On Characteristic Classes of Complex Vector Bundles

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ON CHARACTERISTIC CLASSES OF COMPLEX VECTOR BUNDLES

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1. Introduction

A *vector bundle* is a topological construction which attaches a vector space to a base point in some topological space B. This paper will investigate *characteristic classes* of vector bundles, cohomology classes associated with each vector bundle. These characteristic classes measure the "twistiness" of the bundle; specifically, they measure whether the vector bundle possesses sections. This paper will construction the Chern classes of an arbitrary complex vector bundle using algebraic topology and universal vector bundles.

This paper will first define complex vector bundles and some important background information in Section 2. Section 3 will look at Grassmanians, and construct the universal bundle γ^n. We then introduce oriented vector bundles, the Euler class, and prove the Thom Isomorphism Theorem in Section 4. In Section 5, we use the Gysin sequence to prove our Main Theorem, a calculation of $H^*(G_n;\mathbb{Z})$. Finally, in Section 6 we construct the Chern classes.

Remark. Note that this paper will deal exclusively with complex vector spaces. Thus all isomorphisms and homomorphisms refer explicitly to complex isomorphisms and homomorphisms, unless otherwise stated.

2. Vector Bundles and Bundle Maps

2.1. **Complex Vector Bundles.** Let B be a topological space, called the *base space.*

Definition 1. A *complex vector bundle* ξ over B consists of:

(1) a topological space E, also denoted $E(\xi)$, called the *total space*,
(2) a continuous, surjective map $\pi : E \to B$ called the *projection map*, and
(3) for each $b \in B$, the structure of a vector space over the complex numbers in the set $\pi^{-1}(b)$,

such that the following condition is satisfied:

Local Triviality. For each point $b \in B$ there exists a neighborhood $U \subset B$ of b, an integer $n \in \mathbb{Z}_{\geq 0}$, and a homeomorphism

$$h : U \times \mathbb{C}^n \to \pi^{-1}(U)$$

such that for each $a \in U$, the map $x \mapsto h(a, x)$ determines an isomorphism $\mathbb{C}^n \to \pi^{-1}(a)$.

We call the a pair (U, h) a *local coordinate system of* ξ *about* b. If there is a local coordinate system (B, h), then we say that ξ is a *trivial bundle*. Note that on each connected component of B, the integer n is constant; however, this does not imply that each vector bundle is trivial over connected space. When n is a constant over the entire space B, we call the bundle an *n-plane bundle*. We call the set $\pi^{-1}(b)$ the *fiber* over b, denoted also by $F_b(\xi)$.

For example, the *trivial line bundle* of a space B is the vector bundle with total space $E = \{(b, r) \mid b \in B, r \in \mathbb{C}\}$ and projective maps $\pi : (b, r) \mapsto b$. We denote this bundle by ε_1. In general, we can define the *trivial n-plane bundle*, denoted ε_n, with total space $E(\varepsilon_n) = \{(b, x) \mid b \in B, x \in \mathbb{C}^n\}$, and projection map $\pi : (b, x) \mapsto b$.

We will, on occasion, want to use *cross sections* of vector bundles to prove certain results about vector bundles.

Definition 2. A *cross-section* of a vector bundle ξ with base space B is a continuous function

$$s : B \to E(\xi)$$

which takes each $b \in B$ into the corresponding fiber $F_b(\xi)$. A cross-section is *nowhere zero* if $s(b)$ is a non-zero vector of $F_b(\xi)$ for each $b \in B$. We say that cross-sections $s_1, \dots, s_n$ are *nowhere dependent* if for each $b \in B$, the vectors $s_1(b), \dots, s_n(b)$ are linearly independent.

Then we can use cross-sections to identify trivial n-plane bundles.

Theorem 2.1. *An n-dimensional vector bundle ξ is trivial if and only if ξ admits n cross-sections $s_1, \dots, s_n$ which are nowhere dependent.*

To prove this, we first need some background.

We will want to be able to relate two vector bundles to each other. Suppose that ξ, η are vector bundles over the same base space B.

Definition 3. The vector bundle ξ is *isomorphic* to η, denoted $\xi \cong \eta$, if there exists a homeomorphism

$$f : E(\xi) \to E(\eta)$$

between the total spaces such that each fiber $F_b(\xi)$ is mapped isomorphically to $F_b(\eta)$.

Lemma 2.2. *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$. Then f is a homeomorphism, hence $\xi \cong \eta$.*

Proof. Let $b_0 \in B$ be any point. Choose local coordinate systems (U, g) for ξ and (V, h) for η, such that $b_0 \in U \cap V$. We need to show that the composition

$$(U \cap V) \times \mathbb{C}^n \xrightarrow{h^{-1} \circ f \circ g} (U \cap V) \times \mathbb{C}^n$$

is a homeomorphism. Define $h^{-1}(f(g(b, x))) = (b, y)$. Then the point $y = (y_1, \dots, y_n)$ can be written

$$y_i = \sum_j f_{ij}(b) x_j,$$

where the matrix $[f_{ij}(b)]$ is a non-singular matrix of real numbers. It is clear that the f_{ij} depend continuously on b, since $[f_{ij}(b)]$ as a matrix depends continuously on b. Now let $[F_{ji}(b)]$ be the inverse matrix of $[f_{ij}(b)]$. Then clearly

$$g^{-1} \circ f^{-1} \circ h(b, y) = (b, x),$$

where

$$x_j = \sum_p F_{ji}(b) y_i.$$

Since each $F_{ji}(b)$ depends continuously on the matrix $[f_{ij}(b)]$, they depend continuously on b. Therefore $g^{-1} \circ f^{-1} \circ h$ is continuous, and since g^{-1}, h are continuous, then f^{-1} is continuous. Thus f is a homeomorphism. $\square$

Using this Lemma, we can now prove Theorem 2.1.

Proof. Let $s_1, \ldots, s_n$ be cross-sections of ξ that are nowhere linearly dependent. Define the map $f : B \times \mathbb{R}^n \to E$ by

$$f(b, x) = x_1 s_1(b) + \cdots + x_n s_n(b).$$

Now f is a sum and product of continuous functions, so f is continuous. Furthermore, f maps each fiber of the trivial bundle ε^n_B isomorphically onto the corresponding fiber of ξ. Hence f is a bundle isomorphism, and ξ is trivial.

Now suppose that ξ is trivial, and (B, h) is a coordinate system. Define n sections $s_1, \ldots, s_n$ by the following:

$$s_i(b) = h(b, (0, \ldots, 0, 1, 0, \ldots, 0)) \in F_b(\xi),$$

where 1 is in the i-th place, and all other entries are 0. Then clearly the s_i are nowhere linearly dependent and they are continuous functions $B \to F_b(x) \subset E$. Thus these are nowhere linearly dependent cross-sections. $\square$

We can relate two vector bundles as follows:

Definition 4. A *bundle map* from ξ to η is a continuous function

$$g : E(\xi) \to E(\eta)$$

which carries each fiber $F_b(\xi)$ isomorphically to some fiber $F_{b'}(\eta)$. The induced function $\bar{g} : b \mapsto b'$ gives a continuous map $\bar{g} : B(\xi) \to B(\eta)$.

This motivates the idea of an induced vector bundle. Given a map between two topological spaces $B' \xrightarrow{f} B$, and a vector bundle $E(\xi) \xrightarrow{\pi_\xi} B$, we can induce a vector bundle $f^*(\xi)$ over B'. The total space $E(f^*(\xi))$ is a subset of $B' \times E(\xi)$ consisting of the pairs (b, e) such that $f(b) = \pi_\xi(e)$. The projection map of the induced vector bundle is given by $\pi_{f^*(\xi)} : (b, e) \mapsto b$. This gives a commutative diagram:

$$\begin{array}{ccc} E(f^*(\xi)) & \xrightarrow{\hat{f}} & E(\xi) \\ \Big\downarrow{\scriptstyle \pi_{f^*(\xi)}} & & \Big\downarrow{\scriptstyle \pi_\xi} \\ B' & \xrightarrow{f} & B \end{array}$$

where $\hat{f}(b, e) = e$. The vector space structure of $\pi_{f^*(\xi)}(b)$ is defined by the relation

$$s(b, e) + t(b, e') = (b, se + te').$$

Thus $\hat{f}$ is an isomorphism between $F_b(f^*(\xi))$ and $F_{f(b)}(\xi)$, for each $b \in B'$. Now we need only verify that $f^*(\xi)$ satisfies the local triviality condition. But if (U, h) is a local coordinate system of ξ, let $U' = f^{-1}(U)$, and define

$$H : U' \times \mathbb{C}^n \to \pi_{f^*(\xi)}(U')$$

by the relation $H : (b, x) \mapsto (b, h(f(b), x))$. Thus (U', H) is a local coordinate system of $f^*(\xi)$.

In fact, given the above assumptions, the bundle induced by the continuous map $\bar{g}$ of the above definition is isomorphic to the vector bundle η. This is formalized in the following lemma.

Lemma 2.3. *Let $g : E(\eta) \to E(\xi)$ a bundle map, and let $\bar{g} : B(\eta) \to B(\xi)$ be the corresponding map of base spaces. Then $\eta \cong \bar{g}^*(\xi)$.*

Proof. Let $f : E(\eta) \to E(f^*(\xi))$ be defined by

$$f(e) = (\pi_\eta(e), g(e)).$$

Hence f is continuous and maps each fiber $F_b(\eta)$ isomorphically to $F_b(\bar{g}^*(\xi))$. Thus f is an isomorphism. $\square$

Another construction that we will find useful is the *Whitney sum* of two vector bundles. To describe this, we must first consider the Cartesian product of two vector bundles. The *Cartesian product* of two vector bundles $E(\xi_1) \xrightarrow{\pi_1} B_1, E(\xi_2) \xrightarrow{\pi_2} B_2$, written $\xi_1 \times \xi_2$, is given by the bundle with projection map $\pi_1 \times \pi_2 : E(\xi_1) \times E(\xi_2) \to B_1 \times B_2$ with each fiber $F_{(b_1,b_2)}(\xi_1 \times \xi_2) = F_{b_1}(\xi_1) \times F_{b_2}(\xi_2)$ with the inherited vector space structure. Now suppose that ξ_1 and ξ_2 are two vector bundles over the same base space B. Let $d : B \to B \times B$ be the diagonal embedding, $d : b \mapsto (b, b)$. Then the induced bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and we denote it $\xi_1 \oplus \xi_2$. Thus we see that each fiber $F_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to $F_b(\xi_1) \oplus F_b(\xi_2)$.

Now we can generalize this concept to that of a *sub-bundle*. Notice that if $\xi = \xi_1 \oplus \xi_2$, then we see that each fiber $F_b(\xi_1) \subset F_b(\xi)$. We formalize this idea in the following definition and lemma.

Definition 5. Let ξ, η be two vector bundles over the same base space B, such that $E(\xi) \subset E(\eta)$. Then we say that ξ is a *sub-bundle* of η, written $\xi \subset \eta$, if every fiber $F_b(\xi)$ is a sub-vector space of $F_b(\eta)$.

Lemma 2.4. *Let ξ_1 and ξ_2 be sub-bundles of η over some base space B such that $F_b(\eta) = F_b(\xi_1) \oplus F_b(\xi_2)$ for all $b \in B$. Then η is isomorphic to $\xi_1 \oplus \xi_2$.*

Proof. Let $f : E(\xi_1 \oplus \xi_2) \to E(\eta)$ by $f : (b, (e_1, e_2)) \mapsto e_1 + e_2$. This is an isomorphism. $\square$

This leads to the question: When can a vector bundle be split into the Whitney-sum of two sub-bundles? This is possible whenever we have a Euclidean metric on the vector bundle. Recall that a *Euclidean vector space* is a complex vector space V together with a positive definite quadratic function

$$\mu : V \to \mathbb{R}.$$

Then we define an *inner product* $v \cdot w = \frac{1}{2}(\mu(v+w) - \mu(v) - \mu(w))$. The norm is given by $|v|^2 = v \cdot v$.

Definition 6. A *Euclidean vector bundle* is a complex vector bundle ξ together with a continuous function

$$\mu : E(\xi) \to \mathbb{R}$$

such that the restriction of μ to each fiber F_b of ξ is positive definite and quadratic. This function μ is called a *Euclidean metric* on the vector bundle ξ.

This gives us an immediate Whitney sum decomposition of a vector bundle given a sub-bundle.

Definition 7. Let ξ, η be vector bundles over a base space B such that $\xi \subset \eta$. Then the *orthogonal complement*, $\xi^\perp$, is the vector bundle over base space B such that each fiber $F_b(\xi^\perp)$ is the sub-vector space of $F_b(\eta)$ consisting of vectors v such that $v \cdot w = 0$ for all $w \in F_b(\xi)$. Then the total space $E(\xi^\perp)$ is given by the union of these fibers.

A priori, it is not obvious that this is a vector bundle. The following theorem proves that it is, and that it gives a Whitney sum decomposition of our vector bundle η.

Theorem 2.5. *Let $\xi \subset \eta$ be vector bundles over a base space B. The space $E(\xi^\perp)$ as constructed above is the total space of a sub-bundle $\xi^\perp \subset \eta$. Furthermore, η is isomorphic to the Whitney sum $\xi \oplus \xi^\perp$.*

Proof. Note first of all that each vector space $F_b(\eta)$ is the direct sum of subspaces $F_b(\xi)$ and $F_b(\xi^\perp)$. Thus we need only prove that $\xi^\perp$ satisfies the local triviality condition.

Let $b_0 \in B$ and let U a neighborhood of b_0 which is sufficiently small that both $\xi|_U$ and $\eta|_U$ are trivial. Let $s_1, \ldots, s_m$ be orthonormal cross-sections of $\xi|_U$ and let $s'_1, \ldots, s'_n$ be orthonormal cross-sections of $\eta|_U$, where m and n are the fiber dimensions of $\xi|_U$ and $\eta|_U$. Thus the $m \times n$ matrix given by

$$[s_i(b_0) \cdot s'_j(b_0)]$$

has rank m. Since we can renumber the s'_j so that the first m columns of the matrix are linearly independent, we assume that the first m columns are linearly independent.

Now let $V \subset U$ be the open set of all point b such that the first m columns of the matrix $[s_i(b) \cdot \cdot'_j (b)]$ are linearly independent. Then the cross-sections

$$s_1, s_2 \ldots, s_m, s'_{m+1}, \ldots, s'_n$$

of $\eta|_U$ are not linearly dependent at any point of V. Indeed, if there were a linear combination $c_1 s_1(b) + c_2 s_2(b) + \cdots + c_m s_m(b) + c_{m+1} s'_{m+1} + \cdots + c_n s'_n(b) = 0$, then there would be a non-zero linear combination of $s_1(b), \ldots, s_m(b)$ which is equal to a non-zero linear combination of $s'_{m+1}(b), \ldots, s'_n(b)$, so $s_1(b), \ldots, s_m(b)$ are orthogonal to $s'_1(b), \ldots, s'_n(b)$, a contradiction. Now applying Gram-Schmidt to these cross-sections, we get orthonormal cross-sections, which we will label $s_1, \ldots, s_m, s_{m+1}, \ldots, s_n$ of $\eta|_V$.

Now a local coordinate system $h : V \times \mathbb{C}^{n-m} \to E(\xi^\perp)$ for $\xi^\perp$ is given by:

$$h(b, x) = x_1 s_{m+1}(b) + \cdots + x_{n-m} s_n(b).$$

Then the identity

$$h^{-1}(e) = (\pi(e), (e \cdot s_{m+1}(\pi(e)), \ldots, e \cdot s_n(\pi(e))))$$

is clearly a continuous function and it is an inverse of h, so h is a homeomorphism, as desired. $\square$

2.2. **Characteristic Class of a Complex n-Plane Bundle.**

Definition 8. An *n-dimensional characteristic class (with $\mathbb{Z}$ coefficients) for complex vector bundles of dimension N* is a rule which assigns, to each complex vector bundle ξ^N on a paracompact space X, an n-dimensional cohomology class $c(\xi^N) \in H^n(X;\mathbb{Z})$ such that:

$$\text{If } Y \xrightarrow{f} X \text{ is a continuous map, then } f^*(c(\xi)) = c(f^*(\xi)).$$

Equivalently, the pullback of the class is the class of the pullback.

The cohomology of "classifying spaces" provides a method of constructing an n-dimensional characteristic class for complex vector bundles of dimension k. This will be formalized further below.

3. Grassmannians, Principal Bundles, and the Universal Bundle

3.1. **The Grassmannian.** The Grassmannian will allow us to construct the "classifying space" for complex vector bundles. To define the Grassmanian, we first need to define an *n-frame*.

Definition 9. An *n-frame* in $\mathbb{C}^{n+k}$ is an n-tuple of linearly independent vectors of $\mathbb{C}^{n+k}$.

Notice that we can define a space of orthonormal n-frames in $\mathbb{C}^{n+k}$. We call this space the *Stiefel space*, denoted by

$$V_n(\mathbb{C}^{n+k}) = \{(v_1, \dots, v_n) \mid \{v_i\} \text{ form an orthonormal set in } \mathbb{C}^{n+k}\}.$$

Now consider the equivalence relation on $V_n(\mathbb{C}^{n+k})$ which sets two orthonormal n-frames as equivalent if and only if they span the same subspace. Then we can construct the identification space of $V_n(\mathbb{C}^{n+k})$, which is equivalent to the space of n-planes in $\mathbb{C}^{n+k}$ which pass through the origin.

Definition 10. The *Grassmann manifold* $G_n(\mathbb{C}^{n+k})$ is the set of all n-planes through the origin, lying in $\mathbb{C}^{n+k}$.

We can actually say more. In fact, $V_n(\mathbb{C}^{n+k})$ is a *principal bundle* over the base space $G_n(\mathbb{C}^{n+k})$. We say that a vector bundle $P \xrightarrow{\pi} X$ is G-principal if P admits a free G-action whose orbit space is X. More formally, let G be a topological group, and let P be a right G-space with orbit space X and orbit map $P \xrightarrow{\pi} X$. Let $\mu : P \times G \to P$ denote the G-action.

Definition 11. The five-tuple (P, μ, G, π, X) is a *principal G-bundle* if each $x \in X$ has a neighborhood U for which there exists a homeomorphism

$$U \times G \xrightarrow{\Phi} P|_U$$

making the diagram below commute:

$$\begin{tikzcd} U \times G \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & P|_U \arrow[dl, "\pi|_{P|_U}"] \\ & U & \end{tikzcd}$$

In particular, $V_n \to G_n$ is a principal $O(n)$ bundle.

Now we will want to be able to apply the techniques of vector bundles to the Grassmann, so we must show that $G_n(\mathbb{C}^{n+k})$ is a manifold.

Lemma 3.1. *The Grassmann manifold $G_n(\mathbb{C}^{n+k})$ is a compact topological manifold of dimension nk. The correspondence $X \to X^{\perp}$, which assigns to each n-plane its orthogonal k-plane in $\mathbb{C}^{n+k}$, defines a homeomorphism between $G_n(\mathbb{C}^{n+k})$ and $G_k(\mathbb{C}^{n+k})$.*

Proof. We first show that $G_n(\mathbb{C}^{n+k})$ is a Hausdorff space. To do this, we need only show that any two points can be separated by a continuous, real-valued function. Let $w \in \mathbb{C}^{n+k}$ be fixed, and let $\rho_w(X)$ denote the square of the Euclidean distance from w to X, where X is an n-plane in $\mathbb{C}^{n+k}$. Let $x_1, \ldots, x_n$ be an orthonormal basis of X. Then the function $\rho_w(X)$ is given by:

$$\rho_w(X) = w \cdot w - (w \cdot x_1)^2 - \cdots - (w \cdot x_n)^2.$$

Now define a map $q : V_n(\mathbb{C}^{n+k}) \to G_n(\mathbb{C}^{n+k})$ so that each n-frame is mapped to the n-plane that it spans. Note that this is a continuous map. Since the composition map $\rho_w \circ q$ is continuous, then ρ_w is continuous. Now if $X \neq Y$ are n-planes, suppose $w \in X$ and $w \notin Y$. Then $\rho_w(X) \neq \rho_w(Y)$. Thus $G_n(\mathbb{C}^{n+k})$ is a Hausdorff space.

Now $V_n(\mathbb{C}^{n+k})$ is a closed, bounded subset of $\mathbb{C}^{n+k} \times \ldots \mathbb{C}^{n+k}$, so it is compact. Therefore, $G_n(\mathbb{C}^{n+k})$ is also compact.

Now we need to prove that every point X_0 of $G_n(\mathbb{C}^{n+k})$ has a neighborhood U which is homeomorphic to $\mathbb{C}^{nk}$. Note that $\mathbb{C}^{n+k} = X_0 \oplus X_0^{\perp}$. Now let U be the open set of $G_n(\mathbb{C}^{n+k})$ consisting of all the n-planes Y such that the orthogonal projection

$$p : X_0 \oplus X_0^{\perp} \to X_0$$

maps Y surjectively onto X_0. In other words, U is the set of all Y such that $Y \cap X_0^{\perp} = 0$. Then each $Y \in U$ can be viewed as the graph of a linear map

$$T(Y) : X_0 \to X_0^{\perp}.$$

Thus this defines a one-to-one correspondence

$$T : U \to \mathrm{Hom}(X_0, X_0^{\perp}) \cong \mathbb{C}^{nk}.$$

We want to show that T is a homeomorphism.

Let $x_1, \ldots, x_n$ be some fixed orthonormal basis of X_0. Note that each n-plane $Y \in U$ has a unique basis $y_1, \ldots, y_n$ such that

$$p(y_1) = x_1, \quad \ldots, \quad p(y_n) = x_n.$$

Clearly the n-frame $(y_1, \ldots, y_n)$ depends continuously on Y.

We know that

$$y_i = x_i + T(Y)x_i.$$

Since y_i depends continuously on Y, then $T(Y)x_i \in X_0^\perp$ depends continuously on Y. Therefore $T(Y)$ is continuous, in the sense that T depends continuously on Y.

Furthermore, $y_i = x_i + T(Y)x_i$ shows that the n-frame $(y_1, \ldots, y_n)$ depends continuously on $T(Y)$, so Y depends continuously on $T(Y)$. Therefore, T^{-1} is also continuous. Therefore $G_n(\mathbb{C}^{nk})$ is a manifold.

Finally, we want to prove that $Y^\perp$ depends continuously on Y. Let $(v_1, \ldots, v_k)$ be a fixed basis of $X_0^\perp$. Define a function:

$$f = q^{-1}(U) \to V_k(\mathbb{C}^{n+k})$$

as follows: For each $(y_1, \ldots, y_n) \in q^{-1}(U)$, apply the Gram-Schmidt process to $(y_1, \ldots, y_n, v_1, \ldots, v_k)$, to obtain $(y_1, y_2, \ldots, y_n, v_1', \ldots, v_k')$. Note that this is an orthonormal $(n+k)$-frame. Let $f(y_1, \ldots, y_n) = (v_1', \ldots, v_k')$. Then we see that the following diagram commutes:

$$\begin{tikzcd}
q^{-1}(U) \arrow[r, "f"] \arrow[d, "q"'] & V_k(\mathbb{C}^{n+k}) \arrow[d, "q"] \\
U \arrow[r, "\perp"] & G_k(\mathbb{C}^{n+k})
\end{tikzcd}$$

Since f is continuous, then $q \circ f$ is continuous, so $Y \mapsto Y^\perp$ is continuous. Furthermore, since $(Y^\perp)^\perp = Y$, then we see that the inverse correspondence is also continuous, as desired. $\square$

From this Grassmann, then, we can construct a canonical vector bundle $\gamma^n(\mathbb{C}^{n+k})$ over $G_n(\mathbb{C}^{n+k})$. Let $E = E(\gamma^n(\mathbb{C}^{n+k}))$ be the set of pairs (P, v), where $P \in G_n(\mathbb{C}^{n+k)}$, and v is a vector in the plane P. This is topologized as a subset of $G_n(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k}$. The projection map $\pi : E \to G_n(\mathbb{C}^{n+k})$ is given by $\pi : (P, v) \mapsto P$, and the vector space structure is inherited from the vector space structure of $\mathbb{C}^{n+k}$. Now we need only show that this satisfies the local triviality condition:

Lemma 3.2. *The vector bundle $\gamma_n(\mathbb{C}^{n+k})$ as constructed above satisfies the local triviality condition.*

Proof. Let $X_0 \in G_n(\mathbb{C}^{n+k})$, and let U be the neighborhood of X_0 as constructed in the above proof. Define the coordinate homeomorphism

$$h : U \times X_0 \to \pi^{-1}(U)$$

by the formula $h(P, v) = (P, v')$, where v' is the unique vector in P with image v in the orthogonal projection

$$p : \mathbb{C}^{n+k} \to X_0.$$

Then we have the following two identities:

$$h(P, v) = (P, v + T(P)v), \qquad h^{-1}(P, v') = (P, p(v')),$$

thus h and h^{-1} are continuous. □

3.2. **Universal Bundles.** The purpose of this construction is that most $\mathbb{C}^n$ bundles can be mapped into the bundle $\gamma^n(\mathbb{C}^{n+k})$ as long as k is sufficiently large. Therefore we call $\gamma^n(\mathbb{C}^{n+k})$ a "universal bundle."

Lemma 3.3. *For any n-plane bundle ξ over a* ***compact*** *base space B and for a sufficiently large k, there exists a bundle map $\xi \to \gamma^n(\mathbb{C}^{n+k})$.*

Proof. First notice that to construct a bundle map $f : \xi \to \gamma^n(\mathbb{C}^m)$, it suffices to construct a map $\hat{f} : E(\xi) \to \mathbb{C}^m$ which is linear and injective on each fiber of ξ. Then our desired function f can be defined by

$$f(e) = (\hat{f}(F_e(\xi), \hat{f}(e)).$$

Given that ξ is locally trivial, then f is continuous.

Now choose open sets $U_1, \ldots, U_r$ which cover B such that each $\xi|_{U_i}$ is trivial. Note that this is possible since B is compact, so by covering B with the open sets of the local coordinates, we can take a finite subset of this which covers B. Since B is normal, then there exist open set $V_1, \ldots, V_r$ covering B such that $\overline{V_i} \subset U_i$.

Similarly, we can construct $W_1, \dots, W_r$ such that $\overline{W_i} \subset V_i$. Now let

$$\lambda_i : B \to \mathbb{R}$$

denote a continuous function which takes value 1 on $\overline{W_i}$ and value 0 outside of V_i.

Now $\xi|_{U_i}$ is trivial, so there is a map

$$h_i : \pi^{-1}(U_i) \to \mathbb{C}^n$$

which maps each fiber of $\xi|_{U_i}$ linearly onto $\mathbb{C}^n$. Define a map $h_i' : E(\xi) \to \mathbb{C}^n$ by the formula

$$h_i'(e) = \begin{cases} 0 & \text{if } \pi(e) \notin V_i, \\ \lambda_i(\pi(e)) \cdot h_i(e) & \text{if } \pi(e) \in U_i. \end{cases}$$

Thus h_i' is continuous, and it is linear on each fiber.

Now define

$$\hat{f} : E(\xi) \to \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \cong \mathbb{C}^{rn}$$

by the formula:

$$\hat{f}(e) = (h_1'(e), h_2'(e), \dots, h_r'(e)).$$

Then $\hat{f}$ is continuous, and maps each fiber injectively, as desired. $\square$

This can be extended to *paracompact* base spaces, but we will need to let the dimension of $\mathbb{C}^{n+k}$ approach infinity. This motivates the definition of the infinite Grassmannian: Let $\mathbb{C}^\infty$ be the vector space consisting of sequences $x = (x_1, x_2, \dots)$ of complex numbers such that almost all the x_i are zero. We then identify $\mathbb{C}^k$ with the subspace $\{x = (x_1, x_2, \dots, x_k, 0, 0, 0) | x_i \in C\}$. Therefore we see that $C^1 \subset C^2 \subset C^3 \subset \dots$, and $\bigcup \mathbb{C}^i = \mathbb{C}^\infty$.

Definition 12. The *infinite Grassmannian*

$$G_n = G_n(\mathbb{C}^\infty)$$

is the set of all n-dimensional linear subspaces of $\mathbb{C}^\infty$, topologized as the direct limit of the sequence

$$G_n(\mathbb{C}^n) \subset G_n(\mathbb{C}^{n+1}) \subset G_n(\mathbb{C}^{n+2}) \subset \cdots .$$

Now we construct a canonical bundle γ^n over G_n just as in the finite dimensional case. Let $E(\gamma^n) \subset G_n \times \mathbb{C}^\infty$ be the set of all pairs (P, v), where P is an n-plane in $\mathbb{C}^\infty$ and $v \in P$. We topologize this as a subset of $G_n \times C^\infty$. Define the projection map $\pi : E(\gamma^n) \to G_n$ by $\pi : (P, v) \mapsto P$. Once again, we need only prove that this satisfies the local triviality condition. To do this, we will first need to prove the following lemma.

Lemma 3.4. *Let $A_1 \subset A_2 \subset A_3 \subset \ldots$ and $B_1 \subset B_2 \subset B_3 \subset \ldots$ be sequences of locally compact spaces with direct limits A and B respectively. Then the Cartesian product topology on $A \times B$ coincides with the direct limit topology of the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset A_3 \times B_3 \subset \ldots$.*

Proof. Let W be an open set under the direct limit topology. Let (a, b) be any point of W. Suppose that $(a, b) \in A_i \times B_i$. Choose a compact neighborhoods $K_i \subset A_i$ and $L_i \subset B_i$ such that $(a, b) \in K_i \times L_i \subset W$. Then we can choose compact neighborhoods K_{i+1} of K_i in A_{i+1} and L_{i+1} of L_i in B_{i+1}, so that $K_{i+1} \times L_{i+1} \subset W$. By induction, we can construct neighborhoods $K_i \subset K_{i+1} \subset K_{i+2} \subset \ldots$ with union U, and $L_i \subset L_{i+1} \subset L_{i+2} \subset \ldots$ with union V. Then U and V are open sets, and certainly $(a, b) \in U \times V \subset W$. Thus W is open in the product topology. $\square$

Theorem 3.5. *The vector bundle γ^n as constructed above satisfies the local triviality condition.*

Proof. Let $X_0 \subset \mathbb{C}^\infty$ be a fixed n-plane, and let $U \subset G_n$ be the set of all n-planes Y which project subjectively onto X_0 under the orthogonal projection $p : \mathbb{C}^\infty \to X_0$. This set U is open, since for finite k, the intersection

$$U_k = U \cap G_n(\mathbb{C}^{n+k})$$

is an open set. Now define the map $h : U \times X_0 \to \pi^{-1}(U)$ as above in Lemma 3.2. Thus $h|_{U_k \times X_0}$ is continuous for each k. Thus h is itself continuous, by the above lemma.

As earlier, we know that $h^{-1}(P, v') = (P, p(v'))$ implies that h^{-1} is continuous, so h is a homeomorphism. Thus γ^n is locally trivial. $\square$

Now consider a general paracompact base space B. Recall first the definition of a paracompact space.

Definition 13. A topological space B is called *paracompact* if B is Hausdorff and if, for every open covering $\{U_\alpha\}$ of B, there exists an open covering $\{V_\beta\}$ of B such that

(1) $\{V_\beta\}$ is a refinement of $\{U_\alpha\}$, and
(2) $\{V_\beta\}$ is locally finite; that is, each point of B has a neighborhood which intersects only finitely many of the V_β.

Now we will introduce, without proofs, some classical facts about paracompact spaces:

Theorem 3.6. *(A.H. Stone) Every metric space is paracompact.*

Theorem 3.7. *(Morita) If a regular topological space is the countable union of compact subsets, then it is paracompact.*

Corollary 3.8. *The direct limit of a sequence $K_1 \subset K_2 \subset K_2 \subset \ldots$ of compact spaces is paracompact. In particular, the infinite Grassmann space G_n is paracompact.*

Theorem 3.9. *(Dieudonné) Every paracompact space is normal.*

Given a paracompact base space B, we now have the following universality of γ^n:

Theorem 3.10. *Any $\mathbb{C}^n$-bundle ξ over a paracompact base space admits a bundle map $\xi \to \gamma^n$.*

To prove this, we first need the following lemma:

Lemma 3.11. *For any fiber bundle ξ over a paracompact space B, there exists a locally finite covering of B by countably many open sets $U_1, U_2, \dots$, such that $\xi|_{U_i}$ is trivial for each i.*

Proof. Choose a locally finite open covering $\{V_\alpha\}$ such that each $\xi|_{V_\alpha}$ is trivial, and choose an open covering $\{W_\alpha\}$ such that $\overline{W_\alpha} \subset V_\alpha$ for each α. Let $\lambda_\alpha : B \to \mathbb{R}$ be a continuous function taking value 1 on $\overline{W_\alpha}$ and value 0 outside of V_α. For each non-empty finite subset S of the indexing set $\{\alpha\}$, define the set $U(S) \subset B$ as the set of all $b \in B$ such that

$$\min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \notin S} \lambda_\alpha(b).$$

Let U_k be the union of the sets $U(S)$ such that S has precisely k elements.

Now each U_k is certainly an open set, and

$$B = U_1 \cup U_2 \cup \dots.$$

Indeed, if $b \in B$, then if precisely k of the numbers $\lambda_\alpha(b)$ are positive, then $b \in U_k$. Now note that if $\alpha \in S$, then

$$U(S) \subset V_\alpha.$$

Since $\{V_\alpha\}$ is locally finite, then $\{U_k\}$ is also locally finite. Furthermore, since $\xi|_{V_\alpha}$ is trivial for each α, then $\xi|_{U(S)}$ is trivial. But U_k is equal to the disjoint union of its open subset $U(S)$, so $\xi|_{U_k}$ is also trivial. □

Proof of Theorem 3.10. Now that we have this lemma, we can apply the methods of the proof of Lemma 3.3 to get the desired bundle map $f : \xi \to \gamma^n$. □

Furthermore, we see that given two bundle maps $f, g : \xi \to \gamma^n$, they are *bundle homotopic*, in the sense that there is a family of bundle maps

$$h_t : \xi \to \gamma^n, \qquad t \in [0, 1]$$

such that $h_0 = f, h_1 = g$, and $h : E(\xi) \times [0,1] \to \gamma^n$ is continuous.

Theorem 3.12. *Any two bundle maps $f, g : \xi \to \gamma^n$ are bundle homotopic.*

Proof. Notice that any bundle map $f : \xi \to \gamma^n$ determines a map $\hat{f} : E(\xi) \to \mathbb{C}^\infty$ such that the restriction of $\hat{f}$ to each fiber of ξ is linear and injective. On the other hand, $\hat{f}$ determines f by the identity

$$f(e) = (\hat{f}(F_e(\xi)), \hat{f}(e)).$$

Now let $f, g : \xi \to \gamma^n$ be any two bundle maps. Suppose first that the vector $\hat{f}(e) \in \mathbb{C}^\infty$ is never equal to a negative multiple of $\hat{g}(e)$ for $e \neq 0, e \in E(\xi)$. Then the formula given by

$$\hat{h}_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e), \qquad 0 \leq t \leq 1$$

defines a homotopy between $\hat{f}$ and $\hat{g}$. We need to show that $\hat{h}$ is continuous as a function of e and t. It suffices to show that vector space operations in $\mathbb{C}^\infty$ are continuous, but this is clear from our above lemma. Now $\hat{t}_e \neq 0$ if e is a non-zero vector of $E(\xi)$. Thus we can define a map $h : E(\xi) \times [0,1] \to E(\eta)$ by the formula:

$$h_t(e) = (\hat{h}_t(F_e(\xi)), \hat{h}_t(e)).$$

We want to show that h is continuous. It suffices to prove that the corresponding function $H : B(\xi) \times [0,1] \to G_n$ of base spaces is continuous. Let U be an open subset of $B(\xi)$ such that $\xi|_U$ is trivial, and let $s_1, \ldots, s_n$ be nowhere dependent cross-sections of $\xi|U$. Then $H|_{U\times[0,1]}$ is a composition of the maps ϕ and q, where $\phi : (b,t) \mapsto (\hat{h}_t s_1(b), \ldots, \hat{h}_t s_n(b))$ and q is the Gram-Schmidt map. Thus $H|_{U\times[0,1]}$ maps $U \times [0,1] \to G_n$, and it is continuous. Thus the bundle homotopy h between f and g is continuous.

Now suppose that $f, g : \xi \to \gamma^n$ are arbitrary bundle maps. Let d be the bundle map

$$d : \gamma^n \to \gamma^n$$

induced by the linear transformation $\mathbb{C}^\infty \to \mathbb{C}^\infty$ which carries the i-th basis vector of $\mathbb{C}^\infty$ to the $(2i-1)$-th basis vector. Similarly, define

$$D : \gamma^n \to \gamma^n$$

be the bundle map induced by the linear transformation which carries the i-the basis vector to the $(2i)$-th basis vector. Now notice that there are bundle homotopies

$$f \sim d \circ f \sim D \circ g \sim g$$

by the above paragraph. Hence $f \sim g$. □

3.3. **Constructing a Characteristic Class.** Now these universal results lead us to call the infinite Grassmannian G_n the *classifying space* of complex vector bundles. Notice first that we get an immediate corollary from the above result:

Corollary 3.13. *Any $\mathbb{C}^n$-bundle ξ over a paracompact space B determines a unique homotopy class of maps*

$$\overline{f_\xi} : B \to G_n$$

such that $f_\xi^*(\gamma) = \xi$.

This motivates the following definition:

Definition 14. The *classifying map* of a vector bundle ξ, denoted f_ξ is any of the homotopically equivalent maps ξ^n to γ^n.

Proof. Let $f_\xi : \xi \to \gamma^n$ be a bundle map, and let $\overline{f_\xi}$ be the induced map of base spaces. Since any other bundle map $g : \xi \to \gamma^n$ is bundle homotopic to f_ξ, we see that this is a unique homotopy class, as desired. □

Now let $c \in H^i(G_n; \mathbb{Z})$ be any cohomology class. Then ξ and c determine a cohomology class

$$\overline{f_\xi}^* c \in H^i(B; \mathbb{Z}),$$

which we will denote $c(\xi)$. We call this cohomology class the *characteristic cohomology class* of ξ determined by c.

Notice that this correspondence $\xi \mapsto c(\xi)$ is natural with respect to bundle maps, and furthermore, given any correspondence $\xi \mapsto c(\xi) \in H^i(B;\mathbb{Z})$ which is natural with respect to bundle maps, then

$$c(\xi) = \overline{f_\xi}^* c(\gamma^n).$$

The key here is that the ring of all characteristic classes for $\mathbb{C}^n$-bundles over paracompact base spaces with coefficients in $\mathbb{Z}$ is canonically isomorphic to the cohomology ring $H^*(G_n;\mathbb{Z})$.

Thus our goal is to compute the cohomology ring of G_n.

4. Oriented Vector Bundles, The Euler Class, and the Thom Isomorphism

4.1. **Oriented Vector Bundles.**

We begin by considering an orientation on a real vector space.

Definition 15. An *real orientation* of a vector space V of dimension $n > 0$ is a equivalence class of bases, where two ordered bases are said to be equivalent if and only if the change of basis matrix has positive determinant.

Since a change of basis matrix is necessarily non-singular, we see that its determinant is either positive or negative. Thus every real vector space has precisely two orientations.

Now consider complex vector spaces. We want to show that there is a single standard orientation of complex vector spaces. Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be an ordered basis of an n-dimensional vector space V over $\mathbb{C}$. Then from this, we can obtain a natural ordered basis of V over $\mathbb{R}$, given by $\mathcal{B}' = \{b_1, ib_1, b_2, ib_2, \dots, b_n, ib_n\}$. Now suppose that $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ is another ordered basis of V over $\mathbb{C}$, and $\mathcal{C}' = \{c_1, ic_2, \dots, c_n, ic_n\}$ is the natural ordered basis over $\mathbb{R}$. Now let M be the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Note that $M \in GL(n,\mathbb{C})$. Now form $\hat{M} \in GL(2n,\mathbb{R})$

by replacing each entry $a+ib$ of M with the array $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then $\hat{M}$ is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{C}'$. Furthermore, $\det(\hat{M}) = ||\det(M)|| > 0$, so the orientation of V over $\mathbb{R}$ does not depend on the chosen basis. Thus all complex vector spaces V have a single natural orientation.

The heart of this matter is that a change of basis matrix is non-singular, so it lies in $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. Specifically, two ordered bases have the same orientation if they lie in the same path component of $GL(n,\mathbb{R})$. But notice that $GL(n,\mathbb{R})$ has two path component, whereas $GL(n,\mathbb{C})$ has a single path component. Thus all complex bases are in the same path component of $GL(n,\mathbb{C})$, so there is one orientation.

Notation. Let V be a vector space. Then V_0 denotes the set $V - \{0\}$.

The standard method of specifying an orientation of a simplex in algebraic topology is to order the vertices. We can use this to induce an orientation on a vector space V as follows: Let $\sum^n$ be an n-simplex, linearly embedded in n-dimensional vector space V, with ordered vertices $v_0, v_1, \ldots, v_n$. Then take the vector v_0 to v_1 as the first basis vector, v_1 to v_2 for the second basis vector, and so on. This gives an orientation for V.

Now notice that this choice of orientation for V corresponds to a choice of one of the two generators for the singular homology group $H_n(V, V_0; \mathbb{Z})$. To see this, let Δ^n be the standard n-simplex, with canonically ordered vertices, and let

$$\sigma : \Delta^n \to V$$

be an orientation preserving linear embedding such that the barycenter of Δ^n is mapped to 0. Thus σ maps the boundary of Δ^n into V_0. Then σ is a singular n-simplex which represents a relative n-cycle in the group $Z_n(V, V_0; \mathbb{Z})$. The homology class of σ is then the preferred generator μ_V for the homology group $H_n(V, V_0; \mathbb{Z})$.

We see then that the cohomology group $H^n(V, V_0; \mathbb{Z})$ associated with our oriented vector space V has a preferred generator, denoted u_V, determined by the equation $\langle u_v, \mu_v \rangle = +1$. Therefore, if V is a complex n-dimensional vector space, the group $H^{2n}(V, V_0; \mathbb{Z})$ has a preferred generator.

Now we want to consider a complex vector bundle ξ with fibers of dimension n. Notice that each fiber has the structure of a complex vector space, so each fiber has a natural orientation.

Definition 16. An *orientation* of a complex vector bundle ξ over a base space B is an orientation of each fiber F_b of ξ, viewed as a real vector space.

This has the property that for every point $b_0 \in B$, there exists a local coordinate system (N, h) with $b_0 \in N$ and $h : N \times \mathbb{C}^n \to \pi^{-1}(N)$, so that for each fiber F_b over N the isomorphism $x \mapsto h(b, x)$ is orientation preserving. We call this condition the *local compatibility condition.* We may equivalently define the local compatibility condition as follows: There exist sections $s_1, \ldots, s_n : N \to \pi^{-1}(N)$ such that the basis $s_1(b), \ldots, s_n(b)$ determines the desired orientation of F_b for each $b \in N$.

Thus if $(v_1, \ldots, v_n)$ is any basis of $F_b(\xi)$ over $\mathbb{C}$, the the equivalence class of the $\mathbb{R}$-basis $(v_1, iv_1, \ldots, v_n, iv_n)$ gives a canonical orientation for ξ^n.

Now for any complex vector bundle, this canonical orientation gives a preferred generator of the cohomology group for each fiber F, denoted

$$u_F \in H^{2n}(F, F_0; \mathbb{Z}).$$

The local compatibility condition tells us that for every point $b \in B$ there is a neighborhood N and a cohomology class $u \in H^{2n}(\pi^{-1}(N), \pi^{-1}(N)_0; \mathbb{Z})$ such that for every fiber F over N, the restriction

$$u|_{(F,F_0)} \in H^{2n}(F, F_0; \mathbb{Z})$$

is equal to u_F.

4.2. **The Thom Isomorphism.** This motivates the following theorem, which we will need to prove our main result:

Theorem 4.1. *Let ξ be an complex n-plane bundle with total space E. Then the cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < 2n$, and $H^{2n}(E, E_0; \mathbb{Z})$ contains precisely one cohomology class u whose restriction*

$$u|_{(F,F_0)} \in H^{2n}(F, F_0; \mathbb{Z})$$

is equal to the preferred generator u_F for every fiber F of ξ. Furthermore, the correspondence $\Phi : H^k(E) \to K^{k+2n}(E, E_0)$, defined by the rule $\Phi : y \mapsto y \cup u$ is an isomorphism of $H^k(E; \mathbb{Z})$ onto $H^{k+2n}(E, E_0; \mathbb{Z})$ for every integer k. This isomorphism is called the Thom Isomorphism.

We will break this proof into several parts. First, we shall prove it for coefficients over a ring or field, for a vector bundle of *finite type.* In other words, for a vector bundle for which the base space can be covered by a finite number of neighborhoods over which the restriction of the vector bundle is trivial. Note that this case thus includes all vector bundles over compact spaces. We will prove the case that the coefficients are in a field over an arbitrary space, and finally, the case of ring coefficients over an arbitrary space.

Theorem 4.2. *Let ξ be an finite-type complex n-plane bundle with total space E, and let k be a field or a ring. Then the cohomology group $H^i(E, E_0; k)$ is zero for $i < 2n$, and $H^{2n}(E, E_0; k)$ contains precisely one cohomology class u whose restriction*

$$u|_{(F,F_0)} \in H^{2n}(F, F_0; k)$$

is equal to the preferred generator u_F for every fiber F of ξ. Furthermore, the correspondence $\Phi : H^j(E) \to H^{j+2n}(E, E_0)$, defined by the rule $\Phi : y \mapsto y \cup u$ is an isomorphism of $H^j(E; k)$ onto $H^{j+2n}(E, E_0; k)$ for every integer j.

Proof. We will prove this in an inductive style.

(1) Suppose ξ is a trivial vector bundle. We may as well assume that that $E = B\times\mathbb{C}^n$. Thus we see that $H^n(E, E_0) = H^n(B\times\mathbb{C}^n, B\times\mathbb{C}^n_0)$. But this is isomorphic to $H^0(B)$. Indeed, notice that $H^j(B) \cong H^{j+2n}(B\times\mathbb{C}^n, B\times\mathbb{C}^n_0)$ by the correspondence $y \mapsto y\times e^{2n}$. Thus, to prove existence and uniqueness of u, we need only prove that there is one and only one cohomology class in $H^0(B)$ whose restriction to each point is non-zero. But the the unit element $1 \in H^0(B) \cong H^{2n}(B \times (\mathbb{C}, \mathbb{C}_0))$ exists and satisfies this condition, since it is certainly a unique element. Thus $u = 1$ is our desired class.

(2) Suppose B is the union of two open set B_1 and B_2 such that the theorem holds for $\xi|_{B_1}$ and $\xi|_{B_2}$, and for $\xi|_{B_1\cap B_2}$. Let $B^\cap = B_1 \cap B_2$, and let E_1, E_2, and $E^\cap$ be the inverse images of B_1, B_2, and $B^\cap$, respectively. Then consider the Mayer-Vietoris sequence:

$$\cdots \to H^{i-1}(E^\cap, E^\cap_0) \to H^i(E, E_0) \to H^i(E_1, E_{1_0}) \oplus H^i(E_2, E_{2_0}) \to H^i(E^\cap, E^\cap_0) \to \ldots.$$

Now we have assumed unique cohomology classes $u_1 \in H^{2n}(E_1, E_{1_0})$ and $u_2 \in H^{2n}(E_2, E_{2_0})$ whose restrictions to each fiber is non-zero. Furthermore, we have assumed uniqueness for $\xi|_{B_1\cap B_2}$, thus we see that u_1 and u_2 have the same image in $H^{2n}(E^\cap, E^\cap_0)$. Therefore, they are restrictions of the same cohomology class $u \in H^{2n}(E, E_0)$. Furthermore, this is a unique class u since $H^{2n-1}(E^\cap, E^\cap_0) = 0$.

But now consider the Mayer-Vietoris sequence:

$$\cdots \to H^{j-1}(E^\cap) \to H^j(E) \to H^j(E_1) \oplus H^j(E_2) \to H^j(E^\cap) \to \ldots.$$

Let $j + 2n = i$. Then with the previous Mayer-Vietoris sequence and the correspondence $y \mapsto y \cup u$, we get that

$$H^j(E) \cong H^{j+n}(E, E_0),$$

as desired.

(3) Suppose that B can be covered by finitely many open sets $B_1, \dots, B_\ell$ such that $\xi|_{B_i}$ is trivial for each B_i. Now we use induction. This is clearly true if $i = 1$. If $i > 1$, then we assume by induction that it applies for $\xi|_{B_1 \cup \cdots \cup B_{i-1}}$ and $\xi|_{B_1 \cup \cdots \cup B_{i-1} \cap B_i}$. Then by Case 2, this holds.

□

Now we will prove this for field coefficients over a general base space.

Theorem 4.3. *The above theorem holds for field coefficients over a general base space.*

Proof. Let C be a compact subset of the base space B. Then clearly $\xi|_C$ satisfies the above theorem. Since the union of any two compact subsets is itself compact, then we can form the direct limit of homology groups as C varies over the compact subsets of B:

$$\varprojlim H_j(C),$$

and the corresponding inverse limit of cohomology groups

$$\varinjlim H^j(C).$$

Since we have field coefficients, we have that

$$H^j(B) \cong \varinjlim H^j(C).$$

In particular, we have

$$H^{2n}(E, E_0) \cong \varinjlim H^{2n}(\pi^{-1}(C), \pi^{-1}C_0).$$

But each of these groups contains precisely one class u_C whose restriction to each fiber is non-zero. Thus $H^{2n}(E, E_0)$ contains precisely one class u whose restriction to each fiber is non-zero, as desired.

Now consider the homomorphism

$$\cup u : H^j(E) \to H^{j+2n}(E, E_0).$$

We have the commutative diagram as follows:

$$\begin{tikzcd}
H^j(E) \arrow[r, "\cup u"] \arrow[d] & H^{j+2n}(E, E_0) \arrow[d] \\
H^j(\pi^{-1}(C)) \arrow[r] & H^{j+2n}(\pi^{-1}(C), \pi^{-1}(C)_0)
\end{tikzcd}$$

Then by considering the inverse limit as C varies over all compact subsets, it follows that $\cup u$ is an isomorphism, as desired. □

Now we need only prove the general case for ring coefficients. To do this, we will need several definitions and lemmas.

Lemma 4.4. *The homology group $H_{n-1}(E, E_0; \mathbb{Z})$ is zero.*

Proof. Suppose the base space B is compact. Then we already proved this. Furthermore, since

$$H_{n-1}(E, E_0; \mathbb{Z}) \xrightarrow{\cong} H_{-1}(E; \mathbb{Z}) = 0,$$

and

$$\varinjlim H_i(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbb{Z}) \xrightarrow{\cong} H_i(E, E_0; \mathbb{Z}),$$

where C varies over all compact subset of B, then we have our desired result. □

Now notice that by the unique ring homomorphism $\mathbb{Z} \to R$, for any ring R, the fundamental class in $H^{2n}(E, E_0; \mathbb{Z})$ corresponds to the fundamental class in $H^{2n}(E, E_0; R)$.

Now we need to prove that the cup product $\cup u$ induces cohomology isomorphisms. Thus we will use the following constructions.

Definition 17. A *free chain complex* over $\mathbb{Z}$ is a sequence of free $\mathbb{Z}$-modules K_n and homomorphisms

$$\cdots \to K_n \xrightarrow{\delta} K_{n-1} \xrightarrow{\delta} K_{n-2} \to \ldots$$

with $\delta \circ \delta = 0$. A *chain mapping* $f : K \to K'$ of degree d is a sequence of homomorphisms $K_i \to K'_{i+d}$ satisfying $\delta' \circ f = (-1)^d f \circ \delta$.

Lemma 4.5. *Let $f : K \to K'$ be a chain mapping, where K and K' are free chain complexes over $\mathbb{Z}$. If f induces a cohomology isomorphism*

$$f^* : H^*(K'; R) \to H^*(K; R)$$

for every coefficient field R, then f induces isomorphisms of homology and cohomology with arbitrary coefficients.

Proof. Let K^f denote the mapping cone. Then K^f is a free chain complex constructed as follows: Let

$$K_i^f = K_{i-d-1} \oplus K'_i,$$

with boundary homomorphism $\partial^f : K_i^f \to K_{i-1}^f$, defined by:

$$\partial^f(\kappa, \kappa') = ((-1)^{d+1}\partial\kappa, f(\kappa) + \partial'\kappa').$$

Thus we get a short exact sequence

$$0 \to K' \to K^f \to K \to 0$$

of chain mappings. Furthermore, the boundary homomorphism

$$\partial^f : H_{i-d-1}(K) \to H_{i-1}(K')$$

is the associated homology exact sequence is equal to the map f_*. Therefore, $H_*(K^f)$ is zero if and only if f induces an isomorphism $H_*(K) \to H_*(K')$ of integral homology.

Now by assumption, we know that f induces a cohomology isomorphism $H^*(K'; R) \to H^*(K; R)$ for all coefficient fields R. Using the cohomology exact sequence, then, it follows that $H^*(K^f; R) = 0$. But the cohomology group $H^n(K^f; R)$ is canonically isomorphic to $\mathrm{Hom}_R(H_n(K^f \otimes R), R)$. Therefore, the homology vector space

$H_n(K^f \otimes R)$ is zero, since otherwise, there would be a non-trivial R-linear map from this space to the coefficient field R.

Specifically, we see that the rational homology group $H_n(K^f \otimes \mathbb{Q})$ is zero. Thus, for every cycle $\xi \in Z_n(K^f)$, we see that some integral multiple of ξ is a boundary. Therefore, the integral homology group $H_n(K^f)$ is a torsion group.

Now to prove that this torsion group $H_n(K^f)$ is zero, it suffices to prove that every element of prime order is zero. Let $\xi \in Z_n(K^f)$ be a cycle representing a homology class of order p, a prime. Then

$$p\xi = \partial\kappa$$

for some $\kappa \in K^f_{n+1}$. Thus κ is a cycle modulo p. Since the homology group $H_{n+1}(K^f \otimes \mathbb{Z}_p)$ is zero, then κ is a boundary modulo p, so

$$\kappa = \partial\kappa' + p\kappa''.$$

Therefore $p\xi = \partial\kappa = p\partial\kappa''$, and therefore $\xi = \partial\kappa''$. Thus ξ represents the trivial homology class, so $H_*(K^f) = 0$.

Thus, K^f has trivial homology and cohomology with arbitrary coefficients. In particular, since $Z_{n-1}(K^f)$ is free, then the exact sequence

$$0 \to Z_n(K^f) \to K^f_n \to Z_{n-1}(K^f) \to 0$$

splits, so it is still exact when tensored with an arbitrary group R. Thus the sequence

$$\cdots \to H^f_{n+1} \otimes R \to K^f_n \otimes R \to K^f_{n-1} \otimes R \to \dots$$

is exact too. Therefore, $H_*(K^f \otimes R) = 0$, as desired. $\square$

Now we can prove the final step of the Thom Isomorphism Theorem.

Corollary 4.6. *The correspondence $\eta \mapsto u \cap \eta$ defines an isomorphism from the integral homology group $H_{2n+i}(E, E_0)$ to $H_i(E)$.*

Proof. We will assume the coefficient ring $\mathbb{Z}$ unless otherwise mentioned. Choose a singular cocycle $z \in Z^n(E, E_0)$ which represents the fundamental cohomology class u. Then using the cap product, the correspondence

$$\gamma \mapsto z \cap \gamma$$

from $C_{n+i}(E, E_0)$ to $C_i(E)$ satisfies the identity:

$$\partial(z \cap \gamma) = (-1)^n z \cap (\partial\gamma).$$

Therefore, we see that

$$z\cap : C_{n+i}(E, E_0) \to C_i(E)$$

is a chain mapping of degree $-n$.

Now since

$$\langle c, z \cap \gamma \rangle = \langle c \cup z, \gamma \rangle,$$

we see that the induced chain mapping

$$(z\cap)^{\#} : C^*(E; R) \to C^*(E, E_0; R)$$

is given by $c \mapsto c \cup z$, where R is any ring. If R is a field, then this cochain mapping induces a cohomology isomorphism. Then we can apply the above lemma, to see that

$$u\cap : H_{i+n}(E, E_0; R) \to H_i(E; R)$$

and

$$\cup u : H^i(E; R) \to H^{i+n}(E, E_0; R)$$

are isomorphisms for arbitrary R. In particular, the isomorphism $\cup u : H^0(E; R) \to H^n(E, E_0; R)$ gives the uniqueness of the fundamental cohomology class u with coefficients in R. □

4.3. **The Euler Class.** Let ξ^n be a vector bundle over a base space B.

Definition 18. The *Euler class* of ξ^n is the class

$$\chi(\xi^n) \in H^{2n}(B)$$

which corresponds to $u|_E$ under the canonical isomorphism $\pi^* : H^n(E;\mathbb{Z}) \to H^n(E;\mathbb{Z})$, where $u|_E$ is the preferred generator given by the canonical orientation.

5. The Gysin Sequence for γ_n on $G_n(\mathbb{C}^\infty)$ and the Main Theorem

5.1. **The Gysin Sequence.** We can now obtain the *Gysin Sequence*, a long exact sequence of cohomology groups. Let ξ^n be a n-dimensional complex vector bundle, with projection map $\pi : E(\xi) \to B(\xi)$. As above, let E_0 be the non-zero vectors in E, and let $\pi_0 : E_0 \to B$ be the restriction of π to E_0.

Theorem 5.1. *Let ξ be a complex vector bundle, and let e denoted the Euler class, $\chi(\xi)$. Then there is an exact sequence, called the* Gysin Sequence*,*

$$\cdots \to H^k(B) \xrightarrow{\cup e} H^{k+2n}(B) \xrightarrow{\pi_0^*} H^{k+2n}(E_0) \to H^{k+1}(B) \xrightarrow{\cup e} \cdots .$$

with integer coefficients.

Proof. We begin with the long exact sequence of cohomology groups for the pair (E, E_0):

$$\cdots \to H^j(E, E_0) \to H^j(E) \to H^j(E_0) \xrightarrow{\delta} H^{j+1}(E, E_0) \to \cdots .$$

By the Thom Isomorphism, we have an isomorphism given by:

$$\cup u : H^{j-2n}(E) \to H^j(E, E_0).$$

Substituting the isomorphic group into our sequence, we then get

$$\cdots \to H^{j-2n}(E) \xrightarrow{g} H^j(E) \to H^j(E_0) \xrightarrow{\delta} H^{j-2n+1}(E) \to \cdots ,$$

where $g(x) = (x \cup u)|_E = x \cup (u|_E)$. Now notice that $H^*(B)$ is isomorphic to the ring $H^*(E)$. Since the cohomology class $u|_E \in H^{2n}(E)$ corresponds to the Euler class in $H^*(B)$, we get the long exact sequence:

$$\cdots \to H^{j-2n}(B) \xrightarrow{\cup e} H^j(B) \to H^j(E_0) \to H^{j-2n+1}(B) \to \cdots,$$

as desired. □

5.2. **Application to γ^n on G_n.** Now let γ^n denoted the universal n-plane bundle on G_n, the Grassmanian of n-planes on $\mathbb{C}^\infty$. A point of the total space $E(\gamma^n)$ is given by a pair (P, v), where $P \in G_n$ and $v \in P$. Define the *sphere bundle*

$$S(\gamma^n) = \{(P, v) \in E(\gamma^n) \mid ||v|| = 1\}.$$

Note that this a contractible. Then the projection map $\pi : E \to G_n$ restricts to a map $S(\gamma^n) \xrightarrow{\pi_0} G_n$, where $\pi_0(P, v) = P$. Let q be the map $S(\gamma^n) \xrightarrow{q} G_{n-1}$, given by the rule $q(P, v) = v^\perp \cap P$.

Lemma 5.2. *The space $V_n(\mathbb{C}^\infty)$ is contractible.*

Proof. Note first that $V_1(\mathbb{C}^\infty) = S^\infty$ is contractible, so the claim holds for $n = 1$.

Now assume $V_{n-1}(\mathbb{C}^\infty)$ is contractible. Consider the map $V_n(\mathbb{C}^\infty) \xrightarrow{\pi} V_{n-1}(\mathbb{C}^\infty$ is defined by

$$\pi : (v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_{n-1}).$$

This map is a fibration. Moreover, the fiber F_p over any point p is homeomorphic to the unit sphere $S(W)$, of the vector space $W = \{x \in \mathbb{C}^\infty \mid \langle v_i, x\rangle = 0 \text{ for } i = 1, 2, \ldots, n-1\}$. But this inner product space W is isomorphic to $\mathbb{C}^\infty$. Therefore $S(W)$ is homeomorphic to S^∞ and F_p is contractible for all p.

Therefore $V_n(\mathbb{C}^\infty)$ is the total space of a fibration whose base and fiber are contractible, so it too is contractible. □

Theorem 5.3. *The map $S(\gamma^n) \xrightarrow{q} G_{n-1}$ is a homotopy equivalence. Also,*

$$q^*(\gamma^{n-1}) \oplus \epsilon^1 \cong \pi_0^*(\gamma^n)$$

as bundles on $S(\gamma^n)$.

Proof. Recall that $G_{n-1} \cong V_{n-1}(\mathbb{C}^\infty)/U(n-1)$, where $V_n(\mathbb{C}^\infty) = \{(u_1, \ldots, u_n) \mid u_1, \ldots, u_n\}$ is an orthonormal n-frame in $\mathbb{C}^\infty$. Then $U(n)$ acts on the right on $V_n(\mathbb{C}^\infty)$ by the rule:

$$[u_1, \ldots, u_n]A = [u_1', \ldots, u_n'],$$

where $A \in U(n)$. Now $V_{n-1}(\mathbb{C}^\infty)$ is contractible by the above lemma, and

$$S(\gamma^n) = V_n(\mathbb{C}^\infty)/U(n-1).$$

The map $\tilde{q} : V_n(\mathbb{C}^\infty) \to V_{n-1}(\mathbb{C}^\infty)$ given by

$$\tilde{q} : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_{n-1})$$

is $U(n-1)$ equivarient.

Moreover, the diagram

$$\begin{tikzcd} V_n(\mathbb{C}^\infty) \arrow[r, "\tilde{q}"] \arrow[d] & V_{n-1}(\mathbb{C}^\infty) \arrow[d] \\ S(\gamma^n) \arrow[r, "\mathrm{q}"] & G_{n-1} \end{tikzcd}$$

commutes. Since $\tilde{q}$ is a homotopy equivalence (both domain and range are contractible), then

$$V_n(\mathbb{C}^\infty)/U(n-1) \xrightarrow{\tilde{q}/U(n-1)} V_{n-1}(\mathbb{C}^\infty)/U(n-1)$$

is also a homotopy equivalence. Therefore, $q : S(\gamma^n) \to G_{n-1}$ is a homotopy equivalence. $\square$

Theorem 5.4. *There is a bundle equivalence*

$$q^*(\gamma^{n-1}) \oplus \epsilon^1 \approx \pi_0^*(\gamma^n).$$

Proof. A point of $E(\pi_0^*(\gamma^n))$ is given by a pair $((P,u),(P,v)) \in S(\gamma^n \times E(\gamma^n)$, where u is a unit vector of P and v is any vector of P. A point of $E(q^*(\gamma^{n-1}))$ is a pair $((P,u),(Q,w))$ where $(P,u) \in S(\gamma^n), Q = P \cap u^\perp$, and $w \in Q$.

Therefore, a point of $E(q^*(\gamma^{n-1}) \oplus \epsilon^1)$ is a quintuple (P,u,Q,w,t), where $(P,u) \in S(\gamma^n), Q = P \cap u^\perp, w \in Q$, and $t \in \mathbb{R}$. The bundle isomorphism

$$E(q^*(\gamma^{n-1}) \oplus \epsilon^1) \xrightarrow{J} E(\pi_0^*(\gamma^n))$$

is given by:

$$J(P,u,Q,w,t) = (P,u,P,w+tu),$$

and the inverse isomorphism is given by:

$$J^{-1}((P,u),(P,v)) = (P,u,P \cap u^\perp, v - \langle v,u\rangle u, \langle v,u\rangle).$$

□

The Gysin Sequence applied to the vector bundle γ^n over G_n yields the following:

$$\cdots \to H^{k-2n}(G_n) \xrightarrow{\cup e} H^k(G_n) \xrightarrow{\pi_0^*} H^k(S(\gamma^n)) \to \cdots,$$

and the homotopy equivalence $q : S(\gamma^n) \to G_{n-1}$ then yields the following long exact sequence:

$$\cdots \xrightarrow{\tau} H^{k-2n}(G_n) \xrightarrow{\cup e} H^k(G_n) \xrightarrow{f^*} H^k(G_{n-1}) \xrightarrow{\tau} \cdots. \tag{1}$$

5.3. **The Main Theorem.** We are now ready to prove the following theorem:

Theorem 5.5 (Main Theorem)**.** *For each $n \geq 0$ there are cohomology classes $c_k(\gamma^n) \in H^{2k}(G_n)$, with $0 \leq k < \infty$, uniquely specified by:*

(1) *$c_0(\gamma^n) = 1$, and $c_k(\gamma^n) = 0$ if $k > n$,*
(2) *$c_n = \chi(\gamma^n)$, for all $n > 0$, and*
(3) *$f^*_{\gamma^n \oplus \epsilon^1} c_k(\gamma^n) = c_k(\gamma^{n-1})$ for all $k \geq 0$.*

Moreover,

$$H^*(G_n) = \mathbb{Z}[c_1, c_2, \ldots, c_n].$$

Proof. We will use induction. Let $n = 0$. Then G_0 is a single point, so $H^*(G_0) = \mathbb{Z}$, so this holds for $n = 0$.

Now assume that this is true for G_m, where $m \leq n-1$. Define $c_n(\gamma^n) = \chi(\gamma^n)$. By exactness of (1), $f^*_{\gamma^n \oplus \epsilon^1} c_n(\gamma^n) = 0 = c_n(\gamma^{n-1})$. But we also see that $H^k(G_n) \xrightarrow{f^*_{\gamma^n \oplus \epsilon^1}} H^k(G_{n-1})$ is an isomorphism for all $k \leq 2n-2$. But $H^*(G_{n-1})$ is generated in degrees less than or equal to $2n-2$ and $f^*_{\gamma^n \oplus \epsilon^1}$ is a ring homomorphism, so $f^*_{\gamma^n \oplus \epsilon^1}$ is surjective in *all* dimensions, and therefore the map τ is the 0 map. The we get that the following sequence is exact:

$$0 \to H^{*-2n}(G_n) \xrightarrow{\cup c_n} H^*(G_n) \xrightarrow{f^*_{\gamma^n \oplus \epsilon^1}} H^*(G_{n-1}) \to 0.$$

Now define

$$c_k(\gamma^n) = \begin{cases} 0 & \text{if } k > n, \\ \chi(\gamma^n) & \text{if } k = n, \text{ and} \\ \alpha & \text{if } k < n, \end{cases}$$

where α is the unique element of $H^{2k}(G_n)$ such that $f^*_{\gamma^n \oplus \epsilon^1} c_k = c_k(\gamma^{n-1})$. Then we need only show that $H^*(G_n) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$.

Note first that $H^*(G_n)$ is concentrated in even degrees. Indeed, let a have minimal odd degree in $H^k(G_n)$. Then $f^*_{\gamma^n \oplus \epsilon^1}(a) = 0$, thus $a = a' \cup \chi$ for a lower degree class a' of odd degree, a contradiction. Therefore $H^*(G_n)$ is a commutative ring.

Thus if $c = c_n(\gamma^n)$, we get the following map of exact sequences of graded rings:

$$\begin{tikzcd}
0 \arrow[r] & H^*(G_n) \arrow[r, "\cdot c"] & H^*(G_n) \arrow[r, "f^*_{\gamma^n \oplus \epsilon^1}"] & \mathbb{Z}[c_1, c_2, \ldots, c_{n-1}] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}[c_1, c_2, \ldots, c_n] \arrow[r, "\cdot c_n"] \arrow[u, "\phi"] & \mathbb{Z}[c_1, c_2, \ldots, c_n] \arrow[r] \arrow[u, "\phi"] & \mathbb{Z}[c_1, c_2, \ldots, c_{n-1}] \arrow[r] \arrow[u, "\mathrm{id}"] & 0
\end{tikzcd}$$

where ϕ is defined by $\phi(c_k) = c_k(\gamma^n)$.

We claim that ϕ_j is an isomorphism in degree j for all j. Notice that for $j < 0$ this is clear. Now we will prove by contradiction. Suppose j is the lowest degree for which ϕ_j is not an isomorphism. Then we could have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{j-2n}(G_n) \arrow[r] & H^j(G_n) \arrow[r] & \mathbb{Z}[c_1, c_2, \dots, c_{n-1}]_j \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}[c_1, c_2, \dots, c_n]_{j-2n} \arrow[r] \arrow[u, "\phi_{j-2n}"] & \mathbb{Z}[c_1, c_2, \dots, c_n]_j \arrow[r] \arrow[u, "\phi_j"] & \mathbb{Z}[c_1, c_2, \dots, c_{n-1}]_j \arrow[r] \arrow[u, "\mathrm{id}"] & 0
\end{tikzcd}$$

But this is a contradiction. □

6. Chern Classes of Arbitrary Vector Bundles

Definition 19. Let ξ^n be a complex n-plane vector bundle on a base space B. The *i-th Chern Class* of ξ is the cohomology class

$$c_i(\xi) \in H^{2i}(B; \mathbb{Z})$$

defined as follows:

$$c_i(\xi) = \begin{cases} 1 & \text{if } i = 0, \\ f_\xi^*(c_i) & \text{if } i = 1, 2, \dots, n, \text{ and} \\ 0 & \text{if } j > n, \end{cases}$$

where f_ξ is the classifying map of ξ^n.