On the Topic of Ramsey Theory

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Publication Date

14-12-2023

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Submitted to the Department of Mathematics on

April 12, 2013,

in partial fulfillment of the requirements for graduation with
Honors Senior Thesis

by

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Abstract

Ramsey theory studies conditions that are necessary so that we may preserve "order." Often, we will start with a large structure, and we will break it into finitely many classes. Our typical theorems will say that if the initial structure is large enough, then one of the classes must contain a copy of the original structure. Usually, we will find the smallest number of elements required so that this property is guaranteed to occur; this is often a difficult, but illuminating, problem.

We will begin by looking at finite Ramsey theory, which describes what happens when we divide a finite number of elements into finitely many classes. Our elements will be $k-$sets (we will consider 1-sets, then study 2-sets, and then generalize to $k-$sets for any $k > 2$), and our classes will be colors (beginning with two colors, and then generalizing to additional colors). In particular, we will refer to our $k = 2$ case as "Ramsey theory for graphs." Our analysis will lead up to and conclude with a succinct proof of what is known as the finite version of Ramsey's Theorem, which followed by an application or two.

Next, we will study infinite Ramsey theory, which allows for the set that we are coloring the $k-$sets of a set of infinite cardinality. We will show that we can always find an infinite subset so that all of its $k-$sets are the same color. This is the infinite version of Ramsey's Theorem. We will also show that the infinite version of Ramsey's Theorem implies the finite version, which is a nice result.

Soon after we will generalize further, now coloring infinite sets using infinitely many colors. We'll develop a few definitions to help us find a property so that, when we color the elements of a set of infinite cardinality, we are guaranteed a nice property. This analysis is known as canonical Ramsey theory.

We will also consider graph Ramsey theory, which involves finding the smallest n so that a monochromatic graph G_1 or a monochromatic graph G_2 must result whenever K_n is $2-$colored. As we will see, this is an extension of Ramsey theory for graphs, and we will prove some relevant results.

Finally, we will conclude with an analysis of van der Waerden's Theorem. This Ramsey-type result states that we can always find a large enough natural number $W = W(k, c)$ so that if we $c-$color all of the natural numbers leading up to it, we are assured an arithmetic progression of length k such that all of the natural numbers are the same color.

Contents

1 Introduction

We begin our discussion with the following scenario. Suppose someone is throwing a party, and people gradually arrive, one by one. We ask the question, "What is the minimum number of people that must be present in order to be *guaranteed* either **(A)** a group of 3 people who all know each other or **(B)** a group of 3 people who all don't know each other?" The 3^{rd} Ramsey number, $R(3)$, is the solution to this problem, and one can easily generalize to $R(n)$. We can show that if we have $1, 2, 3, 4$, or even 5 people at the party, it is possible that neither A nor B occurs. However, it turns out that when a 6^{th} person arrives, we are guaranteed either (A) or (B), and so we say that $R(3) = 6$. We simply state this result for now, as we will prove it later.

Theorem 1.1. *R(3) = 6.*

In order to prove Ramsey results, we will often realize the party guests and their relationships as a 2-colored graph. Suppose there are n people at the party. Representing each person as a vertex, we may draw K_n and 2-color the edges red and blue as follows: edge v_iv_j is colored blue if and only if person i knows person j. Thus $R(m)$ describes the smallest value of n such that, given any 2-coloring of $G = K_n$, we either have a blue $K_m \subset G$ or a red $K_m \subset G$.

This is an example of Ramsey theory for graphs, one of the various manifestations of Ramsey theory that we will consider. In each of the upcoming sections, we will observe something similar at work. We will always start out with a large enough structure so that, when we break it up into various classes, we have some copy of our original structure in one of the classes. In Ramsey theory, complete disorder is impossible.

2 Finite Ramsey Theory

Assume we are dividing a finite number of elements (objects, edges, or even $k-$sets) into certain classes. Finite Ramsey theory asserts that, no matter how many classes we are separating into, we can find a large enough number of elements so that, once we divide into various classes, we get a similar structure among one of our classes. We will observe this at work for each of the aforementioned instances.

2.1 1-dimensional Ramsey Theory [5, p. 321-322]

Suppose we are coloring n objects with r different colors. What is the smallest n such that we are guaranteed s objects all of the same color? We will denote this number $n := R_r(1; s)$. In the case of 1-dimensional Ramsey theory, we can easily calculate $R_r(1; s)$. As we will see, this is nothing more than an application of the Dirichlet (or pigeonhole) principle.

Theorem 2.1. $R_r(1; s) = r(s-1) + 1$.

Proof. First, we note that if we have $r(s-1)$ objects, it may not be the case that we have s objects all of the same color. This is because we can break up the $r(s-1)$ objects into r groups of $s-1$ objects. Assigning each group a unique color, we have arrived at a coloring of $r(s-1)$ objects such that we cannot find s objects with the same color. Thus

$R_r(1;s) > r(s-1)$ so that $R_r(1;s) \geq r(s-1)+1$.

We wish to show that $R_r(1;s) \leq r(s-1)+1$. It suffices to show that, given an r-coloring of $r(s-1)+1$ objects, we are guaranteed s objects of the same color. Indeed, if this were not case, then we have at most $s-1$ objects of each color, resulting in a maximum of $r(s-1)$ objects, a contradiction. Therefore $R_r(1;s) \leq r(s-1)+1$ so that $R_r(1;s) = r(s-1)+1$. □

Hence we have shown that we can find an $R_r(1;s)$ so that if we $r-$color $R_r(1;s)$ objects, "order" is preserved among the colors; that is, we can find as many as s objects of one color. We also note that our analysis produced an exact evaluation of $R_r(1;s)$ given particular values of r and s. Because this is not often possible in higher dimensional Ramsey theory, we will usually resort to finding bounds on our Ramsey numbers, as we will see shortly.

2.2 Ramsey Theory for Graphs (2-dimensional Ramsey Theory)

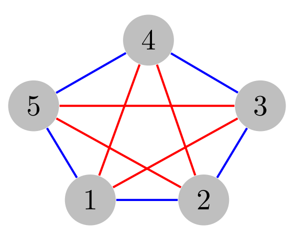
As previously noted, we will view $2-$dimensional Ramsey theory via colorings of K_n. We often seek to find a large enough value of n so that, regardless of what coloring we assign, we are guaranteed a complete monochromatic subgraph of order m. For convenience, we will separate our analysis into two cases: graphs colored using two colors (red and blue), and graphs coloring using multiple (more than two) colors.

2.2.1 Graphs with Two Colors

We first consider the 2-dimensional, two color case of Ramsey theory. Let $R_2(2;m,m) := R(m)$ be the smallest value of n so that any $2-$coloring of $G = K_n$ results in either a blue $K_m \subset G$ or a red $K_m \subset G$. We can use some basic graph theory to deduce that $R(3) = 6$. [4, p. 118-119]

Theorem 1.1. *R(3) = 6.*

Proof. We first note that $R(3) \geq 6$ by producing a 2-coloring of K_5 that contains neither a red K_3 or a blue K_3:



Thus we see that $R(3) > 5$ so that $R(3) \geq 6$. It suffices to show that $R(3) \leq 6$.

Suppose we have a two-coloring of $G = K_6$. Choose some $v_1 \in G$ and note that, by pigeonhole principle, either at least three of the 5 edges incident to v_1 are blue, or at least three of the 5 edges incident to v_1 are red. Without loss of generality, assume three edges $e_1 = v_1v_2, e_2 = v_1v_3, e_3 = v_1v_4$ are blue. We may assume that v_2v_3, v_2v_4, v_3v_4 are all red (if v_iv_j is blue for $2 \leq i < j \leq 4$, then the cycle $v_i - v_j - v_1 - v_i$ produces a blue K_3, and we are done). Since the edges v_2v_3, v_2v_4, v_3v_4 are all red, G contains a red K_3. Since the two-coloring of G was arbitrary, we conclude that any 2-colored, complete graph on 6 vertices contains either a red K_3 or a blue K_3. Thus $R(3) \leq 6$, so we conclude that $R(3) = 6$. □

Note that, in using the pigeonhole principle, this proof applies Theorem 2.1 with $r = 2, s = 3$. This foreshadows the inductive nature behind the proofs of more general finite Ramsey theorems.

What if we were interested in the minimum number people that must be present in order to be guaranteed either a group of s people who all know each other or a group of t people who all don't know each other? We will denote this Ramsey number as $R_2(2; s, t)$), or simply $R(s, t)$. Furthermore, when $s = t$, we will merely write $R_2(2; s, t) = R(s, t) = R(s, s) = R(s)$, which complies with our previous notation. Similar to above, we will instead find the smallest value of n such that, given any 2-coloring of $G = K_n$, we have either a blue $K_s \subset G$ or a red $K_t \subset G$. Unfortunately, there is no known formula for $R(s, t)$; in fact, for only some very small values of s and t is $R(s, t)$ known exactly (see [6, p. 4]). We do, however, know that it always exists and is finite. This fundamental result is known as the finite form of Ramsey's Theorem for two colors, 2-sets [5, p. 322-323]:

Theorem 2.2 (Finite Ramsey's Theorem for two colors, 2-sets)**.** *Let s, t be positive integers such that $s, t \geq 2$. Then $R(s, t)$ exists and is finite.*

Proof. First, for our base case, we note that for all $n \geq 2$, we have $R(n, 1) = R(1, n) = 1$. This is clear by definition of $R(s, t)$. For our induction hypotheses, assume that $R(s-1, t)$ and $R(s, t-1)$ exist and are finite. We will actually show that, for $s, t > 2$, we have

$$R(s, t) \leq R(s-1, t) + R(s, t-1),$$

which will complete the proof.

Consider the complete graph on $R(s-1, t)+R(s, t-1)$ vertices. It suffices to show that, for any red-blue coloring, there exists a complete blue subgraph on s vertices or a complete red subgraph on t vertices.

Let v be a vertex of our graph, and let A, B be such that for each remaining vertex w (distinct from v), $w \in A$ if and only if vw is blue and $w \in B$ if and only if vw is red. Since our graph has $R(s-1, t)+R(s, t-1) = |A|+|B|+1$ vertices, note that either $|A| \geq R(s-1, t)$ or $|B| \geq R(s, t-1)$. Otherwise, $|A| \leq R(s-1, t) - 1, |B| \leq R(s, t-1) - 1 \implies |A| + |B| \leq R(s-1, t) + R(s, t-1) - 2$, a contradiction. Assume $|A| \geq R(s-1, t)$. If A has a red K_t, then so does our original graph and we are done. If A has a blue K_{s-1}, then by construction of A, $A \cup v$ forms a blue K_s. The case when $|B| \geq R(s, t-1)$ is analogous. Thus we have shown that any red-blue coloring of our graph contains either a blue K_s or a red K_t as a subgraph, thus completing the proof. $\square$

As we've said, there is no formula that describes $R(s, t)$ for arbitrary s, t. In fact, $R(s)$ is only known for $s = 1, 2, 3, 4$. We can, however, develop bounds for $R(s, t)$. One bound follows immediately as a corollary of Theorem 2.2. [1, p. 183]

Corollary 2.3. *For $s, t \geq 2$, $R(s,\ t) \leq \binom{s+t-2}{s-1}$.*

Proof. Let $s = 2$. Then $R(s, t) = R(2, t) = t$, whereas $\binom{s+t-2}{s-1} = \binom{t}{1} = t$. Thus the claim holds for $s = 2$. Similarly, the claim holds for $t = 2$. Now assume that $s, t > 2$ and that our claim holds for all s', t' such that $2 \leq s'+t' < s+t$. It suffices to show that $R(s, t) \leq \binom{s+t-2}{s-1}$.

By Theorem 2.2, we have $R(s,t) \leq R(s-1,t) + R(s,t-1)$. Thus

$$\begin{aligned}
R(s,t) &\leq \binom{(s-1)+t-2}{(s-1)-1} + \binom{s+(t-1)-2}{s-1} \\
&= \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} \\
&= \binom{s+t-2}{s-1}
\end{aligned}$$

as desired. $\square$

This bound can be both accurate and inaccurate. For example, through the proof of Corollary 2.3, we have observed equality in the bound $R(s,t) \leq \binom{s+t-2}{s-1}$ when s or t is 2. Corollary 2.3 also tells us that $R(3,3) = R(3) \leq \binom{3+3-2}{3-1} = \binom{4}{2} = 6$, a precise bound. However, it is well known that $R(4) = 18$, yet $\binom{4+4-2}{4-1} = \binom{6}{3} = 20$. This inaccuracy is due to the rapid growth of our bound, since:

$$s = t \implies R(s,s) \leq \binom{2s-2}{s-1}.$$

We can develop another bound for $R(s)$ through the use of Stirling's Formula:

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + O(\frac{1}{n})).$$

Corollary 2.4. *For $s \geq 2$, $R(s) \leq \frac{4^{s-1}}{\sqrt{s-1}}$*

Proof. This follows from Stirling's Formula, since for $\epsilon > 0$ and sufficiently large s, we have

$$(2s-2)! \leq \sqrt{2\pi(2s-2)}(\frac{2s-2}{e})^{2s-2}(1+\epsilon)$$

and

$$(s-1)! \geq \sqrt{2\pi(s-1)}(\frac{s-1}{e})^{s-1}(1-\epsilon).$$

Hence we observe that

$$\begin{aligned}
R(s) &\leq \binom{2s-2}{s-1} \\
&\leq \frac{\sqrt{2\pi(2s-2)}(\frac{2s-2}{e})^{2s-2}}{2\pi(s-1)(\frac{s-1}{e})^{2s-2}} \cdot \frac{1+\epsilon}{(1-\epsilon)^2} \\
&= \frac{2^{2s-2}\sqrt{\pi(s-1)}}{\pi(s-1)} \cdot \frac{1+\epsilon}{(1-\epsilon)^2} \\
&\leq \frac{4^{s-1}}{\sqrt{s-1}} \cdot \frac{1+\epsilon}{(1-\epsilon)^2} \cdot \frac{1}{\sqrt{\pi}}
\end{aligned}$$

and, taking $\epsilon > 0$ sufficiently small, we see that

$$R(s) \leq \frac{4^{s-1}}{\sqrt{s-1}},$$

as desired. $\square$

This is one of the tightest known upper bounds on $R(n)$. Although Corollary 2.4 has recently been improved, it is still unknown whether or not $\limsup \sqrt[s]{R(s)} < 4$.

We may also produce lower bounds on $R(s)$. For example, a trivial lower bound is $R(s) > (s-1)^2$, since for any graph on $(s-1)^2$ vertices, we can color $s-1$ independent copies of K_{s-1} red and the remaining edges blue. Clearly there is no red K_s or blue K_s. We can also provide an exponential bound, $R(s) > \lfloor 2^{s/2} \rfloor$ for all $s \geq 3$. The proof of this lower bound, produced by Paul Erdős, uses a probabilistic argument. Whereas the proof that $R(s) > (s-1)^2$ was constructive, this proof will be existential; that is, the proof doesn't actually give a way of finding a 2-coloring of $K_{\lfloor 2^{s/2} \rfloor}$ without a monochromatic K_s. [5, p. 229-230]

Theorem 2.5. *If $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$, then $R(s) > n$. In particular, $R(s) > \lfloor 2^{s/2} \rfloor$ for all $s \geq 3$.*

Proof. Let n be such that $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$. We assign a probabilistic interpretation to 2-colorings of K_n. Consider a random red-blue coloring of K_n, where each edge is colored independently, and the probability that an edge is colored red is the same as the probability that an edge is colored blue. Let S be a set of s vertices and let A_S be the event that the induced subgraph is monochromatic. Then

$$P(A_S) = 2(\tfrac{1}{2})^{\binom{s}{2}}.$$

Note that the probability that at least one $s-$clique is monochromatic is

$$\begin{aligned} P(\bigcup_S A_S) &\leq \sum_S P(A_S) \\ &= \sum_S 2(\frac{1}{2})^{\binom{s}{2}} \\ &= \binom{n}{s} 2^{1-\binom{s}{2}} \\ &< 1. \end{aligned}$$

Thus the probability of at least one $s-$clique being monochromatic is less than 1. Hence the probability that there does not exist a monochromatic $s-$clique is greater than 0. This shows that there exists a coloring on n vertices such that there is no monochromatic K_s, and so we deduce that $R(s) > n$. In particular, let $s \geq 3$, $n = \lfloor 2^{s/2} \rfloor$. Then

$$\begin{aligned} \binom{n}{s} 2^{1-\binom{s}{2}} &< \frac{n^s}{s!} 2^{1+s/2-s^2/2} \\ &= \frac{n^s}{s!} \frac{2^{1+s/2}}{2^{s^2/2}} \\ &\leq \frac{2^{s^2/2} \cdot 2^{1+s/2}}{s! \cdot 2^{s^2/2}} \\ &< 1 \end{aligned}$$

so that $R(s) > n = \lfloor 2^{s/2} \rfloor$. $\square$

2.2.2 Graphs with Multiple (more than two) Colors [1, p. 183-184]

We can immediately generalize the previous section to graphs with r colors, where $r > 2$. The Ramsey number $R_2(2; s_1, \cdots, s_r) = R_r(s_1, \cdots, s_r)$ will represent the smallest value of n such that, if we r-color $G = K_n$, we are guaranteed that G contains a K_{s_i} of color i for some i, $1 \leq i \leq r$. We arrive at a result similar to Theorem 2.2.

Theorem 2.6 (Finite Ramsey's Theorem for multiple colors, 2-sets)**.** *Let $s_1, \cdots, s_r$ be positive integers such that $s_1, \cdots, s_r \geq 2$. Then $R_r(s_1, \cdots, s_r)$ exists and is finite.*

Proof. Note that we have already observed this in the case of $r = 2$. Let $k \geq 2$. We will assume the result for $r = k-1$ and prove it for $r = k$. If we can show that $R_k(s_1, \cdots, s_k) \leq R_{k-1}(R(s_1, s_2), s_3, \cdots, s_k)$, then the result follows.

Let $G = K_n$ be a complete graph on $n = R_{k-1}(R(s_1, s_2), s_3, \cdots, s_k)$ vertices, and color it with k colors, say colors $1, 2, 3, \cdots, k$. We must show that G contains a K_{s_i} of color i for some i, $1 \leq i \leq k$. Create a new color class "0" by merging colors 1 and 2 (in a sense we go colorblind and view colors 1 and 2 as the same color). Under this new coloring, we have colored G with $k-1$ colors, so we know that there either exists a K_{s_i} of color i for $3 \leq i \leq k-1$, or there exists a K_m of color 0, where $m = R(s_1, s_2)$. If there exists a K_{s_i} of color i, $3 \leq i \leq k$, then we are done. Therefore assume that we have a K_m of color 0. Since $m = R(s_1, s_2)$, we know that our K_m contains either a K_{s_1} of color 1 or a K_{s_2} of color 2. Thus G contains either K_{s_1} of color 1 or a K_{s_2} of color 2, which completes the proof. $\square$

2.3 Ramsey Theory: Coloring k-tuples [1, p. 184-185]

We may generalize Finite Ramsey's Theorem even further, now to coloring k-subsets. Let $r, k, s_1, \cdots, s_r \in \mathbb{Z}_{\geq 1}$ such that $s_1, \cdots, s_r \geq k$. We define the Ramsey number $R_r(k; s_1, \cdots, s_r)$ to be the smallest number n so that, if the k-subsets of an n-set are colored with the r colors $1, \cdots, r$ then there is an s_i-set (i.e. a set of size s_i) all of whose k-subsets have color i for some $i \in [1, \cdots, r]$. If $s_1 = \cdots = s_r = s$, then we will denote this Ramsey number by $R_r(k; s)$. Furthermore, when $k = 2$, we will simply shorten $R_r(2; s_1, \cdots, s_r)$ as $R_r(s_1, \cdots, s_r)$. Also, when $r = 2$, we will write $R_2(k; s_1, s_2)$ as $R(k; s_1, s_2)$. In the instance of two colors, we have that $R(k; s, t)$ is finite.

Theorem 2.7 (Finite Ramsey's Theorem for Two Colors, k-sets)**.** *Let $1 < k < \min\{s, t\}$. Then $R(k; s, t)$ is finite, and $R(k; s, t) \leq R(k-1; R(k; s-1, t), R(k; s, t-1)) + 1$.*

Proof. The proof will be by induction. Note that Theorem 2.2 verifies our base case. We will assume that $R(k-1; u, v)$ is finite for all u, v and that $R(k; s-1, t)$ and $R(s, t-1)$ are also finite. If we can prove the inequality under these assumptions, the result follows.

Let X denote a set with $R(k-1; R(k; s-1, t), R(k; s, t-1)) + 1$ elements, and let c be a $2-$coloring (red, blue) of the $k-$subsets of X. It suffices to show that there is either a s-subset all of whose k-subsets are red (a red $s-$set) or a t-subset all of whose k-subsets are blue (a blue $t-$set).

Let $x \in X$ and define a red-blue coloring c' on the $(k-1)-$subsets of $Y = X \setminus \{x\}$ as follows: $y \in Y^{(k-1)}$ is colored red under c' iff $y \cup \{x\}$ is colored red under c. Since c' is a 2-coloring on the $(k-1)-$subsets of a set with $R(k-1; R(k; s-1, t), R(k; s, t-1))$

elements, there is either a red $R(k; s-1, t)-$set under c' or a blue $R(k; s, t-1))-$set under c'. Without loss of generality, we will assume that Y contains a red $R(k; s-1, t)-$set under c', say Z.

We may assume that Z contains no blue $t-$set, since we are done if there is. Because there is no blue $t-$set, there must be a red $(s-1)-$set, say Z', since Z is a red $R(k; s-1, t)-$set. But by considering $Z' \cup \{x\}$, we arrive at a red $s-$set. This completes the proof. $\square$

In fact, similar to the proof of Theorem 2.7, we can prove our strongest finite version of Ramsey's Theorem:

Theorem 2.8 (Finite Ramsey's Theorem for Multiple Colors, k-sets)**.** *Let $s_1, \cdots, s_r, k$ be positive integers such that $s_1, \cdots, s_r \geq k \geq 2$. Then $R_r(k; s_1, \cdots, s_r)$ exists and is finite.*

Proof. We will in fact prove

$$R_r(k; s_1, \cdots, s_r) \leq R_r(k-1; R_r(k; s_1-1, s_2, \cdots, s_r), \cdots, R_r(s_1, \cdots, s_{r-1}, s_r-1)) + 1.$$

Showing this will complete the proof.

The proof will be by induction. Note that Theorem 2.6 verifies our base case. We will assume that $R_r(k-1; u_1, \cdots, u_r)$ is finite for all $u_1, \cdots, u_r$ and that $R_r(k; s_1-1, s_2, \cdots, s_r)$, $R_r(k; s_1, s_2-1, \cdots, s_r), \cdots, R_r(k; s_1, s_2, \cdots, s_r-1)$ are also finite. If we can prove the inequality under these assumptions, the result follows.

Let X denote a set with $R_r(k-1; R_r(k; s_1-1, s_2, \cdots, s_r), \cdots, R_r(s_1, \cdots, s_{r-1}, s_r-1))+1$ elements, and let c be a $r-$coloring of the $k-$subsets of X. Furthermore, we will refer to the colors as color $c_1, \cdots, c_r$. It suffices to show that there is a s_i-subset all of whose k-subsets are colored c_i (a c_i s_i-set).

Let $x \in X$ and define a $r-$coloring c' on the $(k-1)-$subsets of $Y = X \setminus \{x\}$ as follows: $y \in Y^{(k-1)}$ is colored c_i under c' iff $y \cup \{x\}$ is colored c_i under c. Because of the size of Y and our induction hypotheses, we may assume without loss of generality that Y contains a c_i $R_r(k; s_1-1, s_2, \cdots, s_r)-$set under c', say Z.

We may assume that Z contains no c_j s_j-set for any $j \geq 2$, since we are done if there is. Because there are no such sets, there must be a c_1 $(s_1-1)-$set, say Z'. But by considering $Z' \cup \{x\}$, we arrive at a c_1 s_1-set. This completes the proof. $\square$

2.4 Schur's Theorem [5, p. 326]

As an interesting application of Ramsey theory, we will now prove a classical result known as Schur's Theorem. Let $r \geq 1$ be an integer. We claim that we can find an $n = S(r)$ such that, given any $r-$coloring of $[n]$, one of the color classes contains two numbers x, y such that $x, y, x+y$ are all the same color. We formally state this result:

Theorem 2.9 (Schur's Theorem)**.** *For every r there exists an $n = S(r)$ such that for every partition of the set $[n]$ into r classes, one of the classes contains two numbers x, y together with their sum $x+y$.*

We observe that this is a Ramsey-style result; our special property that $x, y, x+y$ are monochromatic is preserved, regardless of how many colors are used.

Proof. Let r be given. We claim that $n = R_r(2; 3, 3, \cdots, 3)$ is sufficient. For any $r-$coloring $c : [n] \to [r]$, we define a new $r-$coloring, say $c' : [n]^2 \to [r]$ such that $c'(ij) = c(|i-j|)$ for each edge ij. By choice of n, we know that under our coloring c', there exist three vertices x, y, z such that

$$c'(yx) = c'(zy) = c'(zx).$$

Without loss of generality, assume $x < y < z$. Then

$$c(y-x) = c(z-y) = c(z-x),$$

$$(y-x) + (z-y) = (z-x).$$

Thus we have found two monochromatic numbers whose sum is the same color. $\square$

2.5 An Alternate Proof that $R(3) = 6$ [5, p. 328]

Ramsey's Theorem says that if n is large enough and K_n is 2-colored, then one of the color classes contains a triangle. We would expect that as n gets large, a $2-$coloring of K_n contains more and more monochromatic triangles (as a function of n). Goodman (1959) made this precise, proving the following lemma. [3, p. 778-783]

Lemma 2.10. *Let G be a graph on n vertices and m edges. Let $t(G)$ denote the number of triangles contained in the graph G or in its complement. Then*

$$t(G) \geq \binom{n}{3} + \frac{2m^2}{n} - m(n-1).$$

Proof. First, we claim that

$$t(G) = \binom{n}{3} - \frac{1}{2}\sum_i t_i,$$

where t_i denotes the number of triples of vertices $\{i, j, k\}$ such that the vertex i is adjacent to precisely one of vertices j or k. To see this, let G be a graph, and extend G to K_n by filling in missing edges. Let $T = \{\text{triangles in } K_n\}$, $T' = \{\text{triangles in } G \text{ or } \bar{G}\}$, and $S = \{\text{triples } \{i, j, k\} \text{ such that vertex } i \text{ is adjacent (in } G\text{) to precisely one of } j \text{ or } k\}$. It is clear that $T \supset T' \cup S$. Furthermore, note that $T \subset T' \cup S$, since if a triangle in K_n is not a triangle in G or $\bar{G}$, then we can find such a triple $\{i, j, k\}$. Thus $T = T' \cup S$ and

$$\begin{aligned} |T| &= |T'| + |S| - |T' \cap S| \\ &= t(G) - \frac{1}{2}\sum_i t_i \end{aligned}$$

(here we have a factor of $\frac{1}{2}$ due to overcounting, i.e. if vertices $\{i, j, k\}$ form a triple with our desired property, then exactly one other permutation of these vertices will form such a triple as well). Now, we observe that $|T| = \binom{n}{3}$. Hence we have really shown that

$$\binom{n}{3} = t(G) - \frac{1}{2}\sum_i t_i.$$

We now claim that

$$t_i = d_i(n-1-d_i),$$

where d_i is the degree of vertex i. Indeed, in counting the number of triples $\{i, j, k\}$ such that i is attached to precisely one of j or k, we must count the number of vertices adjacent to i and then the number of vertices that are nonadjacent to i. The former is d_i, and the latter is $n-1-d_i$. Thus by the product rule of combinatorics

$$t_i = d_i(n-1-d_i),$$

as desired.

Recall the Cauchy-Schwartz inequality, which states that for real numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$:

$$(\sum_i x_i y_i)^2 \le (\sum_i x_i^2)(\sum_i y_i^2).$$

Taking $x_i = d_i$ and $y_i = 1$ for all i, we see that $(\sum_i d_i)^2 \le (\sum_i d_i^2)(n)$ so that

$$\sum d_i^2 \ge \frac{1}{n}(\sum d_i)^2.$$

Thus we now have the tools to deduce that

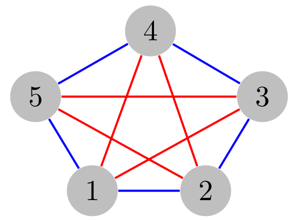
$$\begin{aligned} t(G) &= \binom{n}{3} - \frac{1}{2}\sum_i t_i \\ &= \binom{n}{3} - \frac{1}{2}\sum_i (d_i(n-1-d_i)) \\ &= \binom{n}{3} - \frac{n}{2}\sum_i d_i + \frac{1}{2}\sum_i d_i + \frac{1}{2}\sum_i d_i^2 \\ &\ge \binom{n}{3} - \frac{n}{2}(2m) + \frac{1}{2}(2m) + \frac{1}{2}[\frac{1}{n}(\sum d_i)^2] \\ &= \binom{n}{3} - m(n-1) + \frac{1}{2}[\frac{1}{n}(2m)^2] \\ &= \binom{n}{3} - m(n-1) + \frac{2m^2}{n}. \end{aligned}$$

Hence we have shown that $t(G) \ge \binom{n}{3} - m(n-1) + \frac{2m^2}{n}$, as claimed. $\square$

We can note that when $m = \frac{n(n-1)}{4}$, our lower bound is at a minimum, since $\frac{d}{dm}(\binom{n}{3} - m(n-1) + \frac{2m^2}{n}) = 0 \implies -(n-1) + \frac{4m}{n} = 0 \implies m = \frac{n(n-1)}{4}$ and $\frac{4}{n} > 0$. This bound can help us reprove that the 3^{rd} Ramsey number is, in fact, 6.

Theorem 1.1. *R(3) = 6.*

Proof. Like before, we note that $R(3) \ge 6$ via the graph



It suffices to show that $R(3) \leq 6$.

Let G be a graph on 6 vertices, and 2-color it. We will view $t(G)$ as the number of red and blue triangles in G. Then Lemma 7.1 tells us that

$$t(G) \geq \binom{n}{3} - m(n-1) + \frac{2m^2}{n},$$

where m is the number of red edges. In particular, this bound is worst when $m = \frac{6(5)}{4}$, so when $m = 7.5$. Hence

$$t(G) \geq \binom{6}{3} - 7.5(6-1) + \frac{2(7.5)^2}{6} = 1.25.$$

Thus since $t(G) \geq 1$, we know that there is either a blue triangle or a red triangle when we $2-$color G. Thus $R(3) \leq 6$, so we conclude that $R(3) = 6$. $\square$

Note that this proof actually shows that any $2-$coloring of K_6 must have at least 2 monochromatic triangles. We can also use Lemma 2.10 to show that the coloring of K_5 presented in Theorem 1.1 is the unique coloring (up to isomorphism) of K_5 that contains no monochromatic triangle.

Theorem 2.11. *There is only one coloring (up to isomorphism) of K_5 that contains neither a blue triangle or a red triangle.*

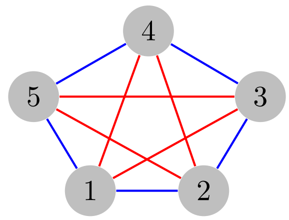
Proof. First, we recall the coloring of K_5 exhibited in the proof of Theorem 1.1. Hence we know that there is at least one such coloring. Now Lemma 2.10 tells us that

$$t(G) \geq \binom{n}{3} - m(n-1) + \frac{2m^2}{n},$$

where m is the number of red edges, and we know that $\binom{n}{3} - m(n-1) + \frac{2m^2}{n}$ achieves its absolute minimum when $m = \frac{5(4)}{4} = 5$. Thus

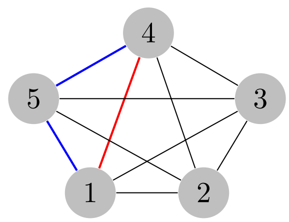
$$t(G) \geq \binom{5}{3} - 5(5-1) + \frac{2(5)^2}{5} = 0.$$

Furthermore, we know that if we increase or decrease m, our bound becomes better. Hence for any $m \neq 5$, $t(G) > 0$ so that $t(G) \geq 1$. Thus if we want to color K_5 so that our coloring has no blue triangle or red triangle, we must use exactly 5 red edges and 5 blue edges. Note that

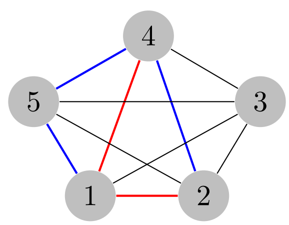


is a 2-coloring with no monochromatic triangle. We claim this is the only one (up to isomorphism).

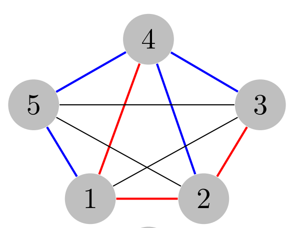
First, we will show that if we assign 5 red edges and 5 blue edges to K_5 such that no red or blue triangle results, we must have a 4-cycle or 5-cycle. Without loss of generality, assume we have the graph

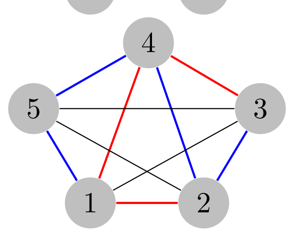


since we know there must be at least two adjacent edges along the outside pentagon of the same color, say v_1v_5 and v_5v_4, and hence v_1v_4 must be red. Consider v_1v_2 and v_2v_4. Exactly one must be red and one must be blue (since both red creates a red triangle, and both blue creates a blue 4-cycle). By symmetry, we may assume the graph

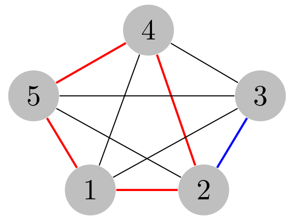


Similarly, we may assume exactly one of the two edges v_2v_3 and v_3v_4 is red, while the other is blue. In either of the two instances, though, we arrive at a blue cycle (in the first instance, via $v_3 - v_1 - v_5 - v_4 - v_3$) or a red cycle (in the second instance, via $v_3 - v_1 - v_5 - v_4 - v_2 - v_3$):

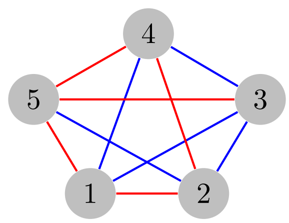




If our graph contains a blue or red 5-cycle, we are done. Therefore, assume our graph has a red 4-cycle, say $v_1 - v_5 - v_4 - v_2 - v_1$. Then either v_2v_3 or v_3v_4 must be blue, say v_2v_3. We have the graph:



Since v_1v_4 and v_5v_2 must be blue, then v_5v_3 must be red. This results in 5 red edges, so the remaining edges must be blue:



But this results in a blue triangle. Thus we cannot have a $4-$cycle, so we must have a $5-$cycle. This forces our graph to be the one from Theorem 1.1, as desired. $\square$

3 Infinite Ramsey Theory [4, p. 282-288]

There is a nice generalization of Ramsey Theory to infinite sets. While one may believe that we can always find a large enough structure so that a finite substructure occurs, it isn't as clear when we move to coloring infinite sets. We ask the question "Given any $c-$coloring of the $n-$tuples of an infinite set, must it be true that we can necessarily find an infinite subset, all of whose $n-$sets are the same color?" As we will soon see, the answer is yes, and the statement of this result is referred to as "Infinite Ramsey's Theorem." We will then go further, showing that Infinite Ramsey's Theorem implies Finite Ramsey's Theorem.

3.1 The Infinite Pigeonhole Principle

We will first extend the pigeonhole principle to infinite dimensions. Simply put, our "Infinite Pigeonhole Principle" states that if we distribute an infinite number of pigeons into a finite number of holes, then at least one hole must be inhabited by an infinite number of pigeons. Let's formalize and prove this result.

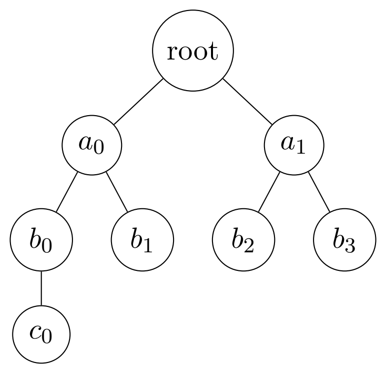
Theorem 3.1 (Infinite Pigeonhole Principle). *Let P be an infinite set, H be a finite set, and $f : P \to H$ be a function. Then we can find an $h \in H$ such that $\{p \in P : f(p) = h\}$ is infinite.*

Proof. Let P be an infinite set, $H = \{h_0, \cdots, h_n\}$ be a finite set, and $f : H \to P$ be such a function. Suppose there is no $h \in H$ such that $\{p \in P : f(p) = h\}$ is infinite. Then for every $h_i \in H$, the set $P_i = \{p \in P : f(p) = h_i\}$ is finite. Let $|P_i| = p_i$. Since $P = P_0 \cup P_1 \cup \cdots \cup P_n$, we know that $|P| = |P_0| + |P_1| + \cdots + |P_n| = p_0 + p_1 + \cdots + p_n = \sum_{i=0}^{n} p_i < \infty$, a contradiction since P is infinite. Hence we can find some $h \in H$ so that $\{p \in P : f(p) = h\}$ is infinite. $\square$

3.2 König's Lemma

We now briefly digress to discuss König's Lemma. Let T be a tree with a distinguished vertex r that we call the root. We are typically concerned with paths stemming from our root. Furthermore, we will often consider the "levels" of our tree. We'll say that vertex v is in level i if and only if vertex v is i adjacencies apart from our root. To make this more visual, we provide the reader with an example.

Example 3.2 Consider the following tree.



This tree has four levels, say L_0, L_1, L_2, L_3. Level 0, or L_0, contains only the root. There are two vertices in level 1: a_0 and a_1. Similarly, $L_2 = \{b_0, b_1, b_2, b_3\}$ and $L_3 = \{c_0\}$.

We will now consider infinite trees, or trees with infinitely many vertices. As it turns out, we can always find an infinite path in an infinite tree whose levels are finite. This result is known as König's Lemma, and it will ultimately prove useful in proving that Infinite Ramsey's Theorem implies Finite Ramsey's Theorem.

Theorem 3.2 (König's Lemma)**.** *Let T be an infinite tree such that each level of T is finite. Then T contains an infinite path.*

Proof. Let T be an infinite tree such that each level of T is finite. Also, let $L_0 = \{r\}, L_1, L_2, \cdots$ be the levels of our tree. We will begin our infinite path with our root r. Note that there are infinitely many vertices below level L_0, and each vertex below L_0 is either in L_1 or below exactly one vertex in L_1. Define $f : \{\text{vertices below } L_0\} \to \{\text{the vertices in } L_1\}$ such that each vertex below L_0 is mapped to the vertex that it is equal to or below. By the infinite pigeonhole principle, we know that there is some vertex $v_1 \in L_1$ above r such that there are infinitely many vertices below it. We can repeat this process to find such a $v_2 \in L_2$, such a $v_3 \in L_3$, and so on. Then $r - v_1 - v_2 - \cdots$ creates an infinite path in T. $\square$

3.3 Application: Alternate Proof That $R(n)$ Exists and is Finite

As an interesting application of what we have shown in this section thus far, we can provide an alternative proof that $R(n)$ exists and is finite. We must first, however, prove a result about $2-$colorings of infinite graphs.

Lemma 3.3. *Let G be a complete infinite graph with vertices $\{v_0, v_1, \cdots\}$. For any $2-$coloring of the edges of G, there exists an infinite complete monochromatic subgraph of G.*

Proof. Let G be a complete infinite graph with vertices $\{v_0, v_1, \cdots\}$, and $2-$color G. Choose some vertex $w_0 = v_0 \in V$. Let V_0 be an infinite set of vertices so that all of the edges of $\{v_0 v : v \in V_0\}$ are the same color (since we have only two colors, such an infinite set must exist). Let w_1 be the smallest-numbered vertex in V_0, and let V_1 be an infinite subset of V_0 so that all of the edges of $\{w_1 v : v \in V_1\}$ are the same color. We can continue this process inductively to obtain the sequence $\langle w_0, w_1, w_2, \cdots \rangle = \langle w_i : i \in \mathbb{N} \rangle$ of vertices of V.

Consider our sequence. Note that if $i < j < k$, then $w_j, w_k \in V_i$ so that both $w_i w_j$ and $w_i w_k$ are the same color. Also by construction, for a given w_i, it must be the case that $w_i w_j$ is the same color for all $j > i$. If this color is red, we will call w_i red-based, and if this color is blue, we will call w_i blue-based. Since each vertex of $\langle w_i : i \in \mathbb{N} \rangle$ must be red-based or blue-based, we deduce that there must be an infinite subsequence of vertices all of the same color base by the infinite pigeonhole principle. Without loss of generality, let $\langle w_{i_0}, w_{i_1}, \cdots \rangle$ be an infinite subsequence where each vertex is blue-based. Then the complete subgraph with vertices $\{w_{i_0}, w_{i_1}, \cdots\}$ is monochromatic (each edge is colored blue), as desired. $\square$

With the aid of König's Lemma and Lemma 3.3, we can now provide an alternate proof that $R(n)$ exists and is finite.

Theorem 3.4. *For any $n \in \mathbb{N}$, $R(n)$ exists and is finite.*

Proof. Suppose not. Then we can find some $n \in \mathbb{N}$ such that for every m, there exists a $2-$coloring of K_m that does not contain a blue or red K_n. Let G be a complete graph with vertices $V = \{v_i : i \in \mathbb{N}\}$ and edges $E = \{e_i : i \in \mathbb{N}\}$. We construct a tree, say T, of edge colorings of G as follows: root$-c_0 - c_1 - c_2 - \cdots - c_k$ is a path in T if and only if whenever e_i is colored using color c_i for all $i \leq k$, the subgraph of G containing the edges $e_0, e_1, \cdots, e_k$ contains no monochromatic K_n. Note that level k contains at most 2^k vertices, and also note that our tree is infinite, since for every m, there exists a $2-$coloring of K_m that does not contain a blue or red K_n. Thus by König's Lemma, we can find an infinite path in T. This infinite path represents a $2-$coloring of G so that no monochromatic K_n occurs. Since G does not even contain a monochromatic K_n as a subgraph, clearly there is no infinite complete monochromatic subgraph of G. This is a contradiction, since Lemma 3.3 assures that any $2-$coloring of the edges of G must contain an infinite complete monochromatic subgraph. Hence we conclude that $R(n)$ exists and is finite. $\square$

3.4 Infinite Ramsey's Theorem

Before we introduce Infinite Ramsey's Theorem, we will adopt new notation in order to make our results cleaner. For ordinals κ, λ and natural numbers n, c, we write $\kappa \to (\lambda)^n_c$ if for any assignment of c colors to the unordered $n-$tuples of $[\kappa]$, there exists some color (say red) and a subset $X \subset [\kappa]$ of size λ so that no matter how we select an $n-$tuple from X, it is colored red.

For example, $\kappa \to (\lambda)^2_c$ means that whenever we $c-$color K_κ, we can find some color and a complete subgraph of size λ so that no matter how we select an edge from the subgraph, it is colored with that color. In particular, we have previously shown that $6 \to (3)^2_2$ and $5 \nrightarrow (3)^2_2$.

Infinite Ramsey's Theorem states that if we assign c colors to the $n-$tuples of an infinite set, then we can find an infinite subset so that all of its $n-$tuples are the same color. The following theorem is exactly the theorem proven by F. P. Ramsey in 1928 that has given this subject its name.

Theorem 3.5 (Infinite Ramsey's Theorem)**.** *Let $n \in \mathbb{N}, c \in \mathbb{N}$. Then $\omega \to (\omega)^n_c$.*

Proof. We will first consider the case when there are only 2 colors, and we will proceed via induction on n:

Let $n = 1$. If we assign two colors to the 1-elements of ω, then either color 1 or color 2 must appear infinitely many times by the pigeonhole principle. Thus $\omega \to (\omega)^1_2$.

Now suppose $\omega \to (\omega)^n_2$. 2-color the $(n+1)$-subsets of $V_0 = \{v_i : i \in \mathbb{N}\}$. We may choose some $w_0 = v_0 \in V_0$ and consider $V_0 \setminus w_0$. Define a coloring on the $n-$subsets of $V_0 \setminus w_0$ by coloring each $n-$subset, say α, the same color as $\alpha \cup v_0$. By our induction hypothesis, we can find an infinite set $V_1 \subset V_0 \setminus w_0$ such that each $n-$subset is monochromatic. Similarly, we can find such a $w_1 \in V_1$ and an infinite $V_2 \subset V_1 \setminus w_1$ with similar properties. Inductively, we obtain an infinite subsequence $\langle w_i : i \in \mathbb{N}\rangle$ of vertices in V_0. Note that the color of each $(n+1)-$subset $(w_{i_0}, \cdots, w_{i_n})$, where $i_0 < i_1 < \cdots < i_n$, depends only on w_{i_0} since $w_{i_1}, \cdots, w_{i_n} \in V_{i_0}$ along with w_{i_0}. By the infinite pigeonhole principle, there exist infinitely many w_{i_k} such that the $(n+1)-$subsets beginning with w_{i_k} are monochromatic (red or blue). Without loss of generality, assume that we can find infinitely many w_{i_k}, say $\{w_{i_{k_1}}, w_{i_{k_2}}, \cdots\}$, such that the $(n+1)-$subsets beginning with each member are colored red. Hence we have arrived at an infinite set $\{w_{i_{k_j}} : j \in \mathbb{N}\}$ such that each $(n+1)-$subset is monochromatic. Thus $\omega \to (\omega)^n_2 \implies \omega \to (\omega)^{n+1}_2$.

So we have inductively shown that, for $c = 2$ and $n \in \mathbb{N}$, $\omega \to (\omega)^n_c$. But observe that the proof is analogous for $c > 2$ (we simply reproduce the proof with more colors). Hence we have actually shown that for $c \in \mathbb{N}$ and $n \in \mathbb{N}$, $\omega \to (\omega)^n_c$, as desired. $\square$

Now that we have proved Infinite Ramsey's Theorem, we will show that it implies Finite Ramsey's Theorem.

Theorem 3.6. *Let $k, n, c \in \mathbb{N}$. Then there exists an $m \in \mathbb{N}$ such that $m \to (k)^n_c$.*

Proof. Suppose not. Then there are k, n, c such that for all $m \in \mathbb{N}$ we have $m \nrightarrow (k)^n_c$; that is, we can find an assignment of c colors to the $n-$tuples of $[m]$ such that for each subset $X \subset [m]$ of size k, the $n-$tuples of this $k-$set are colored using at least two colors. Let S be an infinite set, and enumerate the $n-$tuples of S as $T_1, T_2, \cdots$. We define a tree of partial colorings of $l-$tuples of S as follows: the path $root - c_1 - c_2 - \cdots - c_l$ is in our tree if and only if when we color T_i with color c_i, where $1 \leq i \leq l$, the resulting partial coloring has no $k-$set all of whose $n-$sets get the same color. Note that the l^{th} level contains a maximum of c^l vertices, and our tree must be infinite (otherwise, we have found an m such that $m \to (k)^n_c$). Hence by König's Lemma, our tree contains an infinite path. This infinite path gives a $c-$coloring of $[\omega]$ such that there is no subset $X \subset [\omega]$ of size k which guarantees $n-$sets to be monochromatic. Then, clearly, there is no infinite subset of $[\omega]$ which guarantees $n-$sets to be monochromatic. Thus $\omega \nrightarrow (\omega)^n_c$, which contradicts Theorem 3.5. Hence there exists an $m \in \mathbb{N}$ such that $m \to (k)^n_c$. $\square$

4 Canonical Ramsey Theory [1, p. 189-192]

We now turn to colorings of $\mathbb{N}^{(k)}$ using infinitely many colors. We would like to be able to find an infinite set $M \subset \mathbb{N}$ such that when we color $M^{(k)}$, we are guaranteed a nice property.

For the remainder of this section, let $M, N, M_1, N_1, M_2, N_2, \cdots$ be countable infinite sets. We will call colorings $c_1 : N_1^{(k)} \to C_1$ and $c_2 : N_2^{(k)} \to C_2$ equivalent if we can find a bijective map $\phi : N_1 \to N_2$ such that for any $\rho, \rho' \in N_1^{(k)}$ we have $c_1(\rho) = c_1(\rho') \iff c_2(\phi(\rho)) = c_2(\phi(\rho'))$. So if two colorings are equivalent, we can easily change the discussion from coloring c_1 to coloring c_2, and vice versa.

We will call a coloring $c : N^{(k)} \to C$ irreducible if for every infinite subset N_1 of N, the restriction of c to $N_1^{(k)}$ is equivalent to c. To see this in action, we can very quickly find two irreducible colorings of $\mathbb{N}^{(k)}$. One example is a monochromatic coloring of $\mathbb{N}^{(k)}$, or simply a coloring where every $k-$subset receives the same color. Clearly every for every infinite subset N_1 of $\mathbb{N}$, the restriction of this coloring to $N_1^{(k)}$ is equivalent to our original coloring. Another example is an all-distinct coloring of $\mathbb{N}^{(k)}$, or a coloring where every $k-$subset receives a different color. This coloring is also irreducible, since for every infinite subset N_1 of $\mathbb{N}$, the restriction of this coloring to $N_1^{(k)}$ is equivalent to our original coloring.

We observe another irreducible coloring of $\mathbb{N}^{(k)}$. For $N \subset \mathbb{N}$, $\alpha = \{a_1, \cdots, a_k\}$ a $k-$subset of N with $a_1 < \cdots < a_k$, and $S \subset [k], |S| = s$, let $\alpha_S = \{a_i : i \in S\}$. Define the coloring $c_S : N^{(k)} \to N^{(s)}$ so that $c_S(\alpha) = \alpha_S$. This is often called the $S-$canonical coloring. This irreducible coloring is defined so that two $k-$sets receive the same color if and only if their i^{th} elements are the same for $i \in S$ and are different for $i \notin S$. We quickly observe that the $\emptyset-$canonical coloring $c_\emptyset$ is simply a monochromatic coloring and that the $[k]-$canonical coloring $c_{[k]}$ is simply an all-distinct coloring.

Let $c : \mathbb{N}^{(k)} \to C$ and $T, U \in \mathbb{N}^{(t)}$ for some $t \geq k$. Also, define $\phi : T \to U$ be the unique order-preserving map from T onto U. We will say that the sets T, U have the same pattern if for $\rho, \rho' \in T^{(k)}$ we have $c(\rho) = c(\rho')$ if and only if $c(\phi(\rho)) = c(\phi(\rho'))$. From this definition it is clear that the number of patterns of $t-$sets is finite.

Theorem 4.1. *Let $c : \mathbb{N}^{(2)} \to \mathbb{N}$ be a coloring of $2-$sets. Then there is an infinite subset M of $\mathbb{N}$ such that the restriction of c to $M^{(2)}$ is canonical.*

Proof. It follows from above that the number of patterns of $4-$sets is finite. Applying Infinite Ramsey's Theorem, we know that there is an infinite set $M \subset \mathbb{N}$ such that all $4-$sets of M have the same pattern π. We claim that c restricted to $M^{(2)}$ is canonical.

Assume $M = \{m_1, m_2, \cdots\}$ where $m_1 < m_2 < \cdots$. Note that the colors of $m_i m_j$ and $m_k m_l$ is determined by the relative position of i, j and k, l since all $4-$sets have the same pattern.

From here on out, for ease of notation, we will write c for the restriction of c to $M^{(2)}$. Assume $c \neq c_{\{1,2\}}$; that is, $M^{(2)}$ has two edges of the same color. Hence for $i < j < k < l$ we have one of the following six cases:

$$c(m_i m_j) = c(m_k m_l)$$
$$c(m_i m_k) = c(m_j m_l)$$
$$c(m_i m_l) = c(m_j m_k)$$
$$c(m_i m_j) = c(m_i m_k)$$
$$c(m_i m_k) = c(m_j m_k)$$
$$c(m_i m_j) = c(m_j m_k)$$

The result follows immediately due to the relative position for the edges. For example, in the first instance, if $c(m_i m_j) = c(m_k m_l)$, then $c(m_1 m_2) = c(m_k m_l)$ for any k, l such that $3 \leq k < l$, and then $c(m_1 m_3) = c(m_k m_l)$ for any k, l such that $4 \leq k < l$, and so on. Thus by the relative positions of the vertices, we see that in the first instance, $c = c_\emptyset$. Through similar logic, we see that in the second instance $c = c_\emptyset$, in the third instance $c = c_\emptyset$, in the fourth instance $c = c_{\{1\}}$, in the fifth instance $c = c_{\{2\}}$, and in the sixth instance, $c = c_\emptyset$. Hence since we must have one of the six cases, we see that the restriction of c to $M^{(2)}$ is canonical. □

There is an analogous statement for the coloring of $k-$sets called the full Erdős-Rado canonical theorem. The proof is similar to that of Theorem 4.1.

Theorem 4.2. *Let k be a positive integer, $c : \mathbb{N}^{(k)} \to \mathbb{N}$ a coloring of $k-$sets. Then we can find an infinite subset M of $\mathbb{N}$ such that the restriction of c to $M^{(k)}$ is canonical.*

Proof. We will induct on k. There is nothing to show for $k = 1$, and we have already proved our result for $k = 2$ via Theorem 4.1, so we will now suppose that $k \geq 3$ and that the result holds for smaller values of k. For $c : \mathbb{N}^{(k)} \to \mathbb{N}$, color each $T \in \mathbb{N}^{(2k)}$ with the pattern of the restriction of c to $T^{(k)}$. Since the number of patterns of $2k-$sets is finite, we apply Infinite Ramsey's Theorem to see that there is an infinite set $N \subset \mathbb{N}$ such that all $2k-$sets of N have the same pattern π. During the remainder of this proof, we will let $N = \mathbb{N}$ for ease of notation (we may assume so, via relabeling).

Assume $c \neq c_{[k]}$. Then there exist two $k-$subsets of N with the same color, say $c(\rho) = c(\sigma)$ for $\rho, \sigma \in N^{(k)}$ for $\rho \neq \sigma$. Write $\rho = \{a_1, \cdots, a_k\}$ and $\sigma = \{b_1, \cdots, b_k\}$ for $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$. Since $\rho \neq \sigma$, choose an element $b_i \in \sigma \setminus \rho$. Note that by relative positioning, the sets $\rho_0 = \{2a_1, \cdots, 2a_k\}$, $\sigma_1 = \{2b_1, \cdots, 2b_k\}$, and $\sigma_2 = \{2b_1, \cdots, 2b_{i-1}, 2b_i - 1, 2b_{i+1}, \ldots, 2b_k\}$ all receive the same color.

Since σ_1, σ_2 receive the same color, we know that any two $k-$subsets of N that vary only in the i^{th} place receive the same color as well. Hence for $\tau \in N^{(k)}$, $c(\tau)$ depends solely on $\tau_{[k]-\{i\}}$. We define a coloring $c' : N^{(k-1)} \to \mathbb{N} \cup \infty$ as follows: for a $(k-1)-$subset of N, say v, $c'(v) = c(\tau)$ if $v = \tau_{[k]-\{i\}}$ for some $\tau \in N^{(k)}$ and $c'(v) = \infty$ otherwise.

By our induction hypothesis, we can find an infinite set $M \subset N$ such that c' is canonical on $M^{(k-1)}$. Hence $c'(v) \neq \infty$ for $v \in M^{(k-1)}$, so c is a canonincal coloring of M, as desired. $\square$

5 Graph Ramsey Theory [1, p. 192-195]

In previous sections, we looked at finding sets of vertices in a graph that induce a complete monochromatic subgraph. Now we see what happens if we just ask for a monochromatic copy of a graph that isn't necessarily complete.

Let's quickly introduce some new notation. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. For $k \in \mathbb{Z}_{\geq 1}$, we will let kG_1 represent k *independent* copies of G_1. $G_1 \cup G_2$ will denote the graph union of G_1, G_2; that is, $G_1 \cup G_2 = (V_1 + V_2, E_1 + E_2)$. Finally, $G_1 + G_2$ will denote the graph join of G_1 and G_2, the graph obtained by taking each vertex of G_1 and connecting it to all of the vertices of G_2.

Now let H_1, H_2 be graphs such that $|H_1| = h_1, |H_2| = h_2$. We ask the question, "Is there some n such that, given any red-blue coloring of K_n, we are assured either a red H_1 or a blue H_2 as a subgraph?" The answer, yes, follows immediately from Theorem 2.2, since if we have a complete graph on at least $R(h_1, h_2)$ (which we know exists and is finite) vertices, then we are assured even more: a red K_{h_1} or a blue K_{h_2}. We will denote the smallest such value n as $r(H_1, H_2)$. We expect that if H_1, H_2 are relatively small, then $r(H_1, H_2)$ is significantly less than $R(h_1, h_2)$.

We also note that, instead of working with 2-colorings of graphs, we may choose to look at a graph and its complement. That is to say, if we have a graph, say G, of at least $r(H_1, H_2)$ vertices, we are guaranteed either that H_1 is a subgraph of G or that H_2 is a subgraph of the complement $\overline{G}$. This approach is more visual and, as a result, often more fruitful.

Our observations will usually force some type of bound on our graph Ramsey number. However, when H_1 consists of independent edges and H_2 is a complete graph, we arrive at an exact result for $r(H_1, H_2)$.

Theorem 5.1. *If $l \geq 1, p \geq 2$, then $r(lK_2, K_p) = 2l + p - 2$.*

Proof. First, we develop a graph on $2l + p - 3$ vertices, say G, that contains neither lK_2 as a subgraph of G nor K_p as a subgraph of $\bar{G}$. Indeed, if $G = K_{2l-1} \cup E_{p-2}$, this is the case. Therefore we know that $r(lK_2, K_p) > 2l + p - 3$ so that $r(lK_2, K_p) \geq 2l + p - 2$.

Now let G be a graph of order $2l + p - 2$ that does not contain l independent edges. Suppose G has k independent edges, where $k \leq l - 1$. It suffices to show that, since G does not contain l independent edges, $K_p \subset \bar{G}$. Consider the remaining $(2l + p - 2) - 2k \geq (2l + p - 2) - 2(l - 1) = p$ vertices. There cannot be an edge among any of these vertices, since this would create another independent edge, yet we assumed k to be maximum. Therefore none of these p vertices are connected to each other, so we see that $K_p \subset \bar{G}$. Thus $r(lK_2, K_p) \leq 2l + p - 2$, so we conclude that $r(lK_2, K_p) = 2l + p - 2$, as desired. $\square$

As we will soon see, it is useful to develop a general bound for arbitrary graphs H_1, H_2. Given a graph G, let $c(G)$ denote the maximal order of a component of G, and let $\chi(G)$ denote the chromatic number of G. Also, $u(G)$ will denote the chromatic surplus of G, the minimal number of vertices in a color class, taken among all proper $\chi(G)-$colorings of G. Our bound for $r(H_1, H_2)$ will depend on both of these new quantities.

Theorem 5.2. *Let H_1, H_2 be nonempty graphs. Then*

$$r(H_1, H_2) \geq (\chi(H_1) - 1)(c(H_2) - 1) + u(H_1).$$

Proof. Let $k = \chi(H_1)$, $c = c(H_2)$, $u = u(H_1)$. We want to show that

$$r(H_1, H_2) \geq (k - 1)(c - 1) + u.$$

We break this proof into two cases:

Case 1 ($u \geq c$): For any graphs H_1, H_2, we can quickly observe that

$$\begin{aligned} r(H_1, H_2) &\geq r(H_1, K_2) \\ &= |H_1| \\ &\geq \chi(H_1)u(H_1) \\ &= ku. \end{aligned}$$

Thus if $u \geq c$, we see that

$$\begin{aligned} r(H_1, H_2) &\geq ku \\ &= (k-1)u + u \\ &\geq (k-1)c + u \\ &\geq (k-1)(c-1) + u. \end{aligned}$$

Case 2 ($u < c$): Consider the graph $G = (k-1)K_{c-1} \cup K_{u-1}$. G doesn't contain H_2, since $c = c(H_2)$ but $c(G) \leq c - 1$. Furthermore, $\bar{G}$ doesn't contain H_1, since $u(\bar{G}) = u - 1$ while $u(H_1) = u$. Thus $r(H_2, H_1) \geq |G| + 1$ so that

$$\begin{aligned} r(H_1, H_2) &= r(H_2, H_1) \\ &\geq |G| + 1 \\ &= [(k-1)(c-1) + u - 1] + 1 \\ &= (k-1)(c-1) + u. \end{aligned}$$

This completes the proof. □

In particular, we can examine the situation when one of our two graphs is connected. Theorem 5.2 immediately provides us with a lower bound.

Corollary 5.3. *Let H_1, H_2 be nonempty graphs. If H_2 is connected, then*

$$r(H_1, H_2) \geq (\chi(H_1) - 1)(|H_2| - 1) + 1.$$

Proof. This follows immediately from Theorem 5.2, since for a connected graph H_2, $c(H_2) = |H_2|$. So we have that

$$\begin{aligned} r(H_1, H_2) &\geq (\chi(H_1) - 1)(|H_2| - 1) + u(H_1) \\ &= (\chi(H_1) - 1)(|H_2| - 1) + 1, \end{aligned}$$

as desired. □

The lower bounds presented via Theorem 5.2 and Corollary 5.3 are sometimes the best possible. For example, we have equality when H_1 represents a complete graph and H_2 represents a tree. Before we prove this result, we will provide a fact about a graph's chromatic number that will be of use.

Lemma 5.4. *Let $k = \max_H \delta(H)$, where the maximum is taken over all induced subgraphs of G and $\delta(H)$ denotes the minimum degree of a vertex of H. Then $\chi(G) \leq k+1$.*

Proof. Let G be a graph of order n and let $k = \max_H \delta(H)$. Consider the vertex of largest degree, say vertex x_n. By our hypotheses, we know that $\deg(x_n) \leq k$. Let $H_{n-1} = G - \{x_n\}$. This graph has a vertex of largest degree (at most k), which we label vertex x_{n-1}. We can then consider $H_{n-2} = H_{n-1} - \{x_{n-1}\}$ and reiterate this process. Consider the sequence $x_1, x_2, \cdots, x_n$ of vertices of G. We claim that we can color this using $k+1$ (or less) colors. Indeed, let vertex x_1 receive color 1, let x_2 receive color 2, $\cdots$, and let x_{k+1} receive color $k+1$. Since each vertex has at most k neighbors before it, we can assign one of the $k+1$ colors to x_{k+2} so that it does not conflict with any previous adjacencies. Similarly, we can assign one of the $k+1$ colors to x_{k+3}, and x_{k+4}, and so on. Hence since we have assigned a proper $(k+1)-$coloring to the vertices of G, we see that $\chi(G) \leq k+1$. □

Using this fact, we will now prove that $r(K_s, T) \leq (s-1)(|T|-1)+1$.

Theorem 5.5. *Let $s, t \geq 2$, and let T be a tree on t vertices. Then $r(K_s, T) = (s-1)(t-1)+1$.*

Proof. Note that Corollary 5.3 tells us that

$$\begin{aligned} r(K_s, T) &\geq (\chi(K_s) - 1)(|T| - 1) + 1 \\ &= (s-1)(t-1)+1. \end{aligned}$$

We want to show that $r(K_s, T) \leq (s-1)(t-1)+1$. Let G be a graph on $n = (s-1)(t-1)+1$ vertices so that $\bar{G}$ does not contain K_s. It suffices to show that G contains T. Note that $n \leq (s-1)\chi(G)$, since if we have a proper coloring of G using $\chi(G)$ colors, each color class contains at most $s-1$ vertices (otherwise, we would have s independent vertices, which creates a $K_s \subset \overline{G}$). Hence $\chi(G) \geq \lceil \frac{n}{s-1} \rceil = \lceil t - 1 + \frac{1}{s-1} \rceil = t$. Let $k = \max_H \delta(H)$. Using Lemma 5.4, we know that $t \leq \chi(G) \leq k+1$. Therefore $t-1 \leq k$, so for some induced subgraph H, $\delta(H) \geq t-1$.

We will show that $T \subset H$. Since $\delta(H) \geq t-1$, we can assume that there is a tree of order $t-1$, say T', such that $T' = T - x$, where x is an endvertex of T. Furthermore, let's say that x is adjacent to $y \in T'$. Well, then since $y \in H$, y has a least $t-1$ adjacencies in H. Since T' is a tree with $t-2$ other vertices, we know that y is adjacent to some vertex of H, say z, such that $z \notin T'$. But then $T' \cup z$ provides us with $T \subset H \subset G$. It then follows that $r(K_s, T) \leq (s-1)(t-1)+1$. Thus we conclude that $r(K_s, T) = (s-1)(t-1)+1$. □

The next set of results in graph Ramsey theory that we consider concern the Ramsey numbers of disjoint unions of complete graphs (for which we will just consider the case of edges and triangles). In our treatment of the problem, the following lemma will be useful.

Lemma 5.6. *Let G, H_1, H_2 be nonempty graphs. Then*

$$r(G, H_1 \cup H_2) \leq \max\{r(G, H_1) + |H_2|, r(G, H_2)\}.$$

In particular, $r(sH_1, H_2) \leq r(H_1, H_2) + (s-1)|H_1|$.

Proof. Let $n = \max\{r(G, H_1) + |H_2|, r(G, H_2)\}$ and $2-$color K_n. Assume that our K_n does not contain a red G. We will show that it contains a blue $H_1 \cup H_2$. Since $n \geq r(G, H_2)$, we know that there is a blue H_2. Remove it. Since $n \geq r(G, H_1) + |H_2|$, we know that $n - |H_2| \geq r(G, H_1)$. Thus since we now have $n - |H_2|$ vertices, and our new graph doesn't contain a red G, it must contain a blue H_1. So we have shown that G contains a blue H_1 and a blue H_2. Hence since G contains a blue $H_1 \cup H_2$, we conclude that $r(G, H_1 \cup H_2) \leq \max\{r(G, H_1) + |H_2|, r(G, H_2)\}$.

In particular, $r(H_2, H_1 \cup (s-1)H_1) \leq \max\{r(H_2, H_1) + (s-1)|H_1|, r(H_2, (s-1)H_1)\}$. We can continue applying our lemma until $r(H_2, H_1) + (s-i)|H_1| \geq r(H_2, (s-i)H_1)$ for some i. Then

$$r(H_2, H_1 \cup (s-1)H_1) \leq r(H_2, H_1) + (s-i)|H_1| \leq r(H_2, H_1) + (s-1)|H_1|,$$

as desired. □

Theorem 5.7. *Let $s \geq t \geq 1$. Then*

$$r(sK_2, tK_2) = 2s + t - 1.$$

Proof. Consider the graph $G = K_{2s-1} \cup E_{t-1}$. Clearly this graph does not contain s independent edges, and its complement does not contain t independent edges. Thus $r(sK_2, tK_2) \geq |G| + 1 = [2s - 1 + t - 1] + 1 = 2s + t - 1$. It suffices to show that $r(sK_2, tK_2) \leq 2s + t - 1$.

First, note that Theorem 5.1 tells us that $r(sK_2, K_2) = 2s + 2 - 2 = 2s$. We will show that

$$r((s+1)K_2, (t+1)K_2) \leq r(sK_2, tK_2) + 3.$$

Let G be a graph on $n = r(sK_2, tK_2) + 3 \geq 2s + t + 2$ vertices. If $G = K_n$, then clearly $(s+1)K_2$ is contained in G and we are done. Similarly, if $G = E_n$, then $(t+1)K_2$ is contained in the complement of G. Hence we may assume that G is neither K_n nor E_n. Then we can find three vertices, say vertices x, y, and z, such that $xy \in G$, $xz \notin G$. Consider $G' = G - \{x, y, z\}$. Since $|G'| = r(sK_2, tK_2)$, there are either s independent edges in G' or t independent edges in the complement. If there are s independent edges in G', we can add xy back into G' to form $s+1$ independent edges in G. Similarly, if there are t independent edges in $\bar{G}'$, we add xz back into $\bar{G}'$ to form $t+1$ independent edges in $\bar{G}$. Thus it follows that

$$r((s+1)K_2, (t+1)K_2) \leq r(sK_2, tK_2) + 3.$$

We claim that this completes the proof. Indeed, since $r((s+1)K_2, (t+1)K_2) \leq r(sK_2, tK_2) + 3$ and $s \geq t$, we have that

$$r(sK_2, tK_2) \leq r((s - (t-1))K_2, K_2) + 3(t-1).$$

But the right-hand side is just $2[s - (t-1)] + 3(t-1) = 2s + t - 1$ by Theorem 5.1. So we have shown $r(sK_2, tK_2) \leq 2s + t - 1$

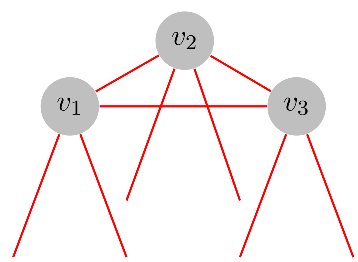
Hence $r(sK_2, tK_2) = 2s + t - 1$ as claimed. □

Lemma 5.8. $R(2K_3, K_3) = 8$.

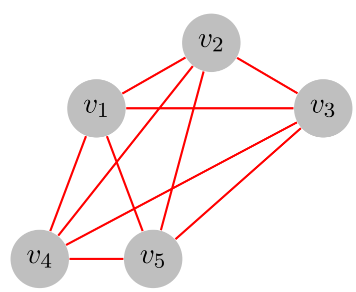
Proof. Consider the graph $G = K_5 \cup (K_1 + E_1)$. Clearly this graph does not contain 2 independent triangles, and its complement does not contain a triangle. Thus $r(2K_3, K_3) \geq |G| + 1 = 7 + 1 = 8$. It suffices to show that $r(2K_3, K_3) \leq 8$.

Let $G = K_8$ be 2−colored. Because $|G| = 8 > 6 = R(K_3, K_3)$, we know that G either contains either a red K_3 or a blue K_3. If it contains a blue K_3, we are done. Thus we may assume that G has a red K_3 as a subgraph. We will label this triangle using the vertices $\{v_1, v_2, v_3\}$, and we will call this triangle T. It suffices to show that there exists a red K_3 disjoint from T, or a blue K_3.

Consider $G_1 = G \setminus \{v_2, v_3\}$. Since $|G_1| = 6 = R(K_3, K_3)$, we have that G_1 contains a red triangle (say T_1) as a subgraph (if it contains a blue K_3, we are done). In fact, two of the edges must be incident with v_3, since if not, we have found a red K_3 disjoint from T, and we are done. Similarly, letting $G_2 = G \setminus \{v_1, v_3\}$ and $G_3 = G \setminus \{v_1, v_2\}$, we can form red triangles T_2 and T_3 and see that there are two more red edges incident with v_2 and v_3 as well. We arrive at the following subgraph:



We consider the unknown edges of T_1, T_2, and T_3. It must be the case that there is only one edge common to all three triangles (if not, we can clearly find two red disjoint copies of K_3). In this instance, let our red edge be labeled using the vertices $\{v_4, v_5\}$. We have the graph:



Call this graph G', and consider the remaining three vertices. There is at least one blue edge between these, for otherwise, we could find a red triangle disjoint from T. Let this blue edge be v_6v_7. Considering the 5 edges from v_6 to G', we see that there can be at most one red edge. Similarly, there can be at most one red edge adjoining v_7 to G'. By the pigeonhole principle, there is at least one vertex of G' that is adjacent to both v_6 and v_7 via blue edges. But then we have arrived at a blue triangle.

Thus G has either a red $2K_3$ or a blue K_3 so that $R(2K_3, K_3) \leq 8$. Hence $R(2K_3, K_3) = 8$. $\square$

Theorem 5.9. *Let $s \geq t$, $t \geq 1$, $s \geq 2$. Then*

$$r(sK_3, tK_3) = 3s + 2t.$$

Proof. Consider the graph $G = K_{3s-1} \cup (K_1 + E_{2t-1})$. Clearly this graph does not contain s independent triangles, and its complement does not contain t independent triangles. Thus $r(sK_3, tK_3) \geq |G| + 1 = [3s - 1 + 1 + 2t - 1] + 1 = 3s + 2t$. It suffices to show that $r(sK_3, tK_3) \leq 3s + 2t$.

First, by repeatedly applying Lemma 5.6, we see that $r(sK_3, K_3) \leq 3s + 2$ since by Lemma 5.8, $R(2K_3, K_3) = 8$. We now claim that

$$r((s+1)K_3, (t+1)K_3) \leq r(sK_3, tK_3) + 5.$$

Let G be a graph on $n = r(sK_3, tK_3) + 5$ vertices. For any red-blue coloring of K_n, find a red triangle, say R, and consider $K_n - R$. If $K_n - R$ contains s independent triangles, then we are done. Since $|K_n - R| = r(sK_3, tK_3) + 2$, we know that $K_n - R$ contains t independent blue triangles. Consider one of these blue triangles, say B. Without loss of generality, assume at least 5 of the edges between R and B are red. Then at least 2 of these red edges are adjoined to the same vertex of B so that we can form a new red triangle, say R'. Then $K_n - R' - B$ has $r(sK_3, tK_3)$ vertices so that we have either a red sK_3 or a blue tK_3. If we have a red sK_3, then adding in R', which is disjoint from our red sK_3, provides us with our red $(s+1)K_3$. If we have a blue tK_3, then adding in B, which is disjoint from our blue tK_3, provides us with our blue $(t+1)K_3$. Thus

$$r((s+1)K_3, (t+1)K_3) \leq r(sK_3, tK_3) + 5.$$

We claim that this completes the proof. Indeed, since $r((s+1)K_3, (t+1)K_3) \leq r(sK_3, tK_3) + 5$ and $s \geq t$, we have that

$$r(sK_3, tK_3) \leq r((s - (t-1))K_3, K_3) + 5(t-1).$$

But $r((s-(t-1))K_3, K_3) + 5(t-1) \leq 3[s-(t-1)] + 2 + 5(t-1) = 3s + 2t$ by Lemma 5.6. So we have shown $r(sK_3, tK_3) \leq 3s + 2t$.

Hence $r(sK_3, tK_3) = 3s + 2t$ as claimed. □

Similar to the methods used in proving Theorem 5.7 and Theorem 5.9, we can obtain bounds on $r(sK_p, tK_q)$. We conclude our analysis of graph Ramsey theory by noting this result.

Theorem 5.10. *For $p, q \geq 2$, choose t_0 so that*

$$t_0 \min\{p, q\} \geq 2r(K_p, K_q),$$

and set $C = r(t_0K_p, t_0K_q)$. Then for $s \geq t \geq 1$, we have

$$ps + (q-1)t - 1 \leq r(sK_p, tK_q) \leq ps + (q-1)t + C.$$

6 Van Der Waerden's Theorem [2, p. 1 - 11]

The purpose of this section is to introduce and prove van der Waerden's Theorem. Throughout this section, we will often refer to an arithmetic progression of length k as a $k-$AP, and given a natural number n we will let $[n] = \{1, 2, \cdots, n\}$. We will begin by stating the result:

Theorem 6.1 (van der Waerden's Theorem). *For every $k \geq 1, c \geq 1$, we can find a $W = W(k, c)$ such that for every $c-$coloring of $[W]$ there exists a monochromatic $k-AP$.*

From now on, by $W = W(k, c)$ we will mean the smallest such W. Note that van der Waerden's Theorem is a Ramey-type result. The initial segment $[W]$ that we color is an arithmetic progression (with common difference 1). Van der Waerden's Theorem says that if we break up a long enough arithmetic progression into finitely many classes, one of the classes contains a long arithmetic progression.

6.1 Immediate Results

We will begin by examining some concrete cases of van der Waerden's Theorem.

Proposition 6.2. *Let $c \geq 1$. Then $W(1, c) = 1$.*

Proof. We ask the question "What is the smallest number W such that, given any $c-$coloring on $[W]$, we can find a monochromatic arithmetic progression of length 1?" Clearly, all we need is one number, as we can then choose that number. Thus $W(1, c) = 1$. □

Proposition 6.3. *Let $c \geq 1$. Then $W(2, c) = c + 1$.*

Proof. First, note that if we have c colors, it is possible to color $[c]$ so that there is no $2-$AP: simply let the i^{th} integer receive the i^{th} color. Because this is a $c-$coloring of $[c]$ with no monochromatic $2-$AP, we note that $W(2, c) > c$.

Note, however, that for any $c-$coloring of $[c + 1]$, we are guaranteed a repeat in color by the pigeonhole principle. Thus (by considering these two numbers) we have a monochromatic arithmetic progression of length 2. Therefore $W(2, c) \leq c + 1$, and, in fact, $W(2, c) = c + 1$. □

Proposition 6.4. *Let $k \geq 1$. Then $W(k, 1) = k$.*

Proof. Clearly, if $W < k$, we cannot arrive at an arithmetic progression of length k, let alone a monochromatic $k - AP$. So we know that $W(k, 1) \geq k$. But for the $1-$coloring of $[k]$, observe that $1, 2, \cdots, k$ is a monochromatic $k-$AP. Thus $W(k, 1) = k$. □

6.2 Finding W(3, 2)

As we have already addressed the instances where $c = 1$ and $k = 1, 2$, we will now consider a more interesting case.

Theorem 6.5. $W(3, 2) = 9$.

Proof. First, we will produce a $2-$coloring of $[8]$ which contains no monochromatic $3-$AP. Color the integers as follows:

$$Blue, Red, Blue, Red, Red, Blue, Red, Blue$$

Since this $2-$coloring contains no monochromatic $3-$AP, we deduce that $W(3,2) > 8$.

It suffices to show that $W(3,2) \leq 9$. $2-$color $[9]$, and, without loss of generality, assume that 1 is colored red. If the numbers 5, 9 are both colored red, a monochromatic $3-$AP results, and we are done. We address the remaining cases:

Case 1 (5 is colored blue, 9 is colored blue): Consider the following sequence:

$$Red\ \_\ \_\ \_\ Blue\ \_\ \_\ \_\ Blue$$

If 7 is colored blue, we are done (5-7-9 forms a monochromatic $3-$AP). Thus, 7 must be colored red. Then through similar logic, 4 must be colored blue (by considering 1-4-7). But then 3, 6 must be red (by considering 3-4-5 and 4-5-6). We are now at the following:

$$Red\ \_\ Red, Blue, Blue, Red, Red\ \_\ Blue$$

Thus 2 and 8 must be colored blue. But then 2-5-8 forms a monochromatic $3-$AP. Thus we are done.

Case 2 (5 is colored red, 9 is colored blue): Consider the following sequence:

$$Red\ \_\ \_\ \_\ Red\ \_\ \_\ \_\ Blue$$

If 3 is colored red, we are done (5-7-9 forms a monochromatic $3-$AP). Thus, 3 must be colored blue. Then through similar logic, 6 must be colored red (by considering 3-6-9). But then 4, 7 must be red (by considering 4-5-6 and 5-6-7). We are now at the following:

$$Red\ \_\ Blue, Blue, Red, Red, Blue\ \_\ Blue$$

Thus 2 and 8 must both be colored red. But then 2-5-8 forms a monochromatic $3-$AP, and we are done.

Case 3 (5 is colored blue, 9 is colored red): Consider the following sequence:

$$Red\ \_\ \_\ \_\ Blue\ \_\ \_\ \_\ Red$$

We will break this case into two sub-cases:

Case 3a (3 is colored blue): Consider the following sequence:

$$Red\ \_\ Blue\ \_\ Blue\ \_\ \_\ \_\ Red$$

If 7 is colored blue, we are done (3-5-7 forms a monochromatic 3-AP). Thus, 7 must be colored red. Also, 4 must be colored red (by considering 3-4-5). But then 1-4-7 forms a monochromatic 3-AP, so we are done.

Case 3b (3 is colored red): Consider the following sequence:

$$Red\ \_\ Red\ \_\ Blue\ \_\ \_\ \_\ Red$$

If 6 is colored red, we are done (3-6-9 forms a monochromatic 3-AP). Thus, 6 must be colored blue. Also, 2 must be colored blue (by considering 1-2-3). We are now at the following:

$$Red, Blue, Red \ \_ \ Blue, Blue \ \_ \ \_ \ Red$$

Thus 4 and 7 must both be colored red. But then 1-4-7 forms a monochromatic $3-$AP, and we are done.

Thus by considering the four possible colorings of 5 and 9, we have shown that a monochromatic $3-$AP always results when we $2-$color [9]. Thus $W(3,2) \leq 9$, and we conclude that $W(3,2) = 9$. □

6.3 Proof of Van Der Waerden's Theorem for k = 3

Let $W = bU$ for some $b, U \in \mathbb{N}$. We view W as U blocks of size b, say $B_1 B_2 \cdots B_U$. Thus it will often be useful to regard a $c-$coloring of W as a c^b-coloring of U blocks, each of size b. Note that, under the induced c^b-coloring of the blocks, two blocks have the same color if and only if their numbers have the same color pattern.

Our proof will rely on the following two facts:

Proposition 6.6. *Let B be a block of size $2c+1$. Let $COL : B \to [c]$ be a $c-$coloring of B. Then there exist a, d such that*

$$a, a+d, a+2d \in B,$$

$$COL(a) = COL(a+d).$$

Proof. Without loss of generality, assume that B is labeled with the numbers $1, 2, \cdots, 2c+1$. Consider the first $c+1$ numbers. For any $c-$coloring of $[c+1]$, by the pigeonhole principle, there will be two numbers in $[c+1]$ with the same color, say $n_1, n_2 \in [c+1]$ where $n_1 < n_2$. It is then clear that

$$n_1, n_1 + (n_2 - n_1), n_1 + 2(n_2 - n_1)$$

is our desired sequence, since $n_1 + 2(n_2 - n_1) = n_2 + n_2 - n_1 \leq n_2 + c \leq (c+1) + c = 2c+1 \in B$ and $COL(n_1) = COL(n_2)$. □

Proposition 6.7. *Let $W = b(2c^b + 1)$. We view $[W]$ as:*

$$B_1 B_2 \cdots B_{2c^b+1}.$$

Let $COL : [W] \to [c]$ be a $c-$coloring of $[W]$ and COL^* be the induced c^b-coloring of the $2c^b + 1$ blocks of size b. Then there exist A, D such that

$$A, A+D, A+2D \in [2c^b + 1],$$

$$COL^*(B_A) = COL^*(B_{A+D}).$$

Proof. Consider the first $c^b + 1$ blocks. Because our induced coloring is a c^b-coloring, we know that two blocks will have the same induced coloring, say B_{n_1}, B_{n_2}, where $n_1 < n_2$. It is then clear that

$$n_1, n_1 + (n_2 - n_1), n_1 + 2(n_2 - n_1)$$

is our desired sequence, since $n_1 + 2(n_2 - n_1) = n_2 + n_2 - n_1 \leq n_2 + c^b \leq (c^b + 1) + c^b = 2c^b + 1 \in [2c^b + 1]$ and $COL^*(B_{n_1}) = COL^*(B_{n_2})$. □

With these two facts in place, we can now begin to address van der Waerden's Theorem for $k = 3$. Our proof will use the following lemma:

Lemma 6.8. *Let $c \geq 1$, $1 \leq r \leq c$. Then we can find a $U = U(c, r)$ such that, for any $c-$coloring of $[U]$, we have one of the following:*

1. *There exists a monochromatic $3-AP$.*
2. *There exist $w \in [U]$, $C \subseteq [c]$ such that*
   (a) *$|C| = r$.*
   (b) *$COL(w) \notin C$.*
   (c) *If the color of w is changed to any color in C, a monochromatic $3-AP$ results.*

Proof. Let $c \geq 1, r = 1$. We claim that $U(c, 1) = 2c + 1$. Let $COL : [U] \to [c]$ be a $c-$coloring of $[U]$. By Proposition 6.6, we can find $a, d \in [2c + 1]$ such that

$$a, a + d, a + 2d \in B,$$

$$COL(a) = COL(a + d).$$

Consider $a+2d$. If $COL(a+2d) = COL(a+d)$, we are done, as we have found a monochromatic $3 - AP$ in $[U]$. If $COL(a + 2d) \neq COL(a + d)$, let $w = a + 2d$, $C = \{COL(a + d)\}$. Note that we have found $w \in [U]$, $C \subseteq [c]$ such that

1. $|C| = 1 = r$.
2. $COL(w) \notin C$.
3. If the color of w is changed to any color in C, a monochromatic $3-$AP results.

Thus, for $r = 1$, we have our result.

Now assume that $U(c, r)$ exists. We will show that $U = U(c, r+1) = U(c, r)[2c^{U(c,r)}+1]$. Let $COL : [U] \to [c]$ be a $c-$coloring of $[U]$, COL^* be the induced $c^{U(c,r)}-$coloring of the $2c^{U(c,r)} + 1$ blocks of size $U(c, r)$. By Proposition 6.7, we can find A, D such that:

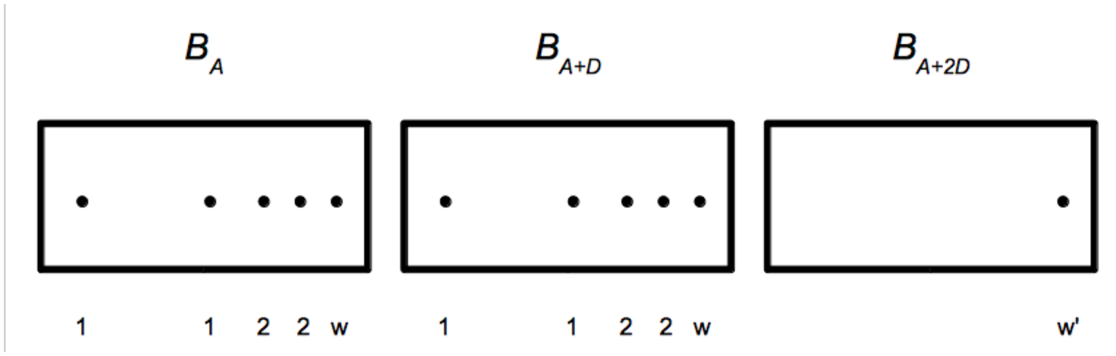
$$A, A + D, A + 2D \in [2c^{U(c,r)} + 1],$$

$$COL^*(B_A) = COL^*(B_{A+D}).$$

Since B_A is of size $U(c, r)$, we know that either B_A contains a monochromatic $3-$AP (if so, we are done) or there are some $w \in B_A$, $C \subset [c]$ such that:

1. $|C| = r$.
2. $COL(w) \notin C$.
3. If the color of w is changed to any color in C, a monochromatic $3-$AP in B_A results.

Note that we have the same situation for the block B_{A+D} since they inherit the same induced coloring. Without loss of generality, let $C = \{1, 2, \cdots, r\}$, and assume $COL(w) = r + 1$. We have the following picture:



Consider $w' = w + 2D(U(c,r)) \in [2c^{U(c,r)} + 1]$, and let $C' = \{1, 2, \cdots, r+1\}$. We claim that $w \notin C'$. If $COL(w') = r+1$, then $w, w + D(U(c,r)), w'$ forms a monochromatic $3-$AP, yet we assumed there are none. If $COL(w') = i$ for $1 \leq i \leq r$, we also arrive at a monochromatic $3-$AP (choosing the first point colored i from B_A and the second from B_{A+D}). Thus $COL(w') \notin C'$. Furthermore, we have shown that if w' is recolored with a color from C', a monochromatic $3-$AP results. So we have shown that either:

1. There exists a monochromatic $3-$AP
2. There exist $w' \in [U(c, r+1)]$, $C' \subseteq [c]$ such that
   (a) $|C'| = r + 1$.
   (b) $COL(w') \notin C'$.
   (c) If the color of w' is changed to any color in C', a monochromatic $3-$AP results.

Thus $U(c, r+1)$ exists via $U(c, r)$. Inductively, we are done. □

With the use of Lemma 6.8, we can now state and prove van der Waerden's Theorem for $k = 3$.

Theorem 6.9. *For every $c \geq 1$, we can find a $W = W(3, c)$ such that for every $c-$coloring of $[W]$ there exists a monochromatic $3-$AP.*

Proof. We apply Lemma 6.8 with $r = c$. Choose $U = U(c, c)$ such that for any $c-$coloring of $[U]$, either

1. There exists a monochromatic $3-$AP.
2. There exist $w \in [U]$, $C \subseteq [c]$ such that
   (a) $|C| = c$.
   (b) $COL(w) \notin C$.
   (c) If the color of w is changed to any color in C, a monochromatic $3-$AP results.

Clearly (a) and (b) cannot both simultaneously occur, since there are only c colors. Thus for any $c-$coloring of $[U]$, there exists a monochromatic $3-$AP. □

6.4 Proof of Van Der Waerden's Theorem

We almost have all of the tools necessary to prove van der Waerden's Theorem. Our next two propositions are the analogs, for general k, of Proposition 6.6 and Proposition 6.7. Let's quickly observe the following:

Proposition 6.10. *Let COL be a $c-$coloring of $[2W(k-1,c)]$. Then there exist a, d such that*

$$a, a+d, \cdots, a+(k-1)d \in [2W(k-1,c)],$$
$$COL(a) = COL(a+d) = \cdots = COL(a+(k-2)d).$$

Proof. Break up $[2W(k-1,c)]$ into two halves, each of size $W(k-1,c)$. Since the first half is of size $W(k-1,c)$, we know that there exist a, d such that:

$$a, a+d, \cdots, a+(k-2)d \in [W(k-1,c)],$$

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-2)d).$$

That is, we know that there is a monochromatic $(k-1)-$AP. We adjoin $a+(k-1)d$ in the second half to arrive at

$$a, a+d, \cdots, a+(k-1)d \in [2W(k-1,c)],$$

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-2)d).$$

as required. □

Proposition 6.11. *Let $W = b(2W(k-1,c^b))$. We view $[W]$ as:*

$$B_1 B_2 \cdots B_{2W(k-1,c)}.$$

Let $COL : [W] \to [c]$ be a $c-$coloring of $[W]$ and COL^* be the induced c^b-coloring of the $2W(k-1,c^b)$ blocks of size b. Then there exist A, D such that

$$A, A+D, \cdots, A+(k-1)D \in [2W(k-1,c^b)],$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

Proof. Consider the first $W(k-1,c^b)$ blocks. We know that there exist A, D such that:

$$A, A+D, \cdots, A+(k-2)D \in [W(k-1,c^b)],$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

We simply adjoin the $(A+(k-1)D)^{th}$ block to arrive at

$$A, A+D, \cdots, A+(k-1)D \in [2W(k-1,c^b)],$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

as required. □

With these two propositions at our disposal, we can prove the full version of van der Waerden's Theorem. It should be noted that in order to complete the proof, we will induct on k, c in an interesting way. We define the ordering:

$$(2,2) < (2,3) < (2,4) < \cdots < (3,2) < (3,3) < (3,4) < \cdots < (4,2) < (4,3) < (4,4) < \cdots$$

We will induct on the ordering so that assuming the result for all $(i', j') < (i, j)$ implies the result for (i, j).

Similar to our $k = 3$ case, our proof will follow immediately from a lemma:

Lemma 6.12. *Let $c \geq 1$, $k \geq 1$. Assume that for all ordered pairs $(k', c') < (k, c)$, $W(k', c')$ exists. Let $1 \leq r \leq c$. Then we can find a $U = U(k, c, r)$ such that, for any $c-$coloring of $[U]$, we have one of the following:*

1. *There exists a monochromatic $k-$AP.*
2. *There exist $w \in [U]$, $C \subseteq [c]$ such that*
   (a) *$|C| = r$.*
   (b) *$COL(w) \notin C$.*
   (c) *If the color of w is changed to any color in C, a monochromatic $k-AP$ results.*

Proof. We will induct on r for $1 \leq r \leq c$.

For $r = 1$, we claim that $U(1, k, c) = 2W(k-1, c)$ suffices. For any $c-$coloring of $[2W(k-1, c)]$, Prop 6.10 tells us that we can find a, d such that

$$a, a+d, \cdots, a+(k-1)d \in [2W(k-1,c)],$$

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-2)d).$$

If $COL(a+(k-1)d) = COL(a)$, then we are done, as we have found a monochromatic $k-$AP. If not, then let $w = a+(k-1)d$, $C = \{COL(a)\}$. Then $|C| = 1 = r$, $COL(w) \notin C$, and if the color of w is changed to any color in C, then a monochromatic $k-$AP results. Thus the lemma holds for $r = 1$.

Now assume that $U(k, c, r)$ exists. We will show that

$$U = U(k, c, r+1) = U(k, c, r)2W(k-1, c^{U(k,c,r)}).$$

We view $[U]$ as:

$$B_1 B_2 \cdots B_{2W(k-1, c^{U(k,c,r)})},$$

where each block is of size $U(k, c, r)$. Let COL^* be the induced $c^{U(k,c,r)}-$coloring of the blocks. By Prop 6.11 we know that there exist A, D such that:

$$A, A+D, \cdots, A+(k-1)D \in [2W(k-1, c^{U(k,c,r)})],$$

$$COL^*(B_A) = COL^*(B_{A+D}) = \cdots = COL^*(B_{A+(k-2)D}).$$

Assume that there is no monochromatic $k-$AP in $[U]$. Since each block is of size $U(k, c, r)$, we know that each block contains a w and a C such that

1. $|C| = r$.

2. $COL(w) \notin C$.

3. If the color of w is changed to any color in C, a monochromatic $k-$AP in B_A results.

In particular, this occurs in $B_A, B_{A+D}, \cdots, B_{A+(k-2)D}$. Furthermore, since these blocks have the same induced $c^{U(k,c,r)}-$coloring, we have the same color pattern occurring at the same positions in each respective block. Without loss of generality, assume that $C = \{1, 2, \cdots, r\}$ and $COL(w) = r+1$. Let w' be the number in block $B_{A+(k-1)D}$ that is in the same relative position as each w, and let $C' = \{1, 2, \cdots, r+1\}$. Note that:

1. $|C'| = r+1$.

2. $COL(w') \notin C'$. If $COL(w') = r+1$, each respective w adjoined with w' would form a monochromatic $k-$AP. If $COL(w') = i$ for $1 \leq i \leq r$, then we could choose our $k-1$ equally spaced numbers of color i from blocks $B_A, \cdots, B_{A+(k-2)D}$ leading to a monochromatic $k-$AP. Thus $COL(w') \notin C'$.

3. If the color of w is changed to any color in C, a monochromatic $k-$AP in B_A results. This is clear from the above discussion.

We have shown that either a monochromatic $k-$AP occurs or the three conditions above are satisfied for w', C'. Thus $U(k, c, r+1)$ exists, and therefore the result follows. $\square$

In a manner similar to the proof of van der Waerden's Theorem for $k = 3$, we can prove our full version of van der Waerden's Theorem as a special case of our lemma.

Theorem 6.1 (van der Waerden's Theorem)**.** *For every $k \geq 1, c \geq 1$, we can find a $W = W(k, c)$ such that for every $c-$coloring of $[W]$ there exists a monochromatic $k-$AP.*

Proof. We apply Lemma 6.12 with $r = c$. Choose $U = U(k, c, c)$ such that for any $c-$coloring of $[U]$, either

1. There exists a monochromatic $k-$AP.

2. There exist $w \in [U]$, $C \subseteq [c]$ such that

   (a) $|C| = c$.

   (b) $COL(w) \notin C$.

   (c) If the color of w is changed to any color in C, a monochromatic $k-$AP results.

Clearly (a) and (b) cannot both simultaneously occur, since there are only c colors. Thus for any $c-$coloring of $[U]$, we have a monochromatic $k-$AP. $\square$

References

[1] Bollobás, Béla. *Modern Graph Theory.* New York: Springer, 1998. Print.

[2] Gasarch, William. *Van Der Waerden's Theorem: Exposition and Generalizations.*

[3] Goodman, A. W. *On Sets of Acquaintances and Strangers at Any Party.* American Mathmatical Monthly, v. 66, no. 9. 778-783.

[4] Harris, John M., Jeffry L. Hirst, and Michael J. Mossinghoff. *Combinatorics and Graph Theory.* New York: Springer, 2008. Print.

[5] Jukna, Stasys. *Extremal combinatorics: With Applications in Computer Science.* Berlin New York: Springer, 2001. Print.

[6] Radziszowski, Stanislaw. *Small Ramsey Numbers.* Electronic Journal of Combinatorics, Dynamic Survey DS1.