright nonsensical, to add that the Fs are not identical with one another. This indicates that the variables used in the context ‘there are at least $n$ Fs’ are exclusive: none of them can have the same value as

\[4\] Since we have not yet shown that the successor operation yields one and only one value for an argument, we cannot at the present stage speak of it as a function. Later we will show that it is a function, and a one-one function at that.
another. The use of exclusive variables is nothing to look askance at. Any sentence involving exclusive variables can be translated into a sentence involving inclusive ones, and vice versa.

Intuitively, to say that there are at least $n'$ $F$s is just to say that there are at least $n$ $F$s and there is another $F$. Here the use of the word ‘another’ makes it redundant to add that the $F$ is not identical with any of the $F$s mentioned before. We can easily duplicate the effect of the word ‘another’ in terms of an exclusive variable as follows:

$$\text{there are at least } n' \text{ } F\text{s} \leftrightarrow \text{there is at least } 1 \text{ } x \text{ such that } (x \text{ is an } F \text{ and there are at least } n \text{ } y\text{'s such that } y \text{ is an } F).$$

Here $y$ is supposed to have distinct values from the one assigned to $x$. In consequence, a sentence of the form ‘there are at least $n$ $F$s’ will be symbolically represented as a series of $n$ nested existential quantifications of the form ‘there is at least $1 \text{ } x \text{ such that } x \text{ is an } F$’ with each of the $n$ elements of the series phrased in terms of a unique variable ‘$x$’.

Lastly, we need a definition of numerical identity. Since numbers are modes of a certain type, it would be helpful to find out under what circumstances modes are recognized as the same. In general, two modes can be recognized as the same if and only if, if things are done or obtain in the one mode, then they must also be done or obtain in the other mode, and vice versa. If we apply this generic criterion of identity of modes to numbers, we can obtain the following criterion of numerical identity: two numbers $m$ and $n$ are the same if and only if, if $F$s exist in the mode

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5The idea of an exclusive variable as opposed to a normal, inclusive variable was first introduced by Wittgenstein (1921, 5.53–5.5352). While we certainly acknowledge the usefulness of the exclusive interpretation of variables in certain contexts like ‘there are at least $n$ $F$s’, we reject his argument that the identity sign must be eliminated in favor of exclusive variables in all contexts. For more on Wittgenstein’s view of identity, see Chapter 3.

6See Hintikka 1956, especially sections 4 and 5.

7In Hintikka’s terms, our variables are strongly exclusive. See Hintikka 1956, 230.
of \(m\), then they must also exist in the mode of \(n\), and vice versa. The modal force of ‘must’ here must be *logical*, because numbers are *logical* entities in that they all can be explained or defined in purely logical terms. So two numbers \(m\) and \(n\) are identical if and only if there being at least \(m\) \(F\)'s *logically* implies there being at least \(n\) \(F\)'s and vice versa. In short:

\[
\text{9 } m = n \iff (\text{there are at least } m \text{ } F\text{'s } \iff \text{there are at least } n \text{ } F\text{'s}),
\]

where ‘\(\iff\)’ means logical equivalence and is to be read ‘(logically) necessarily if and only if’. Unlike 4, our criterion of numerical identity is not affected by how many objects there are in the world or how properties are instantiated in the world. For us, any two numbers \(m\) and \(n\), however huge they might be, are the same if and only if ‘there are at least \(m\) \(F\)'s’ logically follows from ‘there are at least \(n\) \(F\)'s’ and vice versa.

Note that the left-hand side of 9, namely ‘\(m = n\)’, is not intended as a shorthand for the right-hand side, ‘there are at least \(m\) \(F\)'s \(\iff\) there are at least \(n\) \(F\)'s’. It is not a *stipulatory* definition, that is, a definition that introduces a new, simple expression as an abbreviation of a complex expression.\(^8\) Thus, care must be taken to distinguish 9 from the following stipulatory definition:

\[
\text{10 } \text{‘} m = n \text{’ } =_d f \text{ ‘there are at least } m \text{ } F\text{'s } \iff \text{there are at least } n \text{ } F\text{'s’}.
\]

The problem with 10 is that, in effect, it defines the notion of identity specifically for numbers. But that would be a mistake. As Frege correctly pointed out (1884, §63), various types of things admit of one and the same relation of identity and so it ought not to be relativized to each of them.

Unlike 10, 9 gives a criterion for the identity of numbers. The purpose of giving a criterion of identity of \(F\)’s is not to introduce a *new* notion of identity for \(F\)’s but

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\(^8\)For more on definitions, see Chapter 2.
to specify under what conditions \( F \)'s stand in the same old relation of identity to one another. Thus, to make sense of a criterion of identity of \( F \)'s, one has to have a prior grasp of the notion of (absolute) identity. How do we attain a grasp of what it is for two generic individuals to be (not identical \( F \)'s but) identical *simpliciter*? Frege initially claimed in §65 of the *Grundlagen* that the universal substitutability of identicals could be adopted as a *definition* of identity, but as he admitted later, identity is a primitive logical relation and cannot be defined in any simpler terms (1894, 200). We attain a grasp of a primitive logical concept not by definition but by acquiring the logical rules associated with it.\(^9\) So our grasp of the notion of identity consists in our grasp of the substitution rule of identity, namely that if \( a \) has a certain property and \( b \) is identical with \( a \), then \( b \) also has it, and the introduction rule of identity, namely that any individual \( a \) is identical with itself.

Given that \( 9 \) should be understood as involving the absolute notion of identity, a question arises concerning whether numerical identity, as defined in terms of logical equivalence between ‘there are at least \( m \) \( F \)'s’ and ‘there are at least \( n \) \( F \)'s’, is a genuine instance of identity. As Frege pointed out in a similar context, we have to make sure that our definition of numerical identity does not “become involved in conflict with the well-known laws of identity” (1884, §65). To that end, we need to test two things: first, given our definition of numerical identity, is it true that \( n = n \) for every \( n \)? That is, is numerical identity reflexive? The answer is ‘yes’, since logical equivalence is a reflexive relation. Secondly, again given our definition of numerical identity, is it the case that if \( m \) and \( n \) are identical, then they can be substituted for one another without loss of truth in all relevant contexts? To answer the question, we need to determine, of each relevant context, if the substitutability of identicals holds. We introduced two different contexts in which number words

\(^9\)Chapter 2 contains a detailed discussion of this point.
8 and 9. 8 is a trivial case. In the case of 9, namely the context of numerical identity, the question is this: given the way in which numerical identity is defined, is it the case that for any arbitrary number \( l \), if \( m = l \) and \( m = n \), then \( n = l \)? This is to ask whether numerical identity is symmetric and transitive. Again, the answer is ‘yes’, for logical equivalence is a symmetric and transitive relation.\(^{10}\)

One possible objection to our definition of numerical identity could be extracted from David Bostock’s book on the nature of natural numbers. A fictionalist about arithmetic, Bostock seeks a way to eliminate numerical singular terms in favor of numerical quantifiers of the form ‘there are exactly \( n \)’. He first considers the following definition of numerical identity (1974, 12):

\[ \text{‘} m = n \text{’} =_{df} \text{‘for every } F, \text{ there are exactly } m \text{ } F\text{s } \leftrightarrow \text{ there are exactly } n \text{ } F\text{s’}, \]

which coincides with our own 4 except that it is presented as a stipulatory definition. He rejects 11 due to its dependence upon an axiom of infinity (1974, 15–16) and then considers its modalized version (1974, 16–18), namely:

\[ \text{‘} m = n \text{’} =_{df} \text{‘necessarily, (for every } F, \text{ there are exactly } m \text{ } F\text{s } \leftrightarrow \text{ there are exactly } n \text{ } F\text{s’}.} \]

Bostock claims that a construction of arithmetic based on 12 has its own problem but it is “not so much that it is incorrect as that it is incomplete” (1974, 17). He thinks that it is indeed “correct” inasmuch as it “endorses the view that numbers are not objects” but higher-level entities (1974, 17). But he also thinks that it is “incomplete” in that it is “a system in which only objects can be counted, in so far as its only numerical quantifiers are defined by means of variables ranging over

\(^{10}\)In what follows we shall introduce two more contexts in which number words occur, namely 13 and 14. We shall leave the reader to verify that they do not violate the substitution law of identity.
objects and nothing else” (1974, 17). On his view, “this must surely be a mistake, for Frege was obviously right to claim that we can count numbers” (1974, 17).

To elaborate Bostock’s objection to 12, consider a numerical quantifier of the form ‘there are \( n \)’, or more explicitly, ‘there are \( n \) \( x \)’s such that \( \ldots x \ldots \)’. It is “defined by means of” a variable ‘\( x \)’. The difficulty, as Bostock sees it, is that the variable ‘\( x \)’ is supposed to range over “objects and nothing else”. If so, then it would be impossible to say things such as that there are 4 prime numbers less than 10. For the numerical quantifier ‘there are 4’ can only be attached to predicates that can be true of objects and objects only. But since it is obvious that we can count numbers as well as objects, a system of arithmetic based upon a principle like 12 must be “incomplete”.

Bostock’s criticism of 12, if correct, would have devastating consequences for our definition of numerical identity, 9, for it also contains numerical quantifiers defined by means of ordinary variables like ‘\( x \)’. However, the truth is that Bostock is mistaken in believing that those variables range over objects and objects only. What is very significant about this belief, though, is that it is not at all something peculiar to him. His view of first-order variables as object variables is the dominant view of variables espoused by generations of logicians including Frege and Russell, and most ferociously by Quine.

That there must be something wrong with the orthodox view of the variables like ‘\( x \)’, ‘\( y \)’ and ‘\( z \)’ as object variables can be seen from the fact that it would render totally innocent existential claims utterly problematic. For instance, from the self-evident fact that prudence is a virtue follows an equally self-evident fact that there is a virtue. The logical form of the sentence ‘there is a virtue’ could be rendered as ‘there is an \( x \) such that \( x \) is a virtue’. But, given the assumption that ‘\( x \)’ is an object variable, this would mean that there is an object such that it is a virtue. Now, this
is, at the very least, a highly contentious claim. Quine admits, without a qualm, that that is a contentious claim “on which philosophers disagree” (1950, 261). But how could one deny that there is such a thing as prudence? Of course not, and that means that the existential sentence ‘there is an x such that x is a virtue’ should not be construed as saying that there is an object such that it is a virtue.

By the same logic, Quine claims that whether or not there is any number is a contentious issue, “on which philosophers disagree” (1950, 261). Unfortunately, Quine is right about philosophers’ disagreeing over the issue. So many efforts by so many brilliant people have been wasted to argue for or against the existence of numbers conceived as abstract objects, that is, entities that make up part of the world but do not enter into any causal relations with others. The friends and the foes of the realist conception of numbers are both mistaken. That there are numbers is not something one can seriously argue for or against: it is a totally trivial truth. It is self-evident that 7 + 5 = 12, and hence that there is a number identical with 7 + 5, and, a fortiori, that there is a number. It is a truism that to be is to be a value of a variable. The number 12 is a value of the bound variable ‘x’ in ‘there is an x such that 7 + 5 = x’; so it must be. But the fact that there is an x such that 7 + 5 = x does not imply that there is an object in the universe that is identical with 7 + 5. No trivial fact like that can have such an astounding consequence. Famously, Quine turned the truism that to be is to be a value of a bound variable into a highly controversial criterion of ontological commitment, that is, a criterion by which we can assess our commitments as to what things make up the world (1948, 12–19). And his view of variables was the key to how he could magically transform such a truism into a substantial ontological doctrine: to be is to be a value of a bound variable and, by the way, the variable is an object variable!

There are three correlative syntactic notions in logic: identity, singular terms,
and first-order variables. They are mutually related as follows: things can bear the relation of identity to one another if and only if they can be named by singular terms if and only if they can be values of first-order variables. In Chapter 6 we defined an *individual* as an entity of whatever kind with which an identity criterion is associated and argued that identity is a relation between *individuals* and not just objects. This means that singular terms, as terms that can flank the identity sign, can name not only objects but also any individuals. It also means that first-order variables like ‘*x*’, ‘*y*’, and ‘*z*’ are *individual* variables, not *object* variables. Actually, ‘individual variable’ is the standard term for those variables; the term ‘object variable’ is rarely used among logicians and philosophers alike. But this is merely due to the historical accident that in *Principia Mathematica* Whitehead and Russell adopted the term ‘individual’ for the entities of the lowest type, which are objects in our terminology.\(^{11}\)

Once it is realized that first-order variables are *individual* variables in our sense of the term, the alleged difficulty which Bostock found with \(^{12}\) vanishes. In rendering the logical form of ‘there are \(n\) *Fs*’ as ‘there are \(n\) *x*’s such that *x* is an *F*’, we are not restricting the range of countable things to objects. The variable ‘*x*’, as an individual variable, is not supposed to range exclusively over objects. It is a generic individual variable that can take individuals of any type, including numbers, as its values. Of course, I am not saying that it can range over a domain that includes all types of individuals simultaneously. Such a domain is inadmissible. Individuals include objects, their properties, and the properties of those properties, and so on, to mention a few. So if they are all put in a single domain, then self-predication kicks in, immediately resulting in a contradiction. A domain is admissible if and

\(^{11}\)Thus they say: “we will use such letters as *a*, *b*, *c*, *x*, *y*, *z*, *w*, to denote objects which are neither propositions nor functions. Such objects we shall call *individuals*” (Whitehead and Russell 1910, 51).
only if there is a single criterion for the identity of all its members. The important point is that insofar as a domain of individuals is admissible, first-order variables such as ‘x’ can range over it, whether it is a set of persons, a set of numbers, or whatever. Therefore, our own definition of numerical identity, 9, can be used to count not just objects but any individuals whatsoever.

Another possible difficulty one might find with our definition of numerical identity is that it is not clear how we could establish the determinacy of numbers in terms of 9. From the assumption that there are exactly m Fs and there are exactly n Fs, we could easily prove that for every G, there are m Gs if and only if there are n Gs and hence, by 4, that m = n. But 9 allows us to derive ‘m = n’ only if it can be proved from the same assumption that ‘there are m Fs’ and ‘there are n Fs’ are logically equivalent. Is that possible, and if so, how?

The answer to the first question is ‘yes’, and to answer the second question, we need to introduce the numerical relation greater than. Intuitively, our conception of numbers as modes of existence suggests that ‘m > n’ means that if Fs exist in the mode of m, then Fs must also exist in the mode of n but not vice versa. In other words, to say that m > n is to say that if there are at least m Fs then there must be at least n Fs but not vice versa. To use ‘⇒’ for logical implication, we can thus define greater than as follows:

\[ m > n \iff (\text{there are at least } m \text{ Fs} \Rightarrow \text{there are at least } n \text{ Fs}) \text{ and } (\text{it is not the case that there are at least } n \text{ Fs} \Rightarrow \text{there are at least } m \text{ Fs}). \]

In a similar fashion, we can introduce the relation greater than or equal to as follows:

\[ m \geq n \iff (\text{there are at least } m \text{ Fs} \Rightarrow \text{there are at least } n \text{ Fs}). \]

It follows from 9, 13 and 14 that m ≥ n if and only if m > n or m = n.

For detail, see Chapter 6.
For a proof of ‘$m = n$’ from ‘there are exactly $m$ Fs’ and ‘there are exactly $n$ Fs’, suppose the following four conditions hold:

- **a** there are at least $m$ Fs;
- **b** there are not at least $m'$ Fs;
- **c** there are at least $n$ Fs;
- **d** there are not at least $n'$ Fs.

Suppose $m \neq n$. Then, $m > n$ or $n > m$. We have no resources yet to prove this, but shortly we shall provide them. We proceed to show that both disjuncts are false. First, suppose $m > n$. Then it follows, as we shall prove later, that $m \geq n'$. So, by 14, there are at least $m$ Fs ⇒ there are at least $n'$ Fs. By d, it follows that there are not at least $m$ Fs, which contradicts a. Suppose $n > m$. Then, it follows that $n \geq m'$. By 14, there are at least $n$ Fs ⇒ there are at least $m'$ Fs. By b, however, there are not at least $n$ Fs, and this contradicts c. Hence if there are exactly $m$ Fs and there are exactly $n$ Fs, then $m$ must be identical with $n$.

Now we shall show that the set of definitions of the basic concepts of arithmetic we have obtained so far is powerful enough to serve as a foundation of arithmetic. To that end, we shall derive from it the five Peano Axioms, which can be presented for our purposes as follows:

**P1** 1 is a natural number;

**P2** for every $n$, if $n$ is a natural number, then $n'$ is a natural number;

**P3** for every $m$ and $n$, if $m$ and $n$ are natural numbers, then $(m = n \leftrightarrow m' = n')$;

**P4** for every $n$, if $n$ is a natural number, then $1 \neq n'$;

**P5** for every $P$, if 1 is a $P$ and (for every $m$, if $m$ is a $P$, then $m'$ is a $P$), then (for every $n$, if $n$ is a natural number, then $n$ is a $P$).

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13 See the proof for 19.

14 See the proof for 29.
First, as we already observed in Chapter 2, the axioms $P_1$, $P_2$ and $P_5$ are immediate consequences of the following definition of natural number:

15 $n$ is a natural number $\leftrightarrow$ for every $P$, if $1$ is a $P$ and (for every $m$, if $m$ is a $P$, then $m'$ is a $P$), then $n$ is a $P$.

This definition of natural number has been accused of being impredicative. A definition is *impredicative* if it defines a member of a totality in terms of the totality itself. 15 defines the property of being a natural number by quantifying over the totality of numerical properties, of which it is a member. Identifying impredicative definition as the prime suspect for the paradoxes in the foundations of mathematics, Russell adopted the so-called *vicious-circle principle*, namely that “no totality can contain members defined in terms of itself” (Russell 1908, 75). This led him to divide properties of the same type into different orders: a property obtained by quantifying over properties of some order is stipulated to be of the order next above it. So the property of being a natural number, which is obtained by quantifying over first-order properties of numbers, becomes a second-order property of numbers. But since the vicious-circle principle banned speaking of “all properties”, much of ordinary mathematics had to go. Russell attempted to fix the problem by postulating that for any property of whatever order, there is a property of the lowest order that is coextensive with it (Russell 1908, 80–83). But this so-called *axiom of reducibility* is even more objectionable than an axiom of infinity and was eventually renounced by Russell himself, even at the expense of the theory of real numbers (Whitehead and Russell 1910, xiv; xlv–xlv).

It would seem that the ban on impredicative definition implies either sacrificing much of ordinary mathematics or introducing an axiom of reducibility. Since neither of the options is palatable, a natural reaction at this point would be to revoke the
ban, as Ramsey and others did. Ramsey argued that impredicative definition is not viciously circular because the object of definition is not being *constructed* but only *described*. For instance, “we may refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle” (Ramsey 1931, 41). There is no vicious circle, for the man was already in the group and we merely singled him out by the descriptive phrase. In the same way, when we give an impredicative definition of a property, we merely “describe it in a certain way, by reference to a totality of which it may be itself a member” (Ramsey 1931, 41). If Ramsey is right, then 15 should not be understood as illicitly creating the property of being a natural number by reference to the totality of numerical properties but simply as singling it out among all the properties of numbers there are in and by themselves.

I cannot entirely agree with Ramsey. Some impredicative definitions “describe” nothing. Consider the following simple impredicative definition of a rogue numerical property: $n$ is an *anti-number* if and only if for every numerical property $P$, $n$ is not a $P$. Obviously it does not describe any numerical property. For, if a number were an anti-number, then, provided that being an anti-number was itself a numerical property, it would immediately follow that it is not an anti-number. Ramsey showed that those so-called *logical* paradoxes which arise from self-predication (or self-membership) could be removed by adopting a simple theory of types, without ramifying types further into orders (Ramsey 1931, 24–27). Interestingly, however, the above definition of anti-number is unlike those impredicative definitions which give rise to logical paradoxes such as Russell’s in that it does not involve self-predication (or self-membership). The fault clearly lies with its unrestricted

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15See Ramsey 1931, 41–42. Among others, Carnap and Quine joined Ramsey in embracing impredicative definition. See Carnap 1931, 49–52 and Quine 1963, 242–43.
reference to “all properties”. Especially, it does seem to make an illicit attempt to create a certain thing in terms of the totality it belongs to.

So we need a tougher ban on impredicative definition than Ramsey’s. But Russell went too far when he banned all impredicative definitions. Impredicative definition per se is not to blame for contradictions. The fact that the definiendum is itself an element of the totality in terms of which it is defined would be no cause for concern if that totality was already firmly fixed. Let us say that a totality is properly defined if and only if it is defined either by a predicative definition, where the definiendum is defined strictly in terms of a totality whose type is next below its type, or by an impredicative definition in reference to a properly defined totality. Our point is that there is nothing wrong with singling out an entity with reference to the totality it belongs to given that the totality in question was properly defined beforehand. The problematic cases of impredicative definition are only those which appeal to the totalities which have not been properly defined. The above definition of the notion of an anti-number defines a numerical property by making reference to the totality of all numerical properties tout court, which has not been, and cannot be, properly defined.

Let me illustrate the point. Suppose we defined a set of numerical properties $I$ in a predicative way, that is, in terms of a totality of numbers alone and not in terms of any totality of numerical properties. Suppose we then defined a numerical property $N$ in terms of that set as follows:

16 for every $n$, $n$ is an $N$ if and only if for every $I$, $n$ is an $I$.

Since ‘$I$’ means a numerical property $P$ that satisfies a certain condition, 16 can be also put as follows:

17 for every $n$, $n$ is an $N$ if and only if for every $P$, if $(\ldots P \ldots)$, then $n$ is a $P$. 

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where ‘... \(P...\)’ stands for the condition of being a numerical property \(I\). So we defined a numerical property, \(N\), by quantifying over “all numerical properties”, of which \(N\) is a member itself. However, this is entirely harmless since the definition of \(N\) is made in effect only in reference to the properties \(I\). Of course, it may turn out that the property of being an \(N\) is indeed one of the properties \(I\). But we have assumed that the set of the properties \(I\) was given predicatively beforehand and so there should be no fear of illicitly creating a new property.

Let us say that a numerical property \(P\) is inductive if and only if it is possessed by the number 1 and by a successor of any number possessing it. Our definition of a natural number, \(15\), says that for every \(n\), \(n\) is a natural number if and only if for every \(P\), if \(P\) is inductive, then \(n\) is a \(P\). So it is of the same logical form as \(17\). Moreover, as noted above, \(15\) directly implies \(P1\) and \(P2\), which means that the property of being a natural number is an inductive property itself. However, the notion of an inductive property was already fixed by a predicative definition in terms of the totality of numbers alone. So the definition of natural number does not add a new element to the totality of inductive properties but merely singles out one that belongs to it. Our original definition of a natural number, \(15\), stands.

We now turn to a derivation of \(P3\) and \(P4\). First, a proof of \(P4\) can proceed as follows. Observe first by induction that

\[
18 \text{ for every natural number } n, \ n \geq 1.
\]

Given the way the relation ‘\(\geq\)’ is defined in \(14\), it is easy to see from \(18\) that for every natural number \(n\), \(n' \geq 1'\). Now suppose there is a natural number \(n\) such that \(1 = n'\). By the substitutability of identicals, it follows that \(1 \geq 1'\), that is, there is at least \(1 \ F \Rightarrow\) there are at least 2 \(F\)s, which is absurd. Therefore, for no natural number \(n\), \(1 = n'\).
For **P3**, observe that the left-to-right direction of the biconditional is trivial: if $m = n$, then since $m' = m'$ for every natural number $m$, it follows by the substitutability of identicals that $m' = n'$.

For the right-to-left direction of **P3**, we need to establish first the trichotomy principle, namely that

19 for any natural numbers $m$ and $n$, either $m > n$ or $m = n$ or $n > m$.

We shall proceed by induction on $n$. The base step, namely,

20 for any natural number $m$, $m > 1$ or $m = 1$ or $1 > m$,

follows immediately from 18. Now for the induction step, assume that

21 for any natural number $m$, either $m > n$ or $m = n$ or $n > m$.

By the symmetricity of identity, it follows from 21 that

22 for any natural number $m$, either $m > n$ or $n = m$ or $n > m$.

From the way we defined numerical identity and *greater than*, we can easily see that

23 for any natural numbers $n$ and $m$, if $n = m$ or $n > m$, then $n' > m$.

Hence it follows from 22 and 23 that

24 for any natural number $m$, either $m > n$ or $n' > m$.

Now we shall show by induction on $n$ that

25 for any natural numbers $m$ and $n$, if $m > n$, then there is a natural number $l$ such that $m = n + l$.

For the base step, show by induction on $m$ that
for any natural number \( m \), if \( m > 1 \), then there is a natural number \( l \) such that \( m = 1 + l \).

For the induction step, assume that

for any natural number \( m \), if \( m > n \), then there is a natural number \( l \) such that \( m = n + l \).

Then, again by induction on \( m \), observe that

for any natural numbers \( m \) and \( n \), if \( m > n' \), then there is a natural number \( l \) such that \( m = n' + l \).

26, 27 and 28 together prove 25. Now, using 25, we shall show that

for any natural numbers \( m \) and \( n \), if \( m > n \), then \( m > n' \) or \( m = n' \).

Suppose \( m > n \). Then, by 25, there is a natural number \( l \) such that \( m = n + l \). By 18, we know that \( l \geq 1 \), that is, \( l > 1 \) or \( l = 1 \). Suppose \( l > 1 \). It is easy to see by induction that for any natural numbers \( l, m, \) and \( n \), if \( l > m \), then \( n + l > n + m \).

Since \( l > 1 \), it follows that for any natural number \( n, n + l > n + 1 \), that is, \( n + l > n' \).

Since \( m = n + l \), it follows by the substitutability of identicals that \( m > n' \). Suppose \( l = 1 \). Then \( m = n + 1 \), that is, \( m = n' \). This proves 29. Finally, it follows from 24 and 29 that

for any natural number \( m \), either \( m > n' \) or \( m = n' \) or \( n' > m \),

which is our desired result: 20, 21 and 30 together prove the trichotomy principle, 19.

With the trichotomy principle at hand, we can now complete the proof of the right-to-left direction of P3 as follows. We shall proceed by reductio ad absurdum. Suppose \( m' = n' \) yet \( m \neq n \). By the trichotomy principle, it follows that either
Suppose \( m > n \). Given the way we defined the notion of successor and the greater than relation, it follows that \( m' > n' \). But since we have assumed that \( m' = n' \), it follows that \( n' > n' \), that is, (there are at least \( n' \) Fs ⇒ there are at least \( n' \) Fs) and (it is not the case that there are at least \( n' \) Fs ⇒ there are at least \( n' \) Fs), which is absurd. So it cannot be the case that \( m > n \). Parallel reasoning shows that it cannot be the case that \( n > m \), either. Therefore, for any natural numbers \( m \) and \( n \), if \( m' = n' \), then \( m = n \).

The derivation of the Peano Axioms from our definitions of the basic concepts of arithmetic has two major consequences. First, it follows that every truth of arithmetic is analytic. We saw in Chapter 2 that every truth of logic is analytic. Since every truth of arithmetic is a logical consequence of the Peano Axioms\(^{16}\) and each of the axioms logically follows from our definitions of the basic concepts of arithmetic, the question whether every truth of arithmetic is analytic comes down to the question whether our definitions of the basic concepts of arithmetic are analytic. We argued in Chapter 2 that a correct explication of a concept must be analytic. We leave the reader to verify that all of our definitions of the basic concepts of arithmetic explicate them correctly.

Secondly—and this establishes the main thesis of our present inquiry—it follows that every truth of arithmetic is in fact a truth of logic (in a suitably expanded sense of the term). To elaborate, note that since every truth of arithmetic is a logical consequence of the Peano Axioms, every truth of arithmetic would be a truth of logic if each of those axioms is a truth of logic. Now, by way of our definitions of the basic concepts of arithmetic in terms of pure logic, we can convert the Peano Axioms into sentences of pure logic. And the fact that the Peano Axioms logically follow from the definitions of the basic concepts of arithmetic implies that we can

\(^{16}\)If second-order consequence is a type of logical consequence, that is.
prove each Peano Axiom thus converted into a sentence of logic to be a truth of logic. Logicism in the philosophy of arithmetic is established.
By way of concluding remarks, let us summarize some of the major insights underlying our logicist theory of numbers. Firstly, we realized that number words ‘n’ in phrases of the form ‘there are n’ could not be analyzed as higher-level predicates and hence that what they represent, namely numbers, could not be conceived as higher-level properties. Instead, we argued for an adverbial analysis of the basic role of number words: the ‘n’ in ‘there are n Fs’ serves as a modifier of the verb phrase ‘there are’ and represents a certain way in which Fs exist. Thus, we concluded that numbers are modes of existence.

Secondly, we recognized the fundamental importance of the role of the number one in a proper foundation of arithmetic. While every individual number, including the number 0, can be introduced by definition, the number 1 is an irreducible, primitive logical element, since ‘there is at least 1 F’, or in symbols, ‘∃x (Fx)’, is an irreducible, primitive sentential form of quantificational logic. In this regard alone, our logicism sharply diverges from the previous versions of logicism. When the classical logicists such as Frege and Russell espoused logicism, the underlying assumption was that logic is free of any elements of arithmetic, or put conversely, arithmetical elements are foreign to logic. Thus, Frege tried to reduce our discourse about numbers to discourse about supposedly purely logical extensions of concepts, and Russell wanted to eliminate numbers in favor of supposedly purely logical,
higher-level properties, or propositional functions as he called them. Our own theory of numbers is antithetic to this ontological reductionism about arithmetic. For us, numbers are irreducible, *sui generis* elements. Moreover, they are elements of *logic*: the number 1 is no less an element of logic than are truth-functions, quantifiers, or identity, and every other number can be defined in terms of it. In this sense, we can indeed say that logic is itself the home of numbers, as it were, or put differently, numbers are native to logic.

Thirdly, the question arose as to how we can attain a grasp of the concept of the number 1 if it is an indefinable, primitive concept of logic. And our answer was: in the same way as we attain a grasp of the meaning of any other primitive logical constant, that is, by attaining a grasp of the logical rules associated with it. In the case of the number word ‘1’, the relevant rules are the introduction and elimination rule for the quantifier ‘there is at least 1’ (or, in symbols, ‘\(\exists_1 x\)’). The fact that our grasp of the concept of the number 1 and hence of every individual number concept derives ultimately from a grasp of certain logical rules is of major significance in the epistemology of arithmetic. The question how we can have epistemic access to mathematical “objects” has been among the most pressing ones in the philosophy of the subject.\(^1\) In fact, the very same question was the *Urproblem* of Frege’s entire philosophy: “How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?” (1884, §62)\(^2\) The prime problem forced itself upon Frege, only because he had concluded that numbers must be objects. Had he realized that numbers are more or less like such logical entities as identity or truth-functions, he would not have been compelled to give such a strenuous argument for a way of

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\(^1\)For a classic discussion of the issue, see Benacerraf 1973.

\(^2\)Frege refers to the problem as *Urproblem* at the beginning of the last paragraph in the appendix to his *Grundgesetze*: “As the *Urproblem* of arithmetic we can look upon the question: how do we grasp logical objects, especially the numbers?” (1903b, 265)
grasping numbers via grasping extensions of concepts.

Lastly, and most importantly, if numbers are originally given to us not as objects in the context ‘\(m = n\)’ but as modes of existence in the context ‘there are \(n\) \(Fs\)’, how is it possible for us to talk about numbers using numerical singular terms as we do in arithmetic? We answered this question by making a crucial distinction between objects and individuals. Identity is a relation between individuals, that is, any entities with which a criterion of identity is associated. While numbers are modes of existence and originally introduced in the context ‘there are \(n\) \(Fs\)’, they have a criterion of identity associated with them, and to that extent, we are justified in using number words in the context ‘\(m = n\)’. So the fact that we employ numerical singular terms in arithmetic to talk about numbers is perfectly consistent with the fact that numbers are not objects and primarily represented by numerical “adverbs”. Numerical singular terms and numerical “adverbs” refer to one and the same entities, and this is manifested in the definition of numerical identity we gave in the previous chapter, namely:

\[9 \ m = n \leftrightarrow (\text{there are at least } m \ Fs \leftrightarrow \text{there are at least } n \ Fs).\]

Numerical variables here conspicuously occupy at once the places of numerical singular terms (on the left-hand side of the biconditional) and the places of numerical “adverbs” (on the right-hand side of the biconditional).

Our theory of numbers thus suggests a third way between number-theoretic Platonism and fictionalism. The Platonist argues that since singular terms represent objects, the entities that numerical singular terms represent, namely numbers, must be objects. The fictionalist, on the other hand, holds that numerical singular terms are spurious and numbers are mere useful fictions to be eliminated in the ultimate analysis. If we are right, however, both sides of the debate are mistaken. Above all, arithmetic is not about any objects in the Platonist sense of the term. It is rather
about individuals of a certain type, namely modes of existence, and modes in general are not the same type of entities as those that make up the world. Still, inasmuch as arithmetic has numbers as its subject matter, it is a genuine body of truths, not a fictional theory. So our theory of numbers allows us to take arithmetic “at face value”, that is, to take it to be really about what it appears to be about, that is, numbers, without committing ourselves to a dubious and extravagant ontology of number-theoretic Platonism.
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