# APPLICATIONS OF FACTORIZATION HOMOLOGY TO RIEMANNIAN FIELD THEORIES

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Abstract

by

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In this thesis, we relate geometric field theories with classical, homotopical invariants of algebraic objects. We begin by defining an abstract setting in which to model the local observables of field theories depending on a Riemannian structure. We then introduce a family of examples whose input is implicit in quantum mechanical systems. Following standard higher categorical procedures, we produce an extension of these local constructions to general Riemannian manifolds.

Using abstract homotopy theory, factorization algebras, and factorization homology ,we relate the observables of a field theory on a circle (of fixed size) with Hochschild homology with coefficients in a module. The action maps of this module depend explicitly on the geometry on which the field theory habitates. We end with a general discussion of how these techniques may be used in more general contexts.

CONTENTS

FIGURES

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# CHAPTER 1

# INTRODUCTION

The primary goal of this thesis is to establish (in specific examples) relationships between invariants from homotopy theory to field theories which depend on a fixed (Riemannian) geometry. Our primary results proves a result (stated without proof) of [CG1] (Theorem 4.2.2) relating the observables of a Riemannian field theory to Hochschild homology of a certain module.

The primary strategy will be to reduce questions about our geometric field theory to one of a "topological theory", one manifold at a time. Properties of this topological-type theory will reflect the global geometry of that manifold.

## 1.1 Background

Mathematicians and physicists have established connections between field theory and homotopy theory through a variety of routes. This thesis fits under the general program of "factorization algebras." This approach orginates in Belinson and Drinfeld's work in conformal field theory. Lurie, Ayala, and Francis later introduced topological analogs of these methods, relying heavily on abstract homotopical methods. Using a combination of the BV formalism, Wilsonian effective field theory, and homotopy theory, Costello and Gwilliam extended this apporach to general statistical field theories. Although our primary examples are not topological, the results [AF] will be essential to the results which follow.

The general strategy in [AF] is the exploitation of $E_n$-algebras and (co)sheaf theory to construct more refined invariants of n-manifolds. In this framework, one argues that the (local) observables of a topological field theory admit an algebraic structure, whose operations are continuously parametrized by embedded Euclidean neighborhoods. Standard categorical and homotopical constructions extend these local, algebraic structures to global observables of a manifold, referred to as factorization homology. In many cases, these extension admit more conventional homotopical descriptions, the most famous example being Hochschild homology.

A fundamental result shows that factorization homology forms a "multi-linear" homology theory. This "multi-linearity" gives rise to more sensitive invariants of manifolds and $E_n$-type algebraic objects. For example, a standard collar deconstruction of the circle gives an expression of the Hochschild complex of an associative algebra as:

$$\mathrm{HC}_*(A) \simeq A \otimes_{A \otimes A^{\mathrm{op}}} A$$

Work of Costello and Gwilliam extend this operadic approach to statistical field theories which are not "topological." The operations of these algebras are parametrized by the partially ordered set of open sets. Following Beilinson and Drinfeld, they refer to these objects as (pre)factorization algebras.

In [CG1][CG2], they utilize a cohomological approach to the BV formalism (pioneered in the seminal work of Kontsevich and Axelrod-Singer) to construct factorization algebras living over a single fixed manifold. The rigorous existence of these algebraic structures requires deep results in functional analysis and elliptic partial differential equations, most notably paramatrix methods. Moreover, they show that these constructions obey a cosheaf condition with respect to a nonstandard Grothendieck topology.

Unsurprisingly, explicit computations of these algebraic structures within this framework can be extremely difficult for general theories. For example, the mere existence of these multiplication maps rely upon the existence of compactly supported parametrices, which do not necessarily admit easy-to-come-by explicit descriptions.

However, there exists a more tractable class of theories, namely those which obey a "local constancy" condition. This condition will be essential to the results which follow. Loosely, this condition has two facets.

First, it states that the support of any *local* observable can in some way be shrunk to an equivalent observable supported within a smaller region. Second, it states that viewing a *local* observable as an observable in a slightly larger neighborhood results in no loss of information.

If our theory is "classical", we can take the ansatz that observables are functions on the solutions to some equations of motion. In this instance, the local constancy condition asserts that every local solution to the equations of motion admits a "operationally unique" extension to a larger local neighborhood. If a theory admits a description as a "quantization" of a locally constant classical theory, one might hope to somehow inductively exploit the compatibility of the quantization with the equations of motions to extend the classical local constancy to a local constancy at the quantum level. On a formal level, this amounts to a spectral sequence-type argument.

Many well-known field theories satisfy this condition, for example, the Ising model/massive scalar field theory. At the classical level, this should follow from the well-posedness of the Helmholtz/Klein-Gordon equation.

A folk attitude towards these objects asserts the following: if we are thinking of a factorization algebra as describing some field theory, local constancy means that we are modeling a topological field theory. We speculate that this attitude is founded in a result of Lurie, which proves an equivalence of homotopy theories between $E_n$-algebras and locally constant factorization algebras.

In light of the previous discussion, a naive invocation of this attitude suggests that massive scalar field theory describes topological theory. This is clearly nonsense. The author proposes a more conservative interpretation of local constancy: these are field theories which have a hope of some type of algebraic description. For example, Lurie's work demonstrates how one can extract an $E_n$ algebra from a locally constant factorization algebra on $\mathbb{R}^n$.

We now present an example from the standard curriculum in field theory. Everything we say should only be true at the level of physics.

A standard approach to massive scalar field theory amounts to the extraction an associative algebra, a module thereover and a suitable representation of the Poincare group.

Following Dirac, these algebraic structures can (approximately) deconstuct scattering processes as more elementary processes involving the creation and/or annihilation of particles of various types. The relations in this associative algebra are "topological", and admit no explicit reference to geometric quantities such as displacement. The representation then relates processes which occur in different points in spacetime. The geometric dependence of this theory is completely encoded in this representation.

If we restrict our interest to understanding processes which occur over a fixed over time interval with a distinguished point, the relevant computations become purely algebraic. The specific form of these computations (namely the module structure) will depend upon a global geometric property: the duration of the process.

We invite the reader to keep these examples in mind and the thesis progresses.

## 1.2 An Outline of the Approach

We begin by defining a suitably general, abstract model for the local observables of a one dimensional "Riemannian field theory."

*Remark* 1. In the lingua franca of field theory, these describe field theories whose action involve an operator constructed from a Riemannian metric. For example a theory whose action involves a Laplacian or a theory which has been gauge-fixed using a Riemannian structure.

This definition will be categorical in nature, paralleling those in [AF]. These choices are not only meant to provide a rigorous context: they also streamline the importation of homotopical methods.

We then construct our primary object of investigation: a class of "locally constant" examples. This construction takes as input an associative algebra, $A$ equipped with a one parameter family of automorphisms, $\varphi$. On a conceptual level, these theories may be thought of as "probes" for more general theories. For example, those obtained as pushforwards of higher dimensional theories, or those constructed in the BV formalism.

*Remark* 2. This data provides a one-dimensional family of $(A, A)$ bimodules, whose underlying chain complex is $A$, and multiplication maps for every $L \in \mathbb{R}$ are:

$$a_0 \otimes a \otimes a_1 \mapsto \varphi_L(a_0) \cdot a \cdot \varphi_{-L}(a_1)$$

We will denote this module as ${}_L A_{-L}$

After applying categorical methods to extend this local construction to general Riemannian 1-manfolds, we then analyze the global observables for each Riemannian manifold.

The primary result is an identification between the observables on a circle of circumference 2L and Hochschild homology:

$$\mathrm{HH}_*(A,{}_L A_{-L})$$

which was stated (without proof) in [CG1].

We obtain this result by identifying the local observables on this circle with a "topological" theory in the sense of [AF]. This argument only works "one Riemannian manifold at a time," in that the structure of this identification explicitly depends on the geometry of the specific manifold at hand. In other words, we establish the result through the construction of equivalences of *factorization algebras*.

The final two parts and appendix of this thesis contain background information on factorization homology, factorization algebra, and higher category theory.

# CHAPTER 2

## RIEMANNIAN VARIATIONS I: THE LOCAL STORY

We begin with the core of the thesis, leaving much of the relevant background to later chapters. Therefore, those without a particular background will have to either accept certain facts without proof or refer to later chapters, appendices, and/or outside literature.

We begin this chapter with a section setting up certain “algebraic” preliminaries. Paralleling the theory of operads and the approach to topological field theories established in [AF], we will construct categories of geometric objects parametrizing the operations of a certain type of algebraic object, a Riemannian 1-disk algebra.

We then outline a procedure to construct a Riemannian 1-disk algebra from data of a unital associative algebra and a 1-parameter group of automorphisms. Hopefully, this will illuminate the relationship between Remannian 1-disk algebras to field theory, via Dirac-style quantization and the time-ordered product.

The motivation behind this more geometric approach is twofold. First, as outlined in [CG1], the BV formalism constructs examples of such algebras. As the BV approach relies on Feynman’s path integral approach to field theory, this work may be considered as constructing a common setting in which to explore the relationships between these two standard approaches to field theory. More importantly, the work of [AF] will allow us to relate these constructions to abstract homotopy theory.

## 2.1 Monoidal Digression

Before going into the primary content, we pause to make a few technical notes about "ordinary" symmetric monoidal categories, in the sense of Mac Lane, [M].

In this formalism, symmetric monoidal structures are controlled by planar rooted trees whose internal vertices have valency three. On a conceptual level, the planar rooted tree is granting us the structure of a parenthesization of the ordered set of leaves of $T$.

Given such a tree $T$, we will adopt the following notation:

- We let $[T]$ denote the underlying nonempty ordered set of leaves. The ordering exists because we assumed $T$ to be planar
- We will $\langle T \rangle$ denote the underlying set of leaves.

The structure of a symmetric monoidal structure on an ordinary category, $C$, gives an operation with two inputs: a planar rooted tree $T$ along with a labelling of the leaves of $T$ by objects of $C$:

$$A : \langle T \rangle \to \mathrm{ob}(C)$$
$$i \mapsto A_i$$

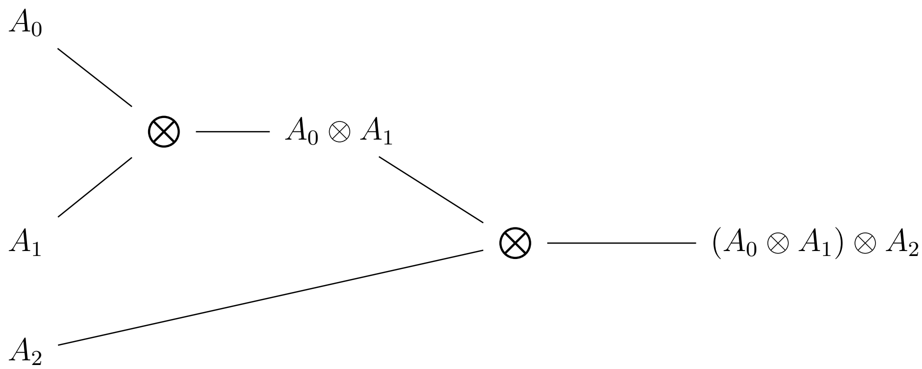
Given such data, the monoidal structure gives a new object of $C$, which we'll denote as:

$$\bigotimes_{i \in T} A_i$$

When $A$ is constant, this object will be denoted as:

$$A^{\otimes T}$$

**Example 3.** We invite the reader to keep the following infographic in mind:




*Remark* 4. We can also think of $T$ as a factorization of the unique map $[n] \to [0]$:

$$[n] \to \dots \to [0]$$

So that for each $[k] \to [k-1]$, the preimage of any point has cardinality one or two. In other words, every such $T$ is (non-uniquely) determined by a simplex in the nerve of $\Delta$ satisfying a property corresponding to the valency property on the planar rooted tree. As the preimage of an degeneracy map is either one or two, every nonempty totally ordered set $[n]$ gives a "standard" planar rooted tree:

$$[n] \xrightarrow{\sigma_n} [n-1] \dots \xrightarrow{\sigma_0} [0]$$

Therefore, we will at times abusively omit reference to the above planar rooted tree structure, and simply write:

$$\bigotimes_{i \in [n]} A_i$$

or

$$A^{\otimes [n]}$$

In the coming discussion, for ease of exposition, we will refer to such a $T$ as an ordered parenthesized set.

Moreover, the symmetric monoidal structure "interpolates" between the various orderings and parenthesizations. More formally, it gives us, for any $(T, A)$, $(\tilde{T}, \tilde{A})$, along with a bijection:

$$\langle T \rangle \overset{\sigma}{\simeq} \langle \tilde{T} \rangle$$

so that:

$$\sigma^*(A) = \tilde{A}$$

an isomorphism:

$$\bigotimes_{i \in T} A_i \simeq \bigotimes_{j \in \tilde{T}} \tilde{A}_j$$

Maclane's coherence theorem asserts that these morphisms assemble into a contractible groupoid. In a more informal language, "all possible diagrams commute". Therefore, for ease of notation, our exposition will not explicitly label these isomorphisms whenever it is clear from the context.

There is an analogous story at the level of morphisms. Given a $T$ , along with a morphism for every $i \in \langle T \rangle$:

$$A_i \xrightarrow{f_i} B_i$$

We obtain a map:

$$\bigotimes_{i \in \langle T \rangle} f_i$$

with source $\bigotimes A_i$ and target $\bigotimes B_i$.

As above, this process is natural with respect to reordering and reparenthesization, in the obvious fashion

## 2.2 Geometric Input

**Definition 5.** Let $\mathrm{Mfld}_1^{\mathcal{R}}$ denote the category of oriented Riemannian 1-manifolds without boundary, and oriented isometric embeddings therebetween. We will view

$\mathrm{Mfld}_1^{\mathcal{R}}$ as a symmetric monoidal category under disjoint union.

We now hone in a certain subcategory containing basic regions.

**Definition 6.** Let $\mathrm{Disk}_1^{\mathcal{R}}$ denote the full symmetric monoidal subcategory of $\mathrm{Mfld}_1^{\mathcal{R}}$ containing finite disjoint unions of open intervals of *finite* diameter. For any $R > 0$ we let:

$$B_R := (-R_i, R_i)$$

denote the (unique up to unique isomorphism in $\mathrm{Disk}_1^{\mathcal{R}}$) Riemannian 1-manifold of volume $2R$. We denote its midpoint as $m_R \in B_R$.

*Remark* 7. The finite diameter condition implies the existence of an essential feature: every object of $\mathrm{Disk}_1^{\mathcal{R}}$ has a notion of a (or a configuration of) midpoint(s).

*Remark* 8. The evaluation at the midpoint (of each connected component) of an embedding $\iota : U \to M$ with $U \in \mathrm{Disk}_1^{\mathcal{R}}$ determines a configuration of points in $M$. Moreover, every finite configuration of points in $M$ may be obtained in this manner by some sufficiently sized $U \in \mathrm{Disk}_1^{\mathcal{R}}$.

*Remark* 9. Every object in $\mathrm{Disk}_1^{\mathcal{R}}$ is equivalent to the data of a finite ordered, parenthesized set labeled by positive real numbers. We will denote such a labeled parenthesized ordered set as $(T, R)$, where $T$ denotes the underlying ordered parenthesized set, and

$$\langle T \rangle \xrightarrow{R} (0, \infty) \hookrightarrow \mathrm{ob}(\mathrm{Disk}_1^{\mathcal{R}})$$
$$i \longmapsto R_i \longmapsto B_{R_i}$$

its labelling by objects. Such data gives rise to an object:

$$B_{(T,R)} := \coprod_{i \in T} B_{R_i} \in \mathrm{Disk}_1^{\mathcal{R}}$$

As the geometry is so rigid, when the source is connected, evaluation at the midpoint permits us to view every map as a point in the target satisfying a condition expressible in terms of its metric geometry. More formally, we have a bijection:

$$\mathrm{Disk}_1^{\mathcal{R}}(B_r, B_R) \simeq \{p \in B_R | d(p, m_R) \leq R - r\}$$

More generally, evaluation at the midpoint of each connected component gives a bijection:

$$\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,r)}, B_R) \simeq \{x : \langle T\rangle \to B_R | d(x_i, x_j) \geq r_i + r_j, d(m_R, x_i) \leq R - r_i, \forall i, j \in \langle T\rangle\}$$

And most generally:

$$\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,r)}, B_{(T',R)}) \simeq \coprod_{f:\langle T\rangle \to \langle T'\rangle} \Big( \coprod_{\substack{j\in\langle T'\rangle \\ f^{-1}(j)\neq\emptyset}} \mathrm{Disk}_1^{\mathcal{R}}(B_{([f^{-1}(j)],r)}, B_{R_j}) \Big)$$

Where we are omitting any parethesization of $[f^{-1}(j)]$ (which has an ordering induced from the $[T]$), as the identification above makes it clear that $\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,R)}, B_R)$ need not make any explicit reference to the ordering or parenthesization.

*Remark* 10. As the previous remark indicates, the orientation and metric give a closed inclusion:

$$\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,r)}, B_R) \hookrightarrow \mathrm{Conf}_{\langle T\rangle}(\mathrm{T}_{m_R} B_R) \simeq \mathrm{Conf}_{\langle T\rangle}(\mathbb{R})$$

into the space of $\langle T\rangle$-labeled configuration spaces. Here, the ordering is induced by the orientation on $B_R$.

**Example 11.** Seeing the set of maps as sitting inside ordered configuration spaces gives the composition a straightforward algebraic description. For example:

$$\Big(\prod_{i\in T}\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,r^i)},B_{R_i})\Big)\times\mathrm{Disk}_1^{\mathcal{R}}(B_{(T,R)},B_{r'})\to\mathrm{Disk}_1^{\mathcal{R}}(B_{\tilde{T}_R},B_r)\Big)$$

$$(t_{j_i},t_i)\longmapsto(t_{j_i}+t_i)$$

Where $\tilde{T}$ is obtained from grafting $(T^i)$ to $T$.

*Remark* 12. This description only holds for a subspace of configurations. In other words, it does not extend to a map on ordered configuration spaces.

## 2.3 Riemannian 1-Disk Algebras

*Remark* 13. Although the definitions in this section will hold for an arbitrary symmetric monoidal $\infty$-category $\mathcal{C}^{\otimes}$, we will restrict our attention to the symmetric monoidal $\infty$-category of chain complexes over $\mathbb{R}$, $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$, as this will be the case relevant to the task at hand. We will refer to the ordinary symmetric monoidal category of chain complexes over $\mathbb{R}$ as $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$. The author does not expect those initiated in the theory of $\infty$-categories to have any difficulties extending these constructions to more general contexts.

**Definition 14.** A *Riemannian 1-disk algebra* (with values in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$) is a symmetric monoidal functor:

$$\mathrm{Disk}_1^{\mathcal{R}}\xrightarrow{\mathcal{A}}\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

*Remark* 15. We invite the reader less aquainted with defining algebraic objects in this manner to consult section 3.2 of [CG1].

*Remark* 16. As the previous remarks indicates, the set of maps in $\mathrm{Disk}_1^{\mathcal{R}}$ admits an enrichment in smooth manifolds with corners, foreshadowing a smoothly varying enhancement of the above definition. We will not pursue this line of reasoning, and regard $\mathrm{Disk}_1^{\mathcal{R}}$ as an ordinary symmetric monoidal category.

*Remark* 17. As the reader can imagine, one can repeat the above definition, replacing 1 by any finite number, $n$. In this case, the value of any such an $\mathcal{A}$ on an open ball admits a natural representation of $\mathrm{SO}(n)$. Therefore, for a general $n$, such an object contains strictly more data than an algebra over the multicategory of open sets of $\mathbb{R}^n$ homeomorphic to a finite disjoint union of $\mathbb{R}^n$.

*Remark* 18. We invite the reader to imagine that some model of a statistical (or quantum) system should give rise to such an algebra. Adopting this perspective, the value of $\mathcal{A}$ on an interval parametrizes some (physically meaningful) observables of that system taking place within that interval.

Following the Wilsonian/renormalization perspective on observables, every observation corresponds to an (or multiple) "external" system(s), localized within a finite region, and interacting weakly with the system of interest in a prescribed manner. This interaction "influences" some aspect of the external system. A description of this alteration via a single numerical quantity constitutes an observation. For more on this perspective, we invite the reader to consult 12-8 of [FH] on "Influence Functionals" and 2.2.3 of [W] on "Effective Field Theory".

We can view multiple of external systems as a constituting an external system localized within a larger region. As it would be unwise to attempt to co-locate these systems, we may assume each of these systems as being contained within disjoint regions of space. As addition admits a similar interpretation, we adopt the perspective that the value of $\mathcal{A}$ on an embedding constitutes a type of algebraic operation.

A general Riemannian 1-disk algebra may be difficult to work with. Therefore, we introduce the following condition, which is satisfied in many interesting examples.

**Definition 19.** A Riemannian 1-disk algebra $\mathcal{A}$ is *locally constant* if the image of any map in $\mathrm{Disk}_1^{\mathcal{R}}$, whose source and target have a single connected component, is an equivalence.

*Remark* 20. A folk attitude towards these objects asserts that if we are thinking of such an object as describing a field theory, the locally constant condition means that we are talking about a topological field theory. To the author's knowledge, this statement is not formally justified anywhere within the literature (the notion of a topological field theory is much more subtle that Atiyah's axiomatization may make it appear).

The author speculates the reason for this mistake lies in a misinterpretation of a (mathematically rigorous and well-defined) result of Lurie's, asserting an equivalence of homotopy theories between "locally constant factorization algebras on $\mathbb{R}^n$" and the classical homotopy theory of $\mathrm{E}_n$-algebras. For sake of continuity, we will postpone our discussion of factorization algebras for later chapters, and present one reason why such a slogan cannot be true.

Suffice it to say that a factorization algebra in this sense is analogous to a (co)sheaf on a *fixed manifold*. This notion, is different than a (co)sheaf on the *site of manifolds*. For example, the maps within the latter category demand that the value of any (co)sheaf on $\mathbb{R}^n$ inherits, an action of the monoid $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$. As any category of open sets is a poset, the values a (co)sheaf on this category need not admit an action of any symmetry groups. Of course, the restriction of a (co)sehaf on manifolds to a fixed manifold still contains a wealth of information.

Most notions of a field theory (which is a notion in physics), on the other hand, demand that its degrees of freedom admit an action of the symmetries of the context at hand. In fact, the construction of irreducible representations of the Poincare group historically coerced the physics community into adopting field-theoretic methods. This requirement becomes even more important in phenomenological/effective field theory approaches to statistical field theory, where equivariance conditions makes the admissible classes of Hamiltonian densities tractable. See chapter 2 of [K] for an elegant treatment of this approach.

Whatever one means by a topological field theory, its local degrees of freedom should come equipped with an action of some "large" subgroup of the relevant diffeomorphism group. Note that a general $\mathrm{E}_n$-algebra does not necessarily come with an action of any such subgroup. Therefore, considering Lurie's equivalence, a factorization algebra is, at best, a shadow containing incomplete (but highly interesting!) data of a field theory.

*Remark* 21. On a concrete level, the local constancy condition first states that every cocycle in $\mathcal{A}(U)$ is equivalent to one contained within an arbrarily small region of any point in $U$, whenever $U$ is connected. This is the surjectivity part of the condition. Second, if the extension of two cocylces to a larger region are equivalent, they must have been equivalent to begin with. This is the injectivity part of the condition.

If we are imagining $\mathcal{A}$ as parametrizing the data of the observables of a field theory, these conditions are dual to the existence and uniqueness of extensions to solutions of the relevant equations of motions (at least at the at the classical level). Therefore, the author asserts that, when the algebra in question is constructed from differential geometry, the question of whether such an algebra is locally constant should be unravelable to a classical analysis question, via some (algebraic) incarnation of the inverse function theorem.

### 2.4 A Family of Examples of such Algebras

In this section, we will construct a broad family of examples of Riemannian 1-disk algebras. This takes as input two pieces of data.

- A unital differential graded associative algebra, $A$ in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$, with multiplications for every finite totally ordered parenthesized set $T$:

  $$\mu_A^T : A^{\otimes T} \to A$$

  Depending on the context, we may omit the dependency of $\mu$ on $T$. For simplicity, we assume all associativity conditions hold on the nose. As our upcoming theorems will be homotopy invariant, standard rigidification results imply this assumption involves no loss of generality. Note that this is implied by stating that $A$ is an algebra in the ordinary category $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$.

- A one-parameter group of automorphisms of $A$:

$$\begin{aligned} \varphi : (\mathbb{R}, +) &\longrightarrow \mathrm{Aut}_{\mathrm{Alg}_{\mathbb{R}}}(A) \\ t &\longmapsto \varphi_t \end{aligned}$$

  In other words, $\varphi_t$ is an algebra automorphism for every $t \in \mathbb{R}$, and

$$\varphi_t \circ \varphi_s = \varphi_{s+t}$$

*Remark* 22. The standard mathematical examples of $A$ to keep in mind are a polynomial algebra on even number of generators (position/momentum), or the universal enveloping algebra of the Heisenberg Lie algebra associated to a symplectic vector space. In other words, the physically meaningful observables of some physical system. In this context, $\varphi$ should be thought of as encoding how outcomes of a fixed measurement change in time. The essential ingredients of such automorphisms are supplied by the solutions to some well-posed ordinary differential equation through the analytic theory of semigroups.

*Remark* 23. Those familar with quantum mechanics can imagine such an input arises from self-adjoint operators on a Hilbert space, along with a group of automorphisms determined by a Hamiltonian, via the Heisenberg equation of motion [SN], [PS]. Therefore, although this input may appear "explicit," when the Hamiltonian involves an interaction/nonlinear term, an explicit description of the automorphism requires solving a potentially intractable ordinary differential equation.

As these problems are conventionally addressed using perturbation theory, one can imagine that an interesting line of inquiry would be to explore examples which are defined in chain complexes over some nilpotent $\mathbb{R}$-algebra.

### 2.4.1 A Technical Categorical Digression

We will eventually be interested in constructing homotopical invariants, and therefore desire a functor with values in the symmetric monoidal infinity category of chain complexes $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$. This $\infty$-category has an ecosystem of equivalent constructions. All of these constructions have the property that there exists a canonical symmetric monoidal functor:

$$\mathrm{Ch}_{\mathbb{R}}^{\otimes} \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

which is a localization sending quasi-isomorphisms in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$ to equivalences in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$.

As this indicates, a convenient way to construct a Riemannian 1-disk algebra with values in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$ is to first construct a symmetric monoidal functor of ordinary categories:

$$\mathrm{Disk}_1^{\mathcal{R}} \to \mathrm{Ch}_{\mathbb{R}}^{\otimes}$$

and compose it with the above functor. This method has the advantage that the data and conditions of constructing a symmetric monoidal functor "a la Maclane" are significantly fewer in number.

### 2.4.2 The Construction

We now combine our geometric and algebraic/analytic input. This is nothing more than the potentially familiar notion of a time-ordered product.

*Remark* 24. For ease of exposition, we pause to establish some simplifying notation. First, note that a map:

$$B_{(T,r)} \xrightarrow{x} B_R$$

along with the orientation on the target, gives the set $T$ a total ordering, which we will denote $[T]_x$.

Second, note that every such map can be labeled by a $T$-labelled collection of numbers, $(t_i)_{i\in T}$. Here, each $t_i \in \mathbb{R}$ is the unique number so that under the Riemannian exponential map (which also identifies $\mathbb{R}$ with the tangent space at the midpoint, $m_R$ of $B_R$):

$$\exp_{m_R}(t_i) = x(m_{r_i})$$

Finally, note that a general map, $B_{(T,R)} \xrightarrow{x} B_{(T',R')}$ decomposes $T$ into:

$$\langle T\rangle \simeq \coprod_{j\in T'} x^{-1}(j)$$

Here, we are conflating $x$ and $\pi_0(x)$. Therefore, $x$ gives an equivalence:

$$B_{(T,R)} \overset{\sigma_x}{\simeq} \coprod_{j\in T'} B_{[x^{-1}(j)]_R}$$

By convention, $B_\emptyset = \emptyset$. Under this equivalence, $x$ decomposes as:

$$x = \sigma_x^*\Big(\coprod_{j\in T'} (B_{[x^{-1}(j)_R]_x} \xrightarrow{x_j} B_{r_j})\Big)$$

as discussed in section 2.1.

**Example 25.** Heuristically, one should interpret $x$ notation as encoding where an embedding sends the midpoint of the source ($B_{(T,r)}$), and the $t$'s encode the displacement from $x$ to the midpoint of the target ($B_R$).

Under this notation, the inclusion $(0,4) \hookrightarrow (0,8) = B_R$ has $t = -2$, as it sends the midpoint of $(0,4)$ two units the to left of the midpoint of $(0,8)$.

With this in place, we can define our primary antagonist.

**Definition 26.** Let

$$\mathcal{A}^\varphi : \mathrm{Disk}_1^{\mathcal{R}} \to \mathrm{Ch}_{\mathbb{R}}^{\otimes}$$

denote the symmetric monoidal functor, whose value on

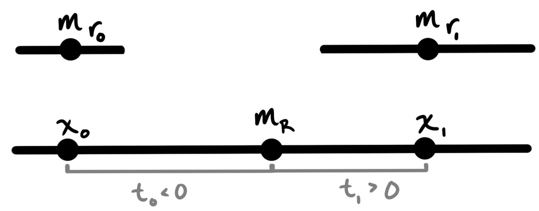



Figure 2.1. An example of our convention for maps in $\mathrm{Disk}_1^{\mathcal{R}}$

1. **Objects** is:
$$\mathcal{A}^{\varphi}(B_{(T,r)}) = A^{\otimes T}$$

2. **Morphisms** of the type, $B_{(T,r)} \xrightarrow{x} B_R$, is defined as the composition:
$$A^{\otimes T} \simeq A^{\otimes [T]_x} \xrightarrow{\bigotimes \varphi_{-t_i}} A^{\otimes [T]_x} \xrightarrow{\mu_A^{[T]_x}} A$$
For a general morphism, $B_{(T,r)} \xrightarrow{x} B_{(T',R)}$, we use the decomposition above, and defined as:
$$\bigotimes_{j \in T'} \mathcal{A}^{\varphi}(x_j)$$

*Remark* 27. The author finds the following narrative helpful. The elements of $\mathcal{A}^{\varphi}(B_R)$ are imagined as being elements of $A$ which sit at the midpoint of $B_R$. The action map evaluates on an inclusion by first using $\varphi$ to move elements of $A$ to the midpoint of the target intervals, and then multiplies everything using $\mu_A$.In particular, the structure maps have no explicit dependence on the diameter of the intervals.

More specifically, given any $a \in A$ and $\iota : B_r \to B_R$, $\mathcal{A}^{\varphi}(\iota)(a)$ is the unique solution to:
$$\varphi_{\iota(0)=t}\Big(\mathcal{A}^{\varphi}(\iota)(a)\Big) = a$$

In order to clarify the above formulas, we now go through some examples of $\mathcal{A}^{\varphi}$'s behavior.

*Remark* 28. Note that the exponential map based at the midpoint of an open interval $(a, b) \subset \mathbb{R}$ as an object of $\mathrm{Disk}_1^{\mathcal{R}}$.

**Example 29.** $\mathcal{A}^\varphi$ evaluates on the inclusion $(0,2) \to (-2,2)$ as:

$$\begin{aligned} A &\longrightarrow A \\ a &\longmapsto \varphi_{-1}(a) \end{aligned}$$

The value on $(-4,-2) \amalg (0,4) \to (-4,4)$ is given by:

$$\begin{aligned} A \otimes A &\longrightarrow A \\ a \otimes b &\longmapsto \varphi_3(a) \cdot \varphi_{-2}(b) \end{aligned}$$

The value on the composition

$$\mathcal{A}^\varphi\Big((-4,-2) \amalg (-2,2) \amalg (2,4) \hookrightarrow (-4,2) \amalg (2,4) \hookrightarrow (-4,4)\Big)$$

can be seen to be:

$$a_0 \otimes a_1 \otimes a_2 \mapsto (\varphi_2(a_0) \cdot \varphi_{-1}(a_1)) \otimes a_2 \mapsto \varphi_1(\varphi_2(a_0) \cdot \varphi_{-1}(a_1)) \cdot \varphi_{-3}(a_2)$$

While the value on the composition

$$\mathcal{A}^\varphi\Big((-4,-2) \amalg (-2,2) \amalg (2,4) \hookrightarrow (-4,2) \amalg (2,4) \hookrightarrow (-4,4)\Big)$$

may be computed as:

$$a_0 \otimes a_1 \otimes a_2 \mapsto a_0 \otimes (\varphi_1(a_1) \cdot \varphi_{-2}(a_2)) \mapsto \varphi_3(a_0) \cdot \varphi_{-1}((\varphi_1(a_1) \cdot \varphi_{-2}(a_2))$$

Both of which agree with the evaluation of $\mathcal{A}^\varphi$ on their composite:

$$\mathcal{A}^\varphi\Big((-4,-2) \amalg (-2,2) \amalg (2,4) \hookrightarrow (-4,4)\Big)$$

$$a_0 \otimes a_1 \otimes a_2 \mapsto \varphi_3(a_0) \cdot a_1 \cdot \varphi_{-3}(a_2)$$

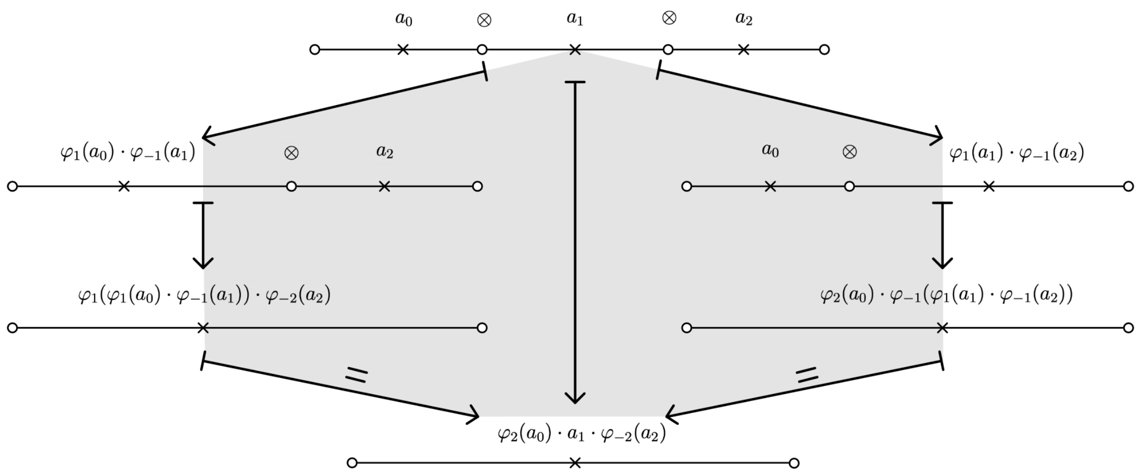



Figure 2.2. A commutative diagram in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$. This illustrates how the properties of $\varphi$ lead to the "associativity" of $\mathcal{A}^\varphi$. Setting $\phi = \mathrm{Id}$ clarifies the motivation behind the term "associativity".

*Remark* 30. The last two examples demonstrate how the algebraic properties of $\varphi$ account for the functoriality, or "associativity" of $\mathcal{A}^\varphi$.

**Example 31.** First, we fix a finite set $J$, equipped with a map:

$$E : J \to \mathbb{R}^{\geq 0}$$

$$j \mapsto E_j$$

From this data, we map obtain a $2|J|+1$ dimensional complex vector space:

$$\mathbb{C}\cdot\{c\}\oplus\bigoplus_{j\in J}\mathbb{C}\cdot\{a(j),a^{\dagger}(j)\}$$

Where $\mathbb{C}\cdot\{S\}$ denote the free complex vector space generated by a set $S$. In particular, $\mathbb{C}\cdot\{a(j),a^{\dagger}(j)\}$ is just the two dimensional complex vector space with distinguished basis vectors $a(j),a^{\dagger}(j)$.

We can endow this vector space with the structure of a Lie algebra, given by:

$$[a(k),a^{\dagger}(j)]=\delta_{kj}c,$$
$$[a(k),a_(j)]=0=[a^{\dagger}(k),a^{\dagger}(j)]$$

and $c$ is central. Those familiar with this approach will recognize the above as a Heisenberg Lie algebra construction.

This Lie algebra has a one parameter family of automorphisms, determined by $E$, which we will denote by $\varphi$. On generators, this behaves as:

$$\varphi_t(a(j))=e^{-iE_jt}a(j)$$
$$\varphi_t(a^{\dagger}(j))=e^{iE_jt}a^{\dagger}(j)$$

In other words, it is the semi-group of operators associated to the ordinary differential equations:

$$\frac{d}{dt}a(j)=-iE_ja(j)$$
$$\frac{d}{dt}a^{\dagger}(j)=iE_ja^{\dagger}(j)$$

Let $\mathcal{W}_J$ denote the universal enveloping algebra the of the above Lie algebra with $c=1$. The above semigroup of operators extends to $\mathcal{W}_J$ by functoriality. Note that we can use the Lie bracket associated to $\mathcal{W}_J$ to write down the following ordinary differential equation for any $O \in \mathcal{W}_J$:

$$\frac{d}{dt}O = \frac{1}{i}\sum_J [O, E_j \cdot a(j)a(j)^\dagger]$$

which agrees with the previous ODE when $O$ is $a(j)$ or $a^\dagger(j)$. This is the familiar Heisenberg equations of motion.

The semigroup of operators on $\mathcal{W}_J$ arising from the above ODE agrees with the one obtained via the functorial extension of $\varphi$. This follows because they agree infinitesmally on generators.

For example a straightforward computation shows:

$$[a(j)a^\dagger(j), (a^\dagger(k))^n] = n \cdot (a^\dagger(k)^n \delta_{jk}$$
$$[a(j)a(j)^\dagger, a(k)^n] = -n \cdot (a(k)^\dagger)^n \delta_{jk}$$

For this reason, we will adopt the notation:

$$N_j = \sum_J a^\dagger(j)a(j)$$

as it "counts (with signs) the number" of powers of the generators. Therefore, the generator of our semigroup is:

$$[H_E, -] = [E_j N_j, -] = \sum_J E_j(a^\dagger(j)\frac{\partial}{\partial a^\dagger(j)} - a(j)\frac{\partial}{\partial a(j)})$$

So that:

$$[H_E, (a^\dagger(j))^n] = (nE_k)a^\dagger(k)$$

This algebra admits another famous basis, given as:

$$a(j)^\dagger = \sqrt{\frac{\omega}{2}}\Big(p_j - i\frac{x_j}{\omega}\Big), \qquad a_(j) = \sqrt{\frac{\omega}{2}}\Big(p_j + i\frac{\omega}{x_j}\Big)$$

where (following De Broglie), $E_j = \omega_j$. A straightforward computation shows that:

$$\begin{aligned} [x_j, p_k] &= \delta_{jk} \\ H_E &= \frac{1}{2}\sum_J p_j^2 + \omega_j^2 x_j - \frac{\omega}{2} \\ [H_E, x_j] &= p \\ [H_E, p_j] &= -\omega x \end{aligned}$$

from which we can see that the above models a quantization of $J$ harmonic oscillators oscillating at frequency $\omega_j$, with $\hbar = 1$. Moreover, in this basis, the automorphism acts as:

$$\varphi_t(x_j) = x_j \cos(\omega t) + \frac{p_j}{\omega}\sin(\omega t)$$

and

$$\varphi_t(p_j) = -\omega x_j \sin(\omega t) + p_j \cos(\omega t)$$

As we have written down an algebra with an automorphism, we obtain an $\mathrm{Disk}_1^{\mathcal{R}}$-algebra by the above construction.

Finally, we adopt a notation for how to twist the action maps of some $(A, A)$-bimodule $(M, \mu_M)$ by $\varphi$. We will use this notation in later chapters.

**Definition 32.** Given any $L, L' \in \mathbb{R}$. Let ${}_L M_{L'}$ be the bimodule in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$, whose underlying object is $M$, and with multiplication maps:

$$\begin{aligned} A^{\otimes[n]} \otimes M \otimes A^{\otimes[k]} &\longrightarrow M \\ a_{[n]} \otimes m \otimes a_{[k]} &\mapsto \varphi_L \mu^{[n]}(a_{[n]}) \cdot m \cdot \varphi_{L'} \mu_A^{[k]}(a_{[k]}) \end{aligned}$$

That is ${}_L M_{L'}$ is the image of $M$ under the pullback functor:

$$(\varphi_L, \varphi_{L'})^* : (A, A)\text{−Mod} \to (A, A)\text{−Mod}$$

We will denote this new action as ${}^L\mu_M^{L'}$.

### 2.5 Associative Algebras and Modules Thereover

The goal of this section is to relate the notions described above to more familiar algebraic structures: unitial associative algebras, and unital left/right modules thereover. The reader may consult [AF] for a detailed treatment.

Although these categories appear elementary, a thorough understanding of their basic features is a necessary prerequisite for anything beyond a formal understanding of how they are used. Therefore, we request the reader to tolerate a slow, convulsive treatment of the subject. A conceptual and honest treatment of this subject requires certain higher categorical notions that may feel unnecessarily technical and opaque at first. However, the author hopes to convey a sense in which these methods are both technically advantageous and conceptually natural. We invite the reader only interested in the formal statements to ignore the upcoming remarks.

#### 2.5.1 One Dimensional Manifolds with Boundary

Written representations and manipulations of expressions within an associative algebra are inherently one dimensional and oriented. The written language of left and right modules over an associative algebra are similarly oriented and one-dimensional, although their intrarelationships are vastly more restricted than in the case of an associative algebra. As we hope to argue, the grammar of these manipulations are controlled by the differential topology of oriented 1-manifolds with boundary.

Although these formal expressions have typographical properties, these characteristics have no mathematical meaning. More specifically, the formal meaning of these written string of symbols are insensitive to kerning and font size. As we hope to argue, this insensitivity may be accounted for by an abstract homotopy theory of oriented 1-manifolds with boundary.

With this in mind, we will first construct an ordinary category overapproximating the theory of associative algebras and unital left and right modules thereover. As we'll see, this ordinary category contains a variety of data which makes this relationship somewhat awkward. We then introduce an $\infty$-category remedying this awkardness.

**Definition 33.** Let $\mathrm{Mfld}_1^{\mathrm{or},\partial}$ denote the category of (possibly empty) smooth, one dimensional oriented manifolds with boundary and smooth oriented embeddings which send boundary components of the source to boundary components of the target. Disjoint union endows this category with a symmetric monoidal structure.

Let $\mathrm{Disk}_1^{\mathrm{or},\partial}$ denote the full symmetric category containing finite disjoint unions of $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, and $\mathbb{R}^{\leq 0}$.

Similiarly, let $\mathrm{Mfld}_1^{\mathrm{or}}$ and $\mathrm{Disk}_1^{\mathrm{or}}$ denote the full symmetric monoidal subcategories of $\mathrm{Mfld}_1^{\mathrm{or},\partial}$ and $\mathrm{Disk}_1^{\mathrm{or},\partial}$ , respectively, containing those oriented manifolds without boundary.

*Remark* 34. For the time being, we are less interested in the specifics of $\mathrm{Disk}_1^{\mathrm{or},\partial}$ per se, as we are in the *data it corepresents*. Therefore, we need to address the questions: what data is contained in a symmetric monoidal functor out of $\mathrm{Disk}_1^{\mathrm{or},\partial}$? How can we interpret this data algebraically? Obviously we have no hope in addressing this question without a thorough understanding of the maps in $\mathrm{Disk}_1^{\mathrm{or},\partial}$.

**Definition 35.** Throughout this section, we will refer to a map in either $\mathrm{Mfld}_1^{\mathrm{or},\partial}$ or $\mathrm{Disk}_1^{\mathrm{or},\partial}$ as an *admissible* embedding.

*Remark* 36. Note that there exists no admissible embeddings from either $\mathbb{R}^{\geq 0}$ or $\mathbb{R}^{\leq 0}$ to $\mathbb{R}$ or between $\mathbb{R}^{\geq 0}$ or $\mathbb{R}^{\leq 0}$. The source of any admissible embedding into an object with boundary must have at most one boundary component. For example, there are no maps from $\mathbb{R}^{\geq 0} \amalg \mathbb{R}^{\geq 0}$ into $\mathbb{R}^{\geq 0}$ or $\mathbb{R}$. There exists unique maps from $\emptyset$ (the symmetric monoidal unit) into $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^{\leq 0}$.

Although there are potentially infinite number of maps between two objects of $\mathrm{Disk}_1^{\mathrm{or},\partial}$, any two maps which agree on $\pi_0$ are isotopic, when considered as habitating within a subspace of the compact-open topology.

*Remark* 37. The image of $\mathbb{R}$ under an admissible embedding of the form:

$$\mathbb{R}^{\geq 0} \amalg \mathbb{R} \to \mathbb{R}^{\geq 0}$$

must be to the right of the image of $\mathbb{R}^{\geq 0}$, while it's image under an admissible embedding of the form :

$$\mathbb{R} \amalg \mathbb{R}^{\leq 0} \to \mathbb{R}^{\leq 0}$$

must be to the left of the image of $\mathbb{R}^{\leq 0}$. Note that any pair of admissible embeddings in $\mathrm{Disk}_1^{\mathrm{or},\partial}$ between $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, $\mathbb{R}^{\leq}$, with a common source and target are isotopic.

*Remark* 38. Note that every admissible embedding of the form:

$$\coprod_I \mathbb{R} \to \mathbb{R}$$

gives $I$ a total ordering. Any two such admissible embeddings are isotopic if and only if their induced linear orderings agree. In other words, the isotopy class of such a map contains precisely the data of a linear ordering of the connected components of the source. Recall that given an $I$-indexed collection of elements of an associative algebra, a unique multiplication of these elements is defined upon a choice of total ordering of $I$.

*Remark* 39. $[-1,1]$ is *not* an object of $\mathrm{Disk}_1^{\mathrm{or},\partial}$. Moreover, every admissible embedding of the form:

$$\coprod_I \mathbb{R} \hookrightarrow [-1,1]$$

factors through an admissible embedding of the form:

$$\mathbb{R}^{\geq 0} \amalg (\coprod_I \mathbb{R}) \amalg \mathbb{R}^{\leq 0} \hookrightarrow [-1,1]$$

Although this extension is not unique, any two such extensions are isotopic.

*Remark* 40. A common typographical representation of an map of the form:

$$\mathbb{R}^{\geq 0} \amalg \Big(\coprod_{i=1}^{2} \mathbb{R}\Big) \amalg \mathbb{R}^{\leq 0} \overset{\iota}{\hookrightarrow} [-1,1]$$

(which is an element of $\mathrm{Disk}_{1/[-1,1]}^{\mathrm{or},\partial}$) is:

$$-|-|-|-$$

where the bars correspond to the connected components of the complement of $\iota$. Although the bar notation does not uniquely determine $\iota$, it does uniquely determine it's isotopy class.

The "$-$" are to be labelled by elements of either a right module (when it is the right most slot), a left module (when it is left most slot), or an associative algebra (when it is in the middle slot). A map in $\mathrm{Disk}_{1/[-1,1]}^{\mathrm{or},\partial}$ that is also a surjection on connected components is then modelled by the removal of a bar, and a concatenation of the slots. When the slots are labelled by the above algebraic data, the labels within concatenated slots multiply in the obvious fashion.

As this suggests, we should think of $\mathbb{R}$ as the canvas labelled by elements of an associative algebra, $\mathbb{R}^{\geq 0}$ as the canvas labelled by elements of an left module, and $\mathbb{R}^{\leq 0}$ as the canvas labelled by elements of a right module. Maps inducing surjections on connected componenst corepresent multiplication maps.

This notation may be formalized by packaging this data as a functor out of $\Delta^{\mathrm{op}}$. On objects,

$$[n] \to N \otimes A^{\otimes n} \otimes M$$

where $A$ is an associative algebra, and $M$ and $N$ are left and right modules over $A$, respectively.

*Remark* 41. In some sense, this (unspoken) relationship between $\mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[-1,1]}$ and associative multiplications precedes the formal Bourbaki approach to algebra, and is arguably the greatest achievement in graphic design.

*Remark* 42. In order to more directly relate $\mathrm{Disk}^{\mathrm{or},\partial}_{1/[-1,1]}$ to bar constructions, one might like to construct some functor:

$$\Delta^{\mathrm{op}} \to \mathrm{Disk}^{\mathrm{or},\partial}_{1/[-1,1]}$$

Unraveling the general "slice" construction (reviewed in A), one recognizes that taking the image of the admissble embedding witnesses the target category as equivalent to the partially ordered set of *proper* subsets of $[-1, 1]$. The previous remark suggests this functor should evaluate as some map of the form:

$$[n] \longmapsto \Big(\mathbb{R}^{\geq 0} \amalg (\coprod_{i=1}^{n} \mathbb{R}) \amalg \mathbb{R}^{\leq 0} \hookrightarrow [-1, 1]\Big)$$

Unfortunately, a cursory examination shows that this assignment has no chance of being functorial, primarily to due either the placement or size of an interval. Moreover, no clever choice of embeddings can address these obstructions to defining a point-set level functor. Somehow, the ordinary category $\mathrm{Disk}_1^{\mathrm{or},\partial}$ fails to account for the insensitivity of the bar notation to our (most likely unintentional) choice of kerning and fontsize (which together determined the placement of the interval).

However, these identities can be coerced into holding, if we allow ourselves to suitably guide our embeddings through isotopies. More precisely, all the desired equations required for it's functoriality do hold, *up to specified homotopy* (which, in this case, is synonymous with isotopy)! Therefore, if we are in a setting in which $\mathrm{Mfld}_1^{\mathrm{or},\partial}$ contains the data of the compact-open topology, and functors need only be natural up to specified homotopy, we'd be able to construct such a functor. This is accomplished using the theory of topologically enriched categories (which will allow us to encode the compact-open topology) and their relationship to $\infty$-categories, via the homotopy coherent nerve construction (which will allow us to rigorously manipulate equations which hold "up to higher coherent homotopy").

**Definition 43.** Let $\mathcal{M}\mathrm{fld}_1^{\mathrm{or},\partial}$ denote the symmetric monoidal topologically enriched category whose underlying symmetric monoidal category is $\mathrm{Mfld}_1^{\mathrm{or},\partial}$, along with the compact-open topology on the space (or a subspace of) the set of embeddings.

Let $\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$ denote the full symmetric monoidal topologically enriched subcategory containing finite disjoint unions of $\mathbb{R}$, $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^{\leq 0}$.

Let $\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$ and $\mathcal{M}\mathrm{fld}_1^{\mathrm{or}}$ denote the obvious subcategories of $\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$ and $\mathcal{M}\mathrm{fld}_1^{\mathrm{or},\partial}$, respectively, containing those oriented manifolds without boundary.

The homotopy coherent nerve functor allow us to view these as $\infty$-categories. Following standard conventions in the literature, our notation will not differentiate between these two perspectives, despite the annoyance the convention generated within focus groups.

*Remark* 44. As reviewed in rapid introductions to $\infty$-categories [Gr], one can assign a topological space of maps to every ordered pair of objects of an $\infty$-category (via a pullback construction). Although there are many different choices for such topological spaces, the abstract theory guarantees the existence of a map relating these choices, all of which are homotopy equivalences.

Moreover, the homotopy coherent nerve construction satisfies the property that the topological space of maps provided by the enrichment is a perfectly suitable model for the mapping spaces of its associated $\infty$- category . Therefore, we do not lose the useful tools of differential topology by viewing $\mathcal{M}\mathrm{fld}_1^{\mathrm{or},\partial}$ or $\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$ as an $\infty$-categories. As we will see, this perspective will streamline the importation (and therefore application) of categorical and algebraic tools.

*Remark* 45. In summary, note that there exists a commutative diagram of $\infty$-categories:

$$\begin{tikzcd} \mathrm{Disk}_1^{\mathrm{or}} \arrow[r] \arrow[d, hook] & \mathrm{Disk}_1^{\mathrm{or},\partial} \arrow[r] \arrow[d, hook] & \mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial} \arrow[d, hook] \\ \mathrm{Mfld}_1^{\mathrm{or}} \arrow[r] & \mathrm{Mfld}_1^{\mathrm{or},\partial} \arrow[r] & \mathcal{M}\mathrm{fld}_1^{\mathrm{or},\partial} \end{tikzcd}$$

where the second horizontal maps are essentially surjective and full, but not faithful. Note this diagram may be rigorously constructed by applying the homotopy coherent nerve construction on the obvious diagram of topologically enriched categories. Viewing this diagram as one of $\infty$-categories should signal the reader that we will transition into think of these as (co)representing *homotopy coherent diagrams.*

**Example 46.** Unraveling the homotopy coherent nerve construction shows that the data of a composition:

$$\Delta^2 \to \mathcal{M}\mathrm{fld}_1^{\mathrm{or},\partial}$$

is the data of three embeddings: $f : M_0 \to M_1$, $g : M_1 \to M_2$, $h : M_0 \to M_2$, along with an isotopy $\iota : f \circ g \simeq h$.

For example, there exists a factorization of

$$(-1, 1) \to [-1, 1]$$

through

$$[-1,-.5)\amalg(-.5,.5)\amalg(.5,1)\to[-1,1]$$

This factorization clearly does not exist in $\mathrm{Mfld}_1^{\mathrm{or},\partial}$.

Moreover, a standard, an elementary differential topology argument shows that the mapping spaces of $\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$ and $\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$ are discrete.

**Definition 47.** Corollary 2.33 of [AFT] states that the data of a symmetric monoidal functor out of $\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$ is precisely the data of a unital associative algebra $A$, and a unital left and right module thereover, $N$ and $M$, which we'll denote as:

$$\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}\overset{(M,A,N)}{\longrightarrow}\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

*Remark* 48. The associative algebra comes from $\mathbb{R}$, and the left and right modules come from $\mathbb{R}^{\leq 0}$ and $\mathbb{R}^{\geq 0}$, respectively. Similiarly, a symmetric monoidal functor out of $\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$ is precisely the data of an unital associative algebra.

We urge those readers having trouble intuitively grasping this relationship to consult Chapter 3, Section 2 and 3, of [G] for a more accessible treatment of these ideas.

*Remark* 49. As the above example suggests, there exists a functor (of $\infty$ categories):

$$\Delta^{\mathrm{op}}\to\mathcal{D}\mathrm{isk}_{1/[-1,1]}^{\mathrm{or},\partial}$$

sending a composition in $\Delta^{\mathrm{op}}$ to a triplet of compatible admissible embeddings which form a composition up to (a specified) isotopy. This isotopy should be viewed as being part of the data of such a functor (although it turns out that the space of such isotopies is contractible).

Given a unital associative algebra $A$ and a unital right and a unital left module, $M$ and $N$ thereover (in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$), incarnated as a symmetric monoidal functor out of

$\mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial}$, the composition:

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}_{1/[-1,1]}^{\mathrm{or},\partial} \to \mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial} \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

represents the classical bar construction, whose colimit computes:

$$M \otimes_A N$$

Again, these statements are to be interpreted at the level of $\infty$-categories. For example, one could instead compute the above colimit at the point-set level by resolving $M$ or $N$. Lemma 3.11 of [AF] gives a rigorous construction of this functor:

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}_{1/[-1,1]}^{\mathrm{or},\partial}$$

along with a proof that it is final (a key ingredient of which follows from reasoning along the lines of 39). In particular, the natural map:

$$M \otimes_A N \to \mathrm{colim}\Big(\mathcal{D}\mathrm{isk}_{1/[-1,1]}^{\mathrm{or},\partial} \to \mathcal{D}\mathrm{isk}_1^{\mathrm{or},\partial} \overset{(M,A,N)}{\longrightarrow} \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}\Big)$$

is an equivalence. A point-set level analog of the above results may be found in Section 4.3 of [G]. This analog is expressed in terms of factorization algebras, which will be defined in coming chapters.

We now explicate one sense in which $\mathcal{A}^{\varphi}$ contains more data than an associative algebra.

**Theorem 50.** *If $\varphi_t$ is not homotopic to $\varphi_0 = \mathrm{Id}_A$ for some $t \in \mathbb{R}$, then $\mathcal{A}^{\varphi}$ does not factor through maps:*

$$\mathrm{Disk}_1^{\mathcal{R}} \longrightarrow \mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$$

*Proof.* We will argue by contradiction.

First, assume that there indeed exists a factorization, which we denote as $\mathcal{A}$. This factorization produces a factorization of one of $\mathcal{A}^{\varphi}$'s action map:

$$\mathrm{Disk}_1^{\mathcal{R}}((-t,t),(-2t,2t)) \to \mathcal{D}\mathrm{isk}_1^{\mathrm{or}}((-t,t),(-2t,2t)) \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}(A,A)$$

By definition the middle space is equivalent to the space oriented self embeddings of $\mathbb{R}$, which contractible. Therefore, the action map factors through a point, i.e. is constant. This contradicts the assumption of the theorem. $\square$

The equivalence between symmetric monoidal functors out of $\mathcal{D}\mathrm{isk}_1^{\mathrm{or}}$ and associative algebras [AFT] gives the following as an immediate corollary of the above:

**Corollary 51.** *$\mathcal{A}^{\varphi}$ is not in the image of the natural functor:*

$$\mathrm{Alg}_{\mathrm{Ass}}(\mathrm{Ch}_{\mathbb{R}}^{\otimes}) \to \mathrm{Fun}^{\otimes}(\mathrm{Disk}_1^{\mathcal{R}}, \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes})$$

We now shift our focus towards Riemannian 1-manifolds which are not disjoint unions of intervals of finite diameter.

# CHAPTER 3

# RIEMANNIAN VARIATIONS II: GLOBALIZATION

We now perform a standard maneuvre in homotopy theory: formally globalizing a locally defined invariant. That is, we extend $\mathcal{A}^\varphi$ to a Riemannian 1-manifold $M$ by taking a colimit of $\mathcal{A}^\varphi$ over the category of isometrically embedded disks in $M$.

We begin by introducing the functor whose colimit defines our desired invariant:

**Definition 52.** Given $M \in \mathrm{Mfld}_1^{\mathcal{R}}$, let $\mathcal{A}^\varphi_{/M}$ denote the induced algebra over $\mathrm{Disk}^{\mathcal{R}}_{1/M}$, given by:

$$\mathcal{A}^\varphi_{/M} : \mathrm{Disk}^{\mathcal{R}}_{1/M} \to \mathrm{Disk}^{\mathcal{R}}_1 \xrightarrow{\mathcal{A}^\varphi} \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}}$$

We will refer to such an object as a *Riemannian prefactorization algebra over M.*

**Definition 53.** Left Kan extension of $\mathcal{A}^\varphi$ along the inclusion $\mathrm{Disk}^{\mathcal{R}}_1 \hookrightarrow \mathrm{Mfld}^{\mathcal{R}}_1$ determines a functor:

$$\begin{aligned} \int_{-} \mathcal{A}^\varphi : \mathrm{Mfld}^{\mathcal{R}}_1 &\longrightarrow \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}} \\ M &\longmapsto \int_M \mathcal{A}^\varphi \end{aligned}$$

whose value on a given $M \in \mathrm{Mfld}^{\mathcal{R}}_1$ can be computed as:

$$\int_M \mathcal{A}^\varphi \simeq \mathrm{colim}\Big(\mathrm{Disk}^{\mathcal{R}}_{1/M} \longrightarrow \mathrm{Disk}^{\mathcal{R}}_1 \xrightarrow{\mathcal{A}^\varphi} \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}}\Big)$$

which we will refer to as *factorization homology of $M$ with coefficients in $\mathcal{A}^\varphi$.*

*Remark* 54. We remind the reader that "colimit" is meant to be interpreted in a homotopical sense. In other words, a colimit in the sense of $\infty$-categories, or as a homotopy colimit.

*Remark* 55. Although $\mathcal{A}^{\varphi}$ was initially constructed as an ordinary functor, it's globalization will be a functor of $\infty$-categories. In other words, it will be functorial "up to specified homotopy." However, it will be an ordinay functor upon passage to homology:

$$\mathrm{Mfld}_1^{\mathcal{R}} \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes} \to \mathrm{Ho}(\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes})$$

Although this composition takes place within the simpler formalism of ordinary categories, it is relatively poorly behaved.

In some sense, this functor is only well-defined up to a contractible choice. Therefore it may be the case that one could construct a left Kan extension which arises from a composition:

$$\mathrm{Mfld}_1^{\mathcal{R}} \to \mathrm{Ch}_{\mathbb{R}}^{\otimes} \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

This is analogous to modeling sheaf cohomology via differential forms.

*Remark* 56. Factorization homology is universal in the following sense. Let's say we have our hands on a functor (which we can imagine as constructed via global analysis):

$$\mathcal{F} : \mathrm{Mfld}_1^{\mathcal{R}} \to \mathrm{Ch}_{\mathbb{R}}^{\otimes}$$

Along with a map (which we can image as arising from the classical analytic techniques):

$$\mathcal{A}^{\varphi} \to \mathcal{F}|_{\mathrm{Disk}_1^{\mathcal{R}}}$$

The universal property of the left Kan extension ensures the existence of a map, for every $M$:

$$\int_M \mathcal{A}^{\varphi} \to \mathcal{F}(M)$$

Even when the locally defined map is an equivalence, in general this map need not be an equivalence , but is instead some "approximation", analogous to:

$$\operatorname{Spec}(\mathbb{R}[\![x]\!]) \hookrightarrow \operatorname{Spec}(\mathbb{R}[x])$$

The theory of factorization algebras provides a cosheaf-type technique for checking whether this map is an equivalence. We will address these ideas in later chapters.

*Remark* 57. These constructions are designed to formalize the "products" found in the Heisenberg approach to canonical quantization. In this formalism, the "products" of exponentials involve a "time-ordering" along with a temporally appropriate application of the time-evolution operator generated by a Hamiltonian.

As we see below,these categorical and homotopical techniques give a manifestly invariant approach to this formalism. This invariance is the essential ingredient in extending these local operations to general Riemannian 1-manifolds.

**Example 58.** The data of:

- a finite ordered configuration of points, $t_0, \ldots, t_n$, in a connected, oriented Riemannian 1-manifold $M$
- a collection of cocycles $\mathcal{O}_0, \ldots, \mathcal{O}_n$ of $\mathcal{A}^{\varphi}\big(B_\epsilon\big)$ of (co)homological degree (minus) $k_i$, with $\epsilon/2 < d_M(t_i, t_j)$ for $i \neq j$,

gives a cocycle in factorization homology of $\mathcal{A}^{\varphi}$ with coefficients in $M$ as follows.

First, as $\epsilon$ is sufficiently small, the (not necessarily order preserving) embedding, $t : \{0, \ldots, n\} \to M$ admits a factorization through some $\epsilon$-neighborhood of the subset $t$:

$$\coprod_{i=0}^{n} B_\epsilon(t_i) \longrightarrow M$$

Second, as we assumed $\mathcal{O}_i$ to be cocyles, they give rise to:

$$\begin{aligned} \mathbb{R}[\tilde{k}] &\longrightarrow A^{\otimes n+1} \\ 1 &\longmapsto \mathcal{O}_0 \otimes \cdots \otimes \mathcal{O}_n \end{aligned}$$

where: $\tilde{k} = \sum k_i$ The functoriality of factorization homology then gives:

$$\mathbb{R}[\tilde{k}] \to A^{\otimes n+1} \simeq \int_{B_\epsilon(t)} \mathcal{A}^\varphi \longrightarrow \int_M \mathcal{A}^\varphi$$

Which one might suggestively denote as:

$$\mathrm{T}\{\mathcal{O}_0(t_0) \cdot \mathcal{O}_1(t_1) \ldots \cdot \mathcal{O}(t_n)\}$$

Note that the $\mathcal{O}_i(x_i)$ must be first reordered using the Koszul sign rule and the ordering on the points $t$ induced by the orientation on $M$. Finally, the automorphisms $\varphi_{t_i}$ must be applied to $\mathcal{O}_i$ before the multiplication.

With this in place, we return to our discussion on how $\mathcal{A}^\varphi$ extends to Riemannian 1-manifolds.

### 3.1 Value on Infinitely Large Objects

As a warm-up to the main result, we will begin by examining the behavior of factorization homology of $\mathcal{A}^\varphi$ on three inequivalent objects of $\mathrm{Mfld}_1^{\mathcal{R}}$ with infinite diameter:

$$\mathbb{E} = (-\infty, \infty), \qquad \mathbb{E}^{>0} = (0, \infty), \qquad \mathbb{E}^{<0} = (-\infty, 0)$$

*Remark* 59. The answer is hardly surprising. However, because $\mathrm{Disk}_1^{\mathcal{R}}$ only contains open intervals of finite diameter, this statement requires a proof.

*Remark* 60. The following proofs will require a fair amount of abstract homotopy theory. The primary technical tool will be Quillen's theorem A, which is reviewed in Appendix A. Heuristically, we will use this theorem to formalizes the intuition that, as factorization homology is built from data contained within finite intervals, every observable is concentrated in some finite region center around some fixed, chosen point.

**Lemma 61.** *A point $q \in \mathbb{E}$ determines an equivalence:*

$$\int_{\mathbb{E}} \mathcal{A}^{\varphi} \simeq A$$

*Proof.* Let $\mathbb{R}^{\geq 1}$ denote the partially ordered set of real numbers greater than or equal to one. Consider the inclusion:

$$\begin{aligned} \mathbb{R}^{\geq 1} &\longrightarrow \mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}} \\ 1 \leq r &\longmapsto \Big(B_r(q) \hookrightarrow \mathbb{E}\Big) \end{aligned}$$

We will argue that this inclusion is final. In particular, pulling back $\mathcal{A}^{\varphi}_{/\mathbb{E}}$ along this inclusion induces an equivalence on colimits.

Recall that $\mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}}$ is equivalent to the category of open subsets of $\mathbb{E}$ diffeomorphic to a finite disjoint union of intervals, all of which happen to lie within some compact subset. Therefore, for every $U \in \mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}}$, ${}^{U/}(\mathbb{R}^{\geq 1})$ is equivalent to the partially ordered set of positive numbers greater 1 and than the distance between $q$ and the point of $x \in \overline{U}$ which is fartherest away from $q$.

This slice category has an inital object: $\mid q - x \mid$ or 1. Therefore, it's classifying space is contractible, and the inclusion is final.

Moreover as the midpoints of the above intervals all coincide, there exists a composition:

$$\begin{tikzcd} \mathbb{R}^{\geq 1} \arrow[rr, "\mathcal{A}^{\varphi}_{/\mathbb{E}}|_{\mathbb{R}^{>0}}"] \arrow[dr] & & \mathrm{Ch}^{\otimes}_{\mathbb{R}} \\ & * \arrow[ur, "A"'] & \end{tikzcd} .$$

This shows:

$$\begin{aligned} \int_{\mathbb{E}} \mathcal{A}^{\varphi} &\simeq \operatorname{colim}\Big(\mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}} \xrightarrow{\mathcal{A}^{\varphi}_{/\mathbb{E}}} \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}}\Big) \\ &\simeq \operatorname{colim}\Big(\mathbb{R}^{\geq 1} \xrightarrow{\mathcal{A}^{\varphi}_{/\mathbb{E}}|_{\mathbb{R}^{\geq 1}}} \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}}\Big) \\ &\simeq \operatorname{colim}\Big(* \xrightarrow{A} \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}}\Big) \\ &\simeq A \end{aligned}$$

The first equality is by definition. The second follows from the finality of $\mathbb{R}^{>0} \hookrightarrow \mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}}$. The equivalence on the third line is provided by the above factorization. It is an equivalence because ${}^{*/}(\mathbb{R}^{\geq 1}) \simeq \mathbb{R}^{\geq 1}$ has an initial object (1), and therefore has a contractible classifying space. The fourth follows from Yoneda Lemma. $\square$

*Remark* 62. The choice of intervals of length greater than or equal to 2 in the above argument is arbitrary, and chosen to avoid an argument about the (weak) contractibility of the "uncountably" long real line.

We now provide describe the behavior of $\mathcal{A}^{\varphi}$ on $\mathbb{E}^{>0}$ and $\mathbb{E}^{<0}$ :

**Theorem 63.** *There exists an identification:*

$$\int_{\mathbb{E}^{>0}} \mathcal{A}^{\varphi} \simeq \int_{\mathbb{E}^{<0}} \mathcal{A}^{\varphi} \simeq A$$

*Proof.* We begin with the case of $\mathbb{E}^{>0}$. The identification of $\mathbb{E}^{<0}$ will be defined in terms of the identification constructed below. First consider the functor:

$$\begin{aligned} \mathbb{R}^{\geq 1} &\longrightarrow \mathrm{Disk}^{\mathcal{R}}_{1/\mathbb{E}^{>0}} \\ 1 \leq r &\longmapsto (0, r) \end{aligned}$$

An argument nearly identical to the one above shows that this inclusion is final. Furthermore, although this restriction of $\mathcal{A}^{\varphi}_{/\mathbb{E}^{>0}}$ to this subcategory doesn't not strictly factor through a point as in the previous case, it does up to homotopy. This homotopy is constructed in terms of $\varphi$, and gives rise to the above equivalence. First, note that $\mathcal{A}^{\varphi}$ evaluates on a morphism $r_0 < r_1$ as:

$$\varphi_{r_1 - r_0} : A \longrightarrow A$$

This indicates that natural isomorphism witnessing the commutativity of the diagram below should "undo" these maps.

$$\begin{tikzcd} \mathbb{R}^{>0} \arrow[rr, "\mathcal{A}^{\varphi}_{/\mathbb{E}}|_{\mathbb{R}^{>\not{0}}}"] \arrow[dr] & & \mathcal{C}\mathrm{h}^{\otimes}_{\mathbb{R}} \\ & * \arrow[ur, "A"'] & \end{tikzcd}.$$

Therefore, we must construct, for each positive real number $r$, an automorphism of $A$. We make the obvious choice:

$$A \xrightarrow{\varphi_{-r}} A$$

The naturality of the collection of the above maps follows from the assumption that $\varphi$ is a one parameter family of automorphisms.

The case of $\mathbb{E}^{<0}$ follows similarly. Loosely, one replaces $r$ by $-r$. $\square$

### 3.2 Translations

Even though no connected object of $\mathrm{Disk}_1^{\mathcal{R}}$ admits any interesting symmetries, $\mathbb{R}$ acts on $\mathbb{E}$ via translations. As factorization homology is functorial with respect to isometries, this implies that for every $\tau \in \mathbb{R}$ (which we view as a translation), factorization homology gives an automorphism, which we'll denote as:

$$\int_{\mathbb{E}} \mathcal{A}^{\varphi} \xrightarrow{\tau_*} \int_{\mathbb{E}} \mathcal{A}^{\varphi}$$

along with equivalences:

$$(\tau + \tau')_* \simeq \tau_* \circ \tau'_*$$

We now analyze this action.

*Remark* 64. The fact that the previous equation doesn't "hold on the nose" is a reflection of the $\infty$-categorical nature of factorization homology. However, the data of the factorization homology (which is a functor) includes a homotopy witnessing that the equation holds on the nose, upon passing to connected components.

Recall that given a point $x \in M$, and a cycle $\mathcal{O} \in A$ of degree $k$, there exists a corresponding cycle, denoted $\mathcal{O}(x)$ This is obtained by evaluating factorization homology on an inclusion of a ball $\mathrm{B}_r(x) \to M$. By construction, this does not depend the choice of a radius.

*Remark* 65. Before proceedinig to the next two lemmas, we briefly comment on the two essential ingredients of their parallel proofs.

The first part is that factorization homology is functorial. In other words, upon the application of factorization homology, any diagram in $\mathrm{Mfld}_1^{\mathcal{R}}$ gives a diagram in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$:

$$\mathcal{J} \to \mathrm{Mfld}_1^{\mathcal{R}} \to \mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$$

For example, when $\mathcal{J} = [3]$.

The second essential ingredient is the uniqueness of the composition of composable maps in any $\infty$ category. In other words, any two candidate compositions of a sequence of composable maps are homotopic. The theory allows one to articulate a notion of a "space of compositions", which is contractible.

Both of these properties (a well behaved theory of Kan extensions and a suitable uniqueness of composition) are part of the foundations of higher category theory.

The following lemma establishes that $\tau_*$ moves the support of $\mathcal{O}(x)$ to $\mathcal{O}(x+\tau)$.

**Theorem 66.** *There exists a chain-homotopy:*

$$\tau_*(\mathcal{O}(x)) \simeq \mathcal{O}(x+\tau)$$

*In particular, they are equal in homology.*

*Proof.* The lemma follows from a straightforward application of the functoriality of factorization homology. We have the following commutative triangle in $\mathrm{Mfld}_1^{\mathcal{R}}$:

$$\begin{tikzcd} B_r \arrow[r, "B_r(x)"'] \arrow[rr, bend left, "B_r(x+\tau)"] & \mathbb{E} \arrow[r, "\tau"'] & \mathbb{E} \end{tikzcd}$$

Application of factorization homology gives a diagram in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$:

$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, "\mathcal{O}"] \arrow[rrr, bend left=40, "\mathcal{O}(x+\tau)"] \arrow[rr, bend right, "\mathcal{O}(x)"] \arrow[rrr, bend right=40, "\tau_*(\mathcal{O}(x))"'] & \int_{B_r} \mathcal{A}^{\varphi} \arrow[r, "\int_{B_r(x)} \mathcal{A}^{\varphi}"'] \arrow[rr, bend left, "\int_{B_r(x+\tau)} \mathcal{A}^{\varphi}"] & \int_{\mathbb{E}} \mathcal{A}^{\varphi} \arrow[r, "\tau_*"] & \int_{\mathbb{E}} \mathcal{A}^{\varphi} \end{tikzcd}$$

Therefore, the top and bottom maps are compositions for the same triplet of maps, and are therefore homotopic, as the space of such compositions is contractible.

Therefore, we obtain a homotopy between the following parallel maps:

$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, bend left, "\mathcal{O}(x+\tau)"] \arrow[r, bend right, "\tau_*(\mathcal{O}(x))"'] & \int_E \mathcal{A}^\varphi \end{tikzcd}$$

As homotopic maps out of $\mathbb{R}[k]$ are homologous when viewed as cycles, the result follows. $\square$

The next lemma relates the action of $\tau$ to the automorphism $\varphi_\tau$.

**Theorem 67.** *There exists a chain homotopy:*

$$\tau_*(\mathcal{O}(x)) \simeq \mathcal{O}(x+\tau) \simeq (\varphi_{-\tau}\mathcal{O})(x)$$

*In particular, they are all equal in homology.*

*Proof.* We begin by fixing some $R > r + |\tau|$. We can apply factorization homology to the following diagram in $\mathrm{Mfld}_1^{\mathcal{R}}$:

$$\begin{tikzcd} B_r \arrow[r, "B_r(\tau)"] \arrow[rr, bend right, "B_r(x+\tau)"'] & B_R \arrow[r, "B_R(x)"] & \mathbb{E} \end{tikzcd}$$

to obtain the following diagram in $\mathcal{C}\mathrm{h}_{\mathbb{R}}^{\otimes}$

$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, "\mathcal{O}"'] \arrow[rrr, bend left=40, "\mathcal{O}(x+\tau)=\tau_*(\mathcal{O}(x))"] \arrow[rr, bend right, "\varphi_{-\tau}(\mathcal{O})"'] \arrow[rrr, bend right=40, "(\varphi_{-\tau}(\mathcal{O})(x)"'] & \int_{B_r} \mathcal{A}^\varphi \arrow[r, "\varphi_{-\tau}=\int_{B_r(\tau)} \mathcal{A}^\varphi"'] \arrow[rr, bend left, "\int_{B_r(x+\tau)} \mathcal{A}^\varphi"] & \int_{B_R} \mathcal{A}^\varphi \arrow[r, "\int_{B_R(x)} \mathcal{A}^\varphi"'] & \int_{\mathbb{E}} \mathcal{A}^\varphi \end{tikzcd}$$

Therefore, the top and bottom maps are compositions for the same triplet of maps, and are therefore homotopic, as the space of such compositions is contractible.

Therefore, we obtain a homotopy between the following parallel maps:

$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, bend left, "\mathcal{O}(x+\tau)=\tau_*(\mathcal{O}(x))"] \arrow[r, bend right, "(\varphi_{-\tau}(\mathcal{O})(x)"'] & \int_{\mathbb{E}} \mathcal{A}^{\varphi} \end{tikzcd}$$

As homotopic maps out of $\mathbb{R}[k]$ are homologous when viewed as cycles, the result follows. $\square$

In summary, we can use factorization homology to recover all of $\varphi$ from the translation action on $\mathbb{E}$.

## 3.3 Value on Circles

We now describe the value of factorization homology of $\mathcal{A}^{\varphi}$ on circles of a fixed circumference $2L$, denoted $\mathrm{S}^1_L$, in terms of a classical homotopical invariant of associative algebras and modules thereover: Hochschild Homology. Therefore, for the sake of continuity, we briefly review one of many standard definitions of Hochschild homology.

*Remark* 68. Those already familiar with Hochschild homology may safely skip the following subsection.

### 3.3.1 Hochschild Homology

*Remark* 69. In the following review, all algebraic descriptors should be interpreted in a manner consistent with the ambient symmetric monoidal ($\infty$-)category. For example, an associative algebra in the ordinary category of chain complexes with tensor product is to be taken as a differential graded associative algebra.

**Definition 70.** Given an associative algebra $A$, and $N$, a bimodule over $A$, then the *Hochschild complex of $M$ over $A$* is defined to be:

$$\mathrm{HC}_*(A, N) := A \otimes_{A \otimes A^{\mathrm{op}}} N$$

In other words, the Hochschild complex may be computed as a bar construction:

$$\mathrm{B}(A, A \otimes A^{\mathrm{op}}, N)$$

When $N = A$, this will be denoted as $\mathrm{HC}_*(A)$

**Example 71.** For simplicity, let's assume $A$ and $N$ is an associative algebra and a bimodule concentrated in degree zero ). In this case it's not difficult to verify that:

$$\mathrm{H}_0(\mathrm{HC}_*(A, N)) \simeq N/[A, N]$$

For example, assume that $A = N$. Then there is a quotient map:

$$A \to \mathrm{HC}_*(A)$$

If we have a representation, $\rho$ of $A$, then it's associated trace map uniquely factors through the Hochschild complex:

$$\begin{tikzcd} A \arrow[r, "\mathrm{Tr}_\rho"] \arrow[d] & \mathbb{R} \\ \mathrm{HC}_*(A) \arrow[ur, dashed] & \end{tikzcd}$$

**Example 72.** A classical application of Hochschild homology is to the study of the homology of free loop spaces (via algebraic methods). For example, given a space $X$, Hochschild homology gives an algebraic description of the cohomology of the free loop space of $X$ in terms of it's singular cochains:

$$\mathrm{C}^*(\mathcal{L}X) \simeq \mathrm{HC}_*(C^*(X))$$

Therefore, at least in this context, the Hochschild complex admits an $S^1$-action.

**Example 73.** A special case of the Hochschild-Konstant-Rosenberg theorem gives a relationship between Hochschild homology and differential forms:

$$\mathrm{HC}(\mathcal{O}_{\mathbb{A}^n}) \simeq \Omega^\bullet(\mathbb{A}^n)$$

We emphasize that $\Omega^\bullet$ is not to be interpreted as a chain complex with the deRham differential and standard grading by weight. These structures should be interpreted in a manner parallel to the $S^1$-action above.

*Remark* 74. As the above examples indicate, Hochschild homology of an associative algebra admits two conceptual interpretations: functions on a free loop space with it's $S^1$ action, and differential forms with it's deRham differential and weight filtration. [B-ZN] give an precise reconciliation of these perspectives (in the case when $A$ is commutative) via language of derived algebraic geometry.

The next section will leverage the relationship between the Hochschild complex and the geometry of the circle (with a distinguished point) to describe factorization homology of $\mathcal{A}^\varphi$ on a circle of finite diameter.

### 3.3.2 Hochschild Homology and Factorization Homology over a Riemannian Circle

Our strategy will be to translate the local geometric dependency of $\mathcal{A}^\varphi_{/\mathrm{S}^1_L}$ to a single point fixed point of the circle, obtaining a bimodule over $A$. This localization process will explicitly depend on a *global* geometric invariant of the circle (it's circumference), the remnants of which are contained in the action maps of this bimodule.

With this identification in place, we will then use standard theorems in the theory of factorization homology to express our desired invariant in terms of Hochschild homology of $A$ with values in the above bimodule. The following theorem and proof make this discussion precise.

**Theorem 75.** *Let* $\mathrm{S}^1_L$ *be the circle of circumference* $2L$. *A point* $x \in \mathrm{S}^1_L$ *determines an equivalence:*

$$\int_{\mathrm{S}^1_L} \mathcal{A}^{\varphi} \simeq HC_*(A,_L A_{-L})$$

*Proof.* Our localization process will be simplified by first pushing $\mathcal{A}^{\varphi}_{/\mathrm{S}^1_L}$ to an closed interval. Our chosen point, $x$ and the metric determines a distance function:

$$\rho^x : \mathrm{S}^1_L \to [0, L]$$
$$p \mapsto \mathrm{d}(x, p)$$

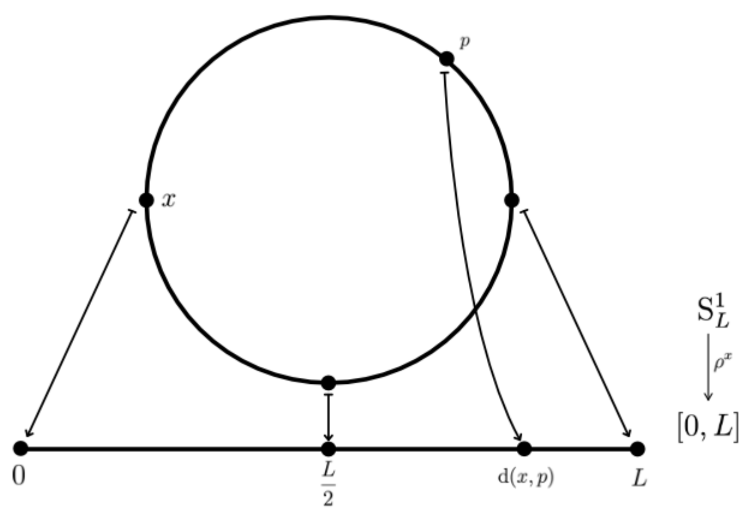



Figure 3.1. An infographic of $\rho^x$, along with it's value on 4 points of $\mathrm{S}^1_L$

Precomposing by $(\rho^x)^{-1}$ gives a functor:

$$(\rho^x)_*\mathcal{A}^{\varphi}_{/\mathrm{S}^1_L} : \mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]} \to \mathrm{Ch}^{\otimes}_{\mathbb{R}},$$

So that the preimage of a disjoint union of intervals not containing $L$:

$$U_0 := [0,\delta) \amalg \big( \coprod_{i=0\ldots n} (t_i - \epsilon_i, t_i + \epsilon_i)\big)$$

with $t_i < t_{i+1}$ can be identified with:

$$\Big(\coprod_{i=n,\ldots,0} \mathrm{B}_{\epsilon_i}(\exp_x(-t_i))\Big)\amalg\Big(\mathrm{B}_x(\delta)\Big)\amalg\Big(\coprod_{i=0\ldots n} \mathrm{B}_{\epsilon_i}(\exp_x(t_i))\Big)$$

*Remark* 76. We are thinking of the ordering of the connected components $(\rho^x)^{-1}(U_0)$ as arising from the orientation on $(\rho^x)^{-1}([0,L))$. This ordering will come into play when analyzing $(\rho^x)_*\mathcal{A}^{\varphi}_{/\mathrm{S}^1_L}$.

Therefore:

$$(\rho^x)_*\mathcal{A}^{\varphi}_{/\mathrm{S}^1_L}(U_0) \simeq \big(\bigotimes_{i=n,\ldots,0} A\big)\otimes A \otimes \big(\bigotimes_{i=0,\ldots,n} A\big)$$

Similarly, the preimage of a disjoint union of intervals not containing 0:

$$U_L := \big(\coprod_{i=0\ldots n} (t_i - \epsilon_i, t_i + \epsilon_i)\big)\amalg(L-\delta, L)$$

can be identified with:

$$\Big(\coprod_{i=n,\ldots,0} \mathrm{B}_{\epsilon_i}(\exp_x(-t_i))\Big)\amalg\Big(\mathrm{B}_x(\delta)\Big)\amalg\Big(\coprod_{i=0\ldots n} \mathrm{B}_{\epsilon_i}(\exp_x(t_i))\Big)$$

so that:

$$(\rho^x)_*\mathcal{A}^{\varphi}_{/\mathrm{S}^1_L}(U_L) \simeq \big(\bigotimes_{i=0,\ldots,n} A\big)\otimes A \otimes \big(\bigotimes_{i=n,\ldots,0} A\big)$$

For ease of notation, we will denote the last factor of $(\rho_x)^{-1}(U_L)$ and the first factor of $(\rho_x)^{-1}(U_0)$ as:

$$A^{\otimes[n]^{\mathrm{op}}}$$

Note that we can view $A$ as an $(A, A)$-bimodule. As described Definition 47 at the end of subsection 2.3.2, we can twist these action maps to obtain a new $(A, A)$-bimodule, ${}_LA_{-L}$. Recall that the action map is given by:

$$\left(\varphi_L^{\otimes[n]} \otimes \mathrm{Id}_A \otimes \varphi_{-L}^{\otimes[n]^{\mathrm{op}}}\right)^*(\mu_A) : A^{\otimes[n]} \otimes A \otimes A^{\otimes[n]^{\mathrm{op}}} \to A$$

In the standard way, we will view $A$ as a right $A \otimes A^{\mathrm{op}}$-module, and ${}_LA_{-L}$ as a left $A \otimes A^{\mathrm{op}}$-module. As noted at the end of section 2.4.1, this data naturally determines a functor:

$$(A, A \otimes A^{\mathrm{op}}, {}_L A_{-L}) : \mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]} \to \mathrm{Ch}^{\otimes}_{\mathbb{R}}$$

whose value on objects coincides with those above.

The primary ingredient of the proof is the following identification:

**Lemma 77.** *There exists an equivalence of functors:*

$$\Psi : (\rho^x)_* \mathcal{A}^{\varphi}_{/\mathrm{S}^1_L} \xrightarrow{\simeq} (A, A \otimes A^{\mathrm{op}}, {}_L A_{-L})$$

Loosely, $\Psi$ will "undo" the effect of $\varphi$. More formally, we define $\Psi$ as follows:

- On half open intervals:
$$\Psi_{[0,\delta)} = \Psi_{(L-\delta,L]} := \mathrm{Id}_A$$
- On open disjoint unions of intervals $U \hookrightarrow (0, L)$:
$$\Psi_U := \varphi_{-t_i}^{\otimes[n]} \otimes \varphi_{t_i}^{\otimes[n]^{\mathrm{op}}} : A^{\otimes[n]} \otimes A^{\otimes[n]^{\mathrm{op}}} \to A^{\otimes[n]} \otimes A^{\otimes[n]^{\mathrm{op}}}$$

We now check that these maps commute with the multiplication maps. It suffices to check the commutativity on the following three classes of maps:

- **Case I: Right Module Compatibility**: $[0,\ell)\amalg U \hookrightarrow [0,\ell')$:

$$\begin{tikzcd}
A^{\otimes[n]^{\mathrm{op}}} \otimes A \otimes A^{\otimes[n]} \arrow[r, "\varphi_{t_i}^{\otimes[n]^{\mathrm{op}}}\otimes \mathrm{Id}_A \otimes \varphi_{-t_i}^{\otimes[n]}"] \arrow[d, "(\varphi_{t_i}^{\otimes[n]^{\mathrm{op}}}\otimes \mathrm{Id}_A \otimes \varphi_{-t_i}^{\otimes[n]})^* \mu_A"'] & A^{\otimes[n]^{\mathrm{op}}} \otimes A \otimes A^{\otimes[n]} \arrow[d, "\mu_A"] \\
A \arrow[r, "\mathrm{Id}_A"] & A
\end{tikzcd}$$

- **Case II: Algebra Compatibility**: $U \hookrightarrow (T-\delta, T+\delta)$

$$\begin{tikzcd}
A^{\otimes[n]} \otimes A^{\otimes[n]^{\mathrm{op}}} \arrow[r, "\varphi_{-t_i}^{\otimes[n]}\otimes \varphi_{t_i}^{\otimes[n]^{\mathrm{op}}}"] \arrow[d, "(\varphi_{(T-t_i)}^{\otimes[n]})^* \mu_A^{[n]} \otimes (\varphi_{(t_i-T)}^{\otimes[n]^{\mathrm{op}}})^* \mu_A^{[n]^{\mathrm{op}}}"'] & A^{\otimes[n]} \otimes A^{\otimes[n]^{\mathrm{op}}} \arrow[d, "\mu_A"] \\
A \otimes A \arrow[r, "\varphi_{-T}\otimes\varphi_T"] & A \otimes A
\end{tikzcd}$$

- **Case III: Left Module Compatibility**: $U \amalg (r, L] \hookrightarrow (r', L]$

$$\begin{tikzcd}
A^{\otimes[n]} \otimes A \otimes A^{\otimes[n]^{\mathrm{op}}} \arrow[r, "\varphi_{-t_i}^{\otimes[n]}\otimes \mathrm{Id}_A \otimes \varphi_{t_i}^{\otimes[n]^{\mathrm{op}}}"] \arrow[d, "(\varphi_{L-t_i}^{\otimes[n]}\otimes \mathrm{Id}_A \otimes \varphi_{-(L-t_i)}^{\otimes[n]^{\mathrm{op}}})^* \mu_A"'] & A^{\otimes[n]} \otimes A \otimes A^{\otimes[n]^{\mathrm{op}}} \arrow[d, "\left(\varphi_{L}^{\otimes[n]}\otimes \mathrm{Id}_A \otimes \varphi_{-L}^{\otimes[n]^{\mathrm{op}}}\right)^* (\mu_A)"] \\
A \arrow[r, "\mathrm{Id}_A"] & A
\end{tikzcd}$$

The commutativity of these squares follow from the fact that $\varphi$ is a one-parameter of automorphisms of the algebra $A$. For example, the fact that $\varphi_\tau$ is a one-parameter family of automorphisms implies that:

$$\big(\varphi_{-T} \otimes \varphi_{-T}\big) \circ \big(\varphi_{T-t_0} \otimes \varphi_{T-t_1}\big) = \varphi_{-t_0} \otimes \varphi_{-t_1}$$

The fact that it is an automorphism of associative algebras gives that:

$$\varphi_{-T} \circ \mu_A^{[1]} = \mu_A^{[1]} \circ \big(\varphi_{-T} \otimes \varphi_{-T}\big)$$

These combine to give an essential ingredient to the commutativity of case II (when

$n = 1$):

$$\varphi_{-T} \circ \mu^{[1]} \circ \left(\varphi_{T-t_0} \otimes \varphi_{T-t_1}\right) = \mu_A^{[1]} \circ \left(\varphi_{-t_0} \otimes \varphi_{-t_1}\right)$$

The general case follows a completely parallel argument. Moreover, $\Psi$ is an equivalence due to the fact that $\varphi_t$ is an equivalence for every $t$. This proves the lemma.

The theorem follows from the following string of equivalences:

$$\begin{aligned}
\int_{S^1_L} A^{\varphi} &\simeq \int_{[0,L]} \rho_* \mathcal{A}^{\varphi} \\
&\simeq \int_{[0,L]} (A, A \otimes A^{\mathrm{op}}, {}_L A_{-L}) \\
&\simeq HC_*(A, {}_L A_{-L})
\end{aligned}$$

The second equivalence is supplied by $\Psi$, and the functoriality of the colimit. The third equivalence follows from theorem 3.19 of [AF]. The first equivalence follows from the finality of the map $\mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]} \xrightarrow{\rho^{-1}} \mathrm{Disk}^{\mathcal{R}}_{1/S^1_L}$. We now verify this is the case.

By Quillen's Theorem A, we must show that for every $U \in \mathrm{Disk}^{\mathcal{R}}_{1/S^1_L}$ the category ${}^{U/}(\mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]})$ has a contractible classifying space. We can recognize this as the partially ordered set of opens in $V \subset [0, L]$ whose preimage in $S^1_L$ contains $U$. As this partially ordered set has an minimal element, namely the image of $U$ under $\rho$, its classifying space is contractible.

$\square$

# CHAPTER 4

# FACTORIZATION HOMOLOGY

We give a brief account of a selection of definitions, results, and examples within the theory of factorization homology.

## 4.1 Algebraic and Geometric Preliminaries

As before, we begin with some geometric preliminaries in order to arrive at our algebraic input. This algebraic input may be conceptualized as the local theory.

**Definition 78.** Let $\mathrm{Mfld}_n^\partial$ denote the category of smooth n-manifolds with boundary and open embeddings which send boundary components to boundary components. This has the structure symmetric monoidal category, $\mathrm{Mfld}_n^{\partial,\amalg}$, under disjoint union.

We let $\mathrm{Disk}_n^\partial$ denote the full symmetric monoidal subcategory of $\mathrm{Mfld}_n^\partial$ containing n-manifolds equivalent to a finite disjoint union of Euclidean spaces.

We let $\mathrm{Mfld}_n$ and $\mathrm{Disk}_n$ denote the full subcategories of $\mathrm{Mfld}_n^\partial$ and $\mathrm{Disk}_n^\partial$ containing those manifolds without boundary.

We now define $\infty$-categorical analogs of the above, so as to more easily discuss isotopy invariants of n-manifolds.

**Definition 79.** The compact open topology gives $\mathrm{Mfld}_n^\partial$ the structure of a symmetric monoidal topologically enriched category. The homotopy coherent nerve construction produces a symmetric monoidal $\infty$-category, $\mathcal{M}\mathrm{fld}_n^\partial$. We let $\mathrm{Disk}_n^\partial$ and $\mathcal{D}\mathrm{isk}_n^\partial$ denote the full symmetric monoidal subcategory of $\mathrm{Mfld}_n^\partial$ and $\mathcal{M}\mathrm{fld}_n^\partial$, respectivley containing n-manifolds equivalent to a finite disjoint union of Euclidean spaces.

**Example 80.** Unraveling the homotopy coherent nerve construction shows that the data of a composition:

$$\Delta^2 \to \mathcal{M}\mathrm{fld}_n$$

is the data of three embeddings: $f : M_0 \to M_1$, $g : M_1 \to M_2$, $h : M_0 \to M_2$, along with an isotopy $\iota : f \circ g \simeq h$.

In particular, when $n = 1$ there exists a factorization of

$$[-2, 0) \amalg (0, 2] \to [-2, 2]$$

through

$$[-2, -1) \amalg (-1, 1) \amalg (1, 2] \to [-2, 2]$$

This factorization clearly does not exist in $\mathrm{Disk}_n^{\partial}$.

**Example 81.** Although topological categories and funtors therebewteen are easy to construct, a general, succint, intrinsic notion of homotopy coherent diagrams is not so easy to come by. For example, one might like the process of taking the connected components to witness an equivalence:

$$\Delta^{\mathrm{op}} \simeq \mathcal{D}\mathrm{isk}^{\partial}_{1/\mathbb{R}},$$

as an embedding of a disjoint union of intervals into a larger interval is uniquely determined, up to isotopy, by an ordering of the connected components of the source of the embedding. This is true, when we interpret the under-category construction in an $\infty$-categorical sense.

*Remark* 82. We emphasize that this is the primary reason we use $\infty$-categories: it provides a convenient language discuss and manipulate *homotopy invariant and homotopy coherent constructions.* The above example illustrates how this can help:

certain geometric constructions simplify into something completely combinatorial. This provides a bridge between certain algebraic structures ($\Delta^{\mathrm{op}}$) and geometric ($\mathcal{D}\mathrm{isk}^{\partial}_{1/\mathbb{R}}$) constructions.

In summary, we obtain the following commutative diagram amongst symmetric monoidal $\infty$-categories:

$$\begin{tikzcd} \mathrm{Disk}^{\partial}_n \arrow[r, hook] \arrow[d] & \mathrm{Mfld}^{\partial}_n \arrow[d] \\ \mathcal{D}\mathrm{isk}^{\partial}_n \arrow[r, hook] & \mathcal{M}\mathrm{fld}^{\partial}_n \end{tikzcd}$$

which rigorously arose from taking the homotopy coherent nerve of the obvious diagram of topologically-enriched categories.

*Remark* 83. One can decorate these categories with geometric structures admitting a description as the lift of a classifying map. For example, a notion of orientation or framing. We will not pursue this avenue, as it would take us too far afield. The interested reader may consult the second section of [AF].

Throughout our discussion, we fix a symmetric monoidal category $\mathcal{C}^{\otimes}$, whose underlying category is presentable, and whose symmetric monoidal structure distributes over colimits in each variable. In other words, $\otimes$-presentable. See definition 3.4 of [AF] for a precise definition.

**Definition 84.** An *n-disk algebra*, $\mathcal{A}$, in $\mathcal{C}^{\otimes}$, is a symmetric monoidal functor:

$$\mathcal{D}\mathrm{isk}_n \xrightarrow{\mathcal{A}} \mathcal{C}^{\otimes}$$

The *category of n-disk algebras* in $\mathcal{C}$ is the category of symmetric functors from $\mathcal{D}\mathrm{isk}_n$ to $\mathcal{C}$:

$$\mathrm{Alg}_{\mathcal{D}\mathrm{isk}_n}(\mathcal{C}^{\otimes}) := \mathrm{Fun}^{\otimes}(\mathcal{D}\mathrm{isk}^{\amalg}_n, \mathcal{C}^{\otimes})$$

*Remark* 85. Note that $\mathcal{A}(\mathbb{R}^n) = A$ inherits a natural action of the orthogonal group.

This is obtained via:

$$\mathrm{O}(n) \hookrightarrow \mathrm{Maps}_{\mathcal{D}\mathrm{isk}_n}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\mathcal{A}} \mathrm{Maps}_{\mathcal{C}}(A, A)$$

*Remark* 86. The data of an $n$-disk algebra is equivalent to an $E_n$ algebra with an action of the orthogonal group, see [AFT] Proposition 2.12.

We now review

### 4.1.1 A Few Classic $n$-Disk Algebras.

Our first example reflects the deep relationship between n-disks algebras and

**Example 87. Unordered Configuration Spaces**

Throughout, we'll let $\mathrm{Conf}_k(M)$ denote the *ordered* configuration space of n points in a manifold M, and

$$\mathrm{B}(M) := \coprod_{n \geq 0} \mathrm{Conf}_k(M)_{\Sigma_k}$$

the disjoint union of all the *unordered* configuration spaces. $\mathrm{B}(-)$ is functorial with respect to open embeddings, and therefore defines a functor into the category of ('good") topological spaces:

$$\mathrm{B} : \mathrm{Disk}_n \to \mathrm{Top}$$

A subset of a disjoint union decomposes as a subsets of each factor. Following this logic, the standard homeomorphism:

$$\mathrm{B}(M \amalg N) \simeq \mathrm{B}(M) \times \mathrm{B}(N)$$

shows that functor is in fact symmetric monoidal. Moreover, one can verify that the above process is continuous with respect the compact open topology on the set of maps of topological spaces. As the $\infty$-category associated to the topologically enriched category of topological spaces is equivalent to the $\infty$-category of spaces, taking the homotopy coherent gives an n-disk algebra of spaces:

$$\mathcal{D}\mathrm{isk}_n^{\amalg} \xrightarrow{\mathcal{B}(-)} \mathcal{S}\mathrm{paces}^{\times}$$

which one can show is free n-disk algebra generated by a point. For obvious reasons, we can compose this functor with chains,

$$\mathcal{D}\mathrm{isk}_n^{\amalg} \xrightarrow{\mathrm{C}_*(\mathcal{B}(-))} \mathrm{Ch}_k^{\otimes}$$

obtaining an n-disk algebra in chain complexes.

**Example 88. n-Fold Loop Spaces**

A similar line of reasoning shows that every pointed topological space $X_*$ gives an $\mathcal{D}\mathrm{isk}_n$-algebra:

$$\begin{aligned} \Omega^n_* X : \mathcal{D}\mathrm{isk}_n &\to \mathcal{S}\mathrm{paces} \\ \coprod_I \mathbb{R}^n &\longmapsto \mathrm{Maps}_c(\coprod_I \mathbb{R}^n, X_*) \end{aligned}$$

The functor evaluates on an embedding by the obvious "extension by zero". This admits an obvious extension to $\mathcal{M}\mathrm{fld}_n$.

**Example 89. Commutative Algebras** The data of a unital commutative algebra in $\mathcal{C}^{\otimes}$ may be encoded equivalently as:

$$\mathrm{Fin}^{\amalg} \xrightarrow{A} \mathrm{Ch}_k^{\otimes}$$

For example, the value on the map: $* \amalg * \to *$ is (by definition) the multiplication map. Precomposing with $\pi_0$ (which is continuous and symmetric monoidal) gives an n-disk algebra:

$$\mathcal{D}\mathrm{isk}_n \xrightarrow{\pi_0} \mathrm{Fin} \xrightarrow{A} \mathrm{Ch}_k^{\otimes}$$

**Example 90. Associative Algebras**

When $n = 1$, the equivalence mentioned in remark 81 gives a method for extracting an associative algebra:

$$\Delta^{\mathrm{op}} \simeq \mathcal{D}\mathrm{isk}_{1/\mathbb{R}} \to \mathcal{D}\mathrm{isk}_1^{\mathrm{or}} \xrightarrow{A} \mathcal{C}^{\otimes}$$

As there are two nonisotopic embeddings of two 1-disks into a larger 1-disk, there are more than one connected components of the space of binary operations. In other words, this associative algebra need not be commutative. Moreover, by the above remark, this associative algebra comes with a natural involution. Therefore, extending an associative algebra to a 1-disk algebra requires additional data.

**Example 91. Ordinary Homology** Every abelian group gives an n-disk algebra:

$$\mathcal{D}\mathrm{isk}_n^{\amalg} \to \mathcal{S}\mathrm{paces}^{\amalg} \xrightarrow{C_*(-,A)} \mathrm{Ch}_{\mathbb{Z}}^{\oplus}$$

reflecting the fact that the notion of "algebra" depends upon the nature of the symmetric monoidal structure of the target. Algebras of this type may be viewed as "many one-bodies", as it is given by the left Kan extension of it's restriction to disks with at most one connected component. See [B] for details.

The work of Arnol'd [A] and Cohen [Co] constructs a deep relationships between Lie theory and the theory of $E_n$- algebras. This relationship was further formalized in [K2]. In this work, he constructs, for every Lie algebra, an

**Example 92. Higher Enveloping Algebras** Fix a lie algebra, $\mathfrak{g}$, in $\mathrm{Ch}_{\mathbb{R}}^{\otimes}$. Recall that the homological Chevalley-Eilenberg construction determines a symmetric monoidal functor:

$$\mathrm{CE}_* : \mathrm{Alg}_{\mathcal{L}\mathrm{ie}}(\mathrm{Ch}_k)^{\times} \to \mathrm{Ch}_k^{\otimes}$$

This determines an n-disk algebra, denoted $U_n(\mathfrak{g})$:

$$\begin{aligned} \mathcal{D}\mathrm{isk}_n^{\amalg} &\to \mathrm{Alg}_{\mathcal{L}\mathrm{ie}}(\mathrm{Ch}_k)^{\times} \to \mathrm{Ch}_k \\ M &\mapsto \Omega_c^{\bullet}(M) \otimes \mathfrak{g} \mapsto \mathrm{CE}_*(\Omega_c^{\bullet}(M) \otimes \mathfrak{g}) \end{aligned}$$

Here, the commutative, nonunital commutative algebra structure on $\Omega_c^*(-)$ gives $\Omega_c^{\bullet}(M) \otimes \mathfrak{g}$ the structure of a Lie algebra. Integration gives an equivalence:

$$\Omega_c^{\bullet}(\mathbb{R}^n) \otimes \mathfrak{g} \simeq \mathfrak{g}[n]$$

One can show, [K2] page 3, that this Lie algebra will always be abelian, so that:

$$U_n(\mathfrak{g})(\mathbb{R}^n) \simeq \mathrm{Sym}(\mathfrak{g}[n-1])$$

as Chevalley-Eilenberg chains on an abelian/trivial Lie algebra is cofree.

Therefore, when $n = 1$, this restricts to the PBW-equivalence:

$$U_1(\mathfrak{g})(\mathbb{R}) \simeq \mathrm{Sym}(\mathfrak{g})$$

In fact, [K2] shows that the composite:

$$\Delta^{\mathrm{op}} \simeq \mathcal{D}\mathrm{isk}_{1/\mathbb{R}} \to \mathcal{D}\mathrm{isk}_1^{\mathrm{or}} \to \mathrm{Ch}_{\mathbb{R}}^{\otimes}$$

coincides with the classical universal enveloping algebra. See section 4.6 of [G] for an illuminating discussion. Loosely, the fundamental relation in the lie algebra is encoded by coupling the Chevalley-Eilenberg differential (which includes the mulitiplication of compactly supported forms) with the geometry of the real line.

*Remark* 93. One can check that the O(1) action coincides with the standard involution on the universal enveloping algebra.

Formal categorical nonsense identifies the first leg of this construction as the n-fold loop space of the Lie algebra $\mathfrak{g}$ allows one to interpret this construction as:

$$\mathrm{CE}_*(\mathrm{Maps}_c(-, \mathfrak{g}))$$

As given any space $X$, the categorical cotensor $\mathfrak{g}^{X_+}$ admits an explicit model as:

$$\Omega_c^*(X) \otimes \mathfrak{g}$$

This construction is especially useful in applying Lie-theoretic methods to study of the rational homology of configuration spaces. For example, theorem A of [K2] states:

$$\mathrm{Free}_{\mathbb{E}_n}(V) \simeq U_n(\mathrm{Free}_{\mathcal{L}\mathrm{ie}}(V[n-1]))$$

There is a similar description, taking into account $O(n)$ actions, which greatly simplifies for $\mathbb{Q}$ with its trivial $O(n)$-action:

$$C_*(\mathcal{B}(-), \mathbb{Q})) \simeq \mathrm{Free}_{\mathcal{D}\mathrm{isk}_n}(\mathbb{Q}) \simeq U_n\Big(\mathrm{Free}_{\mathcal{L}\mathrm{ie}}\big(\mathbb{R}[n-1]\big)\Big)$$

We now demonstrate a common maneuvre in homotopy theory for extending a local invariant to a global one: left Kan extension of a restriction to a comprehensible full subcategory. In our context, this procedure is conventionally referred to as:

### 4.2 Factorization Homology

#### 4.2.1 Factorization Homology: Definitions

*Remark* 94. Throughout, $\mathcal{C}^{\otimes}$ will denote a presentable symmetric monoidal $\infty$-category whose symmetric monoidal structure distributes over colimits in each variable. In the language of [AF] $\mathcal{C}^{\otimes}$ is $\otimes$-presentable. The example to keep in mind is the $\infty$-category of chain complexes with tensor product.

**Definition 95. Factorization Homology** For every n-disk algebra $\mathcal{A}$, there exists a symmetric monoidal functor:

$$\int_{(-)} \mathcal{A} : \mathcal{M}\mathrm{fld}_n^{\amalg} \longrightarrow \mathcal{C}^{\otimes}$$
$$M \mapsto \int_M \mathcal{A}$$

which is a left Kan extension along the inclusion:

$$\mathcal{D}\mathrm{isk}_n \hookrightarrow \mathcal{M}\mathrm{fld}_n$$

evaluating on $M$ as:

$$\int_M A := \mathrm{colim}\Big(\mathcal{D}\mathrm{isk}_{n/M} \to \mathcal{D}\mathrm{isk}_n \xrightarrow{\mathcal{A}} \mathcal{C}\Big)$$

*Remark* 96. This "definition" is really a theorem. It implicitly invokes the existence of a well-behaved homotopy coherent left Kan extensions. That this left Kan extension may in fact be enhanced to a symmetric monoidal functor is the primary reason for our insistence on $\otimes$-presentability.

As the last part of the above suggests, the existence of a factorization homology functor only requires $\mathcal{C}$ to contain (sifted) colimits. One can define factorization homology of an n-disk stack (which involves a limit). When $\mathcal{C}^{\otimes}$ is the $\infty$-category of chain complexes and tensor product, this may not naturally extend to a symmetric monoidal functor, due to the incompatibility of the tensor product with, for example, totalizations.

*Remark* 97. As one can imagine, factorization homology admits an extension to $\mathcal{M}\mathrm{fld}_n^\partial$ given an extension of $\mathcal{A}$ to $\mathcal{D}\mathrm{isk}_n^\partial$

### 4.2.2 Finality Theorems

The fundamental building blocks of this theory are a variety of finality theorems. These types of results are the primary entree of techniques in differential topology. The most famous relates factorization homology to bar constructions, a proof of which may be found in section 3.2 of [AF]:

**Theorem 98.** *There exists a final functor:*

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}^\partial_{1/[-1,1]}$$

*Therefore, factorization homology may be computed as a bar construction.*

*Remark* 99. Writing this functor down on a point set level (e.g. as a map of simplicial sets) is quite laborious, and in some sense superfluous. [AF] construct this map via zig-zags which only requires an existence proof, which can be efficiently accomplish via point-set methods (contractibility of embedding spaces).

**Theorem 100.** *As stated previously, the composition:*

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}^\partial_{1/[-1,1]}$$

*classifies the data of an associative unital algebra, a unital left and right module thereover. It's colimit computes the classical bar construction.*

The a proof of the following may be found in proposition 2.19 of [AF] or proposition 5.5.2.13 of [Lu2]

**Theorem 101.** *The natural functor:*

$$\mathrm{Disk}_{n/M} \to \mathcal{D}\mathrm{isk}_{n/M}$$

*is a localization along the set of maps which induce a bijection on connected components. In particular, it is final.*

*Remark* 102. The above theorem forms a useful bridge from differential geometric methods. For example, one can imagine the efficiently constructing functors out of $\mathrm{Disk}_{n/M}$ via more classical means. The theorem gives a tractable condition to check to produce a factorization through the more intricate $\mathcal{D}\mathrm{isk}_{n/M}$.

### 4.2.3 Factorization Homology: Pushforwards and Formal Properties

As factorization homology is covariant with respect to embeddings, one should expect it to behave as precosheaf. For example, there should be a pushforward operation.

**Definition 103.** Given a smooth map $f : M \to Y$, and an n-disk algebra $\mathcal{A}$, we let:

$$f_*\mathcal{A} : \mathcal{M}\mathrm{fld}^{\partial}_{k/Y} \xrightarrow{f^{-1}} \mathcal{M}\mathrm{fld}^{\partial}_{n/M} \xrightarrow{\int \mathcal{A}} \mathcal{C}^{\otimes}$$

be the pushfoward of $\mathcal{A}$ along $f$. We will denote it's colimit as:

$$\int_Y f_*\mathcal{A}$$

**Example 104.** A classical example to keep in mind is when $f = |r|^2 : \mathbb{R}^n \to \mathbb{R}^{\geq 0}$, the projection: $x^0 : \mathbb{R}^{n+1} \to \mathbb{R}$, a projection $M \times N \to N$, and $f : M \to [-1, 1]$

whose restriction to $\pm 1$ and $(0,1)$ is a smooth fibre bundle. The last example may be obtained from the data of a codimension 1 properly embedded submanifold and a partition of the complement.

The following is lemma 3.18 of [AF]:

**Theorem 105.** *When the restriction of $f$ to the boundary and interior are smooth fibre bundles respectively, the natural map:*

$$\int_Y f_*\mathcal{A} \simeq \int_M \mathcal{A}$$

*is an equivalence.*

These theorems combine to relate factorization homology to bar constructions, the details of which may be found in Section 3 of [AF]:

**Example 106.** A deconstruction of $M \simeq M_- \amalg_{M_0 \times \mathbb{R}} M_+$ can be encoded as some map $f : M \to [-1,1]$. The preimage under $f$ gives a functor:

$$\mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[-1,1]} \xrightarrow{f^{-1}(-)} \mathcal{D}\mathrm{isk}_{n/M}$$

which we can precompose with aany n-disk algebra $\mathcal{A}$ to give:

$$\mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[-1,1]} \xrightarrow{f^{-1}(-)} \mathcal{D}\mathrm{isk}_{n/M} \xrightarrow{\mathcal{A}} \mathcal{C}^{\otimes}$$

By the above remarks, this functor classifies the data of an associative algebra, $(f|_{(-1,1)})_*\mathcal{A}$, a right module $(f|_{(-1,1]})_*\mathcal{A}$, and a left module $(f|_{[-1,1)})_*\mathcal{A}$. The underlying objects are $\int_{M_0 \times \mathbb{R}} \mathcal{A}$, $\int_{M_+} \mathcal{A}$, and $\int_{M_-} \mathcal{A}$, respectively.

Precomposing with the above final functor gives a simplicial object in $\mathcal{C}^{\otimes}$

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[-1,1]} \xrightarrow{f^{-1}(-)} \mathcal{D}\mathrm{isk}_{n/M} \xrightarrow{\mathcal{A}} \mathcal{C}^{\otimes}$$

Combining the above finality theorems gives a string of equivalences:

$$\int_M \mathcal{A} \simeq \int_{[-1,1]} f_*(\mathcal{A}) \simeq \int_{M_+} \mathcal{A} \bigotimes_{\int_{M_0 \times \mathbb{R}} \mathcal{A}} \int_{M_-} \mathcal{A}$$

## 4.3 Computations and Applications

We begin by presenting a result relating a factorization homology with a more classical perspective on ordinary homology:

**Example 107.** Ordinary Homology

[B] shows that:

$$\int_X C_*(-, A) \simeq H_*(X, A)$$

By showing that $C_*(-, A)$ may be obtain by the left Kan extension of its restriction to the category of embedding disks with at most two connected components, one of which is required to contain a fixed point.

**Example 108.** Dold-Thom

Recall that the classical-Dold Thom Theorem relates the homology of a topological space, $X$, with coefficients in an abelian group, $A$, with the homotopy groups of a topological space, called the infinite symmetric product of $X$ with coefficients in $A$, $\mathrm{Sym}(X, A)$. Every point of this space takes the form of an unordered configuration of points (of an arbitrary cardinality) labelled by elements of $A$. Loosely, the topology is so that nearby points can collide, adding their labels, and points labelled by the identity can spontaneously appear. In [B], Bandklayder uses the machinery of factorization homology to prove, (when $X$ is an n-manifold) this theorem:

$$\pi_*(\mathrm{Sym}(X, A)) \simeq H_*(X, A)$$

One appealing aspect of factorization homology is that it does not just give manifold invariants: it also constructs invariants of algebraic objects. This would be akin to viewing a manifold as a potential invariant of a field theory.

**Example 109.** Hochschild Homology

A common heuristic for Hochschild homology is that it produces an invariant of associative algebras by tensoring over the circle. Factorization homology makes this intuition precise:

$$\int_{S^1} A \simeq \mathrm{HC}_*(A)$$

Theorem 5.5.3.11 of[Lu2] proves this by relating factorization homology directly to the cyclic bar construction, while Theorem 3.19 of [AF] use the expression of Hochschild homology as a bar construction:

$$A \otimes_{A^{\mathrm{op}} \otimes A} A$$

When the algebraic input is simple, factorization homology yields a potentially more familiar object: labeled configuration spaces. The following is proposition 5.5 of [AF]

**Example 110.** Configuration Spaces

Given a chain complex $V$, formal nonsense allows one to define the free n-disk algebra generated by $V$, $\mathrm{Free}_n(V)$. In this case, a "hypercover argument" shows that:

$$\int_M \mathrm{Free}_n(V) \simeq \coprod_{k \geq 0} C_*\big(\mathrm{Conf}_k(M)\big) \otimes_{\Sigma_k} V^{\otimes k}$$

In particular, when $V = \mathbb{Q}$, we see that factorization homology computes the rational homology of configuration spaces:

$$\int_M \mathrm{Free}_n(\mathbb{Q}) \simeq C_*(\mathcal{B}(M), \mathbb{Q})$$

A similar result holds for manifold with boundary, but one quotients by configuration with at least one point lying on the boundary:

$$\int_M \mathrm{Free}_n(V) \simeq \coprod_{k\geq 0} C_*\big(\mathrm{Conf}_k(M, \partial M)\big)\otimes_{\Sigma_k} V^{\otimes k}$$

When $M$ is the closed unit ball $D^n$:

$$\coprod_{k\geq 0} \mathrm{Conf}_k(D^n, \partial D^n)_{\Sigma_k} \simeq * \amalg \mathrm{S^n}$$

as

$$\mathrm{Conf}_k(D^n, \partial D^n)_{\Sigma_k} \simeq *$$

for $k \geq 2$ and

$$\mathrm{Conf}_1(D^n, \partial D^n) \simeq S^n$$

Putting this together, we see that:

$$\int_{D^n} \mathrm{Free}_n(V) \simeq \mathbb{Q} \oplus V[n]$$

*Remark* 111. Note the computation of the homology of configuration spaces are not amenable to the standard techniques in algebraic topology. For example, as the diagonal will cut through any cell decomposition of $M$, a cell decomposition of $M$ does not lift to a CW decomposition of its configuration spaces.

*Remark* 112. These examples illustrate dual applications of factorization homology, which we view as a pairing between a "local" algebraic object and a "global" geometric object. When the geometric object is simple (e.g. a circle), factorization homology can be viewed as an invariant of algebraic objects (Hochschild homology). When the algebraic object is simple (e.g. a free algebra) factorization homology may be viewed as an invariant of the geometric object (labelled configuration spaces).

Factorization homology is intimately connected to two forms of duality. The first form is

**Example 113.** (Non-Abelian) Poincare Duality

Broadly speaking, Poincare Duality is a dictionary between homological invariants of an $n$-manifold and cohomological invariants of its one-point compactification. For example, when factorization homology is taken with respect to an $n$-fold loop space, its cohomological counterpart is the space of compactly supported maps into $X$.

As factorization homology is a left Kan extension, one can construct a "scanning map":

$$\int_M \Omega^n(X) \to \mathrm{Maps}_c(M, X)$$

Note that although, this functor can be constructed through purely higher categorical means, one can obtain a more geometric description. Loosely, Pontryagin-Thom collapse maps provide a diagram:

$$\begin{tikzcd}
\mathcal{D}\mathrm{isk}_{n/M} \arrow[r] \arrow[d] & \mathcal{D}\mathrm{isk}_n \arrow[dr, "\Omega^n_*(X)"] & \\
\left(\mathcal{D}\mathrm{isk}_{n/M}\right)^{\triangleright} \arrow[rr] & & \mathrm{Top} \\
* \arrow[u] \arrow[urr, "\mathrm{Maps}_c(M{,}X)"'] & &
\end{tikzcd}$$

that provides the necessary data to induce the above scanning map.

That these two maps agree (up to homotopy) follow from the universal property of left Kan extensions.

In general, this map is not an equivalence. However, it is when $X$ is n-connective. For example, when $M$ is the closed unit ball, factorization homology provides a delooping.

Loosely, the scanning maps surjects onto the subspace of maps which send the $(n-1)$-skeleta of some triangulation of $M$ to the basepoint. In some zany higher categorical sense, the left hand side is a "perturbative approximation" of the right hand side.

The above result passes through to manifolds with boundary, so that when M is a closed unit ball, $D^n$, and $X$ is n-connective:

$$\int_M \Omega^n(X) \simeq X$$

factorization homology provides a delooping of $\Omega^n_*(X)$.

Proposition 3.19 of [K] computes factorization homology of

**Example 114.** Higher Enveloping Algebras

When factorization homology has coefficients in a higher enveloping algebra of a Lie algebra, the result admits a Lie-theoretic description

**Theorem 115.** *There exists an equivalence:*

$$\int_M U_n(\mathfrak{g}) \simeq \mathrm{CE}_*\big(\mathrm{Maps}_c^{O(n)}(\mathrm{Fr}_{M^+}, \mathfrak{g})\big)$$

*Where the right-hand side involves the space of compacting supported $O(n)$-equivariant maps from the frame bundle of the one-point compactication of $M$ into $\mathfrak{g}$. Furthermore, a framing on $M$ induces an equivalence:*

$$\int_M U_n(\mathfrak{g}) \simeq \mathrm{CE}_*\big(\mathrm{Maps}_c(M, \mathfrak{g})\big)$$

In the case of a framing, we can combining this result with the equivalence above gives:

$$
\begin{aligned}
H_*(B(M),\mathbb{Q}) &\simeq \int_M \mathrm{Free}_{\mathcal{D}\mathrm{isk}_n}(\mathbb{Q}) \\
&\simeq \int_M U_n\big(\mathrm{Free}_{\mathcal{L}\mathrm{ie}}(\mathbb{Q}[n-1])\big) \\
&\simeq \mathrm{CE}_*(\Omega_c^*(M)\otimes\mathbb{Q}[n-1])
\end{aligned}
$$

Furthermore, the nilpotence of $\mathrm{Free}_{\mathcal{L}\mathrm{ie}}(\mathbb{Q}[n-1])$ “dampens” any Massey product in $M$. Consequently, $\Omega_c^*(M)\otimes\mathbb{Q}[n-1]$ is in fact formal, so that:

$$
H_*(B(M),\mathbb{Q}) \simeq \mathrm{CE}_*(H_c^*(M)\otimes\mathbb{Q}[n-1])
$$

One can further show that this equivalence is compatible with the cardinality/weight filtrations, allowing one to pick out the homology of the each unordered configuration spaces inside the Chevalley-Eilenberg complex. For a general n-manifold, the rational $O(n)$-equivariant formality of the $n$-sphere produces a similar equivalence, where $\mathbb{Q}$ is replaced by the orientation sheaf.

In summary, this computation splits the computation of the rational homology of configuration spaces of *any manifold* into two pieces: an explicit representation of the rational cohomology ring of $M$, and a formal exercise in Lie homology. In Knudsen's words [K1]:

> Locally, then, configuration spaces enjoy a rich algebraic structure; factorization homology, our primary tool in this work, provides a means of assembling this structure across coordinate patches of a general manifold, globalizing the calculation of Arnold and Cohen.

This was exploited in [D-CK], where they compute the Betti numbers of all surfaces of finite type. They state an interesting feature that comes out of this work, their approach deals with the configuration spaces of all cardinalities simultaneously.

In their words: "It is this simultaneity that renders the computations feasible in practice".

*Remark* 116. We emphasize that, although factorization homology requires a significant background in abstract homotopy theory to define, it enjoys a myriad of powerful formal properties which make it amenable to explicit computations. Moreover, these formal properties have been rigorously established in the literature. In the author's opinion, [D-CK] is the best illustration of this: upon the application of factorization to the relationship between Lie homology and configuration spaces, [D-CK] have no need to make any explicit reference to either factorization homology or $\infty$-categories.

# CHAPTER 5

# FACTORIZATION ALGEBRAS

Whereas factorization homology is defined for every manifold of a fixed dimension and geometry, factorization algebras live on a single manifold. We begin by recalling the definition of a factorization algebra, following [CG1].

## 5.1 Prefactorization Algebras

**Definition 117.** For a topological space $X$, we define a multicategory $\mathrm{Disj}(X)$ as follows. An object of $\mathrm{Disj}(X)$ consists of an open set of $X$. There is a single multimorphism from $(U_1, \ldots, U_n) \to V$ whenever $\{U_i\}$ are disjoint and contained within $V$.

Throughout, we will assume $X$ to be a smooth manifold of dimension $n$.

**Definition 118.** A *prefactorization algebra on* $X$ with values in $\mathcal{C}^{\otimes}$ is a $\mathrm{Disj}(X)$-algebra in $\mathcal{C}^{\otimes}$. We call a prefactorization algebra *locally constant* if it sends any inclusion of open disks to an equivalence in $\mathcal{C}^{\otimes}$.

*Remark* 119. Note that a prefactorization algebra $\mathcal{F}$gives in particular a precosheaf on $X$:

$$\mathcal{F} : \mathrm{Opens}(X) \to \mathcal{C}$$

along with a compatible family of equivalences:

$$\mathcal{F}(\coprod_{i \in I} U_i) \simeq \bigotimes_{i \in I} \mathcal{F}(U_i)$$

in $\mathcal{C}$.

**Definition 120.** A prefactorization algebra $\mathcal{F}$ is *locally constant* if given any inclusion of open sets:

$$U \hookrightarrow V \hookrightarrow X$$

so that $U$ and $V$ are diffeomorphic to $\mathbb{R}^n$, the induced map:

$$\mathcal{F}(V) \to \mathcal{F}(U)$$

is an equivalence.

**Example 121.** The following is an very nonrigorous example, which requires defining a well-behaved symmetric monoidal category of vector spaces decorated with suitable analytic structures, which we'll abusively denote as $\mathrm{Vect}^{\otimes}$. Such a category may be found in appendix B of [CG1].

Fix a vector bundle $E \to X$, the assignment of sections:

$$\mathcal{E} : \mathrm{Opens}(X)^{\mathrm{op}} \to \mathrm{Vect}^{\otimes}$$

is a presheaf. "Formal functions" on this:

$$\begin{aligned} \mathcal{O}_{\mathcal{E}} : \mathrm{Opens}(X) &\to \mathrm{Vect}^{\otimes} \\ U &\mapsto \mathrm{Sym}(\mathcal{E}^{\vee}(U)) \end{aligned}$$

extends to a prefactorization algebra. In fact, one can use this example to reverse engineer many of the analytic details in [CG1].

The essential technical ingredients underlying this theory provide a well-behaved setting to perform standard cohomological and operadic maneuvres. most of which is provided by paramatrix methods in the theory of elliptic operators. We refer the interested reader to the Appendices of [CG1] for details. [Mn] provides a clear exposition on how these techniques relate to more familiar physics approaches to field theory.

*Remark* 122. The primary motivation behind this definition lies in the fact that the observables of a statistical field theory should give rise to a prefactorization algebra. However, notice that when $X = \mathbb{R}^n$, the values of a prefactorization algebra need not carry a representation of the special orthogonal group. Therefore, a prefactorization algebra lacks an essential ingredient of any field theory. However, this is more a reflection of the difficulty of quantizing symmetries than a lack of desire for such a representation. See page 178-179 of [CG1] for more.

### 5.1.1 Examples of Prefactorization Algebras

We know give a few formal examples of prefactorization algebras.

**Example 123.** Massive Scalar Field Theory

A theorem of [CG1] show that massive scalar field theory on an oriented Riemannian n-manifold admits a relatively elementary "differential graded" description. As a graded vector space, it is:

$$\mathrm{Obs}^q = \mathrm{Sym}(\mathrm{C}_c^\infty \oplus \mathrm{C}_c^\infty[1])[\hbar]$$

Its differential is coderivation. It is the sum of a weight one term:

$$\begin{aligned} \mathrm{C}_c^\infty[1] &\xrightarrow{\mathrm{d}^{cl}} \mathrm{C}_c^\infty \\ \phi^\dagger[1] &\mapsto (\Delta_g + m^2)\phi^\dagger \end{aligned}$$

and a weight two term, given as the symmetrization of:

$$\mathrm{C}_c^\infty \otimes \mathrm{C}_c^\infty[1] \xrightarrow{\hbar\Delta_{\mathrm{BV}}} \mathbb{R}\{\hbar\}$$
$$\phi \otimes \phi^\dagger[1] \mapsto \hbar \int \phi \cdot \phi^\dagger$$

The assembly of this data into a rigorous object requires a significant effort, the details of which may be found scattered throughout [CG1]. Note that the functional:

$$S(\phi) = \int \phi \cdot (\Delta_g + m^2)\phi$$

can be expressed using the above data as:

$$\Delta_{\mathrm{BV}}(\phi \otimes \mathrm{d}^{cl}\phi)$$

[CG1] show that that this data assembles into a locally constant prefactorization algebra.

The following shows how factorization homology produces examples of prefactorization algebras.

**Example 124.** Every n-disk algebra, $A$ gives a prefactorization algebra on $X$, $\mathcal{F}_A$:

$$U \subset X \longmapsto \mathcal{F}_A(U) := \int_U A$$

Note that the topology on the mapping spaces of $\mathcal{D}\mathrm{isk}_n$ imply that $\mathcal{F}_A$ is locally constant.

**Example 125.** We now utilize factorization homology to briefly outline how to re-extract the Lie bracket from its higher enveloping algebra, using the concrete definition:

$$\mathrm{CE}(\Omega_c^*(-) \otimes \mathfrak{g})$$

Here, we are thinking of the standard presentation of the Chevalley-Eilenberg chains using the Chevalley-Eilenberg differential. Before doing so, we introduce a convenient notation. We'll denote:

$$\phi \otimes c = c_\phi \in \Omega^*_c \otimes \mathfrak{g}$$

This notation is so that:

$$[c_{\phi_0}, c'_{\phi_1}] = (-1)^{|\phi_1|\cdot|c||}[c, c']_{\phi_0\phi_1}$$

As the purpose of this discussion is purely expository, we'll take $\mathfrak{g}$ to be concentrated in degree 0.

Our Lie bracket will arise from evaluating factorization homology on:

$$B_0(\epsilon) \coprod S^{n-1} \times (\epsilon, 2\epsilon) \hookrightarrow \mathbb{R}^n$$

Which gives a map:

$$\mathrm{CE}_*(\Omega^*_c(B_\epsilon(0)) \otimes \mathfrak{g}) \otimes \mathrm{CE}_*(\Omega^*_c(S^{n-1} \times (\epsilon, 3\epsilon)) \otimes \mathfrak{g}) \to \mathrm{CE}_*(\Omega^*_c(\mathbb{R}^n) \otimes \mathfrak{g})$$

We will use the following forms in our discussion:

- $\rho \in \Omega^n_c(B_\epsilon(0))$ integrating to one and closed. In other word, this form is a Poincare dual to the origin.
- $\eta \in \Omega^1_c(S^{n-1} \times (\epsilon, 2\epsilon))$, a Thom class for the normal bundle of $S^{n-1} \times \{2\epsilon\}$. This form is Poincare Dual to $S^{n-1} \times \{2\epsilon\}$.

Note that both of these forms are closed. Neither are coexact.

By construction, the image of $c_\rho$ and $c'_\eta$ inside of $\mathrm{CE}_*(\Omega^*_c(\mathbb{R}^n) \otimes \mathfrak{g})$ is:

$$c_\rho \cdot c'_\eta$$

Although $\eta$ was not coexact inside of $\Omega_c^*\big(S^{n-1} \times (\epsilon, 3\epsilon)\big)$, it is when extended by zero inside of $\Omega_c^*(\mathbb{R}^n)$, by the Poincare Lemma. That is, there exists $\phi$ so that:

- $\mathrm{d}_{\mathrm{dr}}\phi = \eta$
- $\phi|_{\mathrm{supp}(\eta)} = -1$

We will now use this form to construct a homotopy from $c_\rho \cdot c_\eta$ to a form representing the Lie bracket of $c$ and $c'$.

**Lemma 126.**

$$\mathrm{d}_{\mathrm{CE}}(c_\rho \cdot c'_\phi) = c_\rho \cdot c'_\eta - [c, c']_\rho$$

*Proof.* The proof is a straightforward computation:

$$\begin{aligned} \mathrm{d}_{\mathrm{CE}}(c_\rho \cdot c'_\phi) &= c_\rho \cdot c'_{\mathrm{d}_{\mathrm{dr}}\phi} + [c_\rho, c'_\phi] \\ &= c_\rho \cdot c'_\eta + [c, c']_{\rho\phi} \end{aligned}$$

But the second property of $\phi$ implies that $\rho\phi = -\rho$. Therefore, we can conclude:

$$\mathrm{d}_{\mathrm{CE}}(c_\rho \cdot c'_\phi) = c_\rho \cdot c'_\eta - [c, c']_\rho$$

□

*Remark* 127. We emphasize that, despite the notation, $c_\rho \cdot c'_\phi$ does not arise from factorization homology applied to some embedding, as the support of $\rho$ and $\phi$ are not disjoint (Chevalley-Eilenberg chains do not form a commutative algebra!). The "ordering" of an expression being multiplied is encoded in the support of the forms, *not* the ordering of the "multiplication" inside of Sym. This reflects the general benefit of operads: it frees one from the (at times awkward and oppressive) geometry of conventional type setting.

Passing to homology, we obtain:

**Corollary 128.** *Under the equivalence (given by integration):*

$$c_\rho \cdot c'_\eta = [c, c']_\rho$$

*where the left hand side is the multiplication supplied by the factorization algebra structure.*

*Remark* 129. This reflects one of the advantages of factorization homology. The ability to evaluate an $E_n$-algebra on a general manifold permits“”descent”-type descriptions of the $(1-n)$ shifted Lie bracket of a Lie algebra.

In order to obtain a more tractable object, we articulate a locality condition on a prefactorization algebra.

## 5.2 Weiss CoSheaves: The Appropriate Notion of Locality

One shouldn't expect *observables* of field theory to be local with respect to the ordinary topology. For example:

$$\begin{aligned} \mathrm{Mfld}_0 \simeq \mathrm{Fin} &\to \mathrm{Vect}_{\mathbb{R}} \\ I &\longmapsto \mathrm{Sym}(\mathbb{R}^I) \simeq \mathbb{R}[x_1, \dots, x_{|I|}] \end{aligned}$$

is not determined by a Kan extension of its restriction to $* \in \mathrm{Mfld}_0$, as a polynomial in multiple variables may contain nonzero mixed partial derivatives. Therefore, this functor is not local in with respect to the standard Grothendieck topology. It is however, local with respect to the Weiss topology.

**Definition 130.** A set of open embeddings $\{U_i \hookrightarrow X\}$ is a *Weiss Cover*, if, for any finite subset $S \subset X$, $S \subset U_i$ for some $i$.

**Definition 131.** A prefactorization algebra $\mathcal{F}$ is a *factorization algebra* if its underlying precosheaf:

$$\mathcal{F} : \mathrm{Opens}(X) \to \mathcal{C}$$

is a cosheaf with respect to the Weiss topology on $X$

*Remark* 132. A more precise, general definition of a prefactorization/factorization algebra may be found in section 3 of [AF2].

The locality of factorization algebras can be used to prove the following, the details of which may be found in chapter 7 of [CG1].

**Theorem 133.** *If a map of factorization algebras on an n-manifold $M$:*

$$\mathcal{F} \xrightarrow{f} \mathcal{G}$$

*induces an equivalence on their restriction to:*

$$\mathcal{D}\mathrm{isk}_{n/M} \hookrightarrow \mathrm{Opens}(M) \to \mathcal{C}$$

*then $f$ is an equivalence. In particular, it induces an equivalence on global cosections:*

$$\mathcal{F}(M) \simeq \mathcal{G}(M)$$

*Remark* 134. The most important feature of this theorem is that the condition to be checked involves constructions living on disjoint unions of Euclidean neighborhoods. For example, if $\mathcal{F}$ is given by some PDE-type construction, the above theorem translates a condition about PDE's on manifolds into a condition about PDEs on $\mathbb{R}^n$

The following is Proposition 3.4 of [AF2]:

**Theorem 135.** *Given any n-disk algebra $\mathcal{A}$, the prefactorization algebra*

$$\int_{(-)} \mathcal{A} : \mathrm{Opens}(X) \to \mathcal{C}^{\otimes}$$

*is in fact a locally constant factorization algebra.*

**Example 136.** We imagine the following situation. Say we're given a locally constant factorization algebra, $\mathrm{Obs}^q$ on a connected n-manifold $M$. For example, the observables of field theory constructed in some analytic/BV formalism. For example, [CG1] use a spectral sequence argument to show that massive scalar field theory is in fact a factorization algebra on any Riemannian 1-manifold (whose cohomology depends discontinuously on the circumference).

Furthermore, let's say we have constructed an n-disk algebra $\mathcal{A}$, a map of algebras:

$$\mathcal{A}|_{\mathrm{Disk}_{n/M}} \to \mathrm{Obs}^q|_{\mathrm{Disk}_{n/M}}$$

along with a proof that the map:

$$\mathcal{A}(\mathrm{B}(x)) \to \mathrm{Obs}^q(\mathrm{B}(x))$$

is an equivalence for some euclidean neighborhood of some $x \in M$. Then the natural map:

$$\int_M \mathcal{A} \xrightarrow{\simeq} \mathrm{Obs}^q(M)$$

is an equivalence. Note we still have a comparison map, even if the algebra map is not an equivalence locally.

**Example 137.** What's important about this technique is that factorization homology has tools not easily obtainable through the formalism of [CG1]. For example, [CG1] and [AF2] utilize different filtrations. Although, the approach in [CG1] and [CG2] is inspired by Koszul duality, factorization homology has explicit and rigorous applications of Koszul (and Poincare) duality [K2].

## APPENDIX A

## QUILLEN'S THEOREM A

Before proceeding, we will briefly review a key technical tool in the analysis below: Quillen's Theorem A. Our discussion will be relatively informal. The reader may find more details in [Lu], which contains a rigorous proof in the language of quasicategories. Those familiar with these techniques may safely skip to the next section.

$$\mathcal{D}$$

The basic setup is the following. We are interesting in studying colimits of the form:

$$F : \mathcal{D} \to \mathcal{A}$$

for a wide range of $\mathcal{A}$ and $F$. One might try to understand a piece of this colimit by producing a functor

$$\mathcal{C} \xrightarrow{\iota} \mathcal{D}$$

and analyzing its induced map on colimits:

$$\operatorname{colim}\left(\mathcal{D} \xrightarrow{F} \mathcal{A}\right) \to \operatorname{colim}\left(\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{F} \mathcal{A}\right)$$

Quillen's Theorem A gives a condition under which this map is an equivalence in $\mathcal{A}$.

The condition consists of checking, for every object $d \in \mathcal{D}$, the contractibility of the classifying space of a category (over $\mathcal{C}$) built from the data of $\iota$ and $d$. In particular, the condition is independent of $F$ and $\mathcal{A}$, and therefore holds for the colimit of any functor out of $\mathcal{D}$. In other words, it allows one to replace a colimit over $\mathcal{D}$ with a colimit over (the hopefully simpler) $\mathcal{C}$.

Before stating the theorem, we review the requisite categorical constructions and definitions.

**Definition 138.** We call a functor $\iota : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories **final** if, for any functor $F : \mathcal{D} \to \mathcal{A}$, the natural map:

$$\operatorname{colim}\big(\mathcal{D} \xrightarrow{F} \mathcal{A}\big) \to \operatorname{colim}\big(\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{F} \mathcal{A}\big)$$

is an equivalence.

We now review a standard construction in category theory, extended to $\infty$-categories.

**Definition 139.** Given a functor $\iota : \mathcal{C} \to \mathcal{D}$ and object $d \in \mathcal{D}$, we define the space of $(\mathcal{J} \to \mathcal{C})$-points of ${}^{d/}\mathcal{C} \to \mathcal{C}$ to be the fillers

$$\begin{tikzcd} * \arrow[d, hook] \arrow[dr, "d"] & \\ \mathcal{J}^{\triangleright} \arrow[r, dotted] & \mathcal{D} \\ \mathcal{J} \arrow[u, hook] \arrow[r] & \mathcal{C} \arrow[u, "\iota"'] \end{tikzcd}$$

Where $\mathcal{J}^{\triangleright}$ is defined to be:

$$\mathcal{J}^{\triangleright} = \operatorname{colim}\Big(\mathcal{J} \times \{0 \leq 1\} \hookleftarrow \mathcal{J} \times \{1\} \to *\Big) \in \mathcal{C}\mathrm{at}_\infty$$

One obtains the definition in the setting of ordinary categories by noting that $[n]^{\triangleright} = [n+1]$. For example, an object of ${}^{d/}\mathcal{C}$ is the data of a $c \in \mathcal{C}$ along with a map $\iota(c) \to d$. A map in ${}^{d/}\mathcal{C}$ is the data of a map $f : c \to c'$, along with a commutative triangle in $\mathcal{D}$ of the form:

$$\begin{tikzcd}
\iota(c) \arrow[rr, "\iota(f)"] \arrow[dr] & & \iota(c') \arrow[dl] \\
& d &
\end{tikzcd}$$

**Example 140.** If $\mathcal{D}$ is a partially ordered set, and $\iota$ is a full inclusion of a subset, the ${}^{d/}\mathcal{C}$ is the partially ordered set of elements of $\mathcal{C}$ less than $d$.

**Example 141.** Applying the above construction to $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$ gives ${}^{c/}\mathcal{C}$, the "category of elements of $c$". The Yoneda Lemma states that the inclusion:

$$* \xrightarrow{\mathrm{Id}_{\mathcal{C}}} {}^{c/}\mathcal{C}$$

witness the identity as a terminal object.

We now review the classifying space of an $\infty$-category:

**Definition 142.** Given an $\infty$-category $\mathcal{J}$, we define its **classifying space** as:

$$\mathrm{B}\mathcal{J} \simeq \mathrm{colim}\Big(\mathcal{J} \to * \to \mathcal{S}\mathrm{paces}\Big)$$

The standard nonsense states that this extends to a colimit preserving functor:

$$\mathrm{B} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}\mathrm{paces}$$

whose right adjoint is the functor that views a space as an $\infty$-groupoid. coincides with the familiar "geometric realization of the nerve of $\mathcal{J}$". Moreover, the counit of this adjunction gives a natural map:

$$\mathcal{C} \to \mathrm{B}\mathcal{C}$$

**Example 143.** A standard result states that the classifying space of a category with an initial (or terminal) object is contractible. This can be seen by noting that the pullback of a map of spaces $S \to \mathrm{B}\mathcal{C}$ along the counit map, and factoring the associate map into $\tilde{S} \to \mathcal{C}$ through $\tilde{S}^{\triangleleft} \to \mathcal{C}$ (which is guaranteed to exist by assumption). After proving the classify space of a cone is contractible, taking the classifying space of this diagram gives a factorization of $S \to \mathrm{B}\mathcal{C}$ through a point, thus proving the contractiblity of $\mathrm{B}\mathcal{C}$.

With these, in place, we are now able to state:

**Theorem 144.** ***Quillen's Theorem A***

*Let $\iota : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. Then $\iota$ is final if and only if, for each object $d \in \mathcal{D}$*

$$\mathrm{B}({}^{d/}\mathcal{C}) \simeq *$$

**Example 145.** We can use this to see that the inclusion of a terminal object $* \xrightarrow{x} \mathcal{C}$ is final. This follows from the fact that the category ${}^{d/}*$ is the space of maps from $d$ into $x$, which is equivalent to a point. This reflects the well-known fact that the colimit over a category with a terminal object is the evaluation of functor on the terminal object.

**Example 146.** When $\iota$ is the inclusion: $\{a, a+1, \ldots\} \hookrightarrow \mathbb{N}$, then ${}^{n/}\{a, a+1, \ldots\}$ is subset of natural numbers greater than or equal to $n$. As this always has a minimal element, either $n$ or $a$, this functor is final. This reflects, for example, the obvious fact about unions of topological spaces:

$$\bigcup(U_0 \hookrightarrow U_1 \hookrightarrow \ldots) \simeq \bigcup(U_a \hookrightarrow U_{a+1} \hookrightarrow \ldots)$$

We now give some examples of how this theorem may be applied to the theory of factorization algebras/homology.

**Example 147.** Another important class of final functors come from localizations. In particular, Dwyer-Kan localizations of ordinary categories. For example, Proposition 2.9 of [AF] asserts that the natural functor:

$$\mathrm{Disk}_{n/M} \to \mathcal{D}\mathrm{isk}_{n/M}$$

is a localization along isotopy equivalences. This results provides a means of computing a colimit over the $\infty$-operad of embedded disks in terms of point-set level data provided by differential topology/geometry.

Proving this non-trivial result requires the application of classical methods in differential topology, along with results originally intended for applications in abstract ($\mathbb{A}^1$-)homotopy theory. In some sense, the theory of $\infty$-categories provides one with the conditions one must check with more context-specific methods.

We now outline how these arguments may be used to relate the theory of locally constant factorization algebras (and therefore a large class of field theories) to classical constructions in homological algebra.

**Example 148.** A special case of Proposition 2.22 of [AFT] demonstrates that the functor $\mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]} \to \mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[0,L]}$ is a localization, and is therefore final.

The contractibility of the latter $\infty$-category's mapping spaces allows one to recognize $\mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[0,L]}$ as equivalent to the $\infty$-operad parameterizing the data of a unital associative algebra, and unital left and right $A$-modules $M$ and $N$ (See section 3.2 of [AF]). Therefore, such data may be extracted from the data of a locally constant prefactorization algebra on a closed interval. Lemma 3.11 of [AF] constructs a functor:

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathrm{isk}^{\mathrm{or},\partial}_{1/[0,L]}$$

sending $[n]$ to a union of the left and right half intervals and $n$ open subintervals.

Therefore, the composition of such an $(M, A, N)$ evaluates as:

$$[n] \longmapsto M \otimes A^{\otimes n} \otimes N$$

sends degeneracies maps to "insertion of the identity", and face maps to multiplications. Therefore the precomposition of this functor with $(M, A, N)$ shows the resulting diagram is the simplicial object whose homotopy colimit (when $(M, A, N)$ takes values in an appropriate model category) computes:

$$M \otimes_A^{\mathbb{L}} N$$

As Lemma 3.11 of [AF] proves that this functor is a final, stringing these results together shows that:

$$M \otimes_A N \simeq \operatorname{colim}\Big(\mathrm{Disk}^{\mathrm{or},\partial}_{1/[0,L]} \overset{(M,A,N)}{\longrightarrow} \mathcal{C}^{\otimes}\Big)$$

This is in accord with the relationship between the factorization condition on a locally constant prefactorization algebra on a closed interval in terms of the above tensor product [CG1][G].

# APPENDIX B

# SYMMETRIC MONOIDAL CATEGORIES

We begin by presenting a definition of a symmetric monoidal category. However, before doing so, it will be convenient to recall some of the basic vocabulary around the category parametrizing unital, commutative multiplicative structures:

## B.1 Pointed Finite Sets

Morphisms in $\mathrm{Fin}_*$ factor into two classes, "active" and "inert". These classes serve different purposes.

Note that there is another functor from the category of finite sets and injections:

$$(\mathrm{Fin}^{\mathrm{inj}})^{\mathrm{op}} \overset{(-)^+}{\hookrightarrow} \mathrm{Fin}_*$$

which is a discrete Pontryagin-Thom collapse map. Maps in the image of this functor are referred to as "inert". Inert maps "only throw things away". Here, we are thinking of the basepoint as a trash can.

**Example 149.** $\beta_I := (\emptyset \to I)^+$. This maps "throws everything away"

**Example 150.** $\rho_I^i := (* \xrightarrow{i} I)^+$. This maps "throws away everything but $i \in I$. These are conventionally referred to as the "Segal maps". They give an equivalence:

$$I_+ \overset{\vee\rho_i}{\simeq} \bigvee_{i \in I} *$$

Coming from the covering maps: $\{* \xrightarrow{i} I\}_{i \in I}$.

The active maps are what we are primarily interested. These are in the image of the (essentially surjective, faithful) one-point compactifiction functor:

$$\mathrm{Fin} \xrightarrow{(-)_+} \mathrm{Fin}_*$$

Note that given an active map $f : I_+ \to J_+$, $f^{1}(\emptyset_+) = \emptyset_+$This property characterizes active maps. Active maps "don't throw anything away".

**Example 151.** Let $\mu_I : (I \to *)_+$. This map "collapses everything in $I$ to a point".

**Example 152.** Let $1 : (\emptyset \xrightarrow{i} *)_+$. This map "identifies the basepoint".

*Remark* 153. Note that every map $f : I_+ \to J_+$ produces a decompostion:

$$I \simeq \Big(\coprod_{j \in J} f^{-1}(j)\Big) \coprod f^{-1}(\emptyset_+)$$

Given a pair of sets $I$ and $J$ and a map from a subset of of $I$ into $J$ produces a unique map in $\mathrm{Fin}_*$. Because of this, many think of $\mathrm{Fin}_*$ as a the category of finite sets and partially defined functions.

With these ingredients in places, we may define:

**Definition 154.** A symmetric monoidal $\infty$-category is a functor:

$$\mathcal{C}^{\otimes} : \mathrm{Fin}_* \to \mathcal{C}\mathrm{at}_{\infty}$$

So that:

- The product of the Segal Maps induce equivalences:

  $$\mathcal{C}^{\otimes}(I_+) \simeq \mathcal{C}^{\otimes}(*_+)^{\times I}$$

  for every pointed finite set $I_+$. This is equivalent to the more conceptually appealing condition that

  $$(\mathrm{Fin}^{\mathrm{inj}})^{\mathrm{op}} \xrightarrow{(-)^+} \mathrm{Fin}_* \xrightarrow{\mathcal{C}^{\otimes}} \mathcal{C}\mathrm{at}_{\infty}$$

is a sheaf. This is commonly referred to as the Segal condition.

- The value on $*_+$ is terminal. In other words:

$$\mathcal{C}^{\otimes}(\emptyset_+) \simeq *$$

We let $\mathcal{C}$ denote $\mathcal{C}^{\otimes}(*_+)$, and refer it as the underlying $\infty$-category of $\mathcal{C}^{\otimes}$.

A priori, when $\mathcal{C}$ is an ordinary category the above definition consists of more than the classic definition of a symmetric monoidal category. Fortunately, this is not the case, as is shown in section 2.0 and 2.1 of [Lu2].

*Remark* 155. We note that [Lu2] gives a different, equivalent, definition in terms of coCartesian fibrations. Therefore, one needs to appeal to both chapter 2 of [Lu2] along with section 3.2 of [Lu] to obtain the construction . Heuristically, although the above definition is relatively straightforward to state, it is much easier to construct examples from point-set level data using the language of coCartesian fibrations.

A perk of this definition is the ease with which one can define

**Definition 156.** A symmetrical monoidal functor is a natural transformation.

We now outline how to extract the interesting data present in the above defintion.

*Remark* 157. All classical structures in a symmetric monoidal category arise from evaluating $\mathcal{C}^{\otimes}$ on an appropriate diagram in $\mathrm{Fin}_*$.

Evaluating on $1 \in \mathrm{Arr}(\mathrm{Fin}_*)$ gives an object of $\mathcal{C}$:

$$1_{\mathcal{C}} : \mathcal{C}^{\otimes}((\emptyset \to *)_+) \simeq (* \to \mathcal{C})$$

Which we could define as the symmetric monoidal unit. Evaluating the obvious diagram in $\mathrm{Fin}_*$ shows that this in fact behaves as the monoidal unit.

In particular, a symmetric monoidal category gives, for every finite set $I$, a functor:

$$\mathcal{C}^{\times I} \simeq \mathcal{C}^{\otimes}(I_+) \xrightarrow{\mathcal{C}^{\otimes}(\mu_I)} \mathcal{C}$$

Therefore, given an $I$-indexed copy of objects $\{c_i\}_{i\in I}$, we obtain an object of $\mathcal{C}$, given by:

$$(c_i)^{\otimes I} : * \simeq *^{\times I} \xrightarrow{\{c_i\}} \mathcal{C}^{\times I} \to \mathcal{C}$$

The Segal property states that a permutation of $I$ acts in the obvious way on $\mathcal{C}^{\times I}$. Therefore, given a permutation $\sigma_I \in \Sigma_I$, the evaluation of $\mathcal{C}^{\otimes}$ on the commutative triangle

$$\begin{tikzcd} & I_+ \arrow[dr, "\mu_I"] & \\ I_+ \arrow[ur, "(\sigma_I)_+"] \arrow[rr, "\mu_I"'] & & *+ \end{tikzcd}$$

in $\mathrm{Fin}_*$ gives a commutative diagram in $\mathcal{C}\mathrm{at}_\infty$:

$$\begin{tikzcd} & \mathcal{C}^{\times I} \arrow[dr, "\mathcal{C}^{\otimes}(\mu_I)"] & \\ \mathcal{C}^{\times I} \arrow[ur, "\mathcal{C}^{\otimes}(\sigma_i)"] \arrow[rr, "\mathcal{C}^{\otimes}(\mu_I)"'] & & \mathcal{C} \end{tikzcd}$$

We note that although the above triange commute in $\mathrm{Fin}_*$ due to a property, the commutativity of the above triangle in $\mathcal{C}\mathrm{at}_\infty$ is (a priori) data. For example, it gives a natural isomorphism of functors:

$$\mu_I^*\mathcal{C}^{\otimes}(\mu_I) \simeq \mathcal{C}^{\otimes}(\mu_I)$$

corresponding to the classical braiding ismorphism when $I$ has cardinality two. This in turn gives rise an automorphism of $(c_i)^{\otimes I}$.

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