ON PRESBURGER ARITHMETIC, NONSTANDARD FINITE CYCLIC GROUPS, AND DEFINABLE COMPACTIFICATIONS

A Dissertation

Submitted to the Graduate School

of the University of Notre Dame

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

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April 2016

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Abstract

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It is known that any model of $Th(\mathbb{Z}, +)$ can be decomposed into a direct sum of a torsion-free divisible abelian group and an elementary substructure of the group $\hat{\mathbb{Z}}$. We give a similar result for models of $Th(\mathbb{Z}, +, <)$, the theory of Presburger arithmetic and discuss orderings on direct summands. We show that the torsion-free divisible abelian group is densely ordered and the number of non-isomorphic expansions of the group $\hat{\mathbb{Z}}$ to a model of Presburger arithmetic is $2^{2^{\aleph_0}}$. We also give a description of the f-generic types of saturated models of Presburger arithmetic.

We consider nonstandard analogues of finite cyclic groups as a family of groups defined in an elementary extension of $(\mathbb{Z}, +, <)$. Since the theory of Presburger arithmetic has NIP, any such group H has a smallest type-definable subgroup of bounded index, H^{00}. Each quotient H/H^{00} is a compact group under the logic topology. The main result of this thesis is the classification of these compact groups.

The universal definable compactification of a group G, in a language in which all the subsets of G are definable, coincides with the Bohr compactification bG of G considered as a discrete group. For an abelian group G, in particular the group $\mathbb{Z}$ of integers, we compute the type-connected component. We show that adding predicates for certain subsets of G is enough to get bG as the universal compactification.

To Atena Farghadani

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ACKNOWLEDGMENTS

First, I would like to thank Anand Pillay, my advisor since my fourth year, for giving me an interesting project to work on and for willingly sharing his mathematical insight. After most of our discussions, I would be inspired by new ideas.

My best years as a student were my years at the University of Tehran. Several people helped me progress and prosper while I was there. I would like to especially thank Shahram Mohsenipour, Masoud Pourmahdian, Siamak Yassemi, and Rahim Zaare-Nahandi for their trust in my work, and for all they taught me.

I would like to thank the Notre Dame family, especially Peter Cholak and Julia Knight for all their support. Thanks also to Vincent Guingona, Cameron Hill, and Moshe Kamenky for all I learned from them.

Thank you to the participants of the MSRI model theory program in the spring of 2014, in particular Maryanthe Malliaris for inspirational conversations.

Many thanks are due to Gabriel Conant and Philipp Hieronymi for their detailed reading of an earlier version of this dissertation and for giving me several helpful comments. Especially Gabe's comments on Chapter 2 were extremely helpful.

I would also like to thank the organizers and participants of the model theory seminar at McMaster University and the logic seminar at the University of Illinois at Urbana-Champaign for their helpful comments on my presentation of the material in Chapter 3.

Last but not the least, I would like to thank my amazing friends and my family for their warm support throughout my entire journey.

INTRODUCTION

The main objective of this thesis is to study models of $Th(\mathbb{Z}, +, <)$, the theory of Presburger arithmetic, and groups interpretable in this theory. We give a class of compact groups which are interpretable in $Th(\mathbb{Z}, +, <)$, as quotients (universal definable compactifications) of nonstandard finite cyclic groups. The class contains certain groups which are inverse limits of compact groups which are direct sums of the circle group $\mathbb{S}^1$ with a finite cyclic group.

We also study definable compactifications of abelian groups, in particular the group of integers. Definable compactification is a notion similar to topological compactification, defined for definable groups rather than topological groups. We give a new case where the universal definable compactification of a definable group G coincides with the Bohr compactification of G considered as a discrete topological group.

In Chapter 1, we give preliminaries on model theory and topological groups. Familiarity with the basics of both of these subjects is assumed.

Among model theory background material, of higher importance in our work are the definitions of model theoretic connected components, and the logic topology, and a fact stating the relation between f-generic types and type-connected components in NIP theories.

Among background material on topological groups, of higher importance in our work are the definition and properties of the group $\hat{\mathbb{Z}}$, a fact about compact groups as inverse limits that we use frequently, and the definition and properties of amalgamated direct products of topological groups over their common subgroups.

In Chapter 2, we study models of Presburger arithmetic. Given G a nonstandard model of Presburger arithmetic, since G is also a model of $Th(\mathbb{Z}, +)$, we get a decomposition of G into a direct sum of a divisible torsion-free abelian group and a group isomorphic to an elementary substructure of $\hat{\mathbb{Z}}$. We discuss orderings on the direct summands.

$(\hat{\mathbb{Z}}, +)$ is the largest reduced model of $Th(\mathbb{Z}, +)$ which is also important in the direct sum decopmosition of models of Presburger arithmetic. We consider expansions of this model and show that the number of non-isomorphic expansions of $(\hat{\mathbb{Z}}, +)$ to models of Presburger arithmetic is $2^{2^{\aleph_0}}$. We also give a description of the f-generic types of saturated models of Presburger arithmetic.

In Chapter 3 we consider the groups $H_a = ([0, a), \ + \ mod \ a)$ for nonstandard positive a, a family of groups uniformly definable in a saturated elementary extension of $(\mathbb{Z}, +, <)$. We study the properties of these groups (e.g. their torsion subgroups) for different choices of a. Note that if two elements a and a' have the same divisibility type, then $H_a \cong H_{a'}$.

Given a type-definable equivalence relation E of bounded index on a definable group G, G/E can be equipped with the logic topology making it a compact topological group. In particular, if L is a type-definable subgroup of G of bounded index, then G/L is a compact group under the logic topology.

The main result of this thesis is to classify the compact quotients H/H^{00}. We describe these compact quotients as certain inverse limits. We define K to be the subgroup of infinitesimals of H, and show that $H/K \cong \mathbb{S}^1$, for every choice of a. We give results about the connected component and the type-conned component of H. We show that $H^{00} = \bigcap_n nK$. The method we use depends on the fact that the underlying theory has NIP, and does not generalize to structures on $\mathbb{Z}$ with the independence property. In the general case, the description of the type-connected component can be more challenging.

In Chapter 4, we consider definable compactifications of abelian groups. In [10], Gismatullin, Penazzi and Pillay introduced the notion of definable compactification and proved among other things that the universal definable compactification of a definable group in which every subset is definable, coincides with the Bohr compactification of the group considered as a discrete topological group.

We show that by adding predicates for certain subsets of an abelian group G, the universal definable compactification of the resulting structure is bG, the Bohr compactification of G. In particular, we give an expansion of $(\mathbb{Z}, +)$ for which the universal definable compactification is $b\mathbb{Z}$.

CHAPTER 1

PRELIMINARIES

1.1 Model Theory

In this section we recall some model theory definitions and background. For more on model theory see [21].

We do not distinguish in notation between a model and its universe. We assume T is a complete theory in a first order language L. Given a model $M \models T$, by a set definable in M we mean a subset of M^n definable with parameters. We use $x, y, \cdots$ to denote (finite tuples of) variables and $a, b, \cdots$ to denote (finite tuples of) constants.

Let $A \subseteq M$. Recall that a partial type over A is a consistent set of formulas with parameters from A. A complete type is a maximal such consistent set. The collection of complete types is a topological space with the Stone topology. We use the notation $S_n^M(A)$ for the space of complete types in n variables over A in a model M.

Definition 1.1.1. *Let κ be an infinite cardinal. We say that $M \models T$ is κ-saturated if, for all $A \subseteq M$, if $|A| < \kappa$ and $p \in S_n^M(A)$, then p is realized in M.*

Let $\overline{M} \models T$ be saturated of cardinality κ, where κ is very large. We fix such a model and call it the monster model of T.

All $M \models T$ that we consider are elementary submodels of $\overline{M}$ with $|M| < \kappa$. If $\varphi(v, a)$ is a formula with parameters, we assume $a \in \overline{M}$. We write $tp(a/A)$ for $tp^{\overline{M}}(a/A)$ and $S_n(A)$ for $S_n^{\overline{M}}(A)$. By a global type we mean a complete type over the monster model.

Small and bounded mean of cardinality less than κ, the saturation of the monster model. By a type-definable set (over A) we mean the set of all elements of $\overline{M}$ satisfying a given collection of formulas over a small set (A).

Definition 1.1.2. *Let k be a positive integer. We say that a formula $\varphi(x, b)$ k-divides over a set C, if there is a sequence $\{b_i\}_{i<\omega}$ of realizations of $tp(b/C)$ such that $\{\varphi(x, b) : i < \omega\}$ is k-inconsistent, i.e. every k-element subset of it is inconsistent. We say that $\varphi(x, b)$ divides over C, if it k-divides over C for some positive integer k.*

A formula $\varphi(x, b)$ forks over C if it implies a finite disjunction of formulas each of which divides over C.

A complete type is said to divide (fork) over C if it contains some formula which divides (forks) over C.

Definition 1.1.3. *A complete type $p(x)$ is called definable if for every formula $\varphi(x, y)$, the set $\{b : \varphi(x, b) \in p(x)\}$ is definable.*

Now we recall the definition of stability. For more on stable theories see [27].

Definition 1.1.4. *The model M has an order if there are $n < \omega$, a formula $\varphi(x, y)$ with $|x| = |y| = n$ and n-tuples a_i in M for $i < \omega$ such that for $i, j < \omega$*

$$M \models \varphi(a_i, a_j) \text{ iff } i \leq j.$$

T has an order if there is $M \models T$ where M has an order. T is stable if T has no order.

Example 1.1.5. *$Th(\mathbb{Z}, +)$ is stable.*

Now we recall the following definition and facts from [32].

Definition 1.1.6. *T does not have the finite cover property (does not have f.c.p.) iff for every $\varphi(x, y)$ there exists $k = k(\varphi) < \omega$ such that for every set of parameters A and every $p \subseteq \{\varphi(x, a), \neg\varphi(x, a) : a \in A\}$ the following implication holds.*

If p is inconsistent then $\exists q \subseteq p$ ($|q| \leq k \wedge q$ is inconsistent).

One can think of the above property as a generalization of compactness.

Fact 1.1.7. *If T does not have the f.c.p., then T is stable.*

Fact 1.1.8. *If T is stable, then the following are equivalent:*

1. *T has the f.c.p.*
2. *There exists a formula $\varphi(x, y, z)$ and there exists $\{a_n : n < \omega\}$ such that for every a, $\varphi(x, y, a)$ is an equivalence relation and for every $n < \omega$ we have*

$$n < |\overline{M}/\varphi(x, y, a_n)| < \aleph_0.$$

1.1.1 NIP Theories and Presburger Arithmetic

In this section we recall the definition of NIP theories form [12]. See also [20]. There are alternative equivalent ways of defining NIP theories. See for example [34].

The NIP theory that we are interested in throughout Chapters 2 and 3 is the theory of the ordered group of integers, also known as Presburger arithmetic or Pr.

For a collection $\mathcal{C}$ of subsets of a set X, one can define a dimension called VC-dimension (after Vapnik and Chervonenkis) which measures the combinatorial complexity of $\mathcal{C}$.

Having finite or infinite VC-dimension turns out to be an important difference in terms of complexity. In particular, for the collection of definable subsets of a model, one can ask whether the VC-dimension is finite or infinite. In the following we recall what it means for a theory to have the independence property and how it is related to infinite VC-dimension.

Let X be a set and let $\mathcal{C} \subseteq 2^X$. We call $\mathcal{C}$ a concept class on X. One can either think of $\mathcal{C}$ as a collection of subsets of X, or a collection of functions from X to $\{0, 1\}$. For notational simplicity we assume the second. To define the VC-dimension of $\mathcal{C}$ we need the following:

Definition 1.1.9. *If $\mathcal{C}$ is a concept class on X and $Y \subseteq X$, then*

$$\mathcal{C} \restriction_Y = \{f \restriction_Y : f \in \mathcal{C}\}.$$

We say that $\mathcal{C}$ shatters a set $Y \subseteq X$ if $\mathcal{C} \restriction_Y = 2^Y$.

Definition 1.1.10. *The VC-dimension of a concept class $\mathcal{C}$ on X is the size of the largest finite subset of X that can be shattered by $\mathcal{C}$. If arbitrarily large finite subsets of X can be shattered by $\mathcal{C}$, then we say $\mathcal{C}$ has infinite VC-dimension.*

Definition 1.1.11. *$\mathcal{C}$ is called a VC-class if it has finite VC-dimension.*

Definition 1.1.12. *A formula $\varphi(x, y)$ does not have the independence property (has NIP) if the family of all sets of the form $\varphi(a, \overline{M})$ for $a \in \overline{M}^{|x|}$, is a VC-class. In other words, there exists $n < \omega$ such that, for all $b_0, \cdots, b_{n-1} \in \overline{M}^{|y|}$, there do not exist $a_I \in \overline{M}^{|x|}$ for each $I \subseteq n$ so that for all $i < n$, $I \subseteq n$,*

$$\models \varphi(a_I, b_i) \quad \textit{iff} \quad i \in I.$$

We say that the theory T is NIP if all formulas have NIP.

Example 1.1.13. *Every stable theory has NIP.*

Fact 1.1.14. *Pr is an NIP theory.*

The above fact is a special case of a more general result stating that the theory of any ordered abelian group is NIP. See [13] or [5].

Fact 1.1.15. *Pr has quantifier elimination (QE) in the language*

$$L = \{+, -, <, 0, 1, P_n\}_{n>0},$$

where each P_n is a unary predicate symbol for the set of elements divisible by n.

1.1.2 The Logic Topology

Let G be a group definable in $\overline{M}$. We recall the definition of G^0_A and G^{00}_A, the model theoretic connected components of G over some small set of parameters A. See for example [9].

Definition 1.1.16. *Let A be a small set that includes the parameters over which G is defined. Then*

$$G^0_A = \bigcap\{H < G : H \text{ is } A\text{-definable and } [G : H] < \omega\},$$

and

$$G^{00}_A = \bigcap\{H < G : H \text{ is type-definable over } A \text{ and } [G : H] < \kappa\}.$$

We call G^0_A the connected component of G over A, and G^{00}_A the type-connected component of G over A.

Remark 1.1.17. *G^0_A and G^{00}_A are normal subgroups of G.*

If G^0_A (G^{00}_A) does not depend on A, we say that G^0 (G^{00}) exists. An important example is the case of NIP theories.

Fact 1.1.18. *If T has NIP then G^0 and G^{00} exist.*

A proof of the above fact can be found in [17]. We need to use the following corollary of this fact in Chapter 3:

Corollary 1.1.19. *If H is a group definable in a model of Pr, then H^0 and H^{00} exist.*

Proof. By Fact 1.1.14, Pr is NIP, so we can apply Fact 1.1.18. □

It is well-known that for an arbitrary saturated expansion G of the additive group of integers $\mathbb{Z}$, the connected component G^0 exists and equals $\bigcap_{n\in\mathbb{N}} nG$.

The following fact from [2] gives a sufficient condition for a similar description of G^0 to hold for an arbitrary infinite group.

For the following fact we assume $(G, \cdot)$ is a group definable in a model M, and we denote by G^* the group defined in $\overline{M}$ by the formula defining G.

Fact 1.1.20. *Suppose that for every positive integer n the set $\{g^n : g \in G\}$ generates a subgroup of G of finite index in finitely many steps. Then $(G^*)^0$ exists and*

$$(G^*)^0 = \bigcap_{n \in \mathbb{N}} \langle \{g^n : g \in G^*\} \rangle.$$

Moreover, every subgroup of finite index in a group elementarily equivalent to G is definable in the language of groups.

Now, we recall the definition of the logic topology from [26]. See also [19]. Again, let G be a group definable in $\overline{M}$. Under the logic topology the quotients of G by its type-definable subgroups of bounded index (in particular by its model theoretic connected components) are compact groups. These compact groups are invariants of the definable group G.

The logic topology can be defined in the following more general setting. Let X be a type-definable set. Given a type-definable equivalence relation E on X with a bounded number of classes, consider the quotient X/E.

Definition 1.1.21. *Let $\pi : X \to X/E$ be the natural surjection. We say $C \subseteq X/E$ is closed if $\pi^{-1}(C)$ is type-definable.*

The complements of the closed sets defined as above form open sets of a topology on X/E.

Fact 1.1.22. *Suppose that X is a type-definable subset of G, and E is a bounded equivalence relation on X. If E is type-definable, then X/E with the logic topology is compact Hausdorff.*

Note that in particular, the logic topology can be defined on G/G_A^0 and G/G_A^{00}, making them compact topological groups.

Remark 1.1.23. *Let $S_G(M)$ be the set of complete types over M extending the formula saying $x \in G$. The quotient map $G \to G/G_A^{00}$ factors through the type space $S_G(M)$ for any small model M containing A.*

1.1.3 f-Generic Types and Stabilizers

Let $(G, \cdot)$ be a definable group in a model M. $S_G(A)$ denotes the set of complete types over A extending the formula saying $x \in G$.

A left translate of a formula $\varphi(x, a)$ is defined as $\varphi(g^{-1}x, a)$ for some $g \in G$. A left translate of a type p in $S_G(A)$ is a type containing all the left translates of the formulas in p, by a $g \in G$. Right translates can be defined similarly.

Definition 1.1.24. *We say that a set $X \subseteq G$ is generic if some finitely many left translates of X by elements of G cover G. We say that a formula $\varphi(x)$ is generic if the set $\varphi(G)$ of elements of G realizing φ is generic. Finally, we say that a type $p(x) \in S_G(A)$ is generic if every formula $\varphi(x)$ in $p(x)$ is generic.*

Now we recall the definition of f-generic types. See [4], [16], or [28]. The notion of f-generic is a weaker notion than generic. See [25] for more on weak generic types.

Definition 1.1.25. *Assume T is NIP. Let $p \in S_G(\overline{M})$. We say that p is left f-generic if every left translate of p does not fork over M.*

By a result from [3], forking and dividing over models are the same in NIP theories. So we get:

Fact 1.1.26. *Assume T is NIP. Let $p \in S_G(\overline{M})$. p is left f-generic if every left translate of p does not divide over M.*

Now we recall that under the NIP assumption, a global f-generic type can be used in computing G^{00}. First, we recall the definition of the stabilizer of a type $p \in S_G(M)$.

Definition 1.1.27. *Let $p \in S_G(M)$. The stabilizer of p is the group of all $g \in G$ such that $gp = p$.*

The following fact gives the connection between global f-generic types and type-connected components in NIP theories.

Fact 1.1.28. *Assume T is NIP. Suppose that G has a global left f-generic type p. Then $Stab(p) = G^{00}$.*

1.2 Topological Groups

All the topological spaces we consider in this section are assumed to be Hausdorff. Recall that a topological group is a group G together with a topology on G such that the group operation and the inverse function are continuous maps with respect to the topology. Morphisms in the category of topological groups are continuous homomorphisms. For topological groups G_1 and G_2, a topological isomorphism is a function $\rho : G_1 \to G_2$ which is simultaneously a homeomorphism and a group isomorphism.

See [14] for more and topological groups, and for proofs of the following facts.

Definition 1.2.1. *A topological space X is connected if it is non-empty and there are no non-empty, disjoint open subsets A and B of X such that $X = A \cup B$.*

Thus a non-empty space X is connected iff the only subsets of X which are both open and closed are $\emptyset$ and X.

Fact 1.2.2. *Let G be a topological group. Then every open subgroup of G is closed and every closed subgroup of finite index is open.*

Fact 1.2.3. *A connected topological group has no proper open subgroups. A connected topological group is generated as an abstract group by any neighborhood of identity (or by any non-empty open subset).*

Definition 1.2.4. *For an abelian topological group G we let*

$$div(G) = \bigcup \{H : H \text{ is a divisible subgroup}\},$$

and

$$Div(G) = \bigcap_{n \in \mathbb{N}} nG.$$

For a proof of the following fact see [15].

Fact 1.2.5. *If G is a compact abelian group or a discrete torsion-free abelian group, then $div(G) = Div(G)$.*

Note that in the general case we have $div(G) \subseteq Div(G)$.

Definition 1.2.6. *The circle group denoted by $\mathbb{T}$, is the multiplicative group of all complex numbers with absolute value 1, i.e. the unit circle in the complex plane or simply the unit complex numbers*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Note that $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{T}$, and is also called the circle group.

In this chapter we use the symbol $\mathbb{T}$ to denote the circle group. We also write its group operation as multiplication.

Recall the notion of a character defined for any abstract group.

Definition 1.2.7. *A character on a group G is a homomorphism from G to the circle group $\mathbb{T}$.*

The set of all characters on G is itself a group under pointwise multiplication. For topological groups, one can consider continuous characters.

Definition 1.2.8. *Let G be an abelian topological group. The dual group of G is the group of all continuous characters on G and is denoted by $\hat{G}$.*

One can define a topology on $\hat{G}$ turning it into a topological group. We recall this topology in Chapter 4.

1.2.1 Compact Groups

A compact group is a topological group whose topology is compact.

Example 1.2.9. *The circle group $\mathbb{T}$ is a compact group.*

Definition 1.2.10. *Any topological group isomorphic to an inverse limit of finite groups is called a profinite group.*

Example 1.2.11. *The profinite group $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ is a compact group.*

There are other ways of defining the group $\hat{\mathbb{Z}}$. We recall a more explicit way here. First we recall the definition of p-adic integers. See [1] for more on p-adic integers.

Definition 1.2.12. *Let p be a prime number. A sequence of integers*

$$\{x_n\} = \{x_0, x_1, \cdots, x_n, \cdots\},$$

satisfying

$$x_n \equiv x_{n-1} \ (mod\ p^n)$$

for all $n \geq 1$, determines an object called a p-adic integer. Two sequences $\{x_n\}$ and $\{x'_n\}$ determine the same p-adic integer iff for all $n \geq 0$,

$$x_n \equiv x'_n \ (mod\ p^{n+1}).$$

One can easily see that the operation $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ makes the set of p-adic integers a group. We denote this group by $\mathbb{Z}_p$.

The following fact is well-known.

Fact 1.2.13. $\hat{\mathbb{Z}} \cong \prod_{p\ prime} \mathbb{Z}_p$.

The above fact suggests the following description of the elements of $\hat{\mathbb{Z}}$. Every element x in $\hat{\mathbb{Z}}$ corresponds to an infinite tuple $(x_2, x_3, x_5, \cdots)$ where $x_p \in \mathbb{Z}_p$ for every prime p. Note that for every $y \in \mathbb{Z}_p$, $y \equiv 0 \pmod{q^k}$ for every prime $q \neq p$ and every $k > 0$.

Remark 1.2.14. *The group $\hat{\mathbb{Z}}$ has size $2^{\aleph_0}$.*

A proof of the following fact about compact groups can be found in [30].

Fact 1.2.15. *Any compact group is an inverse limit of compact Lie groups.*

Recall that:

Definition 1.2.16. *A Lie group is a group that is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure.*

Under certain assumptions, a compact group can be approximated in a certain sense by factor groups G/N modulo smaller and smaller normal subgroups N. See [15] for a proof of the following fact.

Fact 1.2.17. *Assume that G is a compact group with a filter basis $\mathcal{N}$ of compact normal subgroups with $\bigcap \mathcal{N} = \{0\}$. For $M \subseteq N$ in $\mathcal{N}$ let $f_{NM} : G/M \to G/N$ denote the natural morphism given by $f_{NM}(gM) = gN$. Then the f_{NM} form an inverse system whose limit is isomorphic to G under the map $G \to \varprojlim_{N \in \mathcal{N}} G/N$ defined by $g \mapsto (gN)_{N \in \mathcal{N}}$. With this isomorphism, the limit maps are equivalent to the quotient maps $G \to G/N$.*

1.2.2 Amalgamated Direct Products

In this section we recall the definition of amalgamated direct products of topological groups. We mostly focus on abelian topological groups.

We first recall the notion of amalgamated products of groups and their universal property. This notion is a generalization of the direct products of groups. See [18].

Amalgamated direct product of two groups over a common subgroup can be defined for subgroups of any group. Here we restrict our attention to its definition for abelian groups.

Definition 1.2.18. *Let G be an abelian group. G is said to be the direct product of its subgroups A and B with amalgamated subgroup C if $A \cap B = C$, and G is generated by $A \cup B$. This amalgamated direct product is denoted by $A \times_C B$.*

The following fact states the universal property of amalgamated direct products.

Fact 1.2.19. *Let the abelian group G be the direct product of its subgroups A and B with amalgamated subgroup C. If H is any group and $\Phi_1 : A \to H$ and $\Phi_2 : B \to H$ are any homomorphisms such that $\Phi_1 \restriction_C = \Phi_2 \restriction_C$, then there exists a homomorphism $\Phi : G \to H$ such that $\Phi \restriction_A = \Phi_1$ and $\Phi \restriction_B = \Phi_2$.*

Remark 1.2.20. *Since $A \times_C B$ is generated by $A \cup B$, each element g in $A \times_C B$ is of the form $g = a + b$, where $a \in A$ and $b \in B$. In the above fact, $\Phi : G \to H$ is defined by $\Phi(ab) = \Phi_1(a) + \Phi_2(b)$. One can show that Φ is a well defined function and a homomorphism.*

Notation 1.2.21. *If A is a subgroup of G, then the centralizer of A in G is denoted by $C_G(A)$.*

The following is the definition of amalgamated direct products of topological groups. Here we do not assume G to be abelian.

Definition 1.2.22. *A topological group G is said to be the (topological) direct product of its topological subgroups A and B with amalgamated subgroup $C = A \cap B$ if it has the following properties.*

1. *G is generated algebraically by $A \cup B$,*
2. *$B \subseteq C_G(A)$, and*
3. *if Φ_1 and Φ_2 are any continuous homomorphisms of A and B, respectively, into any topological group H such that $\Phi_1 \restriction_C = \Phi_2 \restriction_C$ and $\Phi_2(B) \subseteq C_H(\Phi_1(A))$, then there exists a continuous homomorphism $\Phi : G \to H$ such that $\Phi \restriction_A = \Phi_1$ and $\Phi \restriction_B = \Phi_2$.*

Note that amalgamated direct products of topological groups are defined so that they have the same universal property as amalgamated direct products of groups.

CHAPTER 2

PRESBURGER ARITHMETIC

2.1 Models of $Th(\mathbb{Z},+)$

Recall from Section 1.1.1 that Presburger arithmetic, or Pr, is the theory of the ordered group of integers. Any model of Pr is also a model of $Th(\mathbb{Z},+)$, for which a decomposition into a direct sum is known. In the next section, we give a similar decomposition for models of Pr, and we discuss the ordering on the direct summands.

Recall the definition and properties of $\hat{\mathbb{Z}}$ from Section 1.2.1. It is well-known that any saturated model G of $Th(\mathbb{Z},+)$ is isomorphic to a direct sum of $\hat{\mathbb{Z}}$ and a torsion-free divisible abelian group. In this decomposition, the torsion-free divisible group is the largest divisible subgroup of G, called the divisible part of G, but the choice of a subgroup of G isomorphic to $\hat{\mathbb{Z}}$ is not unique. Every model of $Th(\mathbb{Z},+)$ can be decomposed into a direct sum of a torsion-free divisible abelian group and a subgroup isomorphic to an elementary substructure of $\hat{\mathbb{Z}}$.

Definition 2.1.1. *A is called reduced if A has no non-zero divisible subgroups.*

Remark 2.1.2. *$\hat{\mathbb{Z}}$ is the largest reduced model of $Th(\mathbb{Z},+)$.*

The following fact is about divisible subgroups being direct summands, and is used to prove the fact we mentioned about the decomposition of models of $Th(\mathbb{Z},+)$. For more on the material in this section see [23].

Fact 2.1.3. *Suppose G is a torsion-free group, D is a divisible subgroup of G and H is a subgroup of G such that $H \cap D = \{0\}$. Then there is a subgroup F of G such that $F \supseteq H$ and $G = F \oplus D$.*

For a proof of the above fact see [8]. The proof depends on Zorn's lemma and does not specify F uniquely. Although F is in general not unique, it is isomorphic to G/D and hence is unique up to isomorphism. If we take D to be the largest divisible subgroup of G (and use the fact for $H = \{0\}$), the resulting F must be a reduced group (otherwise $F \cap D \neq \{0\}$).

Fact 2.1.4. *Let $\kappa > 2^{\aleph_0}$ be a cardinal and let G be a κ-saturated model of $Th(\mathbb{Z}, +)$ of cardinality κ. Then we have:*

$$G \cong \coprod_{\kappa} \mathbb{Q} \oplus \hat{\mathbb{Z}}.$$

One way to see why the above fact holds is to note that the group of integers is elementarily equivalent to the saturated group $\coprod_{\kappa} \mathbb{Q} \oplus \hat{\mathbb{Z}}$, and elementarily equivalent saturated structures of the same cardinality are isomorphic. See [6].

Remark 2.1.5. *Let G be a saturated model of $Th(\mathbb{Z}, +)$, and D be the divisible part of G. Then any subgroup of G, which is isomorphic to $\hat{\mathbb{Z}}$, is a direct summand of G. Note that there are several subgroups of G isomorphic to $\hat{\mathbb{Z}}$, since every $a \in G \backslash \bigcap_n nG$ can be in a copy of $\hat{\mathbb{Z}}$.*

Proof. The exact sequence

$$0 \to \hat{\mathbb{Z}} \to G \to G/\hat{\mathbb{Z}} \to 0$$

splits since $G/\hat{\mathbb{Z}}$ is isomorphic to D which is a subgroup of G. $\square$

Fact 2.1.6. *Let G be a model of $Th(\mathbb{Z}, +)$. Then there exist an elementary substructure A of $\hat{\mathbb{Z}}$ and a cardinal κ such that $G \cong \coprod_{\kappa} \mathbb{Q} \oplus A$.*

If $(G, 1)$ is a model of $Th(\mathbb{Z}, +, 1)$. Then there exist an elementary substructure A of $\hat{\mathbb{Z}}$ containing 1 and a cardinal κ such that $(G, 1) \cong \coprod_{\kappa} \mathbb{Q} \oplus (A, 1)$.

2.2 Models of Presburger Arithmetic

In this section we consider the decomposition of models of Pr similar to the decomposition of the models of $Th(\mathbb{Z}, +)$, and we discuss the orderings on the direct summands.

Proposition 2.2.1. *Let G be a model of Presburger arithmetic. Then*

$$G = G_1 \oplus G_2,$$

where G_1 is an elementary substructure of G and isomorphic to an elementary substructure of $(\hat{\mathbb{Z}}, +, <)$ (for some expansion of $\hat{\mathbb{Z}}$ to a model of Pr), and G_2 is a densely ordered divisible abelian group.

Proof. Since G is also a model of $Th(\mathbb{Z}, +)$, by Fact 2.1.6 we have the decomposition as groups. In particular, G_1 is an elementary substructure of G containing 1 and also isomorphic to an elementary substructure of $\hat{\mathbb{Z}}$ containing 1.

G has a unique subgroup that works as G_2 here, namely $\bigcap_n nG$. This subgroup is divisible torsion-free abelian and ordered with the order it gets from G. The ordering on G_2 is dense, because if $d_1, d_2 \in G_2$, then $(d_1 + d_2)/2 \in G_2$. $\square$

Let G be a saturated model of Pr. There are several subgroups of G isomorphic to $\hat{\mathbb{Z}}$. Using the following remark, we show that different orderings on copies of $\hat{\mathbb{Z}}$ in G are non-isomorphic.

Remark 2.2.2. *The structure $(\hat{\mathbb{Z}}, +, 1)$ has no non-trivial automorphism.*

Proof. Every two different elements of $\hat{\mathbb{Z}}$ have different types over $\{1\}$, since for every two different elements $c_1, c_2 \in \hat{\mathbb{Z}}$, there is a prime p and a positive integer n so that $c_1 \not\equiv c_2 \pmod{p^n}$. $\square$

Corollary 2.2.3. *Any two different expansions of $(\hat{\mathbb{Z}}, +, 1)$ to models of Pr are non-isomorphic.*

We show that the number of non-isomorphic expansions of $(\hat{\mathbb{Z}}, +)$ to a model of Pr is $2^{2^{\aleph_0}}$. By Corollary 2.2.4 it is enough to show that the number of different orderings on $\hat{\mathbb{Z}}$ which make it a model of Pr is $2^{2^{\aleph_0}}$. Note that this is the maximum number of orderings possible.

Proposition 2.2.4. *There are $2^{2^{\aleph_0}}$ expansions of $(\hat{\mathbb{Z}}, +, 1)$ to a model of Pr, up to isomorphism.*

Proof. Let B be a set containing 1 that generates $\hat{\mathbb{Z}}$ and has the property that for every $b \in B$ and any $b \notin B' \subseteq B$ we have $b \notin \langle B' \rangle$. Then no non-trivial linear combination of elements of B is zero, i.e. for every $k > 0$, $b_1, \cdots, b_k \in B$ distinct, and $n_1, \cdots, n_k \in \mathbb{Z}$ not all equal to 0, we have

$$n_1 b_1 + \cdots + n_k b_k \neq 0.$$

B has size $2^{\aleph_0}$ since it generates $\hat{\mathbb{Z}}$. Note that we need to use Zorn's lemma to show that such a set B exists.

We show that any ordering on the set B can be extended to an ordering on $\hat{\mathbb{Z}}$ that makes it a model of Pr. Let $p = tp(\hat{\mathbb{Z}}/\emptyset)$ in the language $\{+, 1\}$ (with a variable x_a for each $a \in \hat{\mathbb{Z}}$).

Let $\{<_i\}_{i \in 2^{2^{\aleph_0}}}$ be the collection of all linear orderings on the set B. For each $i \in 2^{2^{\aleph_0}}$, let $q_{B,i} \supseteq p$ be the partial type in the language $\{+, <, 1\}$ stating (in addition to the formulas in p) that B is ordered according to $<_i$.

We claim that $q_{B,i}$ is consistent with $Th(\mathbb{Z}, +, <)$, hence $q_{B,i} \cup Th(\mathbb{Z}, +, <)$ has a model (which is an expansion of $(\hat{\mathbb{Z}}, +, 1)$ to a model of Pr, in which B is ordered according to $<_i$).

By compactness, it suffices to show that any finite subset of $q_{B,i}$ can be realized in $(\mathbb{Z}, +, <)$.

Let Σ be a finite subset of $q_{B,i}$. Then Σ has the form:

$$x_{b_1} \equiv k_1 1 \ (mod\ m_1)$$

$$\vdots$$

$$x_{b_k} \equiv k_k 1 \ (mod\ m_k)$$

$$x_{b_1} < \cdots < x_{b_k}$$

$$n_1^1 b_1 + \cdots + n_{t_1}^1 b_{t_1} \neq 0$$

$$\vdots$$

$$n_1^s b_1 + \cdots + n_{t_s}^s b_{t_s} \neq 0,$$

for some $t, k, s, m_1, \cdots, m_l \in \mathbb{N}$, $k_1, \cdots, k_l \in \mathbb{Z}$, and $b_1, \cdots, b_k \in B$, toghether with some congruence relations and linear dependencies and independencies among $x_{c_1}, \cdots, x_{c_l}$, for some $l \in \mathbb{N}$ and $c_1, \cdots, c_l \in \hat{\mathbb{Z}} \setminus B$.

In each inequality $n_1^i b_1 + \cdots + n_{t_i}^i b_{t_i} \neq 0$, we are assuming that $n_1^i, \cdots, n_{t_i}^i \in \mathbb{Z}$ and $n_{t_i}^i \neq 0$.

Using the Chinese Remainder Theorem, we may assume that all the x_{b_i}'s are distinct. Note that there are no equalities among the x_c's since p says $x_c \neq x_{c'}$ for every distinct $c, c' \in \hat{\mathbb{Z}}$.

Now we show that Σ can be realized in $\mathbb{Z}$. First we find realizations for $x_{b_1}, \cdots, x_{b_k}$. Choose an element a_{b_1} for x_{b_1} in $m'\mathbb{Z} + k_1'$.

Assume we have chosen an element $a_{b_{t'-1}}$ for $x_{b_{t'-1}}$. We need to choose an element $a_{b_{t'}}$ for $x_{b_{t'}}$ so that

- $x_{b_1} < \cdots < x_{b_{t'-1}} < x_{b_{t'}}$,
- $x_{b_{t'}} \in m_{t'}\mathbb{Z} + k_{t'}$, and
- $n_1 x_{b_1} + \cdots + n_{t'} x_{b_{t'}} \neq 0$.

For $n_1x_{b_1} + \cdots + n_{t'}x_{b_{t'}}$ to be non-zero, it is enough to be not zero modulo $m_{t'}$. Note that

$$n_1x_{b_1} + \cdots + n_{t'}x_{b_{t'}} \equiv n_1k_1 + \cdots + n_{t'}k_{t'} \pmod{m_t}.$$

Now we find realizations for $x_{c_1}, \cdots, x_{c_l}$. If some c_i is in the subgroup of $\hat{\mathbb{Z}}$ generated by $\{b_1, \cdots, b_k\}$, then we choose a realization for x_{c_i} accordingly. This realization automatically satisfies the congruence relations needed.

For the rest of x_{c_i}'s, if any, there are no dependencies with x_{b_i}'s and we can find realizations one by one, as we did for x_{b_i}'s.

Note that linear independencies among x_{c_i}'s can be easily satisfied since there are infinitely many choices in each congruence class. □

Corollary 2.2.5. *There are $2^{2^{\aleph_0}}$ expansions of $(\hat{\mathbb{Z}}, +)$ to a model of Pr, up to isomorphism.*

Proof. There are countably many choices for 1 in $(\hat{\mathbb{Z}}, +)$. □

2.3 Description of f-Generic Types

In this section we describe f-generic types of a saturated model of $Th(\mathbb{Z}, +, <)$. Throughout this section, we fix G a saturated model of Pr. By a result from [33], the types at $+\infty$ and $-\infty$ in G are f-generic.

Remark 2.3.1. *The only f-generic types of G are the types at $+\infty$ and $-\infty$.*

Proof. We show that any global type other than the types at $+\infty$ and $-\infty$ is not f-generic. Let q be such a type. Then some translate of q contains the formulas $x < a_1$ and $x > a_2$ for some $a_1, a_2 \in G \setminus \mathbb{Z}$. The formula $x < a_1 \wedge x > a_2$ divides over $\mathbb{Z}$, so q is not an f-generic type. □

We get the following well-known fact as a corollary:

Corollary 2.3.2. $G^0 = G^{00}$.

Proof. $G^0 = \bigcap_n nG$ stabalizes any f-generic type, since each type is determined by a cut in G and a coset of $G^0 = \bigcap_n nG$. Now use Fact 1.1.28. $\square$

Remark 2.3.3. *The f-generic types of G are definable.*

Proof. Let p be an f-generic type of G. For a formula $\varphi(x, y)$ of the form $x < y$, the set of $b \in G$ for which $\varphi(x, b) \in p$ is either equal to G or $\emptyset$, depending on whether p is a type at $+\infty$ or $-\infty$, so this set is definable in each case.

For a formula $\varphi(x, y)$ of the form $x \equiv y$ $(mod\ n)$, the set of $b \in G$ for which $\varphi(x, b) \in p$ is a coset of nG and hence definable.

Note that by Fact 1.1.14, Pr has QE up to the above formulas. So we conclude that p is a definable type. $\square$

CHAPTER 3

NONSTANDARD FINITE CYCLIC GROUPS

3.1 Choice of the Language

A finite cyclic group is a group of the form $\mathbb{Z}/a\mathbb{Z}$ for $a \in \mathbb{Z}$ positive. We consider the nonstandard analogues of these groups. The following is the reason why the language of groups is not suitable for this purpose.

Remark 3.1.1. *The family of all finite cyclic groups is not uniformly definable in* $(\mathbb{Z}, +)$.

Proof. $Th(\mathbb{Z}, +)$ is a stable theory that does not have the finite cover property. By Fact 1.1.8, given a family of finite definable sets, there is a bound on the size of the sets. So the family of finite cyclic groups is not uniformly definable. □

Hence we need to consider an expansion of $(\mathbb{Z}, +)$ in order to define analogues of finite cyclic groups in an elementary extension. One natural structure to consider is the ordered group of integers.

In this chapter we work in the theory of the ordered group of integers, Presburger arithmetic. As we said in Section 1.1.1, Pr is an NIP theory.

In the language of Pr, the family of all finite cyclic groups is uniformly definable and hence one can define nonstandard finite cyclic groups in an elementary extension of $(\mathbb{Z}, +, <)$. These groups have the same description as finite cyclic groups, with a positive nonstandard, i.e. with $a > n$ for every $n \in \mathbb{Z}$.

The sets that we call nonstandard finite are the finite sets in the sense of internal set theory. See [24]. Some authors use the word hyperfinite. See for example [7].

Note that nonstandard finite cyclic groups are special cases of pseudofinite groups (ultraproducts of finite groups). See for example [29].

Throughout this chapter, we fix G, a saturated elementary extension of $(\mathbb{Z}, +, <)$.

Definition 3.1.2. *Let $a \in G$ be a nonstandard positive element, i.e. $a > n$ for every $n \in \mathbb{Z}$. We define $H_a = ([0, a), \ + \ mod \ a)$. Wherever a is understood, we simply call this group H.*

Recall that by Corollary 1.1.19, H^{00} exists, for a nonstandard finite cyclic group defined as above in Pr. Recall also that by Fact 1.1.22, H/H^{00} is a compact topological group under the logic topology. Our goal is to classify these compact groups, for different choices of a. We address this problem in the last section of this chapter.

3.2 Properties of Nonstandard Finite Cyclic Groups

In this section, we explore some properties of the nonstandard finite cyclic groups H for different choices of a.

First, we point out that depending on the divisibility type of a, H has different properties (e.g. different torsion subgroups). Since G is saturated, every consistent divisibility type is realized in H, e.g. there is $h \in H$ which is divisible by 2 and not by 4, infinitely divisible by 3, and not divisible by any prime other than 2 and 3. Note that here by h is divisible by n we mean $h \equiv 0 \ (mod \ n)$ which can be stated in the language of Pr. We will also sometimes use the notation $n|h$. The two extreme cases are the case where a is divisible by every natural number, and the case where a is not divisible by any $n > 1$.

Remark 3.2.1. *The torsion subgroup of H is $\{ma/n : \ a \equiv 0 \ (mod \ n), \ m < n\}$ which is isomorphic to the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by $\{1/n + \mathbb{Z} : a \equiv 0 \ (mod \ n)\}$. In particular, if a is not divisible by any $n > 1$, then H has no non-zero torsion element.*

Example 3.2.2. *If a is divisible by 2, then $H/2H \cong \mathbb{Z}/2\mathbb{Z}$, but if a is not divisible by 2, there is $b \in H$ so that $2b = a - 1$, so in H we have: $2b + 2 = (a - 1) + 2 = 1$, which implies $1 \in 2H$. Hence, if a is not divisible by 2, we have $2H = H$.*

The next lemma generalizes the above example. Before stating the lemma, we introduce the following notation.

Notation 3.2.3. *For a positive integer n, let $g_a(n)$ be the largest positive integer m dividing n such that $a \equiv 0 \pmod{m}$.*

Lemma 3.2.4. *$nH = g_a(n)H$.*

Proof. If q_1 and q_2 are powers of different primes, then $g_a(q_1q_2) = g_a(q_1).g_a(q_2)$, so we may assume that n is a power of a prime.

First we show that for every prime p and every $k > 0$, if p does not divide a then $p^kH = H$. It is enough to show that $pH = H$. The index of pG in G is p and $pG \cap H \subseteq pH$, so the index of pH in H is either 1 or p.

In G, we have $a = pb + k$, for some $0 < b < a$ and $1 \leq k \leq p - 1$. Now working in the group H, the cosets pH and $k + pH$ are equal since $pb + k = 0$. We conclude that the index of pH in H is 1.

Now we show that if p^t divides a and p^{t+1} does not divide a, then $p^kH = p^tH$, for every $k > t$. It is enough to show that $p^{t+1}H = p^tH$. Note that the index of $p^{t+1}H$ in p^tH is either 1 or p.

In G, we have $a = p^tc$, for some $0 < c < a$ and $a = p^{t+1}b + k$, for some $0 < b < a$ and $1 \leq k \leq p^{t+1} - 1$. So $k = p^tk'$ for some $1 \leq k' \leq p - 1$. From here we can conclude $p^{t+1}H = k'p^t + p^{t+1}H$, hence the index of $p^{t+1}H$ in p^tH is 1. $\square$

As a corollary of the above lemma, we can give a description of H^0, the connected component of H. Note that by Fact 1.1.20 the definable subgroups of H of finite index are the subgroups nH, and $H^0 = \bigcap_n nH$.

Corollary 3.2.5. $H^0 = \bigcap_{m|a} mH$.

And the next corollary gives a description of the quotients H/nH.

Corollary 3.2.6. $H/nH \cong \mathbb{Z}/g_a(n)\mathbb{Z}$.

Proof. By Lemma 3.2.4, $H/nH = H/g_a(n)H$, so we may assume that a is divisible by m. Let $m|a$. Then we have $i+mH \neq j+mH$ for every distinct $i, j \in \{0, 1, \cdots, m-1\}$, and hence $H/mH \cong \mathbb{Z}/m\mathbb{Z}$. □

3.3 The Subgroup of Infinitesimals

Now we aim to describe H^{00}, the type-connected component of H. First we introduce K, the subgroup of infinitesimals of H, and we show that K is definably isomorphic to an elementary substructure K' of G. Next, we show that K has bounded index in H and hence includes H^{00}. Then we give the description of H^{00} in terms of K.

Consider the case where $a \equiv 0 \ (mod\ n)$ for every $n > 0$. The infinitesimal elements of H (with respect to a) are those elements smaller than all the elements $a/2, a/3, \cdots, a/n, \cdots$.

For the general case, let r_n be the unique natural number for which $a \equiv r_n \ (mod\ n)$ and $0 \leq r_n \leq n-1$, and consider those elements smaller than all the elements $(a-r_n)/n$. The group generated by these infinitesimal elements also includes their additive inverses, the elements bigger than all the elements $a-(a-r_n)/n$. This observation yields us to the following definition.

Definition 3.3.1. *The subgroup of infinitesimals of H is*

$$K = \{x : 0 \leq x < (a-r_n)/n \text{ for all } n\}$$

$$\cup \{x : a-(a-r_n)/n < x < a \text{ for all } n\}.$$

Lemma 3.3.2. *The subgroup K' of G defined by*

$$K' = \{x : -(a - r_n)/n \leq x \leq (a - r_n)/n \text{ for all } n\},$$

is definably isomorphic to K.

Proof. The function $f : K \to K'$, where

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < (a - r_n)/n \text{ for all } n \\ x - a & \text{if } a - (a - r_n)/n < x < a \text{ for all } n \end{cases}$$

is a definable isomorphism between the two groups K and K'. □

Corollary 3.3.3. *For every positive integer n,*

$$K/nK \cong \mathbb{Z}/n\mathbb{Z}.$$

Proof. One can easily check that K' is an elementary substructure of G, hence a model of Presburger arithmetic. □

Now we aim to show that K has bounded index in H. First we need the following:

Remark 3.3.4. *If H is a definable group in a saturated model and K is a subgroup type-defined by a countable intersection of generic sets in H, then K has index at most continuum in H.*

Proof. Let $K = \bigcap_n A_n$. We may assume that $A_i - A_i \subseteq A_{i-1}$. For each A_i, there are finitely many translates of A_i, say $Y_{i,1}, ..., Y_{i,k_i}$, which cover H. Fix b in H. For each i let Y_{i,j_i} be such that $b \in Y_{i,j_i}$. Then the coset $b + K$ is precisely $\bigcap_i Y_{i,j_i}$, because if also $c \in \bigcap_i Y_{i,j_i}$, then $b - c \in A_i - A_i$, for each i.

□

Lemma 3.3.5. *The index of K in H is $2^{\aleph_0}$.*

Proof. Let $A_n = \{x : 0 \leq x < (a - r_n)/n\} \cup \{x : a - (a - r_n)/n < x < a\}$. Then $K = \bigcap_n A_n$.

Note that each set A_n is generic. K is a countable intersection of generic subsets of H. Now we use Remark 3.3.4. □

K is a subgroup of H of bounded index and hence $H^{00} \subseteq K$. These two groups are never equal since K has subgroups of finite index but H^{00} does not. In fact, H^{00} is divisible since $nH^{00} \subseteq H^{00}$ is also a type-definable subgroup of H of bounded index, for every $n > 1$.

By Fact 1.2.5, we have $div(K) = \bigcap_n nK$. Using the fact that H^{00} is divisible, one can show that $H^{00} \subseteq div(K) = \bigcap_n nK$.

We end this section by showing that these two groups are actually equal.

Proposition 3.3.6. $H^{00} = \bigcap_n nK$.

Proof. We only need to show that $Div(K) \subseteq H^{00}$. We use Fact 1.1.25. We point out that $Div(K)$ stabilizes a global f-generic type.

Let $p(x)$ be the type saying that x is divisible by every n, $x > b$ for every $0 < b \in K$ such that $b < (a - r_n)/n$ for all n, and $x < (a - r_n)/n$ for every $n > 0$. p is generic since the formulas in p are of the form $x \equiv 0 \ (mod\ n)$, $x > b,\ b \in K,\ x \leq (a - r_n)/n$, or $x < (a - r_n)/n$. Hence p is f-generic.

$Div(K)$ stabilizes p, since the set

$$\{y : 0 < y < (a - r_n)/n \ \text{ for all } n > 0\}$$

is closed under addition, and also adding by an element of $Div(K)$ does not change the divisibility type. Now we get $Div(K) \subseteq Stab(p) = H^{00}$. □

3.4 Compact Quotients

In the previous sections we introduced H and talked about its type-definable subgroups K, H^0, and H^{00}. In this section we consider the following diagram, and describe the corresponding quotients.

$$
\begin{tikzcd}
& H \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] \arrow[ddr, dash] & & H^0 \arrow[dl, dash] \arrow[ddl, dash] \\
& H^0 \cap K \arrow[d, dash] & \\
& H^{00} &
\end{tikzcd}
$$

First, we determine the structure of H/K and K/H^{00}. Then we describe some other compact quotients.

In the next section we answer our main question about the compact groups H/H^{00} and we classify them using results of this section.

Here we give a summary of what is going to be done in this section. We are going to show that for every choice of a, H/H^{00} is an extension of $\mathbb{S}^1$ by $\hat{\mathbb{Z}}$, i.e. we have the following exact sequence of compact groups:

$$0 \to \hat{\mathbb{Z}} \to H/H^{00} \to \mathbb{S}^1 \to 0.$$

If a is divisible by every n, then the above exact sequence splits and hence we get $H/H^{00} \cong \hat{\mathbb{Z}} \oplus \mathbb{S}^1$.

In the other extreme case, i.e. if a is not divisible by any $n > 1$, H/H^{00} is isomorphic to the inverse limit of the inverse system $(S_i)_{i\in\mathbb{N}}$ where $\mathbb{N}$ is ordered by divisibility, $S_i \cong \mathbb{S}^1$ for every $i \in \mathbb{N}$, and for all $n_1, n_2 \in \mathbb{N}$ with n_1 dividing n_2 there is a map from S_{n_2} to S_{n_1} corresponding to multiplying by n_2/n_1.

In the other cases we both describe H/H^{00} as a certain inverse limit, and as the amalgamated direct product of two subgroups, K/H^{00} and H^0/H^{00}, over their intersection.

Now, we aim to show that for every a, H/K is isomorphic to the circle group. First we need the following two lemmas.

Lemma 3.4.1. *H/K is connected.*

Proof. H has no subgroup of finite index containing K, so H/K has no subgroup of finite index. Hence H/K is a connected topological group. □

Lemma 3.4.2. *There exists $c \in K$ such that $a - c \equiv 0 \ (mod\ n)$ for every positive integer n.*

Proof. We show that the partial type

$$q = \{0 < x < \frac{a - r_n}{n} : n > 0\} \cup \{x \equiv a \ (mod\ n) : n > 0\},$$

is finitely satisfiable in $\mathbb{Z}$. Let $\Sigma \subseteq q$ be finite. Let $\Sigma \cap \{x \equiv a \ (mod\ n) : n > 0\} = \{x \equiv a \ (mod\ n_1), \cdots, x \equiv a \ (mod\ n_l)\}$, and let $m = lcm(n_1, \cdots, n_l)$. Then r_m satisfies Σ. By saturation, we conclude that q is satisfiable in G. Take c to be a realization of q. By definition of q, c will be an element of K with the desired property. □

Now we show that H/K is isomorphic to the circle group.

Proposition 3.4.3. $H/K \cong \mathbb{S}^1$.

Proof. By Lemma 3.4.1, H/K is connected. By Fact 1.2.15 it is enough to show that the torsion subgroup of H/K is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Let $c \in K$ be as in Lemma 3.4.2. $s(a-c)/n$ is a torsion element of H/K for every $0 < s < n$. These elements form a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$. Every other torsion element is in the same class modulo K with one of the elements of $\{s(a-c)/n : 0 < s < n\}$. □

Now we use the material explained in Section 1.2.1 to describe compact groups as inverse limits.

Lemma 3.4.4. *$K/H^{00} \cong \hat{\mathbb{Z}}$.*

Proof. Consider the family $\{nK/H^{00}\}_n$ of compact subgroups of K/H^{00}. We have $\bigcap_n nK/H^{00} = \{0\}$, since by Proposition 3.3.6, $\bigcap_n nK = H^{00}$. Now we apply Fact 1.2.17. Note that by Corollary 3.3.3, $K/nK \cong \mathbb{Z}/n\mathbb{Z}$. $\square$

We can conclude that H/H^{00} is an extension of $\mathbb{S}^1$ by $\hat{\mathbb{Z}}$, as claimed at the beginning of this section.

Corollary 3.4.5. *We have the following exact sequence of compact groups:*

$$0 \to \hat{\mathbb{Z}} \to H/H^{00} \to \mathbb{S}^1 \to 0.$$

Proof. We have the following exact sequence of compact groups:

$$0 \to K/H^{00} \to H/H^{00} \to H/K \to 0.$$

Now, use Proposition 3.4.3 and Lemma 3.4.4. $\square$

The following lemma gives the structure of the compact group H/H^0.

Lemma 3.4.6. *$H/H^0 \cong \varprojlim_{m|a} \mathbb{Z}/m\mathbb{Z}$.*

Proof. We consider the family $\{nH/H^0\}_n$ of compact subgroups of H/H^0 and use Fact 1.2.17. So we get $H/H^0 \cong \varprojlim_n H/nH$. Now, by Corollary 3.2.6 the statement follows. $\square$

Remark 3.4.7. *In particular, if a is divisible by every n, then $H/H^0 \cong \hat{\mathbb{Z}}$.*

Now we describe H^0/H^{00}. This compact group together with K/H^{00} that we have already described, are essential in giving a description of H/H^{00}. Also, we need to describe their intersection, $(H^0 \cap K)/H^{00}$. Note that $(H^0 \cap K)/H^{00}$ is isomorphic to a subgroup of $\hat{\mathbb{Z}}$, since we have the following exact sequence:

$$0 \to (H^0 \cap K)/H^{00} \to K/H^{00} \to K/(H^0 \cap K) \to 0,$$

and also by Lemma 3.4.4, $K/H^{00} \cong \hat{\mathbb{Z}}$.

Lemma 3.4.8. *$H^0 \cap K = \bigcap_{m|a} mK$.*

Proof. By Corollary 3.2.5, $H^0 = \bigcap_{m|a} mH$.

In the general case, $nH \cap K = g_a(n)K$, so if $m|a$ then we have $mK = mH \cap K$. Hence we get $H^0 \cap K = \bigcap_{m|a} mK$. □

In determining the structure of $(H^0 \cap K)/H^{00}$, we need (a corollary of) the following lemma.

Lemma 3.4.9. *$H^0 + nK = g_a(n)H$.*

Proof. First we show that the left side is included in the right side. Let $x \in H^0 + nK$. Then $x = h + nk$ for some $h \in H^0$ and $k \in K$. We have $H^0 \subseteq g_a(n)H$, so there exists $h_1 \in H$ such that $h = g_a(n)h_1$. Now, $x = g_a(n)h_1 + nk$. Also note that $g_a(n)|n$. Hence $x \in g_a(n)H$.

Now we show that right side is included in the left side. Fix $h \in H$. We need to show that $g_a(n)h \in (\bigcap_{m|a} mH) + nK$, or equivalently, there exists $k \in K$ such that $g_a(n)h - nk \in \bigcap_{m|a} mH$. So, we need to show that the following system of congruences has a solution k in K for z.

$$\{g_a(n)h \equiv nz \ (mod\ m)\}_{m|a}.$$

We may assume that m is a power of a prime and $gcd(m, n) = 1$.

The above system is finitely satisfiable in K and hence has a solution by compactness. □

Corollary 3.4.10. $(H^0 \cap K) + nK = g_a(n)K$.

Proof. We have:

$$(H^0 \cap K) + nK = (H^0 + nK) \cap K = g_a(n)H \cap K = g_a(n)K,$$

using Lemma 3.4.9. Note that the last equality holds because $g_a(n)$ divides a. $\square$

In the following proposition, we give a description of $(H^0 \cap K)/H^{00}$ as an inverse limit.

Proposition 3.4.11. *$(H^0 \cap K)/H^{00} \cong \varprojlim_{n\in\mathbb{N}} g_a(n)\mathbb{Z}/n\mathbb{Z}$ where $\mathbb{N}$ is ordered by divisibility, and the maps between $g_a(n)\mathbb{Z}/n\mathbb{Z}$'s are the natural maps.*

Proof. Consider the family $\{(H^0 \cap nK)/H^{00}\}_n$ and use Fact 1.2.17. This gives:

$$(H^0 \cap K)/H^{00} \cong \varprojlim_{n} (H^0 \cap K)/(H^0 \cap nK).$$

Using the second isomorphism theorem for topological groups we have:

$$(H^0 \cap K)/(H^0 \cap nK) \cong ((H^0 \cap K) + nK)/nK$$

$$\cong g_a(n)K/nK \text{ (by Corollary 3.4.10)}$$

$$\cong g_a(n)\mathbb{Z}/n\mathbb{Z} \text{ (by Corollary 3.3.3).}$$

$\square$

Recall the definition of the group of p-adic integers $\mathbb{Z}_p$ from Section 1.2.1, and recall that $\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$.

In the following remark, we give an explicit description of $(H^0 \cap K)/H^{00}$ as a subgroup of $\hat{\mathbb{Z}}$.

Remark 3.4.12. *Let P_1 be the set of all primes that do not divide a, and P_2 the set of all primes that divide a but do not infinitely divide a. Then*

$$(H^0 \cap K)/H^{00} \cong \bigoplus_{p \in P_1} \mathbb{Z}_p \oplus \bigoplus_{q \in P_2} q^{k_q} \mathbb{Z}_q,$$

where k_q is so that q^{k_q} divides a and q^{k_q+1} does not divide a.

We need the following variant of Lemma 3.4.9 and Corollary 3.4.10, in order to give the structure of H^0/H^{00}.

Lemma 3.4.13. *If $n_1|n_2$ then*

$$((H^0 \cap K) \cap n_1 K) + n_2 K = lK,$$

where $l = lcm(n_1, g_a(n_2))$.

Proof. First note that $n_1|n_2$ and $g_a(n_2)|n_2$, so $l|n_2$ and hence $n_2 K \subseteq lK$. Also, $g_a(n_2)|a$, so $(\bigcap_{m|a} mK) \subseteq g_a(n_2)K$. So we get

$$(\bigcap_{m|a} mK) \cap n_1 K \subseteq g_a(n_2)(K) \cap n_1 K = lK.$$

For the other direction, we need to show that the following system of congruences has a solution for z in K:

$$\{lk \equiv n_2 z \ (mod\ m)\}_{m|a \ or \ m=n_1}$$

$n_1|l$ and $n_1|n_2$, so we can remove the condition $m = n_1$. In fact, if we have $lK \subseteq (\bigcap_{m|a} mK) + n_2 K$, then we automatically have $lK \subseteq (\bigcap_{m|a} mK) \cap n_1 K + n_2 K$.

By Lemma 3.4.9, $(\bigcap_{m|a} mK) + n_2 K = g_a(n_2)K$, and since $g_a(n_2)|l$, we have $lK \subseteq g_a(n_2)K$. $\square$

Remark 3.4.14. *We have*

$$H^0/(H^0 \cap nK) \cong g_a(n)H/nK.$$

In fact, the following two exact sequences are isomorphic:

$$0 \to (H^0 \cap K)/(H^0 \cap nK) \to H^0/(H^0 \cap nK) \to H^0/(H^0 \cap K) \to 0,$$

$$0 \to g_a(n)K/nK \to g_a(n)H/nK \to H/K \to 0.$$

Proof. We have:

$$H^0/(H^0 \cap nK) \cong (H^0 + nK)/nK = g_a(n)H/nK.$$

The first isomorphism holds by the second isomorphism theorem for topological groups, and the second holds using Lemma 3.4.9. □

For any a, H^0/H^{00} is a connected compact group and hence an inverse limit of compact connected Lie groups, by Fact 1.2.15.

In the following proposition, we describe the structure of these compact groups, which depends on the divisibility type of a. Note that in the case where a is not divisible by any $n > 1$, $H^0 = H$ and hence $H^0/H^{00} = H/H^{00}$.

Proposition 3.4.15. $H^0/H^{00} \cong \varprojlim_{n\in\mathbb{N}} g_a(n)H/nK$. *For* $n_1, n_2 \in \mathbb{N}$ *with* $n_1|n_2$, *the map* $g_a(n_2)/n_2K \to g_a(n_1)/n_1K$ *has kernel isomorphic to* $l\mathbb{Z}/n_2\mathbb{Z}$, *where* $l = lcm(g_a(n_2), n_1)$.

Proof. Consider the family $\{(H^0 \cap nK)/H^{00}\}_n$ of subgroups of H^0/H^{00}. Using Fact 1.2.17 we get:

$$H^0/H^{00} \cong \varprojlim_{n} H^0/(H^0 \cap nK).$$

Note that $H^0/(H^0 \cap nK) \cong g_a(n)H/nK$ by Remark 3.4.14.

Now we describe the maps $g_a(n_2)/n_2K \to g_a(n_1)/n_1K$ for $n_1|n_2$. Consider the following exact sequence:

$$0 \to (H^0 \cap n_1K)/(H^0 \cap n_2K) \to H^0/(H^0 \cap n_2K) \to H^0/(H^0 \cap n_1K) \to 0.$$

The groups $(H^0 \cap n_1K)/(H^0 \cap n_2K)$ are finite and are the kernels of the maps $g_a(n_2)/n_2K \to g_a(n_1)/n_1K$ in the inverse limit.

Now we compute these kernels. We have:

$$(H^0 \cap n_1K)/(H^0 \cap n_2K) = ((\bigcap_{m|a} mK) \cap n_1K)/((\bigcap_{m|a} mK) \cap n_2K).$$

By the second isomorphism theorem for topological groups we get:

$$((\bigcap_{m|a} mK) \cap n_1K)/((\bigcap_{m|a} mK) \cap n_2K) \cong (n_2K + ((\bigcap_{m|a} mK) \cap n_1K))/n_2K.$$

Now by Lemma 3.4.13,

$$n_2K + ((\bigcap_{m|a} mK) \cap n_1K) = lK.$$

where $l = lcm(g_a(n_2), n_1)$. So,

$$((\bigcap_{m|a} mK) \cap n_1K)/((\bigcap_{m|a} mK) \cap n_2K) \cong lK/n_2K.$$

□

The following lemma is crucial for our main result. It gives the structure of the compact groups H/nK.

Lemma 3.4.16. *For every $n > 0$, we have*

$$H/nK \cong \mathbb{Z}/g_a(n)\mathbb{Z} \oplus \mathbb{S}^1.$$

Proof. Consider the following exact sequence:

$$0 \to g_a(n)H/nK \to H/nK \to H/g_a(n)H \to 0.$$

We have

$$g_a(n)H/nK \cong H/K \cong \mathbb{S}^1,$$

by Remark 3.4.14 and Proposition 3.4.3, and

$$H/nH \cong \mathbb{Z}/g_a(n)\mathbb{Z},$$

by Corollary 3.2.6.

Consider the inclusion homomorphism from $H/g_a(n)H$ to H/nK. Since it is continuous and the composition with $H/nK \to H/g_a(n)H$ is identity of $H/g_a(n)H$, the above exact sequence splits as a sequence of topological groups and the result follows. □

Remark 3.4.17. *In particular, if a is divisible by n, then $H/nK \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{S}^1$, and if no prime divisor of n divides a, then $H/nK \cong \mathbb{S}^1$.*

Before getting to the description of H/H^{00} in the general case in the next section, we describe H/H^{00} in one of the extreme cases.

By Corollary 3.3.6, we have the following exact sequence:

$$0 \to \hat{\mathbb{Z}} \to H/H^{00} \to \mathbb{S}^1 \to 0.$$

Now we restrict our attention to the case where a is divisible by every n.

In this case, $H^{00} = H^0 \cap K = \bigcap_n nK$ and hence we have

$$H^0/H^{00} = H^0/(H^0 \cap K).$$

Note that by the second isomorphism theorem for topological groups,

$$H^0/(H^0 \cap K) \cong (K + H^0)/K.$$

We have $K + H^0 = H$ by Lemma 3.4.9, and hence

$$H^0/H^{00} = H^0/(H^0 \cap K) \cong H/K \cong \mathbb{S}^1.$$

The last congruence holds by Proposition 3.4.3. In this case we also have

$$H/H^0 \cong \hat{\mathbb{Z}}.$$

So we have the following exact sequence:

$$0 \to H^0/H^{00} \to H/H^{00} \to H/H^0 \to 0.$$

The map $H^0/H^{00} \to H/H^{00}$ is inclusion hence continuous. The composition of this map with the map $H/H^{00} \to \mathbb{S}^1$ gives the identity of $\mathbb{S}^1$.

Hence in the case where a is divisible by every n, the exact sequence

$$0 \to \hat{\mathbb{Z}} \to H/H^{00} \to \mathbb{S}^1 \to 0$$

splits as a sequence of compact groups. We conclude that in this case,

$$H/H^{00} \cong \hat{\mathbb{Z}} \oplus \mathbb{S}^1.$$

In this section we classify the compact groups H/H^{00} for different a with different divisibility types.

First we describe each of these groups as a certain inverse limit using Fact 1.2.17 and Lemma 3.4.16. Then we describe them as the amalgamated direct product of the subgroups K/H^{00} and H^0/H^{00} over their intersection.

Theorem 3.5.1. *We have*

$$H/H^{00} \cong \varprojlim_{n\in\mathbb{N}} Y_n,$$

where the set $\mathbb{N}$ *is ordered by divisibility,* $Y_n = \mathbb{Z}/g_a(n)\mathbb{Z} \oplus \mathbb{S}^1$ *for every* $n \in \mathbb{N}$*, and for every* $n_1, n_2 \in \mathbb{N}$ *with* n_1 *dividing* n_2*, the map from* Y_{n_2} *to* Y_{n_1} *is the map sending* $(x + g_a(n_2)\mathbb{Z}, s)$ *to* $(x + g_a(n_1)\mathbb{Z}, (g_a(n_2)/g_a(n_1))s)$*.*

Proof. Consider the family $\{nK/H^{00}\}_n$ of closed subgroups of H/H^{00}. By Fact 1.2.17 we get:

$$H/H^{00} \cong \varprojlim_{n} H/nK.$$

Now, apply Lemma 3.4.16. □

At the end of Section 3.4, we talked about the extreme case where a is divisible by every n.

For the other extreme case where a is not divisible by any $n > 1$, H/H^{00} is isomorphic to the inverse limit of the inverse system $(S_i)_{i\in\mathbb{N}}$ where $\mathbb{N}$ is ordered by divisibility, $S_i \cong \mathbb{S}^1$ for every $i \in \mathbb{N}$, and for all $n_1, n_2 \in \mathbb{N}$ with n_1 dividing n_2 there is a map from S_{n_2} to S_{n_1} corresponding to multiplying by n_2/n_1, as we pointed out at the beginning of section 3.4.

Recall the definition of (topological) amalgamated direct products from Section 1.2.2.

Theorem 3.5.2. *H/H^{00} is the amalgamated direct product of $K/H^{00}(\cong \hat{\mathbb{Z}})$ and H^0/H^{00} over their intersection $L = (H^0 \cap K)/H^{00}$:*

$$H/H^{00} \cong \hat{\mathbb{Z}} \times_L H^0/H^{00}.$$

Proof. H/H^{00} is generated algebraically by $H^0/H^{00} \cup K/H^{00}$ since $H^0 + K = H$. Suppose that D is a topological group and $\Phi_1 : H^0/H^{00} \to D$ and $\Phi_2 : K/H^{00} \to D$ are continuous homomorphisms which agree on $(H^0 \cap K)/H^{00}$ and also assume $\Phi_2(K/H^{00}) \subseteq C_D(\Phi_1((H^0 \cap K)/H^{00}))$. Then by Fact 1.2.19 there exists a homomorphism $\Phi : H/H^{00} \to D$ extending Φ_1 and Φ_2. Φ is continuous since by Remark 1.2.20, it is the sum of two continuous functions. □

CHAPTER 4

DEFINABLE COMPACTIFICATIONS

4.1 The Bohr Compactification

In this section we recall some facts about the Bohr compactification of a topological group, in particular a locally compact abelian group. Later in this chapter we restrict our attention to the case of discrete abelian groups, and after recalling the notion of a definable compactification, we discuss when topological and definable compactifications coincide.

Again, all the topological groups we are considering are Hausdorff, unless stated otherwise.

First, we recall the definition of a compactification of a topological group. For more explanation see [11] or [31].

Definition 4.1.1. *A compactification of a topological group G is a compact topological group C together with a continuous homomorphism $f : G \to C$ with dense image.*

Also recall that the universal compactification of a topological group is called the Bohr compactification, and is denoted by bG.

Definition 4.1.2. *If $f : G \to C$ is a compactification which is universal among compactifications of G, i.e. whenever $g : G \to K$ is another compactification there exists a unique homomorphism $h : C \to K$ such that $g = h \circ f$, then $f : G \to C$ is called the Bohr compactification of G.*

The Bohr compactification of any topological group G exists and is unique up to isomorphism.

Remark 4.1.3. *For a non-abelian topological group G, bG might not be a very useful object. It might even be trivial for non-trivial G.*

Recall from Definition 1.2.8 that for an abelian group G, $\hat{G}$ (the dual of G) is the group of continuous characters of G. One can define a topology on $\hat{G}$, related to the topology on G, turning $\hat{G}$ into a topological group. This topology is defined as follows.

For any open subset $V \subseteq \mathbb{T}$ and any compact subset $K \subseteq G$, define

$$U(K, V) = \{\gamma \in \hat{G} : \gamma(K) \subseteq V\}$$

and let this act as a sub-basis for a topology on $\hat{G}$. This topology is known as the compact-open topology on $\hat{G}$.

In the remainder of this section we restrict our attention to locally compact abelian groups for which nice facts about the Bohr compactification hold.

Definition 4.1.4. *A topological group which is locally compact and abelian is called an LCA group.*

In the case where G is an LCA group, the Bohr compactification of G can be described using characters. For proofs of the following facts see [22].

Remark 4.1.5. *If G is an LCA group, the compact-open topology makes $\hat{G}$ an LCA group.*

Fact 4.1.6. *(Pontryagin duality theorem) Let G be an LCA group. Then the map $\tau : G \to \hat{\hat{G}}$ such that $\tau(g)(\gamma) = \gamma(g)$ for $g \in G$, $\gamma \in \hat{G}$ is an isomorphism of topological groups.*

Remark 4.1.7. *For all $g_1, g_2 \in G$ with $g_1 \neq g_2$, there is a character $\gamma \in \hat{G}$ such that $\gamma(g_1) \neq \gamma(g_2)$. As a consequence, if G is non-trivial then $\hat{G}$ is non-trivial. We say that the set of continuous characters separates points.*

For an LCA group G which is compact, $\hat{\hat{G}}$ ($\cong G$) is also compact. In some sense, $\hat{\hat{G}}$ is compact because $\hat{G}$ is discrete. In general, if we take $\hat{G}$ as a group assigned with the discrete topology (denoted $\hat{G}_d$), then its dual will be compact. For any LCA group G, the dual of $\hat{G}_d$ is the Bohr compactification of G.

G maps injectively into bG in a canonical way. The following fact illustrates the nature of this map.

Fact 4.1.8. *Let G be an LCA group and let $\sigma : G \to bG$ be the natural map from G into bG defined by $\sigma(g)(\gamma) = \gamma(g)$ for $g \in G$, $\gamma \in \hat{G}$. Then σ is an injective continuous homomorphism of groups, and $\sigma(G)$ is dense in bG.*

4.2 Discrete Abelian Groups

Throughout this section we assume that G is a discrete abelian topological group.

Remark 4.2.1. $\hat{G} = Hom(G, \mathbb{T})$*, since every map from a discrete group to a topological space is continuous.*

We describe the compact-open topology on $\hat{G}$. Recall that the sets $U(K, V)$ for $V \subseteq \mathbb{T}$ open and $K \subseteq G$ compact, form a sub-basis for the compact-open topology on $\hat{G}$. Note that every compact subset of G is finite. Let $K = \{g_1, \cdots, g_m\}$. We have:

$$U(K, V) = \{\gamma \in \hat{G} : \gamma(K) \subseteq V\}$$

$$= \{\gamma \in Hom(G, \mathbb{T}) : \gamma(K) \subseteq V\}$$

$$= \bigcap_{i=1}^{m} \{\gamma \in Hom(G, \mathbb{T}) : \gamma(g_i) \in V\}.$$

Remark 4.2.2. $bG = Hom(Hom(G, \mathbb{T}), \mathbb{T})$.

Now we describe the compact-open topology on bG. Note that bG is the dual of $\hat{G}_d$. Since $\hat{G}_d$ is discrete, a subset of $\hat{G}_d$ is compact iff it is finite. The sets of the form

$$U(K,V) = \bigcap_{i=1}^{n} \{\gamma \in Hom(Hom(G,\mathbb{T}),\mathbb{T}) : \gamma(f_i) \in V\},$$

for $V \subseteq \mathbb{T}$ open and $K = \{f_1, \cdots, f_n\} \subseteq Hom(G,\mathbb{T})$ form a sub-basis for the compact-open topology on bG.

In the following example we describe the compact open topology on the dual and the Bohr compactification of the discrete group of integers.

First, note that $Hom(\mathbb{Z},\mathbb{T}) \cong \mathbb{T}$, and hence $b\mathbb{Z} \cong Hom(\mathbb{T},\mathbb{T})$. We identify $Hom(\mathbb{Z},\mathbb{T})$ with $\mathbb{T}$, and $f \in Hom(\mathbb{Z},\mathbb{T})$ with $f(1) \in \mathbb{T}$.

Example 4.2.3. *Consider the discrete abelian group $\mathbb{Z}$ of integers.*

The sets of the form

$$\bigcap_{i=1}^{n} \{\gamma \in Hom(\mathbb{T},\mathbb{T}) : \gamma(t_i) \in V\}$$

for $t_1, \cdots, t_n \in \mathbb{T}$ and $V \subseteq \mathbb{T}$ open, form a sub-basis for the compact-open topology on $b\mathbb{Z}$.

A basic open subset of $b\mathbb{Z}$ is a finite intersection of sets of the form

$$\{\gamma \in Hom(\mathbb{T},\mathbb{T}) : \gamma(\{t_1, \cdots t_m\}) \subseteq V\},$$

where $t_1, \cdots t_m \in T$ and $V \subseteq T$ is open, i.e. a set of the form

$$\bigcap_{j=1}^{m} \bigcap_{i=1}^{n_j} \{\gamma \in Hom(\mathbb{T},\mathbb{T}) : \gamma(t_i^j) \in V_i\},$$

where $V_1, \cdots, V_m \subseteq \mathbb{T}$ are open and $t_i^j \in \mathbb{T}$.

Note that the identity of $b\mathbb{Z}$ is the homomorphism $\mathbf{1} : \mathbb{T} \to \mathbb{T}$, $x \mapsto 1$. We have the following:

$\mathbf{1} \in \bigcap_{j=1}^{m} \bigcap_{i=1}^{n_j} \{\gamma \in Hom(\mathbb{T},\mathbb{T}) : \gamma(t_i^j) \in V_i\}$ *iff* $1 \in V_j$ *for every* $j = 1, \cdots, m$.

Remark 4.2.4. *A basic neighborhood of identity in bG is a set of the form*

$$\bigcap_{j=1}^{m}\bigcap_{i=1}^{n_j}\{\gamma \in Hom(Hom(G,\mathbb{T}),\mathbb{T}) : \gamma(t_i^j) \in V_j\},$$

where $t_i^j \in \mathbb{T}$*, and* $V_1, \cdots, V_m \subseteq \mathbb{T}$ *are basic open neighborhoods of identity.*

Proof. Similar to Example 4.2.3. □

4.3 Topological vs. Definable Compactifications

In this section, first we recall some definitions and facts about definable compactifications from [10]. Then we formulate a question that we answer in the next section.

Let $\overline{M}$ and M be as in Section 1.1. By a set definable in M we mean a set definable in M^n for some $n > 1$.

Definition 4.3.1. *Let Y be a definable set in M and C a compact space. By a definable map f from Y to C we mean a map f such that for any disjoint closed subsets C_1, C_2 of C there is a definable subset Y' of Y such that $f^{-1}(C_1) \subseteq Y'$ and $Y' \cap f^{-1}(C_2) = \emptyset$.*

The following is a reformulation of the definition, in case we are working in a saturated model.

Remark 4.3.2. *Let X be a definable set in $\overline{M}$. Suppose C is a compact space, and $f : X \to C$. Then f is a definable map, if for any closed subset D of C, $f^{-1}(D)$ is type-definable over M.*

The notion of a definable map from a definable group to a topological space can be used to define an analogue of topological compactification for definable groups, as follows.

Definition 4.3.3. *Let G be a group definable in M. By a definable compactification of G we mean a definable homomorphism from G to a compact group C with dense image.*

For a set Y definable in M, we set $Y^* = Y(\overline{M})$.

Fact 4.3.4. *Suppose Y is a definable set in M. Suppose $f : Y \to C$ is a definable map from Y to a compact space C. Then f extends uniquely to an M-definable map $f^* : Y^* \to C$.*

Remark 4.3.5. *In the proof of the above fact, the definition of f^* is as follows. Let $c \in Y^*$ and let $p(y) = tp(c/M)$. For $\varphi(y)$ a formula in p, let $\overline{f(\varphi(M))}$ denote the closure of $f(\varphi(M))$ in C. One can show that $\bigcap_{\varphi\in p} \overline{f(\varphi(M))}$ is a singleton in C. $f^*(c)$ is defined to be the unique element in $\bigcap_{\varphi\in p} \overline{f(\varphi(M))}$.*

Fact 4.3.6. *Let G be a group definable in M. Then there is a (unique) universal definable compactification of G, and it is precisely $G^*/(G^*_M)^{00}$ (where the homomorphism from G to $G^*/(G^*_M)^{00}$ is that induced by the identity embedding of G in G^*).*

Remark 4.3.7. *If G is a definable group with all of its subsets definable, then the universal definable compactification of G and the Bohr compactification of G considered as a discrete topological group coincide.*

Example 4.3.8. *If M is any expansion of $(\mathbb{Z}, +)$ in which every subset of $\mathbb{Z}$ is definable, then $\mathbb{Z}^*/(\mathbb{Z}^*_M)^{00} \cong b\mathbb{Z}$. For example if M is the group of integers with a predicate for every subset of $\mathbb{Z}$.*

Now one can ask the following question.

Question 4.3.9. *What is a sufficient set of predicates to add to $(\mathbb{Z}, +)$ to get $b\mathbb{Z}$ as the universal definable compactification in the resulting structure?*

We give an answer to this question in the next section.

Recall from Fact 4.3.6 that the universal definable compactification of G is related to the type-connected component of G. In [2] an expansion M of $(\mathbb{Z},+)$ by one predicate P is constructed for which the type-connected component is not equal to the connected component, but the type-connected component is not computed. Note that this implies that $Th(\mathbb{Z},+,P)$ is not stable.

We also know that for any expansion of $(\mathbb{Z},+)$ in which every subset of $\mathbb{Z}$ is definable, the universal definable compactification is $b\mathbb{Z}$. See Remark 4.3.7.

For an abelian group G, we ask in what expansions is the universal definable compactification equal to bG.

In the following proposition, we compute the type-connected component of G, in case M is a model in which every subset of G is definable. Our goal is to find out which subsets of G are needed in order for $(G^*)^{00}_M = ker(\sigma^*)$ to be type-definable in a structure.

Proposition 4.4.1. *Let G be an abelian group and let M be an expansion of $(G,+)$ in which every subset of G is definable. Then we have*

$$(G^*)^{00}_M = \bigcap_U (\sigma^*)^{-1}(U),$$

where the intersection is over all the basic neighborhoods U of identity in bG, and $\sigma^* : G^* \to bG$ is the unique extension of the natural homomorphism $\sigma : G \to bG$.

Proof. Let $c \in G^*$. Recall that by Remark 4.3.5 we have $\sigma^*(c) = \bigcap_{\varphi\in p} \overline{\sigma(\varphi(M))}$, where $p(y) = tp(c/M)$, and $\overline{\sigma(\varphi(M))}$ denotes the closure of $\sigma(\varphi(M))$ in bG. Now we compute the kernel of σ^*. Note that $ker(\sigma^*) = (G^*)^{00}_M$ and $G^*/(G^*)^{00}_M \cong bG$.

$\sigma^*(c) = \mathbf{1}$ iff for all $\varphi \in tp(c/M)$, $\mathbf{1} \in \overline{\sigma(\phi(M))}$. Note that for $\varphi \in tp(c/M)$ we have

$$\sigma(\varphi(M)) = \{\sigma(n) \in Hom(Hom(G,\mathbb{T}),\mathbb{T}) : n \in \varphi(M)\},$$

and $\mathbf{1} \in \overline{\sigma(\varphi(M))}$ iff every basic neighborhood of identity in bG intersects $\sigma(\varphi(M))$.

Let $O_{t,V} = \{\gamma \in Hom(Hom(G,\mathbb{T}),\mathbb{T}) : \gamma(t) \in V\}$. By Remark 4.2.4, any basic neighborhood of identity in bG is of the form $\bigcap_{j=1}^{m}\bigcap_{i=1}^{n_j} O_{t_i^j,V_j}$.

For $\sigma(\varphi(M))$ to intersect $\bigcap_{j=1}^{m}\bigcap_{i=1}^{n_j} O_{t_i^j,V_j}$, there must exist $n \in \varphi(M)$ so that

$$\sigma(n) \in \bigcap_{j=1}^{m}\bigcap_{i=1}^{n_j} O_{t_i^j,V_j}.$$

$c \in ker(\sigma^*)$ iff for every $\varphi \in tp(c/M)$, and for every $V_1, \cdots, V_m \subseteq \mathbb{T}$ open all containing 1,

$$\bigcap_{j=1}^{m}\bigcap_{i=1}^{n_j}(\sigma^{-1}(O_{t_i^j,V_j}) \cap \varphi(M)) \neq \emptyset.$$

So $c \in ker(\sigma^*)$ iff for every $\varphi \in tp(c/M)$,

$$(\bigcap_{t,V}(\sigma^{-1}(O_{t,V}))^*) \cap (\varphi(M))^* \neq \emptyset,$$

where $t \in \mathbb{T}$ and $V \subseteq \mathbb{T}$ is a basic open neighborhood of identity, and hence

$$ker(\sigma^*) = \bigcap_{t\in\mathbb{T}, V\text{open}} (\sigma^{-1}(O_{t,V}))^*,$$

where $t \in \mathbb{T}$ and $V \subseteq \mathbb{T}$ is a basic open neighborhood of identity. □

Theorem 4.4.2. *Let M be the expansion of $(G,+)$ by the predicates $\sigma^{-1}(O_{t,V})$ for $t \in \mathbb{T}$ and $V \subseteq \mathbb{T}$ basic open neighborhood of identity. Then the universal definable compactification of M is bG.*

Proof. Since all the sets $\sigma^{-1}(O_{t,V})$ are definable in M, the subgroup $\bigcap_{t,V}(\sigma^{-1}(O_{t,V}))^*$ is type-definable in M.

By proof of Proposition 4.4.1, we know that $\mathbb{Z}^*/\bigcap_{t,V}(\sigma^{-1}(O_{t,V}))^*$ is isomorphic to $b\mathbb{Z}$. Hence $(\mathbb{Z}^*)^{00}_M$ is $\bigcap_{t\in\mathbb{T},V\text{open}}(\sigma^{-1}(O_{t,V}))^*$, and the result follows. □

In the following example we consider the special case of the group of the integers.

Example 4.4.3. *Since $Hom(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T}$, we have*

$$O_{t,V} = \{\gamma \in Hom(\mathbb{T}, \mathbb{T}) : \gamma(t) \in V\}.$$

For $t \in \mathbb{T}$, let $f_t : \mathbb{Z} \to \mathbb{T}$ be defined by $n \mapsto t^n$. One can easily see that $\sigma^{-1}(O_{t,V}) = f_t^{-1}(V)$.

Corollary 4.4.4. *Let M be the expansion of $(\mathbb{Z}, +)$ by the predicates $f_t^{-1}(V)$ for $t \in \mathbb{T}$ and $V \subseteq \mathbb{T}$ basic open neighborhood of identity. Then the universal definable compactification of M is $b\mathbb{Z}$.*

One can ask questions about the properties of the above structure, for example the following:

Question 4.4.5. *Is the theory $Th(\mathbb{Z}, +, f_t^{-1}(V))_{t,V}$ (where $f_t : \mathbb{Z} \to \mathbb{T}, n \mapsto t^n$) NIP?*

Note that $Th(\mathbb{Z}, +, f_t^{-1}(V))_{t,V}$ is an expansion of $Th(\mathbb{Z}, +, P)$ as mentioned from [2] at the beginning of this section, and hence $Th(\mathbb{Z}, +, f_t^{-1}(V))_{t,V}$ is not stable.

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